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(Citation)
Japan Journal of Industrial and Applied Mathematics,36(2):325-355

(Issue Date)
2019-07

(Resource Type)
journal article

(Version)
Accepted Manuscript

(URL)
https://hdl.handle.net/20.500.14094/90006678

JJIAM manuscript No.
(will be inserted by the editor)

Boundary Stabilization of First-Order Hyperbolic Equations with Input Delay

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Received: date / Accepted: date

Abstract This paper is concerned with the boundary stabilization problem of first-order hyperbolic equations with input delay. In our previous work, the same problem for parabolic equations with input delay was addressed. For the hyperbolic equation, the element of time lag is similarly replaced by a transport equation and a backstepping method is employed. However, unlike in the case of the parabolic equation, it is difficult to obtain the eigenvalues and eigenfunctions of the system operator for the hyperbolic equation. Hence, we cannot construct the target system and the controller by using the finite-dimensional control theory together. In this paper, using C_0-semigroups for hyperbolic equations with nonlocal boundary condition, we show that the proposed controller is expressed by an abstract form in a Hilbert space so-called a predictor, and that the predictor makes sense under a condition. Further, for the inverse integral transformation, we obtain an interesting result on its continuity under the same condition. A numerical algorithm is also proposed for solving a non-standard hyperbolic equation appearing in our controller design.

Keywords Hyperbolic equation · boundary control · input delay · Volterra-Fredholm backstepping transformation · predictor · semigroup

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1 Introduction

In this paper, we treat the boundary stabilization problem of first-order hyperbolic partial integro-differential equations with input delay. For example, a mono-tubular heat exchange process is described by a first-order hyperbolic partial differential equation with boundary input. For the process with some special weighted functions, it has been shown that, under zero boundary condition, the system operator generates a C_0-semigroup which satisfies the spectrum determined growth condition [17], by using the result of Huang [10]. Therefore, in such a case, it is possible to conduct the stability analysis based on the distribution of spectrum of the system operator. If the system is unstable, it is important to stabilize the temperature of fluid by controlling the one at the inlet of tube. However, in general, it is a difficult task to regulate the temperature of fluid quickly. So, it is natural to assume that there exists a time lag for the boundary input. In connection with mono-tubular heat exhangers, we consider, in this paper, the following stabilization problem of first-order hyperbolic partial integro-differential equations with input delay:

Problem 1.1. For given non-negative real-valued functions $\gamma \in C[0, L]$, $g \in C^1[0, L]$, and $h \in C(Q)^1$, where $Q := \{(x, y) \in \mathbf{R}^2; 0 \le y \le x \le L\}$, design a control law f such that the closed-loop system consisting of the hyperbolic equation

$$\begin{cases} u_t(t,x) = u_x(t,x) - \gamma(x)u(t,x) + g(x)u(t,0) \\ \qquad\qquad + \displaystyle\int_0^x h(x,y)u(t,y)dy, \quad (t,x) \in (0,\infty) \times (0,L), \\ u(t,L) = f(t-\tau), \quad t > 0, \\ u(0,x) = u_0(x), \quad x \in [0,L], \\ f(\theta) = \phi(\theta), \quad \theta \in [-\tau, 0], \end{cases} \tag{1}$$

and the control law f, becomes asymptotically stable for any initial value $u_0 \in L^2(0, L)$ and $\phi \in H^1(-\tau, 0)$, where $u(t, x)$ denotes the temperature at time t and at point $x \in [0, L]$, and $\tau > 0$ a time lag.

In the case of $\tau = 0$, the problem was solved by using a backstepping method of PDEs developed by Krstic and Smyshlyaev [11]. After that, their result was extended to the boundary stabilization problem for the systems with nonlocal boundary conditions by Nakagiri [15], in which he showed that the system operator with the nonlocal boundary condition generates a C_0-semigroup and the system defines a boundary control system in the sense of Curtain and Zwart [5], and further that the design method could be extended to coupled nonsymmetric first-order hyperbolic systems. Recently, Bribiesca-Argomedo and Krstic constructed stabilizing controllers and observers for hyperbolic PDEs with Fredholm integrals [2], and further their result was extended to the output regulation problem by Xu and Dubljevic [23]. Also, Coron *et al.* proved

[1] The condition of non-negativity for γ, g, h is set for the sake of simplicity. One can consider the problem without the condition.

that, for the systems with Fredholm integrals, the stabilization in finite time is equivalent to the property of the exact controllability [4]. For linear heterodirectional hyperbolic systems without integral terms, the finite-time output regulation problem was solved by Deutscher [6], and the stabilization problem for a linear ODE having linear heterodirectional hyperbolic PDEs in the actuation path was solved by Di Meglio *et al.* [7]. Besides, many kinds of boundary control problems have been studied for linear hyperbolic systems (see e.g. [18, 22, 3, 24, 9] and the references therein).

However, as far as the authors know, the stabilizing problem has not been solved for the hyperbolic system with input delay such as system (1). As is well-known, the element of time lag can be expressed by a transport equation. As a result, system (1) is equivalently written by the coupled hyperbolic equations. In this paper, we apply a backstepping method [12] to the coupled hyperbolic equations. Furthermore, by using the semigroup theory [21, 16, 5, 8], we show that the controller obtained here is expressed by an abstract form in a Hilbert space which is called a predictor, and that the predictor makes sense under an additional condition with respect to $g(x)$. We also have an interesting result that the inverse integral transformation could be continuous under the same condition. Finally, as a way to make the controller feasible, a numerical algorithm is proposed for solving a non-standard hyperbolic equation appearing in our controller design.

On the other hand, for parabolic systems with input delay, Krstic designed the control law using a backstepping method based on two kinds of integral transformations for the Dirichlet boundary stabilization problem [13]. For the same class of distributed parameter systems, backstepping methods based on single integral transformation were developed to solve the Neumann boundary stabilization problem and the stabilization problem by distributed control in our previous studies [19, 20]. As a result, we could successfully derive the stabilizing controller of predictor type in a Hilbert space. From the viewpoint, this paper is a first-order hyperbolic version of the work [19, 20]. However, we emphasize that, unlike in the case of the parabolic equation, we cannot construct the target system and the controller in the hyperbolic case by using the finite-dimensional control theory together, because it is difficult to obtain the eigenvalues and eigenfunctions of the system operator. Also, we cannot use the fractional powers of operators in the hyperbolic case, since the system operator does not generate an analytic semigroup. The difficulty in the hyperbolic case lies in the facts.

2 Systems without Input Delay

2.1 System Description

First, we consider the first-order hyperbolic partial integro-differential equation without input delay[2]. Setting $\tau = 0$ in system (1), we have

$$\begin{cases} u_t(t,x) = u_x(t,x) - \gamma(x)u(t,x) + g(x)u(t,0) \\ \qquad\qquad + \displaystyle\int_0^x h(x,y)u(t,y)dy, \quad (t,x) \in (0,\infty)\times(0,L), \\ u(t,L) = f(t), \quad t > 0, \\ u(0,x) = u_0(x), \quad x \in [0,L], \end{cases} \tag{2}$$

where $\gamma \in C[0,L]$, $g \in C^1[0,L]$, and $h \in C(Q)$ are non-negative real-valued functions, and $Q = \{(x,y) \in \mathbf{R}^2; 0 \le y \le x \le L\}$. According to the method by Krstic and Smyshlyaev [11], one can design a stabilizing control law f, using the information of the state $u(t,x)$, $0 \le x \le L$. The purpose of this section is to review the result of Krstic and Smyshlyaev [11, Section 2] and to summarize it in the framework of the semigroup theory.

2.2 Backstepping

To apply a backstepping method to system (2), we set the target system and the integral transformation as follows:

- Target system

$$\begin{cases} \omega_t(t,x) = \omega_x(t,x) - \gamma(x)\omega(t,x), \quad (t,x) \in (0,\infty)\times(0,L), \\ \omega(t,L) = 0, \quad t > 0, \\ \omega(0,x) = \omega_0(x), \quad x \in [0,L]. \end{cases} \tag{3}$$

- Integral transformation

$$\begin{aligned} &\omega(t,x) = u(t,x) - \int_0^x k(x,y)u(t,y)dy, \\ &D(k) : 0 \le y \le x \le L, \end{aligned} \tag{4}$$

where (4) is called the Volterra transformation. Note that the target system (3) is exponentially stable with any decay rate in the sense of L^2-norm $\|\omega(t,\cdot)\|$, since the solution $\omega(t,x)$ vanishes after $t = L$, where

$$\|\omega(t,\cdot)\| = \left(\int_0^L |\omega(t,x)|^2 dx \right)^{\frac{1}{2}}.$$

[2] When $g(x)$ or $h(x,y)$ or both are large, the open-loop system becomes unstable [11].

Setting $x = L$ in (4), we have

$$\omega(t, L) = u(t, L) - \int_0^L k(L, y)u(t, y)dy,$$

that is, from the boundary condition of (3), we get

$$f(t) = u(t, L) = \int_0^L k(L, y)u(t, y)dy. \tag{5}$$

We can find the kernel $k(x, y)$ of (4) such that the system (2) with the control law (5) is converted to the target system (3) under the integral transformation (4). A set of equations which should be solved for determining the kernel $k(x, y)$ is given by

$$\begin{aligned} k_x(x, y) + k_y(x, y) = (\gamma(x) - \gamma(y))k(x, y) \\ + \int_y^x k(x, z)h(z, y)dz - h(x, y), \end{aligned} \tag{6}$$

$$k(x, 0) = \int_0^x k(x, y)g(y)dy - g(x). \tag{7}$$

Note that (6)–(7) is well-posed (see Appendix A). That is, for given non-negative real-valued functions $\gamma \in C[0, L]$, $g \in C^1[0, L]$ and $h \in C(Q)$, there exists a unique solution $k \in C^1(Q)$ to (6)–(7), where $Q = \{(x, y) \in \mathbf{R}^2; 0 \le y \le x \le L\}$. In this case, the integral transformation (4) is invertible and its inverse mapping $L^2(0, L)$ to $L^2(0, L)$ is continuous (see [11, Proof of Theorem 2] and [14]). Inspired by [11, Theorem 2 and Remark 2], we have the following result:

Theorem 2.1. Let $k \in C^1(Q)$ be the solution of (6)–(7). Then, the closed-loop system (2), (5) is exponentially stable with any decay rate in the sense of L^2-norm for all $u_0 \in L^2(0, L)$.

