Kobe University Repository : Kernel

PDF issue: 2025-02-22

A construction of the fourth order rotatable designs invariant under the hyperoctahedral group

Yamamoto, Hirotaka

Hirao, Masatake

Sawa, Masanori

(Citation)
Journal of Statistical Planning and Inference,200:63-73

(Issue Date)
2019-05

(Resource Type)
journal article

(Version)
Accepted Manuscript

(Rights)
© 2018 Elsevier B.V.
This manuscript version is made available under the CC-BY-NC-ND 4.0 license
http://creativecommons.org/licenses/by-nc-nd/4.0/

(URL)
https://hdl.handle.net/20.500.14094/90005622

A construction of the fourth order rotatable designs invariant under the hyperoctahedral group

Hirotaka Yamamoto

Daifuku Co., Ltd.
3-2-11 Mitejima, Nishiyodogawa-ku, Osaka 550-0012 Japan

Masatake Hirao

Department of Information and Science Technology, Aichi Prefectural University
1522-3 Ibaragabasama, Nagakute, Aichi 480-1198 Japan

Masanori Sawa

Graduate School of System Informatics, Kobe University
1-1 Rokkodai-cho, Nada-ku, Kobe 657-8501 Japan

Abstract

In this paper we establish a construction of Euclidean 9-designs (i.e., the fourth order rotatable designs) on the unit ball. A classical, popular approach for this is to use the corner vectors of the hyperoctahedron such as the vertices, the midpoints of the edges, the barycentres of the faces and so on. As an improvement of this, we propose to use the corner vectors of the hyperoctahedral group, plus their "internally dividing points". We give a classification of Euclidean 9-designs on two spheres, and several examples of the fourth order optimal rotatable designs in low dimensions.

Keywords: Euclidean design, rotatable design, D-optimal design

1. Introduction

Let ξ be a probability measure on the unit ball $B^n = \{\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \|\boldsymbol{x}\| = (\sum_{i=1}^n x_i^2)^{1/2} \leq 1\}$, which we call a *design* on the unit ball.

Email addresses: hirao@ist.aichi-pu.ac.jp (Masatake Hirao), sawa@people.kobe-u.ac.jp (Masanori Sawa)

A design ξ defines the positive semi-definite inner product

$$\langle f, g \rangle_\xi = \int_{\boldsymbol{x} \in B^n} f(\boldsymbol{x}) g(\boldsymbol{x}) \, d\xi(\boldsymbol{x})$$

on the vector space $\mathcal{P}_d(\mathbb{R}^n)$ of all polynomials of degree at most d in n variables.

Let $\boldsymbol{f} = (f_1, \ldots, f_N)'$, where f_i form a basis of $\mathcal{P}_d(\mathbb{R}^n)$ and $N = \dim(\mathcal{P}_d(\mathbb{R}^n)) = \binom{n+d}{d}$. We denote the transpose of $\boldsymbol{f}$ by $\boldsymbol{f}'$. The information matrix of a design ξ is defined as follows:

$$\mathbf{M}_d(\xi) = \int_{\boldsymbol{x} \in B^n} \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{f}(\boldsymbol{x})' \, d\xi(\boldsymbol{x}).$$

We say that a design ξ is *of degree* d if $\mathbf{M}_d(\xi)$ is nonsingular, and hereafter restrict our attention to designs ξ of degree d.

Among many optimality criteria, we are particularly concerned with the *D-optimality criterion*, which is commonly used in practice. We say that a design is *D-optimal* (or *optimal* for short) if the design maximizes the determinant of the information matrix.

Here, we give the concept of Euclidean design introduced by Neumaier and Seidel (1988). Let t be a nonnegative integer, and X be a finite subset in $\mathbb{R}^n$ with a positive weight function w. For $\{r_1, \ldots . r_q\} = \{\|\boldsymbol{x}\| \mid \boldsymbol{x} \in X\}$ with $r_1 > \cdots > r_q \geq 0$, we denote by S_{r_i} the concentric sphere with radius r_i, i.e., $S_{r_i} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\| = r_i\}$. Let $X_i = X \cap S_{r_i}$ and $W_i = \sum_{\boldsymbol{x} \in X_i} w(\boldsymbol{x})$. Let $\sigma_{r_1}, \ldots, \sigma_{r_q}$ be the surface measures on $S_{r_1}, \ldots, S_{r_q}$, respectively. Let $|S_{r_i}| = \int_{S_{r_i}} d\sigma_{r_i}$. We use the convention that $\frac{1}{|S_{r_q}|} \int_{S_{r_q}} f(\boldsymbol{x}) \, d\sigma_{r_q} = f(\mathbf{0})$ for $r_q = 0$. We also use $S^{n-1} := S_1$ and $\sigma = \sigma_1$.

Definition 1.1 (Euclidean design)**.** *A finite weighted pair* (X, w) *is called a Euclidean* t*-design on* q *concentric spheres* $\bigcup_{i=1}^q S_{r_i}$ *if*

$$\sum_{\boldsymbol{x} \in X} w(\boldsymbol{x}) f(\boldsymbol{x}) = \sum_{i=1}^{q} \frac{W_i}{|S_{r_i}|} \int_{\boldsymbol{x} \in S_{r_i}} f(\boldsymbol{x}) \, d\sigma_{r_i}(\boldsymbol{x})$$

for every polynomial $f \in \mathcal{P}_t(\mathbb{R}^n)$.

A D-optimal design with finite support for the d-th degree polynomial regression on the unit ball is well-known to be rotatable. Namely, the d-th order optimal rotatable design on the unit ball is a Euclidean t-design with $t = 2d$, or $t = 2d + 1$; see also Neumaier and Seidel (1992); Bannai and Bannai (2006). Moreover, such a design is supported by $q = (d + 1)/2$ spheres including the surface of B^n; we regard a half as the origin if q is a half integer. Thus, one of the important problems is to construct Euclidean designs supported by "suitably" weighted concentric spheres.

There have been many articles about constructions of Euclidean designs. Among them, we focus on a construction method based on the corner vectors associated with the *hyperoctahedral group* $\mathcal{B}_n$; see, e.g.,Kiefer (1960); Farrell et al. (1967); Pesotchinsky (1978); Gaffke and Heiligers (1995a,b,c, 1998); Bajnok (2007); Hirao et al. (2014). Here, $\mathcal{B}_n$ is the group of all permutations and sign changes of the coordinates of a vector in $\mathbb{R}^n$.

We note that, Euclidean t-designs consisting of only corner vectors have degree $t \leq 7$ (cf. Bajnok (2007); Nozaki and Sawa (2012)). Thus, in this paper, as an extension of the corner vector construction, we propose to use the corner vectors $(\alpha, \ldots, \alpha, 0, \ldots, 0)$ and their internally dividing points $(\beta, \gamma, \ldots, \gamma, 0, \ldots, 0)$, which enable us to find various Euclidean 9-designs. This is an extension of a construction of Euclidean designs on the unit sphere, proposed by Sawa and Xu (2014).

In the next section, we review some basic terminology and explain, in detail, our idea of constructing Euclidean 9-designs supported by one or two concentric spheres. In Section 3, we present classifications of Euclidean 9-designs on two concentric spheres and several examples of the fourth order optimal rotatable designs on the unit ball. In Section 4, we give a proof of our main theorem (see also Theorem 2.5 in the next section).

2. Our method

2.1. Invariant Euclidean design

Let $\mathcal{B}_n$ be the hyperoctahedral group. Given $\gamma \in \mathcal{B}_n$ and $\boldsymbol{x} \in \mathbb{R}^n$, let $\gamma(\boldsymbol{x})$ be the action of $\boldsymbol{x}$ by an element γ of $\mathcal{B}_n$ and $\boldsymbol{x}^{\mathcal{B}_n} = \{\gamma(\boldsymbol{x}) \mid \gamma \in \mathcal{B}_n\}$ be the $\mathcal{B}_n$-orbit of $\boldsymbol{x}$.

For any $f \in \mathcal{P}_d(\mathbb{R}^n)$, we define the action of $\gamma \in \mathcal{B}_n$ on f as follows:

$$(\gamma f)(\boldsymbol{x}) = f(\gamma^{-1}(\boldsymbol{x}))$$

for every $\boldsymbol{x} \in \mathbb{R}^n$. A polynomial f is said to be $\mathcal{B}_n$-*invariant* if $\gamma f = f$ for every $\gamma \in \mathcal{B}_n$.

