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THE CENTRAL LIMIT THEOREM FOR RIESZ-RAIKOV SUMS II

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Dedicated to Professor Norio Kôno on his 80th birthday

ABSTRACT. For a $d \times d$ expanding matrix A, we investigate randomness of the sequence $\{A^k \boldsymbol{x}\}$ and prove the central limit theorem for $\sum f(A^k \boldsymbol{x})$ where f is a periodic function with a mild regularity condition.

1. INTRODUCTION

For $\theta > 1$ and a real valued square integrable function f on $\mathbf{R}$ with period 1 satisfying $\int_0^1 f = 0$, regarded as random variables on $[0, 1]$ Riesz-Raikov sums $\sum_{k=1}^N f(\theta^k x)$ obey the central limit theorem. This fact was first proved by Fortet [5] and Kac [8] when θ is an integer, and was extended to general case by Petit [16] and by the author [6]. In [7] the case when θ is a complex number was investigated. In this note, we consider a multidimensional analogue of this problem.

Let $\mathbf{R}^d$ denote the vector space of d-dimensional real column vectors and $\widehat{\mathbf{R}}^d$ that of real row vectors.

Fan [4] assumed a very mild condition on A and proved that the sequence $\{A^n \boldsymbol{x}\}$ is uniformly distributed mod 1 for a.e. $\boldsymbol{x}$, and Lesigne [11] extended this result. Since the result of Fan implies the law of large numbers for $\{f(A^n \boldsymbol{x})\}$ where f is periodic and continuous, it is very natural to ask whether the central limit theorem holds.

To have the central limit theorem, we here assume that A is *expanding*, i.e., there exists a $q > 1$ such that

$$\|A\boldsymbol{x}\|_2 \geq q\|\boldsymbol{x}\|_2 \quad \text{for} \quad \boldsymbol{x} \in \mathbf{R}^d, \tag{1.1}$$

or equivalently,

$$\|\boldsymbol{\xi} A\|_2 \geq q\|\boldsymbol{\xi}\|_2 \quad \text{for} \quad \boldsymbol{\xi} \in \widehat{\mathbf{R}}^d. \tag{1.2}$$

Let f be a real valued square integrable function on $\mathbf{R}^d$ which satisfies

$$f(\boldsymbol{x} + \boldsymbol{e}_i) = f(\boldsymbol{x}) \quad (i = 1, \dots, d) \quad \text{and} \quad \int_{[0,1)^d} f(\boldsymbol{x})\, d\boldsymbol{x} = 0, \tag{1.3}$$

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where $\boldsymbol{e}_1, \dots, \boldsymbol{e}_d$ is the canonical basis of $\mathbf{R}^d$. We denote the Fourier series of f by

$$f(\boldsymbol{x}) \sim \sum_{\boldsymbol{\xi} \in \mathbf{Z}^d} \widehat{f}(\boldsymbol{\xi}) \exp(2\pi\sqrt{-1}\,\boldsymbol{\xi}\boldsymbol{x}),$$

where $\mathbf{Z}^d$ is regarded as a subset of $\widehat{\mathbf{R}}^d$, and define the subsum f_a by

$$f_a(\boldsymbol{x}) = \sum_{\boldsymbol{\xi} \in R_a} \widehat{f}(\boldsymbol{\xi}) \exp(2\pi\sqrt{-1}\,\boldsymbol{\xi}\boldsymbol{x}), \tag{1.4}$$

where $R_a = \{\boldsymbol{\xi} = (\xi_1, \dots, \xi_d) \in \mathbf{Z}^d \setminus \{\mathbf{0}\} \mid |\xi_1|, \dots, |\xi_d| \le a\}$. Note that we have $\widehat{f}(\mathbf{0}) = 0$. We assume that f satisfies the condition

$$\sum_{m=1}^{\infty} \|f - f_{2^m}\|_{L^2[0,1)^d} < \infty. \tag{1.5}$$

Note that a function of bounded variation over $[0,1)^d$ in the sense of Hardy-Krause satisfies this condition (Cf. [21]). An easy sufficient condition for (1.5) is the L^2-Dini condition

$$\int_0^1 \frac{\omega^{(2)}(H,f)}{H}\,dH < \infty, \tag{1.6}$$

where

$$\omega^{(2)}(H,f) = \sup\left\{ \|f(\,\cdot\, + \boldsymbol{h}) - f(\,\cdot\,)\|_{L^2[0,1]^d} \;\middle|\; \begin{array}{c} \boldsymbol{h} = (h_1, \dots, h_d) \in \mathbf{R}^d, \\ |h_1|, \dots, |h_d| \le H \end{array} \right\}.$$

This fact is stated in Lemma 2.1.

To state our result, we introduce a quantity $\sigma^2(f)$, the limiting variance of the central limit theorem, by

$$\sigma^2(f) = \sum_{l \ge 0} (2 - \delta_{0,l}) \sum_{\boldsymbol{\xi}, \boldsymbol{\xi}' \in \mathbf{Z}^d} \widehat{f}(\boldsymbol{\xi}) \widehat{f}(\boldsymbol{\xi}') \mathbf{1}(\boldsymbol{\xi} + \boldsymbol{\xi}' A^l = \mathbf{0}). \tag{1.7}$$

The series on the right hand side is shown to be absolutely convergent under the conditions (1.1) and (1.5). We denote the Lebesgue measure on $\mathbf{R}^d$ by Leb.

Theorem 1.1. *Let A be a $d \times d$ real matrix satisfying (1.1), and let f be a real valued function on $\mathbf{R}^d$ satisfying (1.3) and (1.5). Then for every bounded measurable $\Gamma \subset \mathbf{R}^d$,*

$$\lim_{N \to \infty} \int_\Gamma \left(\frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(A^k \boldsymbol{x}) \right)^2 d\boldsymbol{x} = \mathrm{Leb}(\Gamma)\sigma^2(f), \tag{1.8}$$

and

$$\lim_{N \to \infty} \mathrm{Leb}\left\{ \boldsymbol{x} \in \Gamma \;\middle|\; \frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(A^k \boldsymbol{x}) \le t \right\} = \mathrm{Leb}(\Gamma)\Phi_{\sigma^2(f)}(t) \tag{1.9}$$

for any $t \ne 0$, where $\Phi_{\sigma^2(f)}$ is the distribution function of $N(0, \sigma^2(f))$. If $\sigma^2(f) > 0$, then (1.9) holds also for $t = 0$.

In the case when every coefficient of A is an integer, that is the case when A^n are endomorphisms on $\mathbf{T}^d$, Leonov [10], Fan [3], Levin [12] and Conze, Le Borgne, and Roger [2] assumed so called the partially expanding condition and proved the central limit theorem.

Löbbe [13] proved the central limit theorem, the law of the iterated logarithm, and the metric discrepancy results for $\{A_n\boldsymbol{x}\}$, where A_n is a $d \times d$ matrix with integer coefficients satisfying

$$\|\boldsymbol{j}A_{n+k}\|_\infty \geq q^k \|A_n\|_\infty \quad \text{if} \quad \boldsymbol{j} \in \mathbf{Z}^d \quad \text{and} \quad k \geq \log_q \|\boldsymbol{j}\|$$

for some $q > 1$.

By putting $\boldsymbol{j} = (1, 0)$ and $1 < c_1 < c_2$, we see that this condition cannot be satisfied even by

$$A_n = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}^n.$$

This example suggests the following inference. For a matrix A, the sequence of matrix $A_n = A^n$ satisfies this condition only if absolute values of characteristic values of A are all equal.

Since the condition (1.1) implies that modulus of all eigenvalues of A are greater than one, we expect that the central limit theorem is valid under this assumption. But we could not prove because of some technical reason.

In the case of endomorphisms on $\mathbf{T}^d$, if $\sigma(f) = 0$, one can express $f(\boldsymbol{x}) = g(A\boldsymbol{x}) - g(\boldsymbol{x})$ by using some locally square integrable periodic g. We could not prove it in our case, since the situation is more complicated. For example if

$$A = \begin{pmatrix} \sqrt[3]{2} & 0 \\ 0 & \sqrt[2]{3} \end{pmatrix},$$

we have $\sigma(f) = 0$ for the function f of the form

$$f(x) = g_1(A^6\boldsymbol{x}) - g_1(\boldsymbol{x}) + g_2(A^3\boldsymbol{x}) - g_2(\boldsymbol{x}) + g_3(A^2\boldsymbol{x}) - g_3(\boldsymbol{x})$$

where g_1, g_2, and g_3 are locally square integrable and periodic, g_2 does not depend on x_2, and g_3 does not depend of x_1. Although by a similar argument as one-dimensional case, we can prove that $\sigma(f) = 0$ implies this expression, we do not have the result for general case so far.

2. Real Jordan form and related estimates

In this section we state some preliminary facts.

Irrespectively of the value of $\gamma \in \mathbf{N}$, for vectors $\boldsymbol{x} = {}^T(x_1, \dots, x_\gamma) \in \mathbf{R}^\gamma$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_\gamma) \in \widehat{\mathbf{R}}^\gamma$, we define $\|\boldsymbol{x}\| = \max_{\delta \leq \gamma} |x_\delta|$ and $\|\boldsymbol{\xi}\| = \max_{\delta \leq \gamma} |\xi_\delta|$. Abusing the notation we denote by $\mathbf{0}$ zero vectors in $\mathbf{R}^\gamma$ and $\widehat{\mathbf{R}}^\gamma$ for any γ.

