Global dynamics of an SIR epidemic model with nonlocal diffusion

Kuniya, Toshikazu

Wang, Jinliang

(Citation)
Nonlinear Analysis: Real World Applications,43:262-282

(Issue Date)
2018-10

(Resource Type)
journal article

(Version)
Accepted Manuscript

Global asymptotic stability of an SIR epidemic model with nonlocal diffusion

Toshikazu Kuniya[a], Jinliang Wang[b]

[a]*Graduate School of System Informatics, Kobe University, 1-1 Rokkodai-cho, Nada-ku, Kobe 657-8501, Japan*
[b]*School of Mathematical Science, Heilongjiang University, Harbin 150080, PR China*

Abstract

In this paper, we are concerned with the global asymptotic stability of each equilibrium of an SIR epidemic model with nonlocal diffusion. Under the assumption of Lipschitz continuity of parameters, the eigenvalue problem associated with the linearized system around the disease-free equilibrium has a principal eigenvalue corresponding to a strictly positive eigenfunction. By setting the eigenfunction as the integral kernel of a Lyapunov function, we prove the global asymptotic stability of the disease-free equilibrium when the basic reproduction number $\mathcal{R}_0$ is less than one. We also prove the uniform persistence of the system when $\mathcal{R}_0 > 1$ by using the persistent theory for dynamical systems. Furthermore, in a special case where the diffusion coefficient for susceptible individuals is equal to zero, we prove the existence, uniqueness and global asymptotic stability of the endemic equilibrium when $\mathcal{R}_0 > 1$ by constructing a suitable Lyapunov function.

Keywords: SIR epidemic model, Nonlocal diffusion, Basic reproduction number, Global asymptotic stability, Lyapunov function

1. Introduction

Since the pioneering work of Kermack and McKendrick [32], differential equations as epidemic models have attracted much attention of many researchers. The heterogeneity (position, age, sex, etc.) of each individual is known to be an important factor in spread of infectious diseases, and hence, motivates more realistic model (reaction-diffusion equations) for disease dynamics. It is well known that spatial heterogeneity of environment and movement of individuals are ubiquitous in the real world. Incorporating these two mechanism would give insights into disease spread and control, and there have been quite a few publications along this line (see, for instance, [2, 7, 12, 21–23, 26, 33–36, 39, 44, 56, 58, 60, 62] and the references therein). Recently, Allen et al. in [2] proposed a susceptible-infective-susceptible (SIS) reaction-diffusion model under heterogeneous environment,

$$\begin{cases} \dfrac{\partial}{\partial t}S(t,x) = d_S\Delta S(t,x) - \dfrac{\beta(x)S(t,x)I(t,x)}{S(t,x)+I(t,x)} + \gamma(x)I(t,x), & t>0, \quad x\in\Omega, \\ \dfrac{\partial}{\partial t}I(t,x) = d_I\Delta I(t,x) + \dfrac{\beta(x)S(t,x)I(t,x)}{S(t,x)+I(t,x)} - \gamma(x)I(t,x), & t>0, \quad x\in\Omega, \\ \dfrac{\partial}{\partial \mathbf{n}}S(t,x) = \dfrac{\partial}{\partial \mathbf{n}}I(t,x) = 0, & t>0, \quad x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $S(t,x)$ and $I(t,x)$ denote the density of susceptible and infective individuals in a given spatial region Ω, which is assumed to be a bounded domain in $\mathbb{R}^n (n \geq 1)$ with smooth boundary $\partial\Omega$. Ω is isolated from

Email addresses: tkuniya@port.kobe-u.ac.jp (Toshikazu Kuniya), jinliangwang@hlju.edu.cn (Jinliang Wang)

outside for the host, implying the homogeneous Neumann boundary condition; $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega$ and $\partial/\partial\mathbf{n}$ means the normal derivative along $\mathbf{n}$ on $\partial\Omega$; d_S and d_I are diffusion coefficients for the susceptible and infective populations, respectively; the positive functions $\beta(x)$ and $\gamma(x)$ represent the rates of disease transmission and recovery at position x, respectively. They studied the existence, uniqueness and particularly the asymptotic behavior of the endemic equilibrium as the diffusion rate of the susceptible individuals goes to zero. Although the theoretical conclusions exhibit the delicacy by defining the low-risk and high-risk sites, which is in terms of rate of disease transmission and rate of disease recovery, due to the mathematical difficulties, they were unable to derive stability result for the endemic equilibrium if it exists. Subsequently, Peng and Liu [48] discussed the global stability of the endemic equilibrium of (1.1) in two special cases: (i) the diffusion rate of susceptible individuals and infective individuals are the same; (ii) the rates of disease transmission is proportional to the rates of disease recovery for any fixed constant. Biological results in [2, 48] revealed that controlling the diffusion rate of the susceptible individuals can help eradicate the infection, while controlling the diffusion rate of the infectious individuals cannot.

Furthermore, Peng [47] studied the asymptotic profiles of the endemic equilibrium of (1.1) when the diffusion rate of either the susceptible individuals or the infective ones goes to infinity or zero, which provide new aspects and understanding of the impacts of diffusion rates on spatial-temporal dynamics of infectious diseases. Note that the rates of disease transmission adopted in [2, 48] obey the standard incidence mechanism. In this circumstance, the basic reproduction number is independent on the total population. Very recently, Wu and Zou [61] continued to study the impacts of spatial heterogeneity of environment and movement of individuals on the persistence and extinction of a disease. They explored the asymptotic profiles of the endemic steady state for large and small diffusion rates by using the mass action mechanism. This leads to more challenges in mathematical analysis that (i) the equilibrium problem is a nonlocal elliptic problem; (ii) the mass action term exhibits an unbounded infection force. Cui and Lou [18] considered circumstances that populations may take passive movement in certain direction due to external environmental forces such as water flow [40, 41], wind [19] and so on. They added an advection term to model in [2, 48], and studied the effects of diffusion and advection in heterogeneous environments.

The above mentioned works considered a fast disease by ignoring the demography of the host. In a recent work [35], the authors studied the following diffusive SIR epidemic model with mass action infection mechanism and homogeneous Neumann boundary condition.

$$\begin{cases} \dfrac{\partial}{\partial t}S(t,x) = k_S\Delta S(t,x) + b(x) - \beta(x)S(t,x)I(t,x) - \mu(x)S(t,x), & t>0, \quad x\in\Omega,\\ \dfrac{\partial}{\partial t}I(t,x) = k_I\Delta I(t,x) + \beta(x)S(t,x)I(t,x) - (\mu(x)+\gamma(x))I(t,x), & t>0, \quad x\in\Omega,\\ \dfrac{\partial}{\partial t}R(t,x) = k_R\Delta R(t,x) + \gamma(x)I(t,x) - \mu(x)R(t,x), & t>0, \quad x\in\Omega,\\ S(0,x)=S_0(x), \quad I(0,x)=I_0(x), \quad R(0,x)=R_0(x), & x\in\Omega,\\ \dfrac{\partial}{\partial \mathbf{n}}S(t,x) = \dfrac{\partial}{\partial \mathbf{n}}I(t,x) = \dfrac{\partial}{\partial \mathbf{n}}R(t,x) = 0, & t>0, \quad x\in\partial\Omega. \end{cases} \tag{1.2}$$

$S(t,x)$, $I(t,x)$ and $R(t,x)$ denote the populations of susceptible, infective and recovered individuals in position x at time t, respectively. k_S, k_I and k_R denote the diffusion coefficients for susceptible, infective and recovered individuals, respectively. $b(x)$, $\beta(x)$, $\mu(x)$ and $\gamma(x)$ denote the birth rate, the disease transmission rate, the mortality rate and the recovery rate at position x, respectively. For two special cases where $k_S=0$, $k_I>0$ or $k_S>0$, $k_I=0$, the basic reproduction number $\mathcal{R}_0$ (see, for instance, Diekmann *et al.* [20] and Inaba [28]) for system (1.2) was obtained and the global asymptotic stability of the disease-free equilibrium when $\mathcal{R}_0\le 1$ and that of the endemic equilibrium when $\mathcal{R}_0>1$ were investigated, respectively. In the proof of the main theoretical results, the authors constructed suitable Lyapunov functions based on those for the corresponding discretized models as in [37].

As well known, the Laplacian operator in model (1.2) essentially accounts for the random diffusion of each individual in adjacent spatial positions. However, the movements of individuals are often free, and should not be limited to a small area. To be a more important and intuitively necessary circumstance, we should include the long range diffusion effect, as stated in [45, Section 9.5]. Coville and Dupaigne [17] studied a one-dimensional non-local variant of Fisher's equation describing the spatial spread of a mutant in a given population. The dispersion of the genetic characters is assumed to follow a non-local diffusion law modeled by a convolution operator,

$$\int_\Omega J(x-y)\varphi(y)dy - \varphi(x).$$

They argued that dispersion of the gene fraction at point $y \in \mathbb{R}^n$ should affect the gene fraction at $x \in \mathbb{R}^n$ by a factor $J(x-y)\varphi(y)dy$, where $J(\cdot)$ is a probability density and is a non-negative even function of mass 1. In a one-dimensional setting and assuming, this diffusion process depends only on the distance between two niches of the population. In the work of García-Melián and Rossi [24], $J(x-y)$ is interpreted as the probability of jumping from position y to position x, the convolution $\int_\Omega J(x-y)\varphi(y)dy$ is the rate at which individuals arrive at position x from all other positions, while $-\int_\Omega J(y-x)\varphi(x)dy = -\varphi(x)$ is the rate at which they leave position x to reach any other position. For example, in ecology, bees often jump from one position to another. Indeed, this nonlocal diffusion problems have attracted many researchers in recent years, and there have been quite a few publications along this line. See, e.g., [1, 3, 4, 6, 8, 9, 11, 13–16, 16, 27, 30, 31, 38, 43, 45, 49, 51, 52, 55, 57, 59] and the references therein.

This paper is a continuation of [35], aiming to explore the following SIR epidemic model with nonlocal diffusion, which is more realistic than the previous model (1.2),

$$\begin{cases} \dfrac{\partial}{\partial t}S(t,x) = k_S\displaystyle\int_\Omega J(x-y)S(t,y)dy - k_S S(t,x) + b(x) - \beta(x)S(t,x)I(t,x) - \mu(x)S(t,x), & t>0, \quad x\in\overline{\Omega}, \\ \dfrac{\partial}{\partial t}I(t,x) = k_I\displaystyle\int_\Omega J(x-y)I(t,y)dy - k_I I(t,x) + \beta(x)S(t,x)I(t,x) - (\mu(x)+\gamma(x))I(t,x), & t>0, \quad x\in\overline{\Omega}, \\ \dfrac{\partial}{\partial t}R(t,x) = k_R\displaystyle\int_\Omega J(x-y)R(t,y)dy - k_R R(t,x) + \gamma(x)I(t,x) - \mu(x)R(t,x), & t>0, \quad x\in\overline{\Omega}, \\ S(0,x)=S_0(x), \quad I(0,x)=I_0(x), \quad R(0,x)=R_0(x), & x\in\overline{\Omega}. \end{cases} \tag{1.3}$$

Here, the meaning of each symbol is similar to that in (1.2), except for the diffusion kernel function $J(\cdot)$. The purpose of this paper is to investigate the global asymptotic stability of the system (1.3). Under the assumption of Lipschitz continuity of parameters, the eigenvalue problem associated with the linearized system of (1.3) around the disease-free equilibrium has a principal eigenvalue corresponding to a strictly positive eigenfunction. By setting the eigenfunction as the integral kernel of a Lyapunov function, we prove the global asymptotic stability of the disease-free equilibrium when the basic reproduction number $\mathcal{R}_0$ is less than one. We also prove the uniform persistence of system (1.3) when $\mathcal{R}_0 > 1$ by using the persistence theory for dynamical systems as in [50, Theorem 3]. Furthermore, in a special case where the diffusion coefficient for susceptible individuals is equal to zero ($k_S = 0$), we prove the existence, uniqueness and global asymptotic stability of the endemic equilibrium when $\mathcal{R}_0 > 1$. For the proof, we construct a suitable Lyapunov function, which has the infective population $I^*(x)$ in the endemic equilibrium as the integral kernel.

