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Global Convergence in an Overlapping Generations Model with Two-Sided Altruism¹

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Abstract

We study the evolutionary dynamics of an overlapping generations economy with two-sided altruism, where each generation cares about the utilities of its parental generation, its offspring, and its own. We assume that the young and old generations in each period act jointly, as a central planner. We then introduce the concepts of a two-sided altruistic path and a two-sided altruistic stationary path. Then, we go on to prove that the former always converges to the latter, in the two-sided altruistic model. This stationary capital level is higher than the steady-state level in the model with a constant discount factor.

Keywords: Two-sided altruism, Global convergence, Euler equation, Principle of optimality.

JEL classification: C61, D91, O41.

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1. Introduction

Since the studies by Muth (1961) and Lucas (1972), the common assumption in neoclassical models is that economic agents form rational expectations and make no forecasting mistakes. In this framework, agents have full knowledge of their economic environment and take perfect account of all the reactions from other agents.³

Although individuals are destined to die, families are more permanent and could live infinitely. If individuals are altruistically linked to their own children, they might, in turn, have altruistic concerns for all of their descendants. In such a model, each generation cares about not only its own consumption, but also the utilities of its successors; thus, an economic agent could be assumed to use all available information to make rational decisions over an infinite time horizon. Barro (1974) found that a chain of positive gifts from each generation to the preceding generation validates the debt neutrality result. Consequently, an altruistic model is similar to the infinite household model. (See Ramsey (1928), Cass (1965) and Koopmans (1965) for the optimal growth model, and also Michel and Thibault (2007) and Lambrecht et al. (2006) for the development of overlapping generations models with one-sided altruism.)

The one-sided altruism model could be extended to that of two-sided altruism, in which each generation cares about not only its own consumption and the utilities of its successors, but also the utility of its own parental generation. Kimball (1987) and Hori and Kanaya (1989) explicitly presented a dynastic utility for a two-sided altruism model, where an infinite stream of altruistic weights extends to both ancestors and descendants.⁴ Later, Hori (1997) introduced capital accumulation into the overlapping generations model with two-sided altruism and took into consideration the subgame perfect Nash equilibrium of the intergenerational game. However, the dynamic property of equilibrium paths has not been extensively examined, except for the special case without time-inconsistency.

Herein, instead of using the intergenerational game-theoretic approach, we examine evolutionary dynamic allocation by simply asking what would happen in the sequence of capital stocks inherited from the parental generation to the next generation if the young and the old in each period jointly behave as a central planner in allocating capital resources over an infinite time horizon. Then, only the capital stock of the first period inherited by the next generation would be actually realized. Consequently, under two-sided altruism, generations have naive expectations and their decisions are time-inconsistent, as the young and the old in the next period do not make decisions as expected by those in the previous period.⁵

Aoki and Nishimura (2014) studied the solution dynamics for the problem in which each generation's period-wise utility is in the Cobb-Douglas form, and in which the planner makes decisions on all allocations in the initial period. In contrast, we assume that the utility function of each generation in each period has a more general form and that not only the young and old generations at the origin but also every successive generation makes decisions

³There exist extensive works in which the economic agents are assumed to be mutually dependent and to make decisions that are not necessarily consistent with rational choice. See Lane, Malerba, Maxfield, and Orsenigo (1996) and Hommes (2009).

⁴See also Abel (1987), Altig and Davis (1993), Blackburn and Cipriani (2005), and Raut (2006) for other examples of two-sided altruism.

⁵In relation to the works by Lane et al. (1996) and Hommes (2009), their "agents" comprise the "generations" in our model, where rationality is also bounded.

in each period. Furthermore, we introduce the concepts of a two-sided altruistic path and a two-sided altruistic stationary path. We then show that two-sided altruistic paths always converge to a two-sided altruistic stationary path.

This paper is organized as follows: Section 2 investigates the basic properties of the overlapping generations model with two-sided altruism; Section 3 derives the dynamic properties of optimal solutions; Section 4 presents the evolutionary dynamics of two-sided altruistic paths and the main results; and Section 5 concludes the study.

2. Model and basic results

In this section, we discuss the basic framework of the overlapping generations model with altruism. Let c_t^y and c_{t+1}^o be consumptions during young and old adulthood of the generation who live in periods t and $t + 1$, and $u(c_t^y)$ and $u(c_{t+1}^o)$ be the corresponding utilities, respectively. The whole life utility of generation t (young adults in period t) is defined as $u_t = u(c_t^y) + u(c_{t+1}^o)$. The total consumption of the young and the old, who live together in period t , is equal to $c_t^y + c_t^o$. $u(\cdot)$ has the following properties:

U1: u is strictly concave and continuous in $c \geq 0$, and smooth for $c > 0$.

U2: $u'(c) > 0$ for $c > 0$.

U3: $\lim_{c \rightarrow 0} u'(c) = \infty$, $\lim_{c \rightarrow \infty} u'(c) = 0$.

Here we impose an additional restriction on the utility function $u(\cdot)$.

U4: $\sigma(x) \equiv \frac{u'(x)u''(x)}{\{u''(x)\}^2}$ is non-decreasing for x .⁶

As in Michel and Thibault (2007), one-sided altruism may be described in the form,

$$V_t = u_t + bV_{t+1}, \quad (1)$$

where V_t is the dynastic utility of generation t . Since the generation is altruistically linked with their own children, it has, in turn, altruistic concerns for all of their descendants, and V_t may be written as

$$V_t = u_t + \sum_{s=1}^{\infty} b^s u_{t+s}. \quad (2)$$

In addition, if each generation cares about the utility of its own parental generation as well as its own consumption and the utilities of its successors, then we would have the two-sided dynastic utility. Following Kimball (1987), we consider the two-sided dynastic utility of generation t (young adults in period t), of the form,

$$V_t = aV_{t-1} + u_t + bV_{t+1}, \quad (3)$$

with $a > 0$, $b > 0$, and $1 > a + b$. Here a is the weight for V_{t-1} , the dynastic utility of the parental generation, and b is the one for V_{t+1} , the dynastic utility of the offsprings.

