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Holographic Wilson's RG

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Abstract

In an earlier paper (arXiv:1706.03371) a holographic form of the Exact Renormalization Group (ERG) evolution operator for a (perturbed) free scalar field (CFT) in D dimensions was formulated. It was shown to be equivalent, after a change of variables, to a free scalar field action in AdS_{D+1} spacetime. We attempt to extend this result to a theory where the scalar field has an anomalous dimension. Instead of the ERG evolution operator, we examine the generating functional with an infrared cutoff, and derive the prescription of alternative quantization by using the change of variables introduced in the previous paper. The anomalous dimension is thus related in the usual way to the mass of the bulk scalar field. Computation of higher point functions remains difficult in this theory, but should be tractable in the large N version.

1. Introduction

The idea of holography has been with us for some time since the publication of the first papers [1,2]. It became a mathematically precise idea with the discovery of the AdS/CFT correspondence [3–6] where an ordinary conformal field theory ($N = 4$ Super Yang Mills) in D flat dimensions is conjectured to be dual to a gravity theory (IIB Superstrings) in AdS$_{D+1}$. By now much evidence has been collected for the correctness of this conjecture. This correspondence has a natural interpretation in string theory where there is a world sheet duality that relates open and

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closed strings. Nevertheless it is worth exploring to what extent string theory is *required* for a holographic AdS description of a CFT. String theory may be required for a UV completion on the gravity side. But the duality itself may be more general if we are only interested in an effective field theory description. It is certainly known that in some limits of the parameter space gravity is sufficient for the correspondence to be correct.

Another intriguing aspect of this correspondence is the possibility of interpreting the extra radial dimension as the renormalization scale of the boundary theory. This gives rise to the idea of "holographic" RG, in which radial evolution in the bulk gravity theory is identified with RG evolution of the boundary theory [7–20]. It is natural to ask whether this identification can be made more precise, i.e., whether it is possible to derive the holographic RG equation from the RG equation of the boundary field theory. In an earlier paper [21] this was answered in the affirmative for the simple case of a free massless scalar field theory. It was shown first that the evolution operator for Wilson's Exact Renormalization Group (ERG) [22–24], in the simpler Polchinski form [25], could be written as a functional integral of a $D+1$ dimensional field theory. (See [26–29] for reviews on ERG.) A change of field variables then transformed this operator into the action for a free scalar field theory in AdS_{D+1}. A contact was thus made with the standard AdS/CFT methods for the calculation of two-point correlators. One important point is that the bulk field took the value of the boundary field at the boundary (rather than the source) so this is more naturally understood as the *alternative quantization* procedure introduced in [16].

In the present paper we change the course of approach a little by considering the generating functional of correlation functions with an infrared cutoff [30–32] instead of the ERG evolution operator. The generating functional is closely related to a Wilson action, and it reduces to the ordinary generating functional in the limit of the vanishing infrared cutoff. We follow section 5.2 of [21] by introducing an elementary scalar field of scale dimension between $(D-2)/2$ and $D/2$ to represent a composite field. We then construct a *quadratic* Wilson action that gives the expected two-point function with an anomalous dimension. Normally anomalous dimensions arise due to interactions. However it is very hard to write down fixed point Wilson actions with interactions and anomalous dimension. Therefore to clarify the role of the anomalous dimension in the map from an ERG equation to an AdS evolution equation we consider a simpler model of a Gaussian theory with anomalous dimension. It solves the standard fixed point ERG equation with anomalous dimension. The mapping techniques used for this simple model should be applicable in the more realistic case of an interacting fixed point also.

Having constructed a fixed point Wilson action, we construct a corresponding generating functional $W_\Lambda[J]$ with an infrared cutoff Λ following a recipe well known in the ERG literature. In the infrared limit $\Lambda \to 0+$, $W_\Lambda[J]$ becomes the generating functional of the connected correlation functions. Since we are ignoring interactions, we obtain

$$\lim_{\Lambda\to 0+} W_\Lambda[J] = \frac{1}{2}\int_{p,q} J(-p)\,\langle\phi(p)\phi(q)\rangle\, J(-q)\,,$$

where the two-point function

$$\langle\phi(p)\phi(q)\rangle = \delta(p+q)\cdot\frac{1}{p^{2-\eta}}$$

has the anomalous dimension η of the scalar field.

Let us sketch briefly how the AdS space arises from ERG without going much into technical details. According to ERG, the cutoff dependence of the generating functional is given by a diffusion equation (in the main text introduced as (15) or more precisely, including an anomalous dimension parameter η, as (23)). The equation is solved by the Gaussian integral formula

$$e^{W_{\Lambda_2}[J]} = \int [dJ'] \exp\left[W_{\Lambda_1}[J+J'] - \frac{1}{2}\int_p \frac{J'(p)J'(-p)}{R_{\Lambda_1}(p) - R_{\Lambda_2}(p)} \right],$$

where $\Lambda_2 < \Lambda_1$, and $R_\Lambda(p)$ is an IR cutoff function. $W_\Lambda[J]$ is quadratic in J, and we can write the above in the form

$$e^{W_{\Lambda_2}[J]} = \int [d\varphi] \exp\left[-\frac{1}{2}\int_p \frac{\varphi(p)\varphi(-p)}{G_{1/\Lambda_2}(p) - G_{1/\Lambda_1}(p)} + \int_p \varphi(p) J(-p) + \cdots \right],$$

where φ is a rescaled J', and we have suppressed a term quadratic in J. $G_{1/\Lambda}(p)$ is the two-point function with an IR cutoff Λ. We can write the quadratic term as a functional integral over the field $y(z,p)$ that interpolates $J'(p)$ at $z = \frac{1}{\Lambda_1}$ and 0 at $z = \frac{1}{\Lambda_2}$:

$$\exp\left[-\frac{1}{2}\int_p \frac{\varphi(p)\varphi(-p)}{G_{1/\Lambda_2}(p) - G_{1/\Lambda_1}(p)} \right] = \int \mathcal{D}y \, \exp\left[-\frac{1}{2}\int_{\frac{1}{\Lambda_1}}^{\frac{1}{\Lambda_2}} dz \int_p \frac{\partial_z y(z,p) \partial_z y(z,-p)}{\partial_z G_z(p)} \right].$$

z gives the radial coordinate of AdS_{D+1}: $z = \frac{1}{\Lambda_1} = \epsilon$ is the radius of the boundary, and $z = \frac{1}{\Lambda_2} = z_0$ is to be taken to infinity. Using the same change of field variables

$$y(z,p) \longrightarrow Y(z,p)$$

that we introduced in 2.3 of [21] (given precisely by (51)), we can rewrite the generating functional in the AdS form:

$$\int \mathcal{D}Y \, \exp\left(S_{\mathrm{AdS}}[Y] \right),$$

where $S_{\mathrm{AdS}}[Y]$, given by (56), is the action of a massive free field in the space AdS_{D+1} with the metric

$$\frac{dz^2 + d\vec{x}\cdot d\vec{x}}{z^2}.$$

We still need to integrate over the field $\varphi(p) \sim Y(\epsilon, p)$ at the boundary; we thus reproduce the prescription of the alternative quantization [16] (reviewed nicely in Appendix of [15]) for computing the two-point function.

Unlike the usual Feynman diagram approach where one integrates over all momenta in a loop, in the Wilsonian approach only modes above a scale Λ are integrated out. In the holographic description, where the radial coordinate is a measure of the scale, this would correspond to integrating out fields beyond a certain value of the radius. This is precisely the notion emphasized by holographic RG (see for e.g. [15]) where the picture is of the boundary moving inward as one proceeds toward the IR. This is the underlying reason why ERG techniques are able to reproduce holographic results.

This paper is organized as follows. In Sec. 2 we overview the ERG formalism to introduce a quadratic Wilson action that gives a two-point function with an anomalous dimension. We then introduce a corresponding generating functional with an infrared cutoff in Sec. 3 to apply the change of variables of [21]. By a judicious adjustment of the change of variables, we can derive the prescription of alternative quantization. We discuss our result and method in Sec. 4. In Sec. 5 we give some background regarding anomalous dimension in ERG and its connection with the change of variables used in [21]. Sec. 6 contains a preliminary discussion of a situation where one might obtain a nontrivial (i.e. cubic and higher order) bulk action starting from a generalized ERG equation. We conclude the paper in Sec. 7.

2. Background

2.1. ERG formalism

For the convenience of the reader, we would like to collect relevant background material from the exact renormalization group formalism (ERG).

Let $S_\Lambda[\phi]$ be a Wilson action of a generic scalar field theory. To preserve physics independent of Λ, we impose the ERG differential equation

$$-\Lambda\frac{\partial}{\partial\Lambda}e^{S_\Lambda[\phi]} = \int_p \left[\frac{\Delta_\Lambda(p)}{K_\Lambda(p)}\phi(p)\frac{\delta}{\delta\phi(p)} + \frac{\Delta_\Lambda(p)}{p^2}\frac{1}{2}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right]e^{S_\Lambda[\phi]}\,, \tag{1}$$

where

$$\Delta_\Lambda(p) \equiv \Lambda\frac{\partial}{\partial\Lambda}K_\Lambda(p)\,. \tag{2}$$

The cutoff function $K_\Lambda(p)$ has three properties:

1. $K_\Lambda(0) = 1$,
2. it is of order 1 for $p^2 < \Lambda^2$,
3. it approaches 0 rapidly for $p^2 \gg \Lambda^2$.

For example, we can take $K_\Lambda(p) = K(p/\Lambda) = \exp\left(-p^2/\Lambda^2\right)$ as shown in Fig. 1. ($k_\Lambda(p) = K(p/\Lambda)\,(1 - K(p/\Lambda))$ and $R_\Lambda(p) = \Lambda^2 R(p/\Lambda) = p^2 K(p/\Lambda)/\,(1 - K(p/\Lambda))$ are respectively defined by Eqs. (5) and (8) below.)

We denote the correlation functions by

$$\langle\phi(p_1)\cdots\phi(p_n)\rangle_{S_\Lambda} \equiv \int[d\phi]\phi(p_1)\cdots\phi(p_n)\,e^{S_\Lambda[\phi]}\,. \tag{3}$$

The ERG differential equation (1) implies that the correlation functions defined below are independent of Λ:

$$\begin{aligned}&\langle\!\langle\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle\\ &\equiv \prod_{i=1}^n\frac{1}{K_\Lambda(p_i)}\cdot\left\langle\exp\left(-\frac{1}{2}\int_p\frac{k_\Lambda(p)}{p^2}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right)\phi(p_1)\cdots\phi(p_n)\right\rangle_{S_\Lambda}\,,\end{aligned} \tag{4}$$

where

$$k_\Lambda(p) \equiv K_\Lambda(p)\,(1 - K_\Lambda(p))\,. \tag{5}$$

The modification of the two-point function by $k_\Lambda(p)/p^2$ does not affect the physics in the infrared as long as $k_\Lambda(p)$ vanishes as p^2 at $p^2 = 0$. (In other words, $k_\Lambda(p)/p^2$ does not correspond

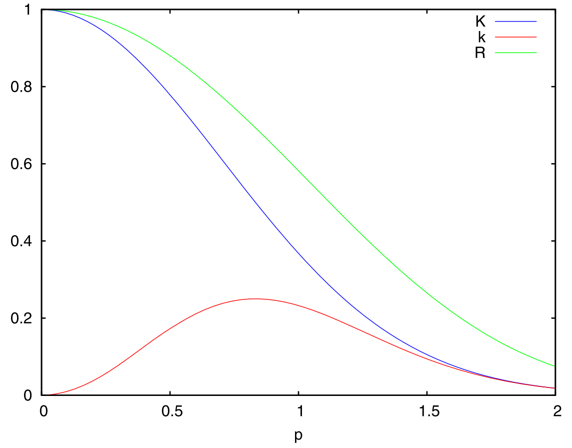

Fig. 1. We plot $K(p) = e^{-p^2}$, $k(p) = e^{-p^2}(1 - e^{-p^2})$, and $R(p) = p^2 \frac{1}{e^{p^2}-1}$.

to the propagation of a free massless particle.) We explain a little more on the correlation functions in double brackets in Appendix C.