The organization of this paper is as follows. Section 2 is devoted to the preliminaries. In Section 3, we prove the global asymptotic stability of the disease-free equilibrium when $\mathcal{R}_0 < 1$. In Section 4, we prove the uniform persistence of system (1.3) when $\mathcal{R}_0 > 1$. In Section 5, we prove the existence, uniqueness and global asymptotic stability of the endemic equilibrium of system (1.3) when $\mathcal{R}_0 > 1$ and $k_S = 0$. In Section 6, we perform numerical simulation to verify the validity of our theoretical results.

2. Preliminaries

2.1. Existence and uniqueness of a positive solution

We make the following assumptions on the parameters of system (1.3).

Assumption 1. (i) $k_S \geq 0$, $k_I > 0$ and $k_R > 0$;

(ii) $b(x)$, $\beta(x)$ and $\mu(x)$ are strictly positive and Lipschitz continuous on $\overline{\Omega}$;
(iii) $\gamma(x)$ are nonnegative and Lipschitz continuous on $\overline{\Omega}$;
(iv) $J(x)$ is Lipschitz on $\overline{\Omega}$ and satisfies the following properties.

$$\int_{\mathbb{R}^n} J(x)dx = 1, \quad J(x) > 0 \ \text{ on } \ \overline{\Omega}, \quad J(x) = J(-x) \geq 0 \ \text{ on } \ \mathbb{R}^n \quad \text{and} \quad J(0) > 0. \tag{2.1}$$

From (ii) of Assumption 1, we see that there exists a positive lower bound $\underline{\mu} > 0$ such that

$$\mu(x) \geq \underline{\mu} \ \text{ for all } \ x \in \overline{\Omega}. \tag{2.2}$$

Since $R(t,x)$ does not appear in the first two equations of (1.3), we can restrict our attention to the following reduced system.

$$\begin{cases} \dfrac{\partial}{\partial t} S(t,x) = k_S \displaystyle\int_\Omega J(x-y)S(t,y)dy - k_S S(t,x) + b(x) - \beta(x)S(t,x)I(t,x) - \mu(x)S(t,x), & t>0, \ \ x \in \overline{\Omega}, \\ \dfrac{\partial}{\partial t} I(t,x) = k_I \displaystyle\int_\Omega J(x-y)I(t,y)dy - k_I I(t,x) + \beta(x)S(t,x)I(t,x) - (\mu(x)+\gamma(x))I(t,x), & t>0, \ \ x \in \overline{\Omega}, \\ S(0,x) = S_0(x), \quad I(0,x) = I_0(x), & x \in \overline{\Omega}. \end{cases} \tag{2.3}$$

Let us consider the following function spaces and positive cones.

$$X := C\left(\overline{\Omega}\right), \quad X_+ := C_+\left(\overline{\Omega}\right), \quad Y := C\left(\overline{\Omega}\right) \times C\left(\overline{\Omega}\right), \quad Y_+ := C_+\left(\overline{\Omega}\right) \times C_+\left(\overline{\Omega}\right).$$

The norms in X and Y are defined as follows, respectively.

$$\|\varphi\|_X := \sup_{x\in\overline{\Omega}} |\varphi(x)|, \quad \varphi \in X, \qquad \|(\varphi,\psi)\|_Y := \sup_{x\in\overline{\Omega}} \sqrt{|\varphi(x)|^2 + |\psi(x)|^2}, \quad (\varphi,\psi) \in Y.$$

Let us define the following linear operators on X.

$$A_S\varphi(x) := k_S \int_\Omega J(x-y)\varphi(y)dy - k_S\varphi(x) - \mu(x)\varphi(x), \quad \varphi \in X, \tag{2.4}$$

$$A_I\varphi(x) := k_I \int_\Omega J(x-y)\varphi(y)dy - k_I\varphi(x) - (\mu(x)+\gamma(x))\,\varphi(x), \quad \varphi \in X. \tag{2.5}$$

Under Assumption 1, A_S and A_I are bounded linear operators. Hence, from the standard theory of semigroups, they are generators of uniformly continuous semigroups $\{T_S(t)\}_{t\geq 0}$ and $\{T_I(t)\}_{t\geq 0}$ on X, respectively (see, for instance, [46, Theorem 1.2]). In particular, from a similar argument as in [30, Section 2.1.1], we see that the semigroups $\{T_S(t)\}_{t\geq 0}$ and $\{T_I(t)\}_{t\geq 0}$ are positive.

On the existence and uniqueness of a positive solution $(S(t,x), I(t,x))$ of system (2.3), we prove the following proposition.

Proposition 2.1. *System* (2.3) *has a unique positive solution* $(S(t,\cdot), I(t,\cdot)) \in Y_+$*, provided* $(S_0, I_0) \in Y_+$.

Proof. Since A_S and A_I are generators of uniformly continuous semigroups $\{T_S(t)\}_{t\geq 0}$ and $\{T_I(t)\}_{t\geq 0}$, respectively, the solution $(S(t,x), I(t,x))$ of system (2.3) can be written as follows.

$$\begin{cases} S(t,x) = T_S(t)S_0(x) + \displaystyle\int_0^t T_S(t-u)\left(b(x) - \beta(x)S(u,x)I(u,x)\right)du, & t>0, \ \ x \in \overline{\Omega}, \\ I(t,x) = T_I(t)I_0(x) + \displaystyle\int_0^t T_I(t-u)\left(\beta(x)S(u,x)I(u,x)\right)du, & t>0, \ \ x \in \overline{\Omega}. \end{cases}$$

Thus, the solution $(S(t,x), I(t,x))$ is continuous. To show the positivity, on the contrary, we suppose that there exists $t_0 \geq 0$ such that

$$S(t,x) \geq 0 \ \text{ for all } \ t \in [0,t_0] \ \text{ and } \ x \in \overline{\Omega}, \quad S(t_0,x^*) = 0 \ \text{ and } \ \frac{\partial}{\partial t}S(t_0,x^*) < 0 \ \text{ for some } \ x^* \in \overline{\Omega}.$$

However, from the first equation of (2.3), we have

$$\frac{\partial}{\partial t}S(t_0, x^*) = k_S \int_\Omega J(x^* - y)S(t_0, y)dy + b(x^*) > 0,$$

which is a contradiction. Therefore, $S(t, x) \geq 0$ for all $t \geq 0$ and $x \in \overline{\Omega}$. The positivity of $I(t, x)$ can be proved in a similar way. The proof is complete. □

From Proposition 2.1, we can define a positive continuous semiflow $\{\Phi_t\}_{t\geq 0} : Y_+ \to Y_+$ for system (2.3) as follows.

$$\Phi_t\left((S_0, I_0)\right) := \left(S(t, \cdot), I(t, \cdot)\right), \quad t \geq 0, \quad (S_0, I_0) \in Y_+. \tag{2.6}$$

2.2. Equilibria

Let $E^0 := (S^0, 0) \in Y_+$ denote the disease-free equilibrium of system (2.3), where $S^0 \in X_+$ is the solution of the following equation.

$$0 = k_S \int_\Omega J(x - y)S^0(y)dy - k_S S^0(x) + b(x) - \mu(x)S^0(x), \quad x \in \overline{\Omega}. \tag{2.7}$$

On the existence of the disease-free equilibrium E^0 and the Lipschitz continuity of $S^0(x)$, we prove the following proposition.

Proposition 2.2. (i) *System* (2.3) *has a unique disease-free equilibrium* $E^0 = (S^0, 0) \in Y_+$*;*
(ii) $S^0(x)$ *is Lipschitz on* Ω*.*

Proof. (i) Using the operator A_S defined by (2.4), we can rewrite (2.7) as follows.

$$0 = A_S S^0(x) + b(x), \quad x \in \overline{\Omega}. \tag{2.8}$$

As stated in Section 2.1, A_S generates the positive uniformly continuous semigroup $\{T_S(t)\}_{t\geq 0}$. Hence, from [53, Theorem 3.12], A_S is a resolvent-positive operator. In particular, it is seen that $s(A_S) < 0$ (see, for instance, [55, Proposition 2.4]), where $s(\cdot)$ denotes the spectral bound of an operator. Hence, taking $\lambda = 0$ in the resolvent operator $(\lambda I_d - A_S)^{-1}$ (where I_d denotes the identity operator), we have from [53, Theorem 3.12] that

$$(-A_S)^{-1}\varphi(x) = \int_0^\infty T_S(t)\varphi(x)dt, \quad \varphi \in X.$$

Hence, from (2.8), we have

$$S^0(x) = (-A_S)^{-1}b(x) = \int_0^\infty T_S(t)b(x)dt, \quad x \in \overline{\Omega},$$

which is positive and uniquely exists.

(ii) From (2.7), we have

$$S^0(x) = \frac{k_S \int_\Omega J(x - y)S^0(y)dy + b(x)}{k_S + \mu(x)}, \quad x \in \overline{\Omega}. \tag{2.9}$$

Under Assumption 1, $b(x)$, $k_S + \mu(x)$ and $J(x)$ are Lipschitz continuous on $\overline{\Omega}$. Hence, we have

$$\begin{aligned}\left\|k_S \int_\Omega J(x - y)S^0(y)dy - k_S \int_\Omega J(\tilde{x} - y)S^0(y)dy\right\|_X \leq& k_S \int_\Omega \|J(x - y) - J(\tilde{x} - y)\|_X S^0(y)dy \\ \leq& k_S \int_\Omega S^0(y)dy \; L_J\,|x - \tilde{x}|\,, \quad x, \tilde{x} \in \overline{\Omega},\end{aligned}$$

where $L_J > 0$ denotes the Lipschitz constant of $J(x)$. Thus, $S^0(x)$ given as (2.9) is the fraction of strictly positive bounded Lipschitz functions and therefore, Lipschitz itself. This completes the proof. □

Let $E^* := (S^*, I^*) \in Y_+$ denote the endemic equilibrium of system (2.3), where $S^* \in X_+$ and $I^* \in X_+ \setminus \{0\}$ must satisfy the following equations.

$$\begin{cases} 0 = k_S \int_\Omega J(x-y)S^*(y)dy - k_S S^*(x) + b(x) - \beta(x)S^*(x)I^*(x) - \mu(x)S^*(x), & x \in \overline{\Omega}, \\ 0 = k_I \int_\Omega J(x-y)I^*(y)dy - k_I I^*(x) + \beta(x)S^*(x)I^*(x) - (\mu(x)+\gamma(x))I^*(x), & x \in \overline{\Omega}. \end{cases} \tag{2.10}$$

The existence and global asymptotic stability of the endemic equilibrium E^* will be discussed in the special case where $k_S = 0$ (see Section 5).

