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# Wilson action for the O(N) model

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# Wilson action for the O(N) model

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#### Abstract

In this paper the fixed-point Wilson action for the critical O(N) model in  $D = 4 - \epsilon$  dimensions is written down in the  $\epsilon$  expansion to order  $\epsilon^2$ . It is obtained by solving the fixed-point Polchinski Exact Renormalization Group equation (with anomalous dimension) in powers of  $\epsilon$ . This is an example of a theory that has scale and conformal invariance despite having a finite UV cutoff. The energy-momentum tensor for this theory is also constructed (at zero momentum) to order  $\epsilon^2$ . This is done by solving the Ward-Takahashi identity for the fixed point action. It is verified that the trace of the energy-momentum tensor is proportional to the violation of scale invariance as given by the exact RG, i.e., the  $\beta$  function. The vanishing of the trace at the fixed point ensures conformal invariance. Some examples of calculations of correlation functions are also given.

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# 1. Introduction

Conformal field theories (CFT) are interesting for a variety of reasons. One of the most important reason is that a theory critical at a continuous phase transition is expected to acquire conformal invariance which imposes strong constraints on the correlation functions [1]. This has motivated the idea of bootstrap [2]. Particularly in two dimensions these ideas have been very fruitful [3]. Reviews of later developments and references are given in [4,5].

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The advent of the AdS/CFT correspondence [6–9] or "holography" between a boundary CFT and a bulk gravity theory opened up another approach to solving CFT's.<sup>1</sup> There is a large amount of literature on this. See, for example, [10] for a review.

In the AdS/CFT correspondence the radial direction can be interpreted as the scale of the boundary field theory. Thus, a radial evolution can be thought of as an RG evolution and has been dubbed "holographic RG" [15–28]. The precise connection between the boundary RG and holographic RG is, however, still an open question.

Recently a connection has been proposed between the Exact Renormalization Group (ERG) equation [11–14] and the Holographic Renormalization Group (Holographic RG) equation. It was shown in [29] that the RG evolution operator for a Wilson action of a D-dimensional field theory obeying the Polchinski ERG equation can be formulated as a D + 1-dimensional functional integral. The extra dimension, corresponding to the moving scale  $\Lambda$  of the ERG, makes it a "holographic" formulation. Furthermore, a change of field variables or field redefinition maps the D + 1 dimensional action for the functional integral to the action of a free massive scalar field in  $AdS_{D+1}$ . It was then shown that the calculation of the two point function reduces to the familiar calculation using the AdS/CFT correspondence.

This proposal is quite general, and detailed calculations were done for the Gaussian theory [29]. The scalar field theory action has a free parameter, i.e., the mass of the scalar field, which is related to the anomalous dimension of the boundary operator in the AdS/CFT context. This parameter appears to come out of nowhere. To understand the origin of the anomalous dimension parameter, an ERG equation with anomalous dimension was analysed in [30]. The same change of variables mapped this to a scalar field theory in the AdS space-time, and this time it was easy to see that the mass parameter is naturally related to the anomalous dimension. Since the exact RG for interacting theories is difficult, a Gaussian theory with an anomalous dimension introduced by hand was studied in [30].

In order to improve our understanding of the connection between ERG and the AdS/CFT correspondence, it is necessary to have an interacting example — one needs a non-trivial boundary CFT and a fixed-point Wilson action for this CFT.<sup>2</sup> Then the RG evolution of small perturbations to this theory can be studied by ERG. Using the ideas of [29,30] this can be mapped to a scalar field theory in D + 1-dimensional AdS space. This would make a contact with more detailed AdS/CFT calculations of higher point correlators. A well studied field theory is the  $\lambda \phi^4$  scalar field theory in  $4 - \epsilon$  dimensions that has the famous Wilson-Fisher fixed point. When there are N scalar fields, this is often referred to as the O(N) model. In this paper, as a first step, we construct a fixed-point Wilson action for this theory to order  $\epsilon^2$ . It is at this order that the anomalous dimension first shows up. The action is obtained by solving the fixed-point ERG equation perturbatively. The fixed-point equation imposes the constraint of scale invariance.

In fact the theory is also conformally invariant. This follows from the properties of the energy momentum tensor — if it is traceless the theory is conformally invariant. Indeed the tracelessness of the energy-momentum tensor defines what we mean by a CFT [31-34]. It is thus important to study the energy-momentum tensor and we construct it in this paper.

The energy-momentum tensor is also important in the context of AdS/CFT: one of the really interesting aspects of the AdS/CFT correspondence is that the D + 1-dimensional bulk theory

<sup>&</sup>lt;sup>1</sup> It also opens up the amazing possibility of rewriting quantum gravity as a quantum field theory in flat space.

 $<sup>^2</sup>$  Note that the "Wilson action" always has a finite UV cutoff — this is a point of departure from the usual CFT actions written in the continuum.

has dynamical gravity. In addition to the scalar field, there is the gravitational field that couples to the energy momentum tensor of the boundary CFT. Thus to extend the ideas of [29,30] to understand bulk gravity in AdS/CFT correspondence, from ERG one has to construct the energy momentum operator.

The energy-momentum tensor for  $\phi^4$  field theory has been worked out in the dimensional regularization scheme [33]. The construction of the energy-momentum tensor from the ERG point of view has been studied in general in [35,37]. The main idea is to solve the Ward Identity associated with coordinate transformations. This can be done in perturbation theory. We construct the leading terms that corresponds to the zero momentum energy momentum tensor. One can also check that the trace of the energy momentum tensor is proportional to the number operator. We apply this prescription here and construct the zero momentum energy momentum tensor to  $O(\lambda^2)$ .

This paper is organized as follows: In Section 2 we give a review of ERG and the fixedpoint equation. We also give some background material on the energy-momentum tensor. In Section 3 we construct the solution to the fixed-point equation and obtain the fixed-point action. In Section 4 we give a different approach to obtaining the fixed point equation and also calculate some correlation functions. In Section 5 the construction of the energy-momentum tensor is given. We conclude the paper in Section 6.

#### 2. Background

#### 2.1. Exact renormalization group and fixed point equation

We review the necessary background in this section. It depends mostly on [38,39].

#### 2.1.1. Exact renormalization group

Renormalization means essentially going from one scale  $\Lambda_0$  to a lower scale  $\Lambda$ , where the initial scale  $\Lambda_0$  is typically called a bare scale. One will want to see how the physics changes with scale. What do we mean by physics at  $\Lambda_0$ ? It means our theory will not be sensitive to momentum  $p > \Lambda_0$ . The partition function of the full theory is given by

$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]}$$

where

$$S = \int_{p} \frac{1}{2} p^2 \phi^2 + S_I[\phi]$$

To make it a partition function at scale  $\Lambda_0$  we will try to suppress the kinetic energy term for  $\infty . To execute this we will put a smooth cutoff in the kinetic energy term to obtain the bare action$ 

$$S_B[\phi] = \frac{1}{2} \int_p \phi \frac{p^2}{K(p^2/\Lambda_0^2)} \phi + S_{I,B}[\phi]$$
(2.1)

and the bare partition function

$$Z_B \equiv \int \mathcal{D}\phi \, e^{-S_B[\phi]} \tag{2.2}$$

We will choose the cutoff function will follow the condition K(0) = 1 and  $K(\infty) = 0$ . In general cutoff functions satisfy stronger properties, but that will not affect the fixed point values of the couplings [40].

Now we want to go to a lower scale  $\Lambda$ . For that, observe the following identity

$$\int \mathcal{D}\phi \exp\left[-\frac{1}{2}\int_{p}\phi(-p)\frac{1}{A(p)+B(p)}\phi(p) - S_{I,B}[\phi]\right]$$
$$=\int \mathcal{D}\phi_{1}\mathcal{D}\phi_{2}\exp\left[-\frac{1}{2}\int_{p}\frac{1}{A(p)}\phi_{1}(-p)\phi_{1}(p)\right.$$
$$\left.-\frac{1}{2}\int_{p}\frac{1}{B(p)}\phi_{2}(-p)\phi_{2}(p) - S_{I,B}[\phi_{1}+\phi_{2}]\right]$$

Using this we can write

$$Z_{B} = \int \mathcal{D}\phi_{l}\mathcal{D}\phi_{h} \exp\left\{-\frac{1}{2}\int_{p} \frac{p^{2}}{K(p^{2}/\Lambda^{2})}\phi_{l}(-p)\phi_{l}(p) - \frac{1}{2}\int_{p} \frac{p^{2}}{K(p^{2}/\Lambda^{2})}\phi_{h}(-p)\phi_{h}(p) - S_{I,B}[\phi_{l} + \phi_{h}]\right\}$$

We can effectively call  $\phi_l(\phi_h)$  as low(high) energy field as it is propagated by low(high) momentum propagator  $\Delta_l(\Delta_h)$  defined below

$$\Delta_l = \frac{K(p^2/\Lambda^2)}{p^2}, \quad \Delta_h = \frac{K(p^2/\Lambda^2) - K(p^2/\Lambda_0^2)}{p^2}$$
(2.3)

So we can write

$$Z_B = \int \mathcal{D}\phi_l \exp\left[-\frac{1}{2}\int_p \phi_l \Delta_l^{-1}\phi_l\right] \int \mathcal{D}\phi_h \exp\left[-\frac{1}{2}\int_p \phi_h \Delta_h^{-1}\phi_h - S_{I,B}[\phi_l + \phi_h]\right]$$
$$= \int \mathcal{D}\phi_l \exp\left[-\frac{1}{2}\int_p \phi_l \Delta_l^{-1}\phi_l\right] \exp\{-S_{I,\Lambda}[\phi_l]\}$$

where

$$\exp\{-S_{I,\Lambda}[\phi_l]\} \equiv \int \mathcal{D}\phi_h \exp\left\{-\frac{1}{2}\int\limits_p \phi_h \Delta_h^{-1}\phi_h - S_{I,B}[\phi_l + \phi_h]\right\}$$
(2.4)

 $S_{I,\Lambda}$  is the interaction part of an effective low energy field theory with a UV cutoff  $\Lambda$ . Let

$$S_{\Lambda}[\phi] \equiv \frac{1}{2} \int_{p} \phi_{l} \Delta_{l}^{-1} \phi_{l} + S_{I,\Lambda}[\phi_{l}]$$

$$(2.5)$$

be the whole action so that

$$Z_B = \int \mathcal{D}\phi_l \, e^{-S_\Lambda[\phi_l]} \tag{2.6}$$

Using (2.4), we obtain

$$e^{-S_{\Lambda}[\phi]} = \int \mathcal{D}\varphi \exp\left[-S_{B}[\varphi] + \frac{1}{2} \int_{p} \frac{p^{2}}{K(p/\Lambda_{0})} \varphi(p)\varphi(-p) - \frac{1}{2} \int_{p} \frac{p^{2}}{K(p/\Lambda)} \phi(p)\phi(-p) - \frac{1}{2} \int_{p} \frac{p^{2}}{K(p/\Lambda_{0}) - K(p/\Lambda)} (\varphi(p) - \phi(p)) (\varphi(-p) - \phi(-p))\right]$$
(2.7)

where we have written  $\phi_l$  as  $\phi$  and  $\phi_h$  as  $\varphi - \phi$ . This will be useful later.

It is to be noted that one can go back to the bare partition function anytime. For this reason this scheme is called "**exact**", i.e. we lose no physical information by varying the scale. It is easy to see this explicitly. Using (2.7), we can calculate the generating functional of  $S_B$  using  $S_{\Lambda}$  as

$$\int \mathcal{D}\phi \exp\left(-S_B[\phi] - \int_p J(-p)\phi(p)\right)$$
  
=  $\exp\left[\frac{1}{2}\int_p J(p)J(-p)\frac{1}{p^2} \{K(p/\Lambda_0)(1 - K(p/\Lambda_0)) - \left(\frac{K(p/\Lambda_0)}{K(p/\Lambda)}\right)^2 K(p/\Lambda)(1 - K(p/\Lambda))\}\right]$   
 $\times \int \mathcal{D}\phi \exp\left(-S_\Lambda[\phi] - \int_p J(-p)\frac{K(p/\Lambda_0)}{K(p/\Lambda)}\phi(p)\right)$  (2.8)

We observe that the correlation functions of  $S_B$  are the same as those of  $S_{\Lambda}$  up to the trivial (shortdistance) contribution to the two-point function and up to the momentum-dependent rescaling of the field by  $\frac{K(p/\Lambda_0)}{K(p/\Lambda)}$  [39]. If we ignore the small corrections to the two-point functions, we can write

$$\prod_{i=1}^{n} \frac{1}{K(p_i/\Lambda)} \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_\Lambda} = \prod_{i=1}^{n} \frac{1}{K(p_i/\Lambda')} \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_{\Lambda'}}$$
(2.9)

#### 2.1.2. Polchinski's ERG equation

We have given an integral formula (2.4) for  $S_{I,\Lambda}$  and (2.7) for  $S_{\Lambda}$ . It is easy to derive differential equations from these. From (2.4), we obtain Polchinski's ERG equation

$$-\Lambda \frac{\partial S_{I,\Lambda}[\phi]}{\partial \Lambda} = \int_{p} (-) \frac{dK(p/\Lambda)}{dp^2} \left( -\frac{\delta S_{I,\Lambda}[\phi]}{\delta \phi(p)} \frac{\delta S_{I,\Lambda}[\phi]}{\delta \phi(-p)} + \frac{\delta^2 S_{I,\Lambda}[\phi]}{\delta \phi(p)\delta \phi(-p)} \right)$$
(2.10)

for  $S_{I,\Lambda}$ . From (2.7) we obtain

$$-\Lambda \frac{\partial S_{\Lambda}[\phi]}{\partial \Lambda} = \int_{p} \left[ -2p^{2} \frac{d \ln K(p/\Lambda)}{dp^{2}} \phi(p) \frac{\delta S_{\Lambda}}{\delta \phi(p)} + \frac{d K(p/\Lambda)}{dp^{2}} \left( -\frac{\delta S_{\Lambda}}{\delta \phi(p)} \frac{\delta S_{\Lambda}}{\delta \phi(-p)} + \frac{\delta^{2} S_{\Lambda}}{\delta \phi(p) \delta \phi(-p)} \right) \right]$$
(2.11)

for the entire Wilson action.

#### 2.1.3. The limit $\Lambda \rightarrow 0+$

In the limit  $\Lambda \to 0+$  we expect  $S_{\Lambda}[\phi]$  approaches something related to the partition function. If we substitute

$$\lim_{\Lambda \to 0+} K(p/\Lambda) = 0 \tag{2.12}$$

into (2.7), we get

$$\lim_{\Lambda \to 0+} e^{-S_{\Lambda}[\phi] + \frac{1}{2} \int_{p} \frac{p^{2}}{K(p/\Lambda)} \phi(p)\phi(-p)} = \lim_{\Lambda \to 0+} e^{-S_{I,\Lambda}[\phi]}$$
$$= e^{-\frac{1}{2} \int_{p} \frac{p^{2}}{K(p/\Lambda_{0})} \phi(p)\phi(-p)} \int \mathcal{D}\varphi \exp\left[-S_{B}[\varphi] + \int_{p} \frac{p^{2}}{K(p/\Lambda_{0})} \varphi(p)\phi(-p)\right]$$
(2.13)

Hence, rewriting  $\phi(p)$  by  $\frac{K(p/\Lambda_0)}{p^2}J(p)$ , we obtain the generating functional of the bare theory as the  $\Lambda \to 0+$  limit of  $S_{I,\Lambda}$ :

$$Z_B[J] \equiv \int \mathcal{D}\varphi \exp\left[-S_B[\varphi] - \int_p \varphi(p)J(-p)\right]$$
$$= e^{-\frac{1}{2}\int_p J(p)J(-p)\frac{K(p/\Lambda_0)}{p^2}} \lim_{\Lambda \to 0+} \exp\left(-S_{I,\Lambda}\left[\frac{K(p/\Lambda_0)}{p^2}J(p)\right]\right)$$
(2.14)

#### 2.1.4. IR limit of a critical theory

For the bare theory at criticality, we expect that the correlation functions

$$\langle \varphi(p_1) \cdots \varphi(p_n) \rangle_B \equiv \int \mathcal{D}\varphi \,\varphi(p_1) \cdots \varphi(p_n) \,e^{-S_B[\varphi]}$$
(2.15)

to become scale invariant in the IR limit, i.e., for small momenta. To be more precise, we can define the limit

$$\mathcal{C}(p_1,\cdots,p_n) \equiv \lim_{t\to\infty} e^{\frac{n}{2}(-(D+2)+\eta)t} \left\langle \varphi(p_1e^{-t})\cdots\varphi(p_ne^{-t}) \right\rangle_B$$
(2.16)

where  $\frac{\eta}{2}$  is the anomalous dimension.