To prove the theorem, we need the following proposition:

Proposition 2.2. Define the unbounded operator $\tilde{A} : D(\tilde{A}) \subset L^2(0, L) \to L^2(0, L)$ by

$$\begin{aligned} &(\tilde{A}\varphi)(x) = \varphi'(x) - \gamma(x)\varphi(x) + g(x)\varphi(0), \quad \varphi \in D(\tilde{A}), \\ &D(\tilde{A}) = \{\, \varphi \in H^1(0, L)\,;\, \varphi(L) = \int_0^L k(L, y)\varphi(y)dy \,\}, \end{aligned} \tag{8}$$

where $H^1(0, L)$ is the Sobolev space of order 1. Then, the operator $\tilde{A}$ generates a C_0-semigroup $T_{\tilde{A}}(t)$ on $L^2(0, L)$.

Proof: Noting that $g \in C^1[0, L]$ and using a way similar to the proof of Proposition 3.6, it is shown that the operator $\tilde{A}$ generates a C_0-semigroup $T_{\tilde{A}}(t)$ on $L^2(0, L)$. □

Proof of Theorem 2.1: Define the bounded operator $\tilde{H} : L^2(0,L) \to L^2(0,L)$ by

$$(\tilde{H}\varphi)(x) = \int_0^x h(x,y)\varphi(y)dy, \quad \varphi \in L^2(0,L). \tag{9}$$

By using the operator $\tilde{A}$ defined by (8) and the operator $\tilde{H}$, we can formulate the closed-loop system consisting of (2) and (5) as follows:

$$\dot{u}(t,\cdot) = (\tilde{A}+\tilde{H})u(t,\cdot), \quad t>0, \quad u(0,\cdot) = u_0. \tag{10}$$

Here, we note that the operator $\tilde{A}+\tilde{H}$ generates a C_0-semigroup $T_{\tilde{A}+\tilde{H}}(t)$ on $L^2(0,L)$ by a perturbation result of semigroups [16, Theorem 3.1.1]. Accordingly, the mild solution of evolution equation (10) is given by

$$u(t,\cdot) = T_{\tilde{A}+\tilde{H}}(t)u_0, \quad t \geq 0.$$

Noting that the solution $\omega(t,x)$ of the target system (3) becomes zero after $t=L$ and that the integral transformation (4) is invertible and its inverse is continuous, we see that the C_0-semigroup $T_{\tilde{A}+\tilde{H}}(t)$ is nilpotent (see [8, pp. 35])[3]. That is, the C_0-semigroup $T_{\tilde{A}+\tilde{H}}(t)$ is exponentially stable with any decay rate, from which the assertion of the theorem follows. □

Remark 2.3. In [15, Theorem 2.1], it is directly proved that the closed-loop operator generates a C_0-semigroup. But, our method is different from his except the point that an auxiliary operator is introduced. We are decomposing the operator as $\tilde{A}+\tilde{H}$ and using two kinds of perturbation results of semigroups ([16, Theorem 3.1.1], [8, Corollary III.1.5]).

3 Systems with Input Delay

3.1 System Description

Next, we consider the first-order hyperbolic partial integro-differential equation with input delay. As is well-known, the time lag of the model can be expressed by a transport equation. That is, eq. (1) is equal to the following coupled hyperbolic equations:

$$\begin{cases} u_t(t,x) = u_x(t,x) - \gamma(x)u(t,x) + g(x)u(t,0) \\ \qquad\qquad + \displaystyle\int_0^x h(x,y)u(t,y)dy, \quad (t,x) \in (0,\infty)\times(0,L), \\ u(t,L) = v(t,0), \quad t>0, \\ u(0,x) = u_0(x), \quad x \in [0,L], \\ v_t(t,x) = v_x(t,x), \quad (t,x) \in (0,\infty)\times(0,\tau), \\ v(t,\tau) = f(t), \quad t>0, \\ v(0,x) = \phi(x-\tau), \quad x \in [0,\tau], \end{cases} \tag{11}$$

[3] In other words, system (2) is stabilizable in finite-time T (see e.g. [4]).

where $\gamma \in C[0,L]$, $g \in C^1[0,L]$ and $h \in C(Q)$ are non-negative real-valued functions, $Q = \{(x,y) \in \mathbf{R}^2; 0 \le y \le x \le L\}$, and $f \in H^1(0,T)$ for any $T > 0$. The goal is to design a stabilizing control law, using the information of the state $u(t,x)$, $0 \le x \le L$ as well as $v(t,x)$, $0 \le x \le \tau$.

Hypothesis 3.1. The function $g \in C^1[0,L]$ satisfies $g(L) = 0$.

3.2 Backstepping

Let $k(x,y)$ be the solution to (6)–(7). To apply a backstepping method to system (11), we set the target system and the integral transformation as follows:

- Target system

$$\begin{cases} u_t(t,x) = u_x(t,x) - \gamma(x)u(t,x) + g(x)u(t,0) \\ \qquad\qquad + \displaystyle\int_0^x h(x,y)u(t,y)dy, \quad (t,x) \in (0,\infty) \times (0,L), \\ u(t,L) = \displaystyle\int_0^L k(L,y)u(t,y)dy + w(t,0), \quad t > 0, \\ u(0,x) = u_0(x), \quad x \in [0,L], \\ w_t(t,x) = w_x(t,x), \quad (t,x) \in (0,\infty) \times (0,\tau), \\ w(t,\tau) = 0, \quad t > 0, \\ w(0,x) = w_0(x), \quad x \in [0,\tau]. \end{cases} \tag{12}$$

- Integral transformation

$$\begin{aligned} &w(t,x) = v(t,x) - \int_0^x q(x,y)v(t,y)dy - \int_0^L \beta(x,y)u(t,y)dy, \\ &D(q) : 0 \le y \le x \le \tau, \quad D(\beta) : 0 \le x \le \tau,\ 0 \le y \le L, \end{aligned} \tag{13}$$

where (13) is called the Volterra-Fredholm transformation. Note that the integral transformation (13) is not always invertible, unlike the Volterra transformation.

Setting $x = \tau$ in (13) and using the boundary condition of (12), we get

$$\begin{aligned} f(t) &= v(t,\tau) \\ &= \int_0^\tau q(\tau,y)v(t,y)dy + \int_0^L \beta(\tau,y)u(t,y)dy. \end{aligned} \tag{14}$$

The problem is to find the kernels $q(x,y)$ and $\beta(x,y)$ of (13) such that the system (11) with the control law (14) is converted to the target system (12) under the integral transformation (13). Here, we have the following theorem with respect to the stability of the target system (12):

Theorem 3.2. Let $u_0 \in L^2(0,L)$ and $w_0 \in H^1(0,\tau)$. Then, the target system (12) is exponentially stable with any decay rate in the sense of norm $(\|w(t,\cdot)\|^2 + \|u(t,\cdot)\|^2)^{\frac{1}{2}}$.

Proof: Note that the subsystem

$$\begin{cases} w_t(t,x) = w_x(t,x), \\ w(t,\tau) = 0, \\ w(0,x) = w_0(x) \end{cases} \tag{15}$$

is exponentially stable with any decay rate in the sense of norm $\|w(t,\cdot)\|$, since the solution $w(t,x)$ vanishes after $t=\tau$. For the first subsystem of (12), let us introduce a new variable $\vartheta(t,x) := u(t,x) - \varphi(x)w(t,0)$, where $\varphi \in C^1[0,L]$ is a solution of the integral equation

$$\int_0^L k(L,y)\varphi(y)dy + 1 = \varphi(L). \tag{16}$$

For the existence of the solution of (16), see Appendix B. Then, the first subsystem of (12) becomes

$$\begin{cases} \vartheta_t(t,x) = \vartheta_x(t,x) - \gamma(x)\vartheta(t,x) + g(x)\vartheta(t,0) + \displaystyle\int_0^x h(x,y)\vartheta(t,y)dy \\ \qquad\qquad + \left\{ \varphi'(x) - \gamma(x)\varphi(x) + g(x)\varphi(0) + \displaystyle\int_0^x h(x,y)\varphi(y)dy \right\} w(t,0) \\ \qquad\qquad -\varphi(x)w_t(t,0), \\ \vartheta(t,L) = \displaystyle\int_0^L k(L,y)\vartheta(t,y)dy, \\ \vartheta(0,x) = u_0(x) - \varphi(x)w_0(0) =: \vartheta_0(x). \end{cases} \tag{17}$$

Here, using the unbounded operator $\tilde{A}$ defined by (8), the bounded operator $\tilde{H}$ defined by (9), and defining the bounded operator $\tilde{C} : \mathbf{R}^2 \to L^2(0,L)$ by

$$\tilde{C}v := [\varphi' - \gamma\varphi + g\varphi(0) + \int_0^{\cdot} h(\cdot,y)\varphi(y)dy, -\varphi]v, \quad v \in \mathbf{R}^2,$$