Similar to Kimball (1987) and Hori (1997), V_t is represented as the weighted sum of the utilities for all generations,

$$V_t = \sum_{s=-\infty}^{\infty} \gamma_s u_{t+s}, \quad (4)$$

⁶U4 is used to guarantee the strict concavity of the total utility of each generation defined in (12) below. The CARA utility function $u(x) = -e^{-ax}$ as well as Cobb-Douglas utility functions satisfy U4.

so that the altruistic weight γ_s for each individual generation depends only on the time distance s from the current generation in period t . Substituting (4) into (3) gives

$$V_t = \sum_{s=1}^{\infty} \alpha^s u_{t-s} + u_t + \sum_{s=1}^{\infty} \beta^s u_{t+s}, \quad (5)$$

with $\alpha = \frac{1-\sqrt{1-4ab}}{2b}$ and $\beta = \frac{1-\sqrt{1-4ab}}{2a}$. Note that $1 > a + b$ implies $1/4 \geq ab$, $1 > \alpha$ and $1 > \beta$, and that $\gamma_s = \beta^s$ for $s \geq 1$, $\gamma_0 = 1$, and $\gamma_s = \alpha^{-s}$ for $-1 \geq s$. This expression is called an “eternity solution” in Hori and Kanaya (1989).

Plugging $u_t = u(c_t^y) + u(c_{t+1}^o)$ into (5) provides the two-sided dynastic utility as

$$V_t = \sum_{s=1}^{\infty} \alpha^s (u(c_{t-s}^y) + u(c_{t+1-s}^o)) + (u(c_t^y) + u(c_{t+1}^o)) + \sum_{s=1}^{\infty} \beta^s (u(c_{t+s}^y) + u(c_{t+1+s}^o)). \quad (6)$$

The utilities prior to period t can be regarded as a term $d_t = \sum_{s=2}^{\infty} \alpha^s (u(c_{t-s}^y) + u(c_{t+1-s}^o)) + \alpha u(c_{t-1}^y)$. Hence, V_t may be rewritten as

$$\begin{aligned} V_t &= d_t + \alpha u(c_t^o) + \sum_{s=0}^{\infty} \beta^s u_{t+s} \\ &= d_t + \alpha u(c_t^o) + \sum_{s=0}^{\infty} \beta^s (u(c_{t+s}^y) + u(c_{t+1+s}^o)). \end{aligned} \quad (7)$$

This expression is called a “mortality solution” in Hori and Kanaya (1989). Hereafter, we denote the current period by $t = 0$. Then the two-sided dynastic utility of the young generation in the current period may be rewritten as

$$\begin{aligned} V_0 &= d_0 + \alpha u(c_0^o) + (u(c_0^y) + u(c_1^o)) + \beta (u(c_1^y) + u(c_2^o)) + \dots \\ &= d_0 + (u(c_0^y) + \alpha u(c_0^o)) + \beta (u(c_1^y) + \beta^{-1} u(c_1^o)) + \dots \\ &= d_0 + (u(c_0^y) + \alpha u(c_0^o)) + \sum_{t=1}^{\infty} \beta^t (u(c_t^y) + \beta^{-1} u(c_t^o)). \end{aligned} \quad (8)$$

Similarly, the two-sided dynastic utility of the old generation in the current period may be written as

$$V_{-1} = d_{-1} + (u(c_{-1}^y) + \alpha u(c_{-1}^o)) + \sum_{t=0}^{\infty} \beta^{t+1} (u(c_t^y) + \beta^{-1} u(c_t^o)). \quad (9)$$

Because the terms prior to period 0 cannot be controlled, d_0 and $d_{-1} + (u(c_{-1}^y) + \alpha u(c_{-1}^o))$ are omitted without loss of generality. We might assume that the young in period t decide consumption allocation, c_t^y and c_t^o , and saving, k_{t+1} . However, as in Hori (1997), we assume that the young and old generations jointly maximize the weighted sum of their two-sided dynastic utilities in each period.⁷ The new objective function in period 0 is now given by

$$(1 - \gamma)V_0 + (\gamma/\beta)V_{-1} = (u(c_0^y) + \tau u(c_0^o)) + \sum_{t=1}^{\infty} \beta^t (u(c_t^y) + \beta^{-1} u(c_t^o)), \quad (10)$$

with $\gamma \in [0, 1)$ and $\tau = (1 - \gamma)\alpha + \beta^{-1}\gamma$. Here γ is the weight placed on the old generation.

Remark 1: Since $1 > \alpha$ and $1 > \beta$, we have $1/\beta > \tau$. The results in Aoki and Nishimura (2014) hold for the case of $\gamma = 0$ or $\tau = \alpha$.

⁷This was suggested by an anonymous referee.

The current young and old allocate the total consumption c_t to c_t^y and c_t^o by solving

$$\max [u(c^y) + \tau u(c^o)] \text{ s.t. } c^y + c^o = c_t, \quad (11)$$

and they assume that each future generation in period $t \geq 1$ will allocate the consumption in the same manner.

Let $c^y(c_t)$ and $c^o(c_t)$ be their respective solutions. The total utility in period t can be expressed as a function of c_t . Define⁸

$$w_\tau(c_t) = u(c^y(c_t)) + \tau u(c^o(c_t)) \quad (12)$$

and

$$w_{1/\beta}(c_t) = u(c^y(c_t)) + \beta^{-1} u(c^o(c_t)). \quad (13)$$

Note that given c_t and from (12) and (13), $1/\beta (= \beta^{-1}) > 1 > \tau$ implies $w_{1/\beta}(c_t) > w_\tau(c_t)$. Similarly, Lemma 2 will demonstrate that $w'_{1/\beta}(c_t) > w'_\tau(c_t)$.

Then, the objective function may be rewritten as

$$\begin{aligned} (1 - \gamma)V_0 + (\gamma/\beta)V_{-1} &= (u(c^y(c_0)) + \tau u(c^o(c_0))) + \sum_{t=1}^{\infty} \beta^t (u(c^y(c_t)) + \beta^{-1} u(c^o(c_t))) \quad (14) \\ &= w_\tau(c_0) + \sum_{t=1}^{\infty} \beta^t w_{1/\beta}(c_t). \end{aligned}$$

Lemma 1: Let $x(c) = \arg \max_{c \geq x \geq 0} [u(x) + \tau u(c - x)]$. Then $0 < c - x(c) < x(c)$.

Proof: $x(c)$ satisfies the first order condition, $u'(x) - \tau u'(c - x) = 0$. Since $0 < \tau < 1$, $u'(x) < u'(c - x)$ holds. Assumption U1 implies $x(c) > c - x(c) > 0$. $\square$

From Lemma 1 and U4, $\sigma(c - x) \leq \sigma(x)$ holds for $x = x(c)$.