For $\lambda \in \mathbf{R}$ a standard Jordan block $J_\gamma(\lambda)$ is a $\gamma \times \gamma$ matrix, and for $\lambda \in \mathbf{C} \setminus \mathbf{R}$ a real Jordan block $C_\gamma(\lambda)$ is a $2\gamma \times 2\gamma$ matrix defined as follows.

$$J_\gamma(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}, \quad C_\gamma(\lambda) = \begin{pmatrix} |\lambda| Z_\theta & Z_0 & & \mathbf{0} \\ & |\lambda| Z_\theta & \ddots & \\ & & \ddots & Z_0 \\ \mathbf{0} & & & |\lambda| Z_\theta \end{pmatrix},$$

where $Z_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $\theta = \arg\lambda$.

It is known (See e.g. Theorem 6.65 of [19]) that a real matrix is similar to the matrix of the form

$$B=\begin{pmatrix} J_{d_1}(\lambda_1) & & & & & 0 \\ & \ddots & & & & \\ & & J_{d_\alpha}(\lambda_\alpha) & & & \\ & & & C_{d_{\alpha+1}/2}(\lambda_{\alpha+1}) & & \\ & & & & \ddots & \\ 0 & & & & & C_{d_\beta/2}(\lambda_\beta) \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_\alpha \in \mathbf{R}$, $\lambda_{\alpha+1}$, $\bar{\lambda}_{\alpha+1}, \ldots, \lambda_\beta$, $\bar{\lambda}_\beta \in \mathbf{C}\setminus\mathbf{R}$ are characteristic values, i.e. there exists a real regular matrix

$$Q=(\boldsymbol{q}_1^{(1)},\ldots,\boldsymbol{q}_{d_1}^{(1)},\boldsymbol{q}_1^{(2)},\ldots,\boldsymbol{q}_{d_2}^{(2)},\ldots,\boldsymbol{q}_1^{(\beta)},\ldots,\boldsymbol{q}_{d_\beta}^{(\beta)})$$

such that $A=QBQ^{-1}$. We simply write $B=\begin{pmatrix} B_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & B_\beta \end{pmatrix}$.

Since Q is regular, there exists a constant $1<D_{1,Q}<\infty$ such that

$$\frac{\|Q\boldsymbol{x}\|}{\|\boldsymbol{x}\|},\frac{\|Q^{-1}\boldsymbol{x}\|}{\|\boldsymbol{x}\|},\frac{\|\boldsymbol{\xi}Q\|}{\|\boldsymbol{\xi}\|},\frac{\|\boldsymbol{\xi}Q^{-1}\|}{\|\boldsymbol{\xi}\|}\in(D_{1,Q}^{-1},D_{1,Q})\quad(\boldsymbol{x}\neq\mathbf{0},\boldsymbol{\xi}\neq\mathbf{0}). \tag{2.1}$$

Denoting the span of $\boldsymbol{q}_1^{(h)}, \ldots, \boldsymbol{q}_{d_h}^{(h)}$ by W_h, we have

$$\mathbf{R}^d=\bigoplus_{h=1}^{\beta}W_h,\quad \dim W_h=d_h\quad\text{and}\quad\sum_{h=1}^{\beta}d_h=d.$$

By calculating $J_\gamma(\lambda)^k$ and $C_\gamma(\lambda)^k$ for $|\lambda|>1$, we have $\|J_\gamma(\lambda)^k\boldsymbol{x}\|\le\gamma k^\gamma|\lambda|^k\|\boldsymbol{x}\|$ and $\|C_\gamma(\lambda)^k\boldsymbol{x}\|\le 2\gamma k^\gamma|\lambda|^k\|\boldsymbol{x}\|$. Hence there exists a constant $D_{2,A}$ depending only on A such that

$$\|A^k\boldsymbol{x}_h\|\le D_{2,A}|\lambda_h|^kk^d\|\boldsymbol{x}_h\|\quad(\boldsymbol{x}_h\in W_h). \tag{2.2}$$

Denote the δ-th component of $\boldsymbol{\xi}$ by $(\boldsymbol{\xi})_\delta$, and the 2-dimensional vector consisting of the δ-th and the $(\delta+1)$-th components of $\boldsymbol{\xi}$ by $(\boldsymbol{\xi})_{\delta,\delta+1}$.

Put $m_1(\mathbf{0})=0$ and $m_1(\boldsymbol{\xi})=|\xi_\delta|$ if $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_\gamma)\in\widehat{\mathbf{R}}^\gamma$ satisfies $\xi_1=\cdots=\xi_{\delta-1}=0\neq\xi_\delta$. In the last case, we have $(\boldsymbol{\xi}J_\gamma(\lambda)^k)_1=\cdots=(\boldsymbol{\xi}J_\gamma(\lambda)^k)_{\delta-1}=0$, $(\boldsymbol{\xi}J_\gamma(\lambda)^k)_\delta=\lambda^k\xi_\delta$, or $m_1(\boldsymbol{\xi}J_\gamma(\lambda)^k)=|\lambda|^km_1(\boldsymbol{\xi})$.

Put $m_2(\mathbf{0})=0$ and $m_2(\boldsymbol{\xi})=\|\boldsymbol{\eta}_\delta\|_2$ if $\boldsymbol{\xi}=(\boldsymbol{\eta}_1,\ldots,\boldsymbol{\eta}_\gamma)\in\widehat{\mathbf{R}}^{2\gamma}$ ($\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_\gamma\in\widehat{\mathbf{R}}^2$) satisfies $\boldsymbol{\eta}_1=\cdots=\boldsymbol{\eta}_{\delta-1}=\mathbf{0}\neq\boldsymbol{\eta}_\delta$. In the last case we have $(\boldsymbol{\xi}C_\gamma(\lambda)^k)_{1,2}=\cdots=(\boldsymbol{\xi}C_\gamma(\lambda)^k)_{2\delta-3,2\delta-2}=\mathbf{0}$ and $(\boldsymbol{\xi}C_\gamma(\lambda)^k)_{2\delta-1,2\delta}=|\lambda|^k\boldsymbol{\eta}_\delta Z_{k\theta}$, or $m_2(\boldsymbol{\xi}C_\gamma(\lambda)^k)=|\lambda|^km_2(\boldsymbol{\xi})$.

For $\boldsymbol{\xi}\in\mathbf{Z}^d\setminus\{\mathbf{0}\}$, we write $\boldsymbol{\xi}Q=(\widetilde{\boldsymbol{\xi}}_1,\ldots,\widetilde{\boldsymbol{\xi}}_\beta)$ where $\widetilde{\boldsymbol{\xi}}_h\in\widehat{\mathbf{R}}^{d_h}$. Put $i(h)=1$ for $h\in[1,\alpha]$ and $i(h)=2$ for $h\in[\alpha+1,\beta]$. The arguments above show that $m_{i(h)}(\widetilde{\boldsymbol{\xi}}_hB_h^k)=|\lambda_h|^km_{i(h)}(\widetilde{\boldsymbol{\xi}}_h)$.

We denote $\Lambda=\min_{h\le\beta}|\lambda_h|>1$. Since Q is regular, there exists an $h\le\beta$ such that $\widetilde{\boldsymbol{\xi}}_h\neq\mathbf{0}$. We denote the smallest such h by $h(\boldsymbol{\xi})$. Denote $D_{3,a}=\min_{\boldsymbol{\xi}\in R_a}m_{i(h(\boldsymbol{\xi}))}(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})})$. The arguments above show that

$$m_{i(h(\boldsymbol{\xi}))}(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B_{h(\boldsymbol{\xi})}^k)\ge D_{3,a}|\lambda_{h(\boldsymbol{\xi})}|^k\ge D_{3,a}\Lambda^k\quad(\boldsymbol{\xi}\in R_a). \tag{2.3}$$

We put

$$D_{4,a}(K)=\min\left\{m_{i(h(\boldsymbol{\xi}))}(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})}+\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})}) \;\middle|\; \begin{array}{l} 0\le k<K,\ \boldsymbol{\xi},\boldsymbol{\xi}'\in R_a, \\ \widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})}+\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})}\neq \mathbf{0}\end{array}\right\},$$

and prove that $D_{4,a}:=D_{4,a}(\infty)$ is positive. Put $\ell_1=\max\{m_{i(h)}(\widetilde{\boldsymbol{\xi}}'_h)\mid \boldsymbol{\xi}'\in R_a, h\le \beta\}$ and $\ell_2=\min\{m_{i(h)}(\widetilde{\boldsymbol{\xi}}'_h)\mid \boldsymbol{\xi}'\in R_a, h\le\beta, m_{i(h)}(\widetilde{\boldsymbol{\xi}}'_h)\neq 0\}>0$. By (2.3), there exists a K_0 such that $k\ge K_0$ implies $m_{i(h(\boldsymbol{\xi}))}(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})})\ge 2\ell_1$.

For $k\ge K_0$, we can verify $m_{i(h(\boldsymbol{\xi}))}(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})}+\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})\ge \ell_1\wedge\ell_2$. Here we prove it in the case $h(\boldsymbol{\xi})=1$. The other case can be proved in the same way. Suppose that $(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})})_1=\cdots=(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})})_{\delta-1}=0$, $|(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})})_\delta|=m_1(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})})$. If $\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})}=\mathbf{0}$, we have $\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})}+\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})}=\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})}$ and $m_1(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})}+\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})=m_1(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})})\ge 2\ell_1$. If not, suppose that $(\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})_1=\cdots=(\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})_{\delta'-1}=0$ and $|(\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})_{\delta'}|=m_1(\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})_{\delta'}$. If $\delta'<\delta$, then we have $(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})}+\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})_1=\cdots=(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})}+\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})_{\delta'-1}=0$ and $|(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})}+\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})_{\delta'}|=|(\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})_{\delta'}|=m_1(\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})_{\delta'}\ge \ell_2$. Hence we have $m_1(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})}+\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})\ge\ell_2$ in this case. If $\delta'=\delta$, then we have $(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})}+\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})_1=\cdots=(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})}+\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})_{\delta-1}=0$ and $|(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})}+\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})_\delta|\ge |(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})})_\delta|-|(\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})_\delta|\ge 2\ell_1-\ell_1$. Hence we have $m_1(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})}+\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})\ge\ell_1$ in this case. If $\delta'>\delta$, then we have $(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})}+\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})_1=\cdots=(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})}+\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})_{\delta-1}=0$ and $|(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})}+\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})_\delta|=|(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})})_\delta|\ge 2\ell_1$. Hence we have $m_1(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})}B^k_{h(\boldsymbol{\xi})}+\widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})})\ge 2\ell_1$ in this case.