What does this mean for  $S_{\Lambda}$  in the limit  $\Lambda \rightarrow 0+?$  As we have seen above, the interaction part  $S_{I,\Lambda}$  becomes the generating functional of the bare theory in this limit. Since only the IR limit of the correlation functions are scale invariant, only the low momentum part of  $\lim_{\Lambda \rightarrow 0+} S_{I,\Lambda}$  corresponds to the scale invariant theory defined by the IR limit (2.16).

To understand the IR limit better, we follow Wilson [11] and reformulate the ERG transformation in two steps:

- introduction of an anomalous dimension (section 2.1.5) the anomalous dimension is an important ingredient of the IR limit. We need to introduce an anomalous dimension of the field within ERG.
- 2. introduction of a dimensionless framework (section 2.1.6) each time we lower the cutoff  $\Lambda$  we have to rescale space to restore the same momentum cutoff. This is necessary to realize scale invariance within ERG.

## 2.1.5. Anomalous dimension in ERG

The cutoff dependent Wilson action  $S_{\Lambda}[\phi]$  has two parts:

$$S_{\Lambda}[\phi] = \frac{1}{2} \int_{p} \frac{p^2}{K(p/\Lambda)} \phi(p)\phi(-p) + S_{I,\Lambda}[\phi]$$
(2.17)

The first term is a kinetic term, but this is not the only kinetic term; part of the interaction quadratic in  $\phi$ 's also contains the kinetic term. The normalization of  $\phi$  has no physical meaning, and it is natural to normalize the field so that  $S_{I,\Lambda}$  contains no kinetic term.

To do this, we modify the ERG differential equation (2.11) by adding a number operator [38,40]:

$$-\Lambda \partial_{\Lambda} S_{\Lambda}[\phi] = \int_{p} \left( -2p^{2} \frac{d}{dp^{2}} \ln K(p/\Lambda) \phi(p) \frac{\delta S_{\Lambda}}{\delta \phi(p)} - \frac{d}{dp^{2}} K(p/\Lambda) \left\{ \frac{\delta^{2} S_{\Lambda}}{\delta \phi(p) \delta \phi(-p)} - \frac{\delta S_{\Lambda}}{\delta \phi(p)} \frac{\delta S_{\Lambda}}{\delta \phi(-p)} \right\} \right)$$
$$- \frac{\eta_{\Lambda}}{2} \mathcal{N}_{\Lambda}[\phi]$$
(2.18)

where the number operator  $\mathcal{N}_{\Lambda}[\phi]$  is defined by

$$\mathcal{N}_{\Lambda}[\phi] \equiv \int_{p} \left[ \phi(p) \frac{\delta S_{\Lambda}}{\delta \phi(p)} + \frac{K(p/\Lambda) \left(1 - K(p/\Lambda)\right)}{p^{2}} \left\{ \frac{\delta^{2} S_{\Lambda}}{\delta \phi(p) \delta \phi(-p)} - \frac{\delta S_{\Lambda}}{\delta \phi(p)} \frac{\delta S_{\Lambda}}{\delta \phi(-p)} \right\} \right]$$
(2.19)

This counts the number of fields:

$$\langle \mathcal{N}_{\Lambda}[\phi]\phi(p_1)\cdots\phi(p_n)\rangle_{S_{\Lambda}} = n\,\langle\phi(p_1)\cdots\phi(p_n)\rangle_{S_{\Lambda}}$$
(2.20)

(Again we are ignoring small corrections to the two-point functions.) Under (2.18) the correlation functions change as

$$\prod_{i=1}^{n} \frac{1}{K(p_i/\Lambda)} \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_\Lambda} = \left(\frac{Z_\Lambda}{Z_{\Lambda'}}\right)^{\frac{n}{2}} \prod_{i=1}^{n} \frac{1}{K(p_i/\Lambda')} \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_{\Lambda'}}$$
(2.21)

where  $Z_{\Lambda}$  is the solution of

$$-\Lambda \frac{\partial}{\partial \Lambda} Z_{\Lambda} = \eta_{\Lambda} Z_{\Lambda} \tag{2.22}$$

satisfying the initial condition

$$Z_{\Lambda_0} = 1 \tag{2.23}$$

We can choose  $\eta_{\Lambda}$  so that  $S_{\Lambda}$  has the same kinetic term independent of  $\Lambda$ . For (2.18), the integral formula (2.7) must be changed to [39]

$$e^{S_{\Lambda}[\phi]} = \int \mathcal{D}\varphi \, e^{S_{0}[\varphi]} \\ \times \exp\left[-\frac{1}{2} \int_{p} \frac{p^{2}}{\frac{1-K(p/\Lambda)}{Z_{\Lambda}K(p/\Lambda)} - \frac{1-K(p/\Lambda_{0})}{K(p/\Lambda_{0})}} \left(\frac{\varphi(p)}{K(p/\Lambda_{0})} - \frac{\phi(p)}{\sqrt{Z_{\Lambda}}K(p/\Lambda)}\right) \right] \\ \times \left(\frac{\varphi(-p)}{K(p/\Lambda_{0})} - \frac{\phi(-p)}{\sqrt{Z_{\Lambda}}K(p/\Lambda)}\right)\right]$$
(2.24)

This reduces to (2.7) for  $Z_{\Lambda} = 1$ .

# 2.1.6. Dimensionless framework

To reach the IR limit (2.16) we must look at smaller and smaller momenta as we lower the cutoff  $\Lambda$ . We can do this by measuring the momenta in units of the cutoff  $\Lambda$ . At the same time we render all the dimensionful quantities such as  $\phi(p)$  dimensionless by using appropriate powers of  $\Lambda$ .

We introduce a dimensionless parameter t by

$$\Lambda = \mu \, e^{-t} \tag{2.25}$$

where  $\mu$  is an arbitrary fixed momentum scale. We then define the dimensionless field with dimensionless momentum by

$$\bar{\phi}(p) \equiv \Lambda^{\frac{D+2}{2}} \phi(p\Lambda) \tag{2.26}$$

and define a Wilson action parametrized by *t*:

$$\bar{S}_t[\bar{\phi}] \equiv S_\Lambda[\phi] \tag{2.27}$$

We can now rewrite (2.18) for  $\bar{S}_t$ :

$$\partial_t \bar{S}_t[\bar{\phi}] = \int_p \left( -2p^2 \frac{d}{dp^2} \ln K(p) + p \cdot \partial_p + \frac{D+2}{2} \right) \bar{\phi}(p) \cdot \frac{\delta \bar{S}_t[\bar{\phi}]}{\delta \bar{\phi}(p)} \\ + \int_p (-) \frac{d}{dp^2} K(p) \left\{ \frac{\delta^2 \bar{S}_t}{\delta \bar{\phi}(p) \delta \bar{\phi}(-p)} - \frac{\delta \bar{S}_t}{\delta \bar{\phi}(p)} \frac{\delta \bar{S}_t}{\delta \bar{\phi}(-p)} \right\} - \frac{\eta_t}{2} \mathcal{N}_t[\bar{\phi}]$$
(2.28)

where we have replaced  $\eta_{\Lambda}$  by  $\eta_t$ , and

$$\mathcal{N}_{t}[\bar{\phi}] = \int_{p} \bar{\phi}(p) \frac{\delta \bar{S}_{t}[\bar{\phi}]}{\delta \bar{\phi}(p)} + \int_{p} \frac{K(p) \left(1 - K(p)\right)}{p^{2}} \left(\frac{\delta^{2} \bar{S}_{t}}{\delta \bar{\phi}(p) \delta \bar{\phi}(-p)} - \frac{\delta \bar{S}_{t}}{\delta \bar{\phi}(p)} \frac{\delta \bar{S}_{t}}{\delta \bar{\phi}(-p)}\right)$$

$$(2.29)$$

is the number operator for  $\bar{S}_t$ .

Rewriting (2.21) in terms of dimensionless fields, we obtain

$$\prod_{i=1}^{n} \frac{1}{K(p_i)} \left\langle \bar{\phi}(p_1) \cdots \bar{\phi}(p_n) \right\rangle_{\bar{S}_t} = \left( \frac{Z_t}{Z_{t'}} \right)^{\frac{n}{2}} e^{-\frac{n}{2}(D-2)(t-t')} \prod_{i=1}^{n} \frac{1}{K(p_i e^{-(t-t')})} \left\langle \bar{\phi}(p_1 e^{-(t-t')}) \cdots \bar{\phi}(p_n e^{-(t-t')}) \right\rangle_{\bar{S}_{t'}}$$
(2.30)

where  $Z_t$  satisfies

 $\partial_t Z_t = \eta_t \, Z_t \tag{2.31}$ 

(The corrections to the two-point functions are ignored.) Comparing (2.30) with (2.16), the existence of the IR limit implies that

$$\lim_{t \to \infty} \eta_t = \eta \tag{2.32}$$

and

$$\lim_{t \to \infty} \prod_{i=1}^{n} \frac{1}{K(p_i)} \left\langle \bar{\phi}(p_1) \cdots \bar{\phi} \right\rangle_{\bar{S}_t} = \mathcal{C}(p_1, \cdots, p_n)$$
(2.33)

In other words  $\bar{S}_t$  approaches a limit as  $t \to +\infty$ :

$$\lim_{t \to +\infty} \bar{S}_t = \bar{S}_\infty \tag{2.34}$$

We call  $\bar{S}_{\infty}$  a fixed point because the right-hand side of (2.28) vanishes for it:

$$0 = \int_{p} \left( -2p^{2} \frac{d}{dp^{2}} \ln K(p) + p \cdot \partial_{p} + \frac{D+2}{2} \right) \bar{\phi}(p) \cdot \frac{\delta \bar{S}_{\infty}[\bar{\phi}]}{\delta \bar{\phi}(p)} + \int_{p} (-) \frac{d}{dp^{2}} K(p) \left\{ \frac{\delta^{2} \bar{S}_{\infty}}{\delta \bar{\phi}(p) \delta \bar{\phi}(-p)} - \frac{\delta \bar{S}_{\infty}}{\delta \bar{\phi}(p)} \frac{\delta \bar{S}_{\infty}}{\delta \bar{\phi}(-p)} \right\} - \frac{\eta}{2} \mathcal{N}_{\infty}[\bar{\phi}]$$
(2.35)

# 2.1.7. Fixed-point equation

Instead of choosing  $\eta$  dependent on t, we may choose  $\eta$  as a constant so that there is a nontrivial fixed-point solution  $\bar{S}_{\infty}$  for which the right-hand side of (2.28) vanishes. With a constant anomalous dimension, the dimensionless ERG equation is given by

$$\partial_t \bar{S}_t[\bar{\phi}] = \int_p \left( -2p^2 \frac{d}{dp^2} \ln K(p) + \frac{D+2}{2} - \frac{\eta}{2} + p \cdot \partial_p \right) \bar{\phi}(p) \cdot \frac{\delta \bar{S}_t[\bar{\phi}]}{\delta \bar{\phi}(p)} \\ + \int_p \left( -2 \frac{d}{dp^2} K(p) - \eta \frac{K(p)\left(1 - K(p)\right)}{p^2} \right) \\ \times \frac{1}{2} \left( \frac{\delta^2 \bar{S}_t[\bar{\phi}]}{\delta \bar{\phi}(p) \delta \bar{\phi}(-p)} - \frac{\delta \bar{S}_t[\bar{\phi}]}{\delta \bar{\phi}(p)} \frac{\delta \bar{S}_t[\bar{\phi}]}{\delta \bar{\phi}(-p)} \right)$$
(2.36)

For the O(N) model with N fields  $\phi^i$  ( $i = 1, \dots, N$ ), the ERG equation becomes

$$\partial_t \bar{S}_t[\bar{\phi}] = \int_p \left( -2p^2 \frac{d}{dp^2} \ln K(p) + \frac{D+2}{2} - \frac{\eta}{2} + p \cdot \partial_p \right) \bar{\phi}^i(p) \cdot \frac{\delta \bar{S}_t[\bar{\phi}]}{\delta \bar{\phi}^i(p)} \\ + \int_p \left( -2 \frac{d}{dp^2} K(p) - \eta \frac{K(p)\left(1 - K(p)\right)}{p^2} \right) \\ \times \frac{1}{2} \left( \frac{\delta^2 \bar{S}_t[\bar{\phi}]}{\delta \bar{\phi}^i(p) \delta \bar{\phi}^i(-p)} - \frac{\delta \bar{S}_t[\bar{\phi}]}{\delta \bar{\phi}^i(p) \delta \bar{\phi}^i(-p)} \right)$$
(2.37)

where the repeated indices i are summed over.

# 2.2. Energy momentum tensor: scale invariance and conformal invariance

#### 2.2.1. Energy momentum tensor in the classical theory

In this paper we will focus on the following Euclidean action whenever a concrete action is required for a calculation

$$S_E = \int d^D x \sqrt{g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right]$$

Using

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu}, \quad \delta \sqrt{g} = \frac{1}{2} \sqrt{g} g^{\mu\nu} \delta g_{\mu\nu}, \quad \delta g_{\mu\nu} = -g_{\mu\rho} \delta g^{\rho\sigma} g_{\sigma\nu}$$

we get

$$\delta S_E = -\int d^D x \, \frac{1}{2} \delta g_{\mu\nu} \sqrt{g} [\partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L}] \equiv -\int d^D x \, \frac{1}{2} \delta g_{\mu\nu} \sqrt{g} T^{\mu\nu} \tag{2.38}$$

where

$$T^{\mu\nu} \equiv -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}} = \partial^{\mu} \phi \partial^{\nu} \phi - g^{\mu\nu} \mathcal{L}$$
(2.39)

One can check that

$$\partial^{\nu} T_{\mu\nu} = -\partial_{\mu} \phi \left[ \frac{\partial \mathcal{L}}{\partial \phi} - \partial^{\rho} \left( \frac{\partial \mathcal{L}}{\partial^{\rho} \phi} \right) \right] = -\partial_{\mu} \phi \frac{\delta S_{E}}{\delta \phi}$$
(2.40)

Thus, classically the energy momentum tensor is conserved on-shell.

Now we rewrite  $T_{\mu\nu}$  in a form that will be useful later. Define the traceless tensor

$$t_{\mu\nu} = D\partial_{\mu}\partial_{\nu} - g_{\mu\nu}\Box \tag{2.41}$$

and the transverse tensor

$$\sigma_{\mu\nu} = (g_{\mu\nu}\Box - \partial_{\mu}\partial_{\nu})\phi^2 \tag{2.42}$$

Using the identity

$$\partial_{\mu}\phi\partial_{\nu}\phi = \partial_{\mu}\partial_{\nu}\frac{1}{2}\phi^2 - \phi\partial_{\mu}\partial_{\nu}\phi$$

one can rewrite

$$T_{\mu\nu} = \frac{1}{4(D-1)} t_{\mu\nu} \phi^2 + \frac{D-2}{4(D-1)} (\partial_\mu \partial_\nu - g_{\mu\nu} \partial^2) \phi^2 - \frac{1}{D} \phi t_{\mu\nu} \phi - \frac{1}{D} g_{\mu\nu} \left[ m^2 \phi^2 + (4-D) \frac{\lambda}{4!} \phi^4 + \frac{D-2}{2} E \right]$$
(2.43)

The trace which is proportional to  $g_{\mu\nu}\frac{\delta S}{\delta g_{\mu\nu}}$  can be written as  $\frac{\partial S}{\partial t}$  when  $g_{\mu\nu} = e^{2t}\delta_{\mu\nu}$  and is the response to scale transformations.

$$T^{\mu}_{\mu} = \frac{(2-D)}{4} \Box \phi^2 - \left[ m^2 \phi^2 + (4-D) \frac{\lambda}{4!} \phi^4 + \frac{D-2}{2} E \right]$$
(2.44)

with

$$E = \phi \frac{\delta S_E}{\delta \phi}$$

proportional to the equation of motion. The terms proportional to  $m^2$  and  $\lambda$  are genuine violations of scale invariance. But the first term can be gotten rid of by defining the improved energy momentum tensor

$$\Theta_{\mu\nu} = T_{\mu\nu} + \frac{D-2}{4(D-1)}\sigma_{\mu\nu}\phi^2$$
(2.45)

which is still conserved. So in a genuinely classically scale invariant theory with  $m^2 = 0$  and  $\lambda = 0$  or D = 4 one expects

$$\Theta^{\mu}_{\mu} = \frac{2-D}{2}E$$

#### 2.2.2. Trace of the energy momentum tensor in the quantum theory: perturbative

When quantum corrections<sup>3</sup> are included the condition for scale invariance is modified. The trace will be defined as before proportional to  $\frac{\partial S}{\partial t}$ . Before we turn to the exact RG let us see what happens in the usual lowest order perturbation theory. Let us start at  $\Lambda_0$  and evolve to  $\Lambda$  with  $\Lambda$  close to  $\Lambda_0$ .

$$S_{\Lambda_0} = \int_{x} \left[ \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{2} m_0^2 \phi^2 + \lambda_0 \frac{\phi^4}{4!} \right]$$
(2.46)

and

$$S_{\Lambda} = \int_{x} \left[ (1 - \delta Z(t)) \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{2} (m_0^2 + \delta m_0(t)^2) \phi^2 + (\lambda_0 + \delta \lambda_0(t)) \frac{\phi^4}{4!} + O(1/\Lambda) \right]$$

Here  $\delta Z$  is the correction to the kinetic term coming from the two loop diagram at  $\mathcal{O}(\lambda^2)$ ,  $\delta m_0^2 \approx O(\lambda)$  and  $\delta \lambda_0 \approx \mathcal{O}(\lambda^2)$  are the corrections starting at one loop.