Lemma 2: Both $w_\tau(c)$ and $w_{1/\beta}(c)$ are strictly increasing, strictly concave, continuously differentiable functions of c with $\lim_{c \rightarrow 0} w'_\tau(c) = \lim_{c \rightarrow 0} w'_{1/\beta}(c) = \infty$ and $w_{1/\beta}(c) > w_\tau(c)$.

Proof: See Appendix A.

We have shown that $w_i(c)$ ($i = \tau, 1/\beta$ with $1/\beta > \tau$) satisfies the following properties:

W1: w_i ($i = \tau, 1/\beta$) is strictly concave and continuous for $c \geq 0$, and smooth for $c > 0$.

W2: $w'_i(c) > 0$ ($i = \tau, 1/\beta$) for $c > 0$.

W3: $w_{1/\beta}(c) > w_\tau(c)$ and $w'_{1/\beta}(c) > w'_\tau(c)$ for $c > 0$.

Now we introduce the production function $f(k)$, where k is the capital with full depreciation in one period. It is assumed that f satisfies the following properties:

F1: f is strictly concave and continuous for $k \geq 0$, and smooth for $k > 0$.

F2: $f(0) = 0$, $\lim_{k \rightarrow \infty} f(k) = \infty$.

⁸The notation $w^\beta(\cdot)$ in Aoki and Nishimura (2014) is the same as $w_{1/\beta}(\cdot)$ in this paper.

F3: $f'(k) > 0$ for $k > 0$.

F4: $\lim_{k \rightarrow 0} f'(k) = \infty$, $\lim_{k \rightarrow \infty} f'(k) = 0$.

In period t , the capital k_t inherited from the previous generation is used in production. The output $f(k_t)$ is allocated to consumption $c^y + c^o = c_t$, and the capital stock k_{t+1} is used in the next period. Generation t gives k_{t+1} to generation $t + 1$ as a bequest.

Consider Problem (I), which the young and the old in the current period 0 jointly solve by maximizing the weighted sum of the two-sided dynastic utilities, $(1 - \gamma)V_0 + (\gamma/\beta)V_{-1}$.

$$(I) \quad \max[w_\tau(c_0) + \sum_{t=1}^{\infty} \beta^t w_{1/\beta}(c_t)], \text{ s.t. } k_0 > 0 \text{ given, } c_t = f(k_t) - k_{t+1}, t \geq 0.$$

Let $W(k_0)$ be the maximum of Problem (I). Because the existence of optimal solutions for any $k_0 \geq 0$, the strict concavity of $W(k_0)$, and the continuity of k_t ($t \geq 1$) in k_0 can be easily proved, we shall omit the proofs. The optimal solution $\{k_t\}_{t=1}^{\infty}$ of Problem (I) for k_0 satisfies

$$W(k_0) = w_\tau(f(k_0) - k_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} w_{1/\beta}(f(k_t) - k_{t+1}). \quad (15)$$

By considering the problem

$$(II) \quad \max \sum_{t=1}^{\infty} \beta^{t-1} w_{1/\beta}(c_t), \text{ s.t. } k_1 > 0 \text{ given, } c_t = f(k_t) - k_{t+1}, t \geq 1,$$

and denoting the maximum by $\bar{W}(k_1)$, $W(k_0)$ can be represented as

$$W(k_0) = w_\tau(f(k_0) - k_1) + \beta \bar{W}(k_1). \quad (16)$$

Additionally, $\bar{W}(k_t)$ for $t \geq 1$ can be rewritten as

$$\bar{W}(k_t) = w_{1/\beta}(f(k_t) - k_{t+1}) + \beta \bar{W}(k_{t+1}). \quad (17)$$

We call $\{k_t\}_{t=0}^{\infty}$ an *interior solution* if $f(k_t) > k_{t+1} > 0$ holds for all $t \geq 0$.

Lemma 3: $\{k_t\}_{t=0}^{\infty}$, the optimal solution of Problem (I) for $k_0 > 0$, is an interior solution.

Proof: See Appendix B.

Lemma 4: $W(k_0)$ is strictly concave and the optimal solution of Problem (I) with a given k_0 is unique.

Proof: See Appendix C.

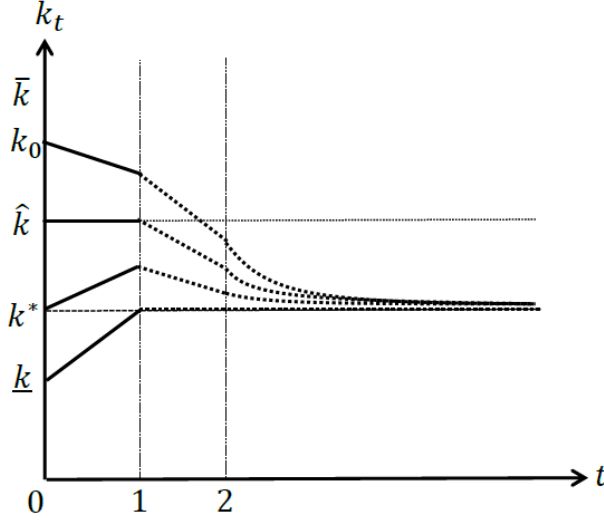


Figure 1 Optimal solutions

3. Dynamics of optimal solutions

In this section, we investigate the dynamic properties the optimal solutions of Problem (I). At first, the next theorem is proved using a method analogous to Dechert and Nishimura (1983).

Theorem 1: Let $\{k_t\}_{t=0}^{\infty}$ and $\{k'_t\}_{t=0}^{\infty}$ be the optimal solutions of Problem (I) for k_0 and k'_0 , respectively. Then $k'_0 > k_0$ implies $k'_t > k_t$ for $t \geq 1$.

Proof: See Appendix D.

Corollary 1: Let $\{k_t\}_{t=0}^{\infty}$ be an optimal solution of Problem (I), starting from $k_0 > 0$. Then either $k_{t+1} \geq k_t$ for all $t \geq 1$ or $k_{t+1} \leq k_t$ for all $t \geq 1$.