Since $D_{4,a}(K_0)$ is positive, we have $D_{4,a}>0$.

Lemma 2.1. *The condition (1.6) implies the condition (1.5).*

Proof. Let $\|\boldsymbol{h}\|\le H$. By

$$f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})=\sum_{\boldsymbol{\xi}\in\mathbf{Z}^d}\widehat{f}(\boldsymbol{\xi})\exp(2\pi\sqrt{-1}\,\boldsymbol{\xi}\boldsymbol{x})\big(\exp(2\pi\sqrt{-1}\,\boldsymbol{\xi}\boldsymbol{h})-1\big),$$

we have

$$\begin{aligned}\big(\omega^{(2)}(H,f)\big)^2&\ge \|f(\,\cdot\,+\boldsymbol{h})-f(\,\cdot\,)\|^2_{L^2[0,1]^d}\\ &=2\sum_{\boldsymbol{\xi}\in\mathbf{Z}^d}|\widehat{f}(\boldsymbol{\xi})|^2\big(1-\operatorname{Re}\exp(2\pi\sqrt{-1}\,\boldsymbol{\xi}\boldsymbol{h})\big).\end{aligned}$$

By integrating both sides over $[\,-H,H\,]^d$ by $\boldsymbol{h}$ and dividing by $(2H)^d$, we have

$$\big(\omega^{(2)}(H,f)\big)^2\ge 2\sum_{\boldsymbol{\xi}\in\mathbf{Z}^d}|\widehat{f}(\boldsymbol{\xi})|^2\left(1-\prod_{i=1}^d\frac{\sin(2\pi\xi_iH)}{2\pi\xi_iH}\right).$$

If $\|\boldsymbol{\xi}\|\ge 1/H$, then there exists a j_0 such that $|\xi_{j_0}|H\ge 1$. Since we have

$$\left|\prod_{i=1}^d\frac{\sin(2\pi\xi_iH)}{2\pi\xi_iH}\right|\le\frac{1}{2\pi|\xi_{j_0}|H}\le\frac{1}{2\pi},$$

we have

$$\big(\omega^{(2)}(H,f)\big)^2 \geq (2-1/\pi) \sum_{\boldsymbol{\xi}\in\mathbf{Z}^d:\|\boldsymbol{\xi}\|\geq 1/H} |\widehat{f}(\boldsymbol{\xi})|^2.$$

Hence we have $\omega^{(2)}(2^{-m},f) \geq \|f - f_{2^m}\|_{L^2[0,1]^d}$. Since the condition (1.6) is equivalent to $\sum_m \omega^{(2)}(2^{-m},f) < \infty$, it implies the condition (1.5). □

3. Fourth moment estimates

In this section, we assume the condition (1.1), Put

$$\rho_0(x) = \Big(\frac{\sin x}{x}\Big)^2 \quad \text{and} \quad \rho_1(x) = \rho_0\big(x/\sqrt{d}\,\big) + \rho_0\big(x/\sqrt{2d}\,\big).$$

By

$$\widehat{\rho}_0(\xi) = \int_{\mathbf{R}} \rho_0(x) e^{2\pi\sqrt{-1}\,\xi x}\,dx = \pi(1-\pi|\xi|)\vee 0,$$

we obtain $\widehat{\rho}_1(\xi) = 0$ if $|\xi| \geq 1/(\pi\sqrt{d}\,)$ and $\rho_1(x) > 0$ for $x \in \mathbf{R}$. By putting

$$\rho(\boldsymbol{x}) = \prod_{s=1}^{d} \rho_1(x_s) > 0 \quad (\boldsymbol{x} = (x_1,\dots,x_d) \in \mathbf{R}^d),$$

by $\sqrt{d}\|\boldsymbol{\xi}\| \geq \|\boldsymbol{\xi}\|_2$, we obtain

$$\widehat{\rho}(\boldsymbol{\xi}) = 0 \quad \text{for} \quad \boldsymbol{\xi} \in \widehat{\mathbf{R}}^d \quad \text{with} \quad \|\boldsymbol{\xi}\| \geq 1/(\pi\sqrt{d}\,) \quad \text{or} \quad \|\boldsymbol{\xi}\|_2 \geq 1/\pi.$$

Lemma 3.1. *Assume that A satisfies (1.1), For any bounded measurable set $\Gamma \subset \mathbf{R}^d$, and for any trigonometric polynomial f_a satisfying (1.3), there exists a constant D_{5,Γ,A,f_a} such that*

$$\int_\Gamma \max_{n\in\Delta} \Big(\sum_{k\in\Delta:k\leq n} f_a(A^k\boldsymbol{x})\Big)^4 d\boldsymbol{x} \leq D_{5,\Gamma,A,f_a}(^\#\Delta)^2 \tag{3.1}$$

for any finite set $\Delta \subset \mathbf{N}$.

Proof. Since we can take a constant $D_{6,\Gamma} < \infty$ such that $\mathbf{1}_\Gamma(\boldsymbol{x}) \leq D_{6,\Gamma}\,\rho(\boldsymbol{x})$, it is sufficient to prove

$$\int_{\mathbf{R}^d} \max_{n\in\Delta} \Big(\sum_{k\in\Delta:k\leq n} f_a(A^k\boldsymbol{x})\Big)^4 \rho(\boldsymbol{x})\,d\boldsymbol{x} \leq D_{7,A,f_a} C(^\#\Delta)^2.$$

Komlós-Révész [9] proved the following: Suppose that X is a non-empty set, m is a σ-finite measure on X, and $\{\varphi_i\}$ is a sequence of real valued measurable functions satisfying

$$\int_X \varphi_{k_1}^4\,dm \leq M, \quad \int_X \varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\,dm = 0 \quad (k_1 < k_2 < k_3 < k_4), \tag{3.2}$$

for some $M < \infty$, then there exists an absolute constant D_8 such that

$$\int_X \Big(\sum_{k=1}^N c_k\varphi_k\Big)^4 dm \leq D_8 \Big(\sum_{k=1}^N c_k^2\Big)^2 \quad (N \in \mathbf{N}). \tag{3.3}$$

Noting this estimate and by applying Erdős-Stečkin Theorem (See [15]), we can derive

$$\int_X \max_{n\leq N} \Big(\sum_{k=1}^n c_k\varphi_k\Big)^4 dm \leq D_9 \Big(\sum_{k=1}^N c_k^2\Big)^2 \quad (N \in \mathbf{N}),$$

where D_9 is an absolute constant. Note that any subsequence of $\{\varphi_k\}$ satisfying (3.2) also satisfies (3.2) and our version

$$\int_X \max_{n\in\Delta}\Big(\sum_{k\in\Delta:k\le n} c_k\varphi_k\Big)^4 dm \le D_9\Big(\sum_{k\in\Delta} c_k^2\Big)^2$$

follows.

We use the expression

$$f_a(\boldsymbol{x}) = \sum_{\boldsymbol{\xi}\in R_a} |\widehat{f}(\boldsymbol{\xi})|\cos(2\pi\boldsymbol{\xi}\boldsymbol{x}+\gamma_{\boldsymbol{\xi}})$$

to have

$$\sum_{k=1}^N f_a(A^k\boldsymbol{x}) = \sum_{\boldsymbol{\xi}\in R_a} |\widehat{f}(\boldsymbol{\xi})|\sum_{k=1}^N \cos(2\pi\boldsymbol{\xi}A^k\boldsymbol{x}+\gamma_{\boldsymbol{\xi}}).$$

Because of $\sum_{\boldsymbol{\xi}\in R_a}|\widehat{f}(\boldsymbol{\xi})|<\infty$, if we have (3.3) for $\varphi_k(\,\cdot\,)=\cos(2\pi\boldsymbol{\xi}A^k\cdot+\gamma_{\boldsymbol{\xi}})$, then we have (3.3) for $\varphi_k(\,\cdot\,)=f_a(A^k\,\cdot\,)$. Hence it is enough to prove that the sequence $\{\cos(2\pi\boldsymbol{\xi}A^k\boldsymbol{x}+\gamma_{\boldsymbol{\xi}})\}_{k\in\mathbf{N}}$ satisfies (3.2) under the measure $\rho(\boldsymbol{x})\,d\boldsymbol{x}$.

Take $p\in\mathbf{N}$ large enough to satisfy

$$\Lambda^{3p}\ge (3D_{3,a}^{-1}D_{1,Q}/\pi)\vee 3^3.$$

For $r=0,\,1,\,\ldots,\,p-1$, we show that $\{\cos(2\pi\boldsymbol{\xi}A^{kp-r}\boldsymbol{x}+\gamma_{\boldsymbol{\xi}})\}_{k\in\mathbf{N}}$ satisfies (3.2) under the measure $\rho(\boldsymbol{x})\,d\boldsymbol{x}$. Then we can see that it satisfies (3.3), and then by using Minkowski's inequality we see that $\{\cos(2\pi\boldsymbol{\xi}A^k\boldsymbol{x}+\gamma_{\boldsymbol{\xi}})\}_{k\in\mathbf{N}}$ itself satisfies (3.3).

