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We show that the observed quark/lepton mass hierarchy can be realized dynamically on an interval extra dimension with point interactions. In our model, the positions of the point interactions play a crucial role in controlling the quark/lepton mass hierarchy and are determined by the minimization of the Casimir energy. By use of the exact extra-dimensional coordinate-dependent vacuum expectation value of a gauge-singlet scalar, we find that there is a parameter set, where the positions of the point interactions are stabilized and fixed, which can reproduce the experimental values of the quark masses precisely enough, while the charged lepton part is less relevant. We also show that possible mixings among the charged leptons will improve the situation significantly.
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Subject Index   B33, B40, B43, B53, B55

1. Introduction

Even after the discovery of the Higgs boson, several issues associated with the Higgs boson still remain to be solved in the Standard Model (SM). One is why the observed quark/lepton mass hierarchy is so drastic. In the SM, quarks and leptons acquire their masses through the Yukawa interactions where they pick up the vacuum expectation value (VEV) of the Higgs field. Here, we should accept the imposition of $\mathcal{O}(10^5)$-order hierarchical values on Yukawa couplings, which look very unnatural.

Another unnatural request from the SM is associated with the matter content of the fermions. In both the quark and lepton sectors, there are three similar but different copies of quarks and leptons, which possess exactly the same quantum numbers except for their masses. The origin of such generational structure is beyond the scope of the SM, and is still to be unveiled.

One fascinating way to address the common origin of the above two issues is to introduce the objects' so-called point interactions on an interval extra dimension in five dimensions.[1] The point interactions provide extra boundary conditions for each of the 5D quarks and 5D leptons, and

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[1] Another way is to introduce the magnetic flux or the magnetized orbifold in the extra dimensions [1–14]. There are also several ways to address the SM problems in the context of 4D gauge theories, by use of noncompact gauge symmetry [15–19].

can induce degenerated chiral zero modes. This is interpreted as spontaneous generation of the three matter generations in the SM when we introduce two point interactions for each 5D fermion. Appealing properties in this direction are that a 5D fermion leads to three chiral zero modes, and they are localized to different segments of each other, whose endpoints are identified by the two corresponding point interactions. Due to these properties, a three-by-three mass matrix is realized as m_{ij} ($i, j = 1, 2, 3$). The generation dependence of the 4D mass matrix m_{ij} is described in the following manner:

$$
m_{ij} \propto \int_0^L dy \, \langle \Phi(y) \rangle \big(g^{(0)}_{\psi_{i\mathrm{L}}}(y) \big)^* f^{(0)}_{\psi'_{j\mathrm{R}}}(y),
\tag{1.1}
$$

where L represents the length of the interval, $\int_0^L dy$ is an integration along the range of the extra-dimensional coordinate shown, and $g^{(0)}_{\psi_{i\mathrm{L}}}(y) \big(f^{(0)}_{\psi'_{j\mathrm{R}}}(y) \big)$ describes mode functions of ith (jth) generation left- (right-)handed chiral zero mode $\psi_{i\mathrm{L}}(x) \big(\psi'_{j\mathrm{R}}(x) \big)$. $\langle \Phi(y) \rangle$ represents the contribution from the classical configuration of the scalar that appears in the 5D Yukawa term and its VEV shows the dependence on y. If not only $g^{(0)}_{\psi_{i\mathrm{L}}}(y)$ and $f^{(0)}_{\psi'_{j\mathrm{R}}}(y)$ but also $\langle \Phi(y) \rangle$ are localized functions, a sizable mass hierarchy can appear through the overlap integrals in Eq. (1.1).

In this paper, based on the above mechanism and the previous research [20–22], we perform a numerical investigation to appraise whether the scenario can reproduce the observed values of the quark/lepton hierarchical masses dynamically (see Refs. [23–26] for dynamical generations of the mass hierarchy). The positions of the point interactions, which play an important role in controlling the magnitude of the overlap integrals, seem to be free parameters. Nonetheless, they are not free but are determined dynamically through the vacuum configuration to minimize the Casimir energy (see Refs. [27,28] and also Refs. [29–32] as related works). This dynamical determination of the positions of the point interactions reduces the number of the free parameters. Also, to accelerate hierarchical values in the overlap integral in Eq. (1.1), we employ a new type of extra-dimensional coordinate-dependent configuration in $\langle \Phi(y) \rangle$, which is also obtained dynamically by solving the equation of motion with the quartic interaction. It should be noted that in the prior research [33], the VEV of the scalar $\langle \Phi(y) \rangle$ is also obtained dynamically. However, a different functional type of the VEV was adopted (see Sect. 3.2 for details). Performing numerical searches brings us to the conclusion that the hierarchical masses of the quarks are fully reproduced, while the charged leptons do not work well. Introduction of flavor mixings may improve our current results, especially for the charged leptons.

The paper is organized as follows. In Sect. 2, we briefly review the extra-dimensional model with point interactions on an interval. In Sect. 3, we discuss the dynamics of the Casimir energy, the potential minimization of the scalar Φ, and their implication for the mass hierarchy. In Sect. 4, we investigate searches for sets of desirable input parameters and show our results. Section 5 is devoted to the conclusion and discussion.

2. The model with point interactions on an interval

2.1. Basic idea

In this subsection, we provide a brief review of the extra-dimensional model, which sheds light on spontaneous generation of the three generations of the fermion matter content, the hierarchical structure of the observed quarks and charged leptons, and their mixing structures in the SM, simultaneously. This kind of scenario was first proposed in Ref. [20] on an interval to explore the

Fig. 1. A schematic figure of the interval extra dimension with the point interactions. The black dots at $y = L_1, L_2$ indicate the point interactions.
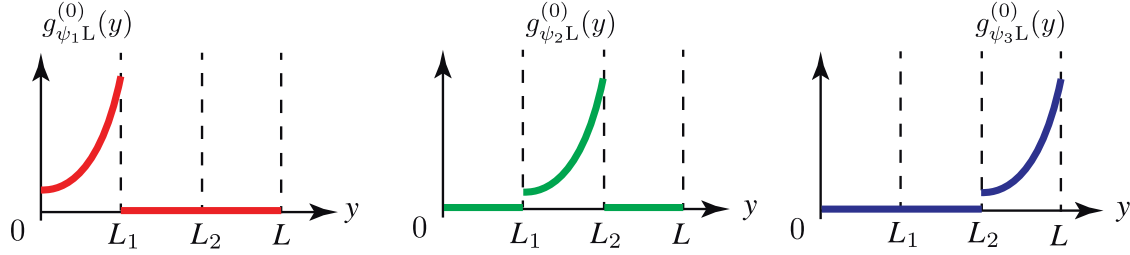

Fig. 2. A schematic figure of the profiles of the three chiral zero modes. Each zero mode localizes towards the different boundary points but in a similar way.

quark sector, where objects' so-called point interactions play a significant role. The point interactions describe possible singularities in 1D quantum mechanics, by which profiles of particles along the extra spacial direction y are represented. This statement is rephrased that we can add "extra (or generalized) boundary conditions" for all kinds of 5D fields in the bulk space, in addition to the boundary points (see Appendix A of Ref. [33] for more details). Three chiral zero modes are generated from a single 5D fermion by imposing the Dirichlet boundary conditions at the point interactions [20]. A key point to realize mass hierarchy is that we can introduce a bulk mass for each of such 5D fermions respectively, which makes the profiles of the three zero modes localized towards the boundary points as shown in Figs. 1 and 2.