2.3. State space

On the boundedness of $S(t,x)$, we prove the following lemma.

Lemma 2.1. *$S(t,x) \leq S^0(x)$ for all $t > 0$ and $x \in \overline{\Omega}$, provided $S_0(x) \leq S^0(x)$ for all $x \in \overline{\Omega}$.*

Proof. The first equation of (2.3) can be evaluated as follows.

$$\frac{\partial}{\partial t}S(t,x) \leq k_S \int_\Omega J(x-y)S(t,y)dy - k_S S(t,x) + b(x) - \mu(x)S(t,x), \quad t > 0, \quad x \in \overline{\Omega}.$$

Let $\bar{S}(t,x)$ be a solution of the following equation.

$$\begin{cases} \frac{\partial}{\partial t}\bar{S}(t,x) = k_S \int_\Omega J(x-y)\bar{S}(t,y)dy - k_S \bar{S}(t,x) + b(x) - \mu(x)\bar{S}(t,x), & t > 0, \quad x \in \overline{\Omega}, \\ \bar{S}(0,x) = S_0(x), & x \in \overline{\Omega}. \end{cases} \tag{2.11}$$

By the comparison principle, we have that $S(t,x) \leq \bar{S}(t,x)$ for all $t > 0$ and $x \in \overline{\Omega}$. From (2.7), we have

$$b(x) = -k_S \int_\Omega J(x-y)S^0(y)dy + k_S S^0(x) + \mu(x)S^0(x), \quad x \in \overline{\Omega}. \tag{2.12}$$

Hence, (2.11) can be rewritten as follows.

$$\begin{cases} \frac{\partial}{\partial t}\bar{S}(t,x) = k_S \int_\Omega J(x-y)\left(\bar{S}(t,y) - S^0(y)\right)dy - k_S\left(\bar{S}(t,x) - S^0(x)\right) - \mu(x)\left(\bar{S}(t,x) - S^0(x)\right), & t > 0, \ x \in \overline{\Omega}, \\ \bar{S}(0,x) = S_0(x), & x \in \overline{\Omega}. \end{cases}$$

Let $u(t,x) := S^0(x) - \bar{S}(t,x)$. Then, by multiplying -1 to both sides, we can rewrite the above equation as follows.

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = k_S \int_\Omega J(x-y)u(t,y)dy - k_S u(t,x) - \mu(x)u(t,x), & t > 0, \quad x \in \overline{\Omega}, \\ u(0,x) = S^0(x) - S_0(x) \geq 0, & x \in \overline{\Omega}. \end{cases} \tag{2.13}$$

Using the linear operator A_S defined by (2.4), we can rewrite (2.13) to the following abstract form in X.

$$\frac{d}{dt}u(t) = A_S u(t), \quad t > 0.$$

As stated in Section 2.1, A_S is the generator of the positive semigroup $\{T_S(t)\}_{t\geq 0}$. Hence, we see from (2.13) that $u(t,x) = S^0(x) - \bar{S}(t,x) \geq 0$ for all $t > 0$ and $x \in \overline{\Omega}$. Thus, we have $S(t,x) \leq \bar{S}(t,x) \leq S^0(x)$ for all $t > 0$ and $x \in \overline{\Omega}$. This completes the proof. □

Let us define the state space for system (2.3) as follows.

$$D := \left\{ (S, I) \in Y_+ : \ S(\cdot, x) \le S^0(x) \ \text{ for all } x \in \overline{\Omega}, \ \int_\Omega S(\cdot, x)dx + \int_\Omega I(\cdot, x)dx \le \frac{\int_\Omega b(x)dx}{\underline{\mu}} \right\},$$

where $\underline{\mu}$ is the positive lower bound of $\mu(x)$ defined in (2.2). On the positive invariance of D, we prove the following proposition.

Proposition 2.3. *State space D is positively invariant for system* (2.3).

PROOF. By virtue of Lemma 2.1, it suffices to prove the inequality $\int_\Omega S(t,x)dx + \int_\Omega I(t,x)dx \le \int_\Omega b(x)dx/\underline{\mu}$ for all $t > 0$. From (2.1) and (2.3), we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega S(t,x)dx =& k_S \int_\Omega \int_\Omega J(x-y)S(t,y)dydx - k_S \int_\Omega S(t,x)dx + \int_\Omega \left(b(x) - \beta(x)S(t,x)I(t,x) - \mu(x)S(t,x)\right) dx \\ \le& k_S \int_\Omega \int_{\mathbb{R}^n} J(x-y)dxS(t,y)dy - k_S \int_\Omega S(t,x)dx + \int_\Omega \left(b(x) - \beta(x)S(t,x)I(t,x) - \mu(x)S(t,x)\right) dx \\ =& \int_\Omega b(x)dx - \int_\Omega \beta(x)S(t,x)I(t,x)dx - \int_\Omega \mu(x)S(t,x)dx \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \frac{d}{dt}\int_\Omega I(t,x)dx =& k_I \int_\Omega \int_\Omega J(x-y)I(t,y)dydx - k_I \int_\Omega I(t,x)dx + \int_\Omega \left(\beta(x)S(t,x)I(t,x) - (\mu(x)+\gamma(x))I(t,x)\right) dx \\ \le& k_I \int_\Omega \int_{\mathbb{R}^n} J(x-y)dxI(t,y)dy - k_I \int_\Omega I(t,x)dx + \int_\Omega \left(\beta(x)S(t,x)I(t,x) - (\mu(x)+\gamma(x))I(t,x)\right) dx \\ =& \int_\Omega \beta(x)S(t,x)I(t,x)dx - \int_\Omega (\mu(x)+\gamma(x))I(t,x)dx. \end{aligned} \tag{2.15}$$

Adding (2.14) and (2.15), we have

$$\begin{aligned} \frac{d}{dt}\left(\int_\Omega S(t,x)dx + \int_\Omega I(t,x)dx\right) &\le \int_\Omega b(x)dx - \int_\Omega \mu(x)S(t,x)dx - \int_\Omega (\mu(x)+\gamma(x))I(t,x)dx \\ &\le \int_\Omega b(x)dx - \underline{\mu}\left(\int_\Omega S(t,x)dx + \int_\Omega I(t,x)dx\right). \end{aligned}$$

Hence, from the variation of constant formula, we have

$$\int_\Omega S(t,x)dx + \int_\Omega I(t,x)dx \le \frac{\int_\Omega b(x)dx}{\underline{\mu}} \quad \text{for all} \quad t > 0,$$

provided $\int_\Omega S_0(x)dx + \int_\Omega I_0(x)dx \le \int_\Omega b(x)dx/\underline{\mu}$. This completes the proof. □

2.4. Basic reproduction number

To define the basic reproduction number $\mathcal{R}_0$ following the definition by Diekmann *et al.* [20], we linearize the second equation of (2.3) around the disease-free equilibrium E^0 as follows.

$$\frac{\partial}{\partial t}I(t,x) = k_I \int_\Omega J(x-y)I(t,y)dy - k_I I(t,x) + \beta(x)S^0(x)I(t,x) - (\mu(x)+\gamma(x))I(t,x), \quad t > 0, \ x \in \overline{\Omega}. \tag{2.16}$$

Let us define the following linear operator on X.

$$F\varphi(x) := \beta(x)S^0(x)\varphi(x), \quad \varphi \in X.$$

Then, using the operator A_I defined by (2.5), we can rewrite (2.16) to the following abstract form in X.

$$\frac{dI(t)}{dt} = A_I I(t) + FI(t), \quad t > 0.$$

Similar as in the proof of Proposition 2.2, we see that A_I is resolvent-positive, $s(A_I) < 0$ and hence,

$$(-A_I)^{-1}\varphi(x) = \int_0^{\infty} T_I(t)\varphi(x)dt, \quad \varphi \in X.$$

Hence, following the definition in [20], the next generation operator $\mathcal{K} := F(-A_I)^{-1}$ is given by

$$\mathcal{K}\varphi(x) = \beta(x)S^0(x)\int_0^{\infty} T_I(t)\varphi(x)dt, \quad \varphi \in X. \tag{2.17}$$

and the basic reproduction number $\mathcal{R}_0$ is defined by

$$\mathcal{R}_0 := r(\mathcal{K}), \tag{2.18}$$

where $r(\cdot)$ denotes the spectral radius of an operator. For $\mathcal{R}_0$ in a similar form, see, for instance, [55, 57, 58].

3. Global asymptotic stability of the disease-free equilibrium

Let us consider the following eigenvalue problem associated with (2.16).

$$\lambda v(x) = k_I \int_\Omega J(x-y)v(y)dy - k_I v(x) + \beta(x)S^0(x)v(x) - (\mu(x)+\gamma(x))v(x), \quad x \in \overline{\Omega}. \tag{3.1}$$

From Assumption 1 and Proposition 2.2, we see that $\beta(x)S^0(x) - (k_I + \mu(x) + \gamma(x))$ is Lipschitz on $\overline{\Omega}$. Hence, from [27, Theorem 3.1], we have the following lemma.

Lemma 3.1. *There exists a principal eigenvalue λ_0 for problem* (3.1)*, which corresponds to a strictly positive continuous eigenfunction $v_0(x)$. More precisely, λ_0 is given by*

$$\lambda_0 = \max_{\|v\|_{L^2(\Omega)}=1} \left(k_I \int_\Omega \int_\Omega J(x-y)v(x)v(y)dydx + \int_\Omega \big(\beta(x)S^0(x) - (k_I + \mu(x) + \gamma(x))\big)\, v(x)^2 dx \right).$$

Since λ_0 is the principal eigenvalue for problem (3.1), we have $\lambda_0 = s(A_I + F)$ (see, for instance, [6, Lemma 2.2]). Since A_I is resolvent-positive and $s(A_I) < 0$ (see Section 2.4), it follows from [53, Theorem 3.5] that $\lambda_0 = s(A_I + F)$ has the same sign as $r(F(-A_I^{-1})) - 1 = r(\mathcal{K}) - 1 = \mathcal{R}_0 - 1$. Consequently, we have the following lemma.

Lemma 3.2. $\mathcal{R}_0 - 1$ *has the same sign as* λ_0.

Using Lemmas 3.1 and 3.2, we prove the following theorem on the global asymptotic stability of the disease-free equilibrium E^0.