We first note that for $h\le\beta$ and $k_1<k_2<k_3<k_4$,

$$\begin{aligned}
&|\lambda_h^{k_4p-r}\pm\lambda_h^{k_3p-r}\pm\lambda_h^{k_2p-r}\pm\lambda_h^{k_1p-r}|\\
&\ge|\lambda_h^{k_4p-r}|-|\lambda_h^{(k_4-1)p-r}|-|\lambda_h^{(k_4-2)p-r}|-\cdots\\
&\ge\Big(1-\frac{1}{|\lambda_h^p|-1}\Big)|\lambda_h^{k_4p-r}|\ge\Big(1-\frac{1}{\Lambda^p-1}\Big)\Lambda^{k_4p-r}\\
&\ge\frac12\Lambda^{k_4p-r}\ge\frac12\Lambda^{3p}.
\end{aligned}$$

By putting $\varsigma_4=1$, we have

$$\begin{aligned}
&\prod_{s=1}^4\cos(2\pi\boldsymbol{\xi}A^{k_sp-r}\boldsymbol{x}+\gamma_{\boldsymbol{\xi}})\\
&=8^{-1}\sum_{\varsigma_1,\varsigma_2,\varsigma_3=\pm1}\exp\Big(2\pi\sqrt{-1}\sum_{s=1}^4\varsigma_s(\boldsymbol{\xi}QB^{k_sp-r}Q^{-1}\boldsymbol{x}+\gamma_{\boldsymbol{\xi}})\Big)\\
&=8^{-1}\sum_{\varsigma_1,\varsigma_2,\varsigma_3=\pm1}\exp(2\pi\sqrt{-1}\,(\varsigma_1+\cdots+\varsigma_4)\gamma_{\boldsymbol{\xi}})\\
&\qquad\times\exp\Big(2\pi\sqrt{-1}\,\Big(\widetilde{\boldsymbol{\xi}}_1\sum_{s=1}^4\varsigma_sB_1^{k_sp-r},\ldots,\widetilde{\boldsymbol{\xi}}_\beta\sum_{s=1}^4\varsigma_sB_\beta^{k_sp-r}\Big)Q^{-1}\boldsymbol{x}\Big).
\end{aligned}$$

Suppose that $h(\boldsymbol{\xi})\le\alpha$. By denoting $h(\boldsymbol{\xi})$ simply by h, denoting $\widetilde{\boldsymbol{\xi}}_h$ by $(\xi_1,\ldots,\xi_{d_h})$, and by taking a δ such that $\xi_1=\cdots=\xi_{\delta-1}=0\ne\xi_\delta$, by $|\xi_\delta|=$

$m_1(\widetilde{\boldsymbol{\xi}}_h) \geq D_{3,a}$, we have

$$\left|\left(\widetilde{\boldsymbol{\xi}}_h \sum_{s=1}^{4} \varsigma_s B_h^{k_s p - r}\right)_\delta\right| = \left|\xi_\delta \sum_{s=1}^{4} \varsigma_s \lambda_h^{k_s p - r}\right| \geq \frac{D_{3,a}\Lambda^{3p}}{2}.$$

Hence we have

$$\left\|\boldsymbol{\xi} Q \sum_{s=1}^{4} \varsigma_s B^{k_s p - r} Q^{-1}\right\| \geq \frac{D_{1,Q}^{-1} D_{3,a}\Lambda^{3p}}{2} \geq \frac{1}{\pi},$$

which implies

$$\int_{\mathbf{R}^d} \prod_{s=1}^{4} \cos(2\pi \boldsymbol{\xi} A^{k_s p - r}\boldsymbol{x} + \gamma_{\boldsymbol{\xi}})\rho(\boldsymbol{x})\, d\boldsymbol{x} = 0. \tag{3.4}$$

Suppose that $h(\boldsymbol{\xi}) > \alpha$. By denoting $h(\boldsymbol{\xi})$ simply by h, denoting $\widetilde{\boldsymbol{\xi}}_h$ by $(\boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_{d_h/2})$, and by taking a δ such that $\boldsymbol{\eta}_1 = \cdots = \boldsymbol{\eta}_{\delta-1} = \mathbf{0} \neq \boldsymbol{\eta}_\delta$, we have

$$\begin{aligned}
&\left\|\left(\widetilde{\boldsymbol{\xi}}_h \sum_{s=1}^{4} \varsigma_s B_h^{k_s p - r}\right)_{2\delta-1,2\delta}\right\|_2 \\
&= \left\|\boldsymbol{\eta}_\delta \sum_{s=1}^{4} \varsigma_s \lambda_h^{k_s p - r} Z_{\theta(k_s p - r)}\right\|_2 \\
&\geq \|\boldsymbol{\eta}_\delta\|_2 \big(|\lambda_h|^{k_4 p - r} - |\lambda_h|^{k_3 p - r} - |\lambda_h|^{k_2 p - r} - |\lambda_h|^{k_1 p - r}\big) \\
&\geq \frac{m_2(\widetilde{\boldsymbol{\xi}}_h)\Lambda^{3p}}{2} \geq \frac{D_{3,a}\Lambda^{3p}}{2}.
\end{aligned}$$

Hence in the same way as before, we can verify (3.4). □

4. Martingale approximation

Take $L_\delta^{(h)} \in \mathbf{R}$ ($\delta = 1, \dots, d_h$, $h = 1, \dots \beta$) and $L > 0$ arbitrarily and put

$$\Omega = \left\{ \sum_{h=1}^{\beta} \sum_{\delta=1}^{d_h} t_\delta^{(h)} \boldsymbol{q}_\delta^{(h)} \;\middle|\; L_\delta^{(h)} \leq t_\delta^{(h)} < L_\delta^{(h)} + L \right\}. \tag{4.1}$$

Let $\mathcal{F}$ be the Borel σ-field on Ω and put

$$P_\Omega(B) = \frac{\mathrm{Leb}(B)}{\mathrm{Leb}(\Omega)} \quad (B \in \mathcal{F}).$$

We consider the sequence $\{f_a(A^k \cdot)\}$ on the probability space $(\Omega, \mathcal{F}, P_\Omega)$. We state the almost sure invariance principle for the sequence. We denote the Lebesgue measure on $[0,1)$ by leb.

Proposition 4.1. *Let A be a $d \times d$ real matrix satisfying (1.1), and let f_a be a trigonometric polynomial on $\mathbf{R}^d$ satisfying (1.3). By taking the product probability space $(\Omega \times [0,1), \mathcal{F} \otimes \mathcal{B}([0,1)), P_\Omega \times \mathrm{leb})$ and regard the sequence $\{f_a(A^k \cdot)\}$ defined on this space. If $\sigma^2(f_a) > 0$, then we can define a sequence $\{Z_i\}$ of standard normal i.i.d. such that*

$$\sum_{k=1}^{N} f_a(A^k \cdot) = \sum_{i \leq N\sigma^2(f_a)} Z_i + o(N^{62/125}) \quad a.s. \tag{4.2}$$

From this proposition, we can derive the central limit theorem.

Corollary 4.2. *Let A be a $d \times d$ real matrix satisfying (1.1), and let f_a be a trigonometric polynomial on $\mathbf{R}^d$ satisfying (1.3). Then for any probability measure P on $\mathbf{R}^d$ which is absolutely continuous with respect to Lebesgue measure, on the probability space $(\mathbf{R}^d, \mathcal{B}^d, P)$ we have the convergence in law*

$$\frac{1}{\sqrt{N}} \sum_{k=1}^{N} f_a(A^k \cdot\,) \xrightarrow{\mathcal{D}} N(0, \sigma^2(f_a)) \quad (N \to \infty). \tag{4.3}$$

Proof of Proposition 4.1. We divide the increasing sequence $\mathbf{N}$ of positive integers into consecutive blocks

$$\mathbf{N} = \Delta_1' \cup \Delta_1 \cup \Delta_2' \cup \Delta_2 \cup \dots$$

where

$$^{\#}\Delta_i = \lfloor i^{2/3} \rfloor \quad \text{and} \quad ^{\#}\Delta_i' = \lfloor 1 + (9 + 5d/3) \log_\Lambda i \rfloor.$$

Put $i^- = \min \Delta_i$ and $i^+ = \max \Delta_i$. Clearly we have

$$i^+ \le (2 + (9+5d/3)/\log\Lambda) i^{5/3} \quad \text{and} \quad i^- - (i-1)^+ = {}^{\#}\Delta_i'.$$

Put

$$\mu_h(i) = \lfloor \log_2 (i^{4+5d/3} |\lambda_h|^{i^+}) \rfloor.$$

For $i \in \mathbf{N}$, $1 \le h \le \beta$, $1 \le \delta \le d_h$, and $j_\delta^{(h)} = 0, \dots, 2^{\mu_h(i)} - 1$, we set

$$\begin{aligned} &J\big(i, (j_1^{(1)}, \dots, j_{d_1}^{(1)}), (j_1^{(2)}, \dots, j_{d_2}^{(2)}), \dots, (j_1^{(\beta)}, \dots, j_{d_\beta}^{(\beta)})\big) \\ &= \left\{ \sum_{h=1}^{\beta} \sum_{\delta=1}^{d_h} (L_\delta^{(h)} + L 2^{-\mu_h(i)} (j_\delta^{(h)} + t_\delta^{(h)})) \boldsymbol{q}_\delta^{(h)} \;\middle|\; 0 \le t_\delta^{(h)} < 1 \right\} \end{aligned}$$

and denote the collection of all such cubes by $\mathcal{J}(i)$. Let $\mathcal{F}_i$ be the σ-field on Ω generated by $\mathcal{J}(i)$. $\{\mathcal{F}_i\}$ forms a filtration on $(\Omega, \mathcal{F}, P_\Omega)$. Let

$$\widetilde{\mathcal{F}}_i = \{ F \times [\,0,1) \mid F \in \mathcal{F}_i \}.$$

Clearly $\{\widetilde{\mathcal{F}}_i\}$ forms a filtration on $(\Omega \times [\,0,1), \mathcal{F} \times \mathcal{B}[\,0,1), P_\Omega \times \mathrm{leb})$.