Let $\mathrm{Hom}_d(\mathbb{R}^n)$ or $\mathrm{Harm}_d(\mathbb{R}^n)$ be the subspace of $\mathcal{P}_d(\mathbb{R}^n)$ which consists of all homogeneous or harmonic homogeneous polynomials of degree d, respectively. Namely,

$$\mathrm{Hom}_d(\mathbb{R}^n) = \{f(x_1,\ldots,x_n) = \sum_{\substack{\lambda_1+\cdots+\lambda_n=d\\ \lambda_1,\ldots,\lambda_n\geq 0}} c_{\lambda_1,\ldots,\lambda_n} x_1^{\lambda_1}\cdots x_n^{\lambda_n} \mid c_{\lambda_1,\ldots,\lambda_n}\in\mathbb{R}\},$$

$$\mathrm{Harm}_d(\mathbb{R}^n) = \{f \in \mathrm{Hom}_d(\mathbb{R}^n) \mid \Delta f = 0\},$$

where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ is the Laplace operator.

By $\mathcal{P}_d(\mathbb{R}^n)^{\mathcal{B}_n}$ and $\mathrm{Harm}_d(\mathbb{R}^n)^{\mathcal{B}_n}$, we denote the set of $\mathcal{B}_n$-invariant polynomials in $\mathcal{P}_d(\mathbb{R}^n)$ and $\mathrm{Harm}_d(\mathbb{R}^n)$, respectively.

There have been many articles concerning constructions of $\mathcal{B}_n$-invariant Euclidean designs; e.g., Pesotchinsky (1978), Gaffke and Heiligers (1995a,b,c, 1998), Hirao et al. (2014).

Definition 2.1. *A finite weighted pair (X, w) is said to be $\mathcal{B}_n$-invariant if X is a union of $\mathcal{B}_n$-orbits and $w(\boldsymbol{y}) = w(\boldsymbol{y}')$ for every $\boldsymbol{y}, \boldsymbol{y}' \in \boldsymbol{x}^{\mathcal{B}_n}$, $\boldsymbol{x} \in X$.*

The following is a special case of a classical theorem by S.L. Sobolev.

Theorem 2.2 (Sobolev (1962))**.** *Let (X, w) be a $\mathcal{B}_n$-invariant weighted pair. Then, (X, w) forms a Euclidean t-design on the unit sphere such that the total weight is equal to 1, i.e, $\sum_{\boldsymbol{x}\in X} w(\boldsymbol{x}) = 1$, if and only if it holds that*

$$\sum_{\boldsymbol{x}\in X} w(\boldsymbol{x}) f(\boldsymbol{x}) = \frac{1}{|S^{n-1}|}\int_{\boldsymbol{x}\in S^{n-1}} f(\boldsymbol{x})\, d\sigma(\boldsymbol{x})$$

for any $f \in \mathcal{P}_t(\mathbb{R}^n)^{\mathcal{B}_n}$.

The Sobolev theorem is extended to that for Euclidean designs, which is useful to construct $\mathcal{B}_n$-invariant Euclidean designs.

Theorem 2.3 (Nozaki and Sawa (2012))**.** *Let (X, w) be a $\mathcal{B}_n$-invariant weighted pair with $X = \cup_{k=1}^{M} r_k \boldsymbol{x}_k^{\mathcal{B}_n}$, where $\boldsymbol{x}_k \in S^{n-1}$ and $r_k > 0$. The following are equivalent:*
(i) *A pair (X, w) is a Euclidean t-design.*
(ii) $\sum_{\boldsymbol{x}\in X} w(\boldsymbol{x})\|\boldsymbol{x}\|^{2j}\phi(\boldsymbol{x}) = 0$ *for any $\phi \in \mathrm{Harm}_l(\mathbb{R}^n)^{\mathcal{B}_n}$ with $1 \leq l \leq t, 0 \leq j \leq \lfloor \frac{t-l}{2} \rfloor$.*

We also obtain the same conclusion for other finite reflection groups, e.g., see Nozaki and Sawa (2012).

2.2. Our method

In our previous work (Hirao et al., 2014), we constructed many $\mathcal{B}_n$-invariant Euclidean 7-designs, i.e., the third order rotatable designs, by using the points

$$\mathcal{X}(\mathcal{R}, J) = \bigcup_{k \in J} r_k \boldsymbol{z}_k^{\mathcal{B}_n}, \quad \boldsymbol{z}_k := \frac{1}{\sqrt{k}} \sum_{i=1}^{k} \boldsymbol{e}_i,$$

where $J \subset \{1, \dots, n\}$ and $\mathcal{R}$ is the set of radii, namely $\mathcal{R} = \{r_k > 0 \mid k \in J\}$. Accordingly, we obtained a number of the third order optimal rotatable designs on the unit ball.

A merit of the corner vectors is simplicity, which, however, cannot produce Euclidean 8-designs, as shown by Bajnok (2007); see also Hirao et al. (2014). Thus, to construct higher degree designs, we propose to examine the *internally dividing points* $\boldsymbol{v}_{a,s}$ *of corner vectors* $\boldsymbol{z}_k$ given by

$$\boldsymbol{v}_{a,s} = \frac{1}{\sqrt{a^2 + s}} \left(a\boldsymbol{e}_1 + \sum_{i=2}^{s+1} \boldsymbol{e}_i \right),$$

where $a > 0$, $a \neq 1$, and $s \in \{1, 2, \dots, n-1\}$. They actually provide a huge number of Euclidean 9-designs on two concentric spheres and accordingly, many examples of the fourth order optimal rotatable designs on the unit ball. The details will be coming up soon later.

Remark 2.4. *The size of* $\boldsymbol{z}_k^{\mathcal{B}_n}$ *and* $\boldsymbol{v}_{a,s}^{\mathcal{B}_n}$ *are equal to*

$$|\boldsymbol{z}_k^{\mathcal{B}_n}| = 2^k \binom{n}{k}, \quad |\boldsymbol{v}_{a,s}^{\mathcal{B}_n}| = 2^{s+1} n \binom{n-1}{s}.$$

2.2.1. Two supporting spheres

The fourth order optimal rotatable design on the unit ball is supported by the origin and two concentric spheres. Moreover, by the definition of Euclidean design, the corresponding weighted set excluding the origin forms a Euclidean 9-design on two concentric spheres. Thus, we consider a Euclidean 9-design $(\mathcal{X}(\{a, s\}, \mathcal{R}, J_1, J_2), w)$ on two concentric spheres $S_{r_1} \cup S_{r_2}$, where $\mathcal{R} = \{r_1, r_2\}$ be the radial set with $r_1 > r_2 > 0$, and $J_1, J_2 \subset \{1, 2, \dots, n\}$. Let

$$\mathcal{X}(\{a, s\}, \mathcal{R}, J_1, J_2) = X_1 \cup X_2, \quad X_1 = r_1 \boldsymbol{v}_{a,s}^{\mathcal{B}_n} \cup \bigcup_{k \in J_1} r_1 \boldsymbol{z}_k^{\mathcal{B}_n}, \quad X_2 = \bigcup_{k \in J_2} r_2 \boldsymbol{z}_k^{\mathcal{B}_n}$$

Let $w_{\boldsymbol{v}} := w(\boldsymbol{x})$ for any $\boldsymbol{x} \in r_1 \boldsymbol{v}_{a,s}^{\mathcal{B}_n}$, $w_k := w(\boldsymbol{x})$ for any $\boldsymbol{x} \in r_1 \boldsymbol{z}_k^{\mathcal{B}_n}$, and $v_k := w(\boldsymbol{x})$ for any $\boldsymbol{x} \in r_2 \boldsymbol{z}_k^{\mathcal{B}_n}$.