We rewrite  $S_{\Lambda}$  in a suggestive way by adding and subtracting some terms proportional to  $\delta Z$ :

$$S_{\Lambda} = \int_{x} \left[ \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{2} \underbrace{(m_{0}^{2} + \delta m_{0}(t)^{2} + \delta Z m_{0}^{2})}_{m^{2}(t) = m_{R}^{2}} \phi^{2} + \underbrace{(\lambda_{0} + \delta \lambda_{0}(t) + 2\delta Z \lambda_{0})}_{\lambda(t) = \lambda_{R}} \frac{\phi'^{4}}{4!} \right]$$

<sup>&</sup>lt;sup>3</sup> We are working in Euclidean space. So "quantum" fluctuations are actually statistical fluctuations.

$$+ \mathcal{O}(1/\Lambda) \Big] - \delta Z \underbrace{\left[\frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi + \frac{1}{2}m_{0}^{2}\phi^{2} + 2\lambda_{0}\frac{\phi^{4}}{4!}\right]}_{\phi\frac{\partial\mathcal{L}}{\partial\phi}}$$
(2.47)

If we think of  $S_{\Lambda_0}$  as the bare action  $S_B$  and  $S_{\Lambda}$  as the renormalized action  $S_R$  so that  $S_B = S_R + S_{counter-term}$ , then  $\lambda_0 = \lambda_B$  and  $\lambda(t) = \lambda_R$ . The relation between renormalized and bare quantities is

$$\lambda_B = \frac{\lambda_R + \delta \lambda_R}{Z^2}$$

Here  $\delta \lambda_R$  is the *counterterm* and is chosen to *cancel* the correction  $\delta \lambda_0$  so  $\delta \lambda_R = -\delta \lambda_0$ . Let us write everything in terms of  $\lambda_B$ :

$$\lambda_B = \lambda_R + \delta\lambda_R - 2\delta Z\lambda_R \approx \lambda_R + \delta\lambda_R - 2\delta Z\lambda_0$$
$$\lambda_B + 2\delta Z\lambda_0 - \delta\lambda_R = \lambda_0 + 2\delta Z\lambda_0 + \delta\lambda_0 = \lambda_R = \lambda(t)$$

Thus for small *t*:

$$\lambda(t) = \lambda_0 + \beta(\lambda_0)t$$
;  $m^2(t) = m^2(0)(1 + \gamma_m t)$ ;  $\delta Z = -2\gamma t$ 

Furthermore define

$$x = \bar{x}\Lambda^{-1} = \bar{x}\Lambda_0 e^t$$

The trace of the energy momentum tensor is given by the dependence on t

$$-T^{\mu}_{\mu} = \frac{\partial S_{\Lambda_0}}{\partial t}$$

$$= \Lambda^{-D} \left\{ \int_{\bar{x}} \left[ \frac{1}{2} m_0^2 \gamma_m(\lambda_0) \phi^2 + \beta(\lambda_0) \frac{\phi^4}{4!} \right] + 2\gamma \int_{x} \frac{1}{2} \phi \frac{\delta S_{\Lambda_0}}{\delta \phi(x)}$$

$$+ D \int_{\bar{x}} \left[ \frac{1}{2} m_0^2 \phi^2 + \lambda_0 \frac{\phi^4}{4!} \right] + (D-2) \int_{\bar{x}} \frac{1}{2} \partial_{\bar{\mu}} \phi \partial^{\bar{\mu}} \phi + O(1/\Lambda_0) \right] \right\}$$
(2.48)

Define dimensionless variables as

$$m_0^2 = \bar{m}^2 \Lambda_0^2 = \bar{m}^2 e^{2t} \Lambda^2$$

and

$$\lambda_0 = (\Lambda_0)^{4-D} \bar{\lambda}_0 = \bar{\lambda}_0 e^{(4-D)t} (\Lambda)^{4-D}$$

and fields

$$\phi = (\Lambda)^{\frac{D-2}{2}} \overline{\phi} = e^{-\frac{D-2}{2}t} \Lambda_0^{\frac{D-2}{2}} \overline{\phi}$$

Now add and subtract

$$(\frac{D-2}{2})\int_{\bar{x}}\bar{\phi}\frac{\delta S_{\Lambda_0}}{\delta\bar{\phi}(x)}$$

to get

$$-T^{\mu}_{\mu} = \int_{\bar{x}} \underbrace{\left[\frac{1}{2}\bar{m}^{2}(2+\gamma_{m}(\lambda_{0}))\bar{\phi}^{2} + (\beta(\lambda_{0}) - (D-4)\lambda_{0})\frac{\bar{\phi}^{4}}{4!}\right]}_{\text{``$\beta-function'''}} + \left(\frac{D-2}{2}+\gamma\right)\int_{\bar{x}} \bar{\phi}\frac{\delta S_{\Lambda_{0}}}{\delta\bar{\phi}(x)} + O(1/\Lambda_{0})$$
(2.49)

LHS can be identified with the trace of the energy momentum tensor in the quantum theory and can be compared with the corresponding classical expression in (2.44). The above gives an idea of how the quantum corrections modify  $T_{\mu\nu}$ . A detailed calculation of the energy momentum tensor in the renormalized theory in terms of composite operators and using dimensional regularization is given in [33]. A systematic and precise treatment is provided by ERG and is given in [35,37] and is summarized below.

#### 2.2.3. Energy momentum tensor in exact RG

We summarize the properties of the energy momentum tensor in ERG, given in [35].

The Ward Identity almost<sup>4</sup> defines the energy momentum. Because of general coordinate invariance

$$\delta x^{\mu} = -\epsilon^{\mu}$$
;  $\phi'(x) = \phi(x) + \epsilon^{\mu} \partial_{\mu} \phi(x)$ 

is equivalent to (Assume that  $g_{\mu\nu} = \eta_{\mu\nu}$ )

$$\delta g_{\mu\nu} = \epsilon_{(\mu,\nu)}$$

and

$$\int \mathcal{D}\phi' = \int \mathcal{D}\phi_{g+\delta g} \quad ; \quad S[\phi, g+\delta g] = S[\phi', g]$$

Thus the following identity must hold

$$Z[J] = \int \mathcal{D}\phi' e^{-S[\phi'(x)] + \int_x J(x)\phi'(x)} = \int \mathcal{D}\phi_{g+\delta g} e^{-S[\phi(x),g+\delta g] + \int_x J(x)(\phi(x) + \epsilon^{\mu}\partial_{\mu}\phi(x))}$$

Then using the definition of the energy momentum tensor, i.e.

$$Z[J=0,g+\delta g] = \int \mathcal{D}\phi_{g+\delta g} e^{-S[\phi,g+\delta g]} \equiv \int \mathcal{D}\phi_g e^{-S[\phi,g]+\frac{1}{2}\int \sqrt{g}\delta g_{\mu\nu}T^{\mu\nu}}$$
(2.50)

we get the Ward identity

$$-\partial_{\mu}\langle T^{\mu}_{\nu}(x)\phi(x_{1})...\phi(x_{n})\rangle + \sum_{i=1}^{n}\delta(x-x_{i})\langle\phi(x_{1})...\partial_{\nu}\phi(x_{i})...\phi(x_{n})\rangle = 0$$
(2.51)

This is a statement of the conservation of  $T_{\mu\nu}$  corresponding to the classical statement (2.40).

In ERG this can be written as a Ward identity for the composite operator  $[T_{\mu\nu}]$ 

$$q^{\mu}[T_{\mu\nu}(q)] = \int_{p} e^{S[\phi]} K(p)(p+q)_{\nu} \frac{\delta}{\delta\phi(p)} ([\phi(p+q)]e^{-S[\phi]})$$
(2.52)

<sup>&</sup>lt;sup>4</sup> up to transverse terms of the form  $\partial_{\mu}\partial_{\nu} - \Box \delta_{\mu\nu}$  that do not contribute.

The equation corresponding to (2.49) and (2.44) is

$$T^{\mu}_{\mu}(0) = -\frac{\partial S}{\partial t} - (\frac{D-2}{2} + \gamma)\mathcal{N}$$
(2.53)

where  $-\frac{\partial S}{\partial t}$  gives the ERG evolution, with anomalous dimension, in terms of dimensionless variables - the " $\beta$ -function". It vanishes at the fixed point. N is the number operator. Note that this equation is obtained for zero momentum or as an integral over space-time in position space. The classical analog of this is (2.44), which was obtained for arbitrary momentum.

Note that in equations (2.52) and (2.53), both LHS and RHS are composite operators. So one strategy will be to evaluate  $T_{\mu\nu}$  using these equations in the bare theory at some scale  $\Lambda_0$  which will be taken to be infinity. The bare theory is very simple so the calculations can be done exactly. Then one can evolve  $T_{\mu\nu}$  down to a scale  $\Lambda << \Lambda_0$  order by order using the ERG evolution operator. If we choose  $\lambda$  and *m* to be on the critical surface we are guaranteed that at  $\Lambda$  the theory flows to the fixed point action. Thus we will have evaluated the energy momentum tensor at the fixed point.

Another approach is to work directly with the known fixed point action and solve the Ward identity order by order. In this paper we follow the second approach.

#### **3.** Wilson-Fisher fixed point for the O(N) model

We will find the fixed-point Wilson action by putting  $\frac{\partial \bar{S}_t}{\partial t} = 0$  in (2.37). As we will work mostly with dimensionless variables we will remove the bar sign from the dimensionless variables unless otherwise mentioned. Also t dependence of actions and fields being readily implied, the subscript t will be omitted too. We give the fixed point action *S* in the following form:

$$S = S_2 + S_4 + S_6$$

where  $S_2$  and  $S_4$  are given by

$$S_2 = \int \frac{d^D p}{(2\pi)^D} U_2(p) \frac{1}{2} \phi^I(p) \phi^I(-p)$$
(3.54)

$$S_4 = \frac{1}{2} \prod_{i=1}^{3} \int \frac{d^D p_i}{(2\pi)^D} U_4(p_1, p_2; p_3, p_4) \frac{1}{2} \phi^I(p_1) \phi^I(p_2) \frac{1}{2} \phi^J(p_3) \phi^J(p_4)$$
(3.55)

where  $p_1 + p_2 + p_3 + p_4 = 0$  is implied. Instead of putting an explicit delta function and integrating over  $p_4$  we will simply impose momentum conservation at every stage. Accordingly  $S_6$  is given by

$$S_{6} = \frac{1}{3!} \prod_{i=1}^{5} \int \frac{d^{D} p_{i}}{(2\pi)^{D}} U_{6}(p_{1}, p_{2}; p_{3}, p_{4}; p_{5}, p_{6}) \\ \times \frac{1}{2} \phi^{I}(p_{1}) \phi^{I}(p_{2}) \frac{1}{2} \phi^{J}(p_{3}) \phi^{J}(p_{4}) \frac{1}{2} \phi^{K}(p_{5}) \phi^{K}(p_{6})$$
(3.56)

#### 3.1. Equations for the vertices

We get the following equations for  $U_2, U_4$  and  $U_6$ :

Equation for  $U_2$ 

$$0 = \int \frac{d^{D}p}{(2\pi)^{D}} \left\{ \left( \frac{-\eta}{2} \frac{K(1-K)}{p^{2}} - K'(p^{2}) \right) \right. \\ \left. \times \frac{1}{8} \left[ 4NU_{4}(p_{1}, -p_{1}; p, -p) + 8U_{4}(p_{1}, p; -p_{1}, -p) \right] \right. \\ \left. - \frac{1}{2!} 2U_{2}(p)U_{2}(p)\delta^{D}(p-p_{1}) \right\} + \left( \frac{-\eta}{2} + 1 - 2\frac{p_{1}^{2}}{K(p_{1}^{2})}K'(p_{1}^{2}) \right) U_{2}(p_{1}) \\ \left. - \frac{1}{2!}p_{1}\frac{dU_{2}(p_{1})}{dp_{1}} \right]$$
(3.57)

Equation for  $U_4$ 

$$0 = \int \frac{d^{D}p}{(2\pi)^{D}} \left( \frac{-\eta}{2} \frac{K(1-K)}{p^{2}} - K'(p^{2}) \right) \frac{1}{48} \\ \times \left\{ 6NU_{6}(p_{1}, p_{2}; p_{3}, p_{4}; p, -p) + 12U_{6}(p_{1}, p; p_{2}, -p; p_{3}, p_{4}) \\ + 12U_{6}(p_{1}, p_{2}; p_{3}, p; p_{4}, -p) \right\} \\ - \sum_{j=1}^{4} \left( \frac{-\eta}{2} \frac{K(1-K)}{p_{j}^{2}} - K'(p_{j}^{2}) \right) U_{2}(p_{j}) \frac{2}{8} U_{4}(p_{1}, p_{2}; p_{3}, p_{4}) \\ + \sum_{j=1}^{4} \left( \frac{-\eta}{2} - 2 \frac{p^{2}}{K(p_{j}^{2})} K'(p_{j}^{2}) \right) \frac{1}{8} U_{4}(p_{1}, p_{2}; p_{3}, p_{4}) \\ + \left[ 4 - D - \sum_{i=1}^{4} p_{i} \frac{d}{dp_{i}} \right] \frac{1}{8} U_{4}(p_{1}, p_{2}; p_{3}, p_{4})$$
(3.58)

Here  $p = p_a + p_b + p_n = -(p_i + p_j + p_m)$ . Equation for  $U_6$ 

$$0 = \frac{2}{48} \sum_{6 \ perm \ of \ (m,n)} \left( \frac{-\eta}{2} \frac{K(1-K)}{(p_i + p_j + p_m)^2} - K'((p_i + p_j + p_m)^2) \right) \\ \times U_4(p_i, p_j; p_m, p) U_4(p_a, p_b; p_n, -p) \\ + \sum_{j=1}^6 \left( K'(p_j^2) - \frac{-\eta}{2} \frac{K(1-K)}{p_j^2} \right) U_2(p_j) \frac{2}{48} U_6(p_1, p_2; p_3, p_4; p_5, p_6) \\ + \sum_{j=1}^6 \left( \frac{-\eta}{2} - 2 \frac{p^2}{K(p_j^2)} K'(p_j^2) \right) \frac{1}{48} U_6(p_1, p_2; p_3, p_4; p_5, p_6) \\ + \left[ 6 - 2D - \sum_{i=1}^6 p_i \frac{d}{dp_i} \right] \frac{1}{48} U_6(p_1, p_2; p_3, p_4; p_5, p_6)$$
(3.59)

#### 3.2. Solving the equations

We know that  $U_4 \approx \mathcal{O}(\epsilon)$  and  $U_6 \approx \mathcal{O}(\epsilon^2)$  and  $\eta \approx \mathcal{O}(\epsilon^2)$ , where  $\epsilon = 4 - D$ .