For $(\boldsymbol{x}, x) \in \Omega \times [\,0,1)$, we here put

$$\widetilde{T}_i(\boldsymbol{x}, x) = T_i(\boldsymbol{x}) = \sum_{k \in \Delta_i} f_a(A^k \boldsymbol{x})$$

and prove

$$\widetilde{E}(\widetilde{T}_i \mid \widetilde{\mathcal{F}}_{i-1})(\boldsymbol{x}, x) = E(T_i \mid \mathcal{F}_{i-1})(\boldsymbol{x}) = O(i^{-4}), \tag{4.4}$$

where $\widetilde{E}(\,\cdot \mid \cdot\,)$ denotes the conditional expectation on $\Omega \times [\,0,1)$ and $E(\,\cdot \mid \cdot\,)$ that on Ω. The first equality is trivial. Take $\boldsymbol{x} \in \Omega$ arbitrarily and take $J \in \mathcal{J}(i-1)$ such that $\boldsymbol{x} \in J$. We note that

$$E(X \mid \mathcal{F}_{i-1})(\boldsymbol{x}) = \frac{1}{\mathrm{Leb}(J)} \int_J X(\boldsymbol{y})\, d\boldsymbol{y}.$$

By putting

$$R_i = \begin{pmatrix} L2^{-\mu_1(i)} E_{d_1} & O & \cdots & O \\ O & L2^{-\mu_2(i)} E_{d_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & L2^{-\mu_\beta(i)} E_{d_\beta} \end{pmatrix},$$

where E_γ is the unit matrix of size $\gamma \times \gamma$, we can write

$$J = \{\boldsymbol{b} + QR_{i-1}\boldsymbol{t} \mid \boldsymbol{t} \in [0,1)^d\}$$

by using some $\boldsymbol{b} \in \mathbf{R}^d$. Changing variables by $\boldsymbol{y} = \boldsymbol{b} + QR_{i-1}\boldsymbol{t}$ and noting

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{t}} = |\det(QR_{i-1})| = \mathrm{Leb}(J),$$

we have

$$\begin{aligned}
&E(\exp(2\pi\sqrt{-1}\,\boldsymbol{\xi}A^k\,\cdot\,) \mid \mathcal{F}_{i-1})(\boldsymbol{x}) \\
&= \frac{1}{\mathrm{Leb}(J)} \int_J \exp(2\pi\sqrt{-1}\,\boldsymbol{\xi}A^k\boldsymbol{y})\,d\boldsymbol{y} \\
&= \int_{[0,1)^d} \exp(2\pi\sqrt{-1}\,\boldsymbol{\xi}QB^kQ^{-1}(\boldsymbol{b} + QR_{i-1}\boldsymbol{t}))\,d\boldsymbol{t} \\
&= \exp(2\pi\sqrt{-1}\,c) \prod_{h=1}^{\beta} \int_{[0,1)^{d_h}} \exp(2\pi\sqrt{-1}\,L2^{-\mu_h(i-1)}\widetilde{\boldsymbol{\xi}}_h B_h^k \boldsymbol{t}_h)\,d\boldsymbol{t}_h,
\end{aligned}$$

where $c = \boldsymbol{\xi}QB^kQ^{-1}\boldsymbol{b}$, and $\widetilde{\boldsymbol{\xi}}_h \in \widehat{\mathbf{R}}^{d_h}$ and $\boldsymbol{t}_h \in \mathbf{R}^{d_h}$ are given by $\boldsymbol{\xi}Q = (\widetilde{\boldsymbol{\xi}}_1, \dots, \widetilde{\boldsymbol{\xi}}_\beta)$ and $\boldsymbol{t} = \begin{pmatrix} \boldsymbol{t}_1 \\ \vdots \\ \boldsymbol{t}_\beta \end{pmatrix}$. If we write $\widetilde{\boldsymbol{\xi}}_h B_h^k = (\zeta_1^{(h)}, \dots, \zeta_{d_h}^{(h)})$, we have

$$\begin{aligned}
&\int_{[0,1)^{d_h}} \exp(2\pi\sqrt{-1}\,L2^{-\mu_h(i-1)}\widetilde{\boldsymbol{\xi}}_h B_h^k \boldsymbol{t}_h)\,d\boldsymbol{t}_h \\
&= \prod_{\delta=1}^{d_h} \int_0^1 \exp(2\pi\sqrt{-1}\,L2^{-\mu_h(i-1)}\zeta_\delta^{(h)} t)\,dt \\
&= \prod_{\delta=1}^{d_h} \phi(\pi L2^{-\mu_h(i-1)}\zeta_\delta^{(h)}) \exp(\pi\sqrt{-1}\,c'),
\end{aligned}$$

where $c' = L2^{-\mu_h(i-1)}\zeta_\delta^{(h)}$, $\phi(x) = (\sin x)/x$ if $x \neq 0$ and $\phi(0) = 1$.

By (2.3), there exists a $\delta(\boldsymbol{\xi})$ such that $|\zeta_{\delta(\boldsymbol{\xi})}^{(h(\boldsymbol{\xi}))}| \geq D_{3,a}|\lambda_{h(\boldsymbol{\xi})}|^k/2$. Hence we have

$$\begin{aligned}
\phi(\pi L2^{-\mu_{h(\boldsymbol{\xi})}(i-1)}\zeta_{\delta(h(\boldsymbol{\xi}))}^{(h(\boldsymbol{\xi}))}) &\leq 2/\pi L2^{-\mu_{h(\boldsymbol{\xi})(i-1)}} D_{3,a}|\lambda_{h(\boldsymbol{\xi})}|^k \\
&\leq 2(i-1)^{4+5d/3}|\lambda_{h(\boldsymbol{\xi})}|^{(i-1)^+}/\pi D_{3,a}|\lambda_{h(\boldsymbol{\xi})}|^{i^-} L \\
&\leq 2i^{4+5d/3}\Lambda^{(i-1)^+ - i^-}/\pi D_{3,a}L = O(i^{-5}).
\end{aligned}$$

By

$$T_i(\boldsymbol{y}) = \sum_{k\in\Delta_i} \sum_{\boldsymbol{\xi}\in R_a} \widehat{f}(\boldsymbol{\xi}) \exp(2\pi\sqrt{-1}\,\xi A^k \boldsymbol{y}),$$

we have (4.4).

Secondly, we prove

$$\widetilde{E}(\widetilde{T}_i \mid \widetilde{\mathcal{F}}_i)(\boldsymbol{x}, x) - \widetilde{T}_i(\boldsymbol{x}, x) = E(T_i \mid \mathcal{F}_i)(\boldsymbol{x}) - T_i(\boldsymbol{x}) = O(i^{-3}). \tag{4.5}$$

Assume that $k \in \Delta_i$ and $\boldsymbol{x} \in J \in \mathcal{J}(i)$. Again the first equality is trivial. We have

$$E(f_a(A^k\,\cdot\,) \mid \mathcal{F}_i)(\boldsymbol{x}) - f_a(A^k\boldsymbol{x}) = \frac{1}{\mathrm{Leb}(J)} \int_J (f_a(A^k\boldsymbol{y}) - f_a(A^k\boldsymbol{x}))\,d\boldsymbol{y}.$$

By $\boldsymbol{x}$, $\boldsymbol{y} \in J$, we have

$$\boldsymbol{y} - \boldsymbol{x} = \sum_{h=1}^{\beta} L2^{-\mu_h(i)} \sum_{\delta=1}^{d_h} t_\delta^{(h)} \boldsymbol{q}_\delta^{(h)}$$

for some $-1 < t_\delta^{(h)} < 1$. By (2.2), we have

$$\|A^k \boldsymbol{q}_\delta^{(h)}\| \le D_{2,A}(\max_{\delta,h} \|\boldsymbol{q}_\delta^{(h)}\|)|\lambda_h|^{i^+}(i^+)^d$$

and

$$\|A^k\boldsymbol{y} - A^k\boldsymbol{x}\| \le \sum_{h=1}^{\beta} L2^{-\mu_h(i)} \sum_{\delta=1}^{d_h} \|A^k \boldsymbol{q}_\delta^{(h)}\| = O(i^{-4}).$$

Lipschitz continuity of f implies

$$\left|E(f_a(A^k\,\cdot\,) \mid \mathcal{F}_i)(\boldsymbol{x}) - f_a(A^k\boldsymbol{x})\right| = O(i^{-4}),$$

and thereby (4.5).