We then define the generating functional $\mathcal{W}[\mathcal{J}]$ of the connected correlation functions by

$$e^{\mathcal{W}[\mathcal{J}]} \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \int_{p_1,\cdots,p_n} \mathcal{J}(-p_1)\cdots\mathcal{J}(-p_n) \langle\!\langle \phi(p_1)\cdots\phi(p_n) \rangle\!\rangle \,. \tag{6}$$

This can be written as a functional integral in the presence of a source term:

$$\begin{aligned} e^{\mathcal{W}[\mathcal{J}]} &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{p_1,\cdots,p_n} \prod_{i=1}^{n} \frac{\mathcal{J}(-p_i)}{K_\Lambda(p_i)} \\ &\quad \cdot \left\langle \exp\left(-\int_p \frac{k_\Lambda(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right) \phi(p_1)\cdots\phi(p_n) \right\rangle_{S_\Lambda} \\ &= \left\langle \exp\left(-\int_p \frac{k_\Lambda(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right) \exp\left(\int_p \frac{\mathcal{J}(-p)}{K_\Lambda(p)} \phi(p) \right) \right\rangle_{S_\Lambda} \\ &= \left\langle \exp\left(\int_p \left(\frac{\mathcal{J}(-p)}{K_\Lambda(p)} \phi(p) - \frac{1}{2} \frac{1}{R_\Lambda(p)} \mathcal{J}(p)\mathcal{J}(-p) \right) \right) \right\rangle_{S_\Lambda} , \end{aligned} \tag{7}$$

where

$$R_\Lambda(p) \equiv \frac{p^2 K_\Lambda(p)^2}{k_\Lambda(p)} = \frac{p^2 K_\Lambda(p)}{1 - K_\Lambda(p)} \,. \tag{8}$$

We now define a field

$$J(p) \equiv \phi(p) \frac{R_\Lambda(p)}{K_\Lambda(p)} = \phi(p) \frac{p^2}{1 - K_\Lambda(p)} \tag{9}$$

and following [30,31] introduce

$$W_\Lambda[J] \equiv S_\Lambda[\phi] + \frac{1}{2}\int_p \frac{J(p)J(-p)}{R_\Lambda(p)}\,. \tag{10}$$

We then obtain

$$\begin{aligned} e^{\mathcal{W}[\mathcal{J}]} &= \int [dJ] \exp\left(W_\Lambda[J] - \frac{1}{2}\int_p \frac{1}{R_\Lambda(p)} \left(J(p) - \mathcal{J}(p)\right)\left(J(-p) - \mathcal{J}(-p)\right)\right) \\ &= \int [dJ] \exp\left(W_\Lambda[J+\mathcal{J}] - \frac{1}{2}\int_p \frac{1}{R_\Lambda(p)} J(p)J(-p)\right)\,. \end{aligned} \tag{11}$$

Since

$$\lim_{\Lambda\to 0+} K_\Lambda(p) = 0\,, \tag{12}$$

we obtain

$$\lim_{\Lambda\to 0+} R_\Lambda(p) = 0\,. \tag{13}$$

Hence, we obtain [30,31,33]

$$\mathcal{W}[\mathcal{J}] = \lim_{\Lambda\to 0+} W_\Lambda[\mathcal{J}]\,. \tag{14}$$

We can think of $W_\Lambda[J]$ as the generating functional with an IR cutoff Λ. Its Λ-dependence can be obtained from the ERG differential equation (1) as

$$-\Lambda\frac{\partial}{\partial\Lambda} e^{W_\Lambda[J]} = \int_p \Lambda \frac{\partial R_\Lambda(p)}{\partial\Lambda}\frac{1}{2}\frac{\delta^2}{\delta J(p)\delta J(-p)} e^{W_\Lambda[J]}\,, \tag{15}$$

which can be solved as

$$e^{W_{\Lambda_2}[J]} = \int [dJ'] \exp\left[W_{\Lambda_1}[J'+J] - \frac{1}{2}\int_p \frac{1}{R_{\Lambda_1}(p) - R_{\Lambda_2}(p)} J'(p)J'(-p)\right]\,. \tag{16}$$

Since $R_{\Lambda_1}(p) - R_{\Lambda_2}(p)$ is non-vanishing mainly for $\Lambda_2 < p < \Lambda_1$, the above equation implies that we obtain $W_{\Lambda_2}[J]$ from $W_{\Lambda_1}[J]$ by integrating fluctuations of momenta between Λ_2 and Λ_1. In the limit $\Lambda \to 0+$, all the momentum modes are integrated to give (14).

2.2. *Quadratic Wilson action with an anomalous dimension*

We now consider a simple quadratic Wilson action with an anomalous dimension:

$$S_\Lambda[\phi] = -\frac{1}{2}\int_p \frac{p^2}{K_\Lambda(p)} \frac{1}{1 + K_\Lambda(p)\left(\left(\frac{p}{\mu}\right)^\eta - 1\right)} \phi(p)\phi(-p)\,, \tag{17}$$

where $\eta = 2\gamma$ $(0 < \gamma < 1)$ is a positive anomalous dimension, and μ is an arbitrary reference momentum scale. This action reproduces a two-point function with an anomalous dimension:

$$\langle\!\langle \phi(p)\phi(q) \rangle\!\rangle = \frac{1}{K_\Lambda(p)^2}\left(\langle \phi(p)\phi(q)\rangle_{S_\Lambda} - \frac{k_\Lambda(p)}{p^2}\delta(p+q) \right)$$
$$= \frac{1}{p^2\left(\frac{p}{\mu}\right)^{-\eta}}\,\delta(p+q)\,. \tag{18}$$

The corresponding generating functional with an IR cutoff Λ is given by

$$W_\Lambda[J] \equiv S_\Lambda[\phi] + \frac{1}{2}\int_p \frac{J(p)J(-p)}{R_\Lambda(p)} = \frac{1}{2}\int_p J(p)J(-p)\frac{1}{p^2\left(\frac{p}{\mu}\right)^{-\eta} + R_\Lambda(p)}\,, \tag{19}$$

where $1/\left(p^2\,(p/\mu)^{-\eta} + R_\Lambda(p)\right)$ is the high-momentum propagator, or the two-point function with an infrared cutoff. Using (13), we get

$$\mathcal{W}[J] = \lim_{\Lambda\to 0} W_\Lambda[J] = \frac{1}{2}\int_p J(p)\frac{1}{p^2\left(\frac{p}{\mu}\right)^{-\eta}}J(-p)\,. \tag{20}$$

Please observe that for small $p \ll \Lambda$ the action is approximately given by

$$S_\Lambda[\phi] \simeq -\frac{1}{2}\int_p \frac{p^2}{\left(\frac{p}{\mu}\right)^{\eta}}\phi(p)\phi(-p)\,, \tag{21}$$

which is a non-analytic (non-local) action. We have two comments:

1. For an elementary field ϕ, we expect the action is analytic at zero momentum, and any non-analyticity comes from interactions. As ϕ, we have composite fields in mind.
2. For example, in the massless free theory, the composite field $\frac{1}{2}\phi^2$ has scale dimension $D-2$ so that the anomalous dimension, compared with the free elementary field, is

   $$\gamma = \frac{D-2}{2} \Longleftrightarrow D-2 = \frac{D-2}{2} + \gamma\,.$$

   $0 < \gamma < 1$ implies $2 < D < 4$. This is an example of our ϕ.

We will discuss the first point further in Sec. 4.

2.3. ERG formalism with explicit dependence on the anomalous dimension

One drawback of our choice W_Λ (19) is that the ERG equation (15) it satisfies shows no sign of the anomalous dimension η contained in W_Λ. In the usual AdS/CFT calculations the anomalous dimension η is introduced as a mass term in the AdS equations. Similarly, we would like to introduce η explicitly in the ERG equation. (More will be discussed later in Sec. 5.)

Given $W_\Lambda[J]$ satisfying (15), let us define

$$\tilde{W}_\Lambda[J] \equiv W_\Lambda\left[\left(\frac{\Lambda}{\mu}\right)^{-\frac{\eta}{2}} J\right]. \tag{22}$$

Then, (15) implies that $\tilde{W}_\Lambda[J]$ satisfies the alternate ERG equation with an explicit dependence on η:

$$-\Lambda \frac{\partial}{\partial \Lambda} e^{\tilde{W}_\Lambda[J]}$$
$$= \int_p \left[\frac{\eta}{2} J(p) \frac{\delta}{\delta J(p)} + \left(\Lambda \frac{\partial \tilde{R}_\Lambda(p)}{\partial \Lambda} - \eta \tilde{R}_\Lambda(p) \right) \frac{1}{2} \frac{\delta^2}{\delta J(p) \delta J(-p)} \right] e^{\tilde{W}_\Lambda[J]}, \tag{23}$$

where

$$\tilde{R}_\Lambda(p) \equiv \left(\frac{\Lambda}{\mu} \right)^\eta R_\Lambda(p) . \tag{24}$$

To derive (23) from (15), we have used

$$\Lambda \frac{\partial R_\Lambda(p)}{\partial \Lambda} = \left(\frac{\Lambda}{\mu} \right)^{-\eta} \left(\Lambda \frac{\partial \tilde{R}_\Lambda(p)}{\partial \Lambda} - \eta \tilde{R}_\Lambda(p) \right) . \tag{25}$$

The integral formula (16) implies the corresponding integral formula:

$$e^{\tilde{W}_\Lambda\left[\left(\frac{\Lambda}{\mu}\right)^{\frac{\eta}{2}} J\right]}$$
$$= \int [dJ'] \exp \left[W_{\Lambda'} \left[\left(\frac{\Lambda'}{\mu} \right)^{\frac{\eta}{2}} (J' + J) \right] - \frac{1}{2} \int_p \frac{J'(p) J'(-p)}{(\mu/\Lambda')^\eta \tilde{R}_{\Lambda'}(p) - (\mu/\Lambda)^\eta \tilde{R}_\Lambda(p)} \right] . \tag{26}$$

(13) can now be written as

$$\lim_{\Lambda \to 0+} \left(\frac{\Lambda}{\mu} \right)^{-\eta} \tilde{R}_\Lambda(p) = 0 , \tag{27}$$

and we obtain, from (14),

$$\mathcal{W}[\mathcal{J}] = \lim_{\Lambda \to 0+} \tilde{W}_\Lambda \left[\left(\frac{\Lambda}{\mu} \right)^{\frac{\eta}{2}} \mathcal{J} \right] . \tag{28}$$

So far, we have not assumed that $W_\Lambda[J]$ is quadratic and given by (19). Let us now assume it so that

$$\tilde{W}_\Lambda[J] = \frac{1}{2} \int_p \frac{J(p) J(-p)}{p^2 \left(\frac{\Lambda}{p} \right)^\eta + \tilde{R}_\Lambda(p)} . \tag{29}$$

This has no more dependence on the reference momentum μ. For the quadratic $\tilde{W}_\Lambda[J]$, (23) reduces to

$$-\Lambda \frac{\partial \tilde{W}_\Lambda[J]}{\partial \Lambda} = \int_p \left[\frac{\eta}{2} J(p) \frac{\delta \tilde{W}_\Lambda[J]}{\delta J(p)} + \left(\Lambda \frac{\partial \tilde{R}_\Lambda(p)}{\partial \Lambda} - \eta \tilde{R}_\Lambda(p) \right) \frac{1}{2} \frac{\delta \tilde{W}_\Lambda[J]}{\delta J(p)} \frac{\delta \tilde{W}_\Lambda[J]}{\delta J(-p)} \right] . \tag{30}$$

(28) gives

$$\mathcal{W}[J] = \lim_{\Lambda\to 0+} \tilde{W}_\Lambda\left[\left(\frac{\Lambda}{\mu}\right)^{\frac{\eta}{2}} J\right] = \frac{1}{2}\int_p J(p)\,\frac{1}{p^2\left(\frac{p}{\mu}\right)^{-\eta}}\,J(-p)\,. \tag{31}$$

