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Predictors for Linear Parabolic Systems with Input Delay

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Abstract In this paper, we study the problem of stabilizing one-dimensional parabolic systems with input delay. Especially, we consider the case of "distributed control". The purpose is to derive the stabilizing controller of predictor type in an abstract space, by using a backstepping method combined with the semigroup theory. The use of the semigroup theory makes the proof of continuity of the inverse transformation easy. Also, it is shown that the abstract controller can be actually implemented by using a finite number of eigenvalues and eigenfunctions of the system operator. Finally, a numerical simulation result is presented to demonstrate our design method.

Keywords Stabilization · input delay · parabolic system · backstepping · semigroup · predictor

1 Introduction

The control problem of the system with input delay has been investigated by many researchers (see e.g. the monograph by Krstic [8] and references therein). The sphere of the control objects treated in [8] is wide, and it ranges from lumped parameter systems to distributed parameter systems, and from linear systems to nonlinear systems. In particular, when we consider the thermal control problem, a time lag exists in the feedback loop, since it is generally

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difficult to adjust heat quickly. So, from the engineering point of view, it is very important to study the control problem of parabolic systems with input delay.

In this paper, we investigate the stabilization problem of one-dimensional parabolic systems with input delay. Especially, we consider the case of "distributed control". Since the element of time lag of this type can be equivalently expressed by using a transport equation, the original system can be expressed as a cascade consisting of the transport equation and the parabolic equation with distributed input. In this paper, we apply a backstepping method of PDEs [6] to the system. Then, an approach within the existing framework of functional analysis is useful when a target system is designed. As a result, it is shown that the control law can be constructed by using the solution to a parabolic equation and the solution to a hyperbolic equation. In particular, by using the semigroup theory [11,9,4], we can give an abstract expression of the control law obtained here.

In Krstic & Smyshlyaev [5], for a finite-dimensional system with input delay described by

$$\dot{X}(t) = AX(t) + Bf(t-\tau), \quad A \in \mathbf{R}^{n\times n},\ B \in \mathbf{R}^{n\times 1},$$

where A is unstable and the pair (A, B) is controllable, the stabilizing control law was designed by a backstepping method using single integral transformation, and further an expression of predictor type such as

$$f(t) = \int_{t-\tau}^{t} Ke^{(t-\theta)A}Bf(\theta)d\theta + Ke^{\tau A}X(t), \ \ K \in \mathbf{R}^{1\times n}\ (A+BK : \text{Hurwitz})$$

was given for the control law. Also, in [7], the control law was designed by a backstepping method based on two kinds of integral transformations for the Dirichlet boundary stabilization problem of an unstable reaction-diffusion process with input delay, but an abstract expression of predictor type was not given for the control law. On the other hand, in this paper, we consider the stabilization problem under distributed control for general parabolic systems with input delay, and use a backstepping method based on single integral transformation of type introduced in [5]. The purpose is to derive the stabilizing controller of predictor type in an abstract space, by using a backstepping method combined with the semigroup theory. Also, the use of the semigroup theory makes the proof of continuity of the inverse transformation easy. Further, it is shown that the abstract controller can be actually implemented by using a finite number of eigenvalues and eigenfunctions of the system operator.

This paper is organized as follows: In Section 2, we introduce a parabolic system with input delay and formulate it in a Hilbert space. In Section 3, we set a target system and assume the form of integral transformation which maps the original system to the target system, and further determine the kernels of the integral transformation as well as the control law. In Section 4, it is shown that the inverse transformation is also continuous. In Section 5, we give a numerical simulation result, and, finally the paper is concluded in Section 6.

2 System Description and Formulation

2.1 System Description

We shall consider the following parabolic system defined on the spatial domain $(0,1)$:

$$\begin{cases} z_t(t,x) = -\mathcal{L}z(t,x) + b(x)f(t-\tau), \quad (t,x) \in (0,\infty)\times(0,1), \\ \alpha(\xi)z(t,\xi) + (1-\alpha(\xi))\frac{\partial}{\partial\nu}z(t,\xi) = 0, \quad (t,\xi) \in (0,\infty)\times\{0,1\}, \\ z(0,x) = z_0(x), \quad x \in [0,1], \\ f(\theta) = \phi(\theta), \quad \theta \in [-\tau, 0], \end{cases} \tag{1}$$

where $\mathcal{L}$ is a Sturm-Liouville operator defined by

$$(\mathcal{L}\varphi)(x) = \frac{1}{h(x)}\left(-\frac{d}{dx}\left(a(x)\frac{d\varphi(x)}{dx}\right) + c(x)\varphi(x)\right),$$

and $0 \le \alpha(\xi) \le 1$, and $\frac{\partial}{\partial\nu}$ is the exterior normal differentiation at $\xi \in \{0,1\}$, that is, $\frac{\partial}{\partial\nu}$ means -$\frac{\partial}{\partial x}$ at $x=0$ and $\frac{\partial}{\partial x}$ at $x=1$. In the above, we assume that $h(x)$, $a(x)$, and $c(x)$ are real-valued, sufficiently smooth functions defined on $[0,1]$, and that $h(x)>0$, $a(x)>0$. $f(t)$ is the control input and $\tau>0$ is a time lag. As the element of time lag can be expressed by using the transport equation, system (1) can be expressed as the cascade consisting of the transport equation and the parabolic equation with distributed input:

$$\begin{cases} z_t(t,x) = -\mathcal{L}z(t,x) + b(x)u(t,0), \quad (t,x) \in (0,\infty)\times(0,1), \\ \alpha(\xi)z(t,\xi) + (1-\alpha(\xi))\frac{\partial}{\partial\nu}z(t,\xi) = 0, \quad (t,\xi) \in (0,\infty)\times\{0,1\}, \\ z(0,x) = z_0(x), \quad x \in [0,1], \\ u_t(t,x) = \frac{1}{\tau}u_x(t,x), \quad (t,x) \in (0,\infty)\times(0,1), \\ u(t,1) = f(t), \quad t>0, \\ u(0,x) = \phi(\tau(x-1)), \quad x \in [0,1]. \end{cases} \tag{2}$$

2.2 Formulation of the System

Let $L^2_h(0,1)$ be the weighted L^2-space with inner product defined by

$$\langle \varphi, \psi \rangle_h = \int_0^1 \varphi(x)\psi(x)h(x)dx \quad \text{for } \varphi, \psi \in L^2_h(0,1),$$

and let us define the operator $A : D(A) \subset L^2_h(0,1) \to L^2_h(0,1)$ by

$$\begin{aligned} &A\varphi = \mathcal{L}\varphi, \quad \varphi \in D(A), \\ &D(A) = \{\, \varphi \in H^2(0,1)\,;\, \alpha(\xi)\varphi(\xi) + (1-\alpha(\xi))\tfrac{\partial}{\partial\nu}\varphi(\xi) = 0,\ \xi = 0,1 \,\}. \end{aligned} \tag{3}$$

Then, the operator A is closed and self-adjoint in $L_h^2(0,1)$, and it has compact resolvent and is bounded from below. Therefore, A has the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ such that

$$-\infty < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots \to \infty,$$

and the corresponding eigenfunctions $\{\varphi_i\}_{i=1}^{\infty}$ forms a complete orthonormal system in $L_h^2(0,1)$. Hence, any $\varphi \in L_h^2(0,1)$ is expressed as

$$\varphi = \sum_{i=1}^{\infty} \langle \varphi, \varphi_i \rangle_h \varphi_i.$$

Especially, in this paper, let $\lambda_1 < 0$ be assumed.

