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S-matrix unitarity and renormalizability in higher-derivative theories

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We investigate the relation between S-matrix unitarity ($SS^\dagger = 1$) and renormalizability in theories with negative-norm states. The relation has been confirmed in many field theories, including gauge theories and Einstein gravity, by analyzing the unitarity bound, which follows from the S-matrix unitarity and the norm positivity. On the other hand, renormalizable theories with a higher-derivative kinetic term do not necessarily satisfy the unitarity bound because of the negative-norm states. In these theories it is not known whether the S-matrix unitarity provides a nontrivial constraint related to the renormalizability. In this paper, by relaxing the assumption of norm positivity we derive a bound on scattering amplitudes weaker than the unitarity bound, which may be used as a consistency requirement for S-matrix unitarity. We demonstrate in scalar field models with a higher-derivative kinetic term that the weaker bound and the renormalizability imply identical constraints.

Subject Index B32, E03, E05

1. Introduction

Unitarity and renormalizability are fundamental principles in quantum field theories (QFTs). Renormalizability means that a theory can be described by a finite number of parameters. Fixing all of these parameters by experiments, we can make nontrivial predictions. On the other hand, unitarity consists of two independent conditions: the S-matrix unitarity,[1] i.e. $SS^\dagger = 1$, and the positivity of physical state norms (norm positivity). S-matrix unitarity guarantees a unique time evolution, whereas the positivity is to make the probability positive. We emphasize that S-matrix unitarity and unitarity are

[1] The condition $SS^\dagger = 1$ in field theories containing negative-norm states is often called "pseudo-unitarity" [1,2]. While the word "pseudo-unitarity" might give the impression that we consider theories with negative norms, the purpose of this paper is to show that the relation to renormalizability is independent of the norm positivity. To avoid giving such a misleading impression, we use "S-matrix unitarity" instead of "pseudo-unitarity."

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carefully distinguished in this paper. Both unitarity and renormalizability are necessary to make the theory predictive, hence it is interesting to explore their possible connections.

Indeed, there are a variety of known examples where the unitarity bound, derived from the S-matrix unitarity and the norm positivity, and renormalizability imply the same constraints on interactions of the theory. Historical arguments include gauge theories and Einstein gravity [3–6].[2] More recently, it was shown that this equivalence holds true in more generic QFTs such as Lifshitz-type nonrelativistic scalar field theories [7,8]. If this equivalence is universal, one may expect, e.g., that quantum gravity theories consistent with the unitarity bound are automatically renormalizable. Since the unitarity bound significantly constrains the particle spectrum and interactions of quantum gravity theories (see, e.g., Ref. [9] for recent discussions), it would be helpful if we could use the unitarity bound as a probe of renormalizability in this way. We would like to explore this possibility by further investigating the possible equivalence.

Let us recall here that the models [3–8] mentioned above do not involve negative-norm states and thus the unitarity bound directly follows from the S-matrix unitarity. On the other hand, one may easily show by explicit calculations that renormalizable theories with higher-derivative terms, such as quadratic gravity, do not necessarily satisfy the unitarity bound. This is because the unitarity bound does not hold in these theories due to the negative-norm states stemming from the higher derivatives. So does this mean that the unitarity bound is not helpful anymore in investigating renormalizability? To answer this question, we raise the following question: Does S-matrix unitarity, without the positive norm assumption, provide nontrivial constraints? In this paper we argue that S-matrix unitarity indeed gives a nontrivial constraint even in theories with negative-norm states. In particular, we demonstrate in a concrete model that the obtained constraint is identical to the one implied by renormalizability.

Among various models with negative-norm states we would like to focus on quadratic gravity, whose action contains quadratic terms in the curvature tensor on top of the Einstein–Hilbert term. In quadratic gravity the graviton propagator is modified as $\sim p^{-4}$ in the UV region and thus renormalizability is achieved at the cost of norm positivity [10,11]. Even though it is not yet clear how to deal with the negative-norm states, the theory is predictive as long as the S-matrix is unitary, so that quadratic gravity could offer an interesting approach to UV completion of gravity. In the present paper we consider a scalar field model with a modified propagator as a toy model for quadratic gravity and demonstrate that S-matrix unitarity and renormalizability lead to identical constraints on this model. Our result provides new evidence for the connection between S-matrix unitarity and renormalizability. It will also be useful for a better understanding of quadratic gravity. In particular, our analysis implies that the unitarity bound is a useful probe for renormalizability in theories without negative-norm states, whereas S-matrix unitarity has to be used instead if the theory contains negative-norm states.

The rest of this paper is organized as follows. In Sect. 2 we introduce a scalar field model with a higher-derivative kinetic term. The scalar propagator in this model is modified as $\sim p^{-4}$ in the UV, so that we can think of it as a toy model for quadratic gravity. We also elaborate on the renormalizability conditions in our model based on Refs. [7,8]. In Sect. 3.1, we review the optical theorem and the unitarity bound. In particular, we argue that the lowest-order approximation of $SS^\dagger = 1$ may be used to constrain the theory, just as the tree-level unitarity of high-energy scattering constrains interactions in unitary theories. The lowest-order approximation of $SS^\dagger = 1$ in the scalar field model

[2] Here, we mention Einstein gravity as one of the examples that possesses neither unitarity nor renormalizability.

is then investigated in Sect. 3.2. There we find that S-matrix unitarity and renormalizability imply identical constraints. A summary and discussion are given in Sect. 4. Technical details are collected in the appendices.

2. Higher-derivative scalar field theory

In this section we introduce a scalar field model with a higher-derivative kinetic term as the simplest example of theories with negative-norm states. Due to the fourth-order derivative term in the kinetic term of the scalar field ϕ, its propagator is modified as $\sim p^{-4}$ at UV and thus negative-norm (ghost) states appear. This model can then be thought of as a toy model for quadratic gravity. We also elaborate on the renormalizability conditions in this higher-derivative theory based on the criterion introduced in Refs. [7,8].[3]

2.1. Quadratic action

The quadratic action for the scalar field ϕ with the fourth-order derivative in four-dimensional spacetime is given by

$$\mathcal{S}_2 = \int d^4x \left[-\frac{1}{2}\phi(\Box - m_1^2)(\Box - m_2^2)\phi \right], \tag{1}$$

where $\Box = \partial^\mu \partial_\mu$ and $m_2 > m_1$ (> 0). The propagator of ϕ is given by $\dfrac{1}{(p^2+m_1^2)(p^2+m_2^2)}$, which has a more convergent UV behavior $\sim p^{-4}$. Also, it has poles at $p^2 = -m_1^2$ and $p^2 = -m_2^2$ representing two species of particles. In order to construct the two sets of creation and annihilation operators, it is better to rewrite the action into that of two scalars with second-order derivatives. This can be done by the following redefinition of the fields (a careful analysis using the Lagrange multiplier is given in Appendix A):

$$\psi_1 := \frac{(\Box - m_2^2)\phi}{M} \qquad \text{and} \qquad \psi_2 := \frac{(\Box - m_1^2)\phi}{M} \qquad \left(\phi = \frac{1}{M}(\psi_2 - \psi_1)\right), \tag{2}$$

where $M := \sqrt{m_2^2 - m_1^2}$. Then, the action in Eq. (1) becomes

$$\mathcal{S}_2 = \int d^4x \left[\frac{1}{2}\psi_1(\Box - m_1^2)\psi_1 - \frac{1}{2}\psi_2(\Box - m_2^2)\psi_2 \right]. \tag{3}$$

The kinetic term of ψ_1 has a positive sign, whereas that of ψ_2 has a negative sign. Thus, the one-particle state of ψ_2 becomes a negative-norm state (see Appendix C). The appearance of negative-norm states is inevitable if the original action has higher derivatives. Note that ϕ is dimensionless, $[\phi] = 0$, and ψ_i has the canonical dimension, $[\psi_i] = 1$, where the bracket means the mass dimension of a parameter or field. We also introduce a canonical scalar field σ by

$$\mathcal{S}_{2\sigma} = \int d^4x \left[\frac{1}{2}\sigma(\Box - m_\sigma^2)\sigma \right] \tag{4}$$

to construct a nonrenormalizable coupling term in the next subsection. This field σ has dimension $[\sigma] = 1$.