Theorem 2.5. *For $a > 0, a \neq 1$, $s \in \{1, 2, \ldots, n-1\}$, $\mathcal{R} = \{r_1, r_2\}$ and $J_1, J_2 \subset \{1, \ldots, n\}$, let $(\mathcal{X}(\{a, s\}, \mathcal{R}, J_1, J_2), w)$ be a $\mathcal{B}_n$-invariant weighted pair defined as in the above paragraph. Then, $(\mathcal{X}(\{a, s\}, \mathcal{R}, J_1, J_2), w)$ is a Euclidean 9-design on two concentric spheres if and only if the following equations hold:*

$$
\begin{aligned}
0 &= \sum_{k \in J_2} v_k \frac{2^k}{k^2} \binom{n-1}{k-1} \left(1 - 3\frac{k-1}{n-1}\right), \\
0 &= w_{\boldsymbol{v}} \frac{2^{s+1}}{(a^2+s)^2} \binom{n-1}{s} \left(a^4 + s - 3(a^2+s-1)\frac{s}{n-1}\right) \\
&\quad + \sum_{k \in J_1} w_k \frac{2^k}{k^2} \binom{n-1}{k-1} \left(1 - 3\frac{k-1}{n-1}\right), \\
0 &= \sum_{k \in J_2} v_k \frac{2^k}{k^3} \binom{n-1}{k-1} \left(1 - 15\frac{k-1}{n-1} + 30\frac{(k-1)(k-2)}{(n-1)(n-2)}\right), \\
0 &= w_{\boldsymbol{v}} \frac{2^{s+1}}{(a^2+s)^3} \binom{n-1}{s} \Bigg(a^6 + s - 15(a^4 + a^2 + s - 1)\frac{s}{n-1} \\
&\quad + 30(a^2 + s - 2)\frac{s(s-1)}{(n-1)(n-2)}\Bigg) \\
&\quad + \sum_{k \in J_1} w_k \frac{2^k}{k^3} \binom{n-1}{k-1} \left(1 - 15\frac{k-1}{n-1} + 30\frac{(k-1)(k-2)}{(n-1)(n-2)}\right), \\
0 &= w_{\boldsymbol{v}} \frac{r_1^8 2^{s+1} n}{(a^2+s)^4} \binom{n-1}{s} \left(a^8 + s + 7(-4a^6 + 3a^4 - 4a^2 + s - 1)\frac{s}{n-1}\right) \\
&\quad + \sum_{k \in J_1} w_k \frac{r_1^8 2^k n}{k^4} \binom{n-1}{k-1} \left(1 + 7\frac{k-1}{n-1}\right) \\
&\quad + \sum_{k \in J_2} v_k \frac{r_2^8 2^k n}{k^4} \binom{n-1}{k-1} \left(1 + 7\frac{k-1}{n-1}\right) \qquad \textit{for } n \geq 3,
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & \frac{w_{\boldsymbol{v}} r_1^8 2^s n}{(a^2+s)^4}\binom{n-1}{s-1}\Bigg(a^4+s-1-(a^6+a^2+s-2)\frac{6(s-1)}{n-2} \\
& + (a^2+s-3)\frac{9(s-1)(s-2)}{(n-2)(n-3)}\Bigg) \\
& + \sum_{k\in J_1} w_k \frac{r_1^8 2^{k-1} n}{k^4}\binom{n-1}{k-2}\left(1-28\frac{k-2}{n-2}+9\frac{(k-2)(k-3)}{(n-2)(n-3)}\right) \\
& + \sum_{k\in J_2} v_k \frac{r_2^8 2^{k-1} n}{k^4}\binom{n-1}{k-2}\left(1-28\frac{k-2}{n-2}+9\frac{(k-2)(k-3)}{(n-2)(n-3)}\right) \quad \textit{for } n \geq 4.
\end{aligned}
$$

Proof. See Section 4. □

In Sections 3.1 and 3.2, we give several examples of $\mathcal{B}_n$-invariant Euclidean 9-designs and some classification results.

Remark 2.6. *In order to find a $\mathcal{B}_n$-invariant Euclidean 9-design, it is necessary to solve five equations in the case $n = 3$, whereas six equations in the case $n \geq 4$. This gap depends on the dimension of* $\mathrm{Harm}_l(\mathbb{R}^n)^{\mathcal{B}_n}$*; see also Lemma 4.1 and some related arguments in Section 4.*

2.2.2. One supporting sphere

We here deal with $\mathcal{B}_n$-invariant Eulcidean 9-designs on the unit sphere. For $J \subset \{1, \ldots, n\}$, let

$$\mathcal{X}(\{a, s\}, J) = \boldsymbol{v}_{a,s}^{\mathcal{B}_n} \cup \bigcup_{k\in J} \boldsymbol{z}_k^{\mathcal{B}_n}.$$

Let $w_{\boldsymbol{v}} := w(\boldsymbol{x})$ for any $\boldsymbol{x} \in \boldsymbol{v}_{a,s}^{\mathcal{B}_n}$, $w_k := w(\boldsymbol{x})$ for any $\boldsymbol{x} \in \boldsymbol{z}_k^{\mathcal{B}_n}$. Moreover, for simplicity of computations, we restrict our attention to Eulcidean 9-designs such that the total weight is equal to 1, i.e.,

$$\sum_{\boldsymbol{x}\in \boldsymbol{v}_{a,s}^{\mathcal{B}_n} \cup \boldsymbol{z}_k^{\mathcal{B}_n}} w(\boldsymbol{x}) = w_{\boldsymbol{v}} 2^{s+1}\binom{n-1}{s} + \sum_{k\in J} w_k 2^k \binom{n}{k} = 1.$$

With the above set up, by the similar argument to Theorem 2.5, we can immediately obtain the following corollary from Theorem 2.2.

Corollary 2.7. *For $a > 0, a \neq 1$, $s \in \{1, 2, \ldots, n-1\}$ and $J \subset \{1, 2, \ldots, n\}$, let $(\mathcal{X}(\{a, s\}, J), w)$ be a $\mathcal{B}_n$-invariant pair such that the total weight is equal to 1. Then, $(\mathcal{X}(\{a, s\}, J), w)$ is a Euclidean 9-design on the unit sphere if and only if the following equations hold:*

$$\begin{aligned}
0 &= w_{\boldsymbol{v}} \frac{2^{s+1}}{(a^2+s)^2} \binom{n-1}{s} \left(a^4 + s - 3(a^2+s-1)\frac{s}{n-1} \right) \\
&\quad + \sum_{k \in J} w_k \frac{2^k}{k^2} \binom{n-1}{k-1} \left(1 - 3\frac{k-1}{n-1} \right), \\
0 &= w_{\boldsymbol{v}} \frac{2^{s+1}}{(a^2+s)^3} \binom{n-1}{s} \Bigg(a^6 + s - 15(a^4 + a^2 + s - 1)\frac{s}{n-1} \\
&\quad + 30(a^2 + s - 2)\frac{s(s-1)}{(n-1)(n-2)} \Bigg) \\
&\quad + \sum_{k \in J} w_k \frac{2^k}{k^3} \binom{n-1}{k-1} \left(1 - 15\frac{k-1}{n-1} + 30\frac{(k-1)(k-2)}{(n-1)(n-2)} \right), \\
0 &= w_{\boldsymbol{v}} \frac{2^{s+1} n}{(a^2+s)^4} \binom{n-1}{s} \left(a^8 + s + 7(-4a^6 + 3a^4 - 4a^2 + s - 1)\frac{s}{n-1} \right) \\
&\quad + \sum_{k \in J} w_k \frac{2^k n}{k^4} \binom{n-1}{k-1} \left(1 + 7\frac{k-1}{n-1} \right) \quad \textit{for } n \geq 3, \\
0 &= \frac{w_{\boldsymbol{v}} 2^s n}{(a^2+s)^4} \binom{n-1}{s-1} \Bigg(a^4 + s - 1 - (a^6 + a^2 + s - 2)\frac{6(s-1)}{n-2} \\
&\quad + (a^2 + s - 3)\frac{9(s-1)(s-2)}{(n-2)(n-3)} \Bigg) \\
&\quad + \sum_{k \in J} w_k \frac{2^{k-1} n}{k^4} \binom{n-1}{k-2} \left(1 - 28\frac{k-2}{n-2} + 9\frac{(k-2)(k-3)}{(n-2)(n-3)} \right) \quad \textit{for } n \geq 4, \\
1 &= w_{\boldsymbol{v}} 2^{s+1} \binom{n-1}{s} + \sum_{k \in J} w_k 2^k \binom{n}{k}.
\end{aligned}$$

By Corollary 2.7, we can find many types of Euclidean 9-designs on S^{n-1}. For example, we obtian

1. 616-points 9-design of S^5 with $J = \{1, 2, 3\}$: $a = 2.46899, s = 5, w_{\boldsymbol{v}} = 0.598477, w_1 = 0.0338246, w_2 = 0.0820219, w_3 = 0.285676$.

2. 1274-points 9-design of S^6 with $J = \{1, 2, 3\}$: $a = 2.48745, s = 6, w_{\boldsymbol{v}} = 0.477909, w_1 = 0.014995, w_2 = 0.103815, w_3 = 0.403281$.
3. 3296-points 9-design of S^7 with $J = \{1, 2, 4\}$: $a = 2.68568, s = 7, w_{\boldsymbol{v}} = 0.523449, w_1 = 0.0122401, w_2 = 0.0749255, w_4 = 0.389386$.