Proof: Suppose $k_2 \geq k_1$. $\{k_2, k_3, \dots\}$ is an optimal solution starting from k_2 . Similar to the last part of the proof for Theorem 1, $k_2 \geq k_1$ implies $k_3 \geq k_2$, and by induction on t , $k_{t+1} \geq k_t$, for all t . For the case where $k_2 \leq k_1$, $k_{t+1} \leq k_t$. The proof is similar. $\square$

We prove the optimal solution of Problem (I) converges to the same value as in the model with a constant discount factor β . Figure 1 shows the dynamics.

Theorem 2: Let $\{k_t\}_{t=0}^{\infty}$ be the optimal solution of Problem (I) from $k_0 > 0$. Then $\{k_t\}_{t=1}^{\infty}$ is a monotone sequence that converges to $k^* > 0$.

Proof: See Appendix E.

Next we state the sufficient condition for a solution to be the optimal solution of Problem (I). Because this condition is well known, we omit the proof.

Lemma 5: $\{k_t\}_{t=0}^{\infty}$ is the optimal solution of Problem (I), if a solution $\{k_t\}_{t=1}^{\infty}$ from $k_0 > 0$ satisfies

- (i) Euler equations: $-w'_\tau(f(k_0) - k_1) + \beta w'_{1/\beta}(f(k_1) - k_2)f'(k_1) = 0$ and $-w'_{1/\beta}(f(k_t) - k_{t+1}) + \beta w'_{1/\beta}(f(k_{t+1}) - k_{t+2})f'(k_{t+1}) = 0$ for all $t \geq 1$, and
- (ii) The transversality condition: $\lim_{t \rightarrow \infty} \beta^t w'_{1/\beta}(f(k_t) - k_{t+1})k_t = 0$.

Note that $1/\beta > \tau$. The next lemma characterizes an idiosyncratic capital investment in the first period, which is higher than that in the model with a constant discount factor β .

Lemma 6: Assume $k_0 = k'_0$, and let $\{k_t\}_{t=0}^\infty$ and $\{k'_t\}_{t=0}^\infty$ be solutions of

$$\max_{f(k_0) \geq k_1 \geq 0} [w_\tau(f(k_0) - k_1) + \beta \bar{W}(k_1)] \quad (18)$$

and

$$\max_{f(k_0) \geq k_1 \geq 0} [w_{1/\beta}(f(k_0) - k_1) + \beta \bar{W}(k_1)], \quad (19)$$

respectively. Then $k_1 > k'_1$.

Proof: See Appendix F.

Remark 2: While (18) is equivalent with Problem (I), (19) proves to be equivalent with the following problem,

$$(II') \quad \max \sum_{t=0}^\infty \beta^t w_{1/\beta}(c_t), \text{ s.t. } k_0 > 0 \text{ given, } c_t = f(k_t) - k_{t+1}, t \geq 0.$$

In case of time-consistency, $\tau\beta = 1$, Problem (I) reduces to Problem (II'). Lemma 6 implies that, under the assumption, $1/\beta > \tau$, Problem (I) attains a higher capital stock, k_1 , in the first (next) period, than the time-consistent model (II') does, k'_1 .

Lemma 7: There is an optimal solution of Problem (I), $\{k_t\}_{t=0}^\infty$, such that $k_0 = k_1 \equiv \hat{k}$ with $k^* < k_0 < \bar{k}$.

Proof: Note that $f(\bar{k}) = \bar{k} > 0$. Let $\{k'_t\}_{t=0}^\infty$ be an optimal solution from $k'_0 = \bar{k}$. Then, by Lemma 3, $k'_1 < f(\bar{k})$. Hence, $k'_0 > k'_1$. If $\{k''_t\}_{t=0}^\infty$ is an optimal solution from $k''_0 = k^*$, then $k''_0 (= k^*) < k'_1$ by Lemma 6. Since k_1 changes continuously with k_0 if k_1 is an optimal choice from k_0 , an optimal solution $\{k_t\}_{t=0}^\infty$ exists such that $\hat{k} \in (k^*, \bar{k})$ and $k_0 = k_1 = \hat{k}$. (See Figure 1.) $\square$

4. Evolutionary dynamics of two-sided altruistic paths

Based on the results in Section 3, in this section, we ask what would happen in the sequence of capital stocks inherited from the parental generation by the next generation if the young and the old in each period jointly behave as a central planner in allocating capital resources over an infinite time horizon by maximizing the weighted sum of their two-sided dynastic utilities. For this purpose, we introduce the concepts of a *two-sided altruistic path* and a *two-sided altruistic stationary path* and show that two-sided altruistic paths always converge to a two-sided altruistic stationary path. Along the two-sided altruistic path, only the capital stock of the first period inherited by the next generation is actually realized, whereas the next generation's decision, in both capital stock and consumption, does not

generation	period 0	period 1	period 2	...
0	k_0^0	k_1^0	k_2^0	...
1	-	k_0^1	k_1^1	...
2	-	-	k_0^2	...
$\vdots$	-	-	-	...

Figure 2 Notations for sequences of capital stocks

meet the parental generation's expectations. In this sense, each generation's expectation is naive and its decision is time-inconsistent.⁹

Let's denote each generation by superscript n instead of subscript t . Hence, k_0^n means the inherited capital held by the young generation n in the initial period 0. See Figure 2.

Definition 1: (Optimal solution for generation n) *Suppose that generation $n \geq 0$ faces the initial capital level, $k_0^n > 0$, in period 0. Then the sequence $\{k_t^n\}_{t=0}^\infty$ is called "the optimal solution for generation n ", if it solves Problem (I), given $k_0 = k_0^n$.*

Consider an optimal solution for generation n . Generation n inherits k_0^n from the parental generation, $n-1$, and leaves k_1^n to the next generation, $n+1$. Thus, $k_1^n = k_0^{n+1}$ must be true. Therefore, the sequence of capital stocks inherited from the parental generation, $\{k_0^n\}_{n=0}^\infty$, is actually realized in a two-sided altruistic economy. We call $\{k_0^n\}_{n=0}^\infty$ a *two-sided altruistic path*. Because, given k_0^n , generation n determines k_1^n and because k_1^n is equal to the initial capital stock k_0^{n+1} of the next generation, it is important to study how k_1^n depends on k_0^n . Formally the two-sided altruistic path can be characterized by the following definition.