# 3.2.1. O(1): retrieving Gaussian theory

We start with (3.57) for  $U_2$ . Neglecting  $U_4$  and  $\eta$  and collecting coefficients of  $\phi^2$  we get

$$0 = K'(p^2)U_2(p)U_2(p) + \left(1 - 2\frac{p^2}{K(p^2)}K'(p^2)\right)U_2(p) - p^2\frac{dU_2(p)}{dp^2}$$
(3.60)

 $U_2(p) = \frac{p^2}{K(p^2)}$  solves this equation. This is expected since the Gaussian theory is expected to be a fixed point - and this ERG was obtained from Polchinski's ERG by adding on the kinetic term  $\frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \phi(p) \frac{p^2}{K(p^2)} \phi(-p)$ . Thus our solution can be written as

$$U_{2}(p) = \frac{p^{2}}{K(p^{2})} + \underbrace{U_{2}^{(1)}(p)}_{\mathcal{O}(\epsilon)} + \mathcal{O}(\epsilon^{2})$$
(3.61)

3.2.2.  $\mathcal{O}(\epsilon)$ : fixed point value of  $m^2$ 

We go back to (3.57) and keep  $U_4$  which is  $\mathcal{O}(\epsilon)$  but drop  $\eta$  which is  $\mathcal{O}(\epsilon^2)$ .

$$0 = \int \frac{d^{D}p}{(2\pi)^{D}} \left( \frac{-\eta}{2} \frac{K(1-K)}{p^{2}} - K'(p^{2}) \right) \times \left\{ \frac{1}{8} \left[ 4NU_{4}(p_{1}, p_{2}; p, -p) + 8U_{4}(p_{1}, p; -p, -p_{1}) \right] - \frac{1}{2!} 2U_{2}(p)U_{2}(p)\delta^{D}(p-p_{1}) \right\} + \left( \frac{-\eta}{2} + 1 - 2\frac{p_{1}^{2}}{K(p_{1}^{2})} K'(p_{1}^{2}) \right) U_{2}(p_{1}) - \frac{1}{2!} p_{1} \frac{dU_{2}(p_{1})}{dp_{1}}$$
(3.62)

We use (3.61) in the above equation and look at the terms of order  $\epsilon$ . To leading order we set  $U_4 = \lambda$ , which is  $\mathcal{O}(\epsilon)$ . The equation for  $U_2^{(1)}$  is given by

$$0 = -\lambda \frac{4N+8}{8} \int \frac{d^{D}p}{(2\pi)^{D}} K'(p^{2}) + 2 \frac{p_{1}^{2}}{K(p_{1}^{2})} U_{2}^{(1)} \left(p_{1}\right) K'(p_{1}^{2}) + (1 - 2\frac{p_{1}^{2}}{K(p_{1}^{2})} K'(p_{1}^{2})\right) U_{2}^{(1)}(p_{1}) - p_{1}^{2} \frac{dU_{2}^{(1)}(p_{1})}{dp_{1}^{2}}$$

To leading order this equation is solved by a constant  $U_2^{(1)}$ , i.e.

$$0 = -\lambda \frac{4N+8}{8} \int \frac{d^D p}{(2\pi)^D} K'(p^2) + U_2^{(1)}$$
(3.63)

Thus

$$U_2^{(1)} = \lambda \frac{N+2}{2} \int \frac{d^D p}{(2\pi)^D} K'(p^2)$$
(3.64)

Here

$$\int \frac{d^D p}{(2\pi)^D} = \frac{1}{2^D \pi^{D/2} \Gamma(D/2)} \int (p^2)^{\frac{D-2}{2}} dp^2$$

To get leading results we can set D = 4:

$$U_2^{(1)} = \lambda \frac{4N+8}{8} \frac{1}{(4\pi)^2} \int_0^\infty dp^2 p^2 K'(p^2) = -\lambda \frac{4N+8}{8} \frac{1}{(4\pi)^2} \int_0^\infty dp^2 K(p^2)$$
(3.65)

We have used K(0) = 1,  $K(\infty) = 0$ . This gives the fixed point value of the dimensionless mass parameter:

$$U_2^{(1)} = m_\star^2 = -\lambda \frac{N+2}{2} \frac{1}{(4\pi)^2} \int_0^\infty dp^2 K(p^2)$$
(3.66)

To evaluate the integral explicitly we need a specific form for K. We use  $K(p^2) = e^{-p^2}$ . Then the integral is equal to 1.

3.2.3.  $\mathcal{O}(\epsilon^2)$ : expression for the six-point vertex

Let us turn to (3.59) reproduced below:

$$0 = -\frac{2}{48} \sum_{\substack{6 \text{ perm of } (i,j,m)}} \left( \frac{-\eta}{2} \frac{K(1-K)}{(p_i + p_j + p_m)^2} - K'((p_i + p_j + p_m)^2) \right) \\ \times U_4(p_i, p_j; p_m, p) U_4(p_a, p_b; p_n, -p) \\ + \sum_{j=1}^6 \left\{ \left( K'(p_j^2) - \frac{-\eta}{2} \frac{K(1-K)}{p_j^2} \right) 2U_2(p_j) + \left( \frac{-\eta}{2} - 2 \frac{p^2}{K(p_j^2)} K'(p_j^2) \right) \right\} \\ \times \frac{1}{48} U_6(p_1, p_2; p_3, p_4; p_5, p_6) \\ + \left[ 6 - 2D - \sum_{i=1}^6 p_i \frac{d}{dp_i} \right] \frac{1}{48} U_6(p_1, p_2; p_3, p_4; p_5, p_6)$$
(3.67)

where  $p = p_a + p_b + p_n = -(p_i + p_j + p_m)$ . In this equation we keep terms of  $\mathcal{O}(\epsilon^2)$ . Since  $\eta$  is  $\mathcal{O}(\epsilon^2)$ , and multiplies terms of  $\mathcal{O}(\epsilon^2)$ , it contributes only at  $\mathcal{O}(\epsilon^4)$  in this equation, so it can be dropped here. Furthermore then, if we use the leading order solution for  $U_2 = \frac{p^2}{K(p^2)}$ , the second and third terms cancel each other. So we are left with

$$0 = -\frac{2}{48} \sum_{6 \ perm \ (i,j,m)} K'((p_i + p_j + p_m)^2) U_4(p_i, p_j; p_n, p) U_4(p_a, p_b; p_n, -p) + \left[ (6 - 2D - \sum_{i=1}^6 p_i \frac{d}{dp_i}) \right] \frac{1}{48} U_6(p_1, p_2; p_3, p_4; p_5, p_6)$$
(3.68)

Since  $U_4 = \lambda$  to this order, we obtain

$$0 = \lambda^2 \frac{2}{48} \sum_{6 \ perm \ (i,j,m)} K'((p_i + p_j + p_m)^2)$$

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$$+\left[6-2D-\sum_{i=1}^{6}p_{i}\frac{d}{dp_{i}}\right]\frac{1}{48}U_{6}(p_{1},p_{2};p_{3},p_{4};p_{5},p_{6})$$
(3.69)

The solution for one permutation is

$$U_6(p_1, p_2; p_3, p_4; p_5, p_6) = \lambda^2 \frac{K((p_1 + p_2 + p_3)^2) - K(0)}{(p_1 + p_2 + p_3)^2}$$

The full solution is given by

$$U_{6}(p_{1}, p_{2}; p_{3}, p_{4}; p_{5}, p_{6}) = -\lambda^{2} \{ h(p_{1} + p_{2} + p_{3}) + h(p_{1} + p_{2} + p_{4}) + h(p_{1} + p_{2} + p_{5}) + h(p_{1} + p_{2} + p_{6}) + h(p_{1} + p_{3} + p_{4}) + h(p_{2} + p_{3} + p_{4}) \}$$

$$(3.70)$$

where  $h(x) = \frac{K(0) - K(x)}{x^2}$ .

# 3.2.4. Fixed point value of $\lambda$ : solution for $U_4$ at $\mathcal{O}(\epsilon)$

The  $U_4$  equation is given by (3.58). In this equation  $\eta$  can be neglected as  $-\eta \approx \mathcal{O}(\epsilon^2)$ . Also we put the value of  $U_2$  upto order of  $\epsilon$  found above. There is a cancellation between the second and third terms on the R.H.S and we obtain

$$\left[ \left( 4 - D - \sum_{i=1}^{4} p_i \frac{d}{dp_i} \right) - \sum_{j=1}^{4} 2K'(p_j^2) \frac{\lambda}{16\pi^2} \frac{N+2}{2} \right] \frac{1}{8} U_4(p_1, p_2; p_3, p_4)$$
  
=  $\int \frac{d^D p}{(2\pi)^D} K'(p^2) \frac{1}{48} \left\{ 6NU_6(p_1, p_2; p_3, p_4; p, -p) + 12U_6(p_1, p; p_2, -p; p_3, p_4) + 12U_6(p_1, p_2; p_3, p; p_4, -p) \right\}$  (3.71)

The solution is given in the Appendix A.1. The fixed point value  $\lambda^*$  given below solves the above equation:

$$\lambda^* = (4 - D) \frac{16\pi^2}{N + 8} \tag{3.72}$$

#### 3.3. Determining anomalous dimension

 $U_2$  equation at  $\mathcal{O}(\epsilon^2)$ 

$$0 = \int \left\{ \frac{d^D p}{(2\pi)^D} \left( \frac{-\eta}{2} \frac{K(1-K)}{p^2} - K'(p^2) \right) \left[ \frac{\delta^2 S_4}{\delta \phi^I(p) \delta \bar{\phi}^I(-p)} - \frac{\delta S_2}{\delta \phi^I(p)} \frac{\delta S_2}{\delta \phi^I(-p)} \right] \right\} \\ + \left\{ -\frac{\eta}{2} - 2 \frac{p^2}{K(p^2)} K'(p^2) \right\} \phi(p) \cdot \frac{\delta S}{\delta \phi(p)} + \mathcal{G}^c_{dil} S_2$$

where we plug in:

$$U_{4}(p_{1}, p_{2}; p_{3}, p_{4}) = \lambda + \underbrace{\tilde{U}_{4}(p_{1}, p_{2}; p_{3}, p_{4})}_{O(\epsilon^{2})}$$

$$U_{2}(p) = \frac{p^{2}}{K} - \lambda \frac{N+2}{2} \int \frac{d^{D}p}{(2\pi)^{D}} K'(p^{2}) + \underbrace{\tilde{U}_{2}(p)}_{O(\epsilon^{2})}$$
(3.73)

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and keep only  $\mathcal{O}(\epsilon^2)$  terms in the above equation to get

$$0 = \int \frac{d^{D} p}{(2\pi)^{D}} \left( \frac{-\eta}{2} \frac{K(1-K)}{p^{2}} - K'(p^{2}) \right) \times \left\{ \frac{1}{8} \left[ 4N \tilde{U}_{4}(p_{1}, -p_{1}; p, -p) + 8 \tilde{U}_{4}(p_{1}, p; -p, -p_{1}) \right] - \frac{1}{2!} 2U_{2}(p) U_{2}(p) \delta^{D}(p-p_{1}) \right] \right\} + \left( \frac{-\eta}{2} + 1 - 2 \frac{p_{1}^{2}}{K(p_{1}^{2})} K'(p_{1}^{2}) \right) U_{2}(p_{1}) - p_{1}^{2} \frac{dU_{2}(p_{1})}{dp_{1}^{2}}$$
(3.74)

On simplification it gives

$$-\frac{-\eta}{2}\frac{(1-K)}{K}p_1^2 - \int \frac{d^D p}{(2\pi)^D} K'(p^2) \frac{1}{8} \bigg[ 4N\tilde{U}_4(p_1, -p_1; p, -p) \\ + 8\tilde{U}_4(p_1, p_1; -p, -p_1) \bigg] + K'(p_1^2)U_2(p_1)U_2(p_1) \\ + \frac{-\eta}{2}\frac{p_1^2}{K} + \tilde{U}_2(p_1) - p_1^2\frac{d\tilde{U}_2(p_1)}{dp_1^2} = 0$$
(3.75)

In the L.H.S the third term will cancel with part of the second term (shown in A.3). Also the raison d'etre for introducing  $\eta$  is to ensure that  $U_2 = p^2 + \mathcal{O}(p^4)$ . So we let  $\tilde{U}_2 = \mathcal{O}(p^4)$ . So the anomalous dimension is given by

$$\frac{\eta}{2} = -\frac{d}{dp_1^2} \int \frac{d^D p}{(2\pi)^D} K'(p^2) \frac{1}{8} \bigg[ 4N \tilde{U}_4^{II}(p_1, -p_1; p, -p) + 8 \tilde{U}_4^{II}(p_1, p; -p_1, -p) \bigg] \bigg|_{p_1^2 = 0}$$
(3.76)

Here the superscript II is explained in Appendix A and refers to a class of Feynman diagrams.

 $\tilde{U}_4$  is determined by solving (3.71). So using (3.76) and (A.151) one can determine  $\eta$ . This is done in the Appendix A.4. The result is of course well known [11]:

$$\frac{\eta}{2} = \lambda^2 \frac{N+2}{4} \frac{1}{(16\pi^2)^2} = \frac{N+2}{(N+8)^2} \frac{\epsilon^2}{4}$$
(3.77)

Collecting results we have (we have put D=4 for  $\mathcal{O}(\epsilon^2)$  terms),

$$U_2(p) = \frac{p^2}{K(p^2)} - \lambda \frac{N+2}{2} \int \frac{d^D p}{(2\pi)^D} K'(p^2) + \tilde{U}_2(p)$$
(3.78)

The expression for  $\tilde{U}_2(p)$  is given in (A.149) (also in the next section a neater expression is presented).

$$U_4(p_1, p_2; p_3, p_4) = (4 - D) \frac{16\pi^2}{N+8} + \frac{(N+2)}{2} \frac{\lambda^2}{16\pi^2} \sum_{j=1}^4 h(p_j) -\lambda^2 \left[ (N+4)F(p_1+p_2) + 2F(p_1+p_3) + 2F(p_1+p_4) \right]$$
(3.79)

where

$$F(p) = \frac{1}{2} \int \frac{d^{D}p}{(2\pi)^{D}} h(q) \bigg[ h(p+q) - h(q) \bigg]$$

and

$$h(p) = \frac{K(0) - K(p^2)}{p^2}$$

$$U_6(p_1, p_2; p_3, p_4; p_5, p_6)$$

$$= -\lambda^2 \left\{ h(p_1 + p_2 + p_3) + h(p_1 + p_2 + p_4) + h(p_1 + p_2 + p_5) + h(p_1 + p_2 + p_6) + h(p_1 + p_3 + p_4) + h(p_2 + p_3 + p_4) \right\}$$
(3.80)

and the anomalous dimension is given by

$$\frac{\eta}{2} = \lambda^2 \frac{N+2}{4} \frac{1}{(16\pi^2)^2} = \frac{N+2}{(N+8)^2} \frac{\epsilon^2}{4}$$
(3.81)

To evaluate the integrals we have put D = 4 and used specific form of  $K(p^2) = e^{-p^2}$ .

This completes the solution of the fixed point ERG equation and determination of the eigenvalue  $\eta$  corresponding to anomalous dimension up to  $O(\epsilon^2)$ . In the next section we give a slightly different approach to obtaining the fixed point action and evaluate correlation functions.

#### 4. Correlation functions

#### 4.1. A more general equation

In the previous section we set  $\frac{\partial S}{\partial t} = 0$  and solved the fixed point equation for the action order by order. One can also solve a more general equation where the LHS is not set to zero but to  $\frac{\partial S}{\partial t} = \beta_J \frac{\partial S}{\partial \lambda_J}$ . The parameters can be chosen so that the beta functions are zero. This has the effect that the equations are modified at each order by terms of higher order. The advantage is that the solutions are easier to write down.