Theorem 3.1. *If $\mathcal{R}_0 < 1$, then the disease-free equilibrium E^0 of system* (2.3) *is globally asymptotically stable in D.*

Proof. Let us construct the following Lyapunov function.

$$V_1 := \int_\Omega v_0(x)I(t,x)dx,$$

where $v_0(x)$ denotes the strictly positive eigenfunction for (3.1), associated with λ_0. It is easy to see that $V_1 \geq 0$ and $V_1 = 0$ if and only if $I \equiv 0$. The derivative of V_1 along the solution trajectory of system (2.3) is calculated as follows.

$$\begin{aligned} V_1' &= \int_\Omega v_0(x)\frac{\partial}{\partial t}I(t,x)dx \\ &= \int_\Omega v_0(x)\left[k_I \int_\Omega J(x-y)I(t,y)dy - k_I I(t,x) + \beta(x)S(t,x)I(t,x) - (\mu(x)+\gamma(x))I(t,x)\right]dx. \end{aligned} \tag{3.2}$$

Now we have from (2.1) and (3.1) that

$$\int_\Omega v_0(x)k_I\int_\Omega J(x-y)I(t,y)dydx = \int_\Omega I(t,y)k_I\int_\Omega J(y-x)v_0(x)dxdy = \int_\Omega I(t,x)k_I\int_\Omega J(x-y)v_0(y)dydx$$
$$= \int_\Omega I(t,x)\left[\lambda_0 v_0(x) + k_I v_0(x) - \beta(x)S^0(x)v_0(x) + (\mu(x)+\gamma(x))\,v(x)\right]dx. \quad (3.3)$$

Hence, substituting (3.3) into (3.2), we have

$$V_1' = \int_\Omega v_0(x)\left[\lambda_0 I(t,x) - \beta(x)(S^0(x)-S(t,x))I(t,x)\right]dx \le \lambda_0\int_\Omega v_0(x)I(t,x)dx.$$

If $\mathcal{R}_0 < 1$, then it follows from Lemma 3.2 that $\lambda_0 < 0$ and hence, $V_1' \le 0$. Since $v_0(x)$ is strictly positive, $V_1' = 0$ if and only if $I \equiv 0$. This implies the global asymptotic stability of the disease-free equilibrium E^0 in D. The proof is complete. □

4. Uniform persistence of the system

Let us define the following spaces.

$$D_0 := \left\{(S,I)\in D:\ I(\cdot,x)>0 \text{ for some } x\in\overline{\Omega}\right\},$$
$$\partial D := D\setminus D_0 = \left\{(S,I)\in D:\ I(\cdot,x)=0 \text{ for all } x\in\overline{\Omega}\right\},$$
$$M_\partial := \left\{(S_0,I_0)\in\partial D:\ \Phi_t\left((S_0,I_0)\right)\in\partial D \text{ for all } t\ge 0\right\},$$

where $\{\Phi_t\}_{t\ge0}$ denotes the semiflow defined by (2.6). System (2.3) is said to be uniformly persistent in D_0 if there exists a positive constant $\xi>0$ such that

$$\liminf_{t\to+\infty} S(t,x)\ge\xi \quad\text{and}\quad \liminf_{t\to+\infty} I(t,x)\ge\xi \quad\text{for all}\quad x\in\overline{\Omega}$$

for any initial condition (S_0,I_0) in D_0. To prove the uniform persistence of system (2.3) for $\mathcal{R}_0>1$, we prove the subsequent three lemmas.

Lemma 4.1. *$\omega\left((S_0,I_0)\right) = \left\{(S^0,0)\right\} = \{E^0\}$ for any $(S_0,I_0)\in M_\partial$, where $\omega\left((S_0,I_0)\right)$ denotes the omega limit set of the positive orbit $\{\Phi_t\left((S_0,I_0)\right):\ t\ge0\}$.*

Proof. For any $(S_0,I_0)\in M_\partial$, the dynamics of $S(t,x)$ is governed by

$$\frac{\partial}{\partial t}S(t,x) = k_S\int_\Omega J(x-y)S(t,y)dy - k_S S(t,x) + b(x) - \mu(x)S(t,x), \quad t>0,\quad x\in\overline{\Omega}.$$

From (2.12), this equation can be rewritten as follows,

$$\frac{\partial}{\partial t}S(t,x) = k_S\int_\Omega J(x-y)\left(S(t,y)-S^0(y)\right)dy - k_S\left(S(t,x)-S^0(x)\right) - \mu(x)\left(S(t,x)-S^0(x)\right), \quad t>0,\quad x\in\overline{\Omega}.$$

Let $U(t,x) := S^0(x) - S(t,x)$. Then, by multiplying -1 to both sides, the above equation can be rewritten as follows,

$$\frac{\partial}{\partial t}U(t,x) = k_S\int_\Omega J(x-y)U(t,y)dy - k_S U(t,x) - \mu(x)U(t,x), \quad t>0,\quad x\in\overline{\Omega}. \quad (4.1)$$

Using the operator A_S defined by (2.4), the equation (4.1) can be rewritten to the following abstract form in X,

$$\frac{d}{dt}U(t) = A_S U(t), \quad t>0.$$

From Assumption 1, it follows as in Lemma 3.1 that there exists a principal eigenvalue $\lambda^* = s(A_S) < 0$ such that $\lambda^* v^*(x) = A_S v^*(x)$, where $v^*(x)$ is a strictly positive continuous eigenfunction. That is, it follows that

$$\lambda^* v^*(x) = k_S \int_\Omega J(x-y) v^*(y) dy - k_S v^*(x) - \mu(x) v^*(x). \tag{4.2}$$

Let us construct the following Lyapunov function.

$$V_2 := \int_\Omega v^*(x) U(t,x) dx.$$

It is easy to see that $V_2 \geq 0$ and $V_2 = 0$ if and only if $U \equiv 0$. The derivative of V_2 along the solution trajectory of (4.1) can be calculated as follows.

$$V_2' = \int_\Omega v^*(x) \frac{\partial}{\partial t} U(t,x) dx = \int_\Omega v^*(x) \left[k_S \int_\Omega J(x-y) U(t,y) dy - k_S U(t,x) - \mu(x) U(t,x) \right] dx. \tag{4.3}$$

Now we have from (2.1) and (4.2) that

$$\begin{aligned} \int_\Omega v^*(x) k_S \int_\Omega J(x-y) U(t,y) dy dx &= \int_\Omega U(t,y) k_S \int_\Omega J(x-y) v^*(x) dx dy = \int_\Omega U(t,x) k_S \int_\Omega J(x-y) v^*(y) dy dx \\ &= \int_\Omega U(t,x) \left[\lambda^* v^*(x) + k_S v^*(x) + \mu(x) v^*(x) \right] dx. \end{aligned} \tag{4.4}$$

Hence, substituting (4.4) into (4.3), we have

$$V_2' = \lambda^* \int_\Omega v^*(x) U(t,x) dx.$$

Since $\lambda^* = s(A_S) < 0$ and $v^*(x)$ is strictly positive, $V_2' \leq 0$ and $V_2' = 0$ if and only if $U \equiv 0$, that is, $S \equiv S^0$. This implies that $\omega((S_0, I_0)) = \{(S^0, 0)\} = \{E^0\}$ and the proof is complete. $\square$

Lemma 4.2. *$S(t,x) > 0$ and $I(t,x) > 0$ for all $t > 0$ and $x \in \overline{\Omega}$, provided $(S_0, I_0) \in D_0$.*

Proof. By integrating the differential equations in (2.3), we have

$$\begin{aligned} S(t,x) =& S_0(x) e^{-\int_0^t (k_S + \beta(x) I(u,x) + \mu(x)) du} + \int_0^t \left(k_S \int_\Omega J(x-y) S(\tau,y) dy + b(x) \right) e^{-\int_\tau^t (k_S + \beta(x) I(u,x) + \mu(x)) du} d\tau, \\ I(t,x) =& I_0(x) e^{-(k_I + \mu(x) + \gamma(x)) t} + \int_0^t \left(k_I \int_\Omega J(x-y) I(\tau,y) dy + \beta(x) S(\tau,x) I(\tau,x) \right) e^{-(k_I + \mu(x) + \gamma(x))(t-\tau)} d\tau. \end{aligned}$$

Hence, from the positivity of $J(x)$ on $\overline{\Omega}$ (see (2.1)), we see that $S(t,x) > 0$ and $I(t,x) > 0$ for all $t > 0$ and $x \in \overline{\Omega}$, provided $(S_0, I_0) \in D_0$. The proof is complete. $\square$

Lemma 4.3. *If $\mathcal{R}_0 > 1$, then the disease-free equilibrium E^0 is a uniform weak repeller for D_0, that is, there exists a sufficiently small positive constant $\epsilon > 0$ such that*

$$\limsup_{t \to +\infty} \left\| \Phi_t((S_0, I_0)) - (S^0, 0) \right\|_Y \geq \epsilon$$

for any $(S_0, I_0) \in D_0$.

Proof. From Lemma 3.2, $\mathcal{R}_0 > 1$ implies that $\lambda_0 = s(A_I + F) > 0$. Hence, there exists a sufficiently small $\epsilon > 0$ such that the eigenvalue problem

$$\eta \phi(x) = k_I \int_\Omega J(x-y) \phi(y) dy - k_I \phi(x) + \beta(x) \left(S^0(x) - \epsilon \right) \phi(x) - (\mu(x) + \gamma(x)) \phi(x), \quad x \in \overline{\Omega} \tag{4.5}$$

has a positive principal eigenvalue $\eta > 0$, which corresponds to a strictly positive continuous eigenfunction $\phi(x)$.

Suppose that

$$\limsup_{t\to+\infty} \left\|\Phi_t\left((S_0, I_0)\right) - (S^0, 0)\right\|_Y < \epsilon \tag{4.6}$$

and show a contradiction. (4.6) implies that there exists a positive constant $T_0 > 0$ such that $S(t,x) > S^0(x) - \epsilon$ for all $t \geq T_0$ and $x \in \overline{\Omega}$. Then, from the second equation in (2.3), we have

$$\frac{\partial}{\partial t} I(t,x) > k_I \int_\Omega J(x-y)I(t,y)dy - k_I I(t,x) + \beta(x)\left(S^0(x) - \epsilon\right) I(t,x) - (\mu(x)+\gamma(x))I(t,x), \quad t \geq T_0, \ \ x \in \overline{\Omega}.$$

The positivity of $I(T_0, x)$ for all $x \in \overline{\Omega}$ implies that there exists a sufficiently small $\tilde{\epsilon} > 0$ such that $I(T_0, x) \geq \tilde{\epsilon}\phi(x)$ for all $x \in \overline{\Omega}$. From (4.5) we see that $e^{\eta(t-T_0)}\tilde{\epsilon}\phi(x)$ is the solution of

$$\frac{\partial}{\partial t} I(t,x) = k_I \int_\Omega J(x-y)I(t,y)dy - k_I I(t,x) + \beta(x)\left(S^0(x) - \epsilon\right) I(t,x) - (\mu(x)+\gamma(x))I(t,x), \quad t \geq T_0, \ \ x \in \overline{\Omega}.$$

Hence, by the comparison principle, we have

$$I(t,x) \geq e^{\eta(t-T_0)}\tilde{\epsilon}\phi(x), \quad t \geq T_0, \quad x \in \overline{\Omega}.$$

Since $\eta > 0$, the right-hand side of the above inequality goes to infinity as $t \to +\infty$. This contradicts with the boundedness proved in Proposition 2.3 and the proof is complete. □

Using Lemmas 4.1, 4.2 and 4.3, we prove the following theorem on the uniform persistence of system (2.3) for $\mathcal{R}_0 > 1$.