In the rest of this subsection, we present a systematic technique for constructing Euclidean designs. The idea is to rescale Euclidean design on the unit sphere with the total weight 1 and then to bring them together. These may have no interest in combinatorics but is of statistical interest since it produces more examples of Euclidean 9-designs on the origin and two concentric spheres.

Proposition 2.8. *Let $(Y_i, w_i), i = 1, 2$, be Euclidean 9-designs on the unit sphere. Let W_1, W_2, W_3 and r be positive real numbers with $\sum_{i=1}^{3} W_i = 1$ and $r < 1$, respectively. Then $(Y_1 \cup rY_2 \cup \{\mathbf{0}\}, w)$ is a Euclidean 9-design on $S_1 \cup S_r \cup \{\mathbf{0}\}$, where $w(\boldsymbol{x}) = W_1 w_1(\boldsymbol{x})$ if $\boldsymbol{x} \in Y_1$, $w(\boldsymbol{x}) = W_2 w_2(r^{-1}\boldsymbol{x})$ if $r^{-1}\boldsymbol{x} \in Y_2$ and $w(\mathbf{0}) = W_3$.*

Proof. Let $f \in \mathrm{Hom}_t(\mathbb{R}^n)$ with $1 \le t \le 9$. Then by noting $f(\mathbf{0}) = 0$, given $W_1, W_2, W_3 > 0$, we have

$$\begin{aligned}
&\frac{W_1}{|S_1|}\int_{\boldsymbol{x}\in S_1} f(\boldsymbol{x})\, d\rho(\boldsymbol{x}) + \frac{W_2}{|S_r|}\int_{\boldsymbol{x}\in S_r} f(\boldsymbol{x})\, d\rho_r(\boldsymbol{x}) + W_3 f(\mathbf{0})\\
&= \frac{W_1}{|S_1|}\int_{\boldsymbol{x}\in S_1} f(\boldsymbol{x})\, d\rho(\boldsymbol{x}) + \frac{r^t W_2}{|S_1|}\int_{\boldsymbol{x}\in S_1} f(\boldsymbol{x})\, d\rho(\boldsymbol{x})\\
&= W_1 \sum_{\boldsymbol{x}\in Y_1} w_1(\boldsymbol{x}) f(\boldsymbol{x}) + r^t W_2 \sum_{\boldsymbol{x}\in Y_2} w_2(\boldsymbol{x}) f(\boldsymbol{x})\\
&= \sum_{\boldsymbol{x}\in Y_1} (W_1 w_1(\boldsymbol{x})) f(\boldsymbol{x}) + \sum_{\boldsymbol{x}\in Y_2} (W_2 w_2(\boldsymbol{x})) f(r\boldsymbol{x}) + W_3 f(\mathbf{0}).
\end{aligned}$$

Next we check the rest case that f is a constant, i.e., $f \in \mathrm{Hom}_0(\mathbb{R}^n)$. Then, we have to consider the case $f \equiv 1$, and

$$\begin{aligned}
&\frac{W_1}{|S_1|}\int_{\boldsymbol{x}\in S_1} f(\boldsymbol{x})\, d\rho(\boldsymbol{x}) + \frac{W_2}{|S_r|}\int_{\boldsymbol{x}\in S_r} f(\boldsymbol{x})\, d\rho_r(\boldsymbol{x}) + W_3 f(\mathbf{0})\\
&= 1 = \sum_{\boldsymbol{x}\in Y_1} (W_1 w_1(\boldsymbol{x})) f(\boldsymbol{x}) + \sum_{\boldsymbol{x}\in Y_2} (W_2 w_2(\boldsymbol{x})) f(r\boldsymbol{x}) + W_3 f(\mathbf{0}).
\end{aligned}$$

Thus we obtain the desired result. □

3. The fourth order rotatable design

3.1. $\mathcal{B}_n$-invariant Euclidean 9-designs on two concentric spheres

Hereafter we only consider Euclidean 9-designs $\mathcal{X}(\{a,s\},\mathcal{R},J_1,J_2)$ corresponding to the fourth order optimal rotatable designs on the unit ball, where

$$\begin{gathered}\mathcal{R}=\{r_1,r_2\} \text{ with } r_1>r_2>0,\\ J_1,J_2\subset\{1,\ldots,n\} \text{ with } J_1\cap J_2=\emptyset \text{ and } |J_1|,|J_2|\le 3.\end{gathered} \tag{1}$$

We remark that, concerning (1), there are not so many candidates for J_1 and J_2. In fact, by using Theorem 2.5 for all possible J_1 and J_2, we know that there exist no Euclidean 9-designs in dimensions $n\le 4$.

Moreover, although we can obtain some examples of Euclidean 9-designs on two concentric spheres in dimension 5, we know that, by tedious case-by-case arguments, all of them do not induce the fourth order optimal rotatable designs on the unit ball. Namely, all possible candidates do not induce Euclidean 9-designs on the origin and two concentric spheres with the optimal weights $W_1=0.81137, W_2=0.175789, W_3=0.0128406$ and radius $r_2=0.70883$; see also Table 1 in the next subsection for these values W_1,W_2,W_3 and r_2.

Thus the smallest nontrivial case to consider is dimension 6. We here consider the cases when $n=6$, plus $n=7$. For convenience we also assume that $r_1=1$.

(a) The 6-dimensional case. We totally obtain the following 4 types of Euclidean 9-designs: Let $A=a^2$ and $R_2=r_2^2>0$.

1. $J_1=\{2,4,5\}$, $J_2=\{1,3,6\}$, $s=3$.

$$\begin{aligned}
w_2&=\tfrac{2(-1+A)^2(-345+42A+34A^2)}{41(3+A)^4}w_{\boldsymbol{v}}, \quad w_4=\tfrac{32(-171+12A-153A^2-30A^3+14A^4)}{41(3+A)^4}w_{\boldsymbol{v}},\\
w_5&=-\tfrac{25(-1+A)^2(-21-6A+A^2)}{41(3+A)^4}w_{\boldsymbol{v}},\\
v_1&=-\tfrac{108(-1+A)^2(-1-12A+2A^2)}{205(3+A)^4R_2^4}w_{\boldsymbol{v}}, \quad v_3=-\tfrac{162(-1+A)^2(-1-12A+2A^2)}{41(3+A)^4R_2^4}w_{\boldsymbol{v}},\\
v_6&=-\tfrac{162(-1+A)^2(-1-12A+2A^2)}{205(3+A)^4R_2^4}w_{\boldsymbol{v}}.
\end{aligned}$$

where $\alpha(\approx 4.60229)\le A<\frac{6+\sqrt{38}}{2}$ and α is the positive root of $-171+12x-153x^2-30x^3+14x^4=0$.

2. $J_1 = \{2,4,5\}$, $J_2 = \{1,3,6\}$, $s = 4$.

$$\begin{aligned}
w_2 &= \tfrac{4(-1+A)^2A(50+51A)W1}{123(4+A)^4}w_{\boldsymbol{v}}, \quad w_4 = \tfrac{64(-1+A)^2(-164-6A+7A^2)}{41(4+A)^4}w_{\boldsymbol{v}},\\
w_6 &= -\tfrac{125A(712-68A-32A^2+3A^3)}{123(4+A)^4}w_{\boldsymbol{v}},\\
v_1 &= -\tfrac{72(-1+A)^2A(-26+3A)}{205(4+A)^4R_2^4}w_{\boldsymbol{v}}, \quad v_3 = -\tfrac{108(-1+A)^2A(-26+3A)}{41(4+A)^4R_2^4}w_{\boldsymbol{v}},\\
v_6 &= -\tfrac{108(-1+A)^2A(-26+3A)}{205(4+A)^4R_2^4}w_{\boldsymbol{v}},
\end{aligned}$$

where $\frac{3+\sqrt{1157}}{7} \leq A < \frac{26}{3}$.

3. $J_1 = \{2,4,6\}$, $J_2 = \{1,3,5\}$, $s = 3$.

$$\begin{aligned}
w_2 &= \tfrac{3(-1+A)^2(-183+26A+15A^2)}{29(3+A)^4}w_{\boldsymbol{v}}, \quad w_4 = \tfrac{8(-369-168A-378A^2-48A^3+35A^4)}{29(3+A)^4}w_{\boldsymbol{v}},\\
w_6 &= -\tfrac{54(-1+A)^2(-18-6A+A^2)}{29(3+A)^4}w_{\boldsymbol{v}},\\
v_1 &= -\tfrac{195(-1+A)^2(-3-30A+5A^2)}{928(3+A)^4R_2^4}w_{\boldsymbol{v}}, \quad v_3 = -\tfrac{675(-1+A)^2(-3-30A+5A^2)}{464(3+A)^4R_2^4}w_{\boldsymbol{v}},\\
v_5 &= -\tfrac{375(-1+A)^2(-3-30A+5A^2)}{928(3+A)^4R_2^4}w_{\boldsymbol{v}}
\end{aligned}$$

where $\alpha(\approx 4.28626) \leq A < \frac{15+4\sqrt{15}}{5}$ and α is the positive root of $-369-168x-378x^2-48x^3+35x^4=0$.