(17) is formulated as

$$\dot{\vartheta}(t,\cdot) = (\tilde{A} + \tilde{H})\vartheta(t,\cdot) + \tilde{C}F(t), \quad \vartheta(0,\cdot) = \vartheta_0, \tag{18}$$

where $F(t) := [w(t,0), w_t(t,0)]^T$. Note that $\tilde{A} + \tilde{H}$ generates a C_0-semigroup $T_{\tilde{A}+\tilde{H}}(t)$ on $L^2(0,L)$ by the proof of Theorem 2.1 and $\vartheta_0 \in L^2(0,L)$ by the assumption $u_0 \in L^2(0,L)$, and that $F \in L^2(0,\tau;\mathbf{R}^2)$ by the assumption $w_0 \in H^1(0,\tau)$. Therefore, (18) has a unique mild solution

$$\vartheta(t,\cdot) = T_{\tilde{A}+\tilde{H}}(t)\vartheta_0 + \int_0^t T_{\tilde{A}+\tilde{H}}(t-s)\tilde{C}F(s)ds, \quad 0 \le t \le \tau,$$

and

$$\vartheta(t,\cdot) = T_{\tilde{A}+\tilde{H}}(t-\tau)\vartheta(\tau,\cdot), \quad \tau < t.$$

Also, note that $\vartheta \in C([0,\infty); L^2(0,L))$ (see [16, pp. 106]). Then, we have the following estimate with any decay rate $\omega > 0$ for $t\,(> \tau)$:

$$\|\vartheta(t,\cdot)\| \le \|T_{\tilde{A}+\tilde{H}}(t-\tau)\|\|\vartheta(\tau,\cdot)\| \le \text{const.}\, e^{-\omega(t-\tau)}\|\vartheta(\tau,\cdot)\|.$$

Therefore, we see that the system (18) with (15) becomes exponentially stable with any decay rate in the sense of norm $(\|w(t,\cdot)\|^2 + \|\vartheta(t,\cdot)\|^2)^{\frac{1}{2}}$. Furthermore, noting that $\vartheta(t,x) = u(t,x)$ for $t\,(> \tau)$, we see that the target system (12) becomes exponentially stable with any decay rate in the sense of norm $(\|w(t,\cdot)\|^2 + \|u(t,\cdot)\|^2)^{\frac{1}{2}}$. □

Remark 3.3. First, $w(t,x)$ vanishes at $t = \tau$, and then $u(t,x)$ vanishes at $t = \tau + L$. That is, the solution $[w(t,x), u(t,x)]^T$ of the target system (12) becomes completely zero after $t = \tau + L$.

In order to get the kernels $q(x,y)$ and $\beta(x,y)$ of (13), differentiating both sides of (13) with respect to t, using (12), and further using integration by parts, we have

$$\begin{aligned} w_t(t,x) &= v_t(t,x) - \int_0^x q(x,y)v_t(t,y)dy - \int_0^L \beta(x,y)u_t(t,y)dy \\ &= v_x(t,x) - q(x,x)v(t,x) + q(x,0)v(t,0) \\ &\quad + \int_0^x q_y(x,y)v(t,y)dy - \beta(x,L)u(t,L) + \beta(x,0)u(t,0) \\ &\quad + \int_0^L \{\beta_y(x,y) + \beta(x,y)\gamma(y)\}u(t,y)dy \\ &\quad - \int_0^L \beta(x,y)g(y)dy \cdot u(t,0) \\ &\quad - \int_0^L \left(\int_y^L \beta(x,z)h(z,y)dz\right)u(t,y)dy. \end{aligned} \tag{19}$$

On the other hand, differentiating both sides of (13) with respect to x, we have

$$\begin{aligned} w_x(t,x) = v_x(t,x) - q(x,x)v(t,x) - \int_0^x q_x(x,y)v(t,y)dy \\ - \int_0^L \beta_x(x,y)u(t,y)dy. \end{aligned} \tag{20}$$

Here, by using (19) and (20), we have

$$\begin{aligned} w_t(t,x) - w_x(t,x) = \{q(x,0) - \beta(x,L)\}v(t,0) \\ + \int_0^x \{q_x(x,y) + q_y(x,y)\}v(t,y)dy \\ + \int_0^L \Big\{\beta_x(x,y) + \beta_y(x,y) + \beta(x,y)\gamma(y) \end{aligned}$$

$$-\int_y^L \beta(x,z)h(z,y)dz\Big\}u(t,y)dy$$
$$+\Big\{\beta(x,0)-\int_0^L \beta(x,y)g(y)dy\Big\}u(t,0). \tag{21}$$

In the above, if there hold

$$\begin{aligned}
&\beta_x(x,y) = -\beta_y(x,y) - \gamma(y)\beta(x,y) + \int_y^L \beta(x,z)h(z,y)dz,\\
&\beta(x,0) = \int_0^L \beta(x,y)g(y)dy,\\
&q_x(x,y) + q_y(x,y) = 0,\\
&q(x,0) = \beta(x,L),
\end{aligned}$$

then, (21) becomes

$$w_t(t,x) - w_x(t,x) = 0.$$

Setting $x = 0$ in (13), we have

$$w(t,0) = v(t,0) - \int_0^L \beta(0,y)u(t,y)dy.$$

On the other hand, from the boundary condition of (12), we have

$$\begin{aligned}
u(t,L) &= \int_0^L k(L,y)u(t,y)dy + w(t,0)\\
&= \int_0^L \{k(L,y) - \beta(0,y)\}u(t,y)dy + v(t,0).
\end{aligned} \tag{22}$$

Comparing the boundary condition of (11) with (22), we have

$$\beta(0,y) = k(L,y).$$

Thus, we obtain the following procedure to compute the kernels $q(x,y)$ and $\beta(x,y)$:

Step 1. For a given function $g(x)$ satisfying Hypothesis 3.1, find the solution $k(x,y)$ to (6)–(7).

Step 2. Using $k(L,y)$, find the solution $\beta(x,y)$ to the following hyperbolic equation:

$$\beta_x(x,y) = -\beta_y(x,y) - \gamma(y)\beta(x,y) + \int_y^L \beta(x,z)h(z,y)dz, \tag{23}$$
$$\beta(x,0) = \int_0^L \beta(x,y)g(y)dy, \tag{24}$$
$$\beta(0,y) = k(L,y). \tag{25}$$

Step 3. Using $\beta(x, L)$, find the solution $q(x, y)$ to the following hyperbolic equation:

$$q_x(x,y) + q_y(x,y) = 0, \tag{26}$$
$$q(x,0) = \beta(x,L). \tag{27}$$

3.3 Predictor

In this subsection, we give an abstract expression of the control law (14). We first formulate (23)–(25) in a Hilbert space $L^2(0, L)$ with inner product defined by

$$\langle \varphi, \psi \rangle = \int_0^L \varphi(y)\overline{\psi(y)}dy, \quad \varphi, \psi \in L^2(0,L).$$

We here define the unbounded operator $A : D(A) \subset L^2(0, L) \to L^2(0, L)$ as

$$\begin{aligned}
&(A\varphi)(y) = -\varphi'(y) - \gamma(y)\varphi(y), \quad \varphi \in D(A), \\
&D(A) = \{\, \varphi \in H^1(0,L)\,;\, \varphi(0) = \int_0^L \varphi(y)g(y)dy\,\}.
\end{aligned} \tag{28}$$

Also, define the bounded operator $H : L^2(0, L) \to L^2(0, L)$ as

$$(H\varphi)(y) := \int_y^L \varphi(z)h(z,y)dz, \quad \varphi \in L^2(0,L). \tag{29}$$

In the above, $\gamma \in C[0, L]$, $g \in C^1[0, L]$, and $h \in C(Q)$, $Q = \{(x, y) \in \mathbf{R}^2; 0 \le y \le x \le L\}$ are non-negative real-valued functions, and g satisfies Hypothesis 3.1. By using the operators A and H, we can formulate (23)–(25) as follows:

$$\beta'(x,\cdot) = (A+H)\beta(x,\cdot), \quad \beta(0,\cdot) = k(L,\cdot). \tag{30}$$

Then, we obtain the following result:

Proposition 3.4. The operator $A : D(A) \subset L^2(0, L) \to L^2(0, L)$ defined by (28) generates a C_0-semigroup $T_A(x)$ on $L^2(0, L)$.

One can show this proposition by using the perturbation theorem of Desch-Schappacher, see [8, Example III.3.5]. In this paper, we provide a more elementary proof based on the following lemma:

Lemma 3.5. Consider the operator $C : D(C) \subset L^2(0, L) \to L^2(0, L)$ defined by

$$\begin{aligned}
&(C\varphi)(y) = \varphi'(y) + g(y)\varphi(0), \quad \varphi \in D(C), \\
&D(C) = \{\, \varphi \in H^1(0,L)\,;\, \varphi(L) = 0\,\},
\end{aligned} \tag{31}$$

where g is the function appearing in (28). Then, the operator C is densely defined and closed. In addition, the adjoint operator of C, $C^* : D(C^*) \subset L^2(0,L) \to L^2(0,L)$ is given by

$$(C^*\varphi)(y) = -\varphi'(y), \quad \varphi \in D(C^*),$$
$$D(C^*) = \{\, \varphi \in H^1(0,L)\,;\, \varphi(0) = \int_0^L g(y)\varphi(y)dy \,\}. \tag{32}$$

Proof: It is clear that $D(C)$ is dense in $L^2(0,L)$. So, we prove that the operator C is closed. In order to apply [5, Theorem A.3.46], we first show that there exists a number $\lambda > 0$ such that the range of $\lambda I - C$ is equal to $L^2(0,L)$, i.e.,

$$R(\lambda I - C) = L^2(0,L). \tag{33}$$

Let λ be a positive number. To prove (33), we need to show that, for any fixed $\sigma \in L^2(0,L)$, there exists a $\rho \in D(C)$ such that

$$(\lambda I - C)\rho = \sigma. \tag{34}$$

Eq. (34) is equivalent to the following equation:

$$\rho'(x) - \lambda\rho(x) + g(x)\rho(0) = -\sigma(x), \quad \rho(L) = 0. \tag{35}$$

From (35), we have

$$\rho(x) = \frac{\int_0^L e^{-\lambda y}\sigma(y)dy}{1-\nu(\lambda)}\left(e^{\lambda x} - \int_0^x e^{\lambda(x-y)}g(y)dy\right) - \int_0^x e^{\lambda(x-y)}\sigma(y)dy. \tag{36}$$

Note that $\nu(\lambda) := \int_0^L e^{-\lambda y}g(y)dy$ takes non-negative values because of $g(x) \geq 0$. In addition, we see that $\nu(\lambda) < 1$ as $\lambda \to \infty$, since

$$\nu(\lambda) \leq \max_{y\in[0,L]} g(y)\frac{1}{\lambda} \to 0 \quad \text{as } \lambda \to \infty. \tag{37}$$

Therefore, there always exists a solution $\rho \in D(C)$ to (34) for a sufficiently large positive number λ.