Definition 2: (Two-sided altruistic path) *Given the initial capital level of generation 0, $k_0^0 > 0$, and the optimal solution for generation n , $\{k_t^n\}_{t=0}^\infty$, the two-sided altruistic path is given by the sequence $\{k_0^n\}_{n=0}^\infty$, which satisfies $k_0^n = k_1^{n-1}$ for $n \geq 1$.*

We state the monotonicity result of a two-sided altruistic path.

Lemma 8: Let $\{k_0^n\}_{n=0}^\infty$ with $k_0^0 > 0$ be a two-sided altruistic path. Then, if $k_0^0 \geq k_0^1$, it holds that $k_0^n \geq k_0^{n+1}$ for any $n \geq 1$.

Proof: If $\{k_t^0\}_{t=0}^\infty$ and $\{k_t^1\}_{t=0}^\infty$ are optimal solutions for generation 0 and 1, given k_0^0 and k_0^1 , respectively, then, by Theorem 1, $k_0^0 \geq k_0^1$ implies $k_1^0 \geq k_1^1$. Since $k_1^0 = k_0^1$ and $k_1^1 = k_0^2$, it follows that $k_0^1 \geq k_0^2$. By induction this also implies $k_0^n \geq k_0^{n+1}$ for any $n \geq 1$. $\square$

⁹While Hori (1997) considers the intergenerational game and the best response function of the next generation, in this paper we are rather interested in agents with a naive expectation and the resulting evolutionary dynamics.

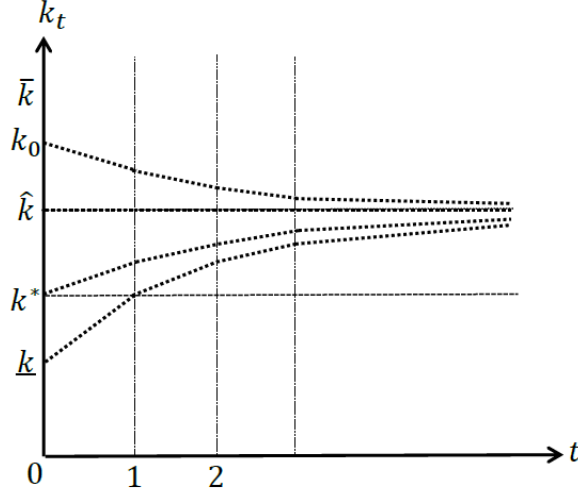


Figure 3 Convergence of two-sided altruistic paths

Then we call a stationary path $\{k_0^n\}_{n=0}^\infty$ with $k_0^0 = \hat{k}$ for $n \geq 0$, a *two-sided altruistic stationary path*, where $\hat{k}$ is defined in the proof of Lemma 7 in Section 3. We prove the final result, Theorem 3.

Theorem 3: Let $\{k_0^n\}_{n=0}^\infty$ with $k_0^0 > 0$ be a two-sided altruistic path. Then $\{k_0^n\}_{n=0}^\infty$ always converges to a two-sided altruistic stationary level, $\hat{k}$.

Proof: See Appendix G.

See Figure 3 for the convergence of two-sided equilibrium paths.

Remark 3: By Lemma 7, the stationary capital level, $\hat{k}$, in the time-inconsistent model is higher than k^* in the time-consistent model.

5. Concluding remarks

Although many authors have discussed models with two-sided altruism, very few have studied the dynamic behaviors of equilibrium paths in the model. Herein, we introduce capital accumulation into the model with altruism toward parents as well as children. Then, by defining the two-sided altruistic paths as capital stock sequences in which each generation inherits from the parental generation, we prove that they converge to a two-sided stationary capital stock.

Compliance with ethical standards

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Appendix

A. Proof of Lemma 2: By definition, $w_\tau(c) = \max_{c \geq x \geq 0} [u(x) + \tau u(c - x)]$. Its solution x is denoted as a function of c by $x(c)$. Because Assumption U3 assures that x is an interior of $[0, c]$, the first order equation is $u'(x) - \tau u'(c - x) = 0$. Totally differentiating it gives $[u''(x) + \tau u''(c - x)] dx - \tau u''(c - x) dc = 0$. Therefore, x is smooth in c by the implicit function theorem, and $\frac{dx}{dc} = \frac{\tau u''(c - x(c))}{u''(x(c)) + \tau u''(c - x(c))}$. Due to strict concavity $u''(\cdot) < 0$, $0 < \frac{dx}{dc} < 1$ holds. Thus, $w'_\tau(c) = \tau u'(c - x(c)) + [u'(x(c)) - \tau u'(c - x(c))] \frac{dx}{dc} = \tau u'(c - x(c)) > 0$, and $w''_\tau(c) = \tau u''(c - x(c)) - \tau u''(c - x(c)) \frac{dx}{dc} = \tau u''(c - x(c))(1 - \frac{dx}{dc}) < 0$. Both prove that $w_\tau(c)$ is strictly increasing and strictly concave in c . $\lim_{c \rightarrow 0} w'_\tau(c) = \infty$ follows from Assumption U3.

Next, again by definition, $w_{1/\beta}(c) = u(x(c)) + \beta^{-1}u(c - x(c))$, where $x(c)$ is already defined as $\arg \max_{c \geq x \geq 0} [u(x) + \tau u(c - x)]$. Then $w_{1/\beta}(c) = u(x(c)) + \tau u(c - x(c)) + (1/\beta - \tau)u(c - x(c)) = w_\tau(c) + (1/\beta - \tau)u(c - x(c))$. Differentiating $w_{1/\beta}(c)$ with c gives $w'_{1/\beta}(c) = w'_\tau(c) + (1/\beta - \tau)u'(c - x(c))(1 - \frac{dx}{dc})$. Since $1/\beta - \tau > 0$, $u'(\cdot) > 0$, and $1 - \frac{dx}{dc} > 0$, we have $w'_{1/\beta}(c) > w'_\tau(c) > 0$. Hence, $w_{1/\beta}(c)$ is strictly increasing, and $\lim_{c \rightarrow 0} w'_{1/\beta}(c) = \infty$ follows from Assumption U3. Again, differentiating $w'_{1/\beta}(c)$ with respect to c gives $w''_{1/\beta}(c) = w''_\tau(c) + (1/\beta - \tau) \left[u''(c - x(c))(1 - \frac{dx}{dc})^2 - u'(c - x(c)) \frac{d^2x}{dc^2} \right]$. Thus, the sufficient condition for the strict concavity of $w_{1/\beta}(c)$, $w''_{1/\beta}(c) < 0$ is $\frac{d^2x}{dc^2} \geq 0$. Denoting $\Delta = \{u''(x(c)) + \tau u''(c - x(c))\}$,