Here, if the scalar appearing in the Yukawa sector of such 5D fermions possesses an extra-dimensional coordinate-dependent VEV, it is clear that sizable mass hierarchy is generated on an interval. It is noted that this scenario has also been developed in the S^1 geometry, where a complex phase for describing the CP-violating phase of the Cabibbo–Kobayashi–Maskawa (CKM) matrix can emerge, originating from a twisted boundary condition [21,22].

In the scenario, an $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge theory is unfolded on an interval, the relevant part of the 5D action of which for our discussion reads

$$S = S_{\text{quark}} + S_{\text{lepton}} + S_{\text{Higgs}} + S_{\text{singlet}} + S_{\text{Yukawa}}, \tag{2.1}$$

$$S_{\text{quark}} = \int d^4x \int_0^L dy \left[\overline{Q} \left(i\Gamma^M D_M^{(Q)} + M_Q \right) Q + \overline{\mathcal{U}} \left(i\Gamma^M D_M^{(\mathcal{U})} + M_{\mathcal{U}} \right) \mathcal{U} + \overline{\mathcal{D}} \left(i\Gamma^M D_M^{(\mathcal{D})} + M_{\mathcal{D}} \right) \mathcal{D} \right], \tag{2.2}$$

$$S_{\text{lepton}} = \int d^4x \int_0^L dy \left[\overline{L} \left(i\Gamma^M D_M^{(L)} + M_L \right) L + \overline{\mathcal{E}} \left(i\Gamma^M D_M^{(\mathcal{E})} + M_{\mathcal{E}} \right) \mathcal{E} \right], \tag{2.3}$$

$$S_{\text{Higgs}} = \int d^4x \int_0^L dy \left[H^\dagger \left(D^M D_M + M_H^2 \right) H - \frac{\lambda_H}{2} (H^\dagger H)^2 \right], \tag{2.4}$$

$$S_{\text{singlet}} = \int d^4x \int_0^L dy \left[\Phi^\dagger \left(\partial^M \partial_M - M_\Phi^2 \right) \Phi - \frac{\lambda_\Phi}{2} (\Phi^\dagger \Phi)^2 \right], \tag{2.5}$$

$$S_{\text{Yukawa}} = \int d^4x \int_0^L dy \left[\Phi \left(-\mathcal{Y}_u \overline{Q} (i\sigma_2 H^*) \mathcal{U} - \mathcal{Y}_d \overline{Q} H \mathcal{D} - \mathcal{Y}_e \overline{L} H \mathcal{E} \right) + (\text{h.c.}) \right], \tag{2.6}$$

which consists of a 5D quark doublet field Q and a 5D lepton doublet field L; singlet fields for an up-type quark $\mathcal{U}$, a down-type quark $\mathcal{D}$, and a charged lepton $\mathcal{E}$; the Higgs doublet H and a gauge-singlet complex scalar Φ; the Pauli matrix σ_2, where we skipped to show the apparent gauge sector of the current scenario; we usually suppress the 5D coordinates as variables of 5D fields. In this manuscript, we do not discuss the minuscule active neutrino mass generation so that the (5D) neutrino singlet field is not introduced (see the discussions on the S^1 geometry without [21]/with [34] Majorana-like mass terms). The capital Latin characters $M, N, \ldots$ represent the 5D Minkowski indices, where the 4D part is described by the Greek characters $\mu, \nu, \ldots$. The 5D coordinate x^M is thus decomposed as $\{x^\mu, y\}$ where x^μ denotes the 4D Minkowski coordinate and y denotes the extra-dimensional one. Our convention for the 5D flat metric reads $\eta_{MN} = \mathrm{diag}(-1, +1, +1, +1, +1)$ and the representation of the 5D Clifford algebra is chosen as $\{\Gamma_M, \Gamma_N\} = -2\eta_{MN}$, where the concrete forms of the 5D gamma matrices are adopted as $\Gamma_\mu = \gamma_\mu$ and $\Gamma_y = -i\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$. Various $D_M^{(\Psi)}$ ($\Psi = Q, \mathcal{U}, \mathcal{D}, L, \mathcal{E}$) symbols represent the corresponding covariant derivatives, M_Ψ ($\Psi = Q, \mathcal{U}, \mathcal{D}, L, \mathcal{E}$) are bulk masses for the 5D fermions, M_H^2 (M_Φ^2) and λ_H (λ_Φ) are squared mass scales and quartic couplings for the Higgs (the gauge-singlet scalar). $\mathcal{Y}_u$, $\mathcal{Y}_d$, and $\mathcal{Y}_e$ are overall Yukawa couplings that do not hold indices to discriminate matter generations since all the 5D fermions are assumed to be one generation. The sign in front of M_H^2 is chosen to ignite the Higgs mechanism. Obedience to the conditions $\lambda_H > 0$ and $\lambda_\Phi > 0$ is compulsory to ensure the stability of the whole potential.[2]

Here we provide several comments.

- As we mentioned above, we introduce two point interactions for each of the 5D fermions to realize three generations at the chiral zero mode sector, but not for the scalars and the gauge bosons.
- We impose the discrete symmetry, $H \to -H$ and $\Phi \to -\Phi$, which prohibits the "ordinary" Yukawa terms like $\overline{Q}(i\sigma_2 H^*)\mathcal{U}$ and $\Phi\overline{Q}Q$ and others likewise. We assume that H has the ordinary constant VEV $\langle H \rangle$ to break the SM gauge group suitably, and Φ acquires a y-dependent VEV $\langle \Phi \rangle$ to generate hierarchical structure in the Yukawa sector. Thus, the above "higher-dimensional" Yukawa terms are suitable for our purpose. The details of the y-dependent gauge-singlet scalar VEV profile will be provided in Sect. 3.2.
- Under the current setup on an interval, no physical CP-violating phase can emerge since only overall complexities can be realized by the Yukawa couplings $\mathcal{Y}_u$, $\mathcal{Y}_d$, and $\mathcal{Y}_e$ and absorbed into field redefinitions. We conquer this obstacle by moving to the S^1 geometry easily [21,22], although we do not discuss this point in this paper.

2.2. Appearance of three-generation mass matrices

As discussed in Ref. [20], the introduction of two suitable Dirichlet boundary conditions by point interactions for a 5D fermion Ψ in the bulk space, in addition to the two endpoints of the interval, as $\Psi_\mathrm{R} \equiv \left(\frac{1+\gamma_5}{2}\right)\Psi = 0$ or $\Psi_\mathrm{L} \equiv \left(\frac{1-\gamma_5}{2}\right)\Psi = 0$ leads to the realization of three localized left- or right-handed chiral zero modes via Ψ, respectively. Concretely, the following conditions are imposed

---

[2] In general, the mixing term $(H^\dagger H)(\Phi^\dagger \Phi)$ can be written down. However, the existence of this term leads to gauge universality violation [20]. Then we set the coefficient as zero by hand.

