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Gravitational Waves in Axion Dark Matter

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Abstract: Axion dark matter is interesting as it allows a natural coupling to the gravitational Chern–Simons term. In the presence of an axion background, the gravitational Chern–Simons term produces parity violating effects in the gravitational sector, in particular on the propagation of gravitational waves. Previously, it has been shown that the coherent oscillation of the axion field leads to a parametric amplification of gravitational waves with a specific frequency. In this paper, we focus on the parity violating effects of the Chern–Simons coupling and show the occurrence of gravitational birefringence. We also find deviation from the speed of light of the velocity of the gravitational waves. We give constraints on the axion-Chern–Simons coupling constant and the abundance of axion dark matter from the observation of GW170817 and GRB170817A.

Keywords: gravitational waves; axion dark matter; Chern–Simons gravity

1. Introduction

The direct detection of gravitational waves in 2015 has widened the frontiers in research in fundamental physics [1]. Indeed, we are now in the era of gravitational wave astrophysics and multi-messenger astronomy, and it is now possible to use gravitational wave signals to discriminate the various models beyond the standard model of particle physics, as well as of cosmology. Moreover, the discovery of gravitational waves, GW170817 [2], from a neutron star binary has sparked new interests in the study of nuclear physics. Remarkably, the observation of the optical counterpart of GW170817, GRB170817A [3], has given a constraint on the velocity of gravitational waves, which killed many modified theories of gravity [4–8].

In gravitational physics, there are three processes to be studied, namely the production, propagation, and detection of gravitational waves. The production process and detection process have been well studied. On the other hand, the propagation process has been mostly regarded as a trivial problem. In fact, in a Minkowski background, the gravitational wave equation is merely a conventional scalar wave equation. Even in the presence of conventional matter, it is easy to solve the propagation problem of gravitational waves in the curved background. However, things can get more interesting with axions. As a pseudoscalar, coupling to a gravitational Chern–Simons term is allowed [9–11]. As a result, parity symmetry is broken in the presence of an axion background. A peculiar feature of axion dark matter is coherent oscillations of the axion field, which may affect the propagation of electromagnetic waves [12] and gravitational waves [13,14].

According to string theory, axions are ubiquitous in the universe [15]. Remarkably, the mass of string axions can take values in the broad range from 10^{-33} eV to 10^{18} GeV. In fact, it has been known that the axion is a natural candidate for the inflaton and induces circularly polarized gravitational waves [16,17]. Recently, the axion has been intensively studied as a candidate for dark matter. As dark matter, we can consider the axion with mass from 10^{-23} eV to 10^{3} eV. The lower bound comes from observations of the cosmic background radiation, and the upper bound comes from observations of X-ray backgrounds [18].

In this paper, we comprehensively investigate gravitational waves propagating in an axion background. First of all, we review the results of the previous work on the parametric resonance of gravitational waves [13,14]. Then, we focus on the gravitational birefringence and the velocity modulation of gravitational waves. In particular, we discuss constraints on the Chern–Simons coupling and the abundance of the axion dark matter from observations of the velocity of gravitational waves.

The organization of the paper is as follows. In Section 2, we introduce basic equations for gravitational waves in axion-Chern–Simons gravity. We also discuss a potential ghost mode and a cutoff scale that is needed in order to avoid the occurrence of this un-physical feature. In Section 3, we analyze the propagation of gravitational waves in a background of coherently oscillating axions. In Section 4, we review the parametric amplification of gravitational waves and present consistency checks. In Section 5, we consider the gravitational birefringence, which can be regarded as a gravitational Faraday rotation. In Section 6, we derive the velocity of gravitational waves. We discuss constraints on the Chern–Simons coupling constant and the abundance of the axion dark matter using the observation of the velocity of gravitational waves. The final section is devoted to the conclusion.