[3] The renormalization group flow is discussed in Ref. [12].

2.2. *Interaction terms*

We then introduce interaction terms. To discuss the relation between the tree-level S-matrix unitarity and renormalizability, we consider both renormalizable and nonrenormalizable interactions, and clarify whether they are consistent with $SS^\dagger = 1$ or not in the perturbative expansion. Among various interactions we focus on the marginal ones, namely $[\lambda] = 0$. It is first because other cases can easily be inferred from the results in the marginal cases. Moreover, if the theory contains a field with a nonpositive dimension (our ϕ field has $[\phi] = 0$), the renormalizability condition is not as simple as in ordinary QFTs [7,8]. As we will explain shortly, the renormalizability condition of marginal interactions does not simply follow from ordinary power-counting, but rather requires an additional condition in our setup. Therefore, we expect to see similar nontrivial conditions for the S-matrix unitarity, the main objective of this paper. We also concentrate on tree-level four-point scattering because it is simple, but it already provides interesting information on the unitarity and renormalizability issues. Thus, we will consider marginal four-point vertex operators.

Before introducing interaction terms, we briefly summarize the results in Refs. [7,8]. Let us start from a general discussion of the following nth-order interaction in four-dimensional spacetime,

$$S_{\text{int}} = \lambda \int d^4x\, (\partial_x^{a_1}\phi_1)\,(\partial_x^{a_2}\phi_2)\cdots(\partial_x^{a_n}\phi_n), \tag{5}$$

where $a_1, \ldots, a_n$ are nonnegative integers. Here, ϕ_i is either ϕ or σ as introduced in the previous subsection. In particular, their kinetic terms are nondegenerate. Each ∂_x denotes spacetime derivatives ∂_μ ($\mu = 0, \ldots, 3$) whose indices are contracted in a Lorentz-invariant way. In the interaction action of Eq. (5), the dimension of the coupling constant λ reads

$$[\lambda] = 4 - \sum_{l=1}^{n}(a_l + [\phi_l]). \tag{6}$$

The usual power-counting renormalizability (PCR) condition requires that the mass dimension of the coupling constant is nonnegative, $[\lambda] \geq 0$. However, an infinite number of interaction terms satisfy this condition if there exists a field with a nonpositive dimension $[\phi_l] \leq 0$. In other words, an infinite number of parameters are required, and thus the theory fails to be renormalizable. For instance, let us look at

$$\lambda \int d^4x \phi^4 (\Box\phi)^2, \tag{7}$$

which satisfies the conventional PCR condition $[\lambda] \geq 0$. This term generates $\int d^4x \phi^n (\Box\phi)^2$ for any n with a divergent coefficient in the one-loop effective action (for a detailed discussion, see Refs. [7,8]). This means that an infinite number of counterterms are required, and thus (7) is not renormalizable. To make the number of parameters finite, we introduce extended PCR conditions given by

$$\sum_{l=1}^{n}(a_l + [\phi_l]) \leq 4, \tag{8}$$

$$\sum (a_l + [\phi_l]) < 4 \quad \text{for an arbitrary partial sum.} \tag{9}$$

The former is nothing but the usual PCR condition, while the latter additional conditions mean that the dimension of $\partial_x^{a_{i_1}}\phi_{a_{i_1}} \cdots \partial_x^{a_{i_k}}\phi_{a_{i_k}}$ ($k \leq n-1$) should be less than the spacetime dimension, 4.

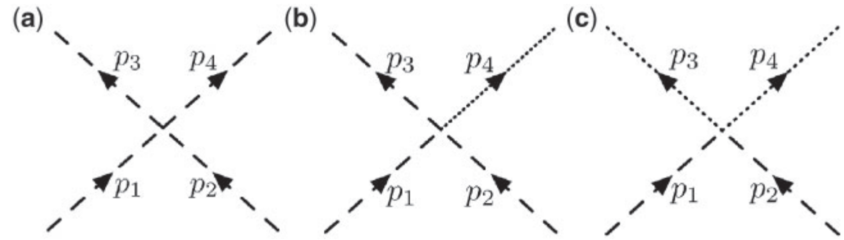

Fig. 1. The scalar four-point vertex functions (a) $\psi_1\psi_1\psi_1\psi_1$, (b) $\psi_1\psi_1\psi_1\psi_2$, and (c) $\psi_1\psi_1\psi_2\psi_2$. The broken and dotted lines stand for the scalar fields ψ_1 and ψ_2, respectively.

In particular, we can see that the second conditions provide nontrivial constraints only on marginal interactions in our setup, which contains a field ϕ with vanishing dimension, but does not contain fields with negative dimensions. We can see that the example in (7) does not satisfy the extended PCR condition of Eq. (9). Based on the extended PCR conditions, we introduce two marginal interaction terms.

2.2.1. Renormalizable interaction terms

We consider the simplest renormalizable marginal interaction term:

$$S_{\text{ren}} = \lambda \int d^4x \left\{(\partial_\mu \phi)^2\right\}^2. \tag{10}$$

This interaction has $a_1 = a_2 = a_3 = a_4 = 1$, and satisfies the extended PCR conditions in Eq. (8). For the later calculations of the scattering amplitudes, it is better to rewrite it in terms of ψ_1 and ψ_2 as defined in Eqs. (2) (the reformulation using the Lagrange multiplier in Appendix A gives the same result):

$$S_{\text{ren}} = \alpha \int d^4x \left\{\left(\partial_\mu(\psi_2 - \psi_1)\right)^2\right\}^2, \tag{11}$$

$$\alpha := \frac{\lambda}{M^4}. \tag{12}$$

This gives the four-point interaction terms for ψ_1 and ψ_2 with derivatives, and they have the same coupling constant α, with $[\alpha] = -4$. Although $[\alpha] < 0$ does not satisfy the PCR condition, the theory turns out to be renormalizable because the common coupling α brings about a cancellation among the ψ_1 loop and the ψ_2 loop. In the next section we use the vertex functions (a) $\psi_1\psi_1\psi_1\psi_1$, (b) $\psi_1\psi_1\psi_1\psi_2$, and (c) $\psi_1\psi_1\psi_2\psi_2$ given by (see Fig. 1)

$$\begin{aligned}
(\text{a}) &= 8i\alpha\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\right],\\
(\text{b}) &= -8i\alpha\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\right],\\
(\text{c}) &= 8i\alpha\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\right],
\end{aligned} \tag{13}$$

where p_i $(i = 1, 2, 3, 4)$ are the four-momenta of scalar fields.

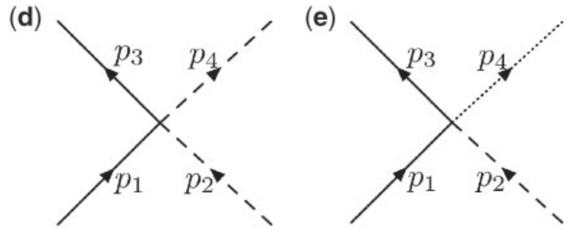

Fig. 2. The scalar four-point vertex functions (d) $\sigma\psi_1\sigma\psi_1$ and (e) $\sigma\psi_1\sigma\psi_2$. The solid, broken, and dotted lines stand for the scalar fields σ, ψ_1, and ψ_2, respectively.