$$\begin{aligned} \frac{d^2x}{dc^2} &= \frac{\tau}{\Delta^2} \left[\begin{aligned} &u'''(c - x(c))(1 - \frac{dx}{dc}) \{u''(x(c)) + \tau u''(c - x(c))\} \\ &- u''(c - x(c)) \{u'''(x(c)) \frac{dx}{dc} + \tau u'''(c - x(c))(1 - \frac{dx}{dc})\} \end{aligned} \right] \\ &= \frac{\tau}{\Delta^2} \left[u'''(c - x(c))u''(x(c))(1 - \frac{dx}{dc}) - u''(c - x(c))u'''(x(c)) \frac{dx}{dc} \right] \\ &= \frac{\tau}{\Delta^3} [u'''(c - x(c))\{u''(x(c))\}^2 - \tau \{u''(c - x(c))\}^2 u'''(x(c))]. \end{aligned} \quad (20)$$

Because $\Delta < 0$, $\frac{d^2x}{dc^2} \geq 0$ is equivalent to $u'''(c - x(c))\{u''(x(c))\}^2 - \tau \{u''(c - x(c))\}^2 u'''(x(c)) \leq 0$, that is, $\frac{u'''(c - x(c))}{\tau \{u''(c - x(c))\}^2} \leq \frac{u'''(x(c))}{\{u''(x(c))\}^2}$. This is satisfied from Lemma 1 and Assumption U4 because $u'(x(c)) - \tau u'(c - x(c)) = 0$ and $x(c) > c - x(c) > 0$. $\square$

B. Proof of Lemma 3: Set $k_0 > 0$. Suppose $k_1 = 0$. Therefore, $k_t = 0$ (all $t \geq 1$). Then $W(k_0) = w_\tau(f(k_0) - 0)$. Consider an alternative solution $k'_1 = \varepsilon > 0$ and $k'_2 = 0$, where the corresponding utility is $w_\tau(f(k_0) - \varepsilon) + \beta w_{1/\beta}(f(\varepsilon) - 0)$. Because $\{k_t\}_{t=0}^\infty$ is an optimal solution,

$$w_\tau(f(k_0)) \geq w_\tau(f(k_0) - \varepsilon) + \beta w_{1/\beta}(f(\varepsilon)). \quad (21)$$

Arranging and dividing both sides by ε gives

$$\frac{w_\tau(f(k_0)) - w_\tau(f(k_0) - \varepsilon)}{\varepsilon} \geq \beta \frac{w_{1/\beta}(f(\varepsilon)) - w_{1/\beta}(f(0))}{\varepsilon}. \quad (22)$$

Due to Assumption F1 and Lemma 2, $f(0) = 0$ and $w_{1/\beta}(0) = 0$. By taking the limits as $\varepsilon \rightarrow 0$, the left-hand side converges to $w'_\tau(f(k_0))$, while the right-hand side diverges to $\lim_{\varepsilon \rightarrow 0} \beta w'_{1/\beta}(f(\varepsilon))f'(\varepsilon) = \infty$ due to Assumption F4 and Lemma 2. The above inequality leads to a contradiction because ε can be arbitrarily small. Therefore, $k_1 > 0$. Similarly $k_{t+1} > 0$ ($t \geq 1$) can be confirmed by replacing w_τ by $w_{1/\beta}$.

Next we shall prove that $c_t = f(k_t) - k_{t+1} > 0$ for all $t \geq 0$. Suppose $k_0 > 0$, but $c_0 = 0$ (i.e., $f(k_0) = k_1$) and $c_1 > 0$ (i.e., $f(k_1) > k_2$). Then

$$W(k_0) = w_\tau(0) + \beta w_{1/\beta}(f(k_1) - k_2) + \beta^2 \bar{W}(k_2). \quad (23)$$

As $k_1 > 0$ and $c_1 = f(k_1) - k_2 > 0$, an alternative solution of $k'_1 = k_1 - \varepsilon > 0$ and $k'_2 = k_2$ is feasible for sufficiently small $\varepsilon > 0$, where the corresponding utility is $w_\tau(\varepsilon) + \beta w_{1/\beta}(f(k_1 - \varepsilon) - k_2) + \beta^2 \bar{W}(k_2)$. Because $\{k_t\}_{t=0}^\infty$ is the optimal solution,

$$w_\tau(0) + \beta w_{1/\beta}(f(k_1) - k_2) + \beta^2 \bar{W}(k_2) \geq w_\tau(\varepsilon) + \beta w_{1/\beta}(f(k_1 - \varepsilon) - k_2) + \beta^2 \bar{W}(k_2). \quad (24)$$

Arranging and dividing both sides by ε gives

$$\beta \frac{w_{1/\beta}(f(k_1) - k_2) - w_{1/\beta}(f(k_1 - \varepsilon) - k_2)}{\varepsilon} \geq \frac{w_\tau(\varepsilon) - w_\tau(0)}{\varepsilon}. \quad (25)$$

By taking the limits as $\varepsilon \rightarrow 0$, the left-hand side converges to $\beta w'_{1/\beta}(f(k_1) - k_2) f'(k_1)$, whereas the right-hand side diverges to $\lim_{\varepsilon \rightarrow 0} w'_\tau(\varepsilon) = \infty$ due to Lemma 2. The above inequality leads to a contradiction because ε can be arbitrarily small. Therefore, $c_0 = c_1 = 0$, or $c_0 > 0$.

