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An infection age-space structured SIR epidemic model with Neumann boundary condition

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ABSTRACT
In this paper, we are concerned with an SIR epidemic model with infection age and spatial diffusion in the case of Neumann boundary condition. The original model is constructed as a nonlinear age structured system of reaction-diffusion equations. By using the method of characteristics, we reformulate the model into a system of a reaction-diffusion equation and a Volterra integral equation. For the reformulated system, we define the basic reproduction number $\mathcal{R}_0$ by the spectral radius of the next generation operator, and show that if $\mathcal{R}_0 < 1$, then the trivial disease-free steady state is globally attractive, whereas if $\mathcal{R}_0 > 1$, then the disease in the system is persistent. Moreover, under an additional assumption that there exists a finite maximum age of infectiousness, we show the global attractivity of a constant endemic steady state for $\mathcal{R}_0 > 1$.

KEYWORDS
SIR epidemic model; infection age; diffusion; basic reproduction number; global attractivity

AMS CLASSIFICATION
35Q92; 37N25; 92D30

1. Introduction

In 1927, Kermack and McKendrick [1] constructed a continuous-time SIR epidemic model, in which the total population is divided into three subpopulations called susceptible (S), infective (I) and recovered (R). Their original model included the infection age, that is, time elapsed since the infection. Recently, in 2010, Magal *et al.* [2] studied an SIR epidemic model with infection age, and proved that if the basic reproduction number $\mathcal{R}_0$ (see, for instance, [3–5]) is less than 1, then the trivial disease-free steady state is globally asymptotically stable, whereas if $\mathcal{R}_0 > 1$, then the nontrivial endemic steady state is so.

Epidemic models with spatial diffusion, which are suitable for diseases such as the rabies ([6]) and the Black Death ([7]), have been studied for decades (see, for instance, [6–9]). However, there are relatively few works on epidemic models with both of the infection age and spatial diffusion. In [10], Webb studied an infection age-space

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structured SEIR epidemic model without birth and death processes, and showed the existence and uniqueness of nonnegative global solution and convergence of the infective population to zero as time goes to infinity. In [11], Fitzgibbon *et al.* studied an infection age-space structured SEIR epidemic model for the crisscross dynamics, which is a generalization of the model in [10]. In [12] and [13], Ducrot and Magal studied the existence of travelling wave solutions in infection age-space structured SIR epidemic models without and with external supplies, respectively. In [14], Zhang and Wang studied a time-periodic infection age-space structured SIR epidemic model, and showed that $\mathcal{R}_0$ is a threshold for the extinction or uniform persistence of the disease.

In this paper, we study an infection age-space structured SIR epidemic model with birth and death processes. It is a generalization of the model in [2] to a spatially diffusive system, and corresponds to the model studied in [13]. Although [13] focused on the existence of travelling wave solutions in a spatially unbounded domain, this study focus on the asymptotic behavior of solutions in a spatially bounded domain. In this study, we derive the basic reproduction number $\mathcal{R}_0$ as the spectral radius of the next generation operator, and show that if $\mathcal{R}_0 < 1$, then the trivial disease-free steady state is globally attractive, whereas if $\mathcal{R}_0 > 1$, then the disease in the system is persistent. Moreover, under an additional assumption that there exists a finite maximum age of infectiousness, we show that a constant positive endemic steady state is globally attractive if $\mathcal{R}_0 > 1$.

The organization of this paper is as follows. In Section 2, we formulate the main model and reformulate it by using the method of characteristics to a coupled system of a reaction-diffusion equation and a Volterra integral equation. In Section 3, we prove the existence and uniqueness of positive global solution by using the Banach-Picard fixed point theorem. In Section 4, we derive the basic reproduction number $\mathcal{R}_0$ and show that if $\mathcal{R}_0 > 1$, then a space-independent endemic steady state for the original model exists, whereas if $\mathcal{R}_0 < 1$, then the trivial disease-free steady state is globally attractive. In Section 5, we prove the persistence of the disease in the system for $\mathcal{R}_0 > 1$. In Section 6, under an additional assumption that there exists a finite maximum age of infectiousness, we prove the global attractivity of a constant positive endemic steady state for $\mathcal{R}_0 > 1$ by constructing a suitable Lyapunov function. Section 7 is devoted to the discussion.

2. The model

Let $I(t,a,x)$ denote the infective population at time $t \geq 0$, infection age $a \geq 0$ and position $x \in \overline{\Omega} \subset \mathbb{R}$, where $\Omega := (\ell_1, \ell_2) \subset \mathbb{R}$ is a spatially bounded domain. Let $S(t,x)$ and $R(t,x)$ denote the densities of susceptible and recovered populations, respectively, at time $t \geq 0$ and position $x \in \Omega$. We assume that all newborns are susceptible and let $b > 0$ be the number of newborns per unit time. Let $\mu > 0$ be the per capita natural death rate per unit time. Let $\gamma(a)$ be the per capita age-specific recovery rate, and let $\beta(a)$ be the per capita age-specific disease transmission rate per unit time. We make the following assumptions,

(A1) $\gamma \in L^\infty_+(0,+\infty)$ and $\beta \in L^\infty_+(0,+\infty)$.
(A2) There exist $0 < a_1 < a_2 < +\infty$ such that $\beta(a) > 0$ for all $a \in (a_1, a_2)$.

Let $\gamma^+ := \text{ess.sup}_{a\geq 0}\, \gamma(a) < +\infty$, $\beta^+ := \text{ess.sup}_{a \geq 0}\, \beta(a) < +\infty$. Let $d_1 > 0$, $d_2 > 0$ and $d_3 > 0$ be the diffusion coefficients for susceptible, infective and recovered individuals, respectively. The boundary condition is Neumann: $\partial_x S(t,\ell_1) = \partial_x S(t,\ell_2) =$

$\partial_x R(t,\ell_1) = \partial_x R(t,\ell_2) = 0$ for all $t > 0$, and $\partial_x I(t,a,\ell_1) = \partial_x I(t,a,\ell_2) = 0$ for all $t > 0$ and $a > 0$. The main model of this paper is formulated as the following system with infection age and spatial diffusion, for $t > 0$, $a > 0$ and $x \in [\ell_1, \ell_2]$,

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = d_1 \dfrac{\partial^2 S(t,x)}{\partial x^2} + b - S(t,x)\displaystyle\int_0^{+\infty} \beta(a) I(t,a,x) \mathrm{d}a - \mu S(t,x), \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) I(t,a,x) = d_2 \dfrac{\partial^2 I(t,a,x)}{\partial x^2} - [\mu + \gamma(a)] I(t,a,x), \\ I(t,0,x) = S(t,x) \displaystyle\int_0^{+\infty} \beta(a) I(t,a,x) \mathrm{d}a, \\ \dfrac{\partial R(t,x)}{\partial t} = d_3 \dfrac{\partial^2 R(t,x)}{\partial x^2} + \displaystyle\int_0^{+\infty} \gamma(a) I(t,a,x)\mathrm{d}a - \mu R(t,x), \end{cases} \tag{1}$$

combined with initial condition $S(0,x) = \phi_1(x)$, $I(0,a,x) = \phi_2(a,x)$, $R(0,x) = \phi_3(x)$, $a \geq 0$, $x \in [\ell_1, \ell_2]$. Without loss of generality, we can make the following change of variable, $x \mapsto \pi(x - \ell_1)/(\ell_2 - \ell_1)$, in which the domain becomes $\Omega = (0, \pi)$ and the diffusion coefficients become $d_i(\pi/(\ell_2 - \ell_1))^2$, $i = 1, 2, 3$ (we denote them again d_i, $i = 1, 2, 3$). Note that the new diffusion coefficients depend on the length of domain.