We want to obtain the fixed-point Wilson action to order  $\lambda^2$  in the following form:

$$S[\phi^{I}] = \int_{p} \frac{1}{2} \phi^{I}(p) \phi^{I}(-p) \left(\frac{p^{2}}{K(p)} + U_{2}(p)\right) + \frac{1}{2} \int_{p_{1}, p_{2}, p_{3}, p_{4}} \frac{1}{2} \phi^{I}(p_{1}) \phi^{I}(p_{2}) \frac{1}{2} \phi^{J}(p_{3}) \phi^{J}(p_{4}) \delta \times \left(\sum_{i=1}^{4} p_{i}\right) \left(\lambda + V_{4}(p_{1}, p_{2}; p_{3}, p_{4})\right) + \frac{1}{3!} \int_{p_{1}, \cdots, p_{6}} \frac{1}{2} \phi^{I}(p_{1}) \phi^{I}(p_{2}) \frac{1}{2} \phi^{J}(p_{3}) \phi^{J}(p_{4}) \frac{1}{2} \phi^{K}(p_{5}) \phi^{K}(p_{6}) \delta\left(\sum_{i=1}^{6} p_{i}\right) \times V_{6}(p_{1}, p_{2}; p_{3}, p_{4}; p_{5}, p_{6})$$

$$(4.82)$$

As we will all vertex function in powers of  $\lambda$  we have to put the general expression for  $\frac{\partial \lambda}{\partial t}$  i.e.

$$\frac{\partial \lambda}{\partial t} = (\epsilon \lambda + \beta_N^{(1)} \lambda^2)$$

Where  $\beta_N^{(1)}$ , the leading term in the beta function, is given by

$$\beta_N^{(1)} = 2(N+8) \int \frac{d^D p}{(2\pi)^D} K'(p) \frac{K(0) - K(p)}{p^2} \equiv -(N+8) \int_p f(p)h(p) dp dp$$

where  $f(p) = -2K'(p^2)$ .

If we assume  $V_2(p) = \lambda v_2^{(1)}(p) + \lambda^2 v_2^{(2)}(p) = \lambda v_2^{(1)}(p) + \left(V_2^I(p) + V_2^{II}(p)\right)$ , where  $V_2^{I(II)}$  is analog of  $\tilde{U}_2^{I(II)}$  in A.3, then

$$\frac{\partial V_2(p)}{\partial t} = \left(\epsilon \lambda + \beta_N^{(1)} \lambda^2\right) v_2^{(1)}(p) + 2\lambda^2 \epsilon v_2^{(2)}(p) + 2\lambda^3 \beta_N^{(1)} v_2^{(2)}(p)$$

Similarly if  $V_4(p_1, p_2; p_3, p_4) = V_4^I(p_1, p_2; p_3, p_4) + V_4^{II}(p_1, p_2; p_3, p_4)$ , where  $V_4^{I(II)}(p_1, p_2; p_3, p_4)$  is equivalent to  $\tilde{U}_4^{I(II)}(p_1, p_2; p_3, p_4)$  in A.2.

$$\frac{\partial}{\partial t} \Big[ \lambda + V_4(p_1, p_2; p_3, p_4) \Big] = \left( \epsilon \lambda + \beta_N^{(1)} \lambda^2 \right) + 2V_4(p_1, p_2; p_3, p_4) \Big( \epsilon + \beta_N^{(1)} \lambda \Big)$$

A. (3.64) is modified to

$$\frac{1}{2}\epsilon v_2^{(1)}(p) = -\frac{4N+8}{8} \int \frac{d^D p}{(2\pi)^D} K'(p^2) + v_2^{(1)}(p),$$

gives

$$v_2^{(1)}(p) = -\frac{N+2}{2-\epsilon} \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} f(p)$$

$$\equiv -(N+2)v_2$$
(4.83)

where  $v_2 = \frac{1}{2-\epsilon} \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} f(p)$ **B**. (A.140) turns into

 $\left[\epsilon + \sum_{j=1}^{4} p_j \frac{d}{dp_j}\right] V_4^{II}(p_1, p_2; p_3, p_4))$ =  $-2\lambda^2 \int_{\bar{p}} K'(p^2) \left[ (N+4)h(p_1+p_2+p) + 2h(p+p_1+p_3) + 2h(p+p_1+p_4) - (N+8)h(p) \right]$  (6)

If we write  $V_4^{II}(p_1, p_2; p_3, p_4) = -\lambda^2 \{ (N+4)F(p_1+p_2) + 2F(p_1+p_3) + 2F(p_1+p_4) \}$ the equation for F(p) can be written as,

$$(p.\partial p + \epsilon)F(p) = \int \frac{d^D p}{(2\pi)^D} f(q)h(q+p) + \frac{1}{3}\beta^{(1)}$$
(4.85)

(4.84)

where

$$\frac{1}{3}\beta^{(1)} = -\int \frac{d^D p}{(2\pi)^D} f(p)h(p)$$

The solution, analytic at p = 0 is,

$$F(q) = \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} h(p) \Big( h(q+p) - h(p) \Big)$$
(4.86)

C. Similarly (A.139a) gets modified to,

$$\left[\epsilon + \sum_{j=1}^{4} p_j \frac{d}{dp_j}\right] \frac{1}{8} V_4^I(p_1, p_2; p_3, p_4) = \lambda^2 (N+2) \int \frac{d^D p}{(2\pi)^D} K'(p^2) \left\{ -\frac{1}{8} \sum_{j=1}^{4} h(p_j) - \frac{1}{4(2-\epsilon)} K'(p_j^2) \right\}$$
(4.87)

whose solution is,

$$V_4^I(p_1, p_2; p_3, p_4) = \lambda^2 \frac{(N+2)}{2-\epsilon} \int \frac{d^D p}{(2\pi)^D} (-K'(p^2)) \sum_{j=1}^4 h(p_j)$$
(4.88)

Also

$$\frac{1}{8} \Big\{ 4NV_4^I(p_1, -p_1; p, -p) + 8V_4^I(p, p_1; -p, -p_1) \Big\} \\= \frac{(N+2)^2}{2-\epsilon} \lambda^2 \int \frac{d^D q}{(2\pi)^D} (-K'(q^2)) \Big[ h(p_1) + h(p) \Big]$$

**D**. (3.75) turns into,

$$(2-2\epsilon)V_2^I - 2p_1^2 \frac{dV_2^I(p_1)}{dp_1^2} = -\frac{2\lambda^2}{2-\epsilon}(N+2)^2 \left\{ \int \frac{d^D p}{(2\pi)^D} (-K'(p^2)) \right\}^2 h(p_1) - 2(v_2^{(1)})^2 K'(p_1^2)$$
(4.89)

The solution is

$$V_2^I(p_1) = -(N+2)^2 \lambda^2 \frac{1}{(2-\epsilon)^2} \frac{1}{4} \left\{ \int \frac{d^D p}{(2\pi)^D} f(p) \right\}^2 h(p_1)$$

E. (A.147) changes to

$$\begin{split} & \left(-2+2\epsilon\right) V_2^{II}(p_1) + \beta_N^{(1)} \lambda^2 v_2^{(1)}(p) + 2p_1^2 \frac{dV_2^{II}(p_1)}{dp_1^2} \\ & = -3\lambda^2 (N+2) \int\limits_{r,p} (-K'(p^2)) h(r) \Big[ h(p_1+p+r) - h(r) \Big] \\ & + \frac{2}{2-\epsilon} \Big[ (N+2)^2 \lambda^2 \int (-K'(q^2)) \Big] \int\limits_p (-K'(p^2)) h(p) - \eta \ p_1^2 \end{split}$$

If we assume

$$V_2^{II}(p) = -3\lambda^2(N+2)G(p)$$

Then G(p) satisfies the following equation,

$$(p.\partial p - 2 + 2\epsilon)G(p) = \int f(q)F(p+q) + \frac{2\nu_2}{3} \int_p f(p)h(p) + \eta^{(2)}p^2$$
(4.90)

From (3.76) we get  $\eta = 3(N+2)\lambda^2 \eta^{(2)}$  where,

$$\eta^{(2)} = -\frac{d}{dp^2} \int f(q) F(q+p) \Big|_{p=0}$$

The solution, analytic at p = 0 is

$$G(p) = \frac{1}{3} \int h(q)(F(p+q) - F(q)) + \frac{1}{\epsilon} \frac{\eta^{(2)}}{2} p^2 - \frac{1}{2 - 2\epsilon} \left( \int f(q)F(q) + \frac{2v_2}{3} \int_p f(p)h(p) \right)$$
(4.91)

 $V_2^I(p) + V_2^{II}(p)$  when calculated in the limit  $\epsilon \to 0$  gives the expression of  $\tilde{U}_2(p)$  mentioned in the previous section.

The solutions are given by,

$$V_2(p) = -\lambda(N+2)v_2 - \lambda^2 \left( 3(N+2)G(p) + (N+2)^2 (v_2)^2 h(p) \right)$$
(4.92a)

$$V_4(p_1, p_2; p_3, p_4) = -\lambda^2 \Big( (N+4)F(p_1+p_2) + 2F(p_1+p_3) + 2F(p_1+p_4) \\ -(N+2)v_2 \sum_{i=1}^4 h(p_i) \Big)$$
(4.92b)

$$V_{6}(p_{1}, p_{2}; p_{3}, p_{4}; p_{5}, p_{6}) = -\lambda^{2} (h(p_{1} + p_{2} + p_{3}) + h(p_{1} + p_{2} + p_{4}) + h(p_{1} + p_{2} + p_{5}) + h(p_{1} + p_{2} + p_{6}) + h(p_{3} + p_{4} + p_{1}) + h(p_{3} + p_{4} + p_{2}))$$

$$(4.92c)$$

where

$$f(p) = -2K'(p^2); \ h(p) = \frac{K(0) - K(p^2)}{p^2}$$

and

$$v_2 = \frac{1}{2 - \epsilon} \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} f(p)$$
(4.93)

If we take the limit  $\epsilon \to 0$  and  $K(p^2) = e^{-p^2}$  we get

$$v_{2} = \frac{1}{2} \int \frac{d^{4}p}{(2\pi)^{4}} e^{-p^{2}} = \frac{1}{2} \frac{1}{16\pi^{2}}$$
$$F(p) = \frac{1}{2} \int \frac{d^{D}p}{(2\pi)^{D}} h(q) \Big[ h(p+q) - h(q) \Big]$$

The coupling constant  $\lambda$  is given, to order  $\epsilon = 4 - D$ , as

$$\lambda = \frac{\epsilon}{-\beta_N^{(1)}} = \frac{(4\pi)^2}{N+8}\epsilon$$
(4.94)

The anomalous dimension is given, to order  $\epsilon^2$ , as

$$\eta = \frac{N+2}{2(N+8)^2} \epsilon^2$$
(4.95)

#### 4.2. Calculation of correlation functions

In this section we will calculate two-, four-, and six-point correlation functions. Recall that our Wilson action has a fixed momentum cutoff of order 1. If we consider the momenta much larger than the cutoff, the vertices of the Wilson action gives the correlation functions [36]. We first rescale the field

$$J^{I}(p) \equiv \frac{1}{h(p)} \phi^{I}(p) \tag{4.96}$$

and define

$$W[J^{I}] \equiv -S[\phi^{I}] + \frac{1}{2} \int_{p} J^{I}(p) J^{I}(-p) \frac{h(p)}{K(p)}$$
(4.97)

For our Wilson action, this is given by

$$W[J^{I}] = \int_{p} \frac{1}{2} J^{I}(p) J^{I}(-p) h(p)^{2} \left(\frac{1}{h(p)} - V_{2}(p)\right) + \frac{1}{2} \int_{p_{1}, p_{2}, p_{3}, p_{4}} \frac{1}{2} J^{I}(p_{1}) J^{I}(p_{2}) \frac{1}{2} J^{J}(p_{3}) J^{J}(p_{4}) \delta\left(\sum_{i=1}^{4} p_{i}\right) \times \prod_{i=1}^{4} h(p_{i}) \cdot (-\lambda - V_{4}(p_{1}, p_{2}; p_{3}, p_{4})) + \frac{1}{3!} \int_{p_{1}, \cdots, p_{6}} \frac{1}{2} J^{I}(p_{1}) J^{I}(p_{2}) \frac{1}{2} J^{J}(p_{3}) J^{J}(p_{4}) \frac{1}{2} J^{K}(p_{5}) J^{K}(p_{6}) \delta\left(\sum_{i=1}^{6} p_{i}\right) \times \prod_{i=1}^{6} h(p_{i}) \cdot (-) V_{6}(p_{1}, p_{2}; p_{3}, p_{4}; p_{5}, p_{6})$$
(4.98)

In the high momentum limit we obtain the generating functional of the connected correlation functions

$$\mathcal{W}[J^{I}] = \lim_{t \to +\infty} W[J_{t}^{I}]$$
(4.99)

where

$$J_t^I(p) \equiv \exp\left(-t\frac{D-2+\eta}{2}\right) J^I(pe^{-t})$$
(4.100)

In our case we obtain

$$W[J_{t}^{I}] = \int_{p} \frac{1}{2} J^{I}(p) J^{I}(-p) \exp(t(2-\eta)) h(pe^{t})^{2} \left(\frac{1}{h(pe^{t})} - V_{2}(pe^{t})\right) + \frac{1}{2} \int_{p_{1}, p_{2}, p_{3}, p_{4}} \frac{1}{2} J^{I}(p_{1}) J^{I}(p_{2}) \frac{1}{2} J^{J}(p_{3}) J^{J}(p_{4}) \delta\left(\sum_{i=1}^{4} p_{i}\right) \times \exp(t(D+4-2\eta) \prod_{i=1}^{4} h(p_{i}e^{t}) \cdot (-\lambda - V_{4}(p_{1}e^{t}, p_{2}e^{t}; p_{3}e^{t}, p_{4}e^{t})) + \frac{1}{3!} \int_{p_{1}, \cdots, p_{6}} \frac{1}{2} J^{I}(p_{1}) J^{I}(p_{2}) \frac{1}{2} J^{J}(p_{3}) J^{J}(p_{4}) \frac{1}{2} J^{K}(p_{5}) J^{K}(p_{6}) \delta\left(\sum_{i=1}^{6} p_{i}\right) \times \exp(t(2D+6-3\eta) \prod_{i=1}^{6} h(p_{i}e^{t}) \cdot (-) V_{6}(p_{1}e^{t}, p_{2}e^{t}; p_{3}e^{t}, p_{4}e^{t}; p_{5}e^{t}, p_{6}e^{t})$$

$$(4.101)$$

In the limit  $t \to +\infty$  we obtain

$$\mathcal{W}[J^{I}] = \int_{p} \frac{1}{2} J^{I}(p) J^{I}(-p) C_{2}(p) + \frac{1}{2} \int_{p_{1}, p_{2}, p_{3}, p_{4}} \frac{1}{2} J^{I}(p_{1}) J^{I}(p_{2}) \frac{1}{2} J^{J}(p_{3}) J^{J}(p_{4}) \delta\left(\sum_{i=1}^{4} p_{i}\right) C_{4}(p_{1}, p_{2}; p_{3}, p_{4}) + \frac{1}{3!} \int_{p_{1}, \cdots, p_{6}} \frac{1}{2} J^{I}(p_{1}) J^{I}(p_{2}) \frac{1}{2} J^{J}(p_{3}) J^{J}(p_{4}) \frac{1}{2} J^{K}(p_{5}) J^{K}(p_{6}) \delta\left(\sum_{i=1}^{6} p_{i}\right) \times C_{6}(p_{1}, p_{2}; p_{3}, p_{4}; p_{5}, p_{6})$$

$$(4.102)$$

# 4.2.1. Two-point function

$$C_{2}(p) = \lim_{t \to +\infty} \exp(t(2-\eta))h(pe^{t})^{2} \left(\frac{1}{h(pe^{t})} - V_{2}(pe^{t})\right)$$
$$= \lim_{t \to +\infty} \frac{1}{(p^{2})^{2}} \left[p^{2}(1-\eta t) + \lambda^{2}3(N+2)e^{-2t}G(pe^{t})\right]$$
(4.103)

Using

$$G(pe^{t}) \xrightarrow{t \to \infty} p^{2} e^{2t} \frac{1}{12(4\pi)^{4}} \ln\left(p^{2} e^{2t}\right)$$

$$(4.104)$$

we obtain

$$C_2(p) = \frac{1}{p^2} \left( 1 + \frac{\eta}{2} \ln p^2 \right) = \frac{1}{p^{2-\eta}}$$
(4.105)