2.2.2. Non-renormalizable interaction terms

We wish to consider the simplest examples of nonrenormalizable marginal interaction terms.[4] One is renormalizable, while the other is not:[5]

$$\int d^4x\phi(\partial_\mu\phi)^2(\Box\phi) \qquad \text{or} \qquad \int d^4x\phi^2(\Box\phi)^2. \tag{14}$$

However, it turns out that we have to analyze six-point scattering amplitudes to check their consistency with the S-matrix unitarity $SS^\dagger = 1$, essentially because these interactions contain the $\Box$ operator (as shown in Appendix B). They are not suitable for our analysis based on four-point scattering amplitudes. Notice here that any marginal fourth-order interaction with four derivatives can be reformulated into the three interaction terms in Eqs. (10) and (14) by partial integrals. We then introduce another scalar field σ defined in Eq. (4) and consider the following interaction as a simple example of nonrenormalizable marginal interactions:

$$S_{\text{non}} = \lambda' \int d^4x\phi^2(\partial_\mu\sigma)^2, \tag{15}$$

where $[(\partial_\mu\sigma)^2] = 4$, and thus this coupling term does not satisfy the extended PCR conditions in Eq. (8). This interaction term can be expressed with ψ_1 and ψ_2 as

$$S_{\text{non}} = \alpha' \int d^4x(\psi_2 - \psi_1)^2(\partial_\mu\sigma)^2, \tag{16}$$

$$\alpha' := \frac{\lambda'}{M^2}. \tag{17}$$

In the next section we use the vertex functions $\sigma\psi_1\sigma\psi_1$ and $\sigma\psi_1\sigma\psi_2$ (see Fig. 2) given by

$$\begin{aligned}(\text{d}) &= -4i\alpha'\,(p_1 \cdot p_3),\\ (\text{e}) &= 4i\alpha'\,(p_1 \cdot p_3),\end{aligned} \tag{18}$$

where p_i $(i = 1, 3)$ are the four-momenta of scalar fields.

[4] Recall that the ordinary PCR condition is not necessarily equivalent to renormalizablitiy in the presence of extra symmetries and/or constraints. The same caveat applies to the extended PCR condition in general. However, the interactions considered in this paper have no such extra symmetries or constraints, so that the extended PCR condition can be used to dictate renormalizability.

[5] The difference between the renormalizable interaction in Eq. (10) and the non-renormalizable interaction in (14) can be also explained by shift symmetry. However, some of the nonrenormalizable interaction terms with shift symmetry could not be excluded by only the nonnegativity of their coefficients. The dimensions of each factor in the interaction terms are indeed crucial to distinguish the renormalizability. Detailed discussion of this is presented in Ref. [8].

3. S-matrix unitarity and renormalizability

We now investigate the relation between renormalizability and S-matrix unitarity for high-energy scattering. In theories without negative-norm states, S-matrix unitarity implies the unitarity bound. We know that the unitarity bound gives identical constraints to renormalizability in various theories. On the other hand, if there exist negative-norm states, the unitarity bound no longer follows from S-matrix unitarity. However, in this section we argue that the optical theorem (which is based on S-matrix unitarity, but does not rely on norm positivity) provides nontrivial consistency conditions even in theories with negative-norm states. Interestingly, we find that the constraint obtained is identical to that required by renormalizability.

In Sect. 3.1 we firstly see the relation between S-matrix unitarity, the optical theorem, and the unitarity bound by carefully reviewing the derivation of the unitarity bound and locating the step in which the positivity of all norms is required. There, we will see that the optical theorem may be used to constrain theories with negative norms.[6] In Sect. 3.2 we then concretely investigate the relation between renormalizability and the optical theorem in the simple scalar field model introduced in Sect. 2. In particular, we will see that the (non)renormalizable interaction term is (in)consistent with the optical theorem in the UV limit.

3.1. *Optical theorem and unitarity bound*

We begin with the S-matrix unitarity,

$$SS^\dagger = 1. \tag{19}$$

Decomposing the S-matrix as $S = 1 + iT$, we can rewrite the above equation as

$$-i(T - T^\dagger) = TT^\dagger. \tag{20}$$

Let us also suppose that we can take a complete orthonormal basis of the Hilbert space $\{|X\rangle\}$ as

$$\sum_X |X\rangle C_X \langle X| = 1 \qquad \text{with} \qquad \langle X|Y\rangle = C_X^{-1}\delta_{XY}, \tag{21}$$

where we have assumed that $\langle X|X\rangle \neq 0$, but it can be either positive or negative. Note that if the theory contains a negative-norm state, its normalization factor C_X is inevitably negative, because the sign of C_X cannot be flipped by normalization of the state $|X\rangle$. We would then like to reformulate the relation in Eq. (20) in terms of the covariant scattering amplitude $\mathcal{M}(i \to f)$ defined as

$$\langle f|T|i\rangle = \delta^4\left(p_i - p_f\right)\mathcal{M}\left(i \to f\right). \tag{22}$$

By substituting Eq. (22) into Eq. (20), we arrive at the cutting rule in terms of covariant amplitudes,

$$-i[\mathcal{M}(i \to f) - \mathcal{M}(f \to i)^*] = \sum_X C_X \delta^4\left(p - p_X\right)\mathcal{M}\left(i \to X\right)\mathcal{M}\left(f \to X\right)^*, \tag{23}$$

where $p\ (= p_i = p_f)$ is the total energy-momentum. By considering forward scattering ($|i\rangle = |f\rangle$) the relation is reduced to the optical theorem,

$$2\mathrm{Im}\,\mathcal{M}\left(i \to i\right) = \sum_X C_X \delta^4\left(p - p_X\right)|\mathcal{M}\left(i \to X\right)|^2, \tag{24}$$

[6] An S-matrix containing a negative-norm state is also discussed in Ref. [13]

where the summation is over all possible intermediate on-shell states. In particular, the optical theorem implies that

$$|\mathcal{M}(i \to i)| \geq |\mathrm{Im}\,\mathcal{M}(i \to i)| = \frac{1}{2}\left|\sum_X C_X \delta^4(p - p_X)\,|\mathcal{M}(i \to X)|^2\right|. \tag{25}$$

The discussion so far is applicable even if the theory contains negative-norm states.

If the theory does not contain negative-norm states, the inequality in Eq. (25) implies the unitarity bound. To explain this, let us consider theories without negative-norm states and set $C_X = 1$ by taking the normal basis. In this case, the right-hand side (RHS) of Eq. (25) is bounded as

$$\text{RHS of (25)} = \frac{1}{2}\sum_X \delta^4(p - p_X)\,|\mathcal{M}(i \to X)|^2 \geq \frac{1}{2}|\mathcal{M}(i \to j)|^2 \tag{26}$$

for an arbitrary on-shell intermediate state j. By setting $j = i$, we obtain the inequality

$$|\mathcal{M}(i \to i)| \leq \text{const.}, \tag{27}$$

which further implies that the left-hand side of Eq. (25) is bounded from above by a constant. By combining Eqs. (25)–(27), we arrive at the following bound on scattering amplitudes with an arbitrary on-shell state j:

$$|\mathcal{M}(i \to j)| \leq \text{const.} \tag{28}$$

This is the unitarity bound.[7] In particular, this bound for high-energy scattering at the tree level (that is, the lowest order in coupling constants) is called the tree-level unitarity condition.