4. $J_1 = \{2,4,6\}$, $J_2 = \{1,3,5\}$, $s = 4$:

$$\begin{aligned}
w_2 &= \tfrac{A(824-228A-16A^2+45A^3)}{29(4+A)^4}w_{\boldsymbol{v}}, \quad w_4 = \tfrac{8(-928+512A-680A^2-64A^3+35A^4)}{29(4+A)^4}w_{\boldsymbol{v}},\\
w_6 &= -\tfrac{18A(604-50A-32A^2+3A^3)}{29(4+A)^4}w_{\boldsymbol{v}},\\
v_1 &= -\tfrac{65A(-112+272A-160A^2+15A^3)}{928(4+A)^4R_2^4}w_{\boldsymbol{v}}, \quad v_3 = -\tfrac{225A(-112+272A-160A^2+15A^3)}{464(4+A)^4R_2^4}w_{\boldsymbol{v}},\\
v_5 &= -\tfrac{125A(-112+272A-160A^2+15A^3)}{928(4+A)^4R_2^4}w_{\boldsymbol{v}},
\end{aligned}$$

where $\alpha_1(\approx 5.20767) \leq A < \alpha_2(\approx 8.67575)$, α_1 is the positive root of $-928+512x-680x^2-64x^3+35x^4=0$ and α_2 is the largest root of $-112+272x-160x^2+15x^3=0$.

(b) The 7-dimensional case. We obtain totally 49 types of Euclidean designs in the following pairs of J_1 and J_2:

(b-1) The case $s = 3$. $\{J_1, J_2\} = \{\{2,4,5\},\{1,3,6\}\}$, $\{\{2,4,5\},\{1,3,7\}\}$, $\{\{2,4,6\},\{1,3,5\}\}$, $\{\{2,4,6\},\{1,3,7\}\}$, $\{\{2,4,7\},\{1,3,5\}\}$, $\{\{2,4,7\},\{1,3,6\}\}$,

$\{\{3,4,6\},\{1,2,5\}\}$, $\{\{3,4,6\},\{1,2,7\}\}$, $\{\{3,4,7\},\{1,2,5\}\}$, $\{\{3,4,7\},\{1,2,6\}\}$, $\{\{3,5,6\},\{1,2,4\}\}$, $\{\{3,5,6\},\{1,2,7\}\}$, $\{\{3,5,7\},\{1,2,4\}\}$, $\{\{3,5,7\},\{1,2,6\}\}$.

(b-2) The case $s=4$. $\{J_1,J_2\}=\{\{2,4,5\},\{1,3,6\}\}$, $\{\{2,4,5\},\{1,3,7\}\}$, $\{\{2,4,6\},\{1,3,5\}\}$, $\{\{2,4,6\},\{1,3,7\}\}$, $\{\{2,4,7\},\{1,3,5\}\}$, $\{\{3,4,7\},\{1,2,5\}\}$, $\{\{3,4,7\},\{1,2,6\}\}$, $\{\{3,5,7\},\{1,2,4\}\}$, $\{\{3,5,7\},\{1,2,6\}\}$, $\{\{3,6,7\},\{1,2,4\}\}$, $\{\{3,6,7\},\{1,2,5\}\}$.

(b-3) The case $s=5$. $\{J_1,J_2\}=\{\{2,4,5\},\{1,3,6\}\}$, $\{\{2,4,5\},\{1,3,7\}\}$, $\{\{2,4,6\},\{1,3,5\}\}$, $\{\{2,4,6\},\{1,3,7\}\}$, $\{\{2,4,7\},\{1,3,5\}\}$, $\{\{2,4,7\},\{1,3,6\}\}$, $\{\{3,4,7\},\{1,2,5\}\}$, $\{\{3,4,7\},\{1,2,6\}\}$, $\{\{3,5,7\},\{1,2,4\}\}$, $\{\{3,5,7\},\{1,2,6\}\}$, $\{\{3,6,7\},\{1,2,4\}\}$, $\{\{3,6,7\},\{1,2,5\}\}$.

(b-4) The case $s=6$. $\{J_1,J_2\}=\{\{2,4,5\},\{1,3,6\}\}$, $\{\{2,4,5\},\{1,3,7\}\}$, $\{\{2,4,6\},\{1,3,5\}\}$, $\{\{2,4,6\},\{1,3,7\}\}$, $\{\{2,4,7\},\{1,3,5\}\}$, $\{\{2,4,7\},\{1,3,6\}\}$, $\{\{3,4,7\},\{1,2,5\}\}$, $\{\{3,4,7\},\{1,2,6\}\}$, $\{\{3,5,7\},\{1,2,4\}\}$, $\{\{3,5,7\},\{1,2,6\}\}$, $\{\{3,6,7\},\{1,2,4\}\}$, $\{\{3,6,7\},\{1,2,5\}\}$.

If we select the first two of the common pairs of J_1 and J_2 in (b-1)–(b-4), by choosing suitable weights and $A=a^2$, we get several Euclidean 9-designs on two concentric spheres as follows:

1. $J_1=\{2,4,5\}, J_2=\{1,3,6\}$:

$$w_2 = \frac{4\{576A^4-409A^3s+2A^2s(-3191+846s)+As(-7081+8316s-1644s^2)+3s(3060-4117s+1386s^2-137s^3)\}}{1089(A+s)^4} w_{\boldsymbol{v}},$$
$$w_4 = \frac{32\{40A^4-20A^3s-8A^2s(-3+8s)+4As(101-143s+37s^2)+s(-270+559s-286s^2+37s^3)\}}{121(A+s)^4} w_{\boldsymbol{v}},$$
$$w_5 = \frac{125\{18A^4+112A^3s+A^2s(-449+189s)-2As(865-1287s+366s^2)-3s(-504+866s-429s^2+61s^3)\}}{2178(A+s)^4} w_{\boldsymbol{v}},$$
$$v_1 = -\frac{9\{72A^4-157A^3s+10A^2s(14+3s)+As(-265+132s-24s^2)-6s(-40+38s-11s^2+s^3)\}}{605(A+s)^4R_2^4} w_{\boldsymbol{v}},$$
$$v_3 = -\frac{63\{72A^4-157A^3s+10A^2s(14+3s)+As(-265+132s-24s^2)-6s(-40+38s-11s^2+s^3)\}}{484(A+s)^4R_2^4} w_{\boldsymbol{v}},$$
$$v_6 = -\frac{36\{72A^4-157A^3s+10A^2s(14+3s)+As(-265+132s-24s^2)-6s(-40+38s-11s^2+s^3)\}}{605(A+s)^4R_2^4} w_{\boldsymbol{v}},$$

where

- $s=3$: $\alpha(\approx 4.41989) \le A < \frac{109+\sqrt{13033}}{48}$ α is the positive root of $-126+15x-126x^2-15x^3+10x^4$.
- $s=4$: $\frac{11}{\sqrt{5}} \le A < \frac{121}{18}$.
- $s=5$: $\alpha_1(\approx 5.88432) \le A < \alpha_2(\approx 8.79757)$, α_1,α_2 are the positive roots of $-5445-17900x+1240x^2+280x^3+9x^4=0$, $-1025+1450x-785x^2+72x^3=0$, respectively.

- $s = 6$: $\alpha_1(\approx 7.79663) \le A < \alpha_2(\approx 10.8711)$, α_1, α_2 are the positives root of $-7272 - 12638x + 685x^2 + 112x^3 + 3x^4 = 0$, and $-48 - 337x + 320x^2 - 157x^3 + 12x^4 - 0$, respectively.