2. Gravitational Waves in Dynamical Chern–Simons Gravity

Let us consider the action of dynamical Chern–Simons gravity:

$$S = \kappa \int \mathrm{d}^4x \sqrt{-g} R - \int \mathrm{d}^4x \sqrt{-g} \left[\frac{1}{2} \nabla^\mu \Phi \nabla_\mu \Phi + V(\Phi) \right] + S_{\mathrm{dCS}}, \tag{1}$$

where the first term of action is the Einstein–Hilbert term, g is the determinant of the metric $g_{\mu\nu}$, and $\kappa = 1/(16\pi G)$. The second term describes the action of an axion field Φ. In this paper, we will take Φ as a dark matter candidate and exploit existing observational constraints on its mass. The last term in (1):

$$S_{\mathrm{dCS}} = \frac{1}{4} \int \mathrm{d}^4x \sqrt{-g} F(\Phi) R\tilde{R}, \qquad R\tilde{R} := \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} R_{\alpha\beta\rho\sigma} R_{\gamma\delta}{}^{\rho\sigma} \tag{2}$$

is the dynamical Chern–Simons action with $\epsilon^{\alpha\beta\gamma\delta}$ being the Levi–Civita tensor density. We allowed a nontrivial coupling $F(\Phi)$ of the axion field to the $R\tilde{R}$ Chern–Simons term. Otherwise, the dynamical Chern–Simons term is topological and will not affect the equation of motion. Note that the propagation of the gravitational wave and the dispersion relation in the dynamical Chern–Simons gravity with a prescribed time-dependent coupling were studied in [19,20].

We are interested in the effect of the dark matter Chern–Simons coupling on the propagation of gravitational wave. Before we start, we need to fix the background. Let us consider a background spacetime with spatial isotropy and homogeneity:

$$ds^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -dt^2 + a^2(t) \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j \,. \tag{3}$$

Due to its structure, the Chern–Simons term does not contribute to the equation of motion of the isotropic and homogeneous universe. As a result, we have the equations of motion,

$$3H^2 = \frac{1}{2\kappa} \left(\frac{1}{2} \dot{\Phi}^2 + V(\Phi) \right) \tag{4}$$

$$\dot{H} + 3H^2 = \frac{1}{2\kappa} V(\Phi) \tag{5}$$

$$\ddot{\Phi} + 3H\dot{\Phi} + V'(\Phi) = 0, \tag{6}$$

where a dot denotes a time derivative and $H = \dot{a}/a$ is the Hubble parameter. Generally, dark matter has a mass:

$$V(\Phi) = \frac{1}{2} m^2 \Phi^2. \tag{7}$$

We will ignore self-interaction as it is not relevant for our analysis.

2.1. Action of Gravitational Waves

Let us now derive the quadratic action of gravitational waves from the action (1). The tensor perturbation reads:

$$ds^2 = g_{\mu\mu} dx^\mu dx^\nu = -dt^2 + a^2(t)(\delta_{ij} + h_{ij}) dx^i dx^j \ , \tag{8}$$

where h_{ij} satisfies the transverse-traceless conditions $h_{ij,j} = h_{ii} = 0$. Substituting the metric into (1), we obtain the quadratic action:

$$S = \frac{\kappa}{4} \int dt d^3x \ a^3 \left[\dot{h}^{ij} \dot{h}_{ij} - \frac{1}{a^2} h^{ij|k} h_{ij|k} + \frac{\dot{F}}{\kappa a} \epsilon^{ijk} \left(\dot{h}_{ai} \dot{h}^a_{k|j} - \frac{1}{a^2} h_{ai|b} h^a{}_k{}^{|b}{}_{|j} \right) \right] , \tag{9}$$

where the stroke | denotes a covariant derivative with respect to the spatial coordinates. Here, we have used the convention $\epsilon^{0ijk} = -\epsilon^{ijk}$ with $\epsilon^{123} \equiv 1$. It is convenient to expand h_{ij} in terms of a circular polarization basis:

$$h_{ij}(\eta, \mathbf{x}) = \sum_{A=\mathrm{R,L}} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} h_A(t, \mathbf{k}) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} p^A_{ij}, \tag{10}$$