As the original ERG equation (15) does not single out a particular η in (19), the ERG equation (23) with an explicit dependence on η has a more general quadratic solution given by

$$\tilde{W}_\Lambda[J] = \frac{1}{2}\int_p \frac{J(p)J(-p)}{p^2\left(\frac{\Lambda}{p}\right)^\eta\left(\frac{\mu}{p}\right)^{\Delta\eta} + \tilde{R}_\Lambda(p)}\,, \tag{32}$$

where $\Delta\eta$ is an arbitrary shift of the anomalous dimension. Given (23), what makes (29) stand out? It is the relation to an RG fixed point. Let us elaborate on this a little. We assume that the cutoff function $\tilde{R}_\Lambda(p)$ has a particular cutoff dependence:

$$\tilde{R}_\Lambda(p) = p^2 r(p/\Lambda)\,. \tag{33}$$

We define a dimensionless field

$$\bar{J}(\bar{p}) \equiv \Lambda^{\frac{D-2}{2}} J(\bar{p}\Lambda) \tag{34}$$

with the dimensionless momentum $\bar{p}$. We then define

$$\bar{W}[\bar{J}] \equiv \tilde{W}_\Lambda[J] = \frac{1}{2}\int_{\bar{p}} \frac{\bar{J}(\bar{p})\bar{J}(-\bar{p})}{\bar{p}^{2-\eta} + R(\bar{p})}\,, \tag{35}$$

where

$$R(\bar{p}) \equiv \bar{p}^2 r(\bar{p})\,. \tag{36}$$

We find that $\bar{W}[\bar{J}]$ is a fixed point action, satisfying the equation for scale invariance

$$\int_{\bar{p}}\left[\bar{J}(-\bar{p})\left(-\bar{p}\cdot\partial_{\bar{p}} - \frac{D+2}{2} + \frac{\eta}{2}\right)\frac{\delta}{\delta\bar{J}(-\bar{p})} e^{\bar{W}[\bar{J}]} + \frac{1}{2}\left(-\bar{p}\cdot\partial_p + 2 - \eta\right)R(\bar{p})\cdot\frac{\delta^2}{\delta\bar{J}(\bar{p})\delta\bar{J}(-\bar{p})} e^{\bar{W}[\bar{J}]}\right] = 0\,. \tag{37}$$

At the same time it also satisfies the equation for special conformal invariance [34]

$$\int_p \bar{J}(-p)\left(-\bar{p}_\nu\frac{\partial^2}{\partial\bar{p}_\mu\partial\bar{p}_\nu} + \frac{1}{2}\bar{p}_\mu\frac{\partial^2}{\partial\bar{p}_\nu\partial\bar{p}_\nu} + \left(-\frac{D+2}{2} + \frac{\eta}{2}\right)\frac{\partial}{\partial\bar{p}_\mu}\right)\frac{\delta}{\delta\bar{J}(-\bar{p})} e^{\bar{W}[\bar{J}]}$$
$$+ \frac{1}{2}\int_{\bar{p}}\left(-\bar{p}\cdot\partial_p + 2 - \eta\right)R(\bar{p})\cdot\frac{\partial}{\partial\bar{p}_\mu}\left(\frac{\delta^2}{\delta\bar{J}(-\bar{p})\delta\bar{J}(\bar{q})} e^{\bar{W}[\bar{J}]}\right)\Big|_{\bar{q}=\bar{p}} = 0\,. \tag{38}$$

In Appendix D we show how to derive (35) either from (37) or from (38).

3. Derivation of the alternative quantization

To simplify our notation, we omit the tilde from $\tilde{W}$ altogether, and consider

$$W_\Lambda[J] = \frac{1}{2}\int_p \frac{J(p)J(-p)}{p^2\left(\frac{\Lambda}{p}\right)^\eta + R_\Lambda(p)}\,. \tag{39}$$

Let $\Lambda_1 = \frac{1}{\epsilon}$ be a large cutoff, and $\Lambda_2 = \frac{1}{z_0}$ be a small cutoff, compared with a reference momentum scale μ. Our goal is to derive the prescription of the alternative quantization of AdS/CFT from the integral formula (26):

$$\exp\left[W_{1/z_0}\left[(\mu z_0)^{-\frac{\eta}{2}} J\right]\right] = \int [dJ'] \exp\left[W_{1/\epsilon}\left[(\mu\epsilon)^{-\frac{\eta}{2}}(J'+J)\right] - \frac{1}{2}\int_p \frac{1}{(\mu\epsilon)^\eta R_{1/\epsilon}(p) - (\mu z_0)^\eta R_{1/z_0}(p)} J'(p)J'(-p)\right]. \tag{40}$$

Since (39) is quadratic, we can expand and rewrite the above as

$$\begin{aligned}&\exp\left[W_{1/z_0}\left[(\mu z_0)^{-\frac{\eta}{2}} J\right]\right]\\ &= \int [dJ'] \exp\Bigg[-\frac{1}{2}\int_p J'(p)J'(-p)\\ &\quad \cdot \left\{\frac{1}{(\mu\epsilon)^\eta R_{1/\epsilon}(p) - (\mu z_0)^\eta R_{1/z_0}(p)} - \frac{1}{p^2\left(\frac{p}{\mu}\right)^{-\eta} + (\mu\epsilon)^\eta R_{1/\epsilon}(p)}\right\}\\ &\quad + \int_p J(p)\frac{J'(-p)}{p^2\left(\frac{p}{\mu}\right)^{-\eta} + (\mu\epsilon)^\eta R_{1/\epsilon}(p)} + \frac{1}{2}\int_p \frac{J(p)J(-p)}{p^2\left(\frac{p}{\mu}\right)^{-\eta} + (\mu\epsilon)^\eta R_{1/\epsilon}(p)}\Bigg].\end{aligned} \tag{41}$$

Rewriting the integration variable as

$$\varphi(p) \equiv \frac{J'(p)}{p^2(p/\mu)^{-\eta} + (\mu\epsilon)^\eta R_{1/\epsilon}(p)}$$

to normalize the $J\varphi$ term, we obtain

$$\exp\left[W_{1/z_0}\left[(\mu z_0)^{-\frac{\eta}{2}} J\right]\right] = \int [d\varphi] \exp\left[-\frac{1}{2}\int_p \frac{\varphi(p)\varphi(-p)}{G_{z_0}(p) - G_\epsilon(p)} + \int_p J(p)\varphi(-p) + \frac{1}{2}\int_p J(p)J(-p)G_\epsilon(p)\right], \tag{42}$$

where we have defined

$$G_z(p) \equiv \frac{1}{p^2\left(\frac{p}{\mu}\right)^{-\eta} + (\mu z)^\eta R_{1/z}(p)}. \tag{43}$$

Note that the parameter

$$z \equiv \frac{1}{\Lambda} \tag{44}$$

is the inverse cutoff, and it will be interpreted as a coordinate of the AdS_{D+1} space. z takes a value between $\frac{1}{\Lambda_1} = \epsilon$ (small) and $\frac{1}{\Lambda_2} = z_0$ (large). With $G_z(p)$, we can write

$$W_{1/z}\left[(\mu z)^{-\frac{\eta}{2}} J\right] = \frac{1}{2}\int_p G_z(p) J(p) J(-p)\,. \tag{45}$$

Using (27)

$$\lim_{z\to+\infty} (\mu z)^{\eta} R_{1/z}(p) = 0\,, \tag{46}$$

we obtain the infrared limit

$$\lim_{z_0\to+\infty} W_{1/z_0}\left[(\mu z_0)^{-\frac{\eta}{2}} J\right] = \frac{1}{2}\int_p \frac{J(p)J(-p)}{p^2\left(\frac{p}{\mu}\right)^{-\eta}}\,. \tag{47}$$

This is obviously independent of ϵ. (This is in fact thanks to the scale invariance (37).)

We wish to rewrite (42) in the AdS form by using the change of variables introduced in [21]. We first rewrite

$$Z_{\epsilon,z_0}[\varphi] \equiv \exp\left[-\frac{1}{2}\int_p \frac{\varphi(p)\varphi(-p)}{G_{z_0}(p) - G_\epsilon(p)}\right] \tag{48}$$

by introducing a field $y(z, p)$ in $D+1$ dimensions as

$$Z_{\epsilon,z_0}[\varphi] = \int \mathcal{D}y\Big|_{\substack{y(\epsilon,p)=\varphi(p)\\ y(z_0,p)=0}} \exp\left[-\frac{1}{2}\int_\epsilon^{z_0} \frac{dz}{z^{D+1}}\int_p \frac{z\partial_z y(z,p)\cdot z\partial_z y(z,-p)}{z^{1-D}\frac{\partial}{\partial z}G_z(p)}\right], \tag{49}$$

where the boundary values of the field $y(z, p)$ are fixed by

$$y(\epsilon, p) = \varphi(p)\,, \tag{50a}$$

$$y(z_0, p) = 0\,. \tag{50b}$$

Then, following 2.3 of [21], we change field variables from $y(z, p)$ to $Y(z, p)$ defined by

$$y(z, p) = f(z, p) Y(z, p)\,, \tag{51}$$

where the positive function $f(z, p)$ is defined by

$$f(z, p)^2 \equiv z^{1-D}\partial_z G_z(p)\,. \tag{52}$$

(Note that the mass dimensions of $y(z, p)$, $f(z, p)$, $Y(z, p)$ are $-\frac{D+2}{2}$, $\frac{D-2}{2}$, $-D$, respectively.) Hence, we obtain

$$\begin{aligned}
&-\frac{1}{2}\int_\epsilon^{z_0}\int_p \frac{z\partial_z y(z,p)\cdot z\partial_z y(z,-p)}{f(z,p)^2}\\
&= -\frac{1}{2}\int_\epsilon^{z_0}\frac{dz}{z^{D+1}}\int_p \Big\{z\partial_z Y(z,p)\cdot z\partial_z Y(z,-p)\\
&\qquad + \left((z\partial_z \ln f(z,p))^2 - z^{D+1}\partial_z\left(z^{1-D}\partial_z \ln f(z,p)\right)\right) Y(z,p)Y(z,-p)\Big\}\\
&\quad + \frac{1}{2}\int_p \epsilon^{1-D}\partial_\epsilon \ln f(\epsilon,p)\cdot Y(\epsilon,p)Y(\epsilon,-p)\,,
\end{aligned} \tag{53}$$

where we have used $Y(z_0, p) = 0$.

We now choose $f(z,p)$ to satisfy

$$(z\partial_z \ln f(z,p))^2 - z^{D+1}\partial_z\left(z^{1-D}\partial_z \ln f(z,p)\right) = p^2 z^2 + \frac{m^2}{\mu^2}\,, \tag{54}$$

where m^2 is a squared mass parameter. This gives

$$\begin{aligned} &-\frac{1}{2}\int_\epsilon^{z_0}\int_p \frac{z\partial_z y(z,p)\cdot z\partial_z y(z,-p)}{f(z,p)^2} \\ &\quad = S_{\mathrm{AdS};\epsilon,z_0}[Y] + \frac{1}{2}\int_p \epsilon^{1-D}\partial_\epsilon \ln f(\epsilon,p)\,\cdot Y(\epsilon,p)Y(\epsilon,-p)\,, \end{aligned} \tag{55}$$

where the action

$$\begin{aligned} S_{\mathrm{AdS};\epsilon,z_0}[Y] \equiv -\frac{1}{2}\int_\epsilon^{z_0}\frac{dz}{z^{D+1}}\int_p \Big\{ & z\partial_z Y(z,p)\cdot z\partial_z Y(z,-p) \\ & + \left(p^2 z^2 + \frac{m^2}{\mu^2}\right) Y(z,p)Y(z,-p)\Big\} \end{aligned} \tag{56}$$

is defined for a massive scalar field Y in the $D+1$-dimensional AdS space with radius of curvature $\frac{1}{\mu}$.

Let us solve (54), which amounts to

$$-\partial_z\left(z^{1-D}\partial_z \frac{1}{f(z,p)}\right) + z^{1-D}\left(p^2 + \frac{m^2}{\mu^2 z^2}\right)\frac{1}{f(z,p)} = 0\,. \tag{57}$$

The general solution is given by

$$\frac{1}{f(z,p)} = A(p) z^{\frac{D}{2}} K_\nu(pz) + B(p) z^{\frac{D}{2}} I_\nu(pz)\,, \tag{58}$$

where

$$\nu \equiv \sqrt{\frac{m^2}{\mu^2} + \frac{D^2}{4}} > 0\,, \tag{59}$$

and I_ν, K_ν are the modified Bessel functions. We will shortly determine the functions of momenta $A(p), B(p)$ (both with the mass dimension 1) and the value of ν (equivalently $\frac{m^2}{\mu^2}$).