We now reformulate model (1). By using the method of characteristics to the second equation in (1), we have

$$I(t,a,x) = \begin{cases} e^{-\int_0^a [\mu + \gamma(\sigma)]\mathrm{d}\sigma} \displaystyle\int_0^{\pi} \Gamma_2(a,x,y) I(t-a,0,y)\mathrm{d}y, & t - a > 0,\ x \in [0,\pi], \\ e^{-\int_0^t [\mu + \gamma(a-t+\sigma)]\mathrm{d}\sigma} \displaystyle\int_0^{\pi} \Gamma_2(t,x,y)\phi_2(a-t,y)\mathrm{d}y, & a - t \geq 0,\ x \in [0,\pi], \end{cases} \tag{2}$$

where

$$\Gamma_2(a,x,y) := \frac{1}{\pi} + \frac{2}{\pi}\sum_{k=1}^{+\infty} \cos(ky)\cos(kx) e^{-k^2 d_2 a}.$$

Let $u(t,x) := I(t,0,x)$. Substituting (2) into the third equation in (1), we obtain the following coupled system of a reaction-diffusion equation of S and a Volterra integral equation of u, for $t > 0$ and $x \in [0,\pi]$,

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = d_1 \dfrac{\partial^2 S(t,x)}{\partial x^2} + b - u(t,x) - \mu S(t,x), \quad \dfrac{\partial S(t,0)}{\partial x} = \dfrac{\partial S(t,\pi)}{\partial x} = 0, \\ u(t,x) = S(t,x)\displaystyle\int_0^t \beta(a) e^{-\int_0^a \{\mu + \gamma(\sigma)\}\mathrm{d}\sigma} \int_0^{\pi} \Gamma_2(a,x,y) u(t-a,y)\mathrm{d}y\mathrm{d}a + S(t,x) F_2(t,x), \\ F_2(t,x) = \displaystyle\int_t^{+\infty} \beta(a) e^{-\int_0^t \{\mu + \gamma(a-t+\sigma)\}\mathrm{d}\sigma} \int_0^{\pi} \Gamma_2(t,x,y)\phi_2(a-t,y)\mathrm{d}y\mathrm{d}a, \end{cases} \tag{3}$$

combined with initial condition $S(0,x) = \phi_1(x)$, $u(0,x) = \phi_1(x)F_2(0,x)$, $x \in [0,\pi]$. Note that we can omit the equation of R since system (3) is independent from R.

3. Existence and uniqueness of the solution

Let $X := C([0,\pi], \mathbb{R})$ with supremum norm $|\cdot|_X$, and let X^+ be its positive cone. Let $Y := L^1(\mathbb{R}_+, X)$ with norm $|\varphi|_Y := \int_0^{+\infty} |\varphi(a)|_X \, \mathrm{d}a$, $\varphi \in Y$, and let Y^+ be its positive cone. The following lemma directly follows from [15, Lemma 2.1].

Lemma 3.1. $0 < \Gamma_2(a,x,y) < \pi^{-1}\left(e^{d_2 a}+1\right)/\left(e^{d_2 a}-1\right)$ *for all* $a > 0$ *and* $x, y \in (0,\pi)$.

Using Lemma 3.1, we next prove the positivity of the solution.

Proposition 3.2. *Let* (S,u) *be a solution of* (3) *corresponding to* $(\phi_1,\phi_2) \in X^+ \times Y^+$ *with an interval of existence* $[0,T)$, $T > 0$. *Then,* $S(t,x) > 0$ *and* $u(t,x) \geq 0$ *for all* $t \in (0,T)$ *and* $x \in [0,\pi]$.

Proof. Note that if $(\phi_1,\phi_2) \in X^+ \times Y^+$, then we have $u(0,x) = S(0,x)F_2(0,x) = \phi_1(x)\int_0^{+\infty}\beta(a)\phi_2(a,x)\mathrm{d}a \geq 0$ for all $x \in [0,\pi]$. We define the linear operator $\Phi : Y \to Y$ as $\Phi(\varphi)(t,x) := \int_0^t \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x,y)\varphi(t-a,y)\mathrm{d}y\mathrm{d}a$, $\varphi \in Y$. Note that Φ is positive, that is, $\Phi(Y^+) \subset Y^+$ by virtue of (A1) and Lemma 3.1. Then, (3) can be rewritten in term of Φ, and implies that, for $t \in [0,T)$ and $x \in [0,\pi]$,

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} > d_1\dfrac{\partial^2 S(t,x)}{\partial x^2} - \left[\mu + \Phi(u)(t,x) + F_2(t,x)\right]S(t,x), \quad \dfrac{\partial S(t,0)}{\partial x} = \dfrac{\partial S(t,\pi)}{\partial x} = 0, \\ u(t,x) = S(t,x)\left[\Phi(u)(t,x) + F_2(t,x)\right]. \end{cases} \tag{4}$$

Since $\mu + \Phi(u)(t,x) + F_2(t,x)$ is continuous and bounded with respect to t and x, it follows from a standard result for PDEs that $S(t,x) > 0$ for all $t \in [0,T)$ and $x \in [0,\pi]$. Now, we focus on u. Suppose by contradiction that there exist $x_1 \in [0,\pi]$ and $t_1 \in (0,T)$ such that $u(t,x) \geq 0$ for all $t \in [0,t_1]$ and $x \in [0,\pi]$, $u(t_1,x_1) = 0$ and $u(t_1+\epsilon,x_1) < 0$ for a small $0 < \epsilon \ll 1$. Since $F_2(t,x) \geq 0$, we have, for small enough ϵ,

$$\begin{aligned} u(t_1+\epsilon,x_1) =& S(t_1+\epsilon,x_1)\int_0^{t_1+\epsilon}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x_1,y)u(t_1+\epsilon-a,y)\mathrm{d}y\mathrm{d}a \\ &+ S(t_1+\epsilon,x_1)F_2(t_1+\epsilon,x_1) \geq 0. \end{aligned}$$

This leads to a contradiction. This completes the proof. □

By using the Banach-Picard fixed point theorem, we prove the following theorem on the existence and uniqueness of solution.

Theorem 3.3. *Let* $(\phi_1,\phi_2) \in X^+ \times Y^+$. *Then, the system* (3) *has a unique positive solution defined on* $[0,+\infty) \times [0,\pi]$.

Proof. We choose $0 < T \ll 1$ to be satisfying $\tilde{h}(T) < 1$, where $\tilde{h}$ is given below by (7) (and clearly $\lim_{\alpha\mapsto 0}\tilde{h}(\alpha) = 0$). Let $Y_T := C([0,T],X)$ with norm $|v|_{Y_T} := \sup_{0\leq t\leq T}|v(t,\cdot)|_X$, $v \in Y_T$. For $(t,x) \in [0,T]\times[0,\pi]$, we have

$$S(t,x) = F_1(t,x) + \int_0^t e^{-\mu(t-a)}\int_0^\pi \Gamma_1(t-a,x,y)\left[b - u(a,y)\right]\mathrm{d}y\mathrm{d}a, \tag{5}$$

where $F_1(t,x) := e^{-\mu t}\int_0^\pi \Gamma_1(t,x,y)\phi_1(y)\mathrm{d}y$, and Γ_1 is defined similarly to Γ_2 by replacing d_2 by d_1. Thus, we can get a single equation in u given, for $(t,x) \in [0,T]\times[0,\pi]$, by

$$\begin{aligned} u(t,x) =& \left[F_1(t,x) + \int_0^t e^{-\mu(t-a)}\int_0^\pi \Gamma_1(t-a,x,y)\left[b-u(a,y)\right]\mathrm{d}y\mathrm{d}a\right] \\ &\times\left[\int_0^t \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x,y)u(t-a,y)\mathrm{d}y\mathrm{d}a + F_{\phi_2}(t,x)\right]. \end{aligned} \tag{6}$$

Define the operator $\mathcal{F} : Y_T \to Y_T$ given by the right side of the above expression of u. Thus, existence and uniqueness of a continuous solution follow as a fixed point of $\mathcal{F}$.