Also, with the number λ, we can express the inverse of $\lambda I - C$ as follows:

$$((\lambda I - C)^{-1}\sigma)(x) = \frac{\int_0^L e^{-\lambda y}\sigma(y)dy}{1-\nu(\lambda)}\left(e^{\lambda x} - \int_0^x e^{\lambda(x-y)}g(y)dy\right)$$
$$- \int_0^x e^{\lambda(x-y)}\sigma(y)dy, \quad \sigma \in L^2(0,L). \tag{38}$$

It is not difficult to see that the operator $(\lambda I - C)^{-1}$ is bounded from $L^2(0,L)$ to $L^2(0,L)$. Hence, by [5, Theorem A.3.46], we see that the operator $\lambda I - C$ is closed. Moreover, noting that the identity operator I is closed since it is a bounded operator, it follows that the operator C is closed.

Next, let us consider the adjoint operator of C. Using the definition of the inner product of $L^2(0,L)$, the adjoint of $(\lambda I - C)^{-1}$ can be calculated as follows:

$$
\begin{aligned}
&((\lambda I - C^*)^{-1}\sigma)(y) \\
&= \frac{e^{-\lambda y}}{1-\nu(\lambda)}\left\{\int_0^L e^{\lambda x}\sigma(x)dx - \int_0^L g(z)\left(\int_z^L e^{\lambda(x-z)}\sigma(x)dx\right)dz\right\} \\
&\quad - \int_y^L e^{\lambda(x-y)}\sigma(x)dx, \quad \sigma \in L^2(0,L),
\end{aligned} \tag{39}
$$

where $\nu(\lambda) = \int_0^L e^{-\lambda y} g(y)dy$.

Now, let us define the operator $R_\lambda : D(R_\lambda) \subset L^2(0,L) \to L^2(0,L)$ by

$$
\begin{aligned}
&(R_\lambda\varphi)(y) = \lambda\varphi(y) + \varphi'(y), \quad \varphi \in D(R_\lambda), \\
&D(R_\lambda) = \{\, \varphi \in H^1(0,L)\,;\, \varphi(0) = \int_0^L g(y)\varphi(y)dy \,\}.
\end{aligned} \tag{40}
$$

We show that the following relation holds when the number $\lambda > 0$ is sufficiently large:

$$
R((\lambda I - C^*)^{-1}) = D(R_\lambda). \tag{41}
$$

Let $\psi \in R((\lambda I - C^*)^{-1})$. Then, there exists a $\sigma \in L^2(0,L)$ such that

$$
\begin{aligned}
(1-\nu(\lambda))\psi(y) &= e^{-\lambda y}\left\{\int_0^L e^{\lambda x}\sigma(x)dx - \int_0^L g(z)\left(\int_z^L e^{\lambda(x-z)}\sigma(x)dx\right)dz\right\} \\
&\quad -(1-\nu(\lambda))\int_y^L e^{\lambda(x-y)}\sigma(x)dx,
\end{aligned} \tag{42}
$$

where $\nu(\lambda) = \int_0^L e^{-\lambda y} g(y)dy$. Note that $\psi \in H^1(0,L)$. Setting $y = 0$ in (42), we have

$$
\begin{aligned}
(1-\nu(\lambda))\psi(0) &= \int_0^L e^{\lambda x}\sigma(x)dx - \int_0^L g(z)\left(\int_z^L e^{\lambda(x-z)}\sigma(x)dx\right)dz \\
&\quad -(1-\nu(\lambda))\int_0^L e^{\lambda x}\sigma(x)dx \\
&= \nu(\lambda)\int_0^L e^{\lambda x}\sigma(x)dx - \int_0^L g(z)\left(\int_z^L e^{\lambda(x-z)}\sigma(x)dx\right)dz.
\end{aligned} \tag{43}
$$

On the other hand, multiplying both sides of (42) by $g(y)$ and integrating from 0 to L with respect to y, we have

$$
\begin{aligned}
(1-\nu(\lambda))\int_0^L g(y)\psi(y)dy &= \nu(\lambda)\int_0^L e^{\lambda x}\sigma(x)dx \\
&\quad - \int_0^L g(z)\left(\int_z^L e^{\lambda(x-z)}\sigma(x)dx\right)dz.
\end{aligned} \tag{44}
$$

From (43)–(44), it follows that $\psi(0) = \int_0^L g(y)\psi(y)dy$ since $1-\nu(\lambda) > 0$, which implies that $\psi \in D(R_\lambda)$. Thus, we have $R((\lambda I - C^*)^{-1}) \subset D(R_\lambda)$.

Next, in order to show the inverse implication, let $\psi \in D(R_\lambda)$. Then, ψ satisfies $\psi \in H^1(0, L)$ and $\psi(0) = \int_0^L g(y)\psi(y)dy$. We need to find a $\sigma \in L^2(0, L)$ such that

$$\begin{aligned} \psi(y) = \frac{e^{-\lambda y}}{1-\nu(\lambda)} \left\{ \int_0^L e^{\lambda x}\sigma(x)dx - \int_0^L g(z) \left(\int_z^L e^{\lambda(x-z)}\sigma(x)dx \right) dz \right\} \\ - \int_y^L e^{\lambda(x-y)}\sigma(x)dx, \end{aligned} \tag{45}$$

where $\nu(\lambda) = \int_0^L e^{-\lambda y} g(y)dy$. Here, multiplying both sides of (45) by $g(y)$ and integrating from 0 to L with respect to y, we have

$$\int_0^L g(z) \left(\int_z^L e^{\lambda(x-z)}\sigma(x)dx \right) dz = \nu(\lambda) \int_0^L e^{\lambda x}\sigma(x)dx - (1-\nu(\lambda))\psi(0). \tag{46}$$

In the above, we used $\psi(0) = \int_0^L \psi(y)g(y)dy$. Substituting (46) into (45), we have

$$\begin{aligned} \psi(y) = \frac{e^{-\lambda y}}{1-\nu(\lambda)} \left\{ \int_0^L e^{\lambda x}\sigma(x)dx - \nu(\lambda) \int_0^L e^{\lambda x}\sigma(x)dx + (1-\nu(\lambda))\psi(0) \right\} \\ - \int_y^L e^{\lambda(x-y)}\sigma(x)dx, \end{aligned}$$

which leads to

$$\int_0^y e^{\lambda x}\sigma(x)dx = e^{\lambda y}\psi(y) - \psi(0). \tag{47}$$

Differentiating both sides of (47) with respect to y yields $\sigma(y) = \lambda\psi(y) + \psi'(y)$ $(\in L^2(0, L))$. Indeed, we can verify that the σ satisfies (45). Therefore, $\psi \in R((\lambda I - C^*)^{-1})$ follows, as the result, we have $D(R_\lambda) \subset R((\lambda I - C^*)^{-1})$. In this way, we see that (41) holds for the number $\lambda > 0$ sufficiently large.

Since (41) holds, by direct calculation, it follows from (39)–(40) that

$$((\lambda I - C^*)^{-1} R_\lambda \varphi)(y) = \varphi(y) \quad \text{for all } \varphi \in D(R_\lambda), \tag{48}$$

$$(R_\lambda (\lambda I - C^*)^{-1} \varphi)(y) = \varphi(y) \quad \text{for all } \varphi \in L^2(0, L), \tag{49}$$

which implies that $R_\lambda = \lambda I - C^*$ (see e.g. [5, Definition A.3.5], [8, Exercises IV.1.21]). Hence, C^* is given by (32). □

Proof of Proposition 3.4: In order to prove that the domain of A, $D(A)$ is dense in $L^2(0, L)$, we use an auxiliary operator $C : D(C) \subset L^2(0, L) \to L^2(0, L)$ introduced in Lemma 3.5. Noting that $D(C^*)$ is dense in $L^2(0, L)$ since $D(C)$ is densely defined [1, Theorem III.21], it follows that $D(A)$ is dense in $L^2(0, L)$ because of $D(C^*) = D(A)$.