Let us formulate system (2) in the Hilbert space $L_h^2(0,1)$. First, we assume that the actuator influence function $b(x)$ is in $L_h^2(0,1)$, and define the bounded operator $B : \mathbf{R} \to L_h^2(0,1)$ by

$$Bv = bv, \quad v \in \mathbf{R}. \tag{4}$$

Then, system (2) is expressed as

$$\begin{cases} \dot{z}(t,\cdot) = -Az(t,\cdot) + Bu(t,0), \\ z(0,\cdot) = z_0, \\ u_t(t,x) = \frac{1}{\tau} u_x(t,x), \\ u(t,1) = f(t), \\ u(0,x) = \phi(\tau(x-1)). \end{cases} \tag{5}$$

Hereafter, we assume that $z_0 \in L_h^2(0,1)$, $\phi \in C_r[-\tau, 0]$ $(0 < r \leq 1)$, where $C_r[-\tau,0]$ denotes the set consisting of Hölder continuous functions with index r. Here, we note that the operator $-A$ generates an analytic semigroup $T_{-A}(t)$ on $L_h^2(0,1)$ and the concrete expression is as follows:

$$T_{-A}(t)\varphi = \sum_{i=1}^{\infty} e^{-\lambda_i t} \langle \varphi, \varphi_i \rangle_h \varphi_i, \quad \varphi \in L_h^2(0,1). \tag{6}$$

Since the growth bound of $T_{-A}(t)$ is equal to $-\lambda_1 > 0$ under the assumption, system (5), namely (2) is unstable. The purpose of this paper is to construct the control law to stabilize system (5) and to give an abstract expression of the controller.

3 Construction of Control Law

3.1 Backstepping Method

First of all, by using a function $k \in L_h^2(0,1)$, we define an operator $K : L_h^2(0,1) \to \mathbf{R}$ by $K\varphi = \langle k, \varphi \rangle_h$, $\varphi \in L_h^2(0,1)$. If the function $k \in L_h^2(0,1)$ is chosen so that the operator $-A + BK$ generates an exponentially stable analytic semigroup, it makes possible to consider the following system as a target system:

$$\begin{cases} \dot{z}(t,\cdot) = (-A+BK)z(t,\cdot) + Bw(t,0), \\ z(0,\cdot) = z_0, \\ w_t(t,x) = \frac{1}{\tau} w_x(t,x), \\ w(t,1) = 0, \\ w(0,x) = w_0(x). \end{cases} \tag{7}$$

The following theorem assures that such a choice of k is actually possible under a condition.

Theorem 1. For a given $\omega > 0$, let the integer n be chosen such that $\omega < \lambda_{n+1}$. If $\langle b, \varphi_i \rangle_h \neq 0$ for $i = 1, 2, \ldots, n$, there exists a function $k \in L^2_h(0,1)$ such that the operator $-A+BK$ generates an analytic semigroup $T_{-A+BK}(t)$ with growth bound $-\omega$, where $K\varphi = \langle k, \varphi \rangle_h$, $\varphi \in L^2_h(0,1)$. Especially, it is possible to choose the function k within the domain of A, $D(A)$.

Proof: By using the orthogonal projection operator

$$P_n \varphi = \sum_{i=1}^{n} \langle \varphi, \varphi_i \rangle_h \varphi_i, \quad \varphi \in L^2_h(0,1),$$

we decompose the state variable $z(t,\cdot)$ and the space $L^2_h(0,1)$ of system (7) as follows:

$$\begin{aligned} & z(t,\cdot) = z_1(t) + z_2(t), \\ & z_1(t) := P_n z(t,\cdot), \quad z_2(t) := (I-P_n)z(t,\cdot), \\ & L^2_h(0,1) = \underbrace{P_n L^2_h(0,1)}_{\dim = n} \oplus \underbrace{(I-P_n)L^2_h(0,1)}_{\dim=\infty}. \end{aligned}$$

Then, the operators A, B, and K are expressed as follows (see e.g. [1]):

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad K = \begin{bmatrix} K_1 \ K_2 \end{bmatrix}, \tag{8}$$

where

$$\begin{aligned} A_1 &:= P_n A P_n, & A_2 &:= (I-P_n)A(I-P_n), \\ B_1 &:= P_n B, & B_2 &:= (I-P_n)B, \\ K_1 &:= K P_n, & K_2 &:= K(I-P_n). \end{aligned}$$

In the above, the operator A_2 is unbounded, whereas all the other operators are bounded[1]. Hereafter, we identify the finite-dimensional Hilbert space $P_n L^2_h(0,1)$ with the Euclidean space $\mathbf{R}^n$ with respect to the basis $\{\varphi_1, \varphi_2, \ldots,$

[1] The projections have been widely used in the field of distributed parameter systems. For example, Byrnes *et al.* solved the output regulation problem for a class of infinite-dimensional systems [2]. Christofides and Daoutidis applied approximate inertial manifolds to the stabilization problem of semilinear distributed parameter systems [3].

$\varphi_n\}$. In this way, each element in $P_n L_h^2(0,1)$ is identified with an n-dimensional vector, and the operators A_1, B_1, and K_1 are identified with matrices with appropriate size.

Here, let us set $K_2 = 0$. Then, the operator $-A+BK$ of system (7) becomes

$$-A + BK = \begin{bmatrix} -A_1 + B_1 K_1 & 0 \\ B_2 K_1 & -A_2 \end{bmatrix}.$$

Since the pair $(-A_1, B_1)$ is controllable under the condition $\langle b, \varphi_i \rangle_h \neq 0$, $i = 1, 2, \ldots, n$, we can choose a matrix K_1 such that the matrix $-A_1 + B_1 K_1$ becomes Hurwitz (e.g. [12]). Especially, it is possible to choose the K_1 such as

$$\sigma(-A_1 + B_1 K_1) \subset \{\lambda \in \mathbf{C}; \mathrm{Re}(\lambda) < -\omega\}.$$

Here, noting that $-A_2$ generates an analytic semigroup $T_{-A_2}(t)$ with norm bound $\|T_{-A_2}(t)\| \leq e^{-\lambda_{n+1} t}$, $t \geq 0$, we see that the operator $-A + BK$ generates an analytic semigroup $T_{-A+BK}(t)$ with norm bound $\|T_{-A+BK}(t)\| \leq me^{-\omega t}$, $t \geq 0$, where m is some positive constant.