# 4.2.2. Four-point function

$$C_{4}(p_{1}, p_{2}; p_{3}, p_{4})$$

$$= \lim_{t \to +\infty} \exp\left(t\left(D + 4 - 2\eta\right)\right) \prod_{i=1}^{4} h(p_{i}e^{t}) \cdot \left(-\lambda - V_{4}(p_{1}e^{t}, p_{2}e^{t}; p_{3}e^{t}, p_{4}e^{t})\right)$$

$$= \prod_{i=1}^{4} \frac{1}{p_{i}^{2}} \lim_{t \to +\infty} (1 - \epsilon t) \left[-\lambda + \lambda^{2} \left((N + 4)F\left((p_{1} + p_{2})e^{t}\right) + 2F\left((p_{1} + p_{3})e^{t}\right) + 2F\left((p_{2} + p_{3})e^{t}\right)\right)\right] \quad (4.106)$$

Using

$$F(pe^{t}) \xrightarrow{t \to +\infty} -\frac{1}{(4\pi)^{2}} \ln\left(pe^{t}\right)$$
(4.107)

we obtain

$$\prod_{i=1}^{4} p_i^2 \cdot C_4(p_1, p_2; p_3, p_4) = -\lambda \left( 1 + \epsilon \frac{1}{N+8} \ln \left\{ (p_1 + p_2)^{N+4} (p_1 + p_3)^2 (p_2 + p_3)^2 \right\} \right)$$
(4.108)

# 4.2.3. Six-point function

Since  $V_6$  is already of order  $\lambda^2$ , we can take D = 4 and  $\eta = 0$  to obtain

$$C_{6}(p_{1}, p_{2}; p_{3}, p_{4}; p_{5}, p_{6})$$

$$= \lim_{t \to +\infty} e^{t(2D+6-3\eta)} \prod_{i=1}^{6} h(p_{i}e^{t}) (-) V_{6}(p_{1}e^{t}, p_{2}e^{t}; p_{3}e^{t}, p_{4}e^{t}; p_{5}e^{t}, p_{6}e^{t})$$

$$= \lim_{t \to +\infty} e^{14t} \prod_{i=1}^{6} \frac{1}{p_{i}^{2}e^{2t}} \cdot \lambda^{2} \left( h\left((p_{1}+p_{2}+p_{3})e^{t}\right) + \cdots \right)$$

$$= \lambda^{2} \prod_{i=1}^{6} \frac{1}{p_{i}^{2}} \left( \frac{1}{(p_{1}+p_{2}+p_{3})^{2}} + \cdots + \frac{1}{(p_{3}+p_{4}+p_{2})^{2}} \right)$$
(4.109)

# 5. Construction of the energy-momentum tensor at the fixed point

Given a fixed-point Wilson action, we wish to construct the energy-momentum tensor  $\Theta_{\mu\nu}(p)$ . It is a **symmetric** tensor implicitly determined by the Ward identity

$$p_{\mu}\Theta_{\mu\nu}(p) = e^{S} \int_{q} K(q)(q+p)_{\nu} \frac{\delta}{\delta\phi^{I}(q)} \left( \left[ \phi^{I}(q+p) \right] e^{-S} \right)$$
(5.110)

where

$$\left[\phi^{I}(p)\right] \equiv \frac{1}{K(p)} \left(\phi^{I}(p) - \frac{K(p)\left(1 - K(p)\right)}{p^{2}} \frac{\delta S}{\delta \phi^{I}(-p)}\right)$$
(5.111)

is the composite operator corresponding to  $\phi^{I}(p)$ . The Ward identity leaves an additive ambiguity of the form

$$\left(p^2\delta_{\mu\nu}-p_{\mu}p_{\nu}\right)\mathcal{O}(p)$$

where  $\mathcal{O}(p)$  is a scalar composite operator. Since  $\Theta_{\mu\nu}$  must have zero scale dimension,  $\mathcal{O}$  must have scale dimension -2. There is no such  $\mathcal{O}$ , since the squared mass operator  $\frac{1}{2}\phi^2$  acquires a positive anomalous dimension at the fixed point. Hence, the Ward identity determines  $\Theta_{\mu\nu}$  unambiguously. In fact we are going to calculate  $\Theta_{\mu\nu}(p)$  only at p = 0; we need not worry about this ambiguity anyway.

It is convenient to expand  $\Theta_{\mu\nu}(p)$  in powers of  $[\phi^I]$ :

$$\Theta_{\mu\nu}(p) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{p_1, \cdots, p_{2n}} \prod_{i=1}^n \frac{1}{2} \left[ \phi^{I_i}(p_{2i-1}) \right] \left[ \phi^{I_i}(p_{2i}) \right] \delta \left( \sum_{i=1}^{2n} p_i - p \right) \\ \times c_{\mu\nu, 2n}(p_1, p_2; \cdots; p_{2n-1}, p_{2n})$$
(5.112)

To order  $\lambda^2$ , we only have three coefficients  $c_{\mu\nu,0}$ ,  $c_{\mu\nu,2}$ ,  $c_{\mu\nu,4}$ . Since the field-independent term (n = 0) is proportional to  $\delta(p)$ , we cannot determine  $c_{\mu\nu,0}$  from the Ward identity. So, we will determine only  $c_{\mu\nu,2}$  and  $c_{\mu\nu,4}$ .

From (4.82), we obtain

$$\begin{bmatrix} \phi^{I}(p) \end{bmatrix} = \phi^{I}(p) - h(p) \Big\{ V_{2}(p)\phi^{I}(p) \\ + \int_{p_{1}, p_{2}, p_{3}} \frac{1}{2} \phi^{J}(p_{1})\phi^{J}(p_{2})\phi^{I}(p_{3})\delta\left(\sum_{i=1}^{3} p_{i} - p\right) (\lambda + V_{4}(p_{1}, p_{2}; p_{3}, -p)) \\ + \frac{1}{2} \int_{p_{1}, \cdots, p_{5}} \frac{1}{2} \phi^{J}(p_{1})\phi^{J}(p_{2}) \frac{1}{2} \phi^{K}(p_{3})\phi^{K}(p_{4})\phi^{I}(p_{5})\delta\left(\sum_{i=1}^{5} p_{i} - p\right) \\ \times V_{6}(p_{1}, p_{2}; p_{3}, p_{4}; p_{5}, -p) \Big\}$$

$$(5.113)$$

Inverting this we obtain, to order  $\lambda^2$ ,

$$\phi^{I}(p) = \left[\phi^{I}(p)\right] + h(p) \left\{ V_{2}^{1PI}(p) \left[\phi^{I}(p)\right] + \int_{p_{1}, p_{2}, p_{3}} \frac{1}{2} \left[\phi^{J}(p_{1})\right] \left[\phi^{J}(p_{2})\right] \left[\phi^{I}(p_{3})\right] \delta \times \left(\sum_{i=1}^{3} p_{i} - p\right) \left(\lambda + V_{4}^{1PI}(p_{1}, p_{2}; p_{3}, -p)\right) \right\}$$
(5.114)

where we have defined the 1PI vertices as

$$V_2^{1PI}(p) = -\lambda(N+2)v_2 - \lambda^2 3(N+2)G(p)$$
(5.115a)

$$V_4^{1PI}(p_1, p_2; p_3, p_4) = -\lambda^2 \left( (N+4)F(p_1+p_2) + 2F(p_1+p_3) + 2F(p_1+p_4) \right)$$
(5.115b)

Note that  $\phi^I$  has no sixth order term expanded in  $[\phi]$ 's to order  $\lambda^2$ .

The rhs of (5.110) gives

$$e^{S} \int_{q} K(q)(q+p)_{\nu} \frac{\delta}{\delta \phi^{I}(q)} \left( \left[ \phi^{I}(q+p) \right] e^{-S} \right)$$
  
= 
$$\int_{q} K(q)(q+p)_{\nu} \left( -\left[ \phi^{I}(q+p) \right] \frac{\delta S}{\delta \phi^{I}(q)} + \frac{\delta}{\delta \phi^{I}(q)} \left[ \phi^{I}(q+p) \right] \right)$$
(5.116)

Expanding this in powers of  $[\phi]$ 's, we obtain from (5.110) the following equations that determine the coefficients  $c_{\mu\nu,2}$  and  $c_{\mu\nu,4}$ .

$$p_{\mu}c_{\mu\nu,2}(p_{1}, p_{2}) = -p_{1\nu}p_{2}^{2} - p_{2\nu}p_{1}^{2}$$

$$+ \lambda(N+2)\left(v_{2}p_{\nu} - \int_{q} (q+p)_{\nu}R(q)h(q)h(q+p)\right)$$

$$+ \lambda^{2}(N+2)\left[3\left(p_{1\nu}G(p_{2}) + p_{2\nu}G(p_{1})\right)\right]$$

$$- (N+2)v_{2}\int_{q} (q+p)_{\nu}R(q)h(q)h(q+p)\left(h(q) + h(q+p)\right)$$

$$+ \frac{1}{2}\int_{q} \left\{(q+p)_{\nu}R(q) - q_{\nu}R(q+p)\right\}h(q)h(q+p)$$

$$\times \left\{(N+2)F(p) + 3F(q+p_{1}) + 3F(q+p_{2})\right\}\right]$$
(5.117a)

and

$$p_{\mu}c_{\mu\nu,4}(p_{1}, p_{2}; p_{3}, p_{4}) = -\lambda p_{\nu} + \lambda^{2} \Big\{ (N+4) \left( F(p_{1}+p_{2})(p_{3}+p_{4})_{\nu} + F(p_{3}+p_{4})(p_{1}+p_{2})_{\nu} \right) + 2p_{1\nu} \left( F(p_{2}+p_{3}) + F(p_{2}+p_{4}) \right) + 2p_{2\nu} \left( F(p_{2}+p_{3}) + F(p_{2}+p_{4}) \right) + 2p_{3\nu} \left( F(p_{4}+p_{1}) + F(p_{4}+p_{2}) \right) + 2p_{4\nu} \left( F(p_{3}+p_{1}) + F(p_{3}+p_{2}) \right) \Big\} + \lambda^{2} \frac{1}{2} \int_{q} \{ (q+p)_{\nu} R(q) - q_{\nu} R(q+p) \} h(q) h(q+p) \times \{ (N+4) \left( h(q+p_{1}+p_{2}) + h(q+p_{3}+p_{4}) \right) + 4 \left( h(q+p_{1}+p_{3}) + h(q+p_{1}+p_{4}) \right) \}$$
(5.117b)

To determine  $c_{\mu\nu,2}(p_1, p_2)$  at p = 0, we substitute  $p_2 = p - p_1$  into the rhs of (5.117a), and expand the result to first order in p. This gives

$$c_{\mu\nu,2}(p_{1},-p_{1}) = -p_{1}^{2}\delta_{\mu\nu} + 2p_{1\mu}p_{1\nu} + \lambda(N+2)\delta_{\mu\nu} \left\{ v_{2} - \int_{q} R(q) \left( h(q)^{2} + \frac{1}{D}h(q)q \cdot \partial_{q}h(q) \right) \right\} + \lambda^{2}(N+2) \left\{ 3 \left( \delta_{\mu\nu}G(p_{1}) - 2p_{1\mu}p_{1\nu}G'(p_{1}) \right) \right\}$$

$$+ \int_{q} \left( \delta_{\mu\nu} R(q) - 2q_{\mu}q_{\nu}R'(q) \right) h(q)^{2} \left( -(N+2)v_{2}h(q) + 3F(q+p_{1}) \right) \right\}$$
(5.118)

Similarly, substituting  $p_4 = p - (p_1 + p_2 + p_3)$  into the rhs of (5.117b) and expanding the result to first order in p, we obtain

$$c_{\mu\nu,4}(p_{1}, p_{2}; p_{3}, -(p_{1}+p_{2}+p_{3})) = -\lambda \delta_{\mu\nu} + \lambda^{2} \Big\{ (N+4) \left( \delta_{\mu\nu} F(p_{1}+p_{2}) - 2(p_{1}+p_{2})_{\mu} (p_{1}+p_{2})_{\nu} F'(p_{1}+p_{2}) \right) + 2 \left( \delta_{\mu\nu} F(p_{1}+p_{3}) - 2(p_{1}+p_{3})_{\mu} (p_{1}+p_{3})_{\nu} F'(p_{1}+p_{3}) \right) + 2 \left( \delta_{\mu\nu} F(p_{2}+p_{3}) - 2(p_{2}+p_{3})_{\mu} (p_{2}+p_{3})_{\nu} F'(p_{2}+p_{3}) \right) + \int_{q} \left( \delta_{\mu\nu} R(q) - 2q_{\mu}q_{\nu} R'(q) \right) h(q)^{2} \times ((N+4)h(q+p_{1}+p_{2}) + 2h(q+p_{1}+p_{3}) + 2h(q+p_{2}+p_{3})) \Big\}$$
(5.119)

Check of the trace anomaly

Using the energy-momentum tensor obtained above, we can verify the trace anomaly

$$\Theta(0) = -\left(\frac{D-2}{2} + \frac{1}{2}\eta\right)\mathcal{N}(0)$$
(5.120)

where the anomalous dimension is given by (4.95) to order  $\epsilon^2$ .

The trace is easily obtained from (5.118, 5.119) as

$$\begin{split} \Theta(0) &= \int_{p} \frac{1}{2} \left[ \phi^{I}(p) \right] \left[ \phi^{I}(-p) \right] \left[ - (D-2)p^{2} \\ &+ \lambda(N+2)D \left\{ v_{2} - \int_{q} R(q) \left( h(q)^{2} + \frac{1}{D} h(q)q \cdot \partial_{q} h(q) \right) \right\} \\ &+ \lambda^{2}(N+2) \left\{ 3(D-p \cdot \partial_{p})G(p) \\ &+ \int_{q} \left( D-q \cdot \partial_{q} \right) R(q) \cdot h(q)^{2} \left( -(N+2)v_{2} + 3F(q+p) \right) \right\} \right] \\ &+ \frac{1}{2} \int_{p_{1}, \cdots, p_{4}} \frac{1}{2} \left[ \phi^{I}(p_{1}) \right] \left[ \phi^{I}(p_{2}) \right] \frac{1}{2} \left[ \phi^{J}(p_{3}) \right] \left[ \phi^{J}(p_{4}) \right] \delta \left( \sum_{i=1}^{4} p_{i} \right) \\ &\times \left[ -\lambda D \\ &+ \lambda^{2} \left\{ (N+4) \left( D-p \cdot \partial_{p} \right) F(p) \right|_{p=p_{1}+p_{3}} + 2 \left( D-p \cdot \partial_{p} \right) F(p) \right|_{p=p_{2}+p_{3}} \end{split}$$

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$$+ \int_{q} (D - q \cdot \partial_{q}) R(q) \cdot h(q)^{2} \times ((N + 4)h(q + p_{1} + p_{2}) + 2h(q + p_{1} + p_{3}) + 2h(q + p_{2} + p_{3})) \Big\} \Big]$$
(5.121)

On the other hand the number operator, defined by

$$\mathcal{N}(0) \equiv -e^{S} \int_{q} K(q) \frac{\delta}{\delta \phi^{I}(q)} \left( \left[ \phi^{I}(q) \right] e^{-S} \right), \qquad (5.122)$$

is calculated as

$$\mathcal{N}(0) = \int_{p} \frac{1}{2} \Big[ \phi^{I}(p) \Big] \Big[ \phi^{I}(-p) \Big] \Big[ 2p^{2} + (N+2)\lambda \left( -2v_{2} + \int_{q} R(q)h(q)^{2} \right) \\ + \lambda^{2}(N+2) \left\{ -6G(p) + 2(N+2)v_{2} \int_{q} R(q)h(q)^{3} - 6 \int_{q} R(q)h(q)^{2}F(q+p) \right\} \Big] \\ + \frac{1}{2} \int_{p_{1}, \cdots, p_{4}} \frac{1}{2} \Big[ \phi^{I}(p_{1}) \Big] \Big[ \phi^{I}(p_{2}) \Big] \frac{1}{2} \Big[ \phi^{J}(p_{3}) \Big] \Big[ \phi^{J}(p_{4}) \Big] \delta \left( \sum_{i=1}^{4} p_{i} \right) \\ \times \Big[ 4\lambda - 4\lambda^{2} \{ (N+4)F(p_{1}+p_{2}) + 2F(p_{1}+p_{3}) + 2F(p_{2}+p_{3}) \} \\ - 2\lambda^{2} \int_{q} R(q)h(q)^{2} \{ (N+4)h(p+p_{1}+p_{2}) \\ + 2h(p+p_{1}+p_{3}) + 2h(p+p_{2}+p_{3}) \} \Big]$$
(5.123)

Using

$$f(q) = (q \cdot \partial_q + 2) h(q) = (2 - q \cdot \partial_q) R(q) \cdot h(q)^2$$
(5.124)

and the equations satisfied by F and G

$$\left(p \cdot \partial_p + \epsilon\right) F(p) = \int_q f(q) \cdot \left(h(q+p) - h(q)\right)$$
(5.125a)

$$\left(p \cdot \partial_p - 2 + 2\epsilon\right) G(p) = \frac{2}{3} v_2 \int_q f(q) \cdot h(q) + \eta \, p^2 + \int_q f(q) \cdot F(q+p)$$
(5.125b)

we obtain

$$\Theta(0) + \left(\frac{D-2}{2} + \gamma_N^{(2)}\lambda^2\right)\mathcal{N}(0)$$
  
=  $\left(\epsilon\lambda + \beta_N^{(1)}\lambda^2\right) \left[\int_p \frac{1}{2} \left[\phi^I(p)\right] \left[\phi^I(-p)\right] (N+2)v_2$ 

$$-\frac{1}{2}\int_{p_1,p_2,p_3,p_4} \frac{1}{2} \Big[\phi^I(p_1)\Big] \Big[\phi^I(p_2)\Big] \frac{1}{2} \Big[\phi^J(p_3)\Big] \Big[\phi^J(p_4)\Big] \delta\left(\sum_{i=1}^4 p_i\right)\Big]$$
(5.126)

where we have dropped  $\epsilon \lambda^2 G(p)$  and  $\epsilon \lambda^2 F(p)$ , which are terms of order  $\epsilon^3$ . This vanishes at the fixed point, where

$$\epsilon \lambda + \beta_N^{(1)} \lambda^2 = 0,$$

to order  $\epsilon^2$ .