Next, for the operator A defined by (28), we estimate $\mathrm{Re}\langle A\varphi, \varphi\rangle$ using a method similar to that of [17]. By the definitions of the inner product of $L^2(0,L)$ and the operator A, we see that, for all $\varphi \in D(A)$,

$$2\mathrm{Re}\langle A\varphi, \varphi\rangle \le |\varphi(0)|^2 - 2\int_0^L \gamma(y)|\varphi(y)|^2 dy \tag{50}$$

holds. Here, noting that $|\varphi(0)| \le \|\varphi\|\|g\|$, it follows from (50) that

$$\begin{aligned} 2\mathrm{Re}\langle A\varphi, \varphi\rangle &\le \|\varphi\|^2\|g\|^2 - 2\min_{y\in[0,L]}\gamma(y)\int_0^L |\varphi(y)|^2 dy \\ &\le \{\sqrt{L}\|g\|\max_{y\in[0,L]} g(y) - 2\min_{y\in[0,L]}\gamma(y)\}\|\varphi\|^2, \end{aligned}$$

which implies that

$$\mathrm{Re}\langle A\varphi, \varphi\rangle \le \omega\|\varphi\|^2, \quad \varphi \in D(A), \tag{51}$$

where

$$\omega := \frac{1}{2}\sqrt{L}\|g\|\max_{y\in[0,L]} g(y) - \min_{y\in[0,L]}\gamma(y).$$

Finally, we show that there exists a number $\lambda > 0$ such that the range of $\lambda I - A$ is equal to $L^2(0,L)$, i.e.,

$$R(\lambda I - A) = L^2(0,L). \tag{52}$$

Let λ be a positive number. To prove (52), we need to show that, for any fixed $\sigma \in L^2(0,L)$, there exists a $\rho \in D(A)$ such that

$$(\lambda I - A)\rho = \sigma. \tag{53}$$

Note that $D(A) = \{\, \varphi \in H^1(0,L)\,;\, \varphi(0) = \int_0^L \varphi(y)g(y)dy \,\}$. Eq. (53) is equivalent to the following equations:

$$\rho'(x) + \gamma_\lambda(x)\rho(x) = \sigma(x), \tag{54}$$

$$\rho(0) = \int_0^L \rho(z)g(z)dz, \tag{55}$$

where $\gamma_\lambda(x) := \gamma(x) + \lambda$. From (54), we have

$$\rho(x) = e^{-\Gamma_\lambda(x)}\rho(0) + e^{-\Gamma_\lambda(x)}\int_0^x e^{\Gamma_\lambda(y)}\sigma(y)dy, \tag{56}$$

where $\Gamma_\lambda(x) := \int_0^x \gamma_\lambda(\xi)d\xi$. Here, substituting (56) into (55) yields

$$(1-\mu(\lambda))\rho(0) = \int_0^L e^{-\Gamma_\lambda(z)}\left(\int_0^z e^{\Gamma_\lambda(y)}\sigma(y)dy\right)g(z)dz. \tag{57}$$

Note that $\mu(\lambda) := \int_0^L e^{-\Gamma_\lambda(z)} g(z)dz$ becomes non-negative because of $g(x) \geq 0$. Also, we see that $\mu(\lambda) \to 0$ as $\lambda \to \infty$, since

$$\begin{aligned}\mu(\lambda) &= \int_0^L e^{-\Gamma_\lambda(z)} g(z)dz = \int_0^L e^{-\int_0^z \gamma(\xi)d\xi} e^{-\lambda z} g(z)dz \\ &\leq \int_0^L e^{-\lambda z} g(z)dz \leq \max_{z\in[0,L]} g(z)\frac{1}{\lambda} \to 0 \quad \text{as } \lambda \to \infty. \qquad (58)\end{aligned}$$

In the above, we used $e^{-\int_0^z \gamma(\xi)d\xi} \leq 1$. Therefore, there exists a number $\lambda_0 > 0$ such that $0 \leq \mu(\lambda) < 1$ for all $\lambda > \lambda_0$. Thus, when λ is chosen such that $\lambda > \lambda_0$, it follows from (56)–(57) that

$$\begin{aligned}\rho(x) &= \frac{e^{-\Gamma_\lambda(x)}}{1-\mu(\lambda)} \int_0^L e^{-\Gamma_\lambda(z)} \left(\int_0^z e^{\Gamma_\lambda(y)} \sigma(y)dy \right) g(z)dz \\ &\quad + e^{-\Gamma_\lambda(x)} \int_0^x e^{\Gamma_\lambda(y)} \sigma(y)dy, \qquad (59)\end{aligned}$$

which gives the solution in $D(A)$ to (53). Here, we remark that the operator A can be shown to be closed, using a method similar to the proof of Lemma 3.5.

Since A is a densely defined closed linear operator as shown in the above, it follows from the fact together with (51)–(52) that the operator $A - \omega I$ generates a contractive C_0-semigroup $T_{A-\omega I}(x)$ on $L^2(0, L)$, i.e., the operator A generates a C_0-semigroup $T_A(x)$ with norm bound $\|T_A(x)\| \leq e^{\omega x}$ on $L^2(0, L)$, by using Lumer-Phillips' Theorem [16, Theorem 1.4.3] (see also the proof of [5, Corollary 2.2.3]). □

As shown above, since the operator A generates a C_0-semigroup $T_A(x)$ on $L^2(0, L)$, the operator $A+H$, where H is defined by (29), does a C_0-semigroup $T_{A+H}(x)$ on $L^2(0, L)$. Therefore, the solution of (30) is written as

$$\beta(x, \cdot) = T_{A+H}(x)k(L, \cdot). \qquad (60)$$

By using this expression, (26)–(27) becomes

$$q_x(x, y) + q_y(x, y) = 0, \qquad (61)$$

$$q(x, 0) = (T_{A+H}(x)k(L, \cdot))(L). \qquad (62)$$

Note that the solution of (61)–(62) can be expressed as $q(x, y) = \phi(x - y)$. Hence, by putting $y = 0$ in it, we have $q(x, 0) = \phi(x) = (T_{A+H}(x)k(L, \cdot))(L)$, that is,

$$q(x, y) = \phi(x - y) = (T_{A+H}(x - y)k(L, \cdot))(L). \qquad (63)$$

Accordingly, using (60) and (63), the control law (14) can be expressed as follows:

$$f(t) = \int_0^\tau q(\tau, y)v(t, y)dy + \langle \beta(\tau, \cdot), u(t, \cdot)\rangle$$

$$
\begin{aligned}
&= \int_0^\tau (T_{A+H}(\tau - y)k(L,\cdot))(L)v(t,y)dy + \langle T_{A+H}(\tau)k(L,\cdot), u(t,\cdot)\rangle \\
&= \int_0^\tau (T_{A+H}(\tau - y)k(L,\cdot))(L)f(t+y-\tau)dy \\
&\quad + \langle T_{A+H}(\tau)k(L,\cdot), u(t,\cdot)\rangle.
\end{aligned}
$$

Further, setting $t + y - \tau = \theta$, we obtain

$$
\begin{aligned}
f(t) = &\int_{t-\tau}^t (T_{A+H}(t-\theta)k(L,\cdot))(L)f(\theta)d\theta \\
&+ \langle T_{A+H}(\tau)k(L,\cdot), u(t,\cdot)\rangle.
\end{aligned} \tag{64}
$$

This is the controller of predictor type. Here, from Hypothesis 3.1 and (7), we see that there holds

$$
k(L,0) = \int_0^L k(L,y)g(y)dy, \quad k(L,\cdot) \in C^1[0,L] \subset H^1(0,L),
$$

which implies that $k(L,\cdot) \in D(A)$. Therefore, under Hypothesis 3.1, the controller (64) (i.e., (14)) makes sense, since $T_{A+H}(t-\theta)k(L,\cdot) \in D(A)$ for $\theta \in [t-\tau, t]$. See [5, Theorem 2.1.10].

3.4 Inverse Integral Transformation

To assure the asymptotical stability of the closed-loop system consisting of system (11) and the control law (64), we need to show that the inverse transformation of (13) exists and that it is continuous. We assume that the inverse integral transformation from the target system (12) to the system (11) with the control law (64) is expressed by

$$
\begin{aligned}
&v(t,x) = w(t,x) + \int_0^x p(x,y)w(t,y)dy + \int_0^L \alpha(x,y)u(t,y)dy, \\
&D(p): 0 \le y \le x \le \tau, \quad D(\alpha): 0 \le x \le \tau,\ 0 \le y \le L,
\end{aligned} \tag{65}
$$

where the kernels $p(x,y)$ and $\alpha(x,y)$ are functions whose existence should be shown. By a discussion similar to that of Subsection 3.2, we can find the kernels $p(x,y)$, $\alpha(x,y)$ of the inverse transformation (65). The concrete steps are as follows:

Step I. Find the solution $\alpha(x,y)$ to the following hyperbolic equation:

$$
\begin{aligned}
\alpha_x(x,y) = &-\alpha_y(x,y) - \gamma(y)\alpha(x,y) + k(L,y)\alpha(x,L) \\
&+ \int_y^L \alpha(x,z)h(z,y)dz,
\end{aligned} \tag{66}
$$

$$
\alpha(x,0) = \int_0^L \alpha(x,y)g(y)dy, \tag{67}
$$

$$
\alpha(0,y) = k(L,y). \tag{68}
$$

Step II. Using $\alpha(x, L)$, find the solution $p(x, y)$ to the following hyperbolic equation:

$$p_x(x, y) + p_y(x, y) = 0, \tag{69}$$

$$p(x, 0) = \alpha(x, L). \tag{70}$$

Finally, we give an abstract expression of the inverse transformation (65). We formulate (66)–(68) in a Hilbert space $L^2(0, L)$. Define the unbounded operator $B : D(B) \subset L^2(0, L) \to L^2(0, L)$ as

$$\begin{aligned}(B\varphi)(y) &= -\varphi'(y) - \gamma(y)\varphi(y) + k(L, y)\varphi(L), \quad \varphi \in D(B),\\ D(B) &= \{ \varphi \in H^1(0, L)\, ;\, \varphi(0) = \int_0^L \varphi(y)g(y)dy \}.\end{aligned} \tag{71}$$

In the above, $\gamma \in C[0, L]$ and $g \in C^1[0, L]$ are non-negative real-valued functions, and g satisfies Hypothesis 3.1. The function k is the solution of (6)–(7). Especially, note that $D(B) = D(A)$. By using the operator B and the bounded operator $H : L^2(0, L) \to L^2(0, L)$ defined by (29), we can formulate (66)–(68) as follows:

$$\alpha'(x, \cdot) = (B + H)\alpha(x, \cdot), \quad \alpha(0, \cdot) = k(L, \cdot). \tag{72}$$

Then, we obtain the following result:

Proposition 3.6. The operator $B : D(B) \subset L^2(0, L) \to L^2(0, L)$ defined by (71) generates a C_0-semigroup $T_B(x)$ on $L^2(0, L)$.

Proof: First, decompose B as $B = A + E$, where $D(A) = D(E)\,(= D(B))$, and

$$\begin{aligned}(A\varphi)(y) &= -\varphi'(y) - \gamma(y)\varphi(y), \quad \varphi \in D(A),\\ (E\varphi)(y) &= k(L, y)\varphi(L), \quad \varphi \in D(E) = D(A).\end{aligned}$$

In Proposition 3.4, it has been shown that the operator A generates a C_0-semigroup on $L^2(0, L)$.