For two functions u_1 and u_2 in Y_T (we set $\tilde{u} := u_1 - u_2$), one can obtain

$$
\begin{aligned}
&\mathcal{F}u_1 - \mathcal{F}u_2 = F_1(t,x)\int_0^t \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x,y)\tilde{u}(t-a,y)\mathrm{d}y\mathrm{d}a \\
&+ B\int_0^t \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x,y)\tilde{u}(t-a,y)\mathrm{d}y\mathrm{d}a \\
&- F_2(t,x)\int_0^t e^{-\mu(t-a)}\int_0^\pi \Gamma_1(t-a,x,y)\tilde{u}(a,y)\,\mathrm{d}y\mathrm{d}a \\
&- \int_0^t e^{-\mu(t-a)}\int_0^\pi \Gamma_1(t-a,x,y)u_1(a,y)\,\mathrm{d}y\mathrm{d}a\int_0^t \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x,y)u_1(t-a,y)\mathrm{d}y\mathrm{d}a \\
&+ \int_0^t e^{-\mu(t-a)}\int_0^\pi \Gamma_1(t-a,x,y)u_2(a,y)\,\mathrm{d}y\mathrm{d}a\int_0^t \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x,y)u_2(t-a,y)\mathrm{d}y\mathrm{d}a,
\end{aligned}
$$

where $B := b\int_0^t e^{-\mu(t-a)}\int_0^\pi \Gamma_1(t-a,x,y)\,\mathrm{d}y\mathrm{d}a$. Then, we get (bellow L is the Lipschitz constant)

$$
\begin{aligned}
&|\mathcal{F}u_1 - \mathcal{F}u_2| \le [F_1(t,x)+B]\int_0^t \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\mathrm{d}a \sup_{0\le s\le t}|\tilde{u}(s,\cdot)|_X \\
&+ F_2(t,x)\int_0^t e^{-\mu(t-a)}\mathrm{d}a \sup_{0\le s\le t}|\tilde{u}(s,\cdot)|_X + L\int_0^t e^{-\mu(t-a)}\int_0^\pi \Gamma_1(t-a,x,y)\,\mathrm{d}y\mathrm{d}a \sup_{0\le s\le t}|\tilde{u}(s,\cdot)|_X \\
&+ L\int_0^t \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x,y)\mathrm{d}y\mathrm{d}a \sup_{0\le s\le t}|\tilde{u}(s,\cdot)|_X.
\end{aligned}
$$

We put

$$
\begin{aligned}
\tilde{h}(T) := \sup_{0\le s\le T}\bigg|&[F_1(s,x)+B]\int_0^s \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\mathrm{d}a + F_2(s,x)\int_0^s e^{-\mu(s-a)}\mathrm{d}a \\
&+L\int_0^s e^{-\mu(s-a)}\mathrm{d}a + L\int_0^s \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\mathrm{d}a\bigg|_X. \qquad (7)
\end{aligned}
$$

This leads to $|\mathcal{F}u_1 - \mathcal{F}u_2|_{Y_T} \le \tilde{h}(T)\,|u_1 - u_2|_{Y_T}$, which implies that the operator $\mathcal{F}$ is a strict contraction in Y_T. Hence, it has a unique fixed point. This implies that the system (3) has a unique local solution. To extend the domain of existence from $[0,T]\times[0,\pi]$ to $[0,+\infty)\times[0,\pi]$, it suffices to show that the solution does not blow up in finite time. In fact, by Proposition 3.2, we have that $\partial_t S(t,x) \le d_1\partial_{xx}S(t,x) + b - \mu S(t,x)$ for all $t > 0$ and $x \in [0,\pi]$, which implies that $S(t,x)$ is bounded above by the upper solution b/μ for all $t > 0$ and $x \in [0,\pi]$. We now claim that $u(t,x) < +\infty$ for all $t > 0$ and $x \in [0,\pi]$. Suppose on the contrary that there exist $t^* > 0$ and $x^* \in [0,\pi]$ such that $\lim_{t\to t^*-0} u(t,x^*) = +\infty$. We then have from the first equation in (3) that $\lim_{t\to t^*-0}\partial_t S(t,x^*) = -\infty$, which implies that $S(t,x^*)$ is negative in the neighborhood of t^*. This contradicts to the positivity of S, which was proved in Proposition 3.2. Thus, blow up never occurs, and we obtain a solution in $C([0,+\infty),X)$. The regularity can be proved as well using the above integral formulations. This completes the proof. □

4. Basic reproduction number

It is easy to see that (3) has the disease-free steady state $(S,u)=(b/\mu,0)\in X^+\times X^+$. The second equation in (3) can be linearized around the disease-free steady state as $u(t,x) = (b/\mu)\int_0^t \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x,y)u(t-a,y)\mathrm{d}y\mathrm{d}a$ for all $t>0$ and $x\in[0,\pi]$. Following the definition by Diekmann *et al.* [3], the basic reproduction number $\mathcal{R}_0$ is given by the spectral radius of the next generation operator,

$$\mathcal{K}\varphi(x) := \frac{b}{\mu}\int_0^{+\infty}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x,y)\varphi(y)\mathrm{d}y\mathrm{d}a, \quad \varphi\in X.$$

That is, $\mathcal{R}_0 := r(\mathcal{K})$, where $r(\cdot)$ denotes the spectral radius of an operator. To obtain explicit $\mathcal{R}_0$, we prove the following lemma on the next generation operator $\mathcal{K}$.

Lemma 4.1. *$\mathcal{K}$ is strictly positive and compact.*

Proof. The strict positivity is obvious by (A2) and Lemma 3.1. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a bounded sequence in X such that $|\varphi_n|_X\le M$, $n\in\mathbb{N}$ for some $M>0$. Let us define a sequence $\{\psi_n\}_{n\in\mathbb{N}}$ by $\psi_n := \mathcal{K}\varphi_n$. Then, we have for all $n\in\mathbb{N}$ and $x\in(0,\pi)$ that $\psi_n(x)\le (b/\mu)\int_0^{+\infty}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi\Gamma_2(a,x,y)\mathrm{d}y\mathrm{d}a\,|\varphi_n|_X \le (b/\mu)\int_0^{+\infty}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\mathrm{d}aM$. That is, $\{\psi_n\}_{n\in\mathbb{N}}$ is uniformly bounded. Next, we prove that $\{\psi_n\}_{n\in\mathbb{N}}$ is equi-continuous. In fact, we have, for $x,\tilde{x}\in[0,\pi]$,

$$\begin{aligned}
&|\psi_n(x)-\psi_n(\tilde{x})| = |\mathcal{K}\varphi_n(x)-\mathcal{K}\varphi_n(\tilde{x})|\,,\\
&\le \frac{b}{\mu}\int_0^{+\infty}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi |\Gamma_2(a,x,y)-\Gamma_2(a,\tilde{x},y)|\,\varphi_n(y)\mathrm{d}y\mathrm{d}a,\\
&\le \frac{b\beta^+}{\mu}\int_0^{+\infty}e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi\frac{2}{\pi}\left|\sum_{k=1}^{+\infty}\{\cos(kx)-\cos(k\tilde{x})\}\cos(ky)e^{-k^2d_2a}\right|\mathrm{d}y\mathrm{d}a\,|\varphi_n|_X\,,\\
&\le \frac{2b\beta^+}{\pi\mu}\int_0^\pi\left|\sum_{k=1}^{+\infty}\{\cos(kx)-\cos(k\tilde{x})\}\cos(ky)\int_0^{+\infty}e^{-(\mu+k^2d_2)a}\mathrm{d}a\right|\mathrm{d}yM,\\
&= \frac{2b\beta^+}{\pi\mu}\int_0^\pi\left|\sum_{k=1}^{+\infty}\frac{\{\cos(kx)-\cos(k\tilde{x})\}\cos(ky)}{\mu+k^2d_2}\right|\mathrm{d}yM, \quad \beta^+ = \operatorname*{ess.sup}_{a\ge0}\beta(a)\in(0,+\infty).
\end{aligned} \tag{8}$$