Suppose that $c_0 = c_1 = 0$ and T is the smallest in $t \geq 1$ such that $c_T = 0$ and $c_{T+1} > 0$. If such a T does not exist, then $c_t = 0$ for all $t \geq 1$ must be true. Clearly, this is not optimal. Therefore, such a T exists. Then, by the same reasoning as above, it leads to a contradiction. Therefore, $c_{T+1} > 0$ must imply $c_T > 0$. Due to recursive induction, $c_t > 0$ for all $t \leq T+1$. Hence, $c_0 > 0$ and $c_1 > 0$ must be true. Because $c_0 > 0$ and $c_1 > 0$, either $c_t > 0$ for all $t \geq 0$ or $c_t = 0$ for some $t \geq 2$. Suppose that T is the smallest in $t \geq 2$ with $c_T = 0$ and $c_{T+1} > 0$. Then the same reasoning as above leads to a contradiction. Hence, $c_t > 0$ for all $t \geq 0$. $\square$

C. Proof of Lemma 4: Let $\{k'_t\}_{t=0}^\infty$ and $\{k_t\}_{t=0}^\infty$ be the optimal solution from k'_0 and k_0 , respectively.

$$\begin{aligned} \frac{1}{2} [W(k'_0) + W(k_0)] &= \frac{1}{2} \sum_{t=0}^\infty \beta^t [\tilde{w}_t(f(k'_t) - k'_{t+1}) + \tilde{w}_t(f(k_t) - k_{t+1})] \\ &\leq \sum_{t=0}^\infty \beta^t \tilde{w}_t \left(\frac{f(k'_t) + f(k_t)}{2} - \frac{k'_{t+1} + k_{t+1}}{2} \right) \\ &\leq \sum_{t=0}^\infty \beta^t \left[\tilde{w}_t \left(f \left(\frac{k'_t + k_t}{2} \right) - \frac{k'_{t+1} + k_{t+1}}{2} \right) \right] \\ &\leq W \left(\frac{k'_0 + k_0}{2} \right), \end{aligned} \quad (26)$$

where $\tilde{w}_t(c) = w_\tau(c)$ for $t = 0$, and $\tilde{w}_t(c) = w_{1/\beta}(c)$ for $t \geq 1$.

The second inequality strictly holds for $k'_0 \neq k_0$. Thus, for $k'_0 \neq k_0$, $\frac{1}{2} [W(k'_0) + W(k_0)] < W \left(\frac{k'_0 + k_0}{2} \right)$ holds, implying that $W(\cdot)$ is strictly concave.

Next, let $k'_0 = k_0$ and suppose that two optimal solutions, $\{k'_t\}_{t=0}^\infty$ and $\{k_t\}_{t=0}^\infty$, exist. Let's assume that T is the smallest, such that $f(k'_T) - k'_{T+1} \neq f(k_T) - k_{T+1}$. Then, the first inequality in (26) becomes strict. Since $\left\{ \frac{k'_t + k_t}{2} \right\}_{t=0}^\infty$ gives a higher utility than $\{k_t\}_{t=0}^\infty$,

there is a contradiction to the optimality of $\{k'_t\}_{t=0}^\infty$ and $\{k_t\}_{t=0}^\infty$. If there is no T such that $f(k'_T) - k'_{T+1} \neq f(k_T) - k_{T+1}$, then $\{k'_t\}_{t=0}^\infty$ and $\{k_t\}_{t=0}^\infty$ coincide. Thus, $k'_t = k_t$ for all $t \geq 0$. $\square$

D. Proof of Theorem 1: Suppose that $k'_0 > k_0$ and $k'_1 < k_1$. Then (k_0, k'_1) and (k'_0, k_1) are feasible because

$$f(k_0) - k'_1 > f(k_0) - k_1 > 0 \text{ and } f(k'_0) - k'_1 > f(k'_0) - k_1 > 0 \quad (27)$$

are true. By the uniqueness of an optimal solution (Lemma 4),

$$w_\tau(f(k_0) - k_1) + \beta \bar{W}(k_1) > w_\tau(f(k_0) - k'_1) + \beta \bar{W}(k'_1) \quad (28)$$

and

$$w_\tau(f(k'_0) - k'_1) + \beta \bar{W}(k'_1) > w_\tau(f(k'_0) - k_1) + \beta \bar{W}(k_1) \quad (29)$$

must be true. Combining (28) and (29) gives

$$w_\tau(f(k_0) - k_1) + w_\tau(f(k'_0) - k'_1) > w_\tau(f(k_0) - k'_1) + w_\tau(f(k'_0) - k_1). \quad (30)$$

This contradicts the strict concavity of $w_\tau(\cdot)$ because $f(k'_0) - k_1$ and $f(k_0) - k'_1$ lie in $(f(k'_0) - k'_1, f(k_0) - k_1)$. Therefore, $k'_1 \geq k_1$ must be true. Suppose that $k'_1 = k_1$. By the Euler equation,

$$-w'_\tau(f(k_0) - k_1) + \beta w'_{1/\beta}(f(k_1) - k_2)f'(k_1) = 0. \quad (31)$$

Since $k'_0 > k_0$ and $k'_1 = k_1$, then both $\{k_t\}_{t=1}^\infty$ and $\{k'_t\}_{t=1}^\infty$ are optimal solutions of Problem (II) from $k'_1 = k_1$. Since the optimal solution of Problem (II) is unique, $k'_2 = k_2$ holds, and $-w'_\tau(x) + \beta w'_{1/\beta}(f(k_1) - k_2)f'(k_1) = 0$ has two distinct solutions, $x = f(k_0) - k_1$ and $x = f(k'_0) - k_1$. This contradicts the strictly decreasing property of $w'_\tau(x)$. Hence, $k'_1 > k_1$.

Next, $\{k_t\}_{t=1}^\infty$ and $\{k'_t\}_{t=1}^\infty$ are the optimal solutions of Problem (II), given k_1 and k'_1 , respectively. By the principle of optimality, $\bar{W}(k_1) = w_{1/\beta}(f(k_1) - k_2) + \beta \bar{W}(k_2)$ and $\bar{W}(k'_1) = w_{1/\beta}(f(k'_1) - k'_2) + \beta \bar{W}(k'_2)$. Therefore, the same argument as above may be applied, and $k'_1 > k_1$ implies $k'_2 > k_2$. Finally, it is proved that $k'_0 > k_0$ implies $k'_t > k_t$ for $t \geq 1$. $\square$

E. Proof of Theorem 2: The Euler equation for $t \geq 1$ is

$$-w'_{1/\beta}(c_t) + \beta w'_{1/\beta}(c_{t+1})f'(k_{t+1}) = 0, \text{ where } c_t = f(k_t) - k_{t+1}. \quad (32)$$

Rewriting this gives $\beta f'(k_{t+1}) = \frac{w'_{1/\beta}(c_t)}{w'_{1/\beta}(c_{t+1})}$. This equation implies that, if $k^* > k_{t+1}$, then $\frac{w'_{1/\beta}(c_t)}{w'_{1/\beta}(c_{t+1})} > 1$ and $c_t < c_{t+1}$. If $k_{t+1} > k^*$, then $c_t > c_{t+1}$. Assuming $k^* > k_1 > 0$, $k^* > k_{t+1} > 0$ for $t \geq 1$ by Theorem 1, assuring that $c_t < c_{t+1}$ for all $t \geq 1$. That is, c_t is increasing, which is impossible if k_t converges to 0. Since $\{k_t\}_{t=1}^\infty$ is a monotone interior solution, it must monotonically increase and converge to some interior point, say $\tilde{k}$. Then $\tilde{k}$ satisfies

$$-w'_{1/\beta}(f(\tilde{k}) - \tilde{k}) + \beta w'_{1/\beta}(f(\tilde{k}) - \tilde{k})f'(\tilde{k}) = 0. \quad (33)$$

Therefore, $\tilde{k}$ must coincide with k^* .