for the 5D fermions with an infinitesimal positive constant ε:

$$Q_R = 0 \qquad \text{at } y = 0, L_1^{(Q)} \pm \varepsilon, L_2^{(Q)} \pm \varepsilon, L, \tag{2.7}$$

$$\mathcal{U}_L = 0 \qquad \text{at } y = 0, L_1^{(\mathcal{U})} \pm \varepsilon, L_2^{(\mathcal{U})} \pm \varepsilon, L, \tag{2.8}$$

$$\mathcal{D}_L = 0 \qquad \text{at } y = 0, L_1^{(\mathcal{D})} \pm \varepsilon, L_2^{(\mathcal{D})} \pm \varepsilon, L, \tag{2.9}$$

$$L_R = 0 \qquad \text{at } y = 0, L_1^{(L)} \pm \varepsilon, L_2^{(L)} \pm \varepsilon, L, \tag{2.10}$$

$$\mathcal{E}_L = 0 \qquad \text{at } y = 0, L_1^{(\mathcal{E})} \pm \varepsilon, L_2^{(\mathcal{E})} \pm \varepsilon, L, \tag{2.11}$$

where $L_1^{(\Psi)}$ and $L_2^{(\Psi)}$ mean the positions of the first and second point interactions ($0 < L_1^{(\Psi)} < L_2^{(\Psi)} < L$) for the field Ψ ($\Psi = Q, \mathcal{U}, \mathcal{D}, L, \mathcal{E}$). We define the conventions $L_0^{(\Psi)} \equiv 0$ and $L_3^{(\Psi)} \equiv L$ for convenience shortly. The Kaluza–Klein (KK) expansions of the 5D fermions are represented as

$$Q(x,y) = \begin{pmatrix} U(x,y) \\ D(x,y) \end{pmatrix} = \sum_{i=1}^{3} \begin{pmatrix} u_{iL}^{(0)}(x) \\ d_{iL}^{(0)}(x) \end{pmatrix} g_{q_{iL}}^{(0)}(y) + \text{(massive modes)}, \tag{2.12}$$

$$\mathcal{U}(x,y) = \sum_{i=1}^{3} u_{iR}^{(0)}(x) f_{u_{iR}}^{(0)}(y) + \text{(massive modes)}, \tag{2.13}$$

$$\mathcal{D}(x,y) = \sum_{i=1}^{3} d_{iR}^{(0)}(x) f_{d_{iR}}^{(0)}(y) + \text{(massive modes)}, \tag{2.14}$$

$$L(x,y) = \begin{pmatrix} N(x,y) \\ E(x,y) \end{pmatrix} = \sum_{i=1}^{3} \begin{pmatrix} v_{iL}^{(0)}(x) \\ e_{iL}^{(0)}(x) \end{pmatrix} g_{l_{iL}}^{(0)}(y) + \text{(massive modes)}, \tag{2.15}$$

$$\mathcal{E}(x,y) = \sum_{i=1}^{3} e_{iR}^{(0)}(x) f_{e_{iR}}^{(0)}(y) + \text{(massive modes)}, \tag{2.16}$$

with the generation indices i ($= 1, 2, 3$). The concrete forms of the zero-mode profiles are easily derived as [20]

$$g_{q_{iL}}^{(0)}(y) = \left[\theta(y - L_{i-1}^{(Q)}) \, \theta(L_i^{(Q)} - y)\right] \cdot \mathcal{N}_i^{(Q)} e^{+M_Q(y - L_{i-1}^{(Q)})}, \tag{2.17}$$

$$f_{u_{iR}}^{(0)}(y) = \left[\theta(y - L_{i-1}^{(\mathcal{U})}) \theta(L_i^{(\mathcal{U})} - y)\right] \cdot \mathcal{N}_i^{(\mathcal{U})} e^{-M_{\mathcal{U}}(y - L_{i-1}^{(\mathcal{U})})}, \tag{2.18}$$

$$f_{d_{iR}}^{(0)}(y) = \left[\theta(y - L_{i-1}^{(\mathcal{D})}) \, \theta(L_i^{(\mathcal{D})} - y)\right] \cdot \mathcal{N}_i^{(\mathcal{D})} e^{-M_{\mathcal{D}}(y - L_{i-1}^{(\mathcal{D})})}, \tag{2.19}$$

$$g_{l_{iL}}^{(0)}(y) = \left[\theta(y - L_{i-1}^{(L)}) \, \theta(L_i^{(L)} - y)\right] \cdot \mathcal{N}_i^{(L)} e^{+M_L(y - L_{i-1}^{(L)})}, \tag{2.20}$$

$$f_{e_{iR}}^{(0)}(y) = \left[\theta(y - L_{i-1}^{(\mathcal{E})}) \, \theta(L_i^{(\mathcal{E})} - y)\right] \cdot \mathcal{N}_i^{(\mathcal{E})} e^{-M_{\mathcal{E}}(y - L_{i-1}^{(\mathcal{E})})}, \quad (i = 1, 2, 3) \tag{2.21}$$

where $\theta(y)$ denotes the Heaviside step function and the individual kinetic normalization constants are expressed

$$\mathcal{N}_i^{(Q)} = \sqrt{\frac{2M_Q}{e^{2M_Q \Delta L_i^{(Q)}} - 1}}, \qquad \mathcal{N}_i^{(\mathcal{U})} = \sqrt{\frac{2M_{\mathcal{U}}}{1 - e^{-2M_{\mathcal{U}} \Delta L_i^{(\mathcal{U})}}}}, \qquad \mathcal{N}_i^{(\mathcal{D})} = \sqrt{\frac{2M_{\mathcal{D}}}{1 - e^{-2M_{\mathcal{D}} \Delta L_i^{(\mathcal{D})}}}},$$

$$\mathcal{N}_i^{(L)} = \sqrt{\frac{2M_L}{e^{2M_L \Delta L_i^{(L)}} - 1}}, \qquad \mathcal{N}_i^{(\mathcal{E})} = \sqrt{\frac{2M_{\mathcal{E}}}{1 - e^{-2M_{\mathcal{E}} \Delta L_i^{(\mathcal{E})}}}}, \qquad (i = 1, 2, 3) \tag{2.22}$$

with the lengths of segments

$$\Delta L_i^{(\Psi)} \equiv L_i^{(\Psi)} - L_{i-1}^{(\Psi)} \qquad (\text{for } i = 1, 2, 3; \Psi = Q, \mathcal{U}, \mathcal{D}, L, \mathcal{E}). \tag{2.23}$$

After the integration along the y direction, we obtain three-by-three mass matrices, e.g., for the up-type quarks as

$$S_{\text{up-type quark mass}} = -\int d^4x \sum_{i,j=1}^{3} \bar{u}_{iL}^{(0)} M_{ij}^{(u)} u_{jR}^{(0)} + (\text{h.c.}). \tag{2.24}$$

It should be emphasized that the ordering of the positions of point interactions governs the form of the mass matrices. For example, when the following ordering is realized,

$$0 < L_1^{(\mathcal{U})} < L_1^{(Q)} < L_2^{(\mathcal{U})} < L_2^{(Q)} < L, \tag{2.25}$$

the corresponding concrete forms of the elements of the up-type quark mass matrix are taken:

$$M^{(u)} = \begin{bmatrix} M_{11}^{(u)} & M_{12}^{(u)} & 0 \\ 0 & M_{22}^{(u)} & M_{23}^{(u)} \\ 0 & 0 & M_{33}^{(u)} \end{bmatrix}, \tag{2.26}$$

with

$$M_{11}^{(u)} = \frac{\mathcal{Y}_u v}{\sqrt{2}} \int_0^{L_1^{(\mathcal{U})}} dy \, \langle \Phi(y) \rangle g_{q1L}^{(0)}(y) f_{u1R}^{(0)}(y), \tag{2.27}$$

$$M_{22}^{(u)} = \frac{\mathcal{Y}_u v}{\sqrt{2}} \int_{L_1^{(Q)}}^{L_2^{(\mathcal{U})}} dy \, \langle \Phi(y) \rangle g_{q2L}^{(0)}(y) f_{u2R}^{(0)}(y), \tag{2.28}$$

$$M_{33}^{(u)} = \frac{\mathcal{Y}_u v}{\sqrt{2}} \int_{L_2^{(Q)}}^{L_3^{(\mathcal{U})}} dy \, \langle \Phi(y) \rangle g_{q3L}^{(0)}(y) f_{u3R}^{(0)}(y), \tag{2.29}$$

$$M_{12}^{(u)} = \frac{\mathcal{Y}_u v}{\sqrt{2}} \int_{L_1^{(\mathcal{U})}}^{L_1^{(Q)}} dy \, \langle \Phi(y) \rangle g_{q1L}^{(0)}(y) f_{u2R}^{(0)}(y), \tag{2.30}$$

$$M_{23}^{(u)} = \frac{\mathcal{Y}_u v}{\sqrt{2}} \int_{L_2^{(\mathcal{U})}}^{L_2^{(Q)}} dy \, \langle \Phi(y) \rangle g_{q2L}^{(0)}(y) f_{u3R}^{(0)}(y), \tag{2.31}$$

where we have assumed the form for the Higgs doublet, $\langle H \rangle = (0, v/\sqrt{2})^{\mathrm{T}}$, where v is the 5D Higgs VEV and can be treated as real without loss of generality as in the SM. If the ordering is different from Eq. (2.25), we can easily obtain the corresponding forms by looking at the profiles of fermion mode functions. See Ref. [20] (and also Refs. [21,22]) for more details.