From the above discussion, by expressing the matrix K_1 as $K_1 = [k_1, \cdots, k_n]$, the function $k(x)$ can be constructed as

$$k(x) = \sum_{i=1}^{\infty} \langle k, \varphi_i \rangle_h \varphi_i(x) = \sum_{i=1}^{n} k_i \varphi_i(x). \tag{9}$$

Hence, it follows that $k \in D(A)$ since each φ_i $(1 \leq i \leq n)$ belongs to $D(A)$. □

Based on Theorem 1, we have the following stability result for system (7):

Theorem 2. Let $z_0 \in L_h^2(0,1)$ and $w_0 \in C_r[0,1]$ $(0 < r \leq 1)$. Suppose that the function $k \in D(A)$ is chosen as stated in Theorem 1. Then, system (7) is asymptotically stable in the sense of norm $(\|w(t,\cdot)\|_h^2 + \|z(t,\cdot)\|_h^2)^{\frac{1}{2}}$, where $\|\cdot\|_h$ denotes the norm of $L_h^2(0,1)$.

Proof: Note that the subsystem

$$\begin{cases} w_t(t,x) = \frac{1}{\tau} w_x(t,x), \\ w(t,1) = 0, \\ w(0,x) = w_0(x) \end{cases}$$

is super-stable in the sense of norm $\|w(t,\cdot)\|_h$, since the solution $w(t,x)$ vanishes after $t = \tau$. Also, note that $w(\cdot, 0) \in C_r[0,\tau]$ by the assumption $w_0 \in C_r[0,1]$. Therefore, the first equation of system (7) has a unique solution

$$z(t,\cdot) = T_{-A+BK}(t) z_0 + \int_0^t T_{-A+BK}(t-s) B w(s,0) ds, \quad 0 \leq t \leq \tau,$$

and

$$z(t,\cdot) = T_{-A+BK}(t-\tau) z(\tau,\cdot), \quad \tau < t.$$

Note that $z(t,\cdot) \in D(A)$ for each $t > 0$ and $z \in C([0,\infty); L_h^2(0,1))$. Then, we have the following estimate for $t\,(> \tau)$:

$$\begin{aligned}\|z(t,\cdot)\|_h &\le \|T_{-A+BK}(t-\tau)\|\|z(\tau,\cdot)\|_h \\ &\le me^{-\omega(t-\tau)}\|z(\tau,\cdot)\|_h.\end{aligned}$$

Therefore, we see that system (7) becomes asymptotically stable in the sense of norm $(\|w(t,\cdot)\|_h^2 + \|z(t,\cdot)\|_h^2)^{\frac{1}{2}}$. □

The above target system (7) is equivalent to the following system:

$$\begin{cases} z_t(t,x) = \frac{1}{h(x)}\big(\big(a(x)z_x(t,x)\big)_x - c(x)z(t,x)\big) \\ \qquad\qquad +b(x)\langle k, z(t,\cdot)\rangle_h + b(x)w(t,0), \\ \alpha(\xi)z(t,\xi) + (1-\alpha(\xi))\frac{\partial}{\partial\nu}z(t,\xi) = 0, \quad \xi = 0,1, \\ w_t(t,x) = \frac{1}{\tau}w_x(t,x), \\ w(t,1) = 0. \end{cases} \tag{10}$$

Of course, system (10) is also asymptotically stable. To map system (2) to the target system (10), we consider the following integral transformation and control law:

$$w(t,x) = u(t,x) - \int_0^x q(x,y)u(t,y)dy - \int_0^1 \gamma(x,y)z(t,y)h(y)dy, \tag{11}$$

$$f(t) = \int_0^1 q(1,y)u(t,y)dy + \int_0^1 \gamma(1,y)z(t,y)h(y)dy, \tag{12}$$

where the kernels $q(x,y)$ and $\gamma(x,y)$ are functions which should be designed.

3.2 Derivation of Kernels

First, differentiating eq. (11) with respect to x and t, we have

$$\begin{aligned} w_x(t,x) &= u_x(t,x) - q(x,x)u(t,x) - \int_0^x q_x(x,y)u(t,y)dy \\ &\quad - \int_0^1 \gamma_x(x,y)z(t,y)h(y)dy, \end{aligned} \tag{13}$$

$$\begin{aligned} w_t(t,x) &= \frac{1}{\tau}u_x(t,x) - \frac{1}{\tau}q(x,x)u(t,x) + \frac{1}{\tau}q(x,0)u(t,0) \\ &\quad + \frac{1}{\tau}\int_0^x q_y(x,y)u(t,y)dy \\ &\quad + a(1)\gamma_y(x,1)z(t,1) - a(0)\gamma_y(x,0)z(t,0) \\ &\quad - a(1)\gamma(x,1)z_y(t,1) + a(0)\gamma(x,0)z_y(t,0) \\ &\quad - \int_0^1 \big(a(y)\gamma_y(x,y)\big)_y z(t,y)dy + \int_0^1 \gamma(x,y)c(y)z(t,y)dy \\ &\quad - \int_0^1 \gamma(x,y)b(y)h(y)dy \cdot u(t,0). \end{aligned} \tag{14}$$

Here, from $w_t(t,x) - \frac{1}{\tau} w_x(t,x) = 0$, we have

$$\begin{aligned}
& w_t(t,x) - \frac{1}{\tau} w_x(t,x) \\
&= \left\{ \frac{1}{\tau} q(x,0) - \int_0^1 \gamma(x,y) b(y) h(y) dy \right\} u(t,0) \\
&\quad + \frac{1}{\tau} \int_0^x \{ q_y(x,y) + q_x(x,y) \} u(t,y) dy \\
&\quad + a(1)\gamma_y(x,1) z(t,1) - a(0)\gamma_y(x,0) z(t,0) \\
&\quad - a(1)\gamma(x,1) z_y(t,1) + a(0)\gamma(x,0) z_y(t,0) \\
&\quad + \int_0^1 \left\{ -\big(a(y)\gamma_y(x,y)\big)_y + c(y)\gamma(x,y) + \frac{1}{\tau} h(y)\gamma_x(x,y) \right\} z(t,y) dy \\
&= 0. \qquad (15)
\end{aligned}$$

In the above, let us set

$$\begin{aligned}
I = {} & a(1)\gamma_y(x,1) z(t,1) - a(0)\gamma_y(x,0) z(t,0) - a(1)\gamma(x,1) z_y(t,1) \\
& + a(0)\gamma(x,0) z_y(t,0).
\end{aligned}$$