2. $J_1 = \{2, 4, 5\}, J_2 = \{1, 3, 7\}$:

$$w_2 = \frac{4\{1584A^4 - 1571A^3s + 2A^2s(-6799 + 1944s) + As(-16859 + 19044s - 3756s^2) + 3s(7140 - 9473s + 3174s^2 - 313s^3)\}}{2421(A+s)^4} w_{\boldsymbol{v}},$$
$$w_4 = \frac{32\{80A^4 - 25A^3s + 2A^2(18 - 73s)s + As(931 - 1288s + 332s^2) + s(-630 + 1271s - 644s^2 + 83s^3)\}}{269(A+s)^4} w_{\boldsymbol{v}},$$
$$w_5 = \frac{125\{90A^4 + 140A^3s + A^2s(-901 + 441s) - 2As(2015 - 2907s + 822s^2) + 3s(1176 - 1978s + 969s^2 - 137s^3)\}}{4842(A+s)^4} w_{\boldsymbol{v}},$$
$$v_1 = -\frac{49\{72A^4 - 157A^3s + 10A^2s(14 + 3s) + As(-265 + 132s - 24s^2) - 6s(-40 + 38s - 11s^2 + s^3)\}}{3228(A+s)^4 R_2^4} w_{\boldsymbol{v}},$$
$$v_3 = -\frac{315\{72A^4 - 157A^3s + 10A^2s(14 + 3s) + As(-265 + 132s - 24s^2) - 6s(-40 + 38s - 11s^2 + s^3)\}}{2152(A+s)^4 R_2^4} w_{\boldsymbol{v}},$$
$$v_7 = -\frac{343\{72A^4 - 157A^3s + 10A^2s(14 + 3s) + As(-265 + 132s - 24s^2) - 6s(-40 + 38s - 11s^2 + s^3)\}}{6456(A+s)^4 R_2^4} w_{\boldsymbol{v}},$$

where

- $s = 3$: $\alpha(\approx 4.41134) \le A < \frac{109 + \sqrt{13033}}{48}$, α is the positive root of $-1116 + 165x - 1206x^2 - 75x^3 + 80x^4 = 0$.
- $s = 4$: $\frac{-15 + \sqrt{43265}}{40} \le A < \frac{121}{18}$
- $s = 5$: $\alpha_1(\approx 6.02267) \le A < \alpha_2(\approx 8.79757)$, α_1, α_2 are the positive roots of $-2421 - 8030x + 652x^2 + 70x^3 + 9x^4 = 0$ and $-1025 + 1450x - 785x^2 + 72x^3 = 0$, respectively.
- $s = 6$: $\alpha_1(\approx 7.9655) \le A < \alpha_2(\approx 10.8711)$, α_1, α_2 are the positive roots of $-3240 - 5666x + 349x^2 + 28x^3 + 3x^4 = 0$ and $-48 - 337x + 320x^2 - 157x^3 + 12x^4 = 0$, respectively.

Remark 3.1. *If we allow the situation $J_1 \cap J_2 \neq \emptyset$, we can, of course, obtain Euclidean 9-designs on two concentric spheres in dimensions 3 and 4. By choosing suitable weights and $A = a^2$, we get, for example,*

1. $(n, s) = (3, 2)$, $J_1 = \{2, 3\}$, $J_2 = \{1, 2, 3\}$*:*

$$w_2 = \frac{8(-5+A)(-1+A)^2}{5(2+A)^3} w_{\boldsymbol{v}}, \quad w_3 = \frac{9A(-29 - 2A + A^2)}{10(2+A)^3} w_{\boldsymbol{v}},$$
$$v_1 = \frac{(13-A)(-1+A)^2 A}{(2+A)^4 R_2^4} w_{\boldsymbol{v}}, \quad v_2 = \frac{8(13-A)(-1+A)^2 A}{5(2+A)^4 R_2^4} w_{\boldsymbol{v}}, \quad v_3 = \frac{9(13-A)(-1+A)^2 A}{10(2+A)^4 R_2^4} w_{\boldsymbol{v}},$$

where $1 + \sqrt{30} \le A < 13$*.*

2. $(n,s)=(4,3),\ J_1=\{1,2,3\},\ J_2=\{1,2,4\}$:

$$w_1=\frac{A(135-15A+9A^2-A^3)}{(3+A)^4}w_{\boldsymbol{v}},\quad w_2=\frac{384A}{(3+A)^4}w_{\boldsymbol{v}},\quad w_3=\frac{27(-3+A)(-1+A)^2}{(3+A)^4}w_{\boldsymbol{v}},$$
$$v_1=\frac{16A(-23-2A+A^2)}{3(3+A)^4R_2^4}w_{\boldsymbol{v}},\quad v_2=\frac{16A(-23-2A+A^2)}{(3+A)^4R_2^4}w_{\boldsymbol{v}},\quad v_4=\frac{32A(-23-2A+A^2)}{3(3+A)^4R_2^4}w_{\boldsymbol{v}},$$

where $1+2\sqrt{6}<A\le 9$.

In the next subsection we translate them into the fourth order optimal rotatable designs on the unit ball.

3.2. The fourth order optimal rotatable designs

The fourth order optimal rotatable designs ξ^* on the unit ball form

$$\int_{\boldsymbol{x}\in B^n} f(\boldsymbol{x})\,d\xi^*(\boldsymbol{x})=\frac{W_1}{|S_1|}\int_{\boldsymbol{x}\in S_1} f(\boldsymbol{x})\,d\sigma_1(\boldsymbol{x})+\frac{W_2}{|S_{r_2}|}\int_{\boldsymbol{x}\in S_{r_2}} f(\boldsymbol{x})\,d\sigma_{r_2}(\boldsymbol{x})+W_3 f(\boldsymbol{0}),$$

where W_1, W_2, W_3 and r_2 are given in Table 1. Thus, to construct the fourth order optimal rotatable designs, it is sufficient to construct the corresponding Euclidean 9-designs on the origin and two concentric spheres.

n	W_1	W_2	W_3	r_2
3	0.700638	0.270568	0.0287936	0.692405
4	0.772754	0.212932	0.0143133	0.701394
5	0.81137	0.175789	0.0128406	0.70883
6	0.848569	0.143399	0.00803164	0.713334
7	0.876652	0.118416	0.0049312	0.716583
8	0.901339	0.0966408	0.0020202	0.718159
9	0.916307	0.0822942	0.0013986	0.720264
10	0.928125	0.0708757	0.000999002	0.721975

Table 1: Optimal values of optimal designs ξ^* of degree 4

We give some more information on the above table. Let $h_d=\dim \mathrm{Harm}_d(\mathbb{R}^n)$, and $\phi_{d,i}, i=1,\ldots,h_d$, be a basis of $\mathrm{Harm}_d(\mathbb{R}^n)$ satisfying $\frac{1}{|S_1|}\int_{\boldsymbol{x}\in S_1}\phi_{d,i_1}(\boldsymbol{x})\phi_{d,i_2}(\boldsymbol{x})\,d\rho_1(\boldsymbol{x})=\delta_{i_1,i_2}$, where $\delta_{i,j}$ is Kronecker's delta. We note that the following set is a basis of $\mathcal{P}_d(\mathbb{R}^n)$:

$$\left\{\|\boldsymbol{x}\|^{2j}\phi_{\ell,i}(\boldsymbol{x}) \mid 0\le\ell\le d, 0\le j\le\left\lfloor\frac{d-\ell}{2}\right\rfloor, 1\le i\le h_d\right\}.$$

We then define the information matrix $\mathbf{M}_d(\xi)$ and calculate "optimal values" listed in Table 1; see Bannai and Bannai (2006); Sawa and Hirao (2017) for the details of numerical calculations.

For the rest of this section, we give several examples of the forth order optimal rotatable designs which correspond to Euclidean designs given in the previous subsection.

We obtain the following tables.

s	J_1	J_2	$A(=a^2)$	$w_{\boldsymbol{v}}$	w_k	v_k	# of points
3	$\{2,4,5\}$	$\{1,3,6\}$	5.98153	0.391897	$\{0.0818476, 0.278636, 0.0961881\}$	$\{0.0143399, 0.107549, 0.0215099\}$	1688
3	$\{2,4,6\}$	$\{1,3,5\}$	5.99611	0.391196	$\{0.0789952, 0.328344, 0.0500346\}$	$\{0.014564, 0.100827, 0.0280076\}$	1688
4	$\{2,4,5\}$	$\{1,3,6\}$	8.53058	0.34168	$\{0.10576, 0.360951, 0.0401793\}$	$\{0.0143399, 0.107549, 0.0215099\}$	1688
4	$\{2,4,6\}$	$\{1,3,5\}$	8.53699	0.341541	$\{0.104633, 0.381262, 0.0211334\}$	$\{0.014564, 0.100827, 0.0280076\}$	1688