Let $\kappa(x,y) := \sum_{k=1}^{+\infty}\cos(kx)\cos(ky)/\left(\mu+k^2d_2\right)$, $(x,y)\in[0,\pi]\times[0,\pi]$. Since $\kappa(x,y)$ is continuous on $[0,\pi]\times[0,\pi]$, it is uniformly continuous. Hence, for any $\epsilon>0$, there exists a $\delta>0$ such that $|\kappa(x,y)-\kappa(\tilde{x},y)|\le \mu\epsilon/(2b\beta^+M)$ for all $|x-\tilde{x}|\le\delta$ and $y\in[0,\pi]$. For such ϵ and δ, we have from (8) that $|\psi_n(x)-\psi_n(\tilde{x})|\le (2b\beta^+/(\pi\mu))\int_0^\pi|\kappa(x,y)-\kappa(\tilde{x},y)|\,\mathrm{d}yM\le\epsilon$, $|x-\tilde{x}|\le\delta$. This implies that $\{\psi_n\}_{n\in\mathbb{N}}$ is equi-continuous. By the Ascoli-Arzela theorem, $\mathcal{K}$ is compact. □

By Lemma 4.1, we see from the Krein-Rutman theorem ([16, Theorem 3.2]) that $\mathcal{R}_0 = r(\mathcal{K})$ is the only eigenvalue of $\mathcal{K}$ having a positive eigenvector. For constant $v\in X^+\setminus\{0\}$, we have $\mathcal{K}v = (b/\mu)\int_0^{+\infty}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi\Gamma_2(a,x,y)\mathrm{d}y\mathrm{d}a\,v = (b/\mu)\int_0^{+\infty}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}da\,v$. That is, $\mathcal{R}_0 = r(\mathcal{K})$ is given by

$$\mathcal{R}_0 = r(\mathcal{K}) = \frac{b}{\mu}\int_0^{+\infty}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\mathrm{d}a. \tag{9}$$

Note that this $\mathcal{R}_0$ is similar to the one obtained in [2,13].

Now, we can easily check that $\mathcal{R}_0$ is a threshold value for the existence of a positive space-independent endemic steady state of the original system (1), which is a solution of the following equations,

$$\begin{cases} 0 = b - u^* - \mu S^*, \quad \dfrac{\mathrm{d}I^*(a)}{\mathrm{d}a} = -(\mu + \gamma(a))I^*(a), \quad a > 0, \\ u^* = I^*(0) = S^* \displaystyle\int_0^{+\infty} \beta(a) I^*(a) \mathrm{d}a, \quad 0 = \displaystyle\int_0^{+\infty} \gamma(a) I^*(a) \mathrm{d}a - \mu R^*. \end{cases} \tag{10}$$

The following theorem directly follows from the argument in [2, Section 1, p.1111].

Theorem 4.2. *Suppose that $\mathcal{R}_0 > 1$. Then, the original system* (1) *has a space-independent endemic steady state $(S^*, I^*(a), R^*)$, which is a solution of* (10).

Next, we prove the global attractivity of the disease-free steady state $(S, u) = (b/\mu, 0)$ of (3) for $\mathcal{R}_0 < 1$. To this end, we prove the existence of upper bounds for S and u in the following set, $\mathcal{C}_M := \{(\phi_1, \phi_2) \in X^+ \times Y^+ : 0 \leq \phi_1(x) \leq b/\mu$ and $0 \leq \phi_2(a, x) \leq M e^{-\int_0^a \{\mu + \gamma(\sigma)\} \mathrm{d}\sigma}$ for all $a \geq 0$ and $x \in [0, \pi]\}$, where $M > 0$ is an arbitrary large positive constant. We next prove the following proposition.

Proposition 4.3. *Suppose that $\mathcal{R}_0 < 1$ and $(\phi_1, \phi_2) \in \mathcal{C}_M$. Then, $0 < S(t, x) \leq b/\mu$ and $0 \leq u(t, x) \leq M$ for all $t > 0$ and $x \in [0, \pi]$.*

Proof. The positivity follows from Proposition 3.2. From the first equation in (3), we have that $\partial_t S(t, x) \leq d_1 \partial_{xx} S(t, x) + b - \mu S(t, x)$ for all $t > 0$ and $x \in [0, \pi]$. We then see that $\overline{S}(t, x) = b/\mu$, $t \geq 0$, $x \in [0, \pi]$ is an upper solution, and thus, $S(t, x) \leq \overline{S}(t, x) = b/\mu$ for all $t > 0$ and $x \in [0, \pi]$, provided $(\phi_1, \phi_2) \in \mathcal{C}_M$. To prove that $u(t, x) \leq M$ for all $t > 0$ and $x \in [0, \pi]$, we suppose by contradiction that there exist $t_2 > 0$ and $x_2 \in [0, \pi]$ such that $u(t, x) \leq M$ for all $t \in [0, t_2]$ and $x \in [0, \pi]$, $u(t_2, x_2) = M$ and $u(t_2 + \epsilon, x_2) > M$ for a small $0 < \epsilon \ll 1$. We then have from the second equation in (3) that, for a small enough $\epsilon > 0$,

$$\begin{aligned} u(t_2 + \epsilon, x_2) \leq & \frac{b}{\mu} \left(\int_0^{t_2 + \epsilon} \beta(a) e^{-\int_0^a \{\mu + \gamma(\sigma)\} \mathrm{d}\sigma} M \mathrm{d}a \right. \\ & \left. + \int_{t_2 + \epsilon}^{+\infty} \beta(a) e^{-\int_0^{t_2 + \epsilon} \{\mu + \gamma(a - t_2 - \epsilon + \sigma)\} \mathrm{d}\sigma} \int_0^\pi \Gamma_2(t_2 + \epsilon, x, y) \phi_2(a - t_2 - \epsilon, y) \mathrm{d}y \mathrm{d}a \right), \\ \leq & \frac{b}{\mu} \left(\int_0^{t_2 + \epsilon} \beta(a) e^{-\int_0^a \{\mu + \gamma(\sigma)\} \mathrm{d}\sigma} M \mathrm{d}a \right. \\ & \left. + \int_{t_2 + \epsilon}^{+\infty} \beta(a) e^{-\int_{a - t_2 - \epsilon}^a \{\mu + \gamma(\sigma)\} \mathrm{d}\sigma} M e^{-\int_0^{a - t_2 - \epsilon} \{\mu + \gamma(\sigma)\} \mathrm{d}\sigma} \mathrm{d}a \right), \\ \leq & \frac{b}{\mu} \int_0^{+\infty} \beta(a) e^{-\int_0^a \{\mu + \gamma(\sigma)\} \mathrm{d}\sigma} \mathrm{d}a M \ = \ \mathcal{R}_0 M \ < \ M, \end{aligned}$$

which is a contradiction. This completes the proof. □

Using Proposition 4.3, we prove the global attractivity of the disease-free steady state for $\mathcal{R}_0 < 1$.

Theorem 4.4. *Suppose that $\mathcal{R}_0 < 1$ and $(\phi_1, \phi_2) \in \mathcal{C}_M$. Then the disease-free steady state $(S, u) = (b/\mu, 0)$ of* (3) *is globally attractive.*

Proof. Let $u^\infty(x) := \limsup_{t \to +\infty} u(t, x)$ and $U^\infty := \sup_{x \in [0, \pi]} u^\infty(x) \leq M$. Suppose that $U^\infty > 0$. Then, since $\mathcal{R}_0 < 1$, there exists an $x^* \in [0, \pi]$ such that $u^\infty(x^*) > \mathcal{R}_0 U^\infty$. We then have

$$\mathcal{R}_0 U^\infty < u^\infty(x^*) \le \frac{b}{\mu} \limsup_{t\to+\infty} \left(\int_0^t \beta(a) e^{-\int_0^a \{\mu+\gamma(\sigma)\}\mathrm{d}\sigma} \int_0^\pi \Gamma_2(a,x^*,y) u(t-a,y)\mathrm{d}y\mathrm{d}a \right)$$
$$\le \frac{b}{\mu} \limsup_{t\to+\infty} \left(\int_0^t \beta(a) e^{-\int_0^a \{\mu+\gamma(\sigma)\}\mathrm{d}\sigma} \int_0^\pi \Gamma_2(a,x^*,y)\mathrm{d}y\mathrm{d}a \right) U^\infty = \mathcal{R}_0 U^\infty,$$

which is a contradiction. Hence, $U^\infty = 0$ and this implies that $\lim_{t\to+\infty} u(t,x) = 0$ for all $x \in [0,\pi]$. For $u = 0$, S obeys the differential equation $\partial_t S(t,x) = d_1 \partial_{xx} S(t,x) + b - \mu S(t,x)$ and it is easy to check that $\lim_{t\to+\infty} S(t,x) = b/\mu$ for all $x \in [0,\pi]$. This completes the proof. □