Then, we can consider nine cases according to the values of $\alpha(0)$ and $\alpha(1)$:

- Case 1: If $0 < \alpha(0) < 1$ and $0 < \alpha(1) < 1$, then

$$\begin{aligned}
I = {} & -\frac{a(1)}{\alpha(1)} \{ \alpha(1)\gamma(x,1) + (1-\alpha(1))\gamma_y(x,1) \} z_y(t,1) \\
& + \frac{a(0)}{\alpha(0)} \{ \alpha(0)\gamma(x,0) - (1-\alpha(0))\gamma_y(x,0) \} z_y(t,0).
\end{aligned}$$

- Case 2: If $0 < \alpha(0) < 1$ and $\alpha(1) = 0$, then

$$I = a(1)\gamma_y(x,1) z(t,1) + \frac{a(0)}{\alpha(0)} \{ \alpha(0)\gamma(x,0) - (1-\alpha(0))\gamma_y(x,0) \} z_y(t,0).$$

- Case 3: If $0 < \alpha(0) < 1$ and $\alpha(1) = 1$, then

$$I = -a(1)\gamma(x,1) z_y(t,1) + \frac{a(0)}{\alpha(0)} \{ \alpha(0)\gamma(x,0) - (1-\alpha(0))\gamma_y(x,0) \} z_y(t,0).$$

- Case 4: If $\alpha(0) = 0$ and $0 < \alpha(1) < 1$, then

$$I = -\frac{a(1)}{\alpha(1)} \{ \alpha(1)\gamma(x,1) + (1-\alpha(1))\gamma_y(x,1) \} z_y(t,1) - a(0)\gamma_y(x,0) z(t,0).$$

- Case 5: If $\alpha(0) = 1$ and $0 < \alpha(1) < 1$, then

$$I = -\frac{a(1)}{\alpha(1)} \{ \alpha(1)\gamma(x,1) + (1-\alpha(1))\gamma_y(x,1) \} z_y(t,1) + a(0)\gamma(x,0) z_y(t,0).$$

- Case 6: If $\alpha(0) = \alpha(1) = 0$, then

$$I = a(1)\gamma_y(x,1) z(t,1) - a(0)\gamma_y(x,0) z(t,0).$$

- Case 7: If $\alpha(0) = \alpha(1) = 1$, then

$$I = -a(1)\gamma(x,1)z_y(t,1) + a(0)\gamma(x,0)z_y(t,0).$$

- Case 8: If $\alpha(0) = 0$ and $\alpha(1) = 1$, then

$$I = -a(0)\gamma_y(x,0)z(t,0) - a(1)\gamma(x,1)z_y(t,1).$$

- Case 9: If $\alpha(0) = 1$ and $\alpha(1) = 0$, then

$$I = a(1)\gamma_y(x,1)z(t,1) + a(0)\gamma(x,0)z_y(t,0).$$

For all cases, in order for eq. (15) to hold for all u and z, $q(x,y)$ and $\gamma(x,y)$ need to satisfy

$$\begin{cases} \gamma_x(x,y) = \frac{\tau}{h(y)}\big(\big(a(y)\gamma_y(x,y)\big)_y - c(y)\gamma(x,y)\big), \\ \alpha(0)\gamma(x,0) - (1-\alpha(0))\gamma_y(x,0) = 0, \\ \alpha(1)\gamma(x,1) + (1-\alpha(1))\gamma_y(x,1) = 0, \end{cases} \tag{16}$$

$$\begin{cases} q_x(x,y) + q_y(x,y) = 0, \\ q(x,0) = \tau\langle\gamma(x,\cdot), b\rangle_h, \end{cases} \tag{17}$$

where $0 \le \alpha(0) \le 1$, $0 \le \alpha(1) \le 1$. Since eq. (16) is of parabolic type, we need the initial condition to solve it. We can determine it in the following way. Setting $x = 0$ in the integral transformation (11), we have

$$w(t,0) = u(t,0) - \langle\gamma(0,\cdot), z(t,\cdot)\rangle_h, \tag{18}$$

and, by substituting (18) to the first equation of system (10), we have

$$\begin{aligned} z_t(t,x) &= -\mathcal{L}z(t,x) + b(x)\langle k, z(t,\cdot)\rangle_h + b(x)w(t,0) \\ &= -\mathcal{L}z(t,x) + b(x)\langle k - \gamma(0,\cdot), z(t,\cdot)\rangle_h + b(x)u(t,0). \end{aligned} \tag{19}$$

Here, comparing (19) with the corresponding equation of system (2), we obtain

$$\gamma(0,y) = k(y) \tag{20}$$

as the initial condition for eq. (16).

From these, we can solve the kernels $q(x,y)$ and $\gamma(x,y)$ of integral transformation (11) in the following steps:

(i) First, we solve the solution $\gamma(x,y)$ $(0 \le x, y \le 1)$ of parabolic equation (16) under the initial condition (20).

(ii) Next, we solve the solution $q(x,y)$ $(0 \le x \le 1, 0 \le y \le x)$ of hyperbolic equation (17) by using the solution obtained by Step (i).

3.3 Abstract Expression of Control Law

By using the operator A, we can formulate (16), (20) as follows:

$$\gamma'(x,\cdot) = -\tau A\gamma(x,\cdot), \quad \gamma(0,\cdot) = k. \tag{21}$$

Since the operator $-\tau A$ generates an analytic semigroup $T_{-\tau A}(x)$ on $L_h^2(0,1)$, the solution of (21) is written as

$$\gamma(x,\cdot) = T_{-\tau A}(x)k. \tag{22}$$

By using this expression, eq. (17) becomes

$$\begin{cases} q_x(x,y) + q_y(x,y) = 0, \\ q(x,0) = \tau\langle T_{-\tau A}(x)k, b\rangle_h. \end{cases} \tag{23}$$

Note that the solution of eq. (23) can be expressed as $q(x,y) = \nu(x-y)$. Therefore, by putting $y = 0$ in it, we have $q(x,0) = \nu(x) = \tau\langle T_{-\tau A}(x)k, b\rangle_h$, that is,

$$q(x,y) = \nu(x-y) = \tau\langle T_{-\tau A}(x-y)k, b\rangle_h. \tag{24}$$

Accordingly, using (22) and (24), the control law (12) can be expressed as follows:

$$\begin{aligned} f(t) &= \langle q(1,\cdot), u(t,\cdot)\rangle + \langle \gamma(1,\cdot), z(t,\cdot)\rangle_h \\ &= \langle \tau\langle T_{-\tau A}(1-\cdot)k, b\rangle_h, u(t,\cdot)\rangle + \langle T_{-\tau A}(1)k, z(t,\cdot)\rangle_h \\ &= \langle \tau\langle T_{-\tau A}(1-\cdot)k, b\rangle_h, f(t+\tau(\cdot-1))\rangle + \langle T_{-\tau A}(1)k, z(t,\cdot)\rangle_h \\ &= \int_0^1 \tau\langle T_{-\tau A}(1-x)k, b\rangle_h f(t+\tau(x-1))dx + \langle T_{-\tau A}(1)k, z(t,\cdot)\rangle_h, \end{aligned} \tag{25}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $L^2(0,1)$. Here, setting $t+\tau(x-1) = \theta$, it follows that

$$f(t) = \int_{t-\tau}^{t} \langle T_{-\tau A}((t-\theta)/\tau)k, b\rangle_h f(\theta)d\theta + \langle T_{-\tau A}(1)k, z(t,\cdot)\rangle_h. \tag{26}$$