Next, we show that the operator E is bounded on $(D(A), \|\cdot\|_1)$, where $\|\psi\|_1 := \|(\lambda I - A)\psi\|$ (λ is a positive constant larger than ω appearing in (51)). For any fixed $\varphi \in D(E) = D(A)$, we express $(\lambda I - A)E\varphi$ as follows:

$$(\lambda I - A)E\varphi = (k'(L, \cdot) + (\gamma(\cdot) + \lambda)k(L, \cdot))\varphi(L), \tag{73}$$

where the φ satisfies $\varphi(0) = \int_0^L \varphi(y)g(y)dy$. Then, noting that

$$\varphi(L) = \varphi(0) + \int_0^L \varphi'(y)dy = \int_0^L \varphi(y)g(y)dy + \int_0^L \varphi'(y)dy,$$

we have

$$|\varphi(L)| \le \text{const.}\, \|\varphi\|_{H^1(0,L)}. \tag{74}$$

Also, it follows from (73) that

$$\begin{aligned}\|E\varphi\|_1 &= \|(\lambda I - A)E\varphi\| = \|k'(L,\cdot) + (\gamma(\cdot)+\lambda)k(L,\cdot)\||\varphi(L)| \\ &\leq \text{const.}\,\|\varphi\|_{H^1(0,L)} \end{aligned} \tag{75}$$

for all $\varphi \in D(E) = D(A)$. Especially, we note that the two norms $\|\cdot\|_1$ and $\|\cdot\|_{H^1(0,L)}$ are equivalent, i.e., there exist positive constants c_1 and c_2 such that

$$c_1\|\varphi\|_{H^1(0,L)} \leq \|\varphi\|_1 \leq c_2\|\varphi\|_{H^1(0,L)} \tag{76}$$

for all $\varphi \in D(E) = D(A)$. Therefore, from (75)–(76), we have

$$\|E\varphi\|_1 \leq \text{const.}\,\|\varphi\|_1 \tag{77}$$

for all $\varphi \in D(E) = D(A)$. Thus, the operator E is bounded on $(D(A), \|\cdot\|_1)$.

Combining this property with the fact that the operator A generates a C_0-semigroup $T_A(x)$ on $L^2(0,L)$, it follows from [8, Corollary III.1.5] that the operator $B\,(= A + E)$ generates a C_0-semigroup $T_B(x)$ on $L^2(0,L)$. □

Since the operator B generates a C_0-semigroup $T_B(x)$ on $L^2(0,L)$, the operator $B+H$ with H being defined by (29) does a C_0-semigroup $T_{B+H}(x)$ on $L^2(0,L)$. So, the solution of (72) is written as

$$\alpha(x,\cdot) = T_{B+H}(x)k(L,\cdot). \tag{78}$$

By using this expression, (69)–(70) becomes

$$p_x(x,y) + p_y(x,y) = 0, \tag{79}$$

$$p(x,0) = (T_{B+H}(x)k(L,\cdot))(L). \tag{80}$$

Since the solution of (79)–(80) is expressed as $p(x,y) = \psi(x-y)$, it follows by setting $y = 0$ that

$$p(x,0) = \psi(x) = (T_{B+H}(x)k(L,\cdot))(L),$$

as a result,

$$p(x,y) = \psi(x-y) = (T_{B+H}(x-y)k(L,\cdot))(L). \tag{81}$$

Therefore, from (78) and (81), the inverse transformation (65) can be expressed as follows:

$$\begin{aligned} v(t,x) &= w(t,x) + \int_0^x p(x,y)w(t,y)dy + \langle \alpha(x,\cdot), u(t,\cdot)\rangle \\ &= w(t,x) + \int_0^x (T_{B+H}(x-y)k(L,\cdot))(L)w(t,y)dy \\ &\quad + \langle T_{B+H}(x)k(L,\cdot), u(t,\cdot)\rangle. \end{aligned} \tag{82}$$

In fact, (82) is the left inverse of the integral transformation (13) with (60) and (63). That is, the integral transformation (13) with (60) and (63) is left invertible. See Appendix C.

Here, we can show that (82) mapping $L^2(0,\tau) \times L^2(0,L)$ to $L^2(0,\tau)$ is continuous under Hypothesis 3.1 as follows: Noting that $k(L,\cdot) \in D(A) = D(B)$ under Hypothesis 3.1, we see that $(T_{B+H}(\xi)k(L,\cdot))(L)$ is continuous on $0 \le \xi \le \tau$. Let us set

$$M_1 := \max_{\xi\in[0,\tau]} |(T_{B+H}(\xi)k(L,\cdot))(L)| \ (< \infty).$$

From (82), we have

$$\begin{aligned} |v(t,x)| &\le |w(t,x)| + M_1 \int_0^\tau |w(t,y)| dy \\ &\quad + \|T_{B+H}(x)k(L,\cdot)\| \|u(t,\cdot)\| \\ &\le |w(t,x)| + \sqrt{\tau} M_1 \|w(t,\cdot)\| + M_2 \|u(t,\cdot)\|, \end{aligned} \tag{83}$$

where

$$M_2 := \max_{x\in[0,\tau]} \|T_{B+H}(x)k(L,\cdot)\|.$$

Further, squaring the both sides of (83) and integrating over $[0,\tau]$ with respect to x, we have

$$\|v(t,\cdot)\|^2 \le (3 + 3\tau^2 M_1^2)\|w(t,\cdot)\|^2 + 3\tau M_2^2 \|u(t,\cdot)\|^2,$$

which implies that

$$\|v(t,\cdot)\| \le \text{const.} \, (\|w(t,\cdot)\|^2 + \|u(t,\cdot)\|^2)^{\frac{1}{2}}. \tag{84}$$

This means that under Hypothesis 3.1 the inverse transformation (82), which maps from $L^2(0,\tau) \times L^2(0,L)$ to $L^2(0,\tau)$, is continuous. Also, it follows from (84) that

$$(\|v(t,\cdot)\|^2 + \|u(t,\cdot)\|^2)^{\frac{1}{2}} \le \text{const.} \, (\|w(t,\cdot)\|^2 + \|u(t,\cdot)\|^2)^{\frac{1}{2}}. \tag{85}$$

Accordingly, system (11) is exponentially stabilized with any decay rate in the sense of norm $(\|v(t,\cdot)\|^2 + \|u(t,\cdot)\|^2)^{\frac{1}{2}}$. In this way, we can summarize what we have discussed as follows:

Theorem 3.7. Let $Q = \{(x,y) \in \mathbf{R}^2; 0 \le y \le x \le L\}$. For given non-negative real-valued functions $\gamma \in C[0,L]$, $g \in C^1[0,L]$, and $h \in C(Q)$, where $g(x)$ satisfies Hypothesis 3.1, two kernels $q(x,y)$, $\beta(x,y)$ can be designed according to the Steps 1–3. Let $u_0 \in L^2(0,L)$ and $\phi \in H^1(-\tau,0)$. Then, the system (11) with the control law (14) (i.e., (64)) is exponentially stable with any decay rate in the sense of norm $(\|v(t,\cdot)\|^2 + \|u(t,\cdot)\|^2)^{\frac{1}{2}}$.

Remark 3.8. From Remark 3.3 and (85), the solution $[v(t,x), u(t,x)]^T$ of the closed-loop system becomes completely zero after $t = \tau + L$.

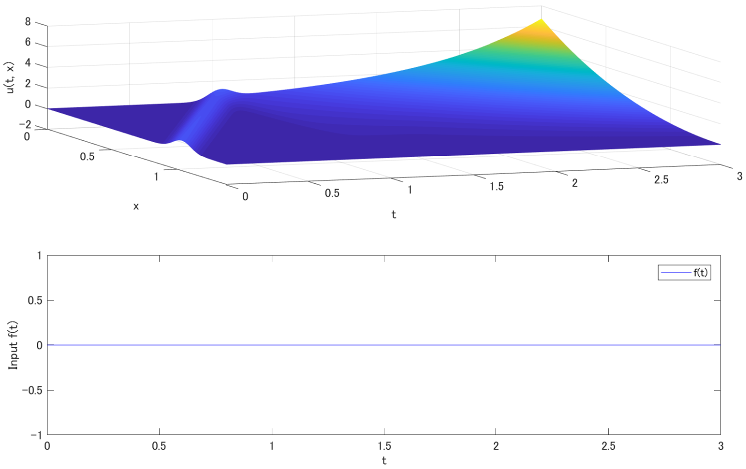

Fig. 1 Open-loop system: Evolution of the state $u(t,x)$ and the control input $f(t)$.

Remark 3.9. Substituting $v(t,y) = f(t+y-\tau)$ into (14) and setting $t+y-\tau = \theta$, we have

$$\begin{aligned} f(t) &= \int_0^\tau q(\tau,y)f(t+y-\tau)dy + \int_0^L \beta(\tau,y)u(t,y)dy \\ &= \int_{t-\tau}^t q(\tau,\theta+\tau-t)f(\theta)d\theta + \int_0^L \beta(\tau,y)u(t,y)dy. \end{aligned} \tag{86}$$

We may use the control law (86) instead of the predictor (64) in the implementation, although we derived the predictor within the abstract framework in Subsection 3.3. Here, we emphasize that the discussions in Subsections 3.3 and 3.4 show that $g(x)$ has to satisfy Hypothesis 3.1 for the controller design.

4 Numerical Simulation

In system (1) (i.e., (11)), we set $\gamma(x) \equiv c$, $g(x) = e^{L-x} - 1$, and $h(x,y) \equiv 0$, where c is a non-negative constant. Then, the system expresses the model of a mono-tubular heat exchange process with internal feedback loop. Note that $g(x)$ satisfies Hypothesis 3.1. For simplicity, let us set $L = \log 4$. According to the Steps 1–3 shown in Subsection 3.2, it is possible to compute the kernels of the control law (86).