3. Dynamics in the scenario and its implications

We first mention that, in Ref. [20], the observed quark mass hierarchy and the three mixing angles of the CKM matrix were successfully reproduced through the current strategy, under the following preconditions:

  ○ All of the positions of the point interactions can be treated as individual free parameters.

- ○ The form of the $\langle \Phi(y) \rangle$ is exponential as $\langle \Phi(y) \rangle \sim e^{M_\Phi y}$ (where the mass scale is not shown), whose form can be materialized by setting the parameters associated with Φ appropriately in the actual form of $\langle \Phi(y) \rangle$ (in Case (II) in Sect. 3.2) [20].

These preconditions would be justified for phenomenological studies. However, the following criticisms will come from various points of view:

(a) **From the stability of the system**: The positions of the point interactions contribute to the background Casimir energy of this extra-dimensional system, which should be minimized (or extremized at least) to ensure the stability of the whole system. If one obeys this line, the positions of the point interactions cannot take arbitrary values.

(b) **From the number of free parameters**: For quarks, in addition to the six parameters as the positions of point interactions for quark fields $\{L_i^{(Q)}, L_i^{(\mathcal{U})}, L_i^{(\mathcal{D})}; i = 1, 2\}$, the following parameters join to describe the quark profiles naively: one as the length of the system L, two as the overall Yukawa couplings $\{\mathcal{Y}_u, \mathcal{Y}_d\}$, three as the bulk masses $\{M_Q, M_\mathcal{U}, M_\mathcal{D}\}$, one as the 5D Higgs VEV v, and four to the gauge-singlet scalar VEV $\langle \Phi(y) \rangle$ (where details are provided in Sect. 3.2); there are 17 free parameters in total (for the lepton sector, where apparently six additional ones $\{L_1^{(L)}, L_2^{(L)}, L_1^{(\mathcal{E})}, L_2^{(\mathcal{E})}, M_L, M_\mathcal{E}\}$ join when we take care of charged leptons).[3] In contrast, only nine digits (six for mass eigenvalues, three for CKM mixing angles) are fitted. Such an unbalanced situation may lead to the criticism that the result of the previous fit is not surprising, and looks trivial.

(c) **From generality in the profile of $\langle \Phi(y) \rangle$**: As partially discussed in Sect. 4 of Ref. [20], the actual form of $\langle \Phi(y) \rangle$ is provided by Jacobi's elliptic function. As we will see in Sect. 3.2, there are two solutions for the VEV $\langle \Phi(y) \rangle$, which may give new insights into the geometric realization of the observed Yukawa texture in the "generalized" setup.

Based on the result of Ref. [33] and the general information of $\langle \Phi(y) \rangle$, we can provide answers for all of the criticisms, at least partially.

3.1. Criterion via stabilization of point interactions

The determination of the positions of the point interactions $L_i^{(\Psi)}$ by the minimization of the Casimir energy has been developed in Ref. [33]. The Casimir energy E at one-loop order as a function of $L_i^{(\Psi)}$ takes the form of

$$E = 2 \sum_{\Psi=Q,L} E^{(\Psi)}[L_1^{(\Psi)}, L_2^{(\Psi)}] + \sum_{\Psi'=\mathcal{U},\mathcal{D},\mathcal{E}} E^{(\Psi')}[L_1^{(\Psi')}, L_2^{(\Psi')}], \tag{3.1}$$

where

$$E^{(\Psi)}[L_1^{(\Psi)}, L_2^{(\Psi)}] = \sum_{i=1}^{3} \frac{|M^{(\Psi)}|^2}{8\pi^2 (L_i^{(\Psi)} - L_{i-1}^{(\Psi)})^2} \sum_{w=1}^{\infty} \frac{e^{-2w|M^{(\Psi)}|(L_i^{(\Psi)} - L_{i-1}^{(\Psi)})}}{w^3}$$

$$\times \left(1 + \frac{3}{2w|M^{(\Psi)}|(L_i^{(\Psi)} - L_{i-1}^{(\Psi)})} + \frac{3}{4w^2|M^{(\Psi)}|^2(L_i^{(\Psi)} - L_{i-1}^{(\Psi)})^2} \right). \tag{3.2}$$

---

[3] As we will see in Sect. 4.1, this counting of parameters is slightly naive. The number of degrees of freedom relevant for mass ratios is less than the digit.

The minimization of the above Casimir energy leads to the results in which the positions of the point interactions for the doublet fermions and the singlet fermions are the same as[4]

$$
\begin{aligned}
L_1^{(\mathcal{Q})} &= L_1^{(\mathcal{U})} = L_1^{(\mathcal{D})} = L_1^{(\mathcal{L})} = L_1^{(\mathcal{E})}, \\
L_2^{(\mathcal{Q})} &= L_2^{(\mathcal{U})} = L_2^{(\mathcal{D})} = L_2^{(\mathcal{L})} = L_2^{(\mathcal{E})},
\end{aligned}
\tag{3.3}
$$

and all of the point interactions should be located at

$$
L_1^{(\Psi)} = \frac{1}{3}L, \qquad L_2^{(\Psi)} = \frac{2}{3}L,
\tag{3.4}
$$

to minimize the Casimir energy in spite of the type of the field Ψ. More concretely,

$$
\begin{aligned}
L_1^{(\mathcal{Q})} &= L_1^{(\mathcal{U})} = L_1^{(\mathcal{D})} = L_1^{(\mathcal{L})} = L_1^{(\mathcal{E})} = \frac{1}{3}L, \\
L_2^{(\mathcal{Q})} &= L_2^{(\mathcal{U})} = L_2^{(\mathcal{D})} = L_2^{(\mathcal{L})} = L_2^{(\mathcal{E})} = \frac{2}{3}L.
\end{aligned}
\tag{3.5}
$$

It is noted that Eq. (3.5) implies no mixing terms in mass matrices. Taking account of this result in parameter fitting would definitely be a reasonable response (despite it being incomplete) to the above criticisms (a) and (b), where the naive counting of the input parameters for quarks is reduced to 11, from 17.