Now, (52) and (54) imply that $G_z(p)/f(z,p)$ satisfies the same differential equation as $1/f(z,p)$. Hence, we obtain

$$\frac{G_z(p)}{f(z,p)} = C(p) z^{\frac{D}{2}} K_\nu(pz) + D(p) z^{\frac{D}{2}} I_\nu(pz)\,. \tag{60}$$

(Note the mass dimension of C, D is -1.) Moreover, (52) gives

$$z\partial_z \frac{1}{f(z,p)}\cdot\frac{G_z(p)}{f(z,p)} - \frac{1}{f(z,p)} z\partial_z \frac{G_z(p)}{f(z,p)} = -z^D\,. \tag{61}$$

Substituting (58) and (60) into the above, and using the Wronskian

$$\frac{d}{dz} I_\nu(z) \cdot K_\nu(z) - I_\nu(z) \frac{d}{dz} K_\nu(z) = \frac{1}{z} , \tag{62}$$

we obtain

$$A(p)D(p) - B(p)C(p) = 1 . \tag{63}$$

We can determine the coefficient functions $A(p)$ to $D(p)$, and the constant ν as follows. From (58) and (60), we obtain

$$G_z(p) = \frac{C(p)K_\nu(pz) + D(p)I_\nu(pz)}{A(p)K_\nu(pz) + B(p)I_\nu(pz)} . \tag{64}$$

Let us consider the limit $z \to 0+$. We must find

$$\lim_{\Lambda \to +\infty} W_\Lambda \left[\left(\frac{\Lambda}{\mu} \right)^{\frac{\eta}{2}} J \right] = \lim_{z \to 0+} W_{1/z}[(\mu z)^{-\frac{\eta}{2}} J] = 0 , \tag{65}$$

since this corresponds to the integration of no momentum mode. Hence, we obtain

$$\lim_{z \to +0} G_z(p) = \frac{C(p)}{A(p)} = 0 . \tag{66}$$

This gives

$$C(p) = 0 . \tag{67}$$

Next, consider the limit $z \to +\infty$. From

$$\lim_{z \to \infty} G_z(p) = \frac{1}{p^2 \left(\frac{p}{\mu} \right)^{-\eta}} \tag{68}$$

we obtain

$$\frac{D(p)}{B(p)} = \frac{1}{p^2 \left(\frac{p}{\mu} \right)^{-\eta}} . \tag{69}$$

Combining the three equations (63, 67, 69), we obtain

$$A(p) = \frac{1}{c}\, p\, (p/\mu)^{-\frac{\eta}{2}} , \tag{70a}$$

$$B(p) = c\, p\, (p/\mu)^{-\frac{\eta}{2}} , \tag{70b}$$

$$C(p) = 0 , \tag{70c}$$

$$D(p) = c\, \frac{1}{p}\, (p/\mu)^{\frac{\eta}{2}} , \tag{70d}$$

where we have taken c to be a constant for simplicity.

To determine c, we must examine $f(z, p)$. From (58), we obtain

$$\frac{1}{f(z, p)} = z^{\frac{D}{2}}\, p \left(\frac{p}{\mu} \right)^{-\frac{\eta}{2}} c \left(\frac{1}{c^2} K_\nu(pz) + I_\nu(pz) \right) . \tag{71}$$

Demanding that the change of variables from $y(z, p)$ to $Y(z, p) = y(z, p)/f(z, p)$ be analytic at $p^2 = 0$, we must first choose

$$c^2 = \frac{\pi}{2}\frac{1}{\sin \pi\nu} \tag{72}$$

so that

$$\frac{1}{c^2} K_\nu(pz) + I_\nu(pz) = I_{-\nu}(pz)\,. \tag{73}$$

Then, since

$$I_{-\nu}(z) \overset{z\to 0}{\longrightarrow} z^{-\nu} \times \left(\frac{1}{\Gamma(1-\nu)} + \text{analytic in } z^2 \right), \tag{74}$$

we must choose

$$\boxed{\nu = 1 - \frac{\eta}{2} = 1 - \gamma < 1} \tag{75}$$

so that

$$\boxed{\frac{1}{f(z,p)} = \left(\frac{\pi}{2}\frac{1}{\sin\pi\nu}\right)^{\frac{1}{2}} z^{\frac{D}{2}} p \left(\frac{p}{\mu}\right)^{-\frac{\eta}{2}} I_{-\nu}(pz)} \tag{76}$$

is analytic at $p^2 = 0$.

Note that the resulting high-momentum propagator

$$\boxed{G_z(p) = \frac{1}{p^2 \left(\frac{p}{\mu}\right)^{-\eta} \frac{I_{-\nu}(pz)}{I_\nu(pz)}}} \tag{77}$$

is analytic at $p^2 = 0$ as long as z is finite. Only as $z \to +\infty$, we find non-analyticity:

$$\lim_{z\to\infty} G_z(p) = \frac{1}{p^2 \left(\frac{p}{\mu}\right)^{-\eta}}\,. \tag{78}$$

To summarize so far, for the choice of the high-momentum propagator (77), we can rewrite (42) as

$$\begin{aligned} e^{W_{1/z_0}[J]} = \int [d\varphi] \exp\left[\frac{1}{2} \int_p J(p)J(-p)G_\epsilon(p) \right] \\ \times \exp\left(\int_p J(p)\varphi(-p) + \frac{1}{2}\epsilon^{-D} \int_p \epsilon\partial_\epsilon \ln f(\epsilon,p) \cdot \frac{1}{f(\epsilon,p)^2} \varphi(p)\varphi(-p) \right) \\ \times \int \mathcal{D}Y \Big|_{\substack{Y(\epsilon,p)=\varphi(p)/f(\epsilon,p) \\ Y(z_0,p)=0}} e^{S_{\mathrm{AdS};\epsilon,z_0}[Y]} \quad , \end{aligned} \tag{79}$$

where the AdS action is given by (56). By construction, (79) is independent of ϵ, but it depends on z_0.

In the limit $z_0 \to +\infty$, we obtain the generating functional:

$$e^{\mathcal{W}[J]} = \int [d\varphi] \exp\left[\frac{1}{2}\int_p J(p)J(-p)G_\epsilon(p)\right]$$
$$\times \exp\left(\int_p J(p)\varphi(-p) + \frac{1}{2}\epsilon^{-D}\int_p \epsilon\partial_\epsilon \ln f(\epsilon,p)\cdot\frac{1}{f(\epsilon,p)^2}\varphi(p)\varphi(-p)\right)$$
$$\times \int \mathcal{D}Y\Big|_{\substack{Y(\epsilon,p)=\varphi(p)/f(\epsilon,p)\\ Y(+\infty,p)=0}} e^{S_{\mathrm{AdS};\epsilon,\infty}[Y]} \tag{80}$$

Since this is independent of ϵ, we can also take the limit $\epsilon \to 0+$. Using

$$\frac{1}{f(\epsilon,p)} \longrightarrow \left(\frac{\pi}{2}\frac{1}{\sin\pi\nu}\right)^{\frac{1}{2}} \epsilon^{\frac{D}{2}-1}(\mu\epsilon)^\gamma \frac{1}{\Gamma(1-\nu)} \propto \epsilon^{\frac{D}{2}-1}(\epsilon\mu)^\gamma\,, \tag{81}$$
$$G_\epsilon(p) \longrightarrow \epsilon^2\,(\mu\epsilon)^{-\eta}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \propto \epsilon^2(\epsilon\mu)^{-\eta}\,, \tag{82}$$

we obtain, for $\epsilon \to 0+$,

$$e^{\mathcal{W}[J]} = \int [d\varphi] \exp\left(\int_p J(p)\varphi(-p) - \frac{\Delta_-}{2}\epsilon^{-2}(\epsilon\mu)^\eta \int_p \alpha^2\varphi(p)\varphi(-p)\right)$$
$$\times \int \mathcal{D}Y\Big|_{\substack{Y(\epsilon,p)=\alpha\mu^\gamma\epsilon^{\Delta_-}\varphi(p)\\ Y(z_0,p)=0}} e^{S_{\mathrm{AdS};\epsilon,\infty}[Y]}\,, \tag{83}$$

where

$$\Delta_- \equiv \frac{D}{2} - \nu = \frac{D-2}{2} + \gamma\,, \tag{84}$$
$$\alpha \equiv \frac{1}{\sqrt{2}}\left(\frac{\Gamma(\nu)}{\Gamma(1-\nu)}\right)^{\frac{1}{2}}\,. \tag{85}$$

In the literature it may be more common to write $\alpha\varphi(p)$ as $\varphi(p)$ and $\frac{1}{\alpha}J(p)$ as $J(p)$ so that

$$e^{\mathcal{W}[J]} = \int [d\varphi] \exp\left(\int_p J(p)\varphi(-p) - \frac{\Delta_-}{2}\epsilon^{-2}(\epsilon\mu)^\eta \int_p \varphi(p)\varphi(-p)\right)$$
$$\times \int \mathcal{D}Y\Big|_{\substack{Y(\epsilon,p)=\mu^\gamma\epsilon^{\Delta_-}\varphi(p)\\ Y(z_0,p)=0}} e^{S_{\mathrm{AdS};\epsilon,\infty}[Y]}\,. \tag{86}$$

We then obtain

$$\mathcal{W}[J] = \frac{1}{2}\int_p \frac{\alpha^2}{p^2\left(\frac{p}{\mu}\right)^{-\eta}} J(p)J(-p)\,. \tag{87}$$

This reproduces the prescription of the alternative quantization of the AdS/CFT correspondence [16] (reviewed nicely in Appendix of [15]) for computing the two-point function.

In [21] it was pointed out that when the ERG equation is mapped to AdS space there remains a boundary term depending on the function $f(p)$. We see this in (86) also. But note that it is analytic in p and therefore does not affect the all important non-analytic piece.

4. Discussion

We have managed to derive the AdS/CFT correspondence from ERG, but our derivation is not without faults. We discuss three issues here.

4.1. Non-analyticity of the Wilson action at zero momentum

As we have already pointed out in Sec. 2, our Wilson action (17) is not analytic at $p^2 = 0$ (hence non-local) due to the anomalous dimension η. This is partially because we are treating a composite field as an elementary field ϕ. Even in the free massless theory in D dimensions, the composite field ϕ^2 has scale dimension $D-2$ so that its anomalous dimension is $(D-2)/2$ compared with the canonical scale dimension $(D-2)/2$ of ϕ.

Another reason for the non-analyticity is that we are not taking interactions into account. Consider a Wilson action whose quadratic part is given by

$$S_\Lambda = -\frac{1}{2}\int_p \frac{p^2}{K(p/\Lambda)} \frac{1}{1+K(p/\Lambda)\left(\left(\frac{p^2+m_\Lambda^2}{\mu^2}\right)^{\frac{\eta}{2}}-1\right)}\phi(p)\phi(-p)\,, \tag{88}$$

where the cutoff dependence of the squared mass m_Λ^2 is determined by four-point interactions. (The cutoff dependence of the four-point interactions is determined by the sixth point interactions, and so on.) The squared mass restores the analyticity of the action at $p^2 = 0$. The corresponding generating functional is

$$W_\Lambda[J] = \frac{1}{2}\int_p \frac{J(p)J(-p)}{p^2\left(\frac{p^2+m_\Lambda^2}{\mu^2}\right)^{\frac{\eta}{2}} + R_\Lambda(p)} + \text{quartic and higher}\,, \tag{89}$$

and it reproduces the same two-point function in the limit $\Lambda \to 0+$, if we assume

$$\lim_{\Lambda\to 0+} m_\Lambda^2 = 0\,. \tag{90}$$

4.2. Non-analyticity of the cutoff function K

In the ERG formulation, the choice of a cutoff function $K(p/\Lambda)$ is totally arbitrary as long as it satisfies $K(0) = 1$, and it decreases rapidly for $p > \Lambda$. But we usually assume $K(p/\Lambda)$ to be analytic at $p^2 = 0$:

$$K(p/\Lambda) = 1 + \text{integral powers of } p^2\,. \tag{91}$$

Now, in Sec. 3 we have chosen a particular cutoff function so that

$$G_{1/\Lambda}(p) \equiv \frac{1}{p^2\left(\frac{p}{\mu}\right)^{-\eta} + \left(\frac{\Lambda}{\mu}\right)^{-\eta} R_\Lambda(p)} = \frac{1}{p^2\left(\frac{p}{\mu}\right)^{-\eta}\frac{I_{-\nu}(p/\Lambda)}{I_\nu(p/\Lambda)}}\,. \tag{92}$$

This is obtained from (77). This implies

$$R_\Lambda(p) = p^2\left(\frac{p}{\Lambda}\right)^{-\eta}\left(\frac{I_{-\nu}(p/\Lambda)}{I_\nu(p/\Lambda)} - 1\right)\,. \tag{93}$$

(Note that this has the form (33).) We then obtain, from (8),

$$K(p/\Lambda) = \frac{1}{1 + \frac{\left(\frac{p}{\Lambda}\right)^\eta \frac{I_\nu(p/\Lambda)}{I_{-\nu}(p/\Lambda)}}{1 - \frac{I_\nu(p/\Lambda)}{I_{-\nu}(p/\Lambda)}}}, \tag{94}$$

where

$$\left(\frac{p}{\Lambda}\right)^\eta \frac{I_\nu(p/\Lambda)}{I_{-\nu}(p/\Lambda)} = \mathrm{O}(p^2/\Lambda^2) \tag{95}$$

is analytic at $p^2 = 0$ since $2\nu = 2 - \eta$. Therefore, we find

$$K(p/\Lambda) = 1 + \mathrm{const}\,(p/\Lambda)^2 + \mathrm{const}\,(p/\Lambda)^{4-\eta} + \cdots . \tag{96}$$

This is not analytic. Since no physics depends on the choice of $K(p/\Lambda)$, one could argue, perhaps, that using a cutoff function non-analytic at $p^2 = 0$ is acceptable.