Step 1. We can find the solution $k(x,y)$ to (6)–(7) as follows:

$$k(x,y) = -\{3 + 2(x-y)\}e^{x-y}.$$

Step 2. We can find the solution $\beta(x,y)$ to (23)–(25) as follows:

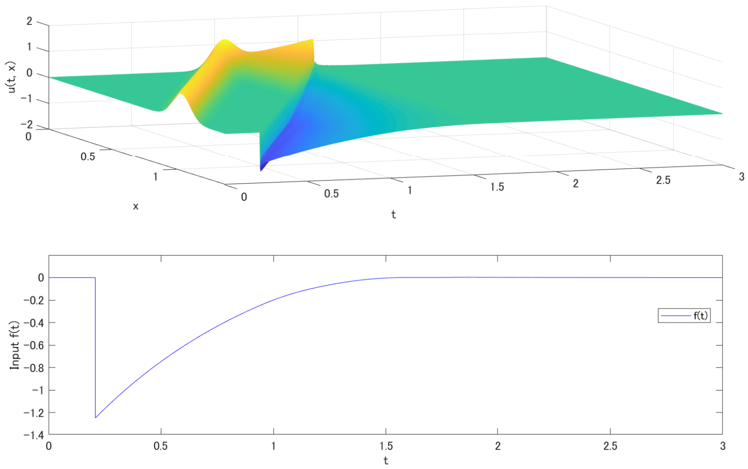

Fig. 2 Closed-loop system: Evolution of the state $u(t, x)$ and the control input $f(t)$.

- In the case of $x \geq y \geq 0$,

$$\beta(x, y) = \Big\{-12 - 8\log 4 - 9(x - y) - \frac{3}{2}(x - y)^2 - \frac{1}{3}(x - y)^3\Big\}e^{(-c+1)x-y}.$$

- In the case of $\log 4 \geq y \geq x$,

$$\beta(x, y) = \{-12 - 8\log 4 + 8(y - x)\}e^{(-c+1)x-y}.$$

Step 3. Setting $y = \log 4$ in Step 2, we have

$$\beta(x, \log 4) = (-12 - 8x)e^{(-c+1)x-\log 4}.$$

Using the $\beta(x, \log 4)$, we can find the solution $q(x, y)$ to (26)–(27) as follows:

$$q(x, y) = \{-12 - 8(x - y)\}e^{(-c+1)(x-y)-\log 4}.$$

Especially, we set $c = 0.1$ and $\tau = 0.15 \times \log 4$. Fig. 1 shows that the open-loop system, i.e. the system (1) with $f \equiv 0$ is unstable. Fig. 2 shows the simulation result of the closed-loop system consisting of (1) and (86). Thus, we see that the control law (86) with gains $\beta(\tau, y)$ and $q(\tau, y)$ works effectively as a stabilizing controller for the unstable system (1).

To solve the hyperbolic equation (1) numerically, we used the finite difference method with mesh width $\Delta x = 0.0001 \times \log 4$, and the Runge-Kutta method of the fourth order with time step $\Delta t = 0.0001$ for its time integration. As initial conditions, we set $u_0(x) = \exp\{-100(x - 0.75 \times \log 4)^2\}$ and $\phi(\theta) \equiv 0$ for (1).

Remark 4.1. In this example, we could find the analytical solutions $\beta(x,y)$ and $q(x,y)$. On the other hand, in the case where one cannot find them analytically, it is possible to compute the solutions approximately by discretizing the PDEs in the Steps 1–3. In these, we need to shape the discretization of the PDE appearing in the Step 1, i.e., in the computation of $k(x,y)$, because it is not standard one. So, we give an algorithm for computing $k(x,y)$ numerically and an example in the next section. The PDEs appearing in the Steps 2–3 are standard ones, to which we can apply a well-known algorithm such as the finite difference method.

Remark 4.2. In the case of linear parabolic systems with input delay, it is shown in [19,20] that the implementation of controller of predictor type is feasible by using a finite number of eigenvalues and eigenfunctions of the system operator.

5 Numerical Solution of the Kernel $k(x,y)$

Finally, we give an algorithm for computing the numerical solution of $k(x,y)$ satisfying

$$k_x(x,y) + k_y(x,y) = (\gamma(x) - \gamma(y))k(x,y) + \int_y^x k(x,z)h(z,y)dz - h(x,y), \tag{87}$$

$$k(x,0) = \int_0^x k(x,y)g(y)dy - g(x), \tag{88}$$

where $0 \le y \le x \le L$. This is the PDE which one should solve in the Step 1 of Subsection 3.2. $k(x,y)$ is computed according to the following procedure:

Algorithm 5.1. Let N be a partition number for numerical computation, and define $p_n := \frac{nL}{N}$, $n = 0, 1, \dots, N$.

Step (i) By (88), $k(p_0, p_0)$ is computed:

$$k(p_0, p_0) = k(0,0) = -g(0) = -g(p_0).$$

Step (ii) Noting that

$$k(p_{n+1}, p_{n+1}) \cong k(p_n, p_n) + \frac{L}{N}(k_x(p_n, p_n) + k_y(p_n, p_n)),$$

it follows from (87) that

$$k(p_{n+1}, p_{n+1}) \cong k(p_n, p_n) - \frac{L}{N}h(p_n, p_n),$$

from which $k(p_n, p_n)$, $n = 1, 2, \dots, N$ are computed.

Step (iii) By (88), $k(p_1, p_0)$ is computed:

$$\begin{aligned} k(p_1, p_0) = k(p_1, 0) &= \int_{p_0}^{p_1} k(p_1, y) g(y) dy - g(p_1) \\ &\cong \frac{L}{N} k(p_1, p_1) g(p_1) - g(p_1). \end{aligned}$$

Step (iv) Similarly to the Step (ii), using

$$k(p_{n+1}, p_n) \cong k(p_n, p_{n-1}) + \frac{L}{N}(k_x(p_n, p_{n-1}) + k_y(p_n, p_{n-1})),$$

and further using

$$\begin{aligned} \int_{p_{n-1}}^{p_n} k(p_n, z) h(z, p_{n-1}) dz \cong \frac{L}{2N}&(k(p_n, p_n) h(p_n, p_{n-1}) \\ &+ k(p_n, p_{n-1}) h(p_{n-1}, p_{n-1})), \end{aligned}$$

it follows from (87) that

$$\begin{aligned} k(p_{n+1}, p_n) \cong k(p_n, p_{n-1}) + \frac{L}{N}\bigg(&(\gamma(p_n) - \gamma(p_{n-1})) k(p_n, p_{n-1}) \\ &+ \frac{L}{2N}(k(p_n, p_n) h(p_n, p_{n-1}) + k(p_n, p_{n-1}) h(p_{n-1}, p_{n-1})) \\ &- h(p_n, p_{n-1})\bigg), \end{aligned}$$

from which $k(p_n, p_{n-1})$, $n = 2, 3, \ldots, N$ are computed.

Step (v) By (88), $k(p_2, p_0)$ is computed:

$$\begin{aligned} k(p_2, p_0) = k(p_2, 0) &= \int_{p_0}^{p_2} k(p_2, y) g(y) dy - g(p_2) \\ &\cong \frac{L}{N}(k(p_2, p_2) g(p_2) + k(p_2, p_1) g(p_1)) - g(p_2). \end{aligned}$$

Continuing the Steps (iv)–(v) in this way, we obtain an approximated kernel $k(p_n, p_m)$ for $0 \le m \le n \le N$. ♢

In Fig. 3, we give numerical results of $k(L, y)$, where we set $\gamma(x) \equiv 0$, $g(x) = (L-x)^2$, $h(x, y) \equiv 1$, $L = 1$, and $N = 100$. The solid line shows a result by the proposed method. On the other hand, dash line does a result after iteration of 4 times by successive approximation [11, Theorem 1], where we set $k_0 \equiv 0$ as the initial data. As for successive approximation, we used Mathematica.

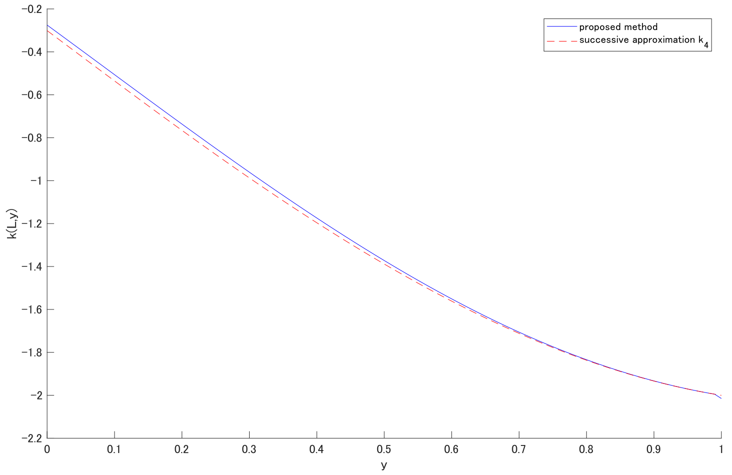

Fig. 3 Numerical results of $k(L, y)$: Proposed method (solid line). Successive approximation k_4 (dash line).

6 Conclusion

In this paper, we studied the problem of constructing boundary stabilizing controllers for first-order hyperbolic equations with input delay. Based on the result for the case without input delay, we could successfully construct infinite-dimensional stabilizing controllers for the system with input delay, by using the backstepping approach for PDEs. Using C_0-semigroups for hyperbolic equations with nonlocal boundary condition, it was shown under Hypothesis 3.1 that the controller makes sense and further that the inverse integral transformation is continuous. Also, we gave an algorithm for the discretization of the PDE with respect to $k(x, y)$ appearing in the Step 1 of Subsection 3.2, which will be a helpful tool when one applies a backstepping design to first-order hyperbolic systems with general coefficients. In the future, we plan to study the similar problem for first-order hyperbolic equations with Fredholm integrals.