Moreover, noting that

$$T_{-\tau A}(x/\tau) = T_{-A}(x), \tag{27}$$

we finally obtain the controller of predictor type as follows:

$$f(t) = \int_{t-\tau}^{t} \langle T_{-A}(t-\theta)k, b\rangle_h f(\theta)d\theta + \langle T_{-A}(\tau)k, z(t,\cdot)\rangle_h. \tag{28}$$

Remark 1. It may seem difficult to implement the control law (28) to system (2), because of the abstract expression. But, it is easy since we have constructed the function $k(x)$ such as $k = \sum_{i=1}^{n} k_i\varphi_i \in D(A)$. By using (6) and (9), we can give the gain functions of (28) as

$$\langle T_{-A}(t-\theta)k, b\rangle_h = \sum_{i=1}^{n} k_i\langle\varphi_i, b\rangle_h e^{-\lambda_i(t-\theta)},$$

$$(T_{-A}(\tau)k)(x) = \sum_{i=1}^{n} k_i e^{-\lambda_i\tau}\varphi_i(x). \tag{29}$$

In this way, we can implement the control law based on a finite number of eigenpairs $\{\lambda_i, \varphi_i\}_{i=1}^{n}$ of the operator A.

Finally, in this section, we show that, if $\phi \in C_r[-\tau, 0]$ $(0 < r \le 1)$ is satisfied in system (1), it follows that $w_0 \in C_r[0, 1]$ as assumed in Theorem 2. This is actually verified as follows: Substituting $t = 0$ to the integral transformation (11), we have

$$w_0(x) = u(0, x) - \int_0^x q(x, y)u(0, y)dy - \langle\gamma(x, \cdot), z_0\rangle_h. \tag{30}$$

In the above, it is clear that $u(0, \cdot) \in C_r[0, 1]$ since $\phi \in C_r[-\tau, 0]$. Also, since $k \in D(A)$, $\langle\gamma(x, \cdot), z_0\rangle_h$ is differentiable and its derivative is expressed as

$$\frac{d}{dx}\langle\gamma(x, \cdot), z_0\rangle_h = \frac{d}{dx}\langle T_{-\tau A}(x)k, z_0\rangle_h = \langle -\tau T_{-\tau A}(x)Ak, z_0\rangle_h,$$

i.e., $\langle\gamma(x, \cdot), z_0\rangle_h$ is in $C^1[0, 1]$. Furthermore, since $k \in D(A)$, $\int_0^x q(x, y)u(0, y)dy$ is differentiable and its derivative is expressed as

$$\begin{aligned}
&\frac{d}{dx}\int_0^x q(x, y)u(0, y)dy = q(x, x)u(0, x) + \int_0^x q_x(x, y)u(0, y)dy \\
&= \tau\langle k, b\rangle_h u(0, x) + \int_0^x \frac{\partial}{\partial x}\tau\langle T_{-\tau A}(x-y)k, b\rangle_h u(0, y)dy \\
&= \tau\langle k, b\rangle_h u(0, x) - \tau^2\int_0^x \langle T_{-\tau A}(x-y)Ak, b\rangle_h u(0, y)dy.
\end{aligned}$$

That is, $\int_0^x q(x, y)u(0, y)dy$ is in $C^1[0, 1]$. Therefore, noting that $C^1[0, 1] \subset C_1[0, 1] \subset C_r[0, 1]$ $(0 < r \le 1)$, from (30) we see that $w_0 \in C_r[0, 1]$.

Remark 2. Let d be a positive constant such that $\lambda_1 + d > 0$. Then, one may consider another target system:

$$\begin{cases}
z_t(t, x) = \frac{1}{h(x)}\big(\big(a(x)z_x(t, x)\big)_x - c(x)z(t, x)\big) - dz(t, x) + b(x)w(t, 0), \\
\alpha(\xi)z(t, \xi) + (1 - \alpha(\xi))\frac{\partial}{\partial\nu}z(t, \xi) = 0, \quad \xi = 0, 1, \\
w_t(t, x) = \frac{1}{\tau}w_x(t, x), \\
w(t, 1) = 0,
\end{cases}$$

which is a simple one. It is clear that this target system is asymptotically stable in the sense of norm $(\|w(t, \cdot)\|_h^2 + \|z(t, \cdot)\|_h^2)^{\frac{1}{2}}$. However, it seems difficult to convert the system (2) with control law (12) to this target system by using the integral transformation (11). The setting of target system such as system (7) (i.e., system (10)) is a key point of this paper.

4 Closed-Loop Stability

4.1 Inverse Integral Transformation

To assure the asymptotical stability of the closed-loop system consisting of system (2) and the control law (28), we need to show that the inverse transformation of (11) exists and that it is continuous. We assume that the inverse integral transformation from the target system (10) to system (2) is expressed as

$$u(t,x) = w(t,x) + \int_0^x p(x,y)w(t,y)dy + \int_0^1 \beta(x,y)z(t,y)h(y)dy, \qquad (31)$$

where the kernels $p(x,y)$ and $\beta(x,y)$ are functions whose existence should be shown.

4.2 Derivation of Kernels

By the similar discussion as in Subsection 3.2, we can solve the kernels $p(x,y)$, $\beta(x,y)$ of the inverse transformation (31). The concrete steps are as follows:

(i') First, we solve the solution $\beta(x,y)$ $(0 \le x, y \le 1)$ of the following parabolic equation with term $\tau k(y)\langle \beta(x,\cdot), b\rangle_h$:

$$\begin{cases} \beta_x(x,y) = \frac{\tau}{h(y)}\big(\big(a(y)\beta_y(x,y)\big)_y - c(y)\beta(x,y)\big) + \tau k(y)\langle \beta(x,\cdot), b\rangle_h, \\ \alpha(0)\beta(x,0) - (1-\alpha(0))\beta_y(x,0) = 0, \\ \alpha(1)\beta(x,1) + (1-\alpha(1))\beta_y(x,1) = 0, \\ \beta(0,y) = k(y). \end{cases} \qquad (32)$$