# Correlation functions

In the previous section we saw how the fixed-point Wilson action gives the correlation functions. Similarly, the coefficient functions  $c_{\mu\nu,2}(p_1, p_2)$  and  $c_{\mu\nu,4}(p_1, p_2; p_3, p_4)$  give the 1PI correlation functions of the energy-momentum tensor at p = 0:

$$\left\langle \Theta_{\mu\nu}(0)\phi^{I}(p)\phi^{J}(q) \right\rangle^{1PI} = p^{2-\eta}q^{2-\eta} \left\langle \Theta_{\mu\nu}(0)\phi^{I}(p)\phi^{J}(q) \right\rangle$$
  
=  $\delta(p+q)\delta^{IJ} \lim_{t \to \infty} e^{(-2+\eta)t} c_{\mu\nu,2}(pe^{t}, -pe^{t})$  (5.127)

and

$$\begin{split} \Theta_{\mu\nu}(0)\phi^{I}(p_{1})\phi^{J}(p_{2})\phi^{K}(p_{3})\phi^{L}(p_{4})\Big\rangle^{1PI} \\ &= \prod_{i=1}^{4} p_{i}^{2-\eta} \cdot \left\langle \Theta_{\mu\nu}(0)\phi^{I}(p_{1})\phi^{J}(p_{2})\phi^{K}(p_{3})\phi^{L}(p_{4}) \right\rangle \\ &= \delta \left( \sum_{i=1}^{4} p_{i} \right) \lim_{t \to \infty} e^{(-\epsilon+4\eta)t} \left[ \delta^{IJ}\delta^{KL}c_{\mu\nu,4}(p_{1}e^{t}, p_{2}e^{t}; p_{3}e^{t}, p_{4}e^{t}) \right. \\ &\left. + \delta^{IK}\delta^{JL}c_{\mu\nu,4}(p_{1}e^{t}, p_{3}e^{t}; p_{2}e^{t}, p_{4}e^{t}) + \delta^{IL}\delta^{JK}c_{\mu\nu,4}(p_{1}e^{t}, p_{4}e^{t}; p_{2}e^{t}, p_{3}e^{t}) \right] \end{split}$$

$$(5.128)$$

We obtain the two-point function as

$$\lim_{t \to \infty} e^{(-2+\eta)t} c_{\mu\nu,2}(pe^{t}, -pe^{t}) = \lim_{t \to \infty} \left\{ (1+\eta t) \left( -p^{2} \delta_{\mu\nu} + 2p_{\mu} p_{\nu} \right) +\lambda^{2} (N+2) 3e^{-2t} \left( \delta_{\mu\nu} G(pe^{t}) - 2p_{\mu} p_{\nu} e^{2t} G'(pe^{t}) \right) \right\}$$
$$= p^{-\eta} \left( -p^{2} \delta_{\mu\nu} + 2p_{\mu} p_{\nu} \right)$$
(5.129)

where we have used the asymptotic form

$$G(p) - 2p_{\mu}p_{\nu}G'(p) \xrightarrow{p \to \infty} \frac{1}{12(4\pi)^4} \left( p^2 \delta_{\mu\nu} - 2p_{\mu}p_{\nu} \right) \ln p^2$$
(5.130)

We obtain the four-point function as

$$\lim_{t \to \infty} e^{(-\epsilon + 4\eta)t} c_{\mu\nu,4}(p_1 e^t, p_2 e^t; p_3 e^t, p_4 e^t)$$
$$= \lambda \lim_{t \to \infty} (1 - \epsilon t) \left[ \delta_{\mu\nu} \left\{ -1 \right\} \right]$$

$$+ \lambda \left( (N+4)F \left( (p_1 + p_2)e^t \right) + 2F \left( (p_1 + p_3)e^t \right) + F \left( (p_2 + p_3)e^t \right) \right) \right\}$$

$$- \lambda \left\{ (N+4) \frac{(p_1 + p_2)_{\mu}(p_1 + p_2)_{\nu}}{(p_1 + p_2)^2} + \frac{(p_1 + p_3)_{\mu}(p_1 + p_3)_{\nu}}{(p_1 + p_3)^2} \right\}$$

$$+ \frac{(p_2 + p_3)_{\mu}(p_2 + p_3)_{\nu}}{(p_2 + p_3)^2} \right\}$$

$$= -\lambda \delta_{\mu\nu} \left[ 1 + \frac{\epsilon}{N+8} \ln \left\{ (p_1 + p_2)^{N+4}(p_1 + p_3)^2(p_2 + p_3)^2 \right\} \right]$$
(5.131)

where we have kept only the logarithms of momenta at order  $\epsilon^2$ .

## 6. Summary and conclusions

In this paper we have studied some aspects of the O(N) model using the Exact RG formalism. We have done two things:

1) We have constructed the Wilson action for the O(N) model at the Wilson Fisher fixed point in  $4 - \epsilon$  dimensions up to order  $\epsilon^2$ . This is done by solving the fixed point equation, order by order in  $\epsilon$ . Some correlation functions have also been calculated.

2) We have constructed the energy momentum tensor for this theory. This is done by solving the Ward Identity for diffeomorphism invariance. The traceless-ness of the energy momentum tensor implies that the Wilson action is scale and conformal invariant. It is important to note that all this is in the presence of a *finite cutoff*  $\Lambda$ .

As mentioned in the introduction, one of the motivations for this construction is the use the ideas in [29,30] and construct the AdS action corresponding to this CFT. A related problem is to construct the AdS action for sources for composite operators such as  $\phi^i \phi^i$ . Even more interesting would be to study the massless spin 2 field that would be the source for the energy momentum tensor. This would give dynamical gravity in the bulk as a consequence of Exact RG in the boundary by a direct change of variables similar to what was done for the scalar field in [29,30].

#### **Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Appendix A. Fixed point action

#### A.1. Evaluation of $U_4$

We need to solve

$$\left[ \left( 4 - D - \sum_{i=1}^{4} p_i \frac{d}{dp_i} \right) + \sum_{j=1}^{4} 2K'(p_j^2) U_2^{(1)}(p_j) \right] \frac{1}{8} U_4(p_1, p_2; p_3, p_4)$$
  
=  $\int \frac{d^D p}{(2\pi)^D} K'(p^2) \frac{1}{48} \left\{ 6NU_6(p_1, p_2; p_3, p_4; p, -p) + 12U_6(p_1, p; p_2, -p; p_3, p_4) + 12U_6(p_1, p_2; p_3, p; p_4, -p) \right\}$ 



Fig. 1. Type I diagram.

$$= \int \frac{d^{D}p}{(2\pi)^{D}} K'(p^{2}) \left\{ -\frac{(N+2)}{8} \left( h(p_{1}) + h(p_{2}) + h(p_{3}) + h(p_{4}) \right) - \frac{(N+4)}{4} \left( h(p+p_{1}+p_{2}) + 2h(p+p_{1}+p_{3}) + 2h(p+p_{1}+p_{4}) \right) \right\}$$
(A.132)

where

$$\int \frac{d^D p}{(2\pi)^D} K'(p^2) \left\{ -\frac{(N+2)}{8} \left( h(p_1) + h(p_2) + h(p_3) + h(p_4) \right) \right\}$$
(A.133)

corresponds to the kind of diagrams shown in Fig. 1. Here the external loop does not involve momenta  $p_i + p_j$ . We will call it Type I diagrams. Considering only leading order terms in  $p_j^2$  the contribution from type I diagram in (A.132) is

$$= -\frac{N+2}{8} \frac{\lambda^2}{16\pi^2} 4K'(p_j^2) \Big|_{p_j=0}$$
(A.134)

Now consider the second term in L.H.S of (A.132). In the limit of small external momenta after putting the value of  $U_2^{(1)}(p) = -\frac{N+2}{2}\frac{\lambda}{16\pi^2}$  (as we are considering terms of  $\mathcal{O}(\epsilon^2)$  we have put D = 4 to find  $U_2^{(1)}$ ) we get

$$-\sum_{j=1}^{4} 2K'(p_j^2) \Big|_{p_j \to 0} \frac{\lambda}{16\pi^2} \frac{N+2}{2} \frac{1}{8} V_4(p_1, p_2; p_3, p_4)$$
  
=  $-4K'(p_j^2) \Big|_{p_j \to 0} \frac{\lambda^2}{16\pi^2} \frac{N+2}{8}$  (A.135)

This cancels exactly with (A.134).

Similarly in (A.132) the term

$$\int \frac{d^{D}p}{(2\pi)^{D}} K'(p^{2}) \left\{ -\frac{(N+4)}{4} \left( h(p+p_{1}+p_{2}) + 2h(p+p_{1}+p_{3}) + 2h(p+p_{1}+p_{4}) \right) \right\}$$
(A.136)



Fig. 2. Type II diagram.

corresponds to the kind of diagram shown in Fig. 2. We will call it Type II diagram. In the limit  $p_i \rightarrow 0$  the above term becomes

$$\lambda^{2} \frac{(N+8)}{4} \frac{1}{16\pi^{2}} \int_{0}^{\infty} dp^{2} K'(p^{2}) \Big( K(p^{2}) - K(0) \Big)$$
$$= \lambda^{2} \frac{(N+8)}{4} \frac{1}{16\pi^{2}} \int_{0}^{\infty} dp^{2} \Big\{ \frac{1}{2} \frac{d(K^{2})}{dp^{2}} - K(0) K'(p^{2}) \Big\}$$

Using  $K(\infty) = 0$  and K(0) = 1, this integral gives  $\frac{1}{2}$ . Equating this contribution with  $\epsilon \frac{\lambda}{4!}$  from L.H.S of (A.132) we obtain

$$\frac{1}{8}(4-D)\lambda = \frac{N+8}{8}\frac{\lambda^2}{(4\pi)^2}$$

Thus in addition to the trivial fixed point  $\lambda = 0$ , we have a non trivial fixed point:

$$\lambda = (4 - D) \frac{16\pi^2}{N + 8} \tag{A.137}$$

A.2. Solving for  $\tilde{U}_4$ 

 $ilde{U}_4$  will have contribution from both type I and II diagram explained above. We write

$$\tilde{U}_4 = \tilde{U}_4^I + \tilde{U}_4^{II}$$

according to contributions from type I(II) diagrams.

(We shall set D = 4 while evaluating integrations in those terms that are already of  $\mathcal{O}(\epsilon^2)$ .)

*Type I diagram* In (A.132) the first term on the LHS and the first terms on the RHS (Type I) cancel only in leading order. In general their difference is

$$\lambda^2 \frac{N+2}{8} \times \frac{1}{(4\pi)^2} \int_0^\infty dp^2 K'(p^2) \left[ \sum_j \frac{K(p_j^2) - K(0)}{p_j^2} - K'(p_j^2) \right]$$

Taylor expanding we find

$$\lambda^2 \frac{N+2}{8} \times \frac{1}{(4\pi)^2} \int dp^2 K'(p^2) K''(0) \frac{1}{2} \sum_j p_j^2 \equiv c \sum_j p_j^2$$

This is a contribution to  $\tilde{U}_4(p_1, p_2; p_3, p_4)$  that we can call  $\Delta U_4^I(p_1, p_2; p_3, p_4)$ . Consider a type I graph where the line at one end has  $p_1$  and lines with momenta  $p_2, p_3, p_4$  are at the other end. This corresponds to the term

$$\lambda^2 \frac{N+2}{8} \times \frac{1}{(4\pi)^2} \int dp^2 K'(p^2) K''(0) \frac{1}{2} p_1^2 \equiv c p_1^2$$

when contracted in a loop in order to contribute to  $\tilde{U}_2$ , so that say  $p_3 = -p_4$ , we have  $p_2 = -p_1$ . It contributes to  $\tilde{U}_2(p_1^2)$  an amount

$$\int dp^2 K'(p^2) \frac{1}{2} \Delta U_4^I(p_1, -p_1, p, -p) = \int dp^2 K'(p^2) \frac{1}{2} c(p_1^2)$$
$$= \left[ c \int dp^2 K'(p^2) \right] p_1^2 \equiv A p_1^2$$

This is just a simple wave function renormalization that does not depend on  $p_1$ . There is no contribution to the mass. The same argument applies to all the other permutations of the type I terms. A simple wave function renormalization  $\phi'^2 = (1 + A)\phi^2$  can ensure the normalization of the kinetic term. They do not affect the physics or contribute to  $\eta$ . However, type-I term contributes to sub-leading order term of  $m^2$  or  $U_2$ .

 $\tilde{U}_4^I$  satisfies the following equation:

$$-\sum_{i=1}^{4} p_i \frac{d}{dp_i} \frac{1}{8} \tilde{U}_4^I(p_1, p_2; p_3, p_4) = \lambda^2 \frac{N+2}{8}$$
$$\times \frac{1}{(4\pi)^2} \int_0^\infty dp^2 K'(p^2) \left[ \sum_j \frac{K(p_j^2) - K(0)}{p_j^2} - K'(p_j^2) \right]$$
(A.138)

The solution is

$$\tilde{U}_{4}^{I}(p_{1}, p_{2}; p_{3}, p_{4}) = -\lambda^{2} \frac{(N+2)}{2} \frac{1}{16\pi^{2}} \sum_{j=1}^{4} \frac{K(p_{j}^{2}) - K(0)}{p_{j}^{2}}$$
(A.139a)

$$=\lambda^2 \frac{(N+2)}{2} \frac{1}{16\pi^2} \sum_{j=1}^4 h(p_j)$$
(A.139b)

where  $K(p) = e^{-p^2}$  is assumed.