Theorem 4.1. *If $\mathcal{R}_0 > 1$, then system* (2.3) *is uniformly persistent in D_0.*

Proof. From Lemmas 4.1, 4.2 and 4.3, we see that $E^0 = (S^0, 0)$ is isolated in D, there is no cycle in M_∂ from $E^0 = (S^0, 0)$ to itself, and $D_0 \cap W^s((S^0, 0)) = \emptyset$, where $W^s(\cdot)$ denotes the stable manifold of a point. Therefore, from [50, Theorem 3], there exists a positive constant $\xi > 0$ such that

$$\min_{(\varphi,\psi)\in\omega((S_0,I_0))} \left[\min\left(\inf_{x\in\overline{\Omega}} \varphi(x), \inf_{x\in\overline{\Omega}} \psi(x)\right)\right] > \xi$$

for all $(S_0, I_0) \in D_0$. This implies that $\liminf_{t\to+\infty} S(t,x) > \xi$ and $\liminf_{t\to+\infty} I(t,x) > \xi$ for all $x \in \overline{\Omega}$. Thus, system (2.3) is uniformly persistent and the proof is complete. □

5. Existence, uniqueness and global asymptotic stability of the endemic equilibrium

In the proof of the existence, uniqueness and global asymptotic stability of the endemic equilibrium E^* for $\mathcal{R}_0 > 1$, we restrict our attention to the special case where $k_S = 0$. This case has been considered for, e.g., the spatial spread of rabies (see [29]). In this case, (2.3) can be rewritten as follows.

$$\begin{cases} \dfrac{\partial}{\partial t} S(t,x) = b(x) - \beta(x)S(t,x)I(t,x) - \mu(x)S(t,x), & t > 0, \ \ x \in \overline{\Omega}, \\ \dfrac{\partial}{\partial t} I(t,x) = k_I \displaystyle\int_\Omega J(x-y)I(t,y)dy - k_I I(t,x) + \beta(x)S(t,x)I(t,x) - (\mu(x)+\gamma(x))I(t,x), & t > 0, \ \ x \in \overline{\Omega}, \\ S(0,x) = S_0(x), \quad I(0,x) = I_0(x), & x \in \Omega. \end{cases} \tag{5.1}$$

Furthermore, $\mathcal{R}_0$ is obtained as the spectral radius $r(\mathcal{K})$ of the following next generation operator.

$$\mathcal{K}\varphi(x) = \beta(x)\frac{b(x)}{\mu(x)} \int_0^{+\infty} T_I(t)\varphi(x)dt, \quad \varphi \in X.$$

(2.10) can be rewritten as follows,

$$\begin{cases} 0 = b(x) - \beta(x)S^*(x)I^*(x) - \mu(x)S^*(x), & x \in \overline{\Omega}, \\ 0 = k_I \displaystyle\int_\Omega J(x-y)I^*(y)dy - k_I I^*(x) + \beta(x)S^*(x)I^*(x) - (\mu(x)+\gamma(x))I^*(x), & x \in \overline{\Omega}. \end{cases} \tag{5.2}$$

From the first equation of (5.2), we have $S^*(x) = b(x)/(\beta(x)I^*(x) + \mu(x)),\ x \in \overline{\Omega}$. Hence, existence of the endemic equilibrium E^* can be shown by finding the positive solution $I^*(x)$ of the following equation.

$$\begin{aligned} 0 =& k_I \int_\Omega J(x-y)I^*(y)dy - k_I I^*(x) + \beta(x)\frac{b(x)}{\beta(x)I^*(x)+\mu(x)}I^*(x) - (\mu(x)+\gamma(x))I^*(x) \\ =& k_I \int_\Omega J(x-y)I^*(y)dy - k_I I^*(x) + \beta(x)S^0(x)\left(1 - \frac{\beta(x)I^*(x)}{\beta(x)I^*(x)+\mu(x)}\right)I^*(x) - (\mu(x)+\gamma(x))I^*(x), \end{aligned} \tag{5.3}$$

where $S^0(x) = b(x)/\mu(x),\ x \in \overline{\Omega}$. Let

$$H(\varphi)(x) := k_I \int_\Omega J(x-y)\varphi(y)dy - k_I\varphi(x) + \beta(x)S^0(x)\left(1 - \frac{\beta(x)\varphi(x)}{\beta(x)\varphi(x)+\mu(x)}\right)\varphi(x) - (\mu(x)+\gamma(x))\varphi(x), \quad x \in \overline{\Omega}.$$

To find the positive root $I^*(x)$ of $H(I^*(x)) = 0$, we will construct a sub-solution $\underline{I}(x)$ and a super-solution $\overline{I}(x)$ such that

$$H(\underline{I}(x)) \ge 0, \quad H(\overline{I}(x)) \le 0 \quad \text{and} \quad 0 < \underline{I}(x) \le \overline{I}(x) \quad \text{for all } x \in \overline{\Omega}.$$

We prove the following proposition on the existence of the endemic equilibrium E^*.

Proposition 5.1. *If $\mathcal{R}_0 > 1$, then system* (5.1) *has at least one endemic equilibrium $E^* = (S^*, I^*)$ in D_0.*

Proof. It follows from Lemma 3.2 that the eigenvalue problem (3.1) has the positive principal eigenvalue $\lambda_0 > 0$ which corresponds to the strictly positive eigenfunction $v_0(x)$. Let $\underline{I}(x) := \epsilon_0 v_0(x)$, where $\epsilon_0 > 0$ is a sufficiently small positive constant. Then, we have

$$\begin{aligned} H\left(\underline{I}(x)\right) =& k_I \int_\Omega J(x-y)\epsilon_0 v_0(y)dy - k_I\epsilon_0 v_0(x) + \beta(x)S^0(x)\left(1 - \frac{\beta(x)\epsilon_0 v_0(x)}{\beta(x)\epsilon_0 v_0(x)+\mu(x)}\right)\epsilon_0 v_0(x) \\ &- (\mu(x)+\gamma(x))\epsilon_0 v_0(x) \\ =& \epsilon_0 v_0(x)\left[\lambda_0 - \beta(x)S^0(x)\frac{\beta(x)\epsilon_0 v_0(x)}{\beta(x)\epsilon_0 v_0(x)+\mu(x)}\right], \quad x \in \overline{\Omega}. \end{aligned}$$

Since $\lambda_0 > 0$, we have that $H(\underline{I}(x)) \ge 0$ for sufficiently small $\epsilon_0 > 0$.

On the other hand, let $\overline{I}(x) := M$, where $M > 0$ is a sufficiently large positive constant such that $\underline{I}(x) = \epsilon_0 v_0(x) \le M = \overline{I}(x)$ for all $x \in \overline{\Omega}$. Then, we have

$$\begin{aligned} H\left(\overline{I}(x)\right) =& k_I \int_\Omega J(x-y)Mdy - k_I M + \beta(x)S^0(x)\left(1 - \frac{\beta(x)M}{\beta(x)M+\mu(x)}\right)M - (\mu(x)+\gamma(x))M \\ \le& k_I M\left(\int_{\mathbb{R}^n} J(x-y)dy - 1\right) + \left(\beta(x)S^0(x)\left(1 - \frac{\beta(x)M}{\beta(x)M+\mu(x)}\right) - (\mu(x)+\gamma(x))\right)M \\ =& \left(\beta(x)S^0(x)\left(1 - \frac{\beta(x)M}{\beta(x)M+\mu(x)}\right) - (\mu(x)+\gamma(x))\right)M, \quad x \in \overline{\Omega}. \end{aligned}$$

Since $1 - \beta(x)M/(\beta(x)M + \mu(x))$ converges to zero as $M \to +\infty$, we have that $H\left(\overline{I}(x)\right) \le 0$ for sufficiently large $M > 0$. Consequently, we see that $\underline{I}(x)$ and $\overline{I}(x)$ are a sub-solution and a super-solution, respectively. Hence, system (5.1) has at least one endemic equilibrium $E^* = (S^*, I^*)$. We have

$$S^*(x) = \frac{b(x)}{\beta(x)I^*(x)+\mu(x)} \le \frac{b(x)}{\mu(x)} = S^0(x), \quad x \in \overline{\Omega}$$

and, by integrating the equations in (5.2) and adding them, we have as in the proof of Proposition 2.3 that

$$\int_\Omega S^*(x)dx + \int_\Omega I^*(x)dx \le \frac{\int_\Omega b(x)dx}{\underline{\mu}}.$$

This implies $E^* = (S^*, I^*) \in D_0$ and the proof is complete. □

We next prove the following theorem on the uniqueness of the endemic equilibrium E^*.

Proposition 5.2. *System* (5.1) *has at most one endemic equilibrium* $E^* = (S^*, I^*)$.

Proof. Let $\tilde{I} \in X_+ \setminus \{0\}$ be a function satisfying (5.3) and $\tilde{I} \not\equiv I^*$. From the continuity, we see that there exists a small positive constant $\zeta > 0$ such that $I^*(x) \ge \zeta \tilde{I}(x)$ for all $x \in \overline{\Omega}$. Let $\zeta^* := \sup\left\{\zeta \in \mathbb{R}_+ : I^*(x) \ge \zeta\tilde{I}(x) \text{ for all } x \in \overline{\Omega}\right\}$. Suppose that $\zeta^* < 1$ and show a contradiction. From the definition of ζ^*, we see that there exists an $\bar{x} \in \overline{\Omega}$ such that $I^*(\bar{x}) = \zeta^*\tilde{I}(\bar{x})$. From (5.3), we have

$$\begin{aligned}
0 =& k_I \int_\Omega J(\bar{x}-y)I^*(y)dy - k_I I^*(\bar{x}) + \frac{b(x)}{\beta(\bar{x})I^*(\bar{x}) + \mu(\bar{x})}\beta(\bar{x})I^*(\bar{x}) - (\mu(\bar{x}) + \gamma(\bar{x}))I^*(\bar{x}) \\
\ge& k_I \int_\Omega J(\bar{x}-y)\zeta^*\tilde{I}(y)dy - k_I \zeta^*\tilde{I}(\bar{x}) + \frac{b(x)}{\beta(\bar{x})\zeta^*\tilde{I}(\bar{x}) + \mu(\bar{x})}\beta(\bar{x})\zeta^*\tilde{I}(\bar{x}) - (\mu(\bar{x}) + \gamma(\bar{x}))\zeta^*\tilde{I}(\bar{x}) \\
>& \zeta^* \left(k_I \int_\Omega J(\bar{x}-y)\tilde{I}(y)dy - k_I\tilde{I}(\bar{x}) + \frac{b(x)}{\beta(\bar{x})\tilde{I}(\bar{x}) + \mu(\bar{x})}\beta(\bar{x})\tilde{I}(\bar{x}) - (\mu(\bar{x}) + \gamma(\bar{x}))\tilde{I}(\bar{x})\right) \\
=& 0,
\end{aligned}$$

which is a contradiction. Therefore, $\zeta^* \ge 1$ and hence, $I^*(x) \ge \zeta^*\tilde{I}(x) \ge \tilde{I}(x)$ for all $x \in \overline{\Omega}$. By changing the role of $I^*(x)$ and $\tilde{I}(x)$, we can show in a similar way that $\tilde{I}(x) \ge I^*(x)$ for all $x \in \overline{\Omega}$. Thus, $I^* \equiv \tilde{I}$ and the proof is complete. □

We finally prove the following theorem on the global asymptotic stability of the endemic equilibrium E^*.