Put

$$Y_i = E(T_i \mid \mathcal{F}_i) - E(T_i \mid \mathcal{F}_{i-1}) \quad \text{and} \quad \widetilde{Y}_i = \widetilde{E}(\widetilde{T}_i \mid \widetilde{\mathcal{F}}_i) - \widetilde{E}(\widetilde{T}_i \mid \widetilde{\mathcal{F}}_{i-1}).$$

Clearly $\{Y_i, \mathcal{F}_i\}$ and $\{\widetilde{Y}_i, \widetilde{\mathcal{F}}_i\}$ are martingale differences and $\widetilde{Y}_i(\boldsymbol{x}, x) = Y_i(\boldsymbol{x})$. By combining (4.4) and (4.5), we have

$$\|\widetilde{Y}_i - \widetilde{T}_i\|_\infty = \|Y_i - T_i\|_\infty = O(i^{-3}). \tag{4.6}$$

By $\|T_i\|_\infty = O(i)$, we have $\|E(T_i \mid \mathcal{F}_i)\|_\infty$, $\|E(T_i \mid \mathcal{F}_{i-1})\|_\infty = O(i)$, and $\|Y_i\|_\infty = O(i)$, which implies $\|Y_i + T_i\|_\infty = O(i)$ and

$$\|Y_i^2 - T_i^2\|_\infty = O(i^{-2}).$$

By $\|Y_i^2 + T_i^2\|_\infty = O(i^2)$, we have

$$\|Y_i^4 - T_i^4\|_\infty = O(1).$$

By the last inequality and (3.1), we have

$$\widetilde{E}\widetilde{Y}_i^4 = \widetilde{E}\widetilde{T}_i^4 + O(1) = ET_i^4 + O(1) = O(i^{4/3}). \tag{4.7}$$

We have

$$\Big(\sum_{k\in\Delta_i} f_a(A^k\boldsymbol{x})\Big)^2 = \sum_{\boldsymbol{\xi}\in R_a} \sum_{\boldsymbol{\xi}'\in R_a} \widehat{f}(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi}') \sum_{k\in\Delta_i} \sum_{k'\in\Delta_i} \exp(2\pi\sqrt{-1}\,(\boldsymbol{\xi}A^k + \boldsymbol{\xi}'A^{k'})\boldsymbol{x}).$$

Put

$$v_i = \sum_{\boldsymbol{\xi}\in R_a} \sum_{\boldsymbol{\xi}'\in R_a} \widehat{f}(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi}') \sum_{k\in\Delta_i} \sum_{k'\in\Delta_i} \mathbf{1}(\boldsymbol{\xi}A^k + \boldsymbol{\xi}'A^{k'} = \mathbf{0}).$$

There exists an l_0 such that for all $\boldsymbol{\xi} \in R_a$ and $\boldsymbol{\xi}' \in R_a$, $\boldsymbol{\xi}A^l + \boldsymbol{\xi}' \ne \mathbf{0}$ and $\boldsymbol{\xi} + \boldsymbol{\xi}'A^l \ne \mathbf{0}$ hold for $l > l_0$. If $l \le l_0$ and $\boldsymbol{\xi} + \boldsymbol{\xi}'A^l = \mathbf{0}$, we have

$$0 \le i - {}^{\#}\{(k,k') \in \Delta_i^2 \mid \boldsymbol{\xi}A^k + \boldsymbol{\xi}'A^{k'} = \mathbf{0}\} = i - (i - l + 1) \vee 0 \le l_0.$$

By noting

$$i\sigma^2(f_a) = i \sum_{\boldsymbol{\xi}\in R_a} \sum_{\boldsymbol{\xi}'\in R_a} \widehat{f}(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi}')\mathbf{1}\begin{pmatrix} \boldsymbol{\xi}A^l + \boldsymbol{\xi}' = \mathbf{0} \text{ or } \boldsymbol{\xi} + \boldsymbol{\xi}'A^l = \mathbf{0} \\ \text{for some } l \ge 0 \end{pmatrix},$$

we have

$$|v_i - i\sigma^2(f_a)| \le l_0 \sum_{\boldsymbol{\xi}\in R_a} \sum_{\boldsymbol{\xi}'\in R_a} |\widehat{f}(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi}')|. \tag{4.8}$$

For $\boldsymbol{x} \in \Omega$, take $J \in \mathcal{F}_{i-1}$ such that $\boldsymbol{x} \in J$. Under the condition $\boldsymbol{\xi}A^k + \boldsymbol{\xi}'A^{k'} \neq \mathbf{0}$, in the same way as before, we have

$$\begin{aligned}
&E(\exp(2\pi\sqrt{-1}\,(\boldsymbol{\xi}A^k + \boldsymbol{\xi}'A^{k'})\cdot) \mid \mathcal{F}_{i-1})(\boldsymbol{x}) \\
&= \frac{1}{\mathrm{Leb}(J)} \int_J \exp(2\pi\sqrt{-1}\,(\boldsymbol{\xi}A^k + \boldsymbol{\xi}'A^{k'})\boldsymbol{y})\,d\boldsymbol{y} \\
&= \exp(2\pi\sqrt{-1}\,c) \prod_{h=1}^{\beta} \int_{[0,1)^{d_h}} \exp(2\pi\sqrt{-1}\,(\widetilde{\boldsymbol{\xi}}_h B_h^{k-k'} + \widetilde{\boldsymbol{\xi}}'_h) B_h^{k'} L 2^{-\mu_h(i-1)}\boldsymbol{t}_h)\,d\boldsymbol{t}_h
\end{aligned}$$

and by $m_{i(h(\boldsymbol{\xi}))}(\widetilde{\boldsymbol{\xi}}_{h(\boldsymbol{\xi})} B_{h(\boldsymbol{\xi})}^{k-k'} + \widetilde{\boldsymbol{\xi}}'_{h(\boldsymbol{\xi})}) \ge D_{4,a}$, as before we have

$$E(\exp(2\pi\sqrt{-1}\,(\boldsymbol{\xi}A^k + \boldsymbol{\xi}'A^{k'})\cdot) \mid \mathcal{F}_{i-1})(\boldsymbol{x}) = O(i^{-5}).$$

Since the number of choices of $(\boldsymbol{\xi}, \boldsymbol{\xi}')$ is finite and the number of choices of (k, k') is at most i, we have

$$E(T_i^2 - v_i \mid \mathcal{F}_{i-1})(\boldsymbol{x}) = O(i^{-4}).$$

By putting

$$\beta_M = \sum_{i=1}^{M} v_i \quad \text{and} \quad l_M = {}^{\#}\Delta_1 + \cdots + {}^{\#}\Delta_M,$$

we have

$$\Big\| \sum_{i=1}^{M} E(T_i^2 \mid \mathcal{F}_{i-1}) - \beta_M \Big\|_\infty = O(1) \quad \text{and} \quad |\beta_M - l_M\sigma^2(f_a)| \le D_{10,f_a} M$$

for some $D_{10,f_a} < \infty$. Since we have

$$\Big\| \sum_{i=1}^{M} \big(E(T_i^2 \mid \mathcal{F}_{i-1}) - E(Y_i^2 \mid \mathcal{F}_{i-1})\big) \Big\|_\infty \le \sum_{i=1}^{M} \|T_i^2 - Y_i^2\|_\infty = O(1),$$

we have

$$\|\widetilde{V}_M - \beta_M\|_\infty \le D_{11}, \tag{4.9}$$

where

$$\widetilde{V}_M = \sum_{i=1}^{M} \widetilde{E}(\widetilde{Y}_i^2 \mid \widetilde{\mathcal{F}}_{i-1}) = \sum_{i=1}^{M} E(Y_i^2 \mid \mathcal{F}_{i-1}) \quad \text{and} \quad D_{11} < \infty.$$

Now we use the theorem which is a version of Strassen's theorem (Theorem 4.4 of [18]).

Theorem 4.3 (Monrad-Philipp [14] Theorem 7. This version is Lemma A.4 in Philipp [17])**.** *Let $\{\widetilde{Y}_i, \widetilde{\mathcal{F}}_i\}$ be a square integrable martingale difference satisfying*

$$\widetilde{V}_M = \sum_{i=1}^{M} \widetilde{E}(\widetilde{Y}_i^2 \mid \widetilde{\mathcal{F}}_{i-1}) \to \infty \ \text{a.s.} \ \text{and} \ \sum_{i=1}^{\infty} \widetilde{E}\left(\frac{\widetilde{Y}_i^2 \mathbf{1}_{\{\widetilde{Y}_i^2 \ge \psi(\widetilde{V}_i)\}}}{\psi(\widetilde{V}_i)} \right) < \infty$$

for some non-decreasing ψ such that $\psi(x)(\log x)^\alpha/x$ is non-increasing for some $\alpha > 50$ and $\lim_{x\to\infty}\psi(x) = \infty$. If there exists a uniformly distributed random *variable* $\widetilde{U}$ *which is independent of* $\{\widetilde{Y}_n\}$, *there exists a sequence* $\{Z_i\}$ *of standard normal i.i.d. such that*

$$\sum_{i\geq 1}\widetilde{Y}_i\mathbf{1}_{\{\widetilde{V}_i\leq t\}} = G_t + o\big(t^{1/2}(\psi(t)/t)^{1/50}\big) \quad (t\to\infty) \quad a.s., \tag{4.10}$$

where

$$G_t = \sum_{i\leq t} Z_i.$$

From now on, we regard $f_a(A^k\boldsymbol{x})$ as a random variable on $\Omega\times[\,0,1)$.

Recall that $\sigma^2(f_a)>0$ and put $\psi(x)=x^{4/5}$. One can see that

$$\beta_M\sim l_M\sigma^2(f_a) \quad\text{and}\quad \widetilde{V}_M\to\infty \quad \text{a.s.}$$

by (4.8) and (4.9).