Acknowledgements

This research is supported by KAKENHI (Grant-in-Aid for Scientific Research (C) No. 15K04999, (S) No. 15H05729), Japan Society for the Promotion of Science.

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A Well-Posedness of (6)–(7)

Set $k(x,y) = e^{\int_y^x \gamma(\xi)d\xi}\overline{k}(x,y)$. Then, $\overline{k}$ satisfies

$$\overline{k}_x(x,y) + \overline{k}_y(x,y) = \int_y^x \overline{k}(x,z)\overline{h}(z,y) - \overline{h}(x,y), \tag{89}$$

$$\overline{k}(x,0) = \int_0^x \overline{k}(x,y)\overline{g}(y)dy - \overline{g}(x), \tag{90}$$

where $\overline{h}(x,y) := e^{-\int_y^x \gamma(\xi)d\xi}h(x,y)$, $\overline{g}(x) := e^{-\int_0^x \gamma(\xi)d\xi}g(x)$. Therefore, it follows from [11, Theorem 1] that (89)–(90), i.e. (6)–(7) is well-posed.

B Existence of the Solution of (16)

Let $Q = \{(x,y) \in \mathbf{R}^2; 0 \le y \le x \le L\}$. We consider the following integral equation instead of (16):

$$\int_0^x k(x,y)\varphi(y)dy + 1 = \varphi(x), \quad x \in [0,L], \tag{91}$$

where $k \in C^1(Q)$ is a unique solution to (6)–(7). By using the method of successive approximations, we see that (91) has a unique solution $\varphi \in C^1[0,L]$ with bound $|\varphi(x)| \le e^{Mx}$, where $M := \max_{(x,y)\in Q} |k(x,y)|$. It is clear that the φ gives the solution to (16).

C Left Invertibility of Backstepping Transformation

In Section 3, we introduced the target system (12) and the integral transformation (13). We rewrite them equivalently as follows:

- Target system

$$\begin{cases} \varphi_t(t,x) = \varphi_x(t,x) - \gamma(x)\varphi(t,x) + g(x)\varphi(t,0) \\ \qquad\qquad + \displaystyle\int_0^x h(x,y)\varphi(t,y)dy, \quad (t,x) \in (0,\infty)\times(0,L), \\ \varphi(t,L) = \displaystyle\int_0^L k(L,y)\varphi(t,y)dy + w(t,0), \quad t>0, \\ \varphi(0,x) = u_0(x), \quad x \in [0,L], \\ w_t(t,x) = w_x(t,x), \quad (t,x) \in (0,\infty)\times(0,\tau), \\ w(t,\tau) = 0, \quad t>0, \\ w(0,x) = w_0(x), \quad x \in [0,\tau]. \end{cases} \tag{92}$$

- Integral transformation

$$\begin{cases} \varphi(t,x) = u(t,x), \\ w(t,x) = \mathcal{P}v(t,x) + \mathcal{Q}u(t,x), \end{cases} \tag{93}$$

where

$$\mathcal{P}v(t,x) := v(t,x) - \int_0^x q(x,y)v(t,y)dy, \quad \mathcal{Q}u(t,x) := -\int_0^L \beta(x,y)u(t,y)dy.$$

Similarly, the inverse integral transformation (65) is rewritten as

- Inverse integral transformation

$$\begin{cases} u(t,x) = \varphi(t,x), \\ v(t,x) = \mathcal{R}w(t,x) + \mathcal{S}\varphi(t,x), \end{cases} \tag{94}$$

where

$$\mathcal{R}w(t,x) := w(t,x) + \int_0^x p(x,y)w(t,y)dy, \quad \mathcal{S}\varphi(t,x) := \int_0^L \alpha(x,y)\varphi(t,y)dy.$$

Now, let us express the two transformations (93) and (94) as

$$\begin{bmatrix} \varphi(t,x) \\ w(t,x) \end{bmatrix} = \mathcal{T}_{\mathcal{P},\mathcal{Q}} \begin{bmatrix} u(t,x) \\ v(t,x) \end{bmatrix}, \quad \mathcal{T}_{\mathcal{P},\mathcal{Q}} := \begin{bmatrix} I & 0 \\ \mathcal{Q} & \mathcal{P} \end{bmatrix}, \tag{95}$$

$$\begin{bmatrix} u(t,x) \\ v(t,x) \end{bmatrix} = \mathcal{T}_{\mathcal{R},\mathcal{S}} \begin{bmatrix} \varphi(t,x) \\ w(t,x) \end{bmatrix}, \quad \mathcal{T}_{\mathcal{R},\mathcal{S}} := \begin{bmatrix} I & 0 \\ \mathcal{S} & \mathcal{R} \end{bmatrix}. \tag{96}$$

First, we calculate $\mathcal{T}_{\mathcal{R},\mathcal{S}}\mathcal{T}_{\mathcal{P},\mathcal{Q}}$ as follows:

$$\mathcal{T}_{\mathcal{R},\mathcal{S}}\mathcal{T}_{\mathcal{P},\mathcal{Q}} = \begin{bmatrix} I & 0 \\ \mathcal{S} + \mathcal{R}\mathcal{Q} & \mathcal{R}\mathcal{P} \end{bmatrix}, \tag{97}$$

where

$$(\mathcal{S} + \mathcal{R}\mathcal{Q})u(t,x) = \int_0^L \left\{ \alpha(x,y) - \beta(x,y) - \int_0^x p(x,z)\beta(z,y)dz \right\} u(t,y)dy,$$

$$(\mathcal{R}\mathcal{P})v(t,x) = v(t,x) + \int_0^x \left\{ p(x,y) - q(x,y) - \int_y^x p(x,z)q(z,y)dz \right\} v(t,y)dy.$$

From this expression, if the kernels p, q, α, and β satisfy the equations

$$\alpha(x,y) - \beta(x,y) = \int_0^x p(x,z)\beta(z,y)dz, \tag{98}$$

$$p(x,y) - q(x,y) = \int_y^x p(x,z)q(z,y)dz, \tag{99}$$

it follows that

$$\mathcal{T}_{\mathcal{R},\mathcal{S}}\mathcal{T}_{\mathcal{P},\mathcal{Q}} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{100}$$

Indeed, we can verify that the kernels p, q, α, and β derived in Section 3 (see (60), (63), (78), and (81)) satisfy the above (98) and (99). It is done as follows: Introducing the variable $\varepsilon(x,y) := \alpha(x,y) - \beta(x,y)$, it follows from (23)–(25) and (66)–(68) that

$$\varepsilon_x(x,y) = -\varepsilon_y(x,y) - \gamma(y)\varepsilon(x,y) + k(L,y)\alpha(x,L) + \int_y^L \varepsilon(x,z)h(z,y)dz, \tag{101}$$

$$\varepsilon(x,0) = \int_0^L \varepsilon(x,y)g(y)dy, \tag{102}$$

$$\varepsilon(0,y) = 0. \tag{103}$$

By using the operators A and H defined by (28)–(29), we can formulate (101)–(103) as

$$\varepsilon'(x,\cdot) = (A+H)\varepsilon(x,\cdot) + k(L,\cdot)\alpha(x,L), \quad \varepsilon(0,\cdot) = 0, \tag{104}$$

from which we have the solution

$$\varepsilon(x,\cdot) = \int_0^x T_{A+H}(x-z)k(L,\cdot)\alpha(z,L)dz. \tag{105}$$

The solution (105) is expressed as

$$\begin{aligned}\varepsilon(x,y) &= \int_0^x (T_{A+H}(x-z)k(L,\cdot))(y)\alpha(z,L)dz \\ &= \int_0^x \beta(x-z,y)\alpha(z,L)dz \\ &= \int_0^x \beta(z,y)\alpha(x-z,L)dz \\ &= \int_0^x \beta(z,y)T_{B+H}(x-z)k(L,\cdot)(L)dz \\ &= \int_0^x p(x,z)\beta(z,y)dz. \end{aligned} \tag{106}$$

This means that p, α, and β of Section 3 actually satisfy (98).

Next, introducing the variable $r(x,y) := p(x,y) - q(x,y)$, from (26)–(27) and (69)–(70), we have

$$r_x(x,y) + r_y(x,y) = 0, \tag{107}$$

$$r(x,0) = \varepsilon(x,L). \tag{108}$$

It is easy to see that eqs. (107)–(108) has the solution

$$r(x,y) = \varepsilon(x-y,L), \tag{109}$$

which leads to

$$\begin{aligned}r(x,y) &= \int_0^{x-y} (T_{A+H}(x-y-z)k(L,\cdot))(L)\alpha(z,L)dz \\ &= \int_0^{x-y} (T_{A+H}(x-y-z)k(L,\cdot))(L)(T_{B+H}(z)k(L,\cdot))(L)dz \\ &= \int_0^{x-y} q(x-z,y)p(x,x-z)dz \\ &= \int_y^x p(x,z)q(z,y)dz. \end{aligned} \tag{110}$$

Therefore, we see that p and q of Section 3 also satisfy (99). In this way, the two transformations (95) and (96) with the kernels (60), (63), (78), and (81) satisfy (100). However, as for $\mathcal{T}_{\mathcal{P},\mathcal{Q}}\mathcal{T}_{\mathcal{R},\mathcal{S}} = \begin{bmatrix} I & 0 \\ \mathcal{Q}+\mathcal{P}\mathcal{S} & \mathcal{P}\mathcal{R} \end{bmatrix}$, we can verify that they do not satisfy the relation

$$\mathcal{T}_{\mathcal{P},\mathcal{Q}}\mathcal{T}_{\mathcal{R},\mathcal{S}} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{111}$$

through a discussion similar to the above. In fact, $\mathcal{P}\mathcal{R} = I$ is satisfied, but $\mathcal{Q}+\mathcal{P}\mathcal{S} = 0$ is not. Hence, the backstepping transformation (95) with the kernels (60) and (63) is left invertible, but not right invertible.