(ii') Next, using the solution obtained in the above step, we solve the solution $p(x,y)$ $(0 \le x \le 1, 0 \le y \le x)$ of the following hyperbolic equation:

$$\begin{cases} p_x(x,y) + p_y(x,y) = 0, \\ p(x,0) = \tau\langle \beta(x,\cdot), b\rangle_h. \end{cases} \qquad (33)$$

4.3 Abstract Expression of Inverse Transformation

First, note that the adjoint operators $B^* : L_h^2(0,1) \to \mathbf{R}$ and $K^* : \mathbf{R} \to L_h^2(0,1)$ of the bounded operators B, K defined in Subsections 2.2 and 3.1 are expressed as follows:

$$B^*\varphi = \langle b, \varphi\rangle_h, \quad \varphi \in L_h^2(0,1), \qquad (34)$$

$$K^*v = kv, \quad v \in \mathbf{R}. \qquad (35)$$

By using these adjoint operators and the self-adjoint operator A defined by (3), eq. (32) can be formulated as

$$\begin{cases} \beta'(x,\cdot) = \tau(-A + K^*B^*)\beta(x,\cdot), \\ \beta(0,\cdot) = k. \end{cases} \tag{36}$$

Since the unbounded operator $\tau(-A+K^*B^*)$ generates an analytic semigroup $T_{\tau(-A+K^*B^*)}(x)$ on $L_h^2(0,1)$, the solution of (36) is written as

$$\beta(x,\cdot) = T_{\tau(-A+K^*B^*)}(x)k. \tag{37}$$

By using this expression, eq. (33) becomes

$$\begin{cases} p_x(x,y) + p_y(x,y) = 0, \\ p(x,0) = \tau\langle T_{\tau(-A+K^*B^*)}(x)k, b\rangle_h. \end{cases} \tag{38}$$

Since the solution of (38) is expressed as $p(x,y) = \mu(x-y)$, it follows by setting $y = 0$ that

$$p(x,0) = \mu(x) = \tau\langle T_{\tau(-A+K^*B^*)}(x)k, b\rangle_h,$$

as a result,

$$\begin{aligned} p(x,y) &= \mu(x-y) \\ &= \tau\langle T_{\tau(-A+K^*B^*)}(x-y)k, b\rangle_h. \end{aligned} \tag{39}$$

Therefore, from (37) and (39), the inverse transformation (31) can be expressed as follows:

$$\begin{aligned} u(t,x) &= w(t,x) + \int_0^x p(x,y)w(t,y)dy + \langle\beta(x,\cdot), z(t,\cdot)\rangle_h \\ &= w(t,x) + \int_0^x \tau\langle T_{\tau(-A+K^*B^*)}(x-y)k, b\rangle_h w(t,y)dy \\ &\quad + \langle T_{\tau(-A+K^*B^*)}(x)k, z(t,\cdot)\rangle_h. \end{aligned} \tag{40}$$

Moreover, noting that

$$T_{\tau(-A+K^*B^*)}(x/\tau) = T_{-A+K^*B^*}(x), \tag{41}$$

we finally obtain the abstract expression

$$\begin{aligned} u(t,x) = w(t,x) &+ \int_0^x \tau\langle T_{-A+K^*B^*}(\tau(x-y))k, b\rangle_h w(t,y)dy \\ &+ \langle T_{-A+K^*B^*}(\tau x)k, z(t,\cdot)\rangle_h. \end{aligned} \tag{42}$$

Using the expression (42) and Theorems 1 and 2, we have the following stability result.

Theorem 3. Let $z_0 \in L_h^2(0,1)$ and $\phi \in C_r[-\tau, 0]$ $(0 < r \leq 1)$. Then, the closed-loop system consisting of system (2) and control law (28) is asymptotically stable in the sense of norm $(\|u(t,\cdot)\|_h^2 + \|z(t,\cdot)\|_h^2)^{\frac{1}{2}}$.

Proof: Noting that the analytic semigroup $T_{-A+BK}(x)$ is exponentially stable and further that $T_{-A+K^*B^*}(x)$ and $T_{-A+BK}(x)$ have the same growth bound, it follows that $T_{-A+K^*B^*}(x)$ is exponentially stable as well. Also, since $k \in D(A)$ by Theorem 1, it is assured that $\langle T_{-A+K^*B^*}(\tau\xi)k, b\rangle_h$ is continuous on $0 \le \xi \le 1$. Let us set

$$M_1 = \max_{\xi\in[0,1]} |\langle T_{-A+K^*B^*}(\tau\xi)k, b\rangle_h| \ (< \infty).$$

From (42), we have

$$\begin{aligned} |u(t,x)| &\le |w(t,x)| + \int_0^x \tau |\langle T_{-A+K^*B^*}(\tau(x-y))k, b\rangle_h||w(t,y)|dy \\ &\quad + |\langle T_{-A+K^*B^*}(\tau x)k, z(t,\cdot)\rangle_h| \\ &\le |w(t,x)| + \tau M_1 \int_0^1 |w(t,y)|dy \\ &\quad + \|T_{-A+K^*B^*}(\tau x)k\|_h \|z(t,\cdot)\|_h \\ &\le |w(t,x)| + \tau M_1 \|w(t,\cdot)\| + M_2\|z(t,\cdot)\|_h, \end{aligned} \tag{43}$$

where $\|\cdot\|$ denotes the usual L^2-norm, and

$$M_2 = \max_{\xi\in[0,1]} \|T_{-A+K^*B^*}(\tau\xi)k\|_h.$$

Here, squaring the both sides of (43) and integrating over $[0,1]$ with respect to x, we have

$$\|u(t,\cdot)\|^2 \le (3 + 3\tau^2 M_1^2)\|w(t,\cdot)\|^2 + 3M_2^2\|z(t,\cdot)\|_h^2. \tag{44}$$

Furthermore, noting that $\|\cdot\|$ and $\|\cdot\|_h$ are equivalent, i.e., there exist positive constants c_1 and c_2 such that

$$c_1\|v\| \le \|v\|_h \le c_2\|v\| \ \text{ for all } v \in L_h^2(0,1) = L^2(0,1), \tag{45}$$

from (44) we obtain

$$\|u(t,\cdot)\|_h \le C(\|w(t,\cdot)\|_h^2 + \|z(t,\cdot)\|_h^2)^{\frac{1}{2}}, \tag{46}$$

where C is some positive constant. This means that the inverse transformation (42), which maps from $L_h^2(0,1) \times L_h^2(0,1)$ to $L_h^2(0,1)$, is continuous. Also, it follows from (46) that

$$(\|u(t,\cdot)\|_h^2 + \|z(t,\cdot)\|_h^2)^{\frac{1}{2}} \le C'(\|w(t,\cdot)\|_h^2 + \|z(t,\cdot)\|_h^2)^{\frac{1}{2}}, \tag{47}$$

where C' is some positive constant. Based on the fact stated in Theorem 2, we can conclude the assertion of this theorem. □