4.3. *Absence of interactions*

We have already discussed the importance of introducing interactions to restore the analyticity or locality of the Wilson action. From the ERG perspectives, it is natural to introduce interactions only at the boundary $z = \epsilon$ of the AdS space. Whether or not interactions are induced in the bulk of the AdS space is left for a future study. We make some preliminary remarks on this in Section 6.

5. Identifying the anomalous dimension from the ERG equation

As mentioned in Section 2.3, in [21] the starting point of the discussion was Polchinski's ERG without any parameter for anomalous dimension. In the AdS version of the evolution operator, however, a new parameter, absent in the original Polchinski evolution operator, made its appearance. This parameter is contained in the specification of the function $f(z, p)$ used in the change of variables. This is the parameter ν and becomes the (anomalous) dimension of the boundary operator as seen from the two point function calculation. In the AdS scalar field equation it shows up as the mass of the scalar field. In this paper we have started with a modified Polchinski ERG *with* a parameter ν, for anomalous dimension. ν was chosen so that the correct high energy propagator G_z of the fixed point theory with anomalous dimension is reproduced. In this section we would like to elucidate the role of this parameter from the point of view of the ERG equation and explain how a choice of variables dictates the equation.

As pointed out originally by Wilson and others (see for instance, [22] or Bell and Wilson, [35,36]), there are two important aspects in an RG — one is the coarse graining and the other is a rescaling of the field variables. In [35,36], they exemplify this with the following simple transformation, T (in their notation):

$$T e^{-H[S]} = \int_\sigma e^{-\frac{1}{2}a \int_q (S_q - b\sigma_{q/2})^2} e^{-H[\sigma]}. \tag{97}$$

The initial field variable is σ and the final field variable if S. The subscript on the field variable has changed from $q/2$ to q. q is dimensionless and this corresponds to a change of the cutoff from $\Lambda \to \Lambda/2$. This is the coarse graining. The factor b is the rescaling. After n steps the rescaling becomes b^n. The factor a can be changed by a scaling of *both* σ and S, and has no effect on the physics.

In a field theory the field variables are integrated over. Thus a constant rescaling does not change the physics. Thus in the above example, $H[S]$ and $H[bS]$, even though they have different mathematical forms, describe the same physics in the sense that one can map correlations calculated with one variable to correlations calculated with the other variable by a change of normalization. The S-matrix is in fact invariant. Thus for instance if $H[S]$ describes a scale invariant theory (i.e. a fixed point) then so does $H[bS]$. Thus if an *exact* RG transformation[1] is done on a critical physical system in the basin of attraction of a fixed point, it will move towards the fixed point regardless of the choice of variables — this is a generic property of all physical systems as we flow to the IR. This does not mean that the Hamiltonian is mathematically form invariant at the fixed point. *A fixed point Hamiltonian is an equivalence class of mathematical expressions for the Hamiltonian related by rescalings of the field variable.* The physical property that defines a fixed point is that physical quantities such as the S-matrix will have a scale invariance — absence of any characteristic scale. All this goes to show that the choice of b does not change the physics.

However, if one wants to make the Hamiltonian *mathematically form invariant* under the ERG, then a particular rescaling parameter (b in the above example) has to be chosen — the precise value depends on the details of the interacting theory. This is a choice of field normalization. A natural choice is to make sure that the kinetic term has a fixed normalization. If this normalization is implemented then the fixed point Hamiltonian is mathematically identical after the transformation. This is convenient in an actual calculation because requiring that the Hamiltonian be mathematically identical after an RG transformation leads to a well defined mathematical "fixed point equation".[2]

Let us now go to the connection between b and the anomalous dimension. In the fixed point equation one works with dimensionless variables. Thus the variable σ has to be scaled by a factor to undo the change of scale from $\Lambda \to \Lambda/2$ and keep the normalization of the kinetic term fixed. In a free theory this is given by a factor $2^{-\frac{(D+2)}{2}}$ which reflects the engineering dimension of the field. So we get

$$b = 2^{-\frac{(D+2)}{2}}.$$

In an interacting theory there are further contributions at each iteration and this modifies the scaling dimension by adding an anomalous dimension to the engineering dimension and gives

$$b = 2^{-\frac{(D+2+\eta)}{2}}.$$

Let us transcribe this to continuous time ERG where $\Lambda(t) = \Lambda(0)e^{-t}$. A factor b becomes $e^{-\frac{(D+2+\eta)}{2}t}$. If we denote by x_i the initial variable (σ in the above example) and x_f the final variable (S in the above example) then the transformation (97) can be written in the form of a time dependent rescaling as (the notation used in [21] where momentum labels are suppressed)

$$\psi(x_f, t_f) = \int_{x_i} e^{-\frac{1}{2}A(e^{\alpha t_f} x_f - e^{\alpha t_i} x_i)^2} \psi(x_i, t_i). \tag{98}$$

[1] If the RG is not exact, the choice of variable becomes important — because a bad choice of variable throws out relevant information. But in an exact RG no information is lost.

[2] If the correct b is not chosen then when approaching a fixed point, after each RG iteration, the normalization of the kinetic term will get multiplied by a constant factor (not equal to 1) and this can lead to complications.

The conclusion we reach is that to determine the anomalous dimension of the evolution equation we can write the integrating kernel in the form (98) and read off the anomalous dimension η from $e^{\alpha t}$.[3] In Appendix A we illustrate this point with two examples of standard ERG equations.

This also suggests that if we do a field redefinition involving a time dependent rescaling, of the form $x(t) = y(t)e^{\mu t}$, the scaling dimension is changed by an amount μ. Thus an ERG with anomalous dimension can be related to one without anomalous dimension, by such a field redefinition. If one wants a mathematical fixed point the anomalous dimension has to be chosen correctly. The precise value will depend on the interactions.

This then answers the question raised at the beginning of this section: In [21] the starting point was Polchinski's ERG without anomalous dimension. The integrating kernel is of the form

$$\psi(x_f, t_f) = \int_{x_i} e^{-\frac{1}{2}\frac{(x_f - x_i)^2}{G_i - G_f}} \psi(x_i, t_i). \tag{99}$$

With the change of variables $x = fy$ we obtain

$$\psi(y_f, t_f) = \int_{y_i} e^{-\frac{1}{2}\frac{(y_f f(t_f) - y_i f(t_i))^2}{G_i - G_f}} \psi(y_i, t_i). \tag{100}$$

We have seen that $f \approx e^{-(\frac{D-2+\eta)}{2}t}$. Clearly the ERG equation obeyed in the new variables will have an anomalous dimension parameter. This explains the appearance of this parameter in the AdS equation in [21]. In Sec. 2.3 we have already seen how (23) is related to (15) by change of variables.

In this section we have shown the role of field redefinitions (or "wave function renormalization" in perturbative calculations) in introducing anomalous dimension in an ERG equation. This is important for locating the mathematical fixed point of the equation. We have also seen that the dimension can be read off from the integral formula. Some other examples of these are given in Appendix A.

6. Nontrivial fixed point action

In this section we consider a nontrivial fixed point action. To begin with we use the usual Polchinski ERG formalism. The kinetic term is $\frac{1}{2}x^2G^{-1}$ and the interacting part is S_0.

$$S_{FP} = \frac{1}{2}x^2G^{-1} + S_0(x)\,.$$

Let the perturbation be S_1 so that the full action is

$$S = \frac{1}{2}x^2G^{-1} + S_0(x) + S_1(x)\,.$$

Then in our earlier notation, the "wave functions" are given by

$$\psi = e^{-S} = e^{-[\frac{1}{2}x^2G^{-1} + S_0(x,t) + S_1(x,t)]}, \qquad \psi' = e^{-[S_0(x,t)+S_1(x,t)]}\,.$$

Polchinski's equation is

[3] As mentioned above the value of η is a property of the action and should be chosen depending on the action one starts with.

$$\frac{\partial\psi'}{\partial t} = -\frac{1}{2}\dot{G}\frac{\partial^2\psi'}{\partial x^2}\,. \tag{101}$$

What is special is that S_0 by itself satisfies Polchinski's equation — eventually it will be taken to be a fixed point solution. Thus we have the following two equations:

$$\frac{\partial S_0}{\partial t} = \frac{1}{2}\dot{G}\left[-\frac{\partial^2 S_0}{\partial x^2} + \left(\frac{\partial S_0}{\partial x}\right)^2\right] \tag{102}$$

and

$$\frac{\partial S_0}{\partial t} + \frac{\partial S_1}{\partial t} = \frac{1}{2}\dot{G}\left[-\frac{\partial^2 S_0}{\partial x^2} + \left(\frac{\partial S_0}{\partial x}\right)^2 - \frac{\partial^2 S_1}{\partial x^2} + \left(\frac{\partial S_1}{\partial x}\right)^2 + 2\left(\frac{\partial S_0}{\partial x}\right)\left(\frac{\partial S_1}{\partial x}\right)\right]. \tag{103}$$

Subtracting (102) from (103) we get

$$\frac{\partial S_1}{\partial t} = \frac{1}{2}\dot{G}\left[-\frac{\partial^2 S_1}{\partial x^2} + \left(\frac{\partial S_1}{\partial x}\right)^2 + 2\left(\frac{\partial S_0}{\partial x}\right)\left(\frac{\partial S_1}{\partial x}\right)\right]. \tag{104}$$

Since S_0 is a solution of (102), its form is (*in principle*) known as a function of time. In the case that S_0 is chosen to be a fixed point solution, its time dependence can be specified very easily: expressed in terms of *rescaled and dimensionless* variables it has no time dependence. This is equivalent to saying that the dimensionless couplings are constant in RG-time, t, i.e., they have vanishing beta functions. One can work backwards and determine the exact t-dependence in terms of the original variables. (104) can be used to define a modified Hamiltonian evolution equation for the wave function $\psi'' = e^{-S_1(x,t)}$:

$$\frac{\partial}{\partial t}\psi'' = -\frac{1}{2}\dot{G}\left[\frac{\partial^2}{\partial x^2} - 2\left(\frac{\partial S_0}{\partial x}\right)\frac{\partial}{\partial x}\right]\psi''\,. \tag{105}$$

Note that the term involving S_0 is like a gauge field coupling – in fact it is "pure gauge". The Action functional corresponding to this Hamiltonian is derived in Appendix B using canonical methods. While it involves more algebra, it can be applied even when S_0 is not a solution of the ERG equation.