5. Persistence of the disease

In this section, we show the persistence of the disease in system (3) for $\mathcal{R}_0 > 1$. First, we consider a semiflow associated with system (3). Based on (2), for solution $u(t,x)$ of system (3), we define

$$i(t,a,x) = \begin{cases} e^{-\int_0^a \{\mu+\gamma(\sigma)\}\mathrm{d}\sigma} \int_0^\pi \Gamma_2(a,x,y) u(t-a,y)\mathrm{d}y, & t-a>0,\ x\in[0,\pi], \\ e^{-\int_{a-t}^a \{\mu+\gamma(\sigma)\}\mathrm{d}\sigma} \int_0^\pi \Gamma_2(t,x,y)\phi_2(a-t,y)\mathrm{d}y, & a-t\ge 0,\ x\in[0,\pi]. \end{cases} \tag{11}$$

We prove the following lemma (see also [17, Section 9.4]).

Lemma 5.1. *Let $(\phi_1,\phi_2) \in X^+ \times Y^+$. There exists a continuous semiflow defined by $\Theta(t,\phi_1,\phi_2) := (S(t,\cdot), i(t,\cdot,\cdot)) \in X^+ \times Y^+$ for all $t \ge 0$, associated with system* (3).

Proof. To show the semiflow property of Θ, we define $S_r(t,x) = S(r+t,x)$, $u_r(t,x) = u(r+t,x)$ and $i_r(t,a,x) = i(r+t,a,x)$ for $r\ge 0$, $t \ge 0$, $a \ge 0$ and $x \in [0,\pi]$. Then, we have, for $r \ge 0$, $t\ge 0$ and $x \in [0,\pi]$,

$$\frac{\partial S_r(t,x)}{\partial t} = d_1 \frac{\partial^2 S_r(t,x)}{\partial x^2} + b - u_r(t,x) - \mu S_r(t,x), \quad S_r(0,x) = S(r,x). \tag{12}$$

From (11) and the second equation in (3), we have, for $t \ge 0$ and $x\in[0,\pi]$,

$$u(t,x) = S(t,x) \int_0^{+\infty} \beta(a) i(t,a,x)\mathrm{d}a. \tag{13}$$

Hence, we have, for $r\ge 0$, $t \ge 0$ and $x \in [0,\pi]$,

$$u_r(t,x) = S_r(t,x) \int_0^{+\infty} \beta(a) i_r(t,a,x)\mathrm{d}a. \tag{14}$$

From (11), we have, for $x\in[0,\pi]$,

$$i_r(t,a,x) = \begin{cases} e^{-\int_0^a \{\mu+\gamma(\sigma)\}\mathrm{d}\sigma} \int_0^\pi \Gamma_2(a,x,y) u_r(t-a,y)\mathrm{d}y, & a < r+t, \\ e^{-\int_{a-r-t}^a \{\mu+\gamma(\sigma)\}\mathrm{d}\sigma} \int_0^\pi \Gamma_2(r+t,x,y)\phi_2(a-r-t,y)\mathrm{d}y, & a \ge r+t. \end{cases} \tag{15}$$

In addition, we have, for $r\ge 0$, $a > t \ge 0$, and $x \in [0,\pi]$,

$$i_r(0,a-t,x) = \begin{cases} e^{-\int_0^{a-t} \{\mu+\gamma(\sigma)\}\mathrm{d}\sigma} \int_0^\pi \Gamma_2(a-t,x,y) u_r(t-a,y)\mathrm{d}y, & a\in[t,r+t), \\ e^{-\int_{a-r-t}^{a-t} \{\mu+\gamma(\sigma)\}\mathrm{d}\sigma} \int_0^\pi \Gamma_2(r,x,y)\phi_2(a-r-t,y)\mathrm{d}y, & a > r+t. \end{cases}$$

Hence, we have, for $r \geq 0$, $a > t \geq 0$, and $x \in [0, \pi]$,

$$
\begin{aligned}
& e^{-\int_{a-t}^{a}\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma} \int_0^\pi \Gamma_2(t,x,y) i_r(0,a-t,y)\mathrm{d}y \\
& = \begin{cases} e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma} \displaystyle\int_0^\pi \Gamma_2(a,x,y) u_r(t-a,y)\mathrm{d}y, & a \in [t, r+t), \\ e^{-\int_{a-r-t}^{a}\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma} \displaystyle\int_0^\pi \Gamma_2(r+t,x,y)\phi_2(a-r-t,y)\mathrm{d}y, & a > r+t. \end{cases}
\end{aligned}
\tag{16}
$$

Comparing (15) and (16), we have, for $r \geq 0$ and $x \in [0,\pi]$,

$$
i_r(t,a,x) = \begin{cases} e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma} \displaystyle\int_0^\pi \Gamma_2(a,x,y) u_r(t-a,y)\mathrm{d}y, & t-a > 0, \\ e^{-\int_{a-t}^{a}\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma} \displaystyle\int_0^\pi \Gamma_2(t,x,y) i_r(0,a-t,y)\mathrm{d}y, & a-t \geq 0. \end{cases}
\tag{17}
$$

From (12), (14) and (17), we see that $\Theta\left(t, S(r,\cdot), i(r,\cdot,\cdot)\right) = \left(S_r(t), i_r(t,\cdot,\cdot)\right) = \Theta\left(r+t, \phi_1, \phi_2\right)$ for all $r \geq 0$ and $t \geq 0$. Hence, the semiflow property of Θ holds. The time-continuity of Θ follows from Theorem 3.3. This completes the proof. □

Let $D := \{(\phi_1, \phi_2) \in X^+ \times Y^+ :\ \phi_1(x) \int_0^{+\infty} \beta(a)\phi_2(a,x)\mathrm{d}a > 0 \text{ for some } x \in [0,\pi]\}$. We now prove the following lemma (see [18, Lemma 6.1] for a similar idea).

Lemma 5.2. *Suppose that* $\mathcal{R}_0 > 1$. *Then, there exists a positive constant* $\epsilon_1 > 0$ *such that* $\limsup_{t\to+\infty} |u(t,\cdot)|_X > \epsilon_1$, *provided* $(\phi_1, \phi_2) \in D$.