Since

$$\widetilde{V}_M\geq \beta_M - D_{11}\geq \sigma^2(f_a)l_M/2$$

holds for large M, we see by (4.7) that

$$\sum \widetilde{E}\big(\widetilde{Y}_i^2\mathbf{1}_{\{\widetilde{Y}_i^2\geq\psi(\widetilde{V}_i)\}}/\psi(\widetilde{V}_i)\big)\leq \sum \widetilde{E}\widetilde{Y}_i^4/\psi^2(\sigma^2(f_a)l_i/2)\ll \sum i^{4/3}/l_i^2<\infty.$$

Because of $\beta_{M+1}-\beta_M = v_{M+1}\to\infty$, we obtain

$$\widetilde{V}_M\leq\beta_M+D_{11}<\beta_{M+1}-D_{11}<\widetilde{V}_{M+1}$$

for large M, and $\widetilde{V}_i\leq\beta_M+D_{11}$ becomes equivalent to $i\leq M$. By putting $t=\beta_M+D_{11}$ in (4.10) and by noting (4.6) we have

$$\sum_{i=1}^M\widetilde{T}_i = \sum_{i=1}^M\widetilde{Y}_i + O(1) = G_{\widetilde{V}_M+D_{11}} + o\big(l_M^{249/250}\big), \quad \text{a.s.} \tag{4.11}$$

Put

$$\Delta_M^\flat=\Delta_1\cup\cdots\cup\Delta_M \quad\text{and}\quad \Delta_M^\natural=\Delta_1'\cup\cdots\cup\Delta_M'.$$

By ${}^\#\Delta_M^\natural = O(M\log M)$, we obtain

$$i_M^+ = l_M + {}^\#\Delta_M^\natural\sim l_M.$$

Note that

$$F_M^\flat := \max_{m\in\Delta_M}\Big|\sum_{k=m}^{i_M^+} f_a(A^k\,\cdot\,)\Big|\leq M^{2/3}\|f\|_\infty = O(l_M^{2/5}). \tag{4.12}$$

We can prove

$$F_M^\natural := \max_{m\in\Delta_M^\natural}\Big|\sum_{k\in\Delta_M^\natural,k\leq m} f_a(A^k\,\cdot\,)\Big| = o(l_M^{19/40}) \quad \text{a.s.}, \tag{4.13}$$

since (3.1) implies

$$E\big((l_M^{-19/40}F_M^\natural)^4\big) = O(M^{-7/6}(\log M)^2),$$

and is summable in M.

Hence for $N\in\Delta_M'\cup\Delta_M$, we obtain

$$\Big|\sum_{k=1}^N f_a(A^k\,\cdot\,) - \sum_{i=1}^M\widetilde{T}_i(\,\cdot\,)\Big|\leq F_M^\flat + F_M^\natural = o(l_M^{19/40}) = o(N^{19/40}). \tag{4.14}$$

Now we apply the following result on the fluctuation of the standard Wiener process $W(t)$ due to Csörgő-Révész ((1.2.4) in Theorem 1.2.1 of [1]). For non-decreasing a_T such that $0 < a_T \le T$ and T/a_T is non-decreasing, we have

$$\overline{\lim_{T\to\infty}} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \frac{|W(t+s)-W(t)|}{\sqrt{2a_T(\log(T/a_T)+\log\log T)}} = 1, \quad \text{a.s.}$$

By putting

$$T = i_M^+\sigma^2(f_a) \quad \text{and} \quad a_T = {}^{\#}(\Delta'_M \cup \Delta_M) + D_{10,f_a}M = O(M\log M),$$

we have

$$\left|G_{\sigma^2(f_a)N} - G_{\beta_M+D_{11}}\right| = O(M^{1/2}\log M) = O(N^{3/10}\log N) \quad \text{a.s.} \tag{4.15}$$

By combining (4.11), (4.14) and (4.15) we have (4.2). □

5. Variance control

We first prove that the series in (1.7) is absolutely convergent. By using the convention $\widehat{f}(\boldsymbol{\xi}) = 0$ for $\boldsymbol{\xi} \notin \mathbf{Z}^d$, we have

$$|\widehat{f}(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi}')|\mathbf{1}(\boldsymbol{\xi}+\boldsymbol{\xi}'A^l = \mathbf{0}) = |\widehat{f}(\boldsymbol{\xi}'A^l)\widehat{f}(\boldsymbol{\xi}')|.$$

Hence by noting

$$\|\boldsymbol{\xi}A^l\| \ge \|\boldsymbol{\xi}A^l\|_2/\sqrt{d} \ge q^l\|\boldsymbol{\xi}\|_2/\sqrt{d} \ge q^l/\sqrt{d}$$

for $\boldsymbol{\xi} \in \mathbf{Z}^d \setminus \{\mathbf{0}\}$, we have

$$\begin{aligned} &\sum_{l\ge 0}\sum_{\boldsymbol{\xi},\boldsymbol{\xi}'\in\mathbf{Z}^d} |\widehat{f}(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi}')|\mathbf{1}(\boldsymbol{\xi}+\boldsymbol{\xi}'A^l = \mathbf{0}) \\ &= \sum_{l\ge 0}\sum_{\boldsymbol{\xi}\in\mathbf{Z}^d} \left|\widehat{f}(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi}A^l)\right| \\ &\le \sum_{l\ge 0}\Big(\sum_{\boldsymbol{\xi}\in\mathbf{Z}^d}\left|\widehat{f}(\boldsymbol{\xi})\right|^2 \sum_{\boldsymbol{\xi}\in\mathbf{Z}^d}\left|\widehat{f}(\boldsymbol{\xi}A^l)\right|^2\Big)^{1/2} \\ &\le \|f\|_{L^2[0,1)^d}\sum_{l\ge 0}\|f - f_{q^l/\sqrt{d}}\|_{L^2[0,1)^d} < \infty. \end{aligned} \tag{5.1}$$

Let $\Gamma \subset \mathbf{R}^d$ be a bounded measurable set. Since we have L^2 convergence $f_a \to f$ on $[0,1)^d$ and by periodicity, we have the convergence on any bounded set Γ'.

By changing variable we have $L^2(\Gamma)$ convergence $f_a(A^k\boldsymbol{x}) \to f(A^k\boldsymbol{x})$. Hence we have $L^1(\Gamma)$ convergence $f_a(A^k\boldsymbol{x})f_a(A^{k'}\boldsymbol{x}) \to f(A^k\boldsymbol{x})f(A^{k'}\boldsymbol{x})$, and hence convergence in measure. Thus we have the convergence

$$\Big(\sum_{k=1}^N f_a(A^k\boldsymbol{x})\Big)^2 \to \Big(\sum_{k=1}^N f(A^k\boldsymbol{x})\Big)^2$$

in measure on Γ under the measure $\rho(\boldsymbol{x})\,d\boldsymbol{x}$. That is why we can apply Fatou's Lemma and have

$$\begin{aligned}
&\int_\Gamma \biggl(\sum_{k=1}^N f(A^k\boldsymbol{x})\biggr)^2 d\boldsymbol{x}\\
&\quad\le D_{6,\Gamma}\int_\Gamma \biggl(\sum_{k=1}^N f(A^k\boldsymbol{x})\biggr)^2 \rho(\boldsymbol{x})\,d\boldsymbol{x}\\
&\quad\le D_{6,\Gamma}\varliminf_{a\to\infty}\int_\Gamma \biggl(\sum_{k=1}^N f_a(A^k\boldsymbol{x})\biggr)^2 \rho(\boldsymbol{x})\,d\boldsymbol{x}\\
&\quad\le 2D_{6,\Gamma}\sum_{\boldsymbol{\xi},\boldsymbol{\xi}'\in\mathbf{Z}^d}\sum_{k\le l\le N}\bigl|\widehat{f}(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi}')\bigr|\mathbf{1}\bigl(\|\boldsymbol{\xi}A^k+\boldsymbol{\xi}'A^l\|\le 1/\pi\sqrt{d}\,\bigr).
\end{aligned}$$

By (1.2), we obtain

$$\bigl\|\boldsymbol{\xi}A^k+\boldsymbol{\xi}'A^l\bigr\|\ge\bigl\|(\boldsymbol{\xi}+\boldsymbol{\xi}'A^{l-k})A^k\bigr\|_2/\sqrt{d}\ge\bigl\|\boldsymbol{\xi}+\boldsymbol{\xi}'A^{l-k}\bigr\|_2/\sqrt{d}.$$

For $\boldsymbol{\eta}\in\mathbf{R}^d$, there is at most one $\boldsymbol{\xi}\in\mathbf{Z}^d$ such that $\|\boldsymbol{\eta}-\boldsymbol{\xi}\|_2\le 1/\pi$. In case such $\boldsymbol{\xi}$ exists, let $\chi(\boldsymbol{\eta})=\boldsymbol{\xi}$, and $\chi(\boldsymbol{\eta})=\mathbf{0}$ otherwise. If $\boldsymbol{\xi}\ne\boldsymbol{\xi}'$, then $\|\boldsymbol{\xi}A^m-\boldsymbol{\xi}'A^m\|_2\ge\|\boldsymbol{\xi}-\boldsymbol{\xi}'\|_2\ge 1$, and $\chi(\boldsymbol{\xi}A^m)\ne\chi(\boldsymbol{\xi}'A^m)$ if $\chi(\boldsymbol{\xi}A^m)\ne\mathbf{0}$ and $\chi(\boldsymbol{\xi}'A^m)\ne\mathbf{0}$. If $\chi(\boldsymbol{\xi}A^m)\ne\mathbf{0}$, then

$$\|\chi(\boldsymbol{\xi}A^m)\|\ge\|\chi(\boldsymbol{\xi}A^m)\|_2/\sqrt{d}\ge(\|\boldsymbol{\xi}A^m\|_2-1/\pi)/\sqrt{d}\ge q^m/2\sqrt{d}.$$