It is known that the unitarity bound in Eq. (28), especially the tree-level unitarity, gives an identical condition to renormalizability in various theories without negative-norm states. A natural question will be if the weaker bound, Eq. (25), which holds true even in theories with negative-norm states, may provide nontrivial constraints useful in analyzing the UV properties such as renormalizability. In the next subsection we use the scalar field model introduced in Sect. 2 to demonstrate that it is indeed the case. In particular, we show that tree-level high-energy scattering induced by the (non)renormalizable interaction is (in)consistent with the bound in Eq. (25) implied by the optical theorem.

3.2. *Weaker bound for amplitudes in higher-derivative scalar models*

We finally come to our main question of whether the weaker unitarity bound of Eq. (25) is satisfied for amplitudes in higher-derivative scalar models of Sect. 2. From the two ways of quantization for ψ_2 (see Appendix C), we choose the one for which negative-norm states exist.[8] For the reason mentioned in Sec.t 3.1, we consider two–two scattering:

$$p_1 + p_2 \to p_3 + p_4. \tag{29}$$

[7] To determine the precise value of the upper bound (which was simply denoted by "const."), we need to carefully evaluate the normalization factor originating from the delta function in Eq. (22). See, e.g., Ref. [8] for details.

[8] Choosing the other quantization creates negative-energy particles, and no negative-norm state appears. The cancellation in Eq. (36), as we will explain, never occurs, and thus the theory does not possess the perturbative S-matrix unitary. The relation between renormalizability and S-matrix unitarity suggests that the perturbative renormalizability property is never better in such quantization with higher-order derivatives.

It is customary to work in the center-of-mass frame, in which the three momenta satisfy

$$\mathbf{p}_2 = -\mathbf{p}_1, \qquad \mathbf{p}_4 = -\mathbf{p}_3. \tag{30}$$

We denote the scattering angle by θ. We consider two–two scattering of ψ_1, ψ_2, σ:

$$\psi_1(\mathbf{p}_1)\psi_1(\mathbf{p}_2) \to \psi_1(\mathbf{p}_3)\psi_1(\mathbf{p}_4), \tag{31a}$$

$$\psi_1(\mathbf{p}_1)\psi_1(\mathbf{p}_2) \to \psi_1(\mathbf{p}_3)\psi_2(\mathbf{p}_4), \tag{31b}$$

$$\psi_1(\mathbf{p}_1)\psi_1(\mathbf{p}_2) \to \psi_2(\mathbf{p}_3)\psi_2(\mathbf{p}_4), \tag{31c}$$

$$\sigma(\mathbf{p}_1)\psi_1(\mathbf{p}_2) \to \sigma(\mathbf{p}_3)\psi_1(\mathbf{p}_4), \tag{31d}$$

$$\sigma(\mathbf{p}_1)\psi_1(\mathbf{p}_2) \to \sigma(\mathbf{p}_3)\psi_2(\mathbf{p}_4). \tag{31e}$$

Note that (31b) and (31e) contain a single negative-norm particle ψ_2. The amplitudes are denoted by

$$\mathcal{M}\big(\psi_1(\mathbf{p}_1)\psi_1(\mathbf{p}_2) \to \psi_1(\mathbf{p}_3)\psi_1(\mathbf{p}_4)\big) \tag{32}$$

for the process in (31a), and similarly for the other four. The calculation of the amplitudes is somewhat technical and is given in Appendix E.

We first give the results for the elastic amplitudes for positive-norm particles given in Eqs. (E.7) and (E.15) as

$$\mathcal{M}\big(\psi_1(\mathbf{p}_1)\psi_1(\mathbf{p}_2) \to \psi_1(\mathbf{p}_3)\psi_1(\mathbf{p}_4)\big) = 8\alpha\big((6+2\cos^2\theta)|\mathbf{p}_1|^4 + 8|\mathbf{p}_1|^2 m_1^2 + 3m_1^4\big), \tag{33}$$

$$\mathcal{M}\big(\sigma(\mathbf{p}_1)\psi_1(\mathbf{p}_2) \to \sigma(\mathbf{p}_3)\psi_1(\mathbf{p}_4)\big) = -4\alpha'\big((1-\cos\theta)|\mathbf{p}_1|^2 + m_\sigma^2\big), \tag{34}$$

where the former is the renormalizable interaction and the latter the nonrenormalizable one. We immediately note that both amplitudes diverge in the limit of $|\mathbf{p}_1| \to \infty$. Hence the unitarity bound of Eq. (28) is apparently violated for both renormalizable and nonrenormalizable marginal interactions, Eqs. (10) and (14), respectively, of Sect. 2. This result is not unexpected because our model contains negative-norm states.

We then discuss whether these two amplitudes are consistent with the weaker bound of Eq. (25) implied by the optical theorem. Similarly to the tree-level unitarity argument, we work in the lowest-order approximation in the coupling constants. In this approximation, only the intermediate two-particle states contribute to the right-hand side of the relation in Eq. (25):

$$\begin{aligned}
&\Big|\mathcal{M}\left(\Psi_1(\mathbf{p}_1), \Psi_2(\mathbf{p}_2) \to \Psi_1(\mathbf{p}_1), \Psi_2(\mathbf{p}_2)\right)\Big| \\
&\geq \Big|\mathrm{Im}\,\mathcal{M}\left(\Psi_1(\mathbf{p}_1), \Psi_2(\mathbf{p}_2) \to \Psi_1(\mathbf{p}_1), \Psi_2(\mathbf{p}_2)\right)\Big| \\
&= \frac{1}{2}\left|\sum_{ij}\int \frac{d^3\mathbf{p}_i}{2E_i}\frac{d^3\mathbf{p}_j}{2E_j}\delta^4(p_1+p_2-p_i-p_j)(-1)^{n_{ij}}\Big|\mathcal{M}\left(\Psi_1(\mathbf{p}_1), \Psi_2(\mathbf{p}_2) \to \Psi_i(\mathbf{p}_i), \Psi_j(\mathbf{p}_j)\right)\Big|^2\right|,
\end{aligned} \tag{35}$$

where $\Psi_i(\mathbf{p}_i)$ means a one-particle state of ψ_1, ψ_2, or σ with the three-dimensional momentum $\mathbf{p}_i$, n_{ij} is the number of negative-norm particles ψ_2 in the state $|\Psi_i(\mathbf{p}_i), \Psi_j(\mathbf{p}_j)\rangle$, and we take all possible two-particle states in the sum with respect to i and j. E_i is the energy of the intermediate on-shell particles i, and similarly for other on-shell momenta. The measure of the integral is fixed by the inner product in Eq. (C.8) in Appendix C. In the center-of-mass frame ($\mathbf{p}_1 = -\mathbf{p}_2$, $\mathbf{p}_i = -\mathbf{p}_j$), the last line of Eq. (35) can be written as (see Appendix D)

$$\propto \left| \sum_{ij} (-1)^{n_{ij}} \frac{|\hat{\mathbf{p}}_i|}{(E_1 + E_2)} \int_0^\pi \sin\theta \left| \mathcal{M}\left(\Psi_1(\mathbf{p}_1), \Psi_2(-\mathbf{p}_1) \to \Psi_i(\hat{\mathbf{p}}_i), \Psi_j(-\hat{\mathbf{p}}_i)\right) \right|^2 d\theta \right|, \quad (36)$$

where θ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_i$. Also, $\hat{\mathbf{p}}_i$ is the on-shell momentum satisfying energy-momentum conservation, so that its absolute value $|\hat{\mathbf{p}}_i|$ is a function of $|\mathbf{p}_1|$ and θ. The proportionality constant is an unimportant numerical factor.