Table 2: Classification of the fourth optimal rotatable designs of B^6

s	J_1	J_2	$A(=a^2)$	$w_{\boldsymbol{v}}$	w_k	v_k	# of points
3	$\{2,4,5\}$	$\{1,3,6\}$	4.58301	0.46367	$\{0.0470192, 0.039162, 0.326801\}$	$\{0.00861211, 0.0753559, 0.0344484\}$	4298
3	$\{2,4,5\}$	$\{1,3,7\}$	4.58614	0.46331	$\{0.0468277, 0.0404505, 0.326064\}$	$\{0.00837288, 0.0807385, 0.0293051\}$	3978
3	$\{2,4,6\}$	$\{1,3,5\}$	4.64046	0.457187	$\{0.0372289, 0.213066, 0.16917\}$	$\{0.00897095, 0.0645908, 0.0448547\}$	4298
4	$\{2,4,5\}$	$\{1,3,6\}$	6.63188	0.375878	$\{0.0758236, 0.195257, 0.229693\}$	$\{0.00861211, 0.0753559, 0.0344484\}$	5418
4	$\{2,4,5\}$	$\{1,3,7\}$	6.63615	0.375671	$\{0.0756124, 0.196255, 0.229114\}$	$\{0.00837288, 0.0807385, 0.0293051\}$	5098
4	$\{2,4,6\}$	$\{1,3,5\}$	6.6858	0.373287	$\{0.0687076, 0.314588, 0.12007\}$	$\{0.00897095, 0.0645908, 0.0448547\}$	5418
5	$\{2,4,5\}$	$\{1,3,6\}$	8.68408	0.338529	$\{0.0862961, 0.266563, 0.185265\}$	$\{0.00861211, 0.0753559, 0.0344484\}$	4746
5	$\{2,4,5\}$	$\{1,3,7\}$	8.68944	0.338378	$\{0.0860823, 0.267449, 0.184742\}$	$\{0.00837288, 0.0807385, 0.0293051\}$	4426
5	$\{2,4,6\}$	$\{1,3,5\}$	8.7375	0.337041	$\{0.080564, 0.361824, 0.0972231\}$	$\{0.00897095, 0.0645908, 0.0448547\}$	4746
6	$\{2,4,5\}$	$\{1,3,6\}$	10.7347	0.318051	$\{0.0914766, 0.307401, 0.159724\}$	$\{0.00861211, 0.0753559, 0.0344484\}$	2954
6	$\{2,4,5\}$	$\{1,3,7\}$	10.7411	0.317929	$\{0.0912628, 0.308231, 0.15923\}$	$\{0.00837288, 0.0807385, 0.0293051\}$	2634
6	$\{2,4,6\}$	$\{1,3,5\}$	10.7889	0.317032	$\{0.0865653, 0.389058, 0.0839969\}$	$\{0.00897095, 0.0645908, 0.0448547\}$	2954

Table 3: Examples of the fourth order optimal rotatable designs of B^7 with $s \in \{3,4,5,6\}$

4. Proof of Theorem 2.5

In order to prove Theorem 2.5, we mainly use Theorem 2.3, i.e., an extended Sobolev theorem.

We first calculate the dimensions of $\mathrm{Harm}_l(\mathbb{R}^n)^{\mathcal{B}_n}$ with $1 \le l \le 9$. We note that, the dimension of $\mathrm{Harm}_l(\mathbb{R}^n)^{\mathcal{B}_n}$ is determined by the so-called *harmonic Molien series* as follows; see, e.g., Goethals and Seidel (1981):

$$\sum_{l=0}^{\infty}(\dim \mathrm{Harm}_l(\mathbb{R}^n)^{\mathcal{B}_n})x^l = \frac{1}{(1-x^4)(1-x^6)\cdots(1-x^{2n})}$$
$$= \begin{cases} 1+x^4+x^6+x^8+\mathcal{O}(x^{10}) & \text{for } n=3, \\ 1+x^4+x^6+2x^8+\mathcal{O}(x^{10}) & \text{for } n\ge 4, \end{cases}$$

as $x \to \infty$. Comparing the both sides, we know that

$$\begin{cases} \dim \mathrm{Harm}_4(\mathbb{R}^n)^{\mathcal{B}_n} = 1 & \text{for } n \ge 3, \\ \dim \mathrm{Harm}_6(\mathbb{R}^n)^{\mathcal{B}_n} = 1 & \text{for } n \ge 3. \\ \dim \mathrm{Harm}_8(\mathbb{R}^n)^{\mathcal{B}_n} = \begin{cases} 1 & \text{for } n=3, \\ 2 & \text{for } n \ge 4, \end{cases} \\ \dim \mathrm{Harm}_l(\mathbb{R}^n)^{\mathcal{B}_n} = 0 & \text{for } l = 1,2,3,5,7,9. \end{cases}$$

Secondary, to give a basis of $\mathrm{Harm}_l(\mathbb{R}^n)^{\mathcal{B}_n}$, we introduce some more notations. The notation $\mathrm{sym}(f)$ stands for a symmetric polynomial defined by

$$\mathrm{sym}(f) = \frac{1}{|(\mathcal{S}_n)_f|}\sum_{\gamma\in\mathcal{S}_n} f(\gamma(\boldsymbol{x})),$$

where $\mathcal{S}_n$ is the symmetric group and $(\mathcal{S}_n)_f := \{\gamma \in \mathcal{S}_n \mid f(\gamma(\boldsymbol{x})) = f(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \in \mathbb{R}^n\}$. Let

$$\begin{cases} f_{4,1}(\boldsymbol{x}) = \mathrm{sym}(x_1^4) - \frac{6}{n-1}\mathrm{sym}(x_1^2x_2^2), \\ f_{6,1}(\boldsymbol{x}) = \mathrm{sym}(x_1^6) - \frac{15}{n-1}\mathrm{sym}(x_1^2x_2^4) + \frac{180}{(n-1)(n-2)}\mathrm{sym}(x_1^2x_2^2x_3^2), \\ f_{8,1}(\boldsymbol{x}) = \mathrm{sym}(x_1^8) - \frac{28}{n-1}\mathrm{sym}(x_1^2x_2^6) + \frac{70}{n-1}\mathrm{sym}(x_1^4x_2^4) \quad \text{for } n \ge 3, \\ f_{8,2}(\boldsymbol{x}) = \mathrm{sym}(x_1^4x_2^4) - \frac{6}{n-2}\mathrm{sym}(x_1^2x_2^2x_3^4) \\ \qquad\qquad + \frac{108}{(n-2)(n-3)}\mathrm{sym}(x_1^2x_2^2x_3^2x_4^2) \quad \text{for } n \ge 4. \end{cases} \tag{2}$$

Then, by direct calculations, it is not difficult to check that $\Delta f = 0$, $f \in \{f_{4,1}, f_{6,1}, f_{8,1}, f_{8,2}\}$. Thus we know that, the above polynomials $f_{4,1}, f_{6,1}$ and

$f_{8,1}, f_{8,2}$ give a basis of $\mathrm{Harm}_4(\mathbb{R}^n)^{\mathcal{B}_n}$, $\mathrm{Harm}_6(\mathbb{R}^n)^{\mathcal{B}_n}$ and $\mathrm{Harm}_8(\mathbb{R}^n)^{\mathcal{B}_n}$, respectively. Moreover, simple but tedious calculations give us the following lemma:

Lemma 4.1. *(i) By substituting $\boldsymbol{z}_k$ into $f_{4,1}, f_{6,1}, f_{8,1}, f_{8,2}$, we obtain*

$$\begin{cases} f_{4,1}(\boldsymbol{z}_k) = \dfrac{1}{k}\big(1-3\dfrac{k-1}{n-1}\big), \\ f_{6,1}(\boldsymbol{z}_k) = \dfrac{1}{k^2}\big(1-15\dfrac{k-1}{n-1}+30\dfrac{(k-1)(k-2)}{(n-1)(n-2)}\big), \\ f_{8,1}(\boldsymbol{z}_k) = \dfrac{1}{k^3}\big(1+7\dfrac{k-1}{n-1}\big) & \text{for } n \geq 3, \\ f_{8,2}(\boldsymbol{z}_k) = \dfrac{k-1}{2k^3}\big(1-6\dfrac{k-2}{n-2}+9\dfrac{(k-2)(k-3)}{(n-2)(n-3)}\big) & \text{for } n \geq 4. \end{cases}$$