Theorem 5.1. *If* $\mathcal{R}_0 > 1$, *then the unique endemic equilibrium* $E^* = (S^*, I^*)$ *of system* (5.1) *is globally asymptotically stable in* D_0.

Proof. From Propositions 5.1 and 5.2, we see that system (5.1) has the unique endemic equilibrium $E^* = (S^*, I^*)$. Let us construct the following Lyapunov function,

$$V_3 := 2\int_\Omega I^*(x)\left[S^*(x)g\left(\frac{S(t,x)}{S^*(x)}\right) + I^*(x)g\left(\frac{I(t,x)}{I^*(x)}\right)\right]dx,$$

where $g(x) = x - 1 - \ln x,\ x > 0$. Note that $g(x) \ge 0$ for all $x > 0$ and $g(x) = 0$ if and only if $x = 1$ (see e.g., [42] for the usage of this function in a Lyapunov function). From Theorem 4.1, we see that this Lyapunov function is well-defined. The derivative of V_3 is calculated as follows,

$$\begin{aligned}
V_3' =& 2\int_\Omega I^*(x)\left[\left(1 - \frac{S^*(x)}{S(t,x)}\right)\frac{\partial}{\partial t}S(t,x) + \left(1 - \frac{I^*(x)}{I(t,x)}\right)\frac{\partial}{\partial t}I(t,x)\right]dx \\
=& 2\int_\Omega I^*(x)\left[\left(1 - \frac{S^*(x)}{S(t,x)}\right)(b(x) - \beta(x)S(t,x)I(t,x) - \mu(x)S(t,x))\right. \\
&\left. + \left(1 - \frac{I^*(x)}{I(t,x)}\right)\left(k_I\int_\Omega J(x-y)I(t,y)dy - k_I I(t,x) + \beta(x)S(t,x)I(t,x) - (\mu(x)+\gamma(x))I(t,x)\right)\right]dx.
\end{aligned} \tag{5.4}$$

From (5.2), we have

$$\begin{cases} b(x) = \beta(x)S^*(x)I^*(x) + \mu(x)S^*(x), & x \in \overline{\Omega}, \\ (k_I + \mu(x) + \gamma(x))I^*(x) = k_I \displaystyle\int_\Omega J(x-y)I^*(y)dy + \beta(x)S^*(x)I^*(x), & x \in \overline{\Omega}. \end{cases}$$

Using these equations, (5.4) can be calculated as follows,

$$\begin{aligned}
V_3' =& 2\int_\Omega I^*(x) \left[\left(1 - \frac{S^*(x)}{S(t,x)}\right) b(x) - \left(1 - \frac{S^*(x)}{S(t,x)}\right) (\beta(x)S(t,x)I(t,x) + \mu(x)S(t,x)) \right. \\
& + \left(1 - \frac{I^*(x)}{I(t,x)}\right) \left(k_I \int_\Omega J(x-y)I(t,y)dy + \beta(x)S(t,x)I(t,x) \right) \\
& \left. + \left(1 - \frac{I(t,x)}{I^*(x)}\right) (k_I + \mu(x) + \gamma(x))I^*(x) \right] dx \\
=& 2\int_\Omega I^*(x) \left[\left(1 - \frac{S^*(x)}{S(t,x)}\right) (\beta(x)S^*(x)I^*(x) + \mu(x)S^*(x)) - \left(1 - \frac{S^*(x)}{S(t,x)}\right) (\beta(x)S(t,x)I(t,x) + \mu(x)S(t,x)) \right. \\
& + \left(1 - \frac{I^*(x)}{I(t,x)}\right) \left(k_I \int_\Omega J(x-y)I(t,y)dy + \beta(x)S(t,x)I(t,x) \right) \\
& \left. + \left(1 - \frac{I(t,x)}{I^*(x)}\right) \left(k_I \int_\Omega J(x-y)I^*(y)dy + \beta(x)S^*(x)I^*(x) \right) \right] dx \\
=& 2\int_\Omega I^*(x) \left[\left(2 - \frac{S^*(x)}{S(t,x)} - \frac{S(t,x)}{S^*(x)}\right) (\beta(x)S^*(x)I^*(x) + \mu(x)S^*(x)) \right. \\
& \left. + \left(1 - \frac{I^*(x)}{I(t,x)}\right) k_I \int_\Omega J(x-y)I(t,y)dy + \left(1 - \frac{I(t,x)}{I^*(x)}\right) k_I \int_\Omega J(x-y)I^*(y)dy \right] dx.
\end{aligned} \tag{5.5}$$

Now we have

$$\begin{aligned}
& \int_\Omega I^*(x) \left[\left(1 - \frac{I^*(x)}{I(t,x)}\right) k_I \int_\Omega J(x-y)I(t,y)dy + \left(1 - \frac{I(t,x)}{I^*(x)}\right) k_I \int_\Omega J(x-y)I^*(y)dy \right] dx \\
&= k_I \int_\Omega \int_\Omega J(x-y)I^*(x)I^*(y) \left(1 - \frac{I(t,x)}{I^*(x)} + \frac{I(t,y)}{I^*(y)} - \frac{I^*(x)I(t,y)}{I(t,x)I^*(y)}\right) dydx.
\end{aligned} \tag{5.6}$$

On the other hand, by changing the order of integration, we have

$$\begin{aligned}
& \int_\Omega I^*(x) \left[\left(1 - \frac{I^*(x)}{I(t,x)}\right) k_I \int_\Omega J(x-y)I(t,y)dy + \left(1 - \frac{I(t,x)}{I^*(x)}\right) k_I \int_\Omega J(x-y)I^*(y)dy \right] dx \\
&= k_I \int_\Omega \int_\Omega J(x-y)I^*(x)I^*(y) \left(1 - \frac{I(t,x)}{I^*(x)} + \frac{I(t,y)}{I^*(y)} - \frac{I^*(x)I(t,y)}{I(t,x)I^*(y)}\right) dxdy \\
&= k_I \int_\Omega \int_\Omega J(y-x)I^*(y)I^*(x) \left(1 - \frac{I(t,y)}{I^*(y)} + \frac{I(t,x)}{I^*(x)} - \frac{I^*(y)I(t,x)}{I(t,y)I^*(x)}\right) dxdy \\
&= k_I \int_\Omega \int_\Omega J(x-y)I^*(x)I^*(y) \left(1 - \frac{I(t,y)}{I^*(y)} + \frac{I(t,x)}{I^*(x)} - \frac{I^*(y)I(t,x)}{I(t,y)I^*(x)}\right) dxdy.
\end{aligned} \tag{5.7}$$

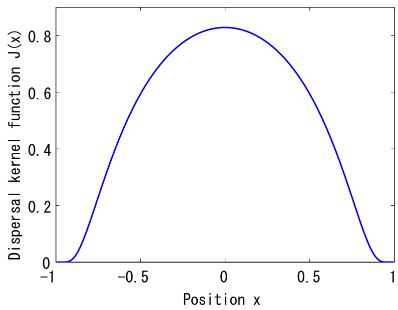


Figure 1: Diffusion kernel function $J(x)$ defined by (6.1) on $(-1, 1)$.

Hence, using (5.6) and (5.7), (5.5) can be calculated as follows.

$$\begin{aligned}
V_3' =& 2\int_\Omega I^*(x)\left(2 - \frac{S^*(x)}{S(t,x)} - \frac{S(t,x)}{S^*(x)}\right)(\beta(x)S^*(x)I^*(x) + \mu(x)S^*(x))\,dx \\
&+ k_I \int_\Omega\int_\Omega J(x-y)I^*(x)I^*(y)\left(1 - \frac{I(t,x)}{I^*(x)} + \frac{I(t,y)}{I^*(y)} - \frac{I^*(x)I(t,y)}{I(t,x)I^*(y)} + 1 - \frac{I(t,y)}{I^*(y)}\right. \\
&\left.+ \frac{I(t,x)}{I^*(x)} - \frac{I^*(y)I(t,x)}{I(t,y)I^*(x)}\right)dxdy \\
=& 2\int_\Omega I^*(x)\left(2 - \frac{S^*(x)}{S(t,x)} - \frac{S(t,x)}{S^*(x)}\right)(\beta(x)S^*(x)I^*(x) + \mu(x)S^*(x))\,dx \\
&+ k_I \int_\Omega\int_\Omega J(x-y)I^*(x)I^*(y)\left(2 - \frac{I^*(x)I(t,y)}{I(t,x)I^*(y)} - \frac{I^*(y)I(t,x)}{I(t,y)I^*(x)}\right)dxdy.
\end{aligned}$$

By using the arithmetic-geometric mean, we see that $V_3' \le 0$ and $V_3' = 0$ if and only if $(S, I) = (S^*, I^*)$. This implies the global asymptotic stability of the endemic equilibrium E^* and the proof is complete. □

6. Numerical simulation

In this section, we perform numerical simulation to verify the validity of our theoretical results. For simplicity, we consider the spatially one-dimensional case where $\Omega = (-1, 1) \subset \mathbb{R}$. We restrict our attention to the special case in Section 5, that is, the case where $k_S = 0$. As in [30], we employ the following diffusion kernel function,

$$J(x) = \begin{cases} C\exp\left(\frac{1}{x^2-1}\right), & -1 < x < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{6.1}$$

where C is a positive constant taken to be 2.2523 so that $\int_{\mathbb{R}} J(x)dx = \int_{-1}^{1} J(x)dx \approx 1$ (see Figure 1). The initial functions are fixed as follows.

$$S_0(x) = 0.99\cos\frac{\pi}{2}x, \quad I_0(x) = 0.01\cos\frac{\pi}{2}x, \quad x \in [-1, 1].$$

For the approximation of the basic reproduction number $\mathcal{R}_0$ defined by (2.18), we consider a corresponding discretized system. Let $m \in \mathbb{N}$ be the number of space subintervals, $\Delta x := 2/m$ be the size of each subinterval and $x_k := -1 + k\Delta x$, $k = 0, 1, 2, \cdots, m$. Then, we have $-1 = x_0 < x_1 < x_2 < \cdots < x_m = 1$. Let us define

$$I_j(t) := I(t, x_j),\quad b_j := b(x_j),\quad \mu_j := \mu(x_j),\quad \gamma_j := \gamma(x_j),\quad \beta_j := \beta(x_j),\quad J_{jk} := J(x_j - x_k),\quad j, k = 1, 2, \cdots, m.$$