Proof. Since $\mathcal{R}_0 > 1$, we can choose $\epsilon_1 > 0$ such that

$$
\frac{b-\epsilon_1}{\mu} \int_0^{+\infty} \beta(a) e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\mathrm{d}a > 1. \tag{18}
$$

Suppose by contradiction that there exists $T_1 > 0$ such that $u(t,x) \leq \epsilon_1$ for all $t \geq T_1$ and $x \in [0,\pi]$. By (18), there exist sufficiently large $T_2 > T_1$ and small $\lambda > 0$ so that

$$
\tilde{\mathcal{R}} := \frac{b-\epsilon_1}{\mu}\left(1 - e^{-\mu h}\right) \int_0^{+\infty} \beta(a) e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma} e^{-\lambda a}\mathrm{d}a > 1. \tag{19}
$$

where $h = T_2 - T_1$. For all $t \geq T_2$ and $x \in [0,\pi]$, we have $\partial_t S(t,x) \geq d_1 \partial_{xx} S(t,x) + b - \epsilon_1 - \mu S(t,x)$. Hence, by constructing a sub solution, we have

$$
\begin{aligned}
S(t,x) \geq & e^{-\mu(t-T_1)} \int_0^\pi \Gamma_1(t-T_1,x,y) S(T_1,y)\mathrm{d}y + \frac{b-\epsilon_1}{\mu}\left(1 - e^{-\mu(t-T_1)}\right) \\
\geq & \frac{b-\epsilon_1}{\mu}\left(1 - e^{-\mu h}\right)
\end{aligned}
$$

for all $t \geq T_2$ and $x \in [0,\pi]$. By Lemma 5.1, without loss of generality, we can assume that $T_2 = 0$ (and thus, $T_1 = -h$) by taking $S(T_2,x)$ and $i(T_2,a,x)$ as a new initial condition. Hence, we have for all $t \geq 0$ and $x \in [0,\pi]$ that

$$
u(t,x) \geq \frac{b-\epsilon_1}{\mu}\left(1 - e^{-\mu h}\right) \int_0^t \beta(a) e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma} \int_0^\pi \Gamma_2(a,x,y) u(t-a,y)\mathrm{d}y\mathrm{d}a. \tag{20}
$$

It is obvious that $\int_0^{+\infty} e^{-\lambda t} u(t,x)\mathrm{d}t < +\infty$ for all $x \in [0,\pi]$. Let $\tilde{x} \in [0,\pi]$ such that $\int_0^{+\infty} e^{-\lambda t} u(t,\tilde{x})\mathrm{d}t = \min_{x\in[0,\pi]} \int_0^{+\infty} e^{-\lambda t} u(t,x)\mathrm{d}t$. By (20), we have

$$
\begin{aligned}
&\int_0^{+\infty} e^{-\lambda t} u(t,\tilde{x})\mathrm{d}t \\
&\geq \frac{b-\epsilon_1}{\mu}\left(1-e^{-\mu h}\right)\int_0^{+\infty} e^{-\lambda t}\int_0^t \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,\tilde{x},y)u(t-a,y)\mathrm{d}y\mathrm{d}a\mathrm{d}t, \\
&\geq \frac{b-\epsilon_1}{\mu}\left(1-e^{-\mu h}\right)\int_0^{+\infty} \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}e^{-\lambda a}\int_0^\pi \Gamma_2(a,\tilde{x},y)\int_a^{+\infty} e^{-\lambda(t-a)}u(t-a,y)\mathrm{d}t\mathrm{d}y\mathrm{d}a, \\
&\geq \frac{b-\epsilon_1}{\mu}\left(1-e^{-\mu h}\right)\int_0^{+\infty} \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}e^{-\lambda a}\int_0^\pi \Gamma_2(a,\tilde{x},y)\int_0^{+\infty} e^{-\lambda t}u(t,y)\mathrm{d}t\mathrm{d}y\mathrm{d}a, \\
&\geq \tilde{\mathcal{R}}\int_0^{+\infty} e^{-\lambda t}u(t,\tilde{x})\mathrm{d}t. \qquad (21)
\end{aligned}
$$

By (19), the inequality (21) implies that $\int_0^{+\infty} e^{-\lambda t}u(t,\tilde{x})\mathrm{d}t > \int_0^{+\infty} e^{-\lambda t}u(t,\tilde{x})\mathrm{d}t$, which is a contradiction. This completes the proof. □

Using Lemma 5.2, we prove the following proposition on the strong $|\cdot|_X$-persistence of the disease in system (3) (see also [19, the proof of Theorem 1]).

Proposition 5.3. *Suppose that $\mathcal{R}_0 > 1$. For any $(\phi_1,\phi_2) \in D$, there exists a positive constant $\epsilon_2 > 0$ such that $\liminf_{t\to+\infty}|u(t,\cdot)|_X > \epsilon_2$.*

Proof. Suppose by contradiction that $\liminf_{t\to+\infty}|u(t,\cdot)|_X > \epsilon_2$ does not hold for any $\epsilon_2 > 0$. Then, by Lemma 5.2, there exist increasing sequences $\{t_k\}_{k=1}^{+\infty}$, $\{\theta_k\}_{k=1}^{+\infty}$, $\{\tau_k\}_{k=1}^{+\infty}$ and a decreasing sequence $\{e_k\}_{k=1}^{+\infty}$ such that $t_k > \theta_k > \tau_k$, $\lim_{k\to+\infty} e_k = 0$ and

$$
\begin{aligned}
&|u(\tau_k,\cdot)|_X > \epsilon_1, \quad |u(\theta_k,\cdot)|_X = \epsilon_1, \quad |u(t_k,\cdot)|_X < e_k < \epsilon_1, \\
&\text{and} \quad |u(t,\cdot)|_X < \epsilon_1 \quad \text{for all} \quad t \in (\theta_k,t_k).
\end{aligned} \qquad (22)
$$

Let $\{S_k\}_{k=1}^{+\infty}$ and $\{u_k\}_{k=1}^{+\infty}$ be functional sequences in X such that $S_k := S(\theta_k,\cdot) \in X$ and $u_k := u(\theta_k,\cdot) \in X$, respectively. From (5) and (6), we can apply the Ascoli-Arzela theorem as in the proof of Lemma 4.1 to obtain that there exist $(S^*,u^*) \in X^+ \times X^+$ such that $\lim_{k\to+\infty} S_k = S^*$ and $\lim_{k\to+\infty} u_k = u^*$ (otherwise we can choose convergent subsequences). Let $(\tilde{S},\tilde{u})$ be a solution of (3) for $\phi_1(x) = S^*(x)$ and $\phi_2(a,x) = e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x,y)u^*(y)\mathrm{d}y$ for all $a \geq 0$ and $x \in [0,\pi]$. Note that the choice of ϕ_2 is based on (11). By Lemma 5.2, there exist $\tau' > 0$, $m > 0$ such that

$$
|\tilde{u}(\tau',\cdot)|_X > \epsilon_1 \quad \text{and} \quad |\tilde{u}(t,\cdot)|_X > m \quad \text{for all} \quad t \in (0,\tau'). \qquad (23)
$$

For each $k \in \mathbb{N}$, let $\tilde{u}_k(t,\cdot) := u(\theta_k + t,\cdot)$. From (23) and the semiflow property of Θ which is proved by Lemma 5.1, we have for sufficiently large k that

$$
|\tilde{u}_k(\tau',\cdot)|_X > \epsilon_1 \quad \text{and} \quad |\tilde{u}_k(t,\cdot)|_X > m > e_k \quad \text{for all} \quad t \in (0,\tau'). \qquad (24)
$$

In contrast, for $\tilde{t}_k := t_k - \theta_k$, we have from (22) that

$$
\left|\tilde{u}_k(\tilde{t}_k,\cdot)\right|_X < e_k < \epsilon_1 \quad \text{and} \quad |\tilde{u}_k(t,\cdot)|_X < \epsilon_1 \quad \text{for all} \quad t \in (0,\tilde{t}_k). \qquad (25)
$$

If $\tau' < \tilde{t}_k$, the second inequality in (25) contradicts to the first inequality in (24). If $\tau' \geq \tilde{t}_k$, (25) contradicts to the second inequality in (24). This completes the proof. □

6. Global attractivity of the endemic steady state

In this section, we show the global attractivity of the positive endemic steady state for $\mathcal{R}_0 > 1$ under the following additional assumption.

(A3) There exists a positive constant $a_\dagger \in (a_2, +\infty)$ such that $\beta(a) = 0$ for all $a > a_\dagger$.