In the end the result can be summarized very simply. Start with the usual RG evolution of ψ':

$$e^{-S_0\left(x(t_f),t_f\right)-S_1\left(x(t_f),t_f\right)} = \int \mathcal{D}x\, e^{\frac{1}{2}\int_{t_i}^{t_f} dt\ \frac{1}{2}\frac{1}{\dot{G}}\left(\frac{dx}{dt}\right)^2} e^{-S_0(x(t_i),t_i)-S_1(x(t_i),t_i)}\,. \tag{106}$$

Take $S_0(x(t_f),t_f)$ into the RHS to get

$$e^{-S_1\left(x(t_f),t_f\right)} = \int \mathcal{D}x\, e^{\frac{1}{2}\int_{t_i}^{t_f} dt\ \frac{1}{2}\frac{1}{\dot{G}}\left(\frac{dx}{dt}\right)^2} e^{S_0\left(x(t_f),t_f\right)-S_0(x(t_i),t_i)-S_1(x(t_i),t_i)}\,. \tag{107}$$

Introduce the evolution of S_0 into the functional integral by writing it as an integral of a total derivative:

$$\Delta S_0 = S_0(x(t_f),t_f) - S_0(x(t_i),t_i) = \int_{t_i}^{t_f} dt\ \frac{dS_0(x(t),t)}{dt}$$

$$= \int_{t_i}^{t_f} dt\ \left[\frac{\partial S_0(x(t),t)}{\partial t} + \frac{dx(t)}{dt}\frac{\partial S_0(x(t))}{\partial x(t)}\right] \tag{108}$$

to get

$$e^{-S_1\left(x(t_f),t_f\right)} = \int \mathcal{D}x\, e^{\frac{1}{2}\int_{t_i}^{t_f} dt\, \left[\frac{1}{2}\frac{1}{\dot{G}}(\frac{dx}{dt})^2 + \frac{\partial S_0(x(t),t)}{\partial t} + \frac{dx(t)}{dt}\frac{\partial S_0(x(t),t)}{\partial x(t)}\right]} e^{-S_1(x(t_i),t_i)}\,. \tag{109}$$

Thus the evolution of the perturbation involves an evolution operator that has a non linear term, but is in fact a *total derivative*.

The total derivative can also be rewritten in other forms using the ERG for S_0. Thus for instance (using field theory notation) it can be rewritten as (see (B)):

$$\begin{aligned} S_{bulk}[\phi(p,t)] = \frac{1}{2}\int dt \int \frac{d^D p}{(2\pi)^D}\Bigg[&-\frac{1}{\dot{G}(p)}\frac{\partial\phi(p,t)}{\partial t}\frac{\partial\phi(-p,t)}{\partial t} \\ &-\dot{G}(p)\left\{\frac{\delta S_0[\phi(p,t),t]}{\delta\phi(p,t)}\frac{\delta S_0[\phi(p,t),t]}{\delta\phi(-p,t)} - \frac{\delta^2 S_0[\phi(p,t),t]}{\delta\phi(-p,t)\delta\phi(p,t)}\right\} \\ &-\frac{\delta S_0[\phi(p,t),t]}{\delta\phi(p,t)}\frac{\partial\phi(p,t)}{\partial t}\Bigg]. \end{aligned} \tag{110}$$

What we have here is a nonlinear action describing ERG evolution — because of the presence of $S_0[\phi]$ in the action, it is no longer quadratic in ϕ. This is unusual in ERG literature: Acting on $\psi = e^{-S_B}$, the evolution given by Polchinski ERG equation (101), is linear. (The equation is of course nonlinear when written in terms of S_B.)

On the other hand in discussions of holographic RG (see for instance [17,15]) the Hamiltonian radial evolution of the boundary action, $S_B[\phi]$, is given by

$$\partial_\epsilon S_B = \int\limits_{z=\epsilon} d^d x \left\{\left(\frac{\delta S_B}{\delta\phi}\right)^2 + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V[\phi]\right\}. \tag{111}$$

Here $V[\phi]$ is the potential for the scalar field in the bulk action. Thus the holographic RG evolution in (111) is nonlinear in ϕ even when acting on $\psi = e^{-S_B[\phi]}$ due to the presence of $V[\phi]$. Indeed in AdS/CFT correspondence, $V[\phi]$ plays an important role in determining higher point correlation functions and also beta functions (see for e.g. [5,19]) of the boundary theory through "Witten diagrams". It is after all in this sense that the bulk action has information about the boundary theory.

In our discussion in this section, nonlinear terms have appeared, but whether the nonlinearity introduced here is of the same ilk as in (111) is not clear to us. In particular it holds here that if S_0 satisfies the RG equation by itself, then this nonlinear piece is a total derivative. The origin of nonlinear terms in holographic RG from the viewpoint of ERG is thus an open question.

7. Conclusion

Some aspects of the mapping of ERG equation to a holographic AdS form that was introduced in [21] were clarified in this paper with a view to extending the results to nontrivial fixed point theories. The ERG written there was for the Wilson action. The bulk AdS field corresponds to a field in the boundary theory rather than a source. This then corresponds to what is called "alternative quantization". The main point of this paper is to clarify how the map from ERG to AdS can be done in the presence of an anomalous dimension parameter. This was done in a concrete example where the generating functional is quadratic but has an anomalous dimension parameter. In general one expects anomalous dimension when there are interactions. Here the focus is on the mapping to AdS, so we simplify calculations by introducing the anomalous dimension into the two point function by hand. The same techniques should work in more realistic theories also. The results reproduce those obtained in the literature using the AdS/CFT conjecture.

Thus many of the results of AdS/CFT calculations for two point functions are obtained directly starting from the ERG of the boundary CFT without invoking any conjecture. Also string theory itself plays no direct role at this level. As mentioned in [21] string theory possibly plays a role in making the bulk theory well defined as a quantum theory.

There are several other open questions. We have studied only the propagator of this theory. Higher order correlators are hard to study because we are dealing with a QFT. In order to obtain concrete results an expansion parameter such as $1/N$ is needed.

This paper discusses only perturbations involving an elementary scalar field. Equally interesting are questions involving composite fields. This was also considered in a general way in [21]. Once again a large N expansion is required to do these computations.

The role of dynamical gravity has not been discussed thus far. This needs to be addressed.

Appendix A. Other ERG equations

Let us study two of the standard ERG equations: We let our Wilson action be (using the simplified notation suppressing momentum labels)

$$S = \frac{1}{2} x G^{-1} x + S^I_\Lambda$$

with G the propagator, $G = \frac{K(\Lambda)}{p^2}$. We take $G_0 = \frac{K(\Lambda_0)}{p^2} = \frac{K_0}{p^2}$. (We take $K_0 = 1$ for convenience.)

Polchinski's equation with anomalous dimension $\frac{\eta}{2}$ in simplified notation, with momentum dependence suppressed is:

$$\frac{\partial \psi}{\partial t} = [-\frac{1}{2}(\dot{G} + \eta \frac{G(G_0 - G)}{G_0})\frac{\partial^2}{\partial x^2} - (\frac{\dot{G}}{G} + \frac{\eta}{2})x\frac{\partial}{\partial x}]\psi \,. \tag{A.1}$$

Another ERG equation with anomalous dimension was written down in [37], [38] which, in simplified notation, is:

$$\frac{\partial \psi}{\partial t} = [-\frac{1}{2}\dot{G}\frac{\partial^2}{\partial x^2} - (\frac{\dot{G}}{G} + \frac{\eta}{2})x\frac{\partial}{\partial x}]\psi \,. \tag{A.2}$$

The integral evolution operator for both these have the same form:

$$\psi(x_f, t_f) = \int dx_i e^{-\frac{1}{2}\frac{(\frac{x_f e^{-\frac{\eta}{2}t_f}}{G_f} - \frac{x_i e^{-\frac{\eta}{2}t_i}}{G_i})^2}{H_f^{-1} - H_i^{-1}}} \psi(x_i, t_i) \,. \tag{A.3}$$

Comparing with the form (99) we see that $e^{\frac{\eta}{2}(t_f - t_i)}$ gives the time dependent relative scaling between x_f and x_i corresponding to anomalous dimension $\frac{\eta}{2}$.

The equations differ in the form of H. Polchinski's equation gives

$$H^{-1} = (\frac{1}{G} - \frac{1}{G_0})e^{-\eta t} \tag{A.4}$$

and (A.2):

$$\frac{dH^{-1}}{dt} = \frac{d(G^{-1})}{dt} e^{-\eta t} . \tag{A.5}$$

We now put back the momentum dependence and go back to more standard notation:

$$G = \frac{K}{p^2}, \quad G_0 = \frac{1}{p^2}, \qquad x_i = \phi_i(p), \quad x_f = \phi_f(p) . \tag{A.6}$$

In the Polchinski case from (A.4)

$$H = \frac{Re^{\eta t}}{p^4}, \qquad R = (\frac{p^2 K}{1-K}), \quad e^{\eta t} = (\frac{\Lambda}{\mu})^{-\eta}$$

we get Polchinski's equation with anomalous dimension by substituting these in (A.1):

$$-\Lambda \frac{\partial}{\partial \Lambda} e^{S_\Lambda[\phi]} = \int_p \left[\left(\frac{\Delta(p/\Lambda)}{K(p/\Lambda)} - \frac{\eta}{2} \right) \phi(p) \frac{\delta}{\delta \phi(p)} + \frac{1}{p^2} \left\{ \Delta(p/\Lambda) - \eta K(p/\Lambda) \left(1 - K(p/\Lambda)\right) \right\} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right] e^{S_\Lambda[\phi]} . \tag{A.7}$$

For $\Lambda_2 < \Lambda_1$, this is solved by the kernel substituting in (A.3):

$$e^{S_{\Lambda_2}[\phi]} = \int [d\phi'] \mathcal{P}_{\Lambda_2,\Lambda_1}[\phi, \phi'] e^{S_{\Lambda_1}[\phi']} , \tag{A.8}$$

where the evolution operator is given by

$$\mathcal{P}_{\Lambda_2,\Lambda_1}[\phi, \phi'] \equiv \exp\left[-\frac{1}{2} \int_p \frac{1}{(\frac{\Lambda_2}{\mu})^\eta \frac{1}{R_{\Lambda_2}(p)} - (\frac{\Lambda_1}{\mu})^\eta \frac{1}{R_{\Lambda_1}(p)}} \times \left\{ \left(\frac{\Lambda_1}{\mu} \right)^{\frac{\eta}{2}} \frac{\phi'(p)}{K(p/\Lambda_1)} - \left(\frac{\Lambda_2}{\mu} \right)^{\frac{\eta}{2}} \frac{\phi(p)}{K(p/\Lambda_2)} \right\} \times \left\{ \left(\frac{\Lambda_1}{\mu} \right)^{\frac{\eta}{2}} \frac{\phi'(-p)}{K(p/\Lambda_1)} - \left(\frac{\Lambda_2}{\mu} \right)^{\frac{\eta}{2}} \frac{\phi(-p)}{K(p/\Lambda_2)} \right\} \right] . \tag{A.9}$$

Similarly (A.2) [37], [38], can be obtained by the same substitution.

$$-\Lambda \frac{\partial}{\partial \Lambda} e^{S_\Lambda[\phi]} = \int_p \left[\left(\frac{\Delta(p/\Lambda)}{K(p/\Lambda)} - \gamma \right) \phi(p) \frac{\delta}{\delta \phi(p)} + \frac{1}{p^2} \left\{ \Delta(p/\Lambda) \right\} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right] e^{S_\Lambda[\phi]} . \tag{A.10}$$

The integrating kernel remains the same as in (A.9), but R is defined by the more complicated relation

$$e^{-\eta t} \frac{d}{dt} (\frac{p^4}{R}) = \frac{d}{dt} (\frac{1-K}{p^2 K} e^{-\eta t}) = \frac{d}{dt} (H^{-1}) .$$

In each case we see that the anomalous dimension can be read off from the powers of Λ multiplying the fields in the exponent in (A.9).

Appendix B. Action for the nontrivial fixed point Hamiltonian

In this appendix we derive (110) using Hamiltonian methods.

Our starting point is

$$\frac{\partial}{\partial t}\psi'' = -\frac{1}{2}\dot{G}\left[\frac{\partial^2}{\partial x^2} - 2(\frac{\partial S_0}{\partial x})\frac{\partial}{\partial x}\right]\psi'' . \tag{B.1}$$

Note that in principle one can postulate this as an ERG equation even if S_0 is not a solution of Polchinki's ERG equation as assumed in Section 6. To that extant the derivation in this Appendix is more general.