3.2. Criterion via the general solution of $\langle \Phi(y) \rangle$

To know the actual solution of Φ means solving the following nonlinear equation derived through the variational principle from S_{singlet} in Eq. (2.5),

$$
\frac{d^2\phi(y)}{dy^2} - M_\Phi^2 \phi(y) - \lambda_\Phi \phi^3(y) = 0,
\tag{3.6}
$$

where we put the ansatz that the VEV of Φ depends on the extra-dimensional coordinate y, i.e.,

$$
\langle \Phi \rangle = \phi(y),
\tag{3.7}
$$

where in general the following form is $\langle \Phi \rangle = e^{i\theta(y)}\phi(y)$, but we have proven that without loss of generality we can set the y-dependent phase part $\theta(y)$ to be zero, where $\phi(y)$ is real. The above equation leads to

$$
\frac{d}{dy}\left[\frac{1}{2}\left(\frac{d\phi(y)}{dy}\right)^2 - \frac{M_\Phi^2}{2}\phi^2(y) - \frac{\lambda_\Phi}{4}\phi^4(y) \right] = 0,
\tag{3.8}
$$

and can be deformed to

$$
\frac{1}{2}\left(\frac{d\phi(y)}{dy}\right)^2 + U(\phi) = E_\Phi, \qquad \left(U(\phi) \equiv -\frac{M_\Phi^2}{2}\phi^2(y) - \frac{\lambda_\Phi}{4}\phi^4(y) \right),
\tag{3.9}
$$

---

[4] In the case of the discussion about the total length L by the Casimir energy, the other field contributions, e.g., gauge fields and the scalar fields, should be included in the minimization condition (see the discussion in Ref. [33]). On the other hand, in the case of the discussion about the positions of the point interactions, only the related fermion's contributions are important.

with an undetermined integration constant E_Φ as a parameter. Here, we should take account of the boundary conditions of $\phi(y)$ at $y = 0$ and $y = L$, where we impose the Robin boundary conditions, which are described with the length parameters $L_\pm$ ($-\infty \le L_\pm \le +\infty$) as below:[5]

$$\phi(0) + L_+\phi'(0) = 0,$$
$$\phi(L) - L_-\phi'(L) = 0, \tag{3.10}$$

where the prime symbol represents the derivative of y (refer to Ref. [35]).

Through elliptic integrals, at least the following *two* solutions are possible in terms of Jacobi's elliptic functions:

(I) When $0 < E_\Phi < M_\Phi^4/(4\lambda_\Phi)$:

$$\phi(y) = \mu_- \frac{\mathrm{sn}\left(\mu_+\sqrt{\frac{\lambda_\Phi}{2}}(y - y_0), k\right)}{\mathrm{cn}\left(\mu_+\sqrt{\frac{\lambda_\Phi}{2}}(y - y_0), k\right)}, \tag{3.11}$$

where

$$\mu_\pm^2 \equiv \frac{M_\Phi^2}{\lambda_\Phi}\left(1 \pm \sqrt{1 - \frac{4\lambda_\Phi E_\Phi}{M_\Phi^4}}\right), \quad k^2 \equiv \frac{\mu_+^2 - \mu_-^2}{\mu_+^2}. \tag{3.12}$$

(II) When $E_\Phi < 0$:

$$\phi(y) = \frac{\nu}{\mathrm{cn}\left(\frac{\mu}{k}\sqrt{\frac{\lambda_\Phi}{2}}(y - y_0), k\right)}, \tag{3.13}$$

where

$$\mu^2 \equiv \frac{M_\Phi^2}{\lambda_\Phi}\left(1 + \sqrt{1 - \frac{4\lambda_\Phi E_\Phi}{M_\Phi^4}}\right), \quad \nu^2 \equiv \frac{M_\Phi^2}{\lambda_\Phi}\left(\sqrt{1 - \frac{4\lambda_\Phi E_\Phi}{M_\Phi^4}} - 1\right), \quad k^2 \equiv \frac{\mu^2}{\mu^2 + \nu^2}. \tag{3.14}$$

We point out that the variables $L_\pm$ in Eq. (3.10) to parametrize the Robin boundary conditions are automatically fixed by the above profiles. y_0 represents the degree of freedom of the translation along the y direction. The parameter k measures ellipticity defined in the range of $0 \le k \le 1$,[6] where the two extremal cases $k = 0$ and 1 correspond to

$$\frac{\mathrm{sn}(x, k)}{\mathrm{cn}(x, k)} = \mathrm{sc}(x, k) \to \begin{cases} \tan x & \text{when } k \to 0, \\ \sinh x & \text{when } k \to 1, \end{cases}$$

$$\frac{1}{\mathrm{cn}(x, k)} \to \begin{cases} 1/\cos x & \text{when } k \to 0, \\ \cosh x & \text{when } k \to 1. \end{cases} \tag{3.15}$$

---

[5] We note that the scalar field Φ is assumed not to feel point interactions.

[6] Actually, the limited region $1/\sqrt{2} \le k \le 1$ can be taken both in the cases of (I) and (II).

When $k \neq 1$, a very nontrivial feature of the two elliptic functions $\text{sc}(x, k)$ and $1/\text{cn}(x, k)$ rises up, where the functions get divergent periodically along the y-direction. The minimal periods between the nearest divergent points of $\phi(y)$ in Case (I) and Case (II) are

$$2\left(\mu_+\sqrt{\frac{\lambda_\Phi}{2}}\right)^{-1} K[k], \quad 2\left(\frac{\mu}{k}\sqrt{\frac{\lambda_\Phi}{2}}\right)^{-1} K[k],$$

respectively, with the complete elliptic integral of the first kind $K[k]$. A possible important difference between Case (I) and Case (II) is that $\text{sc}(x, k)$ increases monotonically from a divergent point x_d $\left(\text{sc}(x_d + \varepsilon, k) \sim -\infty\right)$ to the next divergent point $x_d'\,(> x_d)$ $\left(\text{sc}(x_d' - \varepsilon, k) \sim +\infty\right)$, where the value of $\text{sc}(x, k)$ is negative in the first half part of the region $x_d < x < (x_d + x_d')/2$. This property would be useful to realize gigantic hierarchy in the overlap integrals that express the components of the mass matrices (see Eqs. (2.27)–(2.31)) since negative contributions become possible. This motivates us to have a discussion about exploring the situation with the exact form of $\phi(y)$ in Case (I) in contrast with the prior Case (II) research. This will be the response to the criticism (c) shown above.

4.   Numerical analysis on an interval

Following the discussions in the previous section, we perform a numerical parameter search in the current model on an interval. The differences in method between the analyses on this manuscript and in the prior research [20–22] are summarized:

○ The relative distances of the point interactions are fixed by the minimization of the Casimir energy [33] (see Eq. (3.5)), while previously the positions were treated as completely free parameters [20–22].

○ Due to the minimization of the Casimir energy shown in Eq. (3.3), currently we cannot discuss the flavor mixing in a consistent way. Here, we take the strategy where at first we focus on only mass hierarchies, which may be justified as the first step to know how the geometry constrained by the Casimir energy works well to reproduce the observed magnitudes of the SM mass hierarchy, of course up to the inevitable distortion by mixings. This attitude will be reasonable especially for the quark sector since the observed mixing angles are small. On the other hand, we touch on effects from mixings by introducing off-diagonal terms by hand phenomenologically for the charged leptons.

○ The exact form of the y-dependent VEV of Case (I) is adopted for the singlet Φ, while an approximate form of the Case (II) solution is used in the prior research.