We can now compare the energy dependence of both sides in the inequality of Eq. (35) in the high-energy limit. Since we saw in Eq. (33) that $\mathcal{M}\big(\psi_1(\mathbf{p}_1)\psi_1(\mathbf{p}_2) \to \psi_1(\mathbf{p}_3)\psi_1(\mathbf{p}_4)\big)$ behaves as $|\mathbf{p}_1|^4$ in the UV limit, naive comparison of the energy dependence in each term seems to imply violation of the inequality in Eq. (35) even for the renormalizable interaction. However, we will find that negative contributions in the sum of the optical theorem, Eq. (35), bring about a cancellation that restores the optical theorem for the renormalizable interaction. On the other hand, we will find that the cancellation for the nonrenormalizable case is not enough to restore the inequality of Eq. (35). The argument below is based on the calculation of each scattering amplitude $\mathcal{M}\left(\Psi_1(\mathbf{p}_1), \Psi_2(-\mathbf{p}_1) \to \Psi_i(\hat{\mathbf{p}}_i), \Psi_j(-\hat{\mathbf{p}}_i)\right)$ in Appendix E.

3.2.1. *Renormalizable interaction term*

We first consider the renormalizable interaction in Eq. (11) and investigate the inequality of Eq. (35). The calculation is simple if we take the center-of-mass initial state with two ψ_1, $|i\rangle = |\psi_1, \psi_1\rangle$. The left-hand side of Eq. (35) is obtained by setting $\cos\theta = 1$ in the amplitude in Eq. (33),

$$\mathcal{M}\big(\psi_1(\mathbf{p}_1)\psi_1(-\mathbf{p}_1) \to \psi_1(\mathbf{p}_1)\psi_1(-\mathbf{p}_1)\big) = 8\alpha\big(8|\mathbf{p}_1|^4 + 8|\mathbf{p}_1|^2 m_1^2 + 3m_1^4\big) = \alpha\mathcal{O}\left(|\mathbf{p}_1|^4\right).$$

The right-hand side of the inequality in Eq. (35) is the sum of Eqs. (E.8), (E.11), and (E.14). Here, we have to multiply Eq. (E.11) by 2 because there are two cases, $(\Psi_i, \Psi_j) = (\psi_1, \psi_2), (\psi_2, \psi_1)$. Then, it can easily be seen that the cancellation occurs in the leading order and the next-to-leading order. As a result, the right-hand side of the inequality in Eq. (35) is

$$64\alpha^2 \frac{m_1^4 |\mathbf{p}_1|^5}{E_1 + E_2} \int_0^\pi \mathcal{O}\left(\left(\frac{m_1}{|\mathbf{p}_1|}\right)^0\right) d\theta = \alpha^2 \mathcal{O}\left(|\mathbf{p}_1|^4\right). \quad (37)$$

Hence, the energy dependence of both sides of the inequality in Eq. (35) at high energy is the same. Therefore, the inequality in Eq. (35) is satisfied at high energy, provided we take the coupling constant α sufficiently small.

3.2.2. *Nonrenormalizable interaction term*

Next, we will see that the inequality in Eq. (35) does not hold true for the nonrenormalizable interaction term of Eq. (15). The left-hand side of Eq. (35) is the amplitude for the forward scattering and it can be obtained by setting $\cos\theta = 1$ in Eq. (34),

$$\mathcal{M}\big(\sigma(\mathbf{p}_1)\psi_1(-\mathbf{p}_1) \to \sigma(\mathbf{p}_1)\psi_1(-\mathbf{p}_1)\big) = 4\alpha' m_\sigma^2. \quad (38)$$

Note that the leading-order term in the high-energy limit, that is proportional to $|\mathbf{p}_1|^2$, disappears, and that this approaches a constant in the high-energy limit. The right-hand side is the sum of Eqs. (E.16) and (E.19). The leading order of the sum is cancelled, but the next-to-leading order is not. The exact form of the sum is

$$16\alpha'^2 \frac{|\mathbf{p}_1|^5}{E_1+E_2}\int_0^{\pi}\sin\theta\left[(1-\cos\theta)^2\frac{3M^2}{4|\mathbf{p}_1|^2}+\mathcal{O}\left(\left(\frac{m_1}{|\mathbf{p}_1|}\right)^4\right)\right]d\theta=\alpha'^2\mathcal{O}(|\mathbf{p}_1|^2).$$

This means that the right-hand side diverges in the high-energy limit. Therefore, the inequality in Eq. (35) is not satisfied in the high-energy limit, i.e. the optical theorem is violated at some energy.

4. Summary

We have studied the relation between S-matrix unitarity, $SS^\dagger = 1$, and renormalizability in scalar field models with a higher-derivative kinetic term, which gives negative-norm states. Two cases, a theory with a renormalizable interaction and another with a nonrenormalizable interaction, have been investigated. Tree-level S-matrix unitarity is satisfied in the former, while it is violated in the latter. This provides evidence that S-matrix unitarity and renormalizability are related to each other even in theories with negative-norm states.

Combining our result with past analyses of the relation between the unitarity bound and renormalizability without negative-norm states [3–8], the relation between S-matrix unitarity and renormalizability is expected to hold true in generic QFTs, regardless of whether there are negative-norm states or not. Our results therefore support an expectation that the unitarity bound is useful in investigating the renormalizability of theories with norm positivity. We believe that it is particularly powerful when applied to quantum gravity theories.

The relation to the renormalizability of $R^2_{\mu\nu}$ gravity is a particularly interesting objective. $R^2_{\mu\nu}$ gravity is known to be renormalizable. The matter scattering of canonical fields (i.e. non-ghost fields) with a graviton propagator has been confirmed to satisfy the unitarity bound [14]. It will be interesting to see the optical theorem for graviton scattering. The counter terms in the renormalization can be combined due to the symmetry (i.e. the general covariance), and thus the number of required parameters of the counter terms is finite. Our result suggests that S-matrix unitarity also requires the symmetry. It should be possible to be seen in graviton scattering. From this point of view, it will also be very interesting to analyze the graviton loops to the scalar potential in $R^2_{\mu\nu}$ gravity theory. Actually, in Einstein gravity we know that gravitational corrections give rise to nonrenormalizable divergent terms [15–17]. It is suggested that $R^2_{\mu\nu}$ gravity is a useful attempt at this problem [18,19].

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Appendix A. Decomposition to two scalar modes

In Sect. 2 we rewrite the action in Eq. (1) into the action in Eq. (3) by the redefinition of fields in Eq. (2). One may wonder whether the transformation involving derivatives is allowed. We show in this appendix that this redefinition is consistent with the reformulation by introducing a Lagrange multiplier and an auxiliary field.

We introduce the auxiliary field χ as

$$\chi = \Box\phi. \tag{A.1}$$

This can be done by introducing the Lagrange multiplier λ, and the action in Eq. (1) can be expressed as

$$\mathcal{S}_2 = \int d^4x \left[-\frac{1}{2}(\chi - m_1^2\phi)(\chi - m_2^2\phi) - \lambda(\chi - \Box\phi) \right]. \tag{A.2}$$

We can easily check that the variation of the above action with respect to λ gives Eq. (A.1) and that, substituting Eq. (A.1), the above action is reduced to the original action of Eq. (1). In the action in Eq. (A.2) χ does not have the kinetic term, and thus the variation with respect to χ gives a constraint equation,

$$\chi = \frac{1}{2}\left(m_1^2 + m_2^2\right)\phi - \lambda. \tag{A.3}$$

Substituting this into the action in Eq. (A.2), we have

$$\mathcal{S}_2 = \int d^4x \left[\lambda\Box\phi + \frac{1}{2}\lambda^2 - \frac{1}{2}\left(m_1^2 + m_2^2\right)\lambda\phi + \frac{1}{8}\left(m_1^2 - m_2^2\right)^2\phi^2 \right]. \tag{A.4}$$

This action can be diagonalized by the following field redefinition:

$$\lambda = -\frac{M}{2}(\psi_1 + \psi_2) \qquad \text{and} \qquad \phi = \frac{(\psi_2 - \psi_1)}{M}, \tag{A.5}$$

where $M = \sqrt{m_2^2 - m_1^2}$. We can see that the form of ϕ is the same as that in Eq. (2). Then, the action in Eq. (A.4) becomes

$$\mathcal{S}_2 = \int d^4x \left[\frac{1}{2}\psi_1(\Box - m_1^2)\psi_1 - \frac{1}{2}\psi_2(\Box - m_2^2)\psi_2 \right], \tag{A.6}$$

which is the same as the action in Eq. (3).