*Type II diagram* In (A.132) if we keep terms upto  $\mathcal{O}(\epsilon^2)$ ,

$$\frac{1}{8} \left[ \sum_{j=1}^{4} p_j \frac{d}{dp_j} \right] \tilde{U}_4^{II}(p_1, p_2; p_3, p_4)$$
  
=  $\frac{\lambda^2}{4} \int \frac{d^D p}{(2\pi)^D} K'(p^2) \left\{ (N+4)h(p+p_1+p_2) + 2h(p+p_1+p_3) + 2h(p+p_1+p_4) + 2h($ 

$$-(N+8)h(p)\bigg\}$$
(A.140)

where  $h(p) = \frac{K(0) - K(p)}{p^2}$ . It is to be noted in the momentum independent part  $-\epsilon \frac{\lambda}{4!}$  we have written  $\epsilon$  in terms of  $\lambda$  using the fixed point value of  $\lambda$ . The solution at  $\mathcal{O}(\epsilon^2)$ , analytic at zero external momenta, is given by

$$\tilde{U}_{4}^{II}(p_{1}, p_{2}; p_{3}, p_{4}) = -\frac{\lambda^{2}}{2} \int \frac{d^{D}p}{(2\pi)^{D}} h(p) \Big[ (N+4)h(p_{1}+p_{2}+p) + 2h(p+p_{1}+p_{3}) + 2h(p+p_{1}+p_{4}) \\ - (N+8)h(p) \Big]$$
(A.141a)

$$= -\lambda^{2} \Big[ (N+4)F(p_{1}+p_{2}) + 2F(p_{1}+p_{3}) + 2F(p_{1}+p_{4}) \Big]$$
(A.141b)

where  $F(q) = \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} h(p) \Big( h(p+q) - h(p) \Big).$ 

# A.3. Equation for $\tilde{U}_2$

From (3.75) we get

$$0 = \int \frac{d^{D}p}{(2\pi)^{D}} \left(-K'(p^{2})\right) \times \left\{ \frac{1}{8} \left[ 4N\tilde{U}_{4}^{I}(p_{1},-p_{1};p,-p) + 4N\tilde{U}_{4}^{II}(p_{1},-p_{1};p,-p) + 8\tilde{U}_{4}^{I}(p_{1},p;-p_{1},-p) + 8\tilde{U}_{4}^{II}(p_{1},p;-p_{1},-p) \right] - v_{2}^{(1)}(p)v_{2}^{(1)}(p)\delta^{D}(p-p_{1}) \right\} - \frac{\eta}{2}p_{1}^{2} + \tilde{U}_{2}(p_{1}) - p_{1}^{2}\frac{d\tilde{U}_{2}(p_{1})}{dp_{1}^{2}}$$
(A.142)

From (A.139a)

$$\frac{1}{8} \left\{ 4N \tilde{U}_{4}^{I}(p_{1}, -p_{1}; p, -p) + 8 \tilde{U}_{4}^{I}(p_{1}, p; -p, -p_{1}) \right\}$$
  
$$= \frac{1}{2} (N+2)^{2} \frac{\lambda^{2}}{16\pi^{2}} \left\{ h(p) + h(p_{1}) \right\}$$
(A.143)

and from (A.141a)

$$\frac{1}{8} \left\{ 4N \tilde{U}_{4}^{II}(p_{1}, -p_{1}; p, -p) + 8 \tilde{U}_{4}^{II}(p_{1}, p; -p, -p_{1}) \right\}$$
  
=  $-\frac{3\lambda^{2}}{2}(N+2) \int_{r} \left\{ h(r) \left[ h(r+p_{1}+p) - h(r) \right] \right\}$  (A.144)

If we decompose  $\tilde{U}_2$  in two parts namely  $\tilde{U}_2^I$  and  $\tilde{U}_2^{II}$  respectively, in the following way, 1.

$$\tilde{U}_{2}^{I}(p_{1}) - p_{1}^{2} \frac{d\tilde{U}_{2}^{I}(p_{1})}{dp_{1}^{2}} = \int \frac{d^{D}p}{(2\pi)^{D}} K'(p^{2}) \frac{1}{2} (N+2)^{2} \frac{\lambda^{2}}{16\pi^{2}} h(p_{1}) - \left(U_{2}^{(1)}\right)^{2} K'(p_{1}^{2})$$
(A.145)

which gives

$$\tilde{U}_{2}^{I}(p_{1}) = -\frac{\lambda^{2}}{(16\pi^{2})^{2}} \frac{(N+2)^{2}}{4} h(p_{1})$$
(A.146)

2.

$$-2\tilde{U}_{2}^{II}(p_{1}) + 2p_{1}^{2}\frac{d\tilde{U}_{2}^{II}(p_{1})}{dp_{1}^{2}}$$
  
=  $-6\lambda^{2}(N+2)\int \frac{d^{D}p}{(2\pi)^{D}} \Big(-K'(p^{2})\Big)F(p_{1}+p)$   
+  $(N+2)^{2}\frac{\lambda^{2}}{16\pi^{2}}\int \frac{d^{D}p}{(2\pi)^{D}} \Big(-K'(p^{2})\Big)h(p) - \eta p_{1}^{2}$  (A.147)

which gives

$$\tilde{U}_{2}^{II}(p_{1}) = p_{1}^{2} \int_{p^{2}=0}^{p_{1}^{2}} dp^{2} \frac{\int \frac{d^{D}q}{(2\pi)^{D}} \left\{ -6\lambda^{2}(N+2)(-K'(q^{2}))F(p+q) \right\} - \eta p^{2}}{2p^{4}} - \frac{(N+2)^{2}}{4} \frac{\lambda^{2}}{(16\pi^{2})^{2}}$$
(A.148)

The second term in the expression of  $\tilde{U}_4^{II}$  is evaluated using  $K(p) = e^{-p^2}$ .

Hence the full expression of  $\tilde{U}_2(p_1)$  is given by

$$\begin{split} \tilde{U}_{2}(p_{1}) &= -\frac{\lambda^{2}}{(16\pi^{2})^{2}} \frac{(N+2)^{2}}{4} h(p_{1}) \\ &+ p_{1}^{2} \int_{p^{2}=0}^{p_{1}^{2}} dp^{2} \frac{\int \frac{d^{D}q}{(2\pi)^{D}} \left\{ -6\lambda^{2}(N+2)(-K'(q^{2}))F(p+q) \right\} - \eta \ p^{2}}{2p^{4}} \\ &- \frac{(N+2)^{2}}{4} \frac{\lambda^{2}}{(16\pi^{2})^{2}} \end{split}$$
(A.149)

A.4. Expression for  $\eta$ 

Only Type II diagrams contribute to  $\eta$ . Because we need the external momentum to flow through the loop - to get a momentum dependence in  $U_2$ . This can happen only in Type II terms and that too for **certain contractions**.

(Calculation of this section requires us to go back to bar denoted variable as dimensionless variable. So p's from last section are replaced with  $\bar{p}$ .)

From (3.76) we have

$$\frac{\eta}{2} = -\frac{1}{8} \frac{d}{d\bar{r}^2} \int_{\bar{q}} K'(\bar{q}^2) \left\{ 4N \tilde{U}_4^{II}(\bar{q}, -\bar{q}; \bar{r}, -\bar{r}) + 8 \tilde{U}_4^{II}(\bar{q}, \bar{r}; -\bar{r}, -\bar{q}) \right\} \bigg|_{\bar{r}^2 = 0}$$
(A.150)

We can convert differentiation w.r.t  $p_j$  into that w.r.t  $\Lambda$ , i.e.

$$-\sum_{j=1}^{4}\bar{p}_{j}\frac{d}{d\bar{p}_{j}}=\Lambda\frac{d}{d\Lambda}$$

So (A.140) gives following expression for  $\tilde{U}_4^{II}$ :

$$\frac{1}{8}\tilde{U}_{4}^{II}(\frac{p_{1}}{\Lambda},\frac{p_{2}}{\Lambda};\frac{p_{3}}{\Lambda},\frac{p_{4}}{\Lambda}) = \frac{\lambda^{2}}{4}\int_{0}^{\ln\Lambda} d\ln\Lambda' \int_{\bar{p}} K'(\bar{p}^{2}) \left[ (N+4)h(\bar{p}+\frac{p_{1}}{\Lambda'}+\frac{p_{2}}{\Lambda'}) + 2h(\bar{p}+\frac{p_{1}}{\Lambda'}+\frac{p_{3}}{\Lambda'}) + 2h(\bar{p}+\frac{p_{1}}{\Lambda'}+\frac{p_{4}}{\Lambda'}) - (N+8)h(\bar{p}) \right]$$
(A.151)

Hence

$$\frac{1}{8} \left\{ 4N \tilde{U}_{4}^{II}(\bar{q}, -\bar{q}; \bar{r}, -\bar{r}) + 8 \tilde{U}_{4}^{II}(\bar{q}, \bar{r}; -\bar{r}, -\bar{q}) \right\} \\
= \frac{\lambda^{2}}{4} \int_{0}^{\ln\Lambda} d\ln\Lambda' \int_{\bar{p}, \bar{r}} K'(\bar{p}^{2}) \left\{ (12N + 48)h(\bar{p} + \frac{q}{\Lambda'} + \frac{r}{\Lambda'}) + (12N + 48)h(\bar{p} + \frac{q}{\Lambda'} - \frac{r}{\Lambda'}) - 24(N + 2)h(\bar{p}) \right\}$$
(A.152)

So we need to find the coefficient of  $\bar{r}^2$  in  $\left[h(\bar{p} + \frac{q}{\Lambda'} + \frac{r'}{\Lambda'}) + h(\bar{p} + \frac{q}{\Lambda'} - \frac{r'}{\Lambda'})\right]$  which is calculated as

$$\begin{aligned} & \left. \frac{1}{2} \frac{r^{\mu} r^{\nu}}{\Lambda'^{2}} \frac{d^{2}}{dr'^{\mu} dr'^{\nu}} \left[ h(\bar{p} + \frac{q}{\Lambda'} + \frac{r'}{\Lambda'}) + h(\bar{p} + \frac{q}{\Lambda'} - \frac{r'}{\Lambda'}) \right] \right|_{r'=0} \\ &= \frac{r^{\mu} r^{\nu}}{\Lambda'^{2}} \left( \frac{d^{2}}{d\bar{r}'^{\mu} d\bar{r}'^{\nu}} h(\bar{p} + \frac{q}{\Lambda'} + \bar{r}') \right) \right|_{\bar{r}'=0} \\ &= \frac{\bar{r}^{2}}{4} \left( \frac{d^{2}}{d\bar{r}^{\mu} d\bar{r}_{\mu}} h(\bar{p} + \frac{q}{\Lambda'} + \bar{r}) \right) \right|_{r=0} \\ &= -\frac{\bar{r}^{2}}{4} \frac{d^{2}}{d\bar{r}^{\mu} d\bar{r}_{\mu}} \frac{K(\bar{r}^{2}) - 1}{\bar{r}^{2}} \bigg|_{\bar{r}=\bar{p} + \frac{q}{\Lambda'}} \\ &= \bar{r}^{2} K''((\bar{p} + \frac{q}{\Lambda'})^{2}) \end{aligned}$$
(A.153)

where we have used the facts: in 4 dimensions  $(\frac{d}{dp_{\mu}}\frac{1}{p^2}) = \delta^4(p)$  and K(0) = 1. From (A.150), (A.152) and (A.153) we get

$$\frac{\eta}{2} = 3\lambda^2 (N+2) \int_{\bar{q}} K'(\bar{q}^2) \int_{0}^{\ln\Lambda} d\ln\Lambda' \left(\frac{\Lambda}{\Lambda'}\right)^2 \int_{\bar{p}} K'(\bar{p}^2) K''((\bar{p}+\frac{q}{\Lambda'})^2)$$
(A.154)

**Evaluation of integral:** Let us use  $\bar{q}' = \frac{q}{\Lambda'}$  and  $\Lambda'$  as variables of integration, rather than  $\bar{q} = \frac{q}{\Lambda}$  and  $\Lambda'$ . So change variables:

$$\bar{q} = \bar{q}' \frac{\Lambda'}{\Lambda}; \quad \bar{q}^2 = \bar{q}'^2 \left(\frac{\Lambda'}{\Lambda}\right)^2; \quad \int d^4 \bar{q} = \int d^4 \bar{q} \left(\frac{\Lambda'}{\Lambda}\right)^4$$

to get

$$\frac{\eta}{2} = -3\lambda^2 (N+2) \int_0^{\ln\Lambda} d\ln\Lambda' \int_{\bar{q}'} \left(\frac{\Lambda'}{\Lambda}\right)^{-2} K'(\bar{q}'^2) \left(\frac{\Lambda'}{\Lambda}\right)^2 \int_{\bar{p}} K'(\bar{p}^2) K''((\bar{p}+\frac{q}{\Lambda'})^2)$$

Using  $K'(\bar{q}'^2) = \frac{dK}{d\Lambda'} \frac{d\Lambda'}{d\bar{q}'^2} = -\frac{\Lambda'}{2\bar{q}'^2} \frac{dK}{d\Lambda'}$  we get

$$\frac{\eta}{2} = -3\lambda^2 (N+2) \int_0^{\Lambda} d\Lambda' \, \frac{dK}{d\Lambda'} \int_{\bar{q}'} \frac{1}{2\bar{q}'^2} \int_{\bar{p}} K'(\bar{p}^2) K''((\bar{p}+\bar{q}')^2)$$

Since  $\bar{q}'$  is an independent variable we can write this as

$$\frac{\eta}{2} = -3\lambda^2 (N+2) \int_{\bar{q}'} \int_0^{\Lambda} d\Lambda' \, \frac{dK}{d\Lambda'} \frac{1}{2\bar{q}'^2} \int_{\bar{p}} K'(\bar{p}^2) K''((\bar{p}+\bar{q}')^2)$$

The integral over  $\bar{p}$  is a function of  $\bar{q}'$  and not  $\Lambda'$ . So we can do the  $\Lambda'$  integral easily. Using  $K(\infty) = 0$  we get

$$\frac{\eta}{2} = -\frac{3\lambda^2}{2}(N+2) \underbrace{\int_{\bar{q}'} K(\bar{q}'^2) \frac{1}{\bar{q}'^2} \int_{\bar{p}} K'(\bar{p}^2) K''((\bar{p}+\bar{q}')^2)}_{-\frac{\pi^4}{6(2\pi)^8}} = \frac{1}{4}\lambda^2 (N+2) \frac{1}{(16\pi^2)^2}$$

The integral underbraced above is calculated to give  $-\frac{\pi^4}{6(2\pi)^8}$  for  $K(x) = e^{-x}$ . But it can be shown to give identical result for any smooth K(x) [41]. Using  $\lambda = \frac{16\pi^2}{N+8}\epsilon$  we can write the anomalous dimension as:

$$\frac{\eta}{2} = \frac{1}{4}\lambda^2 (N+2)\frac{1}{(16\pi^2)^2} = \frac{N+2}{(N+8)^2}\frac{\epsilon^2}{4}$$
(A.155)

# Appendix B. Asymptotic behaviors of F(p) and G(p)

The function F(p) is defined by

$$\left(p \cdot \partial_p + \epsilon\right) F(p) = \int_q f(q) \left(h(q+p) - h(q)\right)$$
(B.156)

For large p, we obtain an equation satisfied by the asymptotic form  $F_{asymp}(p)$ :

$$\left(p \cdot \partial_p + \epsilon\right) F_{\text{asymp}}(p) = -\int_q f(q)h(q) = -\frac{1}{(4\pi)^2} + O(\epsilon)$$
(B.157)

This implies

$$F_{\text{asymp}}(p) = -\frac{1}{\epsilon} \int_{q} f(q)h(q) + C_F(\epsilon)p^{-\epsilon}$$
(B.158)

where  $C_F(\epsilon)$  is independent of p. Since F(p) is finite in the limit  $\epsilon \to 0+$ , we must find

$$C_F(\epsilon) = \frac{1}{\epsilon} \frac{1}{(4\pi)^2} + \cdots$$
(B.159)

Hence, expanding in  $\epsilon$ , we obtain

$$F_{\text{asymp}}(p) = -\frac{1}{(4\pi)^2} \ln p + \text{const} + O(\epsilon)$$
(B.160)

We next consider G(p) satisfying

$$\left(p \cdot \partial_p - 2 + 2\epsilon\right) G(p) = \int_q f(q) F(q+p) + 2v_2 \int_q f(q) h(q) + \eta^{(2)} p^2$$
(B.161)

where

$$\eta^{(2)} = -\frac{d}{dp^2} \int_{q} f(q) F(q+p) \Big|_{p=0} = \frac{1}{6(4\pi)^4} + O(\epsilon)$$
(B.162)

The asymptotic form  $G_{asymp}(p)$  satisfies

$$\left(p \cdot \partial_p - 2 + 2\epsilon\right) G_{\text{asymp}}(p) = \eta^{(2)} p^2 \tag{B.163}$$

This gives

$$G_{\text{asymp}}(p) = \frac{1}{2\epsilon} \eta^{(2)} p^2 + C_G(\epsilon) p^{2-2\epsilon}$$
(B.164)

Since G(p) is finite as  $\epsilon \to 0+$ , we obtain

$$C_G(\epsilon) = -\frac{1}{\epsilon} \frac{1}{12(4\pi)^4} + \dots$$
 (B.165)

Hence,

$$G_{\text{asymp}}(p) = p^2 \left( \frac{1}{6(4\pi)^4} \ln p + \text{const} \right) + \mathcal{O}(\epsilon)$$
(B.166)

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