5 Numerical Simulation

We consider the following transport-diffusion equation with input delay:

$$\begin{cases} z_t(t,x) = \varepsilon z_{xx}(t,x) - \tilde{\alpha} z_x(t,x) + \mu z(t,x) + b(x) f(t-\tau), \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in (0,\infty) \times (0,1), \\ z_x(t,0) = z_x(t,1) = 0, \quad t > 0, \\ z(0,x) = z_0(x), \quad x \in [0,1], \\ f(\theta) = \phi(\theta), \quad \theta \in [-\tau, 0]. \end{cases} \tag{48}$$

By defining $\mathcal{L}\varphi = -\varepsilon \dfrac{d^2\varphi}{dx^2} + \tilde{\alpha}\dfrac{d\varphi}{dx} - \mu\varphi$, system (48) is expressed as

$$\begin{cases} z_t(t,x) = -\mathcal{L} z(t,x) + b(x) f(t-\tau), \quad (t,x) \in (0,\infty) \times (0,1), \\ z_x(t,0) = z_x(t,1) = 0, \quad t > 0, \\ z(0,x) = z_0(x), \quad x \in [0,1], \\ f(\theta) = \phi(\theta), \quad \theta \in [-\tau, 0]. \end{cases} \tag{49}$$

Let us define the operator A by

$$\begin{aligned} &A\varphi = \mathcal{L}\varphi, \quad \varphi \in D(A), \\ &D(A) = \{\, \varphi \in H^2(0,1)\,;\, \varphi'(0) = \varphi'(1) = 0 \,\}. \end{aligned} \tag{50}$$

Then, A is expressed as an operator of Sturm-Liouville type as follows:

$$\begin{aligned} &(A\varphi)(x) = \frac{1}{h(x)}\left(-\frac{d}{dx}\left(a(x)\frac{d\varphi(x)}{dx}\right) + c(x)\varphi(x)\right), \\ &h(x) = e^{-\alpha x}, \quad a(x) = \varepsilon e^{-\alpha x}, \quad c(x) = -\mu e^{-\alpha x}, \end{aligned} \tag{51}$$

where $\alpha := \tilde{\alpha}/\varepsilon\,(>0)$. Therefore, A becomes self-adjoint in the weighted L^2-space $L^2_\alpha(0,1)$ whose inner product is defined by

$$\langle \varphi, \psi \rangle_\alpha = \int_0^1 \varphi(x)\psi(x) e^{-\alpha x} dx \quad \text{for } \varphi, \psi \in L^2_\alpha(0,1).$$

A has a set of eigenpairs $\{\lambda_i, \varphi_i\}_{i=1}^\infty$ in $L^2_\alpha(0,1)$ such that $\{\varphi_i\}_{i=1}^\infty$ forms a complete orthonormal system in $L^2_\alpha(0,1)$. The eigenvalues and eigenfunctions of A are given as follows [10]:

$$\begin{cases} \lambda_1 = -\mu, \quad \lambda_{i+1} = i^2\pi^2\varepsilon + \dfrac{\alpha^2}{4}\varepsilon - \mu, \\ \varphi_1(x) \equiv \sqrt{\dfrac{\alpha}{1-e^{-\alpha}}}, \quad \varphi_{i+1}(x) = \nu_i \left(e^{\frac{\alpha}{2}x}\cos i\pi x - \dfrac{\alpha}{2i\pi} e^{\frac{\alpha}{2}x} \sin i\pi x \right), \\ \nu_i := \sqrt{\dfrac{2}{1+\frac{\alpha^2}{4i^2\pi^2}}}\ (\le \sqrt{2}), \quad \text{for } i \ge 1. \end{cases} \tag{52}$$

In (49), by defining the bounded operator $B : \mathbf{R} \to L^2_\alpha(0,1)$ as

$$Bv = bv, \quad v \in \mathbf{R},$$

we have

$$\begin{cases} \dot{z}(t,\cdot) = -Az(t,\cdot) + Bf(t-\tau), \quad z(0,\cdot) = z_0, \\ f(\theta) = \phi(\theta), \quad \theta \in [-\tau, 0], \end{cases} \tag{53}$$

which is equivalent to system (5).

Here, we show the result of a numerical simulation. In system (48), we set $\varepsilon = 0.1$, $\tilde{\alpha} = 0.25$, $\alpha = \tilde{\alpha}/\varepsilon = 2.5$, $\mu = 5$, $\tau = 0.1$, and $b(x) = \frac{e^{\frac{\alpha}{2}x}}{\sqrt{0.2}} \mathbf{1}_{[0.1,0.3]}(x)$, where $\mathbf{1}_{[\cdot,\cdot]}(\cdot)$ is a characteristic function. First, we give $\omega = 3.5$ and choose an integer $n\,(n \geq 1)$ as $n = 3$. In fact, the inequality $\omega < \lambda_{n+1}$ holds with $n = 3$. Then, we can calculate the matrices A_1 and B_1 in (8) as follows:

$$A_1 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -5. & 0 & 0 \\ 0 & -3.85679 & 0 \\ 0 & 0 & -0.895908 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} \langle b, \varphi_1 \rangle_\alpha \\ \langle b, \varphi_2 \rangle_\alpha \\ \langle b, \varphi_3 \rangle_\alpha \end{bmatrix} = \begin{bmatrix} 0.576289 \\ 0.33245 \\ 0.0695239 \end{bmatrix}.$$

In this case, since the pair $(-A_1, B_1)$ is controllable, we can choose a matrix K_1 such that a set of eigenvalues of $-A_1 + B_1K_1$ is equal to $\{-3, -4, -5\}$. By using MATLAB Control System Toolbox, the matrix K_1 is solved as follows:

$$K_1 = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} = \begin{bmatrix} -266.2864 & 424.0030 & -133.1130 \end{bmatrix}.$$

Fig. 1 shows the simulation result of the closed-loop system consisting of (48) and (28). Thus, we see that the control law (28) with (29) works effectively as a stabilizing controller for the original system (48). On the other hand, the control law $f(t) = \langle k, z(t,\cdot) \rangle_\alpha$ constructed without taking into account the time lag destabilizes the system as shown in Fig. 2.

To solve the linear transport-diffusion equation (48) numerically, we used the finite difference method with mesh width $\Delta x = 0.02$, and the Runge-Kutta method of the fourth order with time step $\Delta t = 0.0001$ for its time integration. As initial conditions, we set $z_0(x) = \exp\{-50(x-0.3)^2\}$ and $\phi(\theta) \equiv 0$ for (48).