For the interaction term $\mathcal{L}_{\mathrm{int}}(\partial, \phi)$, we can substitute Eq. (A.5). The reason is as follows. When we introduce the auxiliary field χ, we can leave ϕ in the interaction term untouched. Then, the elimination of χ can be done in the same way and we do the field redefinition of Eq. (A.5). This is nothing but substituting Eq. (A.5).

Appendix B. Interaction with $\Box$ operator

In this paper we discuss the relation between renormalizability and S-matrix unitarity. However, in the latter analysis, if the interaction term has $\Box$ operators attached to external lines then the amplitude is suppressed. Therefore, in this order, we cannot see the relation. If we go to the higher-point amplitude we would see the correspondence, but calculation of the higher-point amplitude is involved. We briefly show why $\Box$ operators make the discussion different.

We naively expect that the $\Box$ operator gives the $\mathcal{O}(E^2)$ contribution. However, if the $\Box$ operator acts on an external line, it becomes m^2 $(= \mathcal{O}(E^0))$ by the on-shell condition; that is, the UV behavior becomes mild. This is also true even in the canonical theory (without negative-norm states). For instance, we consider the following action for a canonical scalar field ζ:

$$\mathcal{S} = \int d^4x \left[\frac{1}{2}\zeta\left(\Box - m^2\right)\zeta + \alpha_\zeta \zeta^3 \Box \zeta\right]. \tag{B.1}$$

Based on the power-counting argument, the interaction term is not renormalizable. Meanwhile, the tree-level four-point scattering amplitude is constant, and thus it satisfies the unitarity bound. To check the violation of the unitarity bound, we have to consider a tree diagram that has internal lines where the $\Box$ operator is assigned. If we consider the six-point amplitude an internal line appears, and thus it is expected to violate the unitarity bound.

The situation is the same in the case with higher derivatives that we analyze in this paper. We have seven kinds of four-point marginal vertices. Nevertheless, they are related to the integration by parts, and actually all of them can be expressed by the following three vertex functions:

$$(\Box^2\phi)\phi^3, \qquad (\Box\phi)^2\phi^2, \qquad \left((\partial_\mu\phi)^2\right)^2. \tag{B.2}$$

The last term is considered in this paper as a renormalizable vertex function. The other two are nonrenormalizable vertex terms, but because of the $\Box$ operator we have to go the six-point scattering amplitude to see violation of the S-matrix unitarity.

Another possible way to see the correspondence for a nonrenormalizable operator would be considering a three-point marginal vertex term and constructing the four-point scattering amplitude with two vertices. However, the situation is the same as the case with the four-point vertex. All three-point marginal operators can be expressed by

$$(\Box\phi)^2\phi, \qquad (\Box\phi)(\partial_\mu\phi)^2. \tag{B.3}$$

The nonrenormalizable term is the former, but the $\Box$ operator operates on a different ϕ. In each vertex of tree-level four-point amplitudes only one line is internal, and thus we still have one $\Box$ operator in the external line. Although it would be interesting to check the six-point amplitude, the calculation is complicated. Meanwhile, we can see the relation by the four-point vertex in the simple model that we introduce in Sect. 2. Therefore, we do not analyze the nonrenormalizable interaction terms appearing in (B.3) and (B.2).

Appendix C. Quantization

We show the quantization for the fields ψ_1, ψ_2, and σ. The quantization for the canonical fields ψ_1 and σ is done in the usual manner:

$$\psi_1 = \int \frac{d^3p}{\sqrt{(2\pi)^3E^{(1)}}}\left\{a_1(\mathbf{p})e^{ipx} + a_1^\dagger(\mathbf{p})e^{-ipx}\right\}, \quad \left(E^{(1)} := \sqrt{\mathbf{p}^2 + m_1^2}(= -p_0^{(1)})\right), \tag{C.1}$$

$$\sigma = \int \frac{d^3p}{\sqrt{(2\pi)^3E^{(\sigma)}}}\left\{a_\sigma(\mathbf{p})e^{ipx} + a_\sigma^\dagger(\mathbf{p})e^{-ipx}\right\}, \quad \left(E^{(\sigma)} := \sqrt{\mathbf{p}^2 + m_\sigma^2}(= -p_0^{(\sigma)})\right). \tag{C.2}$$

The commutation relations of the creation and annihilation operators for the positive-norm fields ψ_1 and σ are

$$\left[a_1(\mathbf{p}), a_1^\dagger(\mathbf{k})\right] = \delta^3(\mathbf{p} - \mathbf{k}), \qquad \left[a_\sigma(\mathbf{p}), a_\sigma^\dagger(\mathbf{k})\right] = \delta^3(\mathbf{p} - \mathbf{k}). \tag{C.3}$$

The quantization of ψ_2 is unsettled due to the negative sign of the kinetic term. We have two natural choices to define creation and annihilation operators for ψ_2:

$$\psi_2 = \int \frac{d^3p}{\sqrt{(2\pi)^3 E^{(2)}}} \left\{ a_2(\mathbf{p}) e^{ipx} + a_2^\dagger(\mathbf{p}) e^{-ipx} \right\}, \quad \left(E^{(2)} := \sqrt{\mathbf{p}^2 + m_2^2} (= -p_0^{(2)}) \right) \tag{C.4}$$

or

$$\psi_2 = \int \frac{d^3p}{\sqrt{(2\pi)^3 E^{(2)}}} \left\{ \tilde{a}_2^\dagger(\mathbf{p}) e^{ipx} + \tilde{a}_2(\mathbf{p}) e^{-ipx} \right\}, \quad \left(E^{(2)} := \sqrt{\mathbf{p}^2 + m_2^2} (= -p_0^{(2)}) \right). \tag{C.5}$$

With the former definition, the creation operator creates a positive-energy particle, but the sign of the commutation relation is flipped, $\left[a_2(\mathbf{p}), a_2^\dagger(\mathbf{k})\right] = -\delta^3(\mathbf{p} - \mathbf{k})$. On the other hand, the latter definition gives the usual commutation relation, $\left[\tilde{a}_2(\mathbf{p}), \tilde{a}_2^\dagger(\mathbf{k})\right] = \delta^3(\mathbf{p} - \mathbf{k})$, but particle states have negative energy. We often choose the latter definition for physical fields (especially in cosmology), because the flipped sign of the kinetic term would mean negativity of the kinetic energy and any state satisfies positivity of the norm. However, the negative energy leads to instability of the vacuum state (the ghost instability), and its decay rate is indeed infinity! Thus, the perturbative approach breaks down.[9] The renormalizability of the theory with higher-order derivative is discussed with the former definition. Although it is unclear how to interpret the negative-norm state, without the negative-energy state the perturbative approach is applicable because of the absence of the ghost instability. Therefore, we choose the former definition here.