That is, $a_\dagger$ represents the maximum age of infectiousness. Under (A3), we have that $F_2(t,x) = 0$ for all $t > a_\dagger$ and $x \in [0,\pi]$. Hence, for all $t > a_\dagger$ and $x \in [0,\pi]$, we have

$$\begin{aligned} u(t,x) =& S(t,x)\int_0^t \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x,y)u(t-a,y)\mathrm{d}y\mathrm{d}a, \\ =& S(t,x)\int_0^{a_\dagger} \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x,y)u(t-a,y)\mathrm{d}y\mathrm{d}a. \end{aligned} \tag{26}$$

In addition, from (13), we have $u(a_\dagger+\tau,x) = S(a_\dagger+\tau,x)\int_0^{a_\dagger}\beta(a)i(a_\dagger+\tau,a,x)\mathrm{d}a$ for all $\tau \in [-a_\dagger, 0]$ and $x \in [0,\pi]$, where i is defined by (11). Hence, by the semiflow property of Θ (Lemma 5.1), we can take the following values as the new initial condition,

$$\tilde{\phi}_1(x) := S(a_\dagger, x), \quad \tilde{\phi}_2(\tau,x) := u(a_\dagger+\tau, x), \quad \tau\in[-a_\dagger,0],\ x\in[0,\pi]. \tag{27}$$

From (26) and (27), regarding $t = a_\dagger$ as $t = 0$, the system (3) can be rewritten to the following time-delayed system, for $t > 0$ and $x \in [0,\pi]$,

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = d_1\dfrac{\partial^2 S(t,x)}{\partial x^2} + b - u(t,x) - \mu S(t,x), \quad \dfrac{\partial S(t,0)}{\partial x} = \dfrac{\partial S(t,\pi)}{\partial x} = 0, \\ u(t,x) = S(t,x)\displaystyle\int_0^{a_\dagger}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x,y)u(t-a,y)\mathrm{d}y\mathrm{d}a. \end{cases} \tag{28}$$

with initial condition $S(0,x) = \tilde{\phi}_1(x)$ and $u(\tau,x) = \tilde{\phi}_2(\tau,x)$, $\tau\in[-a_\dagger,0]$, $x\in[0,\pi]$. The constant endemic steady state $(S^*,u^*) \in (X^+\setminus\{0\})\times(X^+\setminus\{0\})$ of (28) satisfy

$$b = u^* + \mu S^*, \quad u^* = S^*\int_0^{a_\dagger}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\mathrm{d}a\, u^*. \tag{29}$$

By Theorem 4.2, we immediately obtain the following theorem.

Theorem 6.1. *Suppose that $\mathcal{R}_0 > 1$. Then, system* (28) *has a constant endemic steady state $(S^*,u^*) \in (X^+\setminus\{0\})\times(X^+\setminus\{0\})$.*

To prove the global attractivity of (S^*,u^*) for $\mathcal{R}_0 > 1$, we construct a Lyapunov function $V(S,u_t) := V_1(S) + V_2(u_t)$, where $V_1(S) := \int_0^\pi g\left(S(t,x)/S^*\right)\mathrm{d}x$ and

$$V_2(u_t) := \int_0^\pi\int_0^{a_\dagger}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^a\int_0^\pi \Gamma_2(\theta,x,y)u^* g\left(\frac{u(t-\theta,y)}{u^*}\right)\mathrm{d}y\mathrm{d}\theta\mathrm{d}a\mathrm{d}x,$$

where $g(r) = r - 1 - \ln r$, $r > 0$. Since $g(r) \geq 0$ for all $r > 0$ and $g(r) = 0$ if and only if $r = 1$ (see, e.g., [20]), $V(S,u_t)$ is nonnegative and equals to zero if and only if $(S,u) = (S^*,u^*)$. By Proposition 3.2, $V_1(S)$ is bounded for all $t > 0$, provided $(\phi_1,\phi_2) \in D$. To show the boundedness of $V_2(u_t)$, we prove the following lemma.

Lemma 6.2. *Suppose that $\mathcal{R}_0 > 1$ and $(\phi_1,\phi_2) \in D$. Then, there exists a $\hat{T} > 0$ such that $u(t,x) > 0$ for all $t > \hat{T}$ and $x \in [0,\pi]$.*

Proof. By Proposition 5.3, there exists a $\tilde{T} > 0$ such that $|u(t,\cdot)|_X > \epsilon_2$ for all $t \geq \tilde{T}$. By (A2), Lemma 3.1 and Proposition 3.2, we then have that $u(t,x) \geq$

$S(t,x)\int_{a_1}^{a_2}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi\Gamma_2(a,x,y)u(t-a,y)\mathrm{d}y\mathrm{d}a>0$ for all $t>\hat{T}:=\tilde{T}+a_2$ and $x\in[0,\pi]$. This completes the proof. $\square$

By Lemma 6.2, we see that if $\mathcal{R}_0>1$ and $(\phi_1,\phi_2)\in D$, then $V_2(u_t)$ is bounded for all $t>0$ (taking $t=\hat{T}$ as a new initial time), and thus, $V(S,u_t)$ is so. Using $V(S,u_t)$, we finally prove the following theorem,

Theorem 6.3. *Suppose that $\mathcal{R}_0>1$ and $(\phi_1,\phi_2)\in D$. Then, the constant endemic steady state $(S^*,u^*)\in(X^+\setminus\{0\})^2$ is globally attractive.*

Proof. In what follows, for the sake of simplicity, we write $S(t,x)$ and $u(t,x)$ as S and u, respectively. From the Neumann boundary condition and the first equation in (29), the derivative of $V_1(S)$ along the solution trajectory of (28) is calculated as

$$\begin{aligned}\dot{V}_1(S)&=\int_0^\pi\left(\frac{1}{S^*}-\frac{1}{S}\right)\frac{\partial S}{\partial t}\mathrm{d}x=\int_0^\pi\left(\frac{1}{S^*}-\frac{1}{S}\right)\left(d_1\frac{\partial^2 S}{\partial x^2}+b-u-\mu S\right)\mathrm{d}x,\\&=d_1\left[\left(\frac{1}{S^*}-\frac{1}{S}\right)\frac{\partial S}{\partial x}\right]_0^\pi-d_1\int_0^\pi\left(\frac{1}{S}\frac{\partial S}{\partial x}\right)^2\mathrm{d}x+\int_0^\pi\left(\frac{1}{S^*}-\frac{1}{S}\right)(u^*+\mu S^*-u-\mu S)\,\mathrm{d}x,\\&=-d_1\int_0^\pi\left(\frac{1}{S}\frac{\partial S}{\partial x}\right)^2\mathrm{d}x-\int_0^\pi\frac{\mu}{S^*S}(S-S^*)^2\,\mathrm{d}x+\int_0^\pi\left(\frac{u^*}{S^*}-\frac{u^*}{S}-\frac{u}{S^*}+\frac{u}{S}\right)\mathrm{d}x.\end{aligned}\tag{30}$$

On the other hand, from the second equation in (29), the derivative of $V_2(u_t)$ along the solution trajectory of (28) is calculated as