Now, the quark and lepton mass matrices are diagonal, where the elements of the matrices are formulated for $\psi\,(= u, d, e)$ as

$$M_{ii}^{(\psi)} = \widetilde{\mathcal{Y}}_\psi \cdot \frac{v_\text{H}}{\sqrt{2}} \sqrt{\frac{\widetilde{M}_\Phi^2}{\widetilde{\lambda}_\Phi}} (1 - X)^{1/2}$$

$$\times \int_{\widetilde{L}_{i-1}}^{\widetilde{L}_i} d\widetilde{y}\, \text{sc}\left(\sqrt{\frac{\widetilde{M}_\Phi^2(1 + X)}{2}}(\widetilde{y} - \widetilde{y}_0), \sqrt{\frac{2X}{1 + X}}\right) \widetilde{g}_{\psi_{i\text{L}}}^{(0)}(\widetilde{y}) \widetilde{f}_{\psi_{i\text{R}}}^{(0)}(\widetilde{y}) \quad (i = 1, 2, 3), \quad (4.1)$$

where

$$X \equiv \sqrt{1 - \frac{4\widetilde{E}_\Phi \widetilde{\lambda}_\Phi}{\widetilde{M}_\Phi^4}}, \tag{4.2}$$

and the formula is factorized as the product of the flavor-independent and -dependent parts, and here all of the parameters are scaled to the dimensionless ones by the total length of the system L, e.g., $M_\Phi = \widetilde{M}_\Phi \cdot L^{-1}$ except for the 4D Higgs VEV $v_{\mathrm{H}} (= 246\,\mathrm{GeV})$ defined as $v_{\mathrm{H}} \equiv v\sqrt{L}$ from the 5D Higgs VEV v. We adopted the notation that variables accompanying the tilde symbol are dimensionless, where the normalization parts of the fermion wave functions are made dimensionless following the instruction. The uniformly located positions of the point interactions are represented as (cf. Eq. (3.5))

$$\widetilde{L}_0 = 0, \quad \widetilde{L}_1 = \frac{1}{3}, \quad \widetilde{L}_2 = \frac{2}{3}, \quad \widetilde{L}_3 = 1. \tag{4.3}$$

We comment that the form in Eq. (4.1) does not depend on L itself since it describes phenomena of zero modes. The independent parameters for describing the deformation of the flavor-dependent part via Φ are taken as X, $\widetilde{M}_\Phi$, and $\widetilde{y}_0$. We remind ourselves that the five bulk masses of the fermions $\{\widetilde{M}_Q, \widetilde{M}_\mathcal{U}, \widetilde{M}_\mathcal{D}, \widetilde{M}_L, \widetilde{M}_\mathcal{E}\}$ and the three overall Yukawa couplings $\{\widetilde{\mathcal{Y}}_u, \widetilde{\mathcal{Y}}_d, \widetilde{\mathcal{Y}}_e\}$ also contribute to the flavor-dependent part, where the 10 parameters in Eqs. (4.4) and (4.5) are relevant when we debate with the flavor dependence up to the overall scale (where one of the Yukawa couplings is included in the overall scale). The fit observables are the eight mass ratios of the nine SM fermion masses like $m_{\mathrm{others}}/m_{\mathrm{top}}$. We should mention that, as was pointed out in Ref. [22], more parameters than the experimental values of the quark/lepton masses do not mean that we can always reproduce the quark and lepton masses in the present scenario. The geometry of the extra dimension tightly restricts the form of the mass matrices to four-zero texture, see, e.g., Eq. (2.26). Therefore, it is quite nontrivial whether our model can reproduce the quark and lepton masses even though two extra parameters (out of 10 for fitting the eight observables) seem to remain.

4.1. Results for the numerical research

We found that, even in the current limited setup, when we take the following benchmark,[7]

$$\widetilde{M}_\Phi = 4.3, \quad \widetilde{y}_0 = 0.33, \quad X = 0.74,$$
$$\widetilde{M}_Q = 40, \quad \widetilde{M}_\mathcal{U} = 320, \quad \widetilde{M}_\mathcal{D} = 0.1, \quad \widetilde{M}_L = 100, \quad \widetilde{M}_\mathcal{E} = 0.01, \tag{4.4}$$

and take the degrees of freedom of the Yukawa couplings to adjust the third-generation fermions as

$$\frac{\widetilde{\mathcal{Y}}_d}{\widetilde{\mathcal{Y}}_u} \simeq 0.124, \quad \frac{\widetilde{\mathcal{Y}}_e}{\widetilde{\mathcal{Y}}_u} \simeq 0.0454, \tag{4.5}$$

the result comes to

$$\frac{M_{11}^{(u)}}{m_{\mathrm{up}}} \simeq 0.746, \quad \frac{M_{22}^{(u)}}{m_{\mathrm{charm}}} \simeq 1.08, \quad \frac{M_{33}^{(u)}}{m_{\mathrm{top}}} = 1,$$

$$\frac{M_{11}^{(d)}}{m_{\mathrm{down}}} \simeq -1.07, \quad \frac{M_{22}^{(d)}}{m_{\mathrm{strange}}} \simeq 0.966, \quad \frac{M_{33}^{(d)}}{m_{\mathrm{bottom}}} = 1,$$

$$\frac{M_{11}^{(e)}}{m_{\mathrm{electron}}} \simeq -0.775, \quad \frac{M_{22}^{(e)}}{m_{\mathrm{muon}}} \simeq 0.253, \quad \frac{M_{33}^{(e)}}{m_{\mathrm{tauon}}} = 1, \tag{4.6}$$

---

[7] Here, the nearest divergent point of the Jacobi's elliptic function is located at $\widetilde{y} \simeq 1.002\,49$, which is outside the region where the interval geometry is elaborated.

where we compare the obtained result with the derived pole masses at the leading order from the PDG digits [36].[8] We reached a good fit for quarks as ratios,[9] while the fit of the charged lepton is quite far from being precise. We do not pay attention to the overall flavor-independent part since we may adjust this part by use of $\widetilde{\mathcal{Y}}_u$ and $\widetilde{\lambda}_\Phi$. The negativity for $M_{11}^{(d)}$, $M_{11}^{(e)}$ originates from the lower tail part of the sc function, where now the absolute values of them physically make sense.

4.2. An extra phenomenological study of the mixing effect in charged leptons

A possible origin of the deviated result in the charged leptons may be that no suitably relevant degree of freedom remains for charged leptons after we fix the parameters to reproduce the observed mass hierarchy of quarks under the constraints in Eq. (4.3).

To look into possible improvements of the charged lepton part, we do a phenomenological trial for the charged lepton part, which seems to have difficulty regenerating the observed patterns of their mass eigenvalues. Now we focus on the speculation that the charged leptons can be largely mixed, because the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix, which describes the mixings among the left-handed charged leptons and active neutrinos, contains large mixing angles. Under such circumstances mass eigenvalues can alter remarkably by the introduction of sizable non-diagonal terms of the mass matrix. The existence of non-diagonal elements might be realized when we evaluate the Casimir energy at two-loop order or introduce exotic fermions, which have no chiral zero modes and only affect the Casimir energy. Therefore, to investigate such a case with phenomenologically introduced terms by hand would be fruitful to identify further possibilities of the current scenario.

In what follows, we bring the following phenomenological form for a charged-lepton mass matrix into focus:

$$\mathcal{M}^{(e)} = \begin{bmatrix} M_{11}^{(e)} & m_{12}^{(e)} & 0 \\ 0 & M_{22}^{(e)} & m_{23}^{(e)} \\ m_{31}^{(e)} & 0 & M_{33}^{(e)} \end{bmatrix}, \tag{4.7}$$

where we assume that the diagonal inputs take the same digits as in Eq. (4.6) without rounding, and $m_{12}^{(e)}$, $m_{23}^{(e)}$, and $m_{31}^{(e)}$ are phenomenological parameters.[10] We found a desirable configuration to reproduce the measurements, where $m_{12}^{(e)}$, $m_{23}^{(e)}$, and $m_{31}^{(e)}$ are chosen as 104 MeV, 100 MeV, and 11.2 MeV, respectively (see Fig. 3). Here, we achieved good accuracy: $(m_{\text{eigen}}/m_{\text{exp.}})_{\text{electron}} \simeq 1.00052$, $(m_{\text{eigen}}/m_{\text{exp.}})_{\text{muon}} \simeq 1.01612$, and $(m_{\text{eigen}}/m_{\text{exp.}})_{\text{tauon}} \simeq 1.0016$. The current scope of this issue is just within a naive discussion founded on phenomenological assumptions so it would

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[8] Concrete digits of the central values are as follows: $m_{\text{up}} = 2.5$ MeV, $m_{\text{down}} = 5.2$ MeV, $m_{\text{strange}} = 110$ MeV, $m_{\text{charm}} = 1.448$ GeV, $m_{\text{bottom}} = 4.557$ GeV, $m_{\text{top}} = 173.0$ GeV, $m_{\text{electron}} = 0.5109989461$ MeV, $m_{\text{muon}} = 105.6583745$ MeV, $m_{\text{tauon}} = 1.77686$ GeV, respectively.