The particle states are usually expressed by the covariant creation operators,

$$A_i^\dagger(\mathbf{p}) = a_i^\dagger(\mathbf{p})(2\pi)^{3/2}\sqrt{2E^{(i)}(\mathbf{k})} \qquad (i = 1, 2, \sigma), \tag{C.6}$$

as

$$|\mathbf{p}_{1_1}, \ldots, \mathbf{p}_{1_l}; \mathbf{p}_{2_1}, \ldots, \mathbf{p}_{2_m}; \mathbf{p}_{\sigma_1}, \ldots, \mathbf{p}_{\sigma_n}\rangle := \\ A_1^\dagger(\mathbf{p}_{1_1}) \cdots A_1^\dagger(\mathbf{p}_{1_l}) A_2^\dagger(\mathbf{p}_{2_1}) \cdots A_2^\dagger(\mathbf{p}_{2_m}) A_\sigma^\dagger(\mathbf{p}_{\sigma_1}) \cdots A_\sigma^\dagger(\mathbf{p}_{\sigma_m})|0\rangle. \tag{C.7}$$

Based on the covariant creation operators $A_i^\dagger(\mathbf{p})$, the usual Feynman rule gives the scattering amplitude. Then, the norm of the one-particle state becomes

$$\langle 0|A_i(\mathbf{k}) A_j^\dagger(\mathbf{p})|0\rangle = \eta_i (2\pi)^3 2E^{(i)} \delta^3(\mathbf{k} - \mathbf{p}) \delta_{ij}, \tag{C.8}$$

with

$$\eta_1 = 1 = \eta_\sigma, \qquad \eta_2 = -1. \tag{C.9}$$

The normalization factor C_X for particle states defined in Eq. (21) is fixed by Eq. (C.8).

Appendix D. Derivation of Eq. (36)

Here, we show the derivation of Eq. (36) from the right-hand side of the inequality in Eq. (35). Three of the four delta functions represent the momentum conservation law, and there each component of $\mathbf{p}_j$ appears linearly. Therefore, the integration with respect to $\mathbf{p}_j$ can be done easily and in the

[9] If Lorentz symmetry is broken, the decay rate of vacuum can be mild and we would be able to live with the ghost modes [20,21].

center-of-mass frame it gives the replacement of $\mathbf{p}_j$ with $-\mathbf{p}_i$ in the integrand. The remaining delta function, namely the energy conservation, can be deformed as follows:

$$\begin{aligned}\delta(E_1+E_2-E_i-E_j) &= \delta\left(\sqrt{|\mathbf{p}_1|^2+\mu_1^2}+\sqrt{|\mathbf{p}_1|^2+\mu_2^2}-\sqrt{|\mathbf{p}_i|^2+\mu_i^2}-\sqrt{|\mathbf{p}_i|^2+\mu_j^2}\right)\\ &= \frac{E_iE_j}{|\mathbf{p}_1|(E_i+E_j)}\delta\big(|\mathbf{p}_i|-|\hat{\mathbf{p}}_i|(|\mathbf{p}_1|)\big)\\ &= \frac{E_iE_j}{|\mathbf{p}_i|(E_1+E_2)}\delta\big(|\mathbf{p}_i|-|\hat{\mathbf{p}}_i|(|\mathbf{p}_1|)\big),\end{aligned} \tag{D.1}$$

where μ_k ($k = 1, 2, i, j$) shows the mass corresponding to the particle with momentum $\mathbf{p}_k$. Taking spherical coordinates for the momentum space of $\mathbf{p}_i$, $d^3\mathbf{p}_i$ is written as $|\mathbf{p}_i|^2 \sin\theta_p d|\mathbf{p}_i| d\theta_p d\phi_p$. Substituting this and Eq. (D.1), integrating with respect to ϕ_p, and ignoring the unimportant numerical factor, we can obtain the form in Eq. (36).

Appendix E. Calculation of the scattering amplitudes

In Sect. 3.2 we considered five amplitudes of two–two scattering of ψ_1, ψ_2, and σ: (31a)–(31e). The momentum assignment of scattering is as

$$p_1+p_2 \to p_3+p_4 \tag{E.1}$$

for all five processes. Here, p_i are four-momenta. The amplitudes are denoted by

$$\mathcal{M}\big(\psi_1(\mathbf{p}_1)\psi_1(\mathbf{p}_2) \to \psi_1(\mathbf{p}_3)\psi_1(\mathbf{p}_4)\big) \tag{E.2}$$

for the process in (31a), and similarly for the other four. After taking account of the momentum conservation we write

$$\mathcal{M}\big(\psi_1(\mathbf{p}_1)\psi_1(\mathbf{p}_2) \to \psi_1(\hat{\mathbf{p}}_3)\psi_1(\hat{\mathbf{p}}_4)\big), \tag{E.3}$$

and similarly for the other processes. Here, $\hat{\mathbf{p}}_i$ is the momentum satisfying both the on-shell condition and energy-momentum conservation. The momentum variables $\mathbf{p}_1, \ldots, \mathbf{p}_4$ are often omitted when it is understood. Throughout this paper we take the center-of-mass frame, hence

$$\mathbf{p}_2 = -\mathbf{p}_1, \qquad \mathbf{p}_4 = -\mathbf{p}_3. \tag{E.4}$$

The scattering angle is denoted by θ. $|\hat{\mathbf{p}}_i|$ is expressed in terms of $|\mathbf{p}_1|$, but its expression is different depending on the processes in (31a)–(31e). The first three processes arise from the interaction term in Eq. (14) and hence their amplitudes are the same except for the all-over sign ϵ,

$$\mathcal{M} = 8\epsilon\alpha\,[(p_1\cdot p_2)\,(p_3\cdot p_4)+(p_1\cdot p_3)\,(p_2\cdot p_4)+(p_1\cdot p_4)\,(p_2\cdot p_3)], \tag{E.5}$$

where $\epsilon = 1$ if the final state gives a positive norm, otherwise $\epsilon = -1$. The last two processes arise from the interaction term in Eq. (16), and their amplitudes are given by

$$\mathcal{M} = 4\epsilon\alpha'\,(p_1\cdot p_3), \tag{E.6}$$

where $\epsilon = 1$ if the final state gives a positive norm, otherwise $\epsilon = -1$. However, the sign ϵ is immaterial in computing Eqs. (35) and (36), since the absolute values $|\mathcal{M}|^2$ are used there.

E.1. $\psi_1 + \psi_1 \to \psi_1 + \psi_1$

The two particles of the initial state and the two in the final state all have the same mass m_1 and hence $|\hat{\mathbf{p}}_i| = |\mathbf{p}_1|$. Using this in the $\mathcal{M}$ of Eq. (E.5), we have

$$\mathcal{M}\big(\psi_1(\mathbf{p}_1)\psi_1(-\mathbf{p}_2) \to \psi_1(\hat{\mathbf{p}}_i)\psi_1(-\hat{\mathbf{p}}_i)\big) = 8\alpha\big((6+2\cos^2\theta)|\mathbf{p}_1|^4 + 8m_1^2|\mathbf{p}_1|^2 + 3m_1^4\big). \tag{E.7}$$

Equation (36) of Sect. 3.2 now takes the form

$$64\alpha^2 \frac{m_1^4|\mathbf{p}_1|^5}{E_1+E_2}\int_0^\pi \sin\theta(6+2\cos^2\theta)^2\left[\left(\frac{|\mathbf{p}_1|}{m_1}\right)^4 + \frac{16}{6+2\cos^2\theta}\left(\frac{|\mathbf{p}_1|}{m_1}\right)^2 + \mathcal{O}\left(\left(\frac{m_1}{|\mathbf{p}_1|}\right)^0\right)\right]d\theta. \tag{E.8}$$