We rotate to Minkowski space: $it_M = t_E$. Hence,

$$e^{iS_M} = e^{i\int dt_M(-V)} = e^{-\int dt_E V} = e^{-S_E} .$$

Thus, $S_E = -iS_M$. Thus in the above case $S_0 = -iS_{0M}$. Let $-\tau_E = G$ so that

$$\frac{1}{\dot{G}}\frac{\partial}{\partial t_E} = \frac{\partial}{\partial G} = -\frac{\partial}{\partial \tau_E} .$$

Write (105) as

$$\frac{\partial}{\partial t_E} = -\frac{1}{2}\dot{G}\left[\frac{\partial^2}{\partial x^2} - 2(\frac{\partial S_0}{\partial x})\frac{\partial}{\partial x}\right] .$$

This implies

$$\begin{aligned}\frac{\partial}{\partial G} &= -\frac{\partial}{\partial \tau_E} = i\frac{\partial}{\partial \tau_M} \\ &= -\frac{1}{2}\left[\frac{\partial^2}{\partial x^2} - 2(\frac{\partial S_0}{\partial x})\frac{\partial}{\partial x}\right] = -\frac{1}{2}\left[\frac{\partial^2}{\partial x^2} - 2(\frac{-i\partial S_M}{\partial x})\frac{\partial}{\partial x}\right] \\ &= \frac{1}{2}\left[-\frac{\partial^2}{\partial x^2} + 2(\frac{\partial S_M}{\partial x})(-i\frac{\partial}{\partial x})\right] = \frac{1}{2}\Big[P^2 + 2\underbrace{(\frac{\partial S_M}{\partial x})}_{A_x} P\Big].\end{aligned}$$

Writing in terms of $P = -i\frac{\partial}{\partial x}$, we get finally for the Hamiltonian in Minkowski spacetime

$$\begin{aligned}H &= \frac{1}{2}[P^2 + 2\frac{\partial S_M}{\partial x}P] = \frac{1}{2}[P^2 + PA_x + A_xP + i\frac{\partial A_x}{\partial x}] \\ &= \frac{1}{2}[P^2 + PA_x + A_xP + A_x^2] + \frac{1}{2}i\frac{\partial A_x}{\partial x} - \frac{1}{2}A_x^2 \\ &= \frac{1}{2}(P + A_x)^2 + \frac{1}{2}i\frac{\partial A_x}{\partial x} - \frac{1}{2}A_x^2 .\end{aligned} \tag{B.2}$$

This gives

$$\dot{x} = P + A_x = P + \frac{\partial S_M}{\partial x} ,$$

and one can obtain using $L = \dot{x}P - H$ an action

$$L = \frac{1}{2}\dot{x}^2 - \dot{x}A_x + \frac{1}{2}A_x^2 - \frac{1}{2}i\frac{\partial A_x}{\partial x} .$$

Here $A_x(x)$ is to be understood as $A_x(x(t))$, and $\frac{\partial A_x}{\partial x} = \frac{\partial A_x(x(t))}{\partial x(t)}$. In Minkowski spacetime we obtain

$$\begin{aligned} iA_x &= i\frac{\partial S_M}{\partial x} \implies i\frac{\partial A_x}{\partial x} = i\frac{\partial^2 S_M}{\partial x^2}\,, \\ L &= \frac{1}{2}\dot{x}^2 - \dot{x}\frac{\partial S_M}{\partial x} + \frac{1}{2}\left(\frac{\partial S_M}{\partial x}\right)^2 - \frac{1}{2}i\frac{\partial^2 S_M}{\partial x^2}\,, \\ iS_M &= i\int d\tau_M \left[\frac{1}{2}\dot{x}^2 - \dot{x}\frac{\partial S_M}{\partial x} + \frac{1}{2}\left(\frac{\partial S_M}{\partial x}\right)^2 - \frac{1}{2}i\frac{\partial^2 S_M}{\partial x^2}\right]. \end{aligned}$$

In Euclidean space this becomes

$$-S_E = \int d\tau_E \left[-\frac{1}{2}\dot{x}^2 - \frac{dx}{d\tau_E}\frac{i\partial S_M}{\partial x} + \frac{1}{2}\left(\frac{\partial S_M}{\partial x}\right)^2 - \frac{1}{2}i\frac{\partial^2 S_M}{\partial x^2}\right].$$

Hence,

$$S_E = \int d\tau_E \left[\frac{1}{2}\dot{x}^2 - \frac{dx}{d\tau_E}\frac{\partial S_0}{\partial x} + \frac{1}{2}\left(\frac{\partial S_0}{\partial x}\right)^2 - \frac{1}{2}\frac{\partial^2 S_0}{\partial x^2}\right].$$

Now reintroduce G by $dG = dt_E \frac{dG}{dt_E} = dt_E \dot{G}$:

i)

$$\frac{1}{2}\int d\tau_E \left(\frac{dx}{d\tau_E}\right)^2 = -\frac{1}{2}\int dG\left(\frac{dx}{dG}\right)^2 = -\frac{1}{2}\int dt\,\frac{1}{\dot{G}}\left(\frac{dx}{dt_E}\right)^2.$$

ii)

$$\int d\tau_E \frac{1}{2}\left[\left(\frac{\partial S_0}{\partial x}\right)^2 - \frac{1}{2}\frac{\partial^2 S_0}{\partial x^2}\right] = -\int dt_E \dot{G}\frac{1}{2}\left[\left(\frac{\partial S_0}{\partial x}\right)^2 - \frac{1}{2}\frac{\partial^2 S_0}{\partial x^2}\right].$$

iii)

$$-\int d\tau_E\, \frac{dx}{d\tau_E}\frac{\partial S_0}{\partial x} = \int dt_E\, \frac{dx}{dt_E}\frac{\partial S_0}{\partial x}\,.$$

So

$$S_E = \int_{t_i}^{t_f} dt\,\left[-\frac{1}{2}\frac{1}{\dot{G}}\left(\frac{dx}{dt}\right)^2 - \underbrace{\dot{G}\frac{1}{2}\left[\left(\frac{\partial S_0}{\partial x}\right)^2 - \frac{1}{2}\frac{\partial^2 S_0}{\partial x^2}\right]}_{\frac{\partial S_0(x,t)}{\partial t}} - \frac{dx(t)}{dt}\frac{\partial S_0(x(t))}{\partial x(t)}\right]. \tag{B.3}$$

Note that in writing $\frac{\partial S_0(x,t)}{\partial t}$, it is understood that the explicit t dependence of S_0 due to RG evolution is being differentiated. The implicit t dependence due to $x(t)$ is not considered here. The next term $\frac{dx}{dt}\frac{\partial S_0}{\partial x}$ differentiates this implicit t-dependence. Adding them we get $\frac{dS_0}{dt}$. So we have

$$S_E = \int_{t_i}^{t_f} dt\,\left[-\frac{1}{2}\frac{1}{\dot{G}}\left(\frac{dx}{dt}\right)^2\right] - \left\{S_0(x(t_f),t_f) - S_0(x(t_i),t_i)\right\} \tag{B.4}$$

Thus, we get for the evolution of ψ'':

$$e^{-S_1\left(x(t_f),t_f\right)} = \int \mathcal{D}x\, e^{\frac{1}{2}\int_{t_i}^{t_f} dt\ \frac{1}{2}\frac{1}{G}(\frac{dx}{dt})^2} e^{-S_0(x(t_i),t_i)+S_0\left(x(t_f),t_f\right)} e^{-S_1(x(t_i),t_i)}\,.$$

Hence, we obtain

$$e^{-S_0\left(x(t_f),t_f\right)-S_1\left(x(t_f),t_f\right)} = \int \mathcal{D}x\, e^{\frac{1}{2}\int_{t_i}^{t_f} dt\ \frac{1}{2}\frac{1}{G}(\frac{dx}{dt})^2} e^{-S_0(x(t_i),t_i)-S_1(x(t_i),t_i)}\,. \tag{B.5}$$

This is of course the usual RG evolution of ψ'! Thus in writing (B.3) what really has been done is that the evolution of S_0 has been introduced into the functional integral by writing

$$\begin{aligned}\Delta S_0 = S_0\left(x(t_f),t_f\right) - S_0\left(x(t_i),t_i\right) &= \int_{t_i}^{t_f} dt\ \frac{dS_0(x(t),t)}{dt}\\ &= \int_{t_i}^{t_f} dt\ \left[\frac{\partial S_0(x(t),t)}{\partial t} + \frac{dx(t)}{dt}\frac{\partial S_0(x(t))}{\partial x(t)}\right].\end{aligned} \tag{B.6}$$

It is important to point out that (B.3) is more general than (B.4) because it does not require S_0 to satisfy the ERG equation.

Thus, in conclusion, the action governing the evolution of ψ'' is (B.3).

Appendix C. Correlation functions in double brackets

In (4) we have defined correlation functions in double brackets as

$$\begin{aligned}&\langle\!\langle \phi(p_1)\cdots\phi(p_n)\rangle\!\rangle\\ &\equiv \prod_{i=1}^n \frac{1}{K_\Lambda(p_i)}\cdot\left\langle \exp\left(-\frac{1}{2}\int_p \frac{K_\Lambda(p)\left(1-K_\Lambda(p)\right)}{p^2}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right)\phi(p_1)\cdots\phi(p_n)\right\rangle_{S_\Lambda}\,.\end{aligned} \tag{C.1}$$

Intuition for this definition partly comes from perturbation theory. Let us imagine we split the total Wilson action into two parts:

$$S_\Lambda[\phi] = S_{F,\Lambda}[\phi] + S_{I,\Lambda}[\phi]\,, \tag{C.2}$$

where

$$S_{F,\Lambda}[\phi] = -\frac{1}{2}\int_p \frac{p^2}{K_\Lambda(p)}\phi(p)\phi(-p) \tag{C.3}$$

is the action for the free massless theory. The second term contains the rest of the action. The massless free propagator given by $S_{F,\Lambda}$ is

$$\langle\phi(p)\phi(q)\rangle_{S_{F,\Lambda}} = \frac{K_\Lambda(p)}{p^2}\delta(p+q)\,. \tag{C.4}$$

Consider a connected part of the correlation functions higher than two-point. We find

$$\langle\phi(p_1)\cdots\phi(p_n)\rangle_{S_\Lambda}^{\text{connected}} = \prod_{i=1}^n K_\Lambda(p_i)\cdot(\Lambda\text{-independent})\,. \tag{C.5}$$

Each factor of a cutoff function is due to the external propagator. The Λ-dependence of the internal propagators is canceled by the Λ-dependence of $S_{I,\Lambda}$: this is how the ERG differential equation (1) is derived. Hence,

$$\langle\!\langle \phi(p_1)\cdots\phi(p_n)\rangle\!\rangle^{\text{connected}} \equiv \prod_{i=1}^{n} \frac{1}{K_\Lambda(p_i)} \cdot \langle \phi(p_1)\cdots\phi(p_n)\rangle_{S_\Lambda}^{\text{connected}} \tag{C.6}$$

is independent of Λ.

The Λ-dependence of the two-point function is a little tricky:

$$\langle \phi(p)\phi(q)\rangle_{S_\Lambda} = \frac{K_\Lambda(p)}{p^2}\delta(p+q) + K_\Lambda(p)\,(\Lambda\text{-independent})\,K_\Lambda(q)\,. \tag{C.7}$$

The multiplication by $1/(K_\Lambda(p)K_\Lambda(q))$ does not work due to the tree level propagator. We first make a subtraction at high momentum:

$$\begin{aligned}&\langle \phi(p)\phi(q)\rangle_{S_\Lambda} - \frac{K_\Lambda(p)\,(1-K_\Lambda(p))}{p^2}\delta(p+q)\\ &\quad = K_\Lambda(p)\left(\frac{1}{p^2}\delta(p+q) + \Lambda\text{-independent}\right)K_\Lambda(q)\,.\end{aligned} \tag{C.8}$$

We then obtain

$$\begin{aligned}\langle\!\langle \phi(p)\phi(q)\rangle\!\rangle &\equiv \frac{1}{K_\Lambda(p)K_\Lambda(q)}\left(\langle \phi(p)\phi(q)\rangle_{S_\Lambda} - \frac{K_\Lambda(p)\,(1-K_\Lambda(p))}{p^2}\delta(p+q)\right)\\ &= \frac{1}{p^2}\delta(p+q) + \Lambda\text{-independent}\,,\end{aligned} \tag{C.9}$$

which is independent of Λ.