[9] On the other hand, these results are sensitive to the modulation of the input parameters in sub-leading magnitudes. For example, if we change $\widetilde{M}_\Phi$ from 4.3 to 4.31, the accurate fit for quarks turns out to be disrupted as $M_{11}^{(u)}/m_{\text{up}} \simeq 0.416$, $M_{22}^{(u)}/m_{\text{charm}} \simeq 0.605$, $M_{11}^{(d)}/m_{\text{down}} \simeq -0.761$, $M_{22}^{(d)}/m_{\text{strange}} \simeq 0.687$, $M_{11}^{(e)}/m_{\text{electron}} \simeq -0.504$, $M_{22}^{(e)}/m_{\text{muon}} \simeq 0.165$, respectively, where the other ratios are still unities after suitable adjustments of $\mathcal{Y}_d/\mathcal{Y}_u$ and $\mathcal{Y}_e/\mathcal{Y}_u$. The position of the nearest divergent point of the elliptic function becomes $\widetilde{y}_0 \simeq 1.00093$. This point is considered as an inevitable difficulty of the present scenario, where a suitable tuning of inputs is required.

[10] It is pointed out that no $1 \leftrightarrow 3$ mixing term can be realized on an interval, but it is easy to activate such a mixing term after we switch the geometry to S^1 (see Ref. [22]).
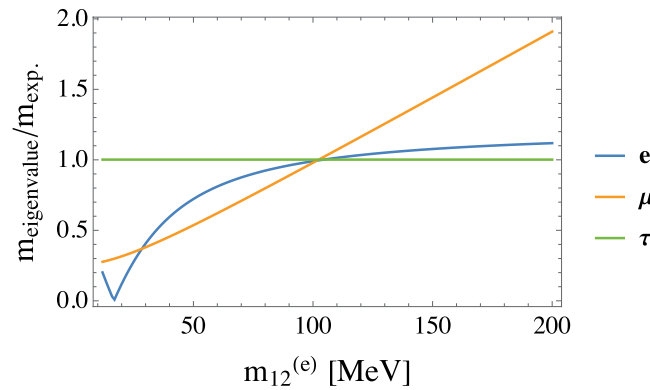
Fig. 3. Shifts of mass eigenvalues of the phenomenological charged-lepton mass matrix $\mathcal{M}^{(e)}$ as functions of $m_{12}^{(e)}$, where $m_{23}^{(e)}$ and $m_{31}^{(e)}$ are pinned down as 100 MeV and 11.2 MeV, respectively. The three eigenvalues become very close to the measured digits when $m_{12}^{(e)} \simeq 104$ MeV simultaneously.

be worth investigating the full situation grounded on higher-loop calculations or modification of the matter contents.

5. Conclusion and discussion

In this paper, we have performed numerical research in our dynamical model to reproduce the quark/lepton mass hierarchy with experimental values of the quark/lepton masses. Our model consists of 10 point interactions, the positions of which were treated as free parameters in the prior research [20–22]. Their positions have been determined by the minimization condition of the Casimir energy. Because of that, the parameters relevant for fixing the ratios of the fermion masses have been reduced to 10 from 20 in our model and only the remnant 10 parameters are tools for reemergence of the eight experimental values of the quark and lepton mass ratios. As was pointed out in Ref. [22], more parameters than the experimental values of the quark/lepton masses do not mean that we can always reproduce the quark and lepton masses in our model. The geometry of the extra dimension tightly restricts the form of the mass matrices to four-zero texture; see Eq. (2.26). Therefore, it is quite nontrivial whether our model can reproduce the quark and lepton masses.

  To obtain the desired mass hierarchy, we introduced the extra-dimension coordinate-dependent VEV of the gauge-singlet scalar $\langle \Phi(y) \rangle$, which is also obtained dynamically by minimizing the potential of the scalar Φ (refer to Eq. (2.5)). As a result of our numerical analysis, we found that there is a parameter set that can reproduce the experimental values of the quark well, while some difficulties remain for the charged leptons. On the other hand, we pointed out that the mixing terms among the charged leptons will resolve the remaining discrepancies, even though this is not justified fully from the theoretical point of view within the current scope.

  A definite next step is to contemplate how to realize non-diagonal terms in the quarks and the charged leptons, which are necessary ingredients to generate the observed textures of the mixings represented by the CKM and PMNS matrices. The calculation of the Casimir energy was achieved at one-loop order, and consequently the positions of the point interactions are determined so as to divide the interval extra dimension equally among three, i.e., the mass matrices of quarks and leptons automatically become diagonal ones. If off-diagonal components appear after introducing a suitable mechanism, the eigenvalues of the mass matrices deviate from the diagonal components of the mass

matrices and the experimental values might be recovered with flavor mixings, as is discussed in Sect. 4.2.

One possible idea to generate off-diagonal components in the mass matrices is to introduce exotic fermions. Boundary conditions for the exotic fermions should be chosen to produce no massless chiral zero modes, and then they will contribute only to the Casimir energy at low energies. A possible boundary condition for the exotic fermion $\Psi^{\text{ex}}(x, y)$ is given by

$$\Psi^{\text{ex}}_{\text{L}}(x, y) = 0 \quad \text{at} \quad y = 0, L_1 + \varepsilon, \qquad \Psi^{\text{ex}}_{\text{R}}(x, y) = 0 \quad \text{at} \quad y = L_1 - \varepsilon, L. \tag{5.1}$$

It is noted that the exotic fermion feels the point interaction only at $y = L_1$ but not the one at $y = L_2$. The boundary condition (5.1) can prohibit the presence of chiral zero modes [35], and will force the point interaction at $y = L_1$ to move from $L_1 = L/3$ to $L_1 > L/3$. This is because the exotic fermion contributes to the position $y = L_1$ of the point interaction so as to divide the interval extra dimension equally among two. After combining the contributions of the exotic fermion with the (5D) SM fermions, the minimization condition of the Casimir energy is modified and the off-diagonal component M_{12} (and probably other off-diagonal components) can appear due to $L_1 > L/3$. It should be mentioned that there is a way to obtain $L_1 < L/3$ by extending the mechanism. We can introduce extra three point interactions only for the exotic fermion and impose the boundary conditions at $y = L'_2, L'_3, L'_4$ in addition to $y = L_1$. In this case, the exotic fermion may contribute to the position $y = L_1$ of the point interaction so as to divide the interval extra dimension equally among four, in which $L_1 < L/3$ can be realized from the modified minimization condition of the Casimir energy. The key ingredient is that this mechanism accomplishes the appearance of off-diagonal components dynamically and the contribution of the exotic fermion might be of the same order as the (5D) SM fermions, so that large off-diagonal components could emerge. If bulk masses of some exotic fermions are chosen to be much larger than those of the (5D) SM fermions, their effects on the Casimir energy are expected to be limited and will give a small contribution to the off-diagonal components of the mass matrices. Therefore we can expect that both the small/large mixings in the quark/lepton sector might be obtained dynamically by use of this mechanism.