By

$$\begin{aligned}
\bigl|\widehat{f}(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi}')\bigr|\mathbf{1}\bigl(\bigl\|\boldsymbol{\xi}A^k+\boldsymbol{\xi}'A^l\bigr\|\le 1/\pi\sqrt{d}\bigr)&\le\bigl|\widehat{f}(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi}')\bigr|\mathbf{1}\bigl(\bigl\|\boldsymbol{\xi}A^k+\boldsymbol{\xi}'A^l\bigr\|_2\le 1/\pi\bigr)\\
&=\bigl|\widehat{f}(\chi(\boldsymbol{\xi}'A^{k-l}))\widehat{f}(\boldsymbol{\xi}')\bigr|,
\end{aligned}$$

we have

$$\begin{aligned}
&\sum_{\boldsymbol{\xi},\boldsymbol{\xi}'\in\mathbf{Z}^d}\sum_{k\le l\le N}\bigl|\widehat{f}(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi}')\bigr|\mathbf{1}\bigl(\bigl\|\boldsymbol{\xi}A^k+\boldsymbol{\xi}'A^l\bigr\|\le 1/\pi\sqrt{d}\bigr)\\
&\le\sum_{k\le l\le N}\sum_{\boldsymbol{\xi}\in\mathbf{Z}^d}\bigl|\widehat{f}(\boldsymbol{\xi})\widehat{f}(\chi(\boldsymbol{\xi}A^{k-l}))\bigr|\le N\sum_{m\le N}\sum_{\boldsymbol{\xi}\in\mathbf{Z}^d}\bigl|\widehat{f}(\boldsymbol{\xi})\widehat{f}(\chi(\boldsymbol{\xi}A^m))\bigr|\\
&\le N\sum_{m\le N}\biggl(\sum_{\boldsymbol{\xi}\in\mathbf{Z}^d}|\widehat{f}(\boldsymbol{\xi})|^2\sum_{\boldsymbol{\xi}\in\mathbf{Z}^d}|\widehat{f}(\chi(\boldsymbol{\xi}A^m))|^2\biggr)^{1/2}\\
&\le N\|f\|_{L^2[0,1)^d}\sum_{m\le N}\|f-f_{q^m/2\sqrt{d}}\|_{L^2[0,1)^d}.
\end{aligned}$$

We have proved

$$\int_\Gamma\biggl(\sum_{k=1}^N f(A^k\boldsymbol{x})\biggr)^2 d\boldsymbol{x}\le\widehat{C}N\|f\|_{L^2[0,1)^d},\tag{5.2}$$

where $\widehat{C}=D_{6,\Gamma}\sum_{m=0}^\infty\|f-f_{q^m/2\sqrt{d}}\|_{L^2[0,1)^d}$.

If $\boldsymbol{\xi}A^{k'}+\boldsymbol{\xi}'\ne 0$, by Riemann-Lebesgue Lemma we have

$$\int_\Gamma\exp(2\pi\sqrt{-1}\,(\boldsymbol{\xi}A^{k'}+\boldsymbol{\xi}')A^k\boldsymbol{x})\,d\boldsymbol{x}=\widehat{\mathbf{1}}_\Gamma((\boldsymbol{\xi}A^{k'}+\boldsymbol{\xi}')A^k)\to 0$$

as $k \to \infty$. Hence for a trigonometric polynomial f_a, we have (1.8) as below:

$$\begin{aligned}
&\frac{1}{N\,\mathrm{Leb}(\Gamma)} \int_\Gamma \Big(\sum_{k=1}^{N} f_a(A^k \boldsymbol{x}) \Big)^2 d\boldsymbol{x} \\
&= \sum_{\boldsymbol{\xi} \in R_a} \sum_{\boldsymbol{\xi}' \in R_a} \widehat{f}(\boldsymbol{\xi}) \widehat{f}(\boldsymbol{\xi}') \sum_{l=0}^{N-1} \frac{2-\delta_{l,0}}{N\,\mathrm{Leb}(\Gamma)} \sum_{k=1}^{N-l} \int_\Gamma \exp(2\pi\sqrt{-1}\,(\boldsymbol{\xi} + \boldsymbol{\xi}' A^l) A^k \boldsymbol{x})\, d\boldsymbol{x} \\
&\to \sum_{\boldsymbol{\xi} \in R_a} \sum_{\boldsymbol{\xi}' \in R_a} \widehat{f}(\boldsymbol{\xi}) \widehat{f}(\boldsymbol{\xi}') \sum_{l=0}^{\infty} (2-\delta_{l,0}) \mathbf{1}(\boldsymbol{\xi} + \boldsymbol{\xi}' A^l = \mathbf{0}) = \sigma^2(f_a).
\end{aligned}$$

Because of the absolute convergence (5.1), we obtain

$$\sigma(f_a) \to \sigma(f).$$

Proof of Corollary 4.2. Put

$$X_N^a(\boldsymbol{x}) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} f_a(A^k \boldsymbol{x}).$$

First we assume $\sigma^2(f_a) > 0$ and prove the central limit theorem under the measure P_Ω. By using Proposition 4.1 we have

$$X_N^a - G_{N\sigma^2(f_a)}/\sqrt{N} \to 0 \quad \text{a.s.}$$

and the law of $G_{N\sigma^2(f_a)}/\sqrt{N}$ is $N(0, \sigma^2(f_a))$, we see that the limit law under $P_\Omega \times \mathrm{leb}$ of X_N^a is $N(0, \sigma^2(f_a))$. Since the law of X_N^a under $P_\Omega \times \mathrm{leb}$ is identical with the law under P_Ω, we see that $N(0, \sigma^2(f_a))$ is also the limit law under P of X_N^a.

Now, denote by $\mathcal{G}$ the class of integrable functions g on $\mathbf{R}^d$ satisfying

$$\lim_{N\to\infty} \int_{\{X_N^a \le t\}} g(\boldsymbol{x})\, d\boldsymbol{x} = \Phi_{\sigma^2(f)}(t) \int_{\mathbf{R}^d} g(\boldsymbol{x})\, d\boldsymbol{x} \quad (t \in \mathbf{R}), \tag{5.3}$$

and denote by $\mathcal{H}$ the collection of $\mathbf{1}_\Omega \in \mathcal{G}$ given by (4.1) using arbitrary $L > 0$ and $L_\delta^{(h)} \in \mathbf{R}$ ($\delta = 1, \ldots, d_h$, $h = 1, \ldots, \beta$). We can easily show

$$g_1, g_2 \in \mathcal{G}, \quad \alpha_1, \alpha_2 \in \mathbf{R} \implies \alpha_1 g_1 + \alpha_2 g_2 \in \mathcal{G}, \tag{5.4}$$

$$g_1, g_2, \cdots \in \mathcal{G}, \quad \lim_{k\to\infty} \|g - g_k\|_{L^1(\mathbf{R}^d)} = 0 \implies g \in \mathcal{G}. \tag{5.5}$$

By the above argument we have already proved $\mathcal{H} \subset \mathcal{G}$, and by (5.4) we can see that any simple function which is given as a linear combination of indicator functions with supports in $\mathcal{H}$ belongs to $\mathcal{G}$. Since any continuous function with compact support can be arbitrarily approximated in the sense of $L^1(\mathbf{R}^d)$ by such simple function, we see that it belongs to $\mathcal{G}$. Since any integrable function with compact support can be arbitrarily approximated in the sense of $L^1(\mathbf{R}^d)$ by a continuous function with compact support, we see that it belongs to $\mathcal{G}$. Hence we can see that (4.3) holds under any probability measure P on $\mathbf{R}^d$ which is absolutely continuous with respect to the Lebesgue measure.

In case when $\sigma^2(f_a) = 0$, by (1.8) we see that the limit law of X_N^a is the delta measure concentrated on $\mathbf{0}$, that is $N(0, 0)$. It proves (4.3) for $\sigma^2(f_a) = 0$. □

Proof of Theorem 1.1. Put

$$X_N(\boldsymbol{x}) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(A^k \boldsymbol{x}) \quad \text{and} \quad Y_N^a = X_N - X_N^a.$$

We have proved

$$E|Y_N^a|^2 \le \widetilde{C} \|f - f_a\|_{L^2[0,1)^d},$$

which implies

$$P\big(|Y_N^a| \ge b_a\big) \le \widetilde{C} b_a,$$

where $b_a = \|f - f_a\|_{L^2[0,1)^d}^{1/3}$. By

$$\begin{aligned} P(X_N^a \le t - b_a) - P(|Y_N^a| \ge b_a) &\le P(X_N \le t) \\ &\le P(X_N^a \le t + b_a) + P(|Y_N^a| \ge b_a) \end{aligned}$$

we have

$$\begin{aligned} \Phi_{\sigma^2(f_a)}(t - b_a) - \widetilde{C} b_a &\le \varliminf_{N\to\infty} P(X_N \le t) \\ &\le \varlimsup_{N\to\infty} P(X_N \le t) \\ &\le \Phi_{\sigma^2(f_a)}(t + b_a) + \widetilde{C} b_a. \end{aligned}$$

By letting $a \to \infty$, we have (1.9).

By putting

$$\|X\|_{\widetilde{L}^2(\Gamma)} = \left(\frac{1}{\mathrm{Leb}(\Gamma)} \int_\Gamma X^2(\boldsymbol{x})\, d\boldsymbol{x} \right)^{1/2},$$

we have

$$\|X_N^a\|_{\widetilde{L}^2(\Gamma)} - \|Y_N^a\|_{\widetilde{L}^2(\Gamma)} \le \|X_N\|_{\widetilde{L}^2(\Gamma)} \le \|X_N^a\|_{\widetilde{L}^2(\Gamma)} + \|Y_N^a\|_{\widetilde{L}^2(\Gamma)}$$

and hence by letting $N \to \infty$,

$$\sigma(f_a) - (\widetilde{C} b_a^3)^{1/2} \le \varliminf_{N\to\infty} \|X_N\|_{\widetilde{L}^2(\Gamma)} \le \varlimsup_{N\to\infty} \|X_N\|_{\widetilde{L}^2(\Gamma)} \le \sigma(f_a) + (\widetilde{C} b_a^3)^{1/2}.$$

By letting $a \to \infty$, we have (1.8). □

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