By the rectangle method, we have the following approximation.

$$\int_{-1}^{1} J(x_j-y)I(t,y)dy \approx \sum_{k=1}^{m} J(x_j-x_k)I(t,x_k)\Delta x = \sum_{k=1}^{m} \frac{2}{m} J_{jk} I_k(t), \quad t>0, \quad j=1,2,\cdots,m.$$

Then, the linearized equation (2.16) can be rewritten to the following multi-group system (see, for instance, [25, 37]).

$$\frac{d}{dt}I_j(t) = k_I \sum_{k=1}^{m} \frac{2}{m} J_{jk} I_k(t) - k_I I_j(t) + \beta_j S_j^0 I_j(t) - (\mu_j+\gamma_j) I_j(t), \quad t>0, \quad j=1,2,\cdots,m, \tag{6.2}$$

where $S_j^0 := S^0(x_j) = b_j/\mu_j,\ j=1,2,\cdots,m$. From [54], the next generation matrix $\mathcal{K}_m$ of the multi-group system (6.2) can be calculated as $\mathcal{K}_m := F_m\,(-A_{I,m})^{-1}$, where

$$F_m = \begin{pmatrix} \beta_1 S_1^0 & 0 & \cdots & 0 \\ 0 & \beta_2 S_2^0 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \beta_m S_m^0 \end{pmatrix},$$

$$A_{I,m} = \begin{pmatrix} k_I \frac{2}{m} J_{11} - (k_I+\mu_1+\gamma_1) & k_I \frac{2}{m} J_{12} & \cdots & k_I \frac{2}{m} J_{1m} \\ k_I \frac{2}{m} J_{21} & k_I \frac{2}{m} J_{22} - (k_I+\mu_2+\gamma_2) & & \vdots \\ \vdots & & \ddots & \\ k_I \frac{2}{m} J_{m1} & \cdots & & k_I \frac{2}{m} J_{mm} - (k_I+\mu_m+\gamma_m) \end{pmatrix}.$$

Under Assumption 1, $\sum_{k=1}^{m}(2/m)J_{kj} \lessapprox 1$ for all $j=1,2,\cdots,m$ and hence, $-A_{I,m}$ is a nonsingular M-matrix. Therefore, the inverse $(-A_{I,m})^{-1}$ is positive and $\mathcal{K}_m = F_m\,(-A_{I,m})^{-1}$ is positive and irreducible. From the Perron-Frobenius theorem (see, for instance, [5]), $r(\mathcal{K}_m)$ is the positive dominant eigenvalue. Let $\mathcal{R}_{0,m} := r(\mathcal{K}_m)$. From the definition, we can expect that $\mathcal{R}_{0,m} \to \mathcal{R}_0$ as $m \to +\infty$. To prove it rigorously, we should show not only the pointwise convergence but also the strong stability of $\{\mathcal{K}_m\}_{m\in\mathbb{N}}$ (see, for instance, [10]). In this paper, we leave this as a future work and use $\mathcal{R}_{0,m}$ for sufficiently large m as the approximated threshold value instead of $\mathcal{R}_0$.

First we consider the case of constant parameters. Fix

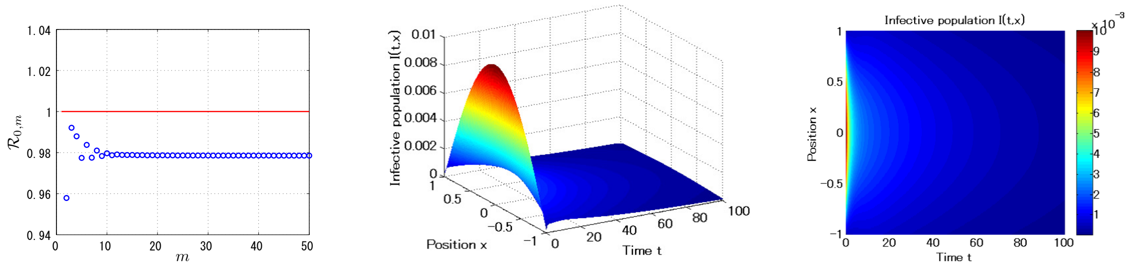
$$b(x)=1, \quad \mu(x)=0.5, \quad \gamma(x)=0.5, \quad k_I=0.5, \quad x\in[-1,1] \tag{6.3}$$

and vary $\beta(x)$ to observe the stability change with different $\mathcal{R}_{0,m}$. When $\beta(x)=0.52$, $\mathcal{R}_{0,m}$ converges approximately to $0.9785<1$ as m increases (see (a) of Figure 2). In this case, the infective population $I(t,x)$ converges to 0 as time evolves (see (b) of Figure 2). This is consistent with the global asymptotic stability of the disease-free equilibrium E^0, which was proved in Theorem 3.1. When $\beta(x)=0.54$, $\mathcal{R}_{0,m}$ converges approximately to $1.0162>1$ as m increases (see (a) of Figure 3). In this case, the infective population $I(t,x)$ converges to a positive distribution $I^*(x)$ as time evolves (see (b) of Figure 3). This is consistent with the global asymptotic stability of the endemic equilibrium E^*, which was proved in Theorem 5.1.

Next we consider the case where some parameters are nonconstant. Fix

$$b(x)=1+0.1x^2, \quad \mu(x)=0.5, \quad \gamma(x)=0.5, \quad k_I=0.5, \quad m=100, \quad x\in[-1,1] \tag{6.4}$$

and vary $\beta(x)$ to observe the stability change with different $\mathcal{R}_{0,m}$. When $\beta(x)=0.51(1+0.1\cos 5\pi x)$, $\mathcal{R}_{0,m}$ converges approximately to $0.9924<1$ as m increases (see (a) of Figure 4). In this case, the infective population $I(t,x)$ converges to 0 as time evolves (see (a)of Figure 4). This is consistent with the global

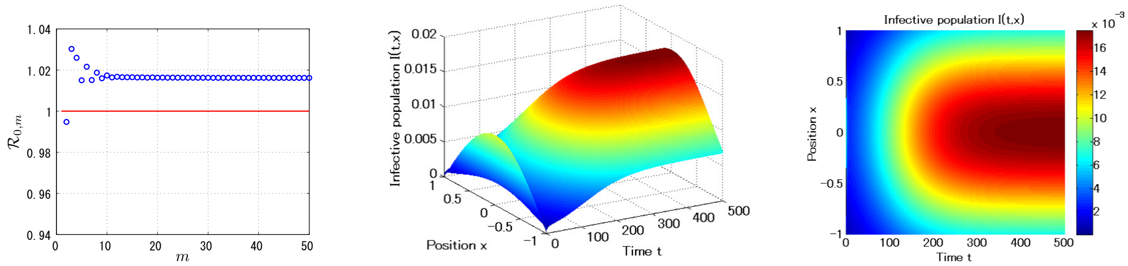


(a) The approximated threshold value $\mathcal{R}_{0,m}$ versus the number of space subintervals m ($\mathcal{R}_{0,50} \approx 0.9785 < 1$).

(b) Time evolution of infective population $I(t,x)$ in position x, $-1 \le x \le 1$.

(c) Time evolution of infective population $I(t,x)$ in position x, $-1 \le x \le 1$.

Figure 2: Numerical simulation result with constant parameters (6.3) and $\beta(x) = 0.52$.



(a) The approximated threshold value $\mathcal{R}_{0,m}$ versus the number of space subintervals m ($\mathcal{R}_{0,50} \approx 1.0162 > 1$).

(b) Time evolution of infective population $I(t,x)$ in position x, $-1 \le x \le 1$.

(c) Time evolution of infective population $I(t,x)$ in position x, $-1 \le x \le 1$.

Figure 3: Numerical simulation result with constant parameters (6.3) and $\beta(x) = 0.54$.

asymptotic stability of the disease-free equilibrium E^0, which was proved in Theorem 3.1. When $\beta(x) = 0.52(1 + 0.1\cos 5\pi x)$, $\mathcal{R}_{0,50}$ converges approximately to $1.0119 > 1$ (see (a) of Figure 5). In this case, the infective population $I(t,x)$ converges to a positive distribution $I^*(x)$ as time evolves (see (b) of Figure 5). This is consistent with the global asymptotic stability of the endemic equilibrium E^*, which was proved in Theorem 5.1.

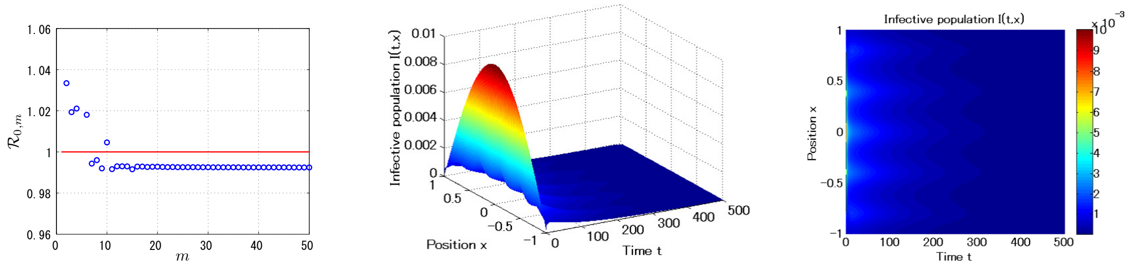
Acknowledgments

T. Kuniya is supported by Grant-in-Aid for Young Scientists (B) of Japan Society for the Promotion of Science (Grant No.15K17585) and the Japan Initiative for Global Research Network on Infectious Diseases (J-GRID) from Mistry of Education, Culture, Sport, Science and Technology in Japan, and Japan Agency for Medical Research and Development (AMED). J. Wang is supported by National Natural Science Foundation of China (No.11401182, 11471089), and Science and Technology Innovation Team in Higher Education Institutions of Heilongjiang Province (No.2014TD005).

References

[1] G. Alberti, G. Bellettini, A nonlocal anisotropic model for phase transitions Part I: the optimal profile problem, Mathematische Annalen 310 (1998) 527–560.

[2] L.J.S. Allen, B.M. Bolker, Y. Lou, A.L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic reaction-diffusion model, Disc. Cont. Dyn. Syst. 21 (2008) 1–20.

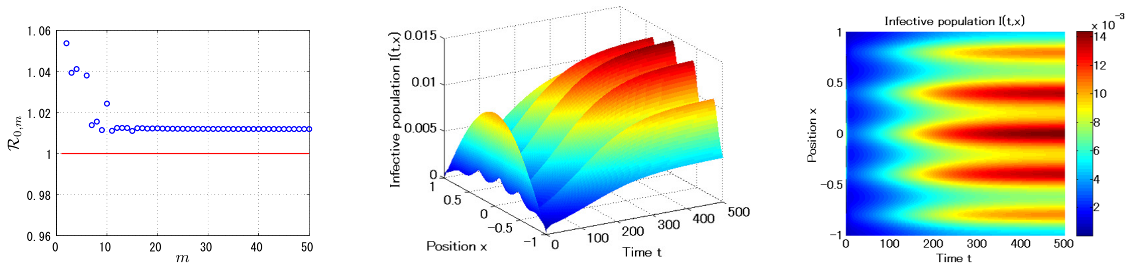


(a) The approximated threshold value $\mathcal{R}_{0,m}$ versus the number of space subintervals m ($\mathcal{R}_{0,50} \approx 0.9924 < 1$).