Another idea is to take account of the higher-order effects of the Casimir energy. Since the flavor structures of the quarks and leptons are nontrivial, the effects of the matter contents at two-loop order might produce corrections to the minimization conditions of the Casimir energy. That may lead to off-diagonal components of the mass matrices through modulations of the positions of the point interactions. The origin of modulations may also be the (zero-mode) (KK-mode) mixing, which can cause small mass perturbations. The issues stated above remain to be pursued in future work.

References

[1] D. Cremades, L. E. Ibáŋez, and F. Marchesano, J. High Energy Phys. **0405**, 079 (2004) [arXiv:hep-th/0404229] [Search INSPIRE].

[2] H. Abe, T. Kobayashi, and H. Ohki, J. High Energy Phys. **0809**, 043 (2008) [arXiv:0806.4748 [hep-th]] [Search INSPIRE].

[3] H. Abe, K.-S. Choi, T. Kobayashi, and H. Ohki, Nucl. Phys. B **814**, 265 (2009) [arXiv:0812.3534 [hep-th]] [Search INSPIRE].

[4] T.-h. Abe, Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, and M. Sakamoto, Nucl. Phys. B **890**, 442 (2014) [arXiv:1409.5421 [hep-th]] [Search INSPIRE].

[5] H. Abe, T. Kobayashi, K. Sumita, and Y. Tatsuta, Phys. Rev. D **90**, 105006 (2014) [arXiv:1405.5012 [hep-ph]] [Search INSPIRE].

[6] T.-h. Abe, Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, M. Sakamoto, and Y. Tatsuta, Nucl. Phys. B **894**, 374 (2015) [arXiv:1501.02787 [hep-ph]] [Search INSPIRE].

[7] Y. Fujimoto, T. Kobayashi, K. Nishiwaki, M. Sakamoto, and Y. Tatsuta, Phys. Rev. D **94**, 035031 (2016) [arXiv:1605.00140 [hep-ph]] [Search INSPIRE].

[8] Y. Sakamura, J. High Energy Phys. **1610**, 083 (2016) [arXiv:1607.07152 [hep-th]] [Search INSPIRE].

[9] T. Kobayashi, K. Nishiwaki, and Y. Tatsuta, J. High Energy Phys. **1704**, 080 (2017) [arXiv:1609.08608 [hep-th]] [Search INSPIRE].

[10] M. Ishida, K. Nishiwaki, and Y. Tatsuta, Phys. Rev. D **95**, 095036 (2017) [arXiv:1702.08226 [hep-th]] [Search INSPIRE].

[11] W. Buchmuller and J. Schweizer, Phys. Rev. D **95**, 075024 (2017) [arXiv:1701.06935 [hep-ph]] [Search INSPIRE].

[12] M. Ishida, K. Nishiwaki, and Y. Tatsuta, J. High Energy Phys. **1807**, 125 (2018) [arXiv:1802.06646 [hep-ph]] [Search INSPIRE].

[13] H. Abe, T. Kobayashi, S. Takada, S. Tamba, and T. H. Tatsuishi, Phys. Rev. D **98**, 106017 (2018) [arXiv:1807.02063 [hep-th]] [Search INSPIRE].

[14] H. Abe, M. Ishida, and Y. Tatsuta, Nucl. Phys. B **947**, 114732 (2019) [arXiv:1806.10369 [hep-th]] [Search INSPIRE].

[15] K. Inoue, Prog. Theor. Phys. **93**, 403 (1995) [arXiv:hep-ph/9410220] [Search INSPIRE].

[16] K. Inoue and N.-a. Yamashita, Prog. Theor. Phys. **104**, 677 (2000) [arXiv:hep-ph/0005178] [Search INSPIRE].

[17] K. Inoue and N.-a. Yamashita, Prog. Theor. Phys. **110**, 1087 (2003) [arXiv:hep-ph/0305297] [Search INSPIRE].

[18] K. Inoue and N. Yamatsu, Prog. Theor. Phys. **119**, 775 (2008) [arXiv:0712.2938 [hep-ph]] [Search INSPIRE].

[19] N. Yamatsu, Prog. Theor. Exp. Phys. **2013**, 023B03 (2013) [arXiv:1209.6318 [hep-ph]] [Search INSPIRE].

[20] Y. Fujimoto, T. Nagasawa, K. Nishiwaki, and M. Sakamoto, Prog. Theor. Exp. Phys. **2013**, 023B07 (2013) [arXiv:1209.5150 [hep-ph]] [Search INSPIRE].

[21] Y. Fujimoto, K. Nishiwaki, M. Sakamoto, and R. Takahashi, J. High Energy Phys. **1410**, 191 (2014) [arXiv:1405.5872 [hep-ph]] [Search INSPIRE].

[22] Y. Fujimoto, K. Nishiwaki, and M. Sakamoto, Phys. Rev. D **88**, 115007 (2013) [arXiv:1301.7253 [hep-ph]] [Search INSPIRE].

[23] G. Panico and A. Pomarol, J. High Energy Phys. **1607**, 097 (2016) [arXiv:1603.06609 [hep-ph]] [Search INSPIRE].

[24] K. Agashe, P. Du, S. Hong, and R. Sundrum, J. High Energy Phys. **1701**, 016 (2017) [arXiv:1608.00526 [hep-ph]] [Search INSPIRE].

[25] L. Da Rold, J. High Energy Phys. **1710**, 120 (2017) [arXiv:1708.08515 [hep-ph]] [Search INSPIRE].

[26] A. Ahmed, A. Carmona, J. C. Ruiz, Y. Chung, and M. Neubert, J. High Energy Phys. **1908**, 045 (2019) [arXiv:1905.09833 [hep-ph]] [Search INSPIRE].

[27] E. Pontón and E. Poppitz, J. High Energy Phys. **0106**, 019 (2001) [arXiv:hep-ph/0105021 [hep-ph]] [Search INSPIRE].

[28] L. C. de Albuquerque and R. M. Cavalcanti, J. Phys. A: Math. Gen. **37**, 7039 (2004) [arXiv:hep-th/0311052] [Search INSPIRE].

[29] J. Garriga, O. Pujolás, and T. Tanaka, Nucl. Phys. B **605**, 192 (2001) [arXiv:hep-th/0004109] [Search INSPIRE].

[30] W. D. Goldberger and I. Z. Rothstein, Phys. Lett. B **491**, 339 (2000) [arXiv:hep-th/0007065] [Search INSPIRE].

[31] R. Hofmann, P. Kanti, and M. Pospelov, Phys. Rev. D **63**, 124020 (2001) [arXiv:hep-ph/0012213] [Search INSPIRE].

[32] Y. Abe, T. Inami, Y. Kawamura, and Y. Koyama, Prog. Theor. Exp. Phys. **2014**, 073B04 (2014) [arXiv:1404.5125 [hep-th]] [Search INSPIRE].

[33] Y. Fujimoto, T. Miura, K. Nishiwaki, and M. Sakamoto, Phys. Rev. D **97**, 115039 (2018) [arXiv:1709.05693 [hep-th]] [Search INSPIRE].

[34] C. Cai and H.-H. Zhang, Phys. Rev. D **93**, 036003 (2016) [arXiv:1503.08805 [hep-ph]] [Search INSPIRE].

[35] Y. Fujimoto, T. Nagasawa, S. Ohya, and M. Sakamoto, Prog. Theor. Phys. **126**, 841 (2011) [arXiv:1108.1976 [hep-th]] [Search INSPIRE].

[36] M. Tanabashi et al. [Particle Data Group], Phys. Rev. D **98**, 030001 (2018).