Assume $k_1 > k^*$. Then $k_t > k^*$ for all $t \geq 1$. If k_t converges to $\bar{k}$ where $f(\bar{k}) = \bar{k} > 0$, then c_t converges to zero. Clearly, accumulating capital without consuming is not optimal. Therefore, k_t does not converge to $\bar{k}$. Because $\{k_t\}_{t=1}^\infty$ is a monotone interior solution, it must converge to k^* . $\square$

F. Proof of Lemma 6: Consider a function of e , $w_e(f(k_0) - k_1) + \beta\bar{W}(k_1)$. Let k_0 be given. Furthermore, both $\{k_t\}_{t=1}^\infty$ and $\{k'_t\}_{t=1}^\infty$ are unique optimal solutions for $e = \tau$ and $1/\beta$, respectively. Suppose that $k'_1 > k_1$. Then, by the uniqueness of an optimal solution,

$$w_\tau(f(k_0) - k_1) + \beta\bar{W}(k_1) > w_\tau(f(k_0) - k'_1) + \beta\bar{W}(k'_1) \quad (34)$$

and

$$w_{1/\beta}(f(k_0) - k'_1) + \beta\bar{W}(k'_1) > w_{1/\beta}(f(k_0) - k_1) + \beta\bar{W}(k_1). \quad (35)$$

Combining (28) and (29),

$$w_\tau(f(k_0) - k_1) + w_{1/\beta}(f(k_0) - k'_1) > w_\tau(f(k_0) - k'_1) + w_{1/\beta}(f(k_0) - k_1). \quad (36)$$

From definitions (12) and (13),

$$(1/\beta - \tau) [u(c^o(f(k_0) - k'_1)) - u(c^o(f(k_0) - k_1))] > 0, \quad (37)$$

where $u(c^o(c))$ is, by the proof of Lemma 2, increasing in c . Therefore, $1/\beta > \tau$ implies that $f(k_0) - k'_1 > f(k_0) - k_1$. Hence, $k_1 > k'_1$, which is a contradiction. Consequently, $k_1 \geq k'_1$.

Suppose that $k_1 = k'_1$. Then the Euler equation,

$$-w'_e(f(k_0) - k_1) + \beta w'_{1/\beta}(f(k_1) - k_2) f'(k_1) = 0, \quad (38)$$

must hold for $e = \tau, 1/\beta$. However, by $1/\beta > \tau$ and Property W3, it holds that $w'_{1/\beta}(f(k_0) - k_1) > w'_\tau(f(k_0) - k_1)$. This contradicts the Euler equation above. Hence, $k_1 > k'_1$ must hold. $\square$

G. Proof of Theorem 3: Let $\hat{K}$ be the set of positive initial capital stocks of two-sided altruistic stationary paths. Since an optimal solution $\{\bar{k}_t\}_{t=0}^\infty$ from $\bar{k}_0 = \bar{k}$ where $f(\bar{k}) = \bar{k} > 0$ satisfies $\bar{k}_1 < \bar{k}$ by Lemma 3, $\bar{x} \equiv \sup\{\hat{k} | \hat{k} \in \hat{K}\} < \bar{k}$ must be true.

Let $\{k_t\}_{t=0}^\infty$ and $\{k'_t\}_{t=0}^\infty$ with $k_0 = k'_0$ be solutions of

$$\max_{f(k_0) \geq k_1 \geq 0} [w_\tau(f(k_0) - k_1) + \beta\bar{W}(k_1)] \quad (39)$$

and

$$\max_{f(k_0) \geq k_1 \geq 0} [w_{1/\beta}(f(k_0) - k_1) + \beta\bar{W}(k_1)], \quad (40)$$

respectively. If $0 < k_0 (= k'_0) < k^*$, $\{k'_t\}_{t=0}^\infty$ is monotonically increasing and converges to k^* by Theorem 2. Thus, $k_0 < k'_1$ is true. Since $k_1 > k'_1$ by Lemma 6, $k_1 > k_0$ holds. Hence, $k_0 < \inf\{\hat{k} | \hat{k} \in \hat{K}\} \equiv \underline{x}$.

Consider a two-sided altruistic path $\{x_0^n\}_{n=0}^\infty$ from $x_0^0 \in (0, \bar{k})$. For each generation n , the optimal solution, $\{x_t^n\}_{t=0}^\infty$, satisfies the Euler equation

$$-w'_\tau(f(x_0^n) - x_1^n) + \beta w'_{1/\beta}(f(x_1^n) - x_2^n)f'(x_1^n) = 0. \quad (41)$$

Since $\{x_0^n\}_{n=0}^\infty$ is a monotone sequence by Lemma 8 that lies in $(0, \bar{k})$, x_0^n converges to some $\hat{k} \in (0, \bar{k})$. $x_1^n (= x_0^{n+1})$ also converges to $\hat{k}$. Since (x_0^n, x_1^n, x_2^n) satisfies the Euler equation, the limit as $n \rightarrow \infty$ should also satisfy it. Therefore, x_2^n converges to some value $\hat{k}_2$ satisfying

$$-w'_\tau(f(\hat{k}) - \hat{k}) + \beta w'_{1/\beta}(f(\hat{k}) - \hat{k}_2)f'(\hat{k}) = 0. \quad (42)$$

This implies $\hat{k}_2 = \hat{k}$. Consequently, the two-sided altruistic path $\{x_0^n\}_{n=0}^\infty$ converges to a two-sided altruistic stationary path $\{\hat{k}, \hat{k}, \hat{k}, \dots\}$. $\square$