Appendix D. Derivation of (35) from (37) or (38)

Assuming that $\bar{W}$ is quadratic in $\bar{J}$, we can derive (35) either from scale invariance (37) or from conformal invariance (38). For simplicity, we omit the bars from $\bar{W}$, $\bar{J}$, and dimensionless momenta in the following.

Let us consider a quadratic functional (action)

$$W[J] = \frac{1}{2}\int_p J(p)J(-p)w(p)\,, \tag{D.1}$$

where $w(p)$ is a function of p^2. Ignoring the terms independent of J, (37) and (38) reduce to

$$\begin{aligned}\int_p \Bigg[J(-p)\left(-p\cdot\partial_p + \frac{-D-2+\eta}{2}\right)\frac{\delta W}{\delta J(-p)}\\ + \frac{1}{2}(-p\cdot\partial_p + 2 - \eta)R(p)\cdot\frac{\delta W}{\delta J(-p)}\frac{\delta W}{\delta J(p)}\Bigg] = 0\,,\end{aligned} \tag{D.2a}$$

and

$$\int_p \Big[J(-p) \left(-p_\nu \frac{\partial^2}{\partial p_\mu \partial p_\nu} + \frac{1}{2} p_\mu \frac{\partial^2}{\partial p_\nu \partial p_\nu} + \frac{-D-2+\eta}{2} \frac{\partial}{\partial p_\mu} \right) (w(p) J(p)) \\ + \frac{1}{2} \int_p \big(-p \cdot \partial_p + 2 - \eta \big) R(p) \cdot \frac{\partial}{\partial p_\mu} \frac{\delta W}{\delta J(-p)} \cdot \frac{\delta W}{\delta J(p)} \Big] = 0 \,, \tag{D.2b}$$

respectively.

D.1. Scale invariance

Substituting (D.1) into (D.2a), we obtain

$$\int_p J(-p) \left(-p \cdot \partial_p + \frac{-D-2+\eta}{2} \right) \frac{\delta W}{\delta J(-p)} \\ = \frac{1}{2} \int_p J(p) J(-p) \big(-p \cdot \partial_p - 2 + \eta \big) w(p) \,, \tag{D.3a}$$

and

$$\frac{1}{2} \int_p (-p \cdot \partial_p + 2 - \eta) R(p) \cdot \frac{\delta W}{\delta J(-p)} \frac{\delta W}{\delta J(p)} \\ = \frac{1}{2} \int_p \big(-p \cdot \partial_p + 2 - \eta \big) R(p) \cdot w(p)^2 J(p) J(-p) \,. \tag{D.3b}$$

Hence, we obtain

$$\big(p \cdot \partial_p - 2 + \eta \big) \left(\frac{1}{w(p)} - R(p) \right) = 0 \,. \tag{D.4}$$

This gives

$$w(p) = \frac{1}{\text{const}\, p^{2-\eta} + R(p)} \,. \tag{D.5}$$

(35) corresponds to a particular choice of the constant.

D.2. Conformal invariance

Substituting (D.1) into (D.2b), we obtain

$$\int_p J(-p) \left(-p_\nu \frac{\partial^2}{\partial p_\mu \partial p_\nu} + \frac{1}{2} p_\mu \frac{\partial^2}{\partial p_\nu \partial p_\nu} + \frac{-D-2+\eta}{2} \frac{\partial}{\partial p_\mu} \right) (w(p) J(p)) \\ = \frac{1}{2} \int_p J(-p) \frac{\partial J(p)}{\partial p_\mu} \big(-p \cdot \partial_p - 2 + \eta \big) w(p) \,, \tag{D.6a}$$

and

$$\frac{1}{2}\int_p \left(-p\cdot\partial_p + 2 - \eta\right) R(p)\cdot \frac{\partial}{\partial p_\mu}\frac{\delta W}{\delta J(-p)}\cdot\frac{\delta W}{\delta J(p)}$$
$$= -\frac{1}{2}\int_p J(-p)\frac{\partial J(p)}{\partial p_\mu}\left(p\cdot\partial_p - 2 + \eta\right) R(p)\cdot w(p)^2 . \tag{D.6b}$$

Hence, we obtain (D.4) again.

References

[1] Gerard 't Hooft, Dimensional reduction in quantum gravity, in: Conference on Highlights of Particle and Condensed Matter Physics, SALAMFEST, Trieste, Italy, March 8–12, 1993, in: Conf. Proc., vol. C930308, 1993, pp. 284–296, arXiv:gr-qc/9310026 [gr-qc].

[2] Leonard Susskind, The world as a hologram, J. Math. Phys. 36 (1995) 6377–6396, arXiv:hep-th/9409089 [hep-th].

[3] Juan Martin Maldacena, The large N limit of superconformal field theories and supergravity, Int. J. Theor. Phys. 38 (1999) 1113–1133, Adv. Theor. Math. Phys. 2 (1998) 231, arXiv:hep-th/9711200 [hep-th].

[4] S.S. Gubser, Igor R. Klebanov, Alexander M. Polyakov, Gauge theory correlators from noncritical string theory, Phys. Lett. B 428 (1998) 105–114, arXiv:hep-th/9802109 [hep-th].

[5] Edward Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. 2 (1998) 253–291, arXiv:hep-th/9802150 [hep-th].

[6] Edward Witten, Anti-de Sitter space, thermal phase transition, and confinement in gauge theories, Adv. Theor. Math. Phys. 2 (1998) 505–532, [(1998)], arXiv:hep-th/9803131 [hep-th], 1998.

[7] Emil T. Akhmedov, A remark on the AdS/CFT correspondence and the renormalization group flow, Phys. Lett. B 442 (1998) 152–158, arXiv:hep-th/9806217 [hep-th].

[8] E.T. Akhmedov, I.B. Gahramanov, E.T. Musaev, Hints on integrability in the Wilsonian/holographic renormalization group, JETP Lett. 93 (2011) 545–550, arXiv:1006.1970 [hep-th].

[9] Enrique Alvarez, Cesar Gomez, Geometric holography, the renormalization group and the C theorem, Nucl. Phys. B 541 (1999) 441–460, arXiv:hep-th/9807226 [hep-th].

[10] L. Girardello, M. Petrini, M. Porrati, A. Zaffaroni, Novel local CFT and exact results on perturbations of $N = 4$ super Yang Mills from AdS dynamics, J. High Energy Phys. 12 (022) (1998), arXiv:hep-th/9810126 [hep-th].

[11] Jacques Distler, Frederic Zamora, Nonsupersymmetric conformal field theories from stable anti-de Sitter spaces, Adv. Theor. Math. Phys. 2 (1999) 1405–1439, arXiv:hep-th/9810206 [hep-th].

[12] D.Z. Freedman, S.S. Gubser, K. Pilch, N.P. Warner, Renormalization group flows from holography supersymmetry and a C theorem, Adv. Theor. Math. Phys. 3 (1999) 363–417, arXiv:hep-th/9904017 [hep-th].

[13] Jan de Boer, Erik P. Verlinde, Herman L. Verlinde, On the holographic renormalization group, J. High Energy Phys. 08 (2000) 003, arXiv:hep-th/9912012 [hep-th].

[14] J. de Boer, The holographic renormalization group, Fortschr. Phys. 49 (2001) 339–358, arXiv:hep-th/0101026 [hep-th].

[15] Thomas Faulkner, Hong Liu, Mukund Rangamani, Integrating out geometry: holographic Wilsonian RG and the membrane paradigm, J. High Energy Phys. 08 (2011) 051, arXiv:1010.4036 [hep-th].

[16] Igor R. Klebanov, Edward Witten, AdS/CFT correspondence and symmetry breaking, Nucl. Phys. B 556 (1999) 89–114, arXiv:hep-th/9905104 [hep-th].

[17] Idse Heemskerk, Joseph Polchinski, Holographic and Wilsonian renormalization groups, J. High Energy Phys. 06 (2011) 031, arXiv:1010.1264 [hep-th].

[18] J.M. Lizana, T.R. Morris, M. Perez-Victoria, Holographic renormalisation group flows and renormalisation from a Wilsonian perspective, J. High Energy Phys. 03 (2016) 198, arXiv:1511.04432 [hep-th].

[19] Adam Bzowski, Paul McFadden, Kostas Skenderis, Scalar 3-point functions in CFT: renormalisation, beta functions and anomalies, J. High Energy Phys. 03 (2016) 066, arXiv:1510.08442 [hep-th].

[20] Sebastian de Haro, Sergey N. Solodukhin, Kostas Skenderis, Holographic reconstruction of space-time and renormalization in the AdS/CFT correspondence, Commun. Math. Phys. 217 (2001) 595–622, arXiv:hep-th/0002230 [hep-th].

[21] B. Sathiapalan, Hidenori Sonoda, A holographic form for Wilson's RG, Nucl. Phys. B 924 (2017) 603–642, arXiv:1706.03371 [hep-th].

[22] K.G. Wilson, John B. Kogut, The renormalization group and the epsilon expansion, Phys. Rep. 12 (1974) 75–199.

[23] Franz J. Wegner, Anthony Houghton, Renormalization group equation for critical phenomena, Phys. Rev. A 8 (1973) 401–412.
[24] K.G. Wilson, The renormalization group and critical phenomena, Rev. Mod. Phys. 55 (1983) 583–600.
[25] Joseph Polchinski, Renormalization and effective Lagrangians, Nucl. Phys. B 231 (1984) 269–295.
[26] C. Becchi, On the construction of renormalized gauge theories using renormalization group techniques, arXiv: hep-th/9607188 [hep-th], 1996.
[27] C. Bagnuls, C. Bervillier, Exact renormalization group equations and the field theoretical approach to critical phenomena, in: The Exact Renormalization Group. Proceedings, 2nd Conference, Rome, Italy, September 18–22, 2000, Int. J. Mod. Phys. A 16 (2001) 1825, arXiv:hep-th/0101110 [hep-th].
[28] Yuji Igarashi, Katsumi Itoh, Hidenori Sonoda, Realization of symmetry in the ERG approach to quantum field theory, Prog. Theor. Phys. Suppl. 181 (2010) 1–166, arXiv:0909.0327 [hep-th].
[29] Oliver J. Rosten, Fundamentals of the exact renormalization group, Phys. Rep. 511 (2012) 177–272, arXiv:1003.1366 [hep-th].
[30] T.R. Morris, The exact renormalization group and approximate solutions, Int. J. Mod. Phys. A 9 (1994) 2411–2450, arXiv:hep-ph/9308265 [hep-ph].
[31] T.R. Morris, Derivative expansion of the exact renormalization group, Phys. Lett. B 329 (1994) 241–248, arXiv: hep-ph/9403340 [hep-ph].
[32] Y. Igarashi, K. Itoh, H. Sonoda, On the wave function renormalization for Wilson actions and their one particle irreducible actions, PTEP 2016 (2016) 093B04, arXiv:1607.01521 [hep-th].
[33] H. Sonoda, The generating functional of correlation functions as a high momentum limit of a Wilson action, PTEP 2017 (2017) 123B01, arXiv:1706.00198 [hep-th].
[34] Hidenori Sonoda, Conformal invariance for Wilson actions, PTEP 2017 (2017) 083B05, arXiv:1705.01239 [hep-th].
[35] Thomas L. Bell, Kenneth G. Wilson, Finite-lattice approximations to renormalization groups, Phys. Rev. B 11 (1975) 3431.
[36] Thomas L. Bell, Kenneth G. Wilson, Nonlinear renormalization groups, Phys. Rev. B 10 (1974) 3935–3944.
[37] Richard D. Ball, Peter E. Haagensen, Jose I. Latorre, Enrique Moreno, Scheme independence and the exact renormalization group, Phys. Lett. B 347 (1995) 80–88, arXiv:hep-th/9411122 [hep-th].
[38] H. Osborn, D.E. Twigg, Remarks on exact RG equations, Ann. Phys. 327 (2012) 29–73, arXiv:1108.5340 [hep-th].