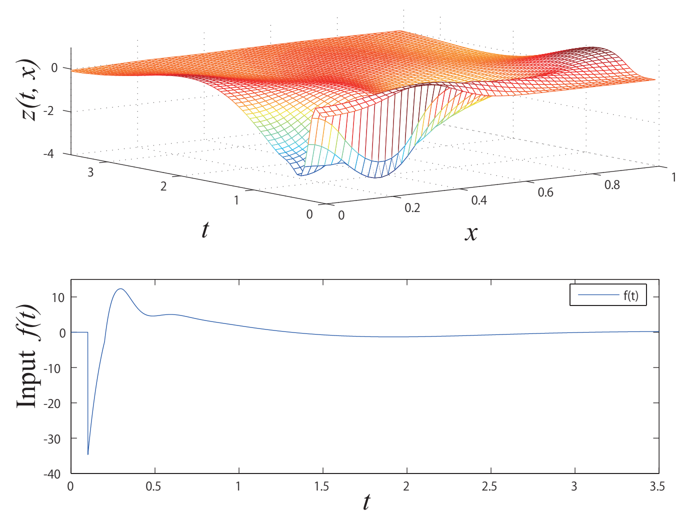

Fig. 1. The case of the proposed control law (28): Evolution of the state $z(t, x)$ and the input $f(t)$.

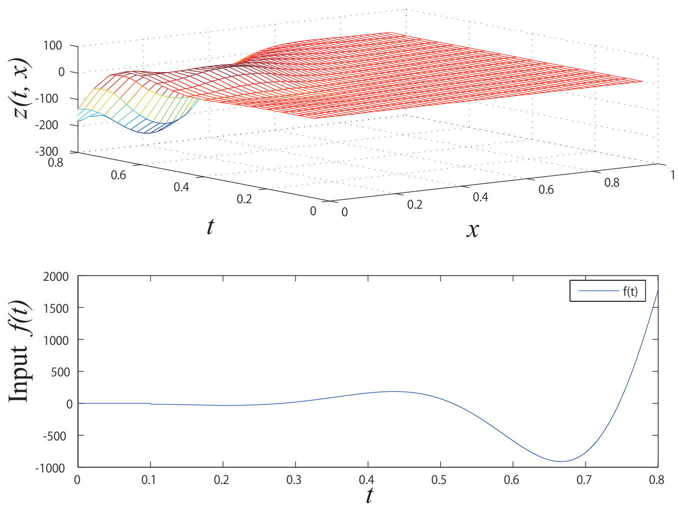

Fig. 2. The case of the control law $f(t) = \langle k, z(t, \cdot)\rangle_\alpha$: Evolution of the state $z(t, x)$ and the input $f(t)$.

6 Conclusion

In this paper, we treated the stabilization problem of unstable parabolic systems with input delay. After this system was expressed as a cascade consisting of the transport equation and the parabolic equation with distributed input, it was shown that the control law was constructed by using the solution to a parabolic equation and the solution to a hyperbolic equation, based on a backstepping method using single integral transformation. Also, by using the semigroup theory, we gave an abstract expression of the control law and further showed the continuity of the inverse transformation. Especially, the fact that the function $k(x)$ contained in the target system could be designed based on a finite number of eigenfunctions $\{\varphi_i\}_{i=1}^{n}$ of the system operator A was essential in this work. Based on this fact, we could give a concrete expression for the abstract controller. As shown in Appendix A, this work generalizes the result by Krstic & Smyshlyaev [5].

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Appendix A. Overhead view

Table A.1. Stabilizing controllers (predictors).

	ODE (Krstic & Smyshlyaev[5])	PDE with distributed input (This work)
Control object	$\dot{X} = AX + Bf(t-\tau)$ where A: an unstable matrix the pair (A, B): controllable	$z_t(t,x) = -\mathcal{L}z(t,x) + b(x)f(t-\tau)$ $\alpha(\xi)z(t,\xi) + (1-\alpha(\xi))\frac{\partial z}{\partial \nu}(t,\xi) = 0$ where $\mathcal{L}$: a Sturm-Liouville operator
Equivalent expression	$\dot{X} = AX + Bu(t,0)$ $u_t(t,x) = \frac{1}{\tau}u_x(t,x)$ $u(t,1) = f(t)$	$z_t(t,x) = -\mathcal{L}z(t,x) + b(x)u(t,0)$ $\alpha(\xi)z(t,\xi) + (1-\alpha(\xi))\frac{\partial z}{\partial \nu}(t,\xi) = 0$ $u_t(t,x) = \frac{1}{\tau}u_x(t,x)$ $u(t,1) = f(t)$
Target system	$\dot{X} = (A+BK)X + Bw(t,0)$ $w_t(t,x) = \frac{1}{\tau}w_x(t,x)$ $w(t,1) = 0$	$z_t(t,x) = -\mathcal{L}z(t,x) + b(x)\langle k, z(t,\cdot)\rangle_h$ $+b(x)w(t,0)$ $\alpha(\xi)z(t,\xi) + (1-\alpha(\xi))\frac{\partial z}{\partial \nu}(t,\xi) = 0$ $w_t(t,x) = \frac{1}{\tau}w_x(t,x)$ $w(t,1) = 0$
Integral transfor-mation	$w(t,x)$ $= u(t,x) - \int_0^x q(x,y)u(t,y)dy$ $-\gamma(x)^T X(t)$	$w(t,x)$ $= u(t,x) - \int_0^x q(x,y)u(t,y)dy$ $-\langle \gamma(x,\cdot), z(t,\cdot)\rangle_h$
Equations of kernels	$\gamma'(x) = \tau A^T \gamma(x),\ \gamma(0) = K^T$ $q_x(x,y) + q_y(x,y) = 0$ $q(x,0) = \tau\gamma(x)^T B$	$\gamma_x(x,y) = -\tau\mathcal{L}\gamma(x,y)$ $\alpha(\eta)\gamma(x,\eta) + (1-\alpha(\eta))\frac{\partial \gamma}{\partial \nu}(x,\eta) = 0$ $\gamma(0,y) = k(y)$ $q_x(x,y) + q_y(x,y) = 0$ $q(x,0) = \tau\langle \gamma(x,\cdot), b\rangle_h$
Stabilizing controller (predictor)	$f(t) = \int_{t-\tau}^t Ke^{(t-\theta)A}Bf(\theta)d\theta$ $+Ke^{\tau A}X(t)$ where e^{tA}: the state transition matrix generated by A	$f(t) = \int_{t-\tau}^t \langle T_{-A}(t-\theta)k, b\rangle_h f(\theta)d\theta$ $+\langle T_{-A}(\tau)k, z(t,\cdot)\rangle_h$ where $T_{-A}(t)$: the analytic semi-group generated by $-A$ $A\varphi = \mathcal{L}\varphi,\ \varphi \in D(A)$ $D(A) = \{\varphi \in H^2(0,1);$ $\alpha(\xi)\varphi(\xi) + (1-\alpha(\xi))\frac{\partial \varphi}{\partial \nu}(\xi) = 0\}$

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