E.2. $\psi_1 + \psi_1 \to \psi_1 + \psi_2$

The expression for $|\hat{\mathbf{p}}_i|$ is slightly complicated, containing $M_2 = {m_2}^2 - {m_1}^2$:

$$\begin{aligned} |\hat{\mathbf{p}}_i| &= |\mathbf{p}_1|\sqrt{1 - \frac{M^2}{2|\mathbf{p}_1|^2} + \frac{M^4}{16|\mathbf{p}_1|^2(|\mathbf{p}_1|^2+m_1^2)}} \\ &= |\mathbf{p}_1|\left[1 - \frac{M^2}{4|\mathbf{p}_1|^2} + \mathcal{O}\left(\left(\frac{M}{|\mathbf{p}_1|}\right)^4\right)\right]. \end{aligned} \tag{E.9}$$

Using this $|\hat{\mathbf{p}}_i|$ in the $\mathcal{M}$ of Eq. (E.5), we have

$$\begin{aligned} &\mathcal{M}\big(\psi_1(\mathbf{p}_1)\psi_1(-\mathbf{p}_2) \to \psi_1(\hat{\mathbf{p}}_i)\psi_2(-\hat{\mathbf{p}}_i)\big) \\ &\quad = 8\alpha\left((6+2\cos^2\theta)|\mathbf{p}_1|^4 + \left[8m_1^2 - M^2(1+\cos^2\theta)\right]|\mathbf{p}_1|^2 + m_1^2\left(3m_1^2 - \frac{M^2}{2}\right)\right. \\ &\qquad\qquad \left. - \frac{M^4}{8} + \frac{M^4|\mathbf{p}_1|^2}{8(|\mathbf{p}_1|^2+m_1^2)}\cos^2\theta\right). \end{aligned} \tag{E.10}$$

Equation (36) now takes the form

$$\begin{aligned} &-64\alpha^2\frac{m_1^4|\mathbf{p}_1|^5}{E_1+E_2}\int_0^\pi \sin\theta(6+2\cos^2\theta)^2 \\ &\times\left[\left(\frac{|\mathbf{p}_1|}{m_1}\right)^4 + \frac{1}{6+2\cos^2\theta}\left(16 - \frac{M^2}{2m_1^2}(7+5\cos^2\theta)\right)\left(\frac{|\mathbf{p}_1|}{m_1}\right)^2 + \mathcal{O}\left(\left(\frac{m_1}{|\mathbf{p}_1|}\right)^0\right)\right]d\theta. \end{aligned} \tag{E.11}$$

E.3. $\psi_1 + \psi_1 \to \psi_2 + \psi_2$

The two particles in the final state have the same mass m_2 and hence $|\hat{\mathbf{p}}_i|$ is easily found:

$$\begin{aligned} |\hat{\mathbf{p}}_i| &= |\mathbf{p}_1|\sqrt{1 - \frac{M^2}{|\mathbf{p}_1|^2}} \\ &= |\mathbf{p}_1|\left[\left(1 - \frac{M^2}{2|\mathbf{p}_1|^2}\right) + \mathcal{O}\left(\left(\frac{M}{|\mathbf{p}_1|}\right)^4\right)\right]. \end{aligned} \tag{E.12}$$

Using this $|\hat{\mathbf{p}}_1|$ in the $\mathcal{M}$ of Eq. (E.5), we have

$$\mathcal{M}\big(\psi_1(\mathbf{p}_1)\psi_1(-\mathbf{p}_2) \to \psi_2(\hat{\mathbf{p}}_i)\psi_2(-\hat{\mathbf{p}}_i)\big) = 8\alpha\left((6+2\cos^2\theta)|\mathbf{p}_1|^4 + \left[8m_1^2 - 2M^2(1+\cos^2\theta)\right]|\mathbf{p}_1|^2 + m_1^2(3m_1^2 - M^2)\right). \quad \text{(E.13)}$$

Equation (36) now takes the form

$$64\alpha^2 \frac{m_1^4|\mathbf{p}_1|^5}{E_1+E_2}\int_0^\pi \sin\theta(6+2\cos^2\theta)^2 \times \left[\left(\frac{|\mathbf{p}_1|}{m_1}\right)^4 + \frac{1}{6+2\cos^2\theta}\left(16 - \frac{M^2}{m_1^2}(7+5\cos^2\theta)\right)\left(\frac{|\mathbf{p}_1|}{m_1}\right)^2 + \mathcal{O}\left(\left(\frac{m_1}{|\mathbf{p}_1|}\right)^0\right)\right] d\theta. \quad \text{(E.14)}$$

We note that if we set $M^2 = 0$, all three amplitudes in Eqs. (E.7), (E.10), and (E.13) coincide.

E.4. $\sigma + \psi_1 \to \sigma + \psi_1$

Since the two final-state particles are the same as the two initial ones, $|\hat{\mathbf{p}}_i| = |\mathbf{p}_1|$. Using this $|\hat{\mathbf{p}}_i|$ in the $\mathcal{M}$ of Eq. (E.6), we have

$$\mathcal{M}\big(\sigma(\mathbf{p}_1)\psi_1(\mathbf{p}_2) \to \sigma(\hat{\mathbf{p}}_i)\psi_1(\hat{\mathbf{p}}_j)\big) = -4\alpha'\big((1-\cos\theta)|\mathbf{p}_1|^2 + m_\sigma^2\big). \quad \text{(E.15)}$$

Equation (36) now takes the form

$$16\alpha'^2 \frac{|\mathbf{p}_1|^5}{E_1+E_2}\int_0^\pi \sin\theta\left[(1-\cos\theta)^2 + (1-\cos\theta)\frac{2m_\sigma^2}{|\mathbf{p}_1|^2} + \mathcal{O}\left(\left(\frac{m_1}{|\mathbf{p}_1|}\right)^4\right)\right] d\theta. \quad \text{(E.16)}$$

E.5. $\sigma + \psi_1 \to \sigma + \psi_2$

This process involves three unequal masses, and $|\hat{\mathbf{p}}_i|$ in terms of $|\mathbf{p}_1|$ is more involved; we only show the high-energy approximation,

$$|\hat{\mathbf{p}}_1| = |\mathbf{p}_1|\left[1 - \frac{M^2}{4|\mathbf{p}_1|^2} + \mathcal{O}\left(\left(\frac{m_1}{|\mathbf{p}_1|}\right)^4\right)\right]. \quad \text{(E.17)}$$

Using this $|\hat{\mathbf{p}}_i|$ in the $\mathcal{M}$ of Eq. (E.6), we have

$$\begin{aligned}\mathcal{M}\big(\sigma(\mathbf{p}_1)\psi_1(\mathbf{p}_2) \to \sigma(\hat{\mathbf{p}}_i)\psi_2(\hat{\mathbf{p}}_j)\big) &= 4\alpha'\left(\sqrt{|\mathbf{p}_1|^2 + m_\sigma^2}\sqrt{|\hat{\mathbf{p}}_i|^2 + m_\sigma^2} - |\mathbf{p}_1||\hat{\mathbf{p}}_i|\cos\theta\right) \\ &= 4\alpha'|\mathbf{p}_1|^2\left[(1-\cos\theta)\left(1 - \frac{M^2}{4|\mathbf{p}_1|^2}\right)\frac{m_\sigma^2}{|\mathbf{p}_1|^2} + \mathcal{O}\left(\left(\frac{m_1}{|\mathbf{p}_1|}\right)^4\right)\right]. \quad \text{(E.18)}\end{aligned}$$

Equation (36) now takes the form

$$-16\alpha'^2 \frac{|\mathbf{p}_1|^5}{E_1+E_2}\int_0^\pi \sin\theta\left[(1-\cos\theta)^2\left(1 - \frac{3M^2}{4|\mathbf{p}_1|^2}\right) + \frac{2m_\sigma^2}{|\mathbf{p}_1|^2} + \mathcal{O}\left(\left(\frac{m_1}{|\mathbf{p}_1|}\right)^4\right)\right] d\theta. \quad \text{(E.19)}$$

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