(ii) By substituting $\boldsymbol{v}_{a,s}$ into $f_{4,1}, f_{6,1}, f_{8,1}, f_{8,2}$, we obtain

$$\begin{cases} f_{4,1}(\boldsymbol{v}_{a,s}) = \dfrac{1}{(a^2+s)^2}\big(a^4+s-\dfrac{6}{n-1}\big(sa^2+\dfrac{s(s-1)}{2}\big)\big), \\ f_{6,1}(\boldsymbol{v}_{a,s}) = \dfrac{1}{(a^2+s)^3}\big(a^6+s-\dfrac{15}{n-1}(a^2s+sa^4+s(s-1)) \\ \qquad +\dfrac{180}{(n-1)(n-2)}\big(a^2\dfrac{s(s-1)}{2}+\dfrac{s(s-1)(s-2)}{3!}\big)\big), \\ f_{8,1}(\boldsymbol{v}_{a,s}) = \dfrac{1}{(a^2+s)^4}\big(a^8+s-\dfrac{28}{n-1}(a^2s+sa^6+s(s-1)) \\ \qquad +\dfrac{70}{n-1}\big(a^4s+\dfrac{s(s-1)}{2}\big)\big) \qquad \text{for } n \geq 3, \\ f_{8,2}(\boldsymbol{v}_{a,s}) = \dfrac{1}{(a^2+s)^4}\big(a^4s+\dfrac{(s-1)s}{2}-\dfrac{6}{n-2}\big(a^2(s-1)s+\dfrac{a^4(s-1)s}{2} \\ \qquad +\dfrac{(s-2)(s-1)s}{2}\big)+\dfrac{9s(-3+4a^2+s)(s-2)(s-1)}{2(n-3)(n-2)}\big) \qquad \text{for } n \geq 4. \end{cases}$$

Proof of Theorem 2.5. By combining (2) with Theorem 2.3, $\mathcal{X}(\{a,s\},\mathcal{R},J_1,J_2)$ is a $\mathcal{B}_n$-invariant Euclidean 9-design on two concentric spheres if and only if

$$\begin{aligned} &\sum_{\boldsymbol{x}\in\mathcal{X}(\{a,s\},\mathcal{R},J_1,J_2)} w(\boldsymbol{x})\|\boldsymbol{x}\|^{2j} f_{l,k}(\boldsymbol{x}) \\ &\qquad = \sum_{\boldsymbol{x}\in X_1} w(\boldsymbol{x}) r_1^{2j} f_{l,k}(\boldsymbol{x}) + \sum_{\boldsymbol{x}\in X_2} w(\boldsymbol{x}) r_2^{2j} f_{l,k}(\boldsymbol{x}) \\ &\qquad = r_1^{2j+l} \sum_{\boldsymbol{x}\in X_1} w(\boldsymbol{x}) f_{l,k}(r_1^{-1}\boldsymbol{x}) + r_2^{2j+l} \sum_{\boldsymbol{x}\in X_2} w(\boldsymbol{x}) f_{l,k}(r_2^{-1}\boldsymbol{x}) = 0, \end{aligned} \tag{3}$$

for any $(l,k) \in \{(4,1),(6,1),(8,1)\}$ (when $n=3$) or $(l,k) \in \{(4,1),(6,1),(8,1),(8,2)\}$ (when $n \geq 4$) and nonnegative integer j with $0 \leq j \leq \lfloor \frac{9-l}{2} \rfloor$.

Now we consider the common cases $(l,k) \in \{(4,1),(6,1)\}$. We first let $(l,k)=(4,1)$ (i.e., $j=0,1,2$). If the equations (3) hold, i.e.,

$$\begin{cases} r_1^4 \sum_{\boldsymbol{x}\in X_1} w(\boldsymbol{x}) f_{4,1}(r_1^{-1}\boldsymbol{x}) + r_2^4 \sum_{\boldsymbol{x}\in X_2} w(\boldsymbol{x}) f_{4,1}(r_2^{-1}\boldsymbol{x}) = 0, \\ r_1^6 \sum_{\boldsymbol{x}\in X_1} w(\boldsymbol{x}) f_{4,1}(r_1^{-1}\boldsymbol{x}) + r_2^6 \sum_{\boldsymbol{x}\in X_2} w(\boldsymbol{x}) f_{4,1}(r_2^{-1}\boldsymbol{x}) = 0, \\ r_1^8 \sum_{\boldsymbol{x}\in X_1} w(\boldsymbol{x}) f_{4,1}(r_1^{-1}\boldsymbol{x}) + r_2^8 \sum_{\boldsymbol{x}\in X_2} w(\boldsymbol{x}) f_{4,1}(r_2^{-1}\boldsymbol{x}) = 0, \end{cases}$$

then by noting $r_1 > r_2 > 0$, we have

$$\sum_{\boldsymbol{x}\in X_1} w(\boldsymbol{x}) f_{4,1}(r_1^{-1}\boldsymbol{x}) = \sum_{\boldsymbol{x}\in X_2} w(\boldsymbol{x}) f_{4,1}(r_2^{-1}\boldsymbol{x}) = 0,$$

and vice verse. Secondary, let $(l,k)=(6,1)$ (i.e., $j=0,1$). If the equations (3) hold, i.e.,

$$\begin{cases} r_1^6 \sum_{\boldsymbol{x}\in X_1} w(\boldsymbol{x}) f_{6,1}(r_1^{-1}\boldsymbol{x}) + r_2^6 \sum_{\boldsymbol{x}\in X_2} w(\boldsymbol{x}) f_{6,1}(r_2^{-1}\boldsymbol{x}) = 0, \\ r_1^8 \sum_{\boldsymbol{x}\in X_1} w(\boldsymbol{x}) f_{6,1}(r_1^{-1}\boldsymbol{x}) + r_2^8 \sum_{\boldsymbol{x}\in X_2} w(\boldsymbol{x}) f_{6,1}(r_2^{-1}\boldsymbol{x}) = 0, \end{cases}$$

then we have

$$\sum_{\boldsymbol{x}\in X_1} w(\boldsymbol{x}) f_{6,1}(r_1^{-1}\boldsymbol{x}) = \sum_{\boldsymbol{x}\in X_2} w(\boldsymbol{x}) f_{6,1}(r_2^{-1}\boldsymbol{x}) = 0,$$

and vice verse.

Thus by combining this with Lemma 4.1, we obtain the desired result.

References

Bajnok, B., 2007. Orbits of the hyperoctahedral group as Euclidean designs. Journal of Algebraic Combinatorics 25, 375–397.

Bannai, E., Bannai, E., 2006. On optimal tight 4-designs on 2 concentric spheres. European Journal of Combinatorics 27, 179–192.

Farrell, R. H., Kiefer, J., Walbran, A., 1967. Optimum multivariate designs. In: Proceedings of the Berkeley Symposium. Vol. 1. pp. 113–138.

Gaffke, N., Heiligers, B., 1995a. Algorithms for optimal design with application to multiple polynomial regression. Metrika 42, 173–190.

Gaffke, N., Heiligers, B., 1995b. Computing optimal approximate invariant designs for cubic regression on multidimensional balls and cubes. Journal of Statistical Planning and Inference 47, 347–376.

Gaffke, N., Heiligers, B., 1995c. Optimal and robust invariant designs for cubic multiple regression. Metrika 42, 29–48.

Gaffke, N., Heiligers, B., 1998. Minimum support invariant designs for multiple cubic regression. Journal of Statistical Planning and Inference 72, 229–245.

Goethals, J. M., Seidel, J. J., 1981. Cubature formulae, polytopes, and spherical designs. In: The geometric vein. Springer, New York-Berlin, pp. 203–218.

Hirao, M., Sawa, M., Jimbo, M., 2014. Constructions of Φ_p-optimal rotatable designs on the ball. Sankhyâ Series A 77, 212–236.

Kiefer, J., 1960. Optimum experimental designs V, with applications to systematic and rotatable designs. In: Proceedings of the Berkeley Symposium. Vol. 1. pp. 381–405.

Neumaier, A., Seidel, J. J., 1988. Discrete measures for spherical designs, eutactic stars and lattices. Nederl. Akad. Wetensch. Indag. Math. 50, 321–334.

Neumaier, A., Seidel, J. J., 1992. Measures of strength $2e$ and optimal designs of degree e. Sankhyâ Series A 54, 299–309.

Nozaki, H., Sawa, M., 2012. Note on cubature formulae and designs obtained from group orbits. Canadian Journal of Mathematics 64, 1359–1377.

Pesotchinsky, L., 1978. Φ_p-optimal second order designs for symmetric regions. Journal of Statistical Planning and Inference 2, 173–188.

Sawa, M., Hirao, M., 2017. Characterizing D-optimal rotatable designs with finite reflection groups. Sankhya A 79, 101–132.

Sawa, M., Xu, Y., 2014. On positive cubature rules on the simplex and isometric embeddings. Math. Comp. 83, 1251–1277.

Sobolev, S. L., 1962. Cubature formulas on the sphere which are invariant under transformations of finite rotation groups. Doklady Akademii Nauk SSSR 146, 310–313.