(b) Time evolution of infective population $I(t, x)$ in position x, $-1 \leq x \leq 1$.

(c) Time evolution of infective population $I(t, x)$ in position x, $-1 \leq x \leq 1$.

Figure 4: Numerical simulation result with parameters (6.4) and $\beta(x) = 0.51(1 + 0.1 \cos 5\pi x)$.



(a) The approximated threshold value $\mathcal{R}_{0,m}$ versus the number of space subintervals m ($\mathcal{R}_{0,m} \approx 1.0119 > 1$).

(b) Time evolution of infective population $I(t, x)$ in position x, $-1 \leq x \leq 1$.

(c) Time evolution of infective population $I(t, x)$ in position x, $-1 \leq x \leq 1$.

Figure 5: Numerical simulation result with parameters (6.4) and $\beta(x) = 0.52(1 + 0.1 \cos 5\pi x)$.

[3] P.W. Bates, A. Chmaj, An integrodifferential model for phase transitions: stationary solutions in higher space dimensions, J. Stat. Phys. 95 (1999) 1119–1139.

[4] P.W. Bates, P.C. Fife, X. Ren, X. Wang, Traveling waves in a convolution model for phase transitions, Arch. Rational Mech. Anal. 138 (1997) 105–136.

[5] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Academic Press, New York, 1979.

[6] P.W. Bates, G. Zhao, Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal, J. Math. Anal. Appl. 332 (2007) 428–440.

[7] F. Capone, V.D. Cataldis, R.D. Luca, On the nonlinear stability of an epidemic SEIR reaction-diffusion model, Ricerche Mat. 62 (2013) 161–181.

[8] A.J.J. Chamaj, X. Ren, The nonlocal bistable equation: stationary solutions on a bounded interval, Electron. J. Differential Equations 2 (2002) 1–12.

[9] E. Chasseigne, M. Chaves, J.D. Rossi, Asymptotic behavior for nonlocal diffusion equations, J. Math. Pures Appl. 86 (2006) 271–291.

[10] F. Chatelin, The spectral approximation of linear operators with applications to the computation of eigenelements of differential and integral operators, SIAM Review 23 (1981) 495–522.

[11] X. Chen, Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations, Adv. Differential Equations 2 (1997) 125–160.

[12] S. Chinviriyasit, W. Chinviriyasit, Numerical modelling of an SIR epidemic model with diffusion, Appl. Math. Comput. 216 (2010) 395–409.

[13] C. Cortázar, J. Coville, M. Elgueta, S. Martínez, A nonlocal inhomogeneous dispersal process, J. Differential Equations 241 (2007) 332–358.

[14] C. Cortazar, M. Elgueta, J.D. Rossi, A nonlocal diffusion equation whose solutions develop a free boundary, Ann. Henri Poincaŕe 6 (2005) 269–281.

[15] J. Coville, On uniqueness and monotonicity of solutions of non-local reaction diffusion equation, Annali di Matematica Pura ed Applicata 185 (2006) 461–485.

[16] J. Coville, On a simple criterion for the existence of a principal eigenfunction of some nonlocal operators, J. Differential Equations 249 (2010) 2921–2953.
[17] J. Coville, L. Dupaigne, On a non-local equation arising in population dynamics, Proc. Roy. Soc. Edinburgh Sect. A 137 (2007) 727–755.
[18] R. Cui and Y. Lou, A spatial SIS model in advective heterogeneous environments, J. DifferentialEquations, 261(2016), 3305–3343.
[19] K.A. Dahmen, D.R. Nelson, N.M. Shnerb, Life and death near a windy oasis, J. Math. Biol. 41(1) (2000) 1–23.
[20] O. Diekmann, J.A.J. Heesterbeek, J.A.J. Metz, On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations, J. Math. Biol. 28 (1990) 365–382.
[21] A. Ducrot, Travelling wave solutions for a scalar age-structured equation, Disc. Cont. Dyn. Syst. 7 (2007) 251–273.
[22] A. Ducrot, P. Magal, Travelling wave solutions for an infection-age structured epidemic model with external supplies, Nonlinearity 24 (2011) 2891–2911.
[23] W.E. Fitzgibbon, M.E. Parrott, G.F. Webb, Diffusion epidemic models with incubation and crisscross dynamics, Math. Biosci. 128 (1985) 131–155.
[24] García-Melián and Rossi, On the principal eigenvalue of some nonlocal diffusion problems, J. Differential Equations 246 (2009) 21–38.
[25] H. Guo, M.Y. Li, Z. Shuai, Global stability of the epidemic equilibrium of multi-group SIR epidemic models, Canad. Appl. Math. Quart. 14 (2006) 259–284.
[26] Y. Hosono, B. Ilyas, Traveling waves for a simple diffusive epidemic model, Math. Mod. Meth. Appl. Sci. 5 (1995) 935–966.
[27] V. Hutson, S. Martinez, K. Mischaikow, G.T. Vickers, The evolution of dispersal, J. Math. Biol. 47 (2003) 483–517.
[28] H. Inaba, On a new perspective of the basic reproduction number in heterogeneous environments, J. Math. Biol. 65 (2012) 309–348.
[29] A. Källén, P. Arcuri, J.D. Murray, A simple model for the spatial spread and control of rabies, J. Theor. Biol. 116 (1985) 377–393.
[30] C.Y. Kao, Y. Lou, W. Shen, Random dispersal vs non-local dispersal, Disc. Cont. Dyn. Syst. 26 (2010) 551–596.
[31] C.Y. Kao, Y. Lou, W. Shen, Evolution of mixed dispersal in periodic environments, Disc. Cont. Dyn. Syst. Ser. B 17 (2012) 2047–2072.
[32] W.O. Kermack, A.G. McKendrick, A contribution to the mathematical theory of epidemics, Proc. Roy. Soc. Lond. A 115 (1927) 700–721.
[33] M. Kubo, M. Langlais, Periodic solutions for a population dynamics problem with age-dependence and spatial structure, J. Math. Biol. 29 (1991) 363–378.
[34] T. Kuniya, R. Oizumi, Existence result for an age-structured SIS epidemic model with spatial diffusion, Nonlinear Anal. RWA 23 (2015) 196–208.
[35] T. Kuniya, J. Wang, Lyapunov functions and global stability for a spatially diffusive SIR epidemic model, Appl. Anal. 13 (2016) DOI:10.1080/00036811.2016.1199796.
[36] M. Langlais, S. Busenberg, Global behaviour in age structured S.I.S. models with seasonal periodicities and vertical transmission, J. Math. Anal. Appl. 213 (1997) 511–533.
[37] M.Y. Li, Z. Shuai, Global stability of an epidemic model in a patchy environment, Canad. Appl. Math. Quart. 17 (2009) 175–187.
[38] W.T. Li, Y.J. Sun, Z.C. Wang, Entire solutions in the Fisher-KPP equation with nonlocal dispersal, Nonlinear Anal. RWA 11 (2010) 2302–2313.
[39] Y. Lou, X.Q. Zhao, A reaction-diffusion malaria model with incubation period in the vector population, J. Math. Biol. 62 (2011) 543–568.
[40] F. Lutscher, M.A. Lewis, E. McCauley, Effects of heterogeneity on spread and persistence in rivers, Bull. Math. Biol. 68(8) (2006) 2129–2160.
[41] F. Lutscher, E. McCauley, M.A. Lewis, Spatial patterns and coexistence mechanisms in systems with unidirectional flow, Theor. Popul. Biol. 71(3) (2007) 267–277.
[42] C.C. McCluskey, Complete global stability for an SIR epidemic model with delay - distributed or discrete, Nonlinear Anal. RWA 11 (2010) 55–59.
[43] J. Medlock, M. Kot, Spreading disease: integro-diffrential equations old and new, Math. Biosci. 184 (2003) 201–222.
[44] I. Melkemi, A.Z. Morkrane, A. Youkana, Boundedness and large-time behavior results for a diffusive epidemic model, J. Appl. Math. (2007) Article ID 17930, doi: 10.1155/2007/17930.
[45] J.D. Murray, Mathematical Biology, Springer-Verlag, Berlin, 1989.
[46] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag New York Berlin Heidelberg Tokyo, 1983.
[47] R. Peng, Asymptotic profiles of the positive steady state for an SIS epidemic reaction-diffusion model. Part I, J. Differential Equations 247 (2009) 1096–1119.
[48] R. Peng and S. Liu, Global stability of the steady states of an SIS epidemic reaction-diffusion model, Nonlinear Analysis, 71 (2009) 239–247.
[49] N. Rawal, W. Shen, Criteria for the existence and lower bounds of principal eigenvalues of time periodic nonlocal dispersal operators and applications, J. Dyn. Diff. Equat. 24 (2012) 927–954.
[50] H.L. Smith, X.Q. Zhao, Robust persistence for semidynamical systems, Nonlinear Anal. 47 (2001) 6169–6179.
[51] J.W. Sun, F.Y. Yang, W.T. Li, A nonlocal dispersal equation arising from a selection-migration model in genetics, J. Differential Equations 257 (2014) 1372–1402.
[52] Y.J. Sun, W.T. Li, Z.C. Wang, Entire solutions in nonlocal dispersal equations with bistable nonlinearity, J. Differential

Equations 251 (2011) 551–581.
[53] H.R. Thieme, Spectral bound and reproduction number for infinite-dimensional population structure and time heterogeneity, SIAM J. Appl. Math. 70 (2009) 188–211.
[54] P. van den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, Math. Biosci. 180 (2002) 29–48.
[55] F.Y. Yang, W.T. Li, Dynamics of a nonlocal dispersal SIS epidemic model, (2016) ArXiv:1601.05188.
[56] J. Wang, J. Yang, T. Kuniya, Dynamics of a PDE viral infection model incorporating cell-to-cell transmission, J. Math. Anal. Appl. 444 (2016) 1542–1564.
[57] W. Wang, X. Zhao, A nonlocal and time-delayed reaction-diffusion model of dengue transmission, SIAM J. Appl. Math. 71 (2011) 147–168.
[58] W. Wang, X. Zhao, Basic reproduction numbers for reaction-diffusion epidemic models, SIAM J. Appl. Dyn. Syst. 11 (2012) 1652–1673.
[59] X. Wang, Metastability and stability of patterns in a convolution model for phase transitions, J. Differential Equations 183 (2002) 434–461.
[60] G.F Webb, A reaction-diffusion model for a deterministic diffusive model, J. Math. Anal. Appl. 84 (1981) 150–161.
[61] Y. Wu and X. Zou, Asymptotic profiles of steady states for a diffusive SIS epidemic model with mass action infection mechanism, J. DifferentialEquations, 261(2016), 4424–4447.
[62] T. Zhang, W. Wang, Existence of traveling wave solutions for influenza model with treatment, J. Math. Anal. Appl. 419 (2014) 469–495.