$$\begin{aligned}\dot{V}_2(u_t)&=\int_0^\pi\int_0^{a_\dagger}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^a\int_0^\pi\Gamma_2(\theta,x,y)u^*\frac{\partial}{\partial t}g\left(\frac{u(t-\theta,y)}{u^*}\right)\mathrm{d}y\mathrm{d}\theta\mathrm{d}a\mathrm{d}x,\\&=-\int_0^\pi\int_0^{a_\dagger}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^a\int_0^\pi\Gamma_2(\theta,x,y)u^*\frac{\partial}{\partial\theta}g\left(\frac{u(t-\theta,y)}{u^*}\right)\mathrm{d}y\mathrm{d}\theta\mathrm{d}a\mathrm{d}x,\\&=\int_0^\pi\int_0^{a_\dagger}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi\left[\Gamma_2(0,x,y)u^*g\left(\frac{u(t,y)}{u^*}\right)\right.\\&\quad\left.-\Gamma_2(a,x,y)u^*g\left(\frac{u(t-a,y)}{u^*}\right)+\int_0^a\frac{\partial}{\partial\theta}\Gamma_2(\theta,x,y)u^*g\left(\frac{u(t-\theta,y)}{u^*}\right)\mathrm{d}\theta\right]\mathrm{d}y\mathrm{d}a\mathrm{d}x,\\&=\int_0^\pi\int_0^{a_\dagger}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\left[u^*g\left(\frac{u}{u^*}\right)-\int_0^\pi\Gamma_2(a,x,y)u^*g\left(\frac{u(t-a,y)}{u^*}\right)\mathrm{d}y\right]\mathrm{d}a\mathrm{d}x\\&\quad+\int_0^\pi\int_0^{a_\dagger}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^a\int_0^\pi d_2\frac{\partial^2\Gamma_2(\theta,x,y)}{\partial x^2}\mathrm{d}xu^*g\left(\frac{u(t-\theta,y)}{u^*}\right)\mathrm{d}\theta\mathrm{d}y\mathrm{d}a,\\&=\int_0^\pi\int_0^{a_\dagger}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi\Gamma_2(a,x,y)u^*\left[g\left(\frac{u}{u^*}\right)-g\left(\frac{u(t-a,y)}{u^*}\right)\right]\mathrm{d}y\mathrm{d}a\mathrm{d}x\\&\quad+\int_0^\pi\int_0^{a_\dagger}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^a d_2\left[\frac{\partial\Gamma_2(\theta,x,y)}{\partial x}\right]_0^\pi u^*g\left(\frac{u(t-\theta,y)}{u^*}\right)\mathrm{d}\theta\mathrm{d}y\mathrm{d}a,\\&=\int_0^\pi\int_0^{a_\dagger}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi\Gamma_2(a,x,y)u^*\left[\frac{u}{u^*}-\frac{u(t-a,y)}{u^*}+\ln\frac{u(t-a,y)}{u}\right]\mathrm{d}y\mathrm{d}a\mathrm{d}x,\\&=\int_0^\pi\left(\frac{u}{S^*}-\frac{u}{S}\right)\mathrm{d}x+\int_0^\pi\int_0^{a_\dagger}\beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi\Gamma_2(a,x,y)u^*\ln\frac{u(t-a,y)}{u}\mathrm{d}y\mathrm{d}a\mathrm{d}x.\end{aligned}\tag{31}$$

Hence, combining (30) and (31), we calculate the derivative of $V(S,u_t)$ along the solution trajectory of (28) as

$$
\begin{aligned}
\dot V(S,u_t) &= \dot V_1(S) + \dot V_2(u_t),\\
&= -d_1\int_0^\pi \left(\frac{1}{S}\frac{\partial S}{\partial x}\right)^2 \mathrm{d}x - \int_0^\pi \frac{\mu}{S^*S}(S-S^*)^2\,\mathrm{d}x + \int_0^\pi \left(\frac{u^*}{S^*} - \frac{u^*}{S}\right)\mathrm{d}x\\
&\quad + \int_0^\pi\int_0^{a_\dagger} \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x,y)u^*\ln\frac{u(t-a,y)}{u(t,x)}\mathrm{d}y\mathrm{d}a\mathrm{d}x,\\
&= -d_1\int_0^\pi \left(\frac{1}{S}\frac{\partial S}{\partial x}\right)^2 \mathrm{d}x - \int_0^\pi \frac{\mu}{S^*S}(S-S^*)^2\,\mathrm{d}x + \int_0^\pi \left(2\frac{u^*}{S^*} - \frac{u}{S^*}\frac{u^*}{u} - \frac{u^*}{S}\right)\mathrm{d}x\\
&\quad + \int_0^\pi\int_0^{a_\dagger} \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x,y)u^*\ln\frac{u(t-a,y)}{u}\mathrm{d}y\mathrm{d}a\mathrm{d}x,\\
&= -d_1\int_0^\pi \left(\frac{1}{S}\frac{\partial S}{\partial x}\right)^2 \mathrm{d}x - \int_0^\pi \frac{\mu}{S^*S}(S-S^*)^2\,\mathrm{d}x + \int_0^\pi\int_0^{a_\dagger} \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x,y)u^*\\
&\quad \times\left[2 - \frac{Su(t-a,y)}{S^*u} - \frac{S^*}{S} + \ln\frac{u(t-a,y)}{u}\right]\mathrm{d}y\mathrm{d}a\mathrm{d}x,\\
&= -d_1\int_0^\pi \left(\frac{1}{S}\frac{\partial S}{\partial x}\right)^2 \mathrm{d}x - \int_0^\pi \frac{\mu}{S^*S}(S-S^*)^2\,\mathrm{d}x + \int_0^\pi\int_0^{a_\dagger} \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x,y)u^*\\
&\quad \times\left[1 - \frac{Su(t-a,y)}{S^*u} + \ln\frac{Su(t-a,y)}{S^*u} + 1 - \frac{S^*}{S} + \ln\frac{S^*}{S}\right]\mathrm{d}y\mathrm{d}a\mathrm{d}x,\\
&= -d_1\int_0^\pi \left(\frac{1}{S}\frac{\partial S}{\partial x}\right)^2 \mathrm{d}x - \int_0^\pi \frac{\mu}{S^*S}(S-S^*)^2\,\mathrm{d}x - \int_0^\pi\int_0^{a_\dagger} \beta(a)e^{-\int_0^a\{\mu+\gamma(\sigma)\}\mathrm{d}\sigma}\int_0^\pi \Gamma_2(a,x,y)u^*\\
&\quad \times\left[g\left(\frac{Su(t-a,y)}{S^*u}\right) + g\left(\frac{S^*}{S}\right)\right]\mathrm{d}y\mathrm{d}a\mathrm{d}x \;\le\; 0.
\end{aligned}
$$

Hence, we have $\dot V(S,u_t) \le 0$ and the equality holds if and only if $S(t,x) = S^*$ and $u(t,x) = u(t-a,y)$ for all $t > a_\dagger$, $0 < a < a_\dagger$ and $x, y \in (0,\pi)$. From the first equation in (28), when $S(t,x) = S^*$ for all $t > a_\dagger$ and $x \in (0,\pi)$, we have $u(t,x) = d_1\partial_{xx}S^* + b - \mu S^* = u^*$. Hence, the largest invariant set such that $\dot V(S,u_t) = 0$ is the constant endemic steady state $\{(S^*,u^*)\} \in (X^+ \setminus \{0\})^2$. By the invariance principle (see [21, p.168, Theorem 4.2]), we can conclude that the constant endemic steady state is globally attractive. This completes the proof. □

7. Discussion

In this paper, we proposed and analyzed an age-space-structured SIR epidemic model (1), which is a system of partial differential equations in one-dimensional spatially bounded domain with homogeneous Neumann boundary conditions. This model is a generalization of the model studied in [2] to the spatially heterogeneous system. By using the method of the characteristics, we derived a coupled system (3) of a reaction-diffusion equation and a Volterra integral equation. The problem of existence, positivity and uniqueness of the solution was treated by using the Banach-Picard fixed point theorem in an appropriate Banach space (Theorem 3.3). Moreover, we studied the asymptotic behavior of the system. The basic reproduction number $\mathcal{R}_0$ was derived based on the classical definition by Diekmann *et al.* [3] as the spectral radius of the next generation operator. First, we focused on the behavior of the trivial steady state and we gave a necessary and sufficient condition ($\mathcal{R}_0 < 1$) for global attractivity of the disease-free steady state (Theorem 4.4). This situation corresponds to the eradication of disease from the population. Related again to the basic reproduction number $\mathcal{R}_0$, we then proved the persistence of the disease in system (3) for $\mathcal{R}_0 > 1$ (Proposition 5.3) using the persistence theory (see [17,19]). Moreover, under the additional assumption (A3) that there exists a maximum age $a_\dagger$ of infectiousness, we proved the global attractivity of the positive constant endemic steady state (S^*, u^*) for $\mathcal{R}_0 > 1$. In the proof, we constructed a suitable Lyapunov function. In conclusion, it is clarified that $\mathcal{R}_0$ plays the role of the threshold for the asymptotic behavior of the solution, that is, the eradication or persistence of the disease after a long time.

In the forthcoming works, we will continue to analyze the influence of the diffusion of the susceptible and infected population on the asymptotic behaviors in the case of Dirichlet boundary conditions and more general n-dimensional spatial domains ($n \geq 2$). In such cases, we guess that the model will provide more various insights on the spatial effects to the epidemic problem.

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