where p^A_{ij} is the circular polarization tensor defined by:

$$p^{\mathrm{R}}_{ij} \equiv \frac{1}{\sqrt{2}} \left(p^+_{ij} + i p^\times_{ij} \right) , \quad p^{\mathrm{L}}_{ij} \equiv \frac{1}{\sqrt{2}} \left(p^+_{ij} - i p^\times_{ij} \right) \tag{11}$$

and the polarization tensors p^+_{ij} and $p^\times_{ij}$ are the plus and cross modes, respectively. The polarization tensors are normalized as:

$$p^{*A}_{ij} p^B_{ij} = 2\delta^{AB}, \quad A, B = \mathrm{R, L} \tag{12}$$

and satisfy the helicity condition:

$$i\epsilon_i{}^{sj} \hat{k}_s p^A_{jk} = \rho_A p^A_{ik} \quad \text{with} \quad \rho_{\mathrm{R}} = +1, \ \rho_{\mathrm{L}} = -1, \tag{13}$$

where $\hat{k^i} := k^i/k$ is the unit vector in the direction of the propagation of the gravitational wave. The circular polarization modes are special as they diagonalize the dynamical Chern–Simons action term, and we obtain the quadratic action (9):

$$S = \frac{\kappa}{2} \sum_{A=\mathrm{R,L}} \int dt \int \frac{d^3 k}{(2\pi)^3} \ a^3 B_A \left(\dot{h_A}^2 - \frac{k^2}{a^2} {h_A}^2 \right) \ . \tag{14}$$

where the function B_A is defined by:

$$B_A(t, k) := 1 - \frac{k\rho_A}{\kappa} \frac{\dot{F}}{a}. \tag{15}$$

In order for the perturbation to be stable, it is required that:

$$B_A > 0 \tag{16}$$

for both polarizations A. One can derive from this a natural upper bound to the energy scale of the gravitational wave in the theory [21]. This is in contrast to the parity violating gravitational waves in Lorentz violating gravity [22].

3. Gravitational Waves Propagating in Axion Dark Matter

In this section, we suppose that a source in our cosmological horizon emits gravitational waves propagating in the axion dark matter background. In this situation, we can neglect the effect of cosmic expansion of the Universe in the dynamical Equation (6) of the axion. To see this, let us note that the Friedmann Equation (4) set the Hubble parameter to be of the order of $H \sim \sqrt{\rho/\kappa} \sim \sqrt{\rho}/M_{\rm pl}$, where $M_{\rm pl} := \sqrt{2\kappa}$ is the reduced Planck mass. On the other hand, the scalar field changes at a rate determined by the mass scale: $\dot{\Phi} \sim m\Phi$. The cosmological expansion in (6) is negligible if:

$$H/m \sim \sqrt{\rho}/(mM_{\rm pl}) \ll 1. \tag{17}$$

Since an upper bound of the dark matter density is given by:

$$\rho = 0.3\,{\rm GeV/cm^3} = 2.3 \times 10^{-6}\,{\rm eV}^4\,, \tag{18}$$

the condition (17) is always satisfied for axion dark matter with a mass $m > 10^{-30}$ eV.

For simplicity, let us consider a linear dynamical Chern–Simons coupling:

$$F(\Phi) = \alpha\Phi \tag{19}$$

Following [23], we express the Chern–Simons coupling constant α in terms of a length ℓ as:

$$\alpha = \frac{M_{\rm pl}}{2}\ell^2, \tag{20}$$

where $M_{\rm pl} = \sqrt{2\kappa}$ is the reduced Planck mass. The coupling constant ℓ is experimentally constrained by the gravity probe B as [24]:

$$\ell \leq 10^8\,{\rm km}. \tag{21}$$

Now, we can write down the condition (16) in the present context. The axion satisfies the equation of motion:

$$\ddot{\Phi} + m^2\Phi = 0. \tag{22}$$

This can be solved as

$$\Phi = \Phi_0 \cos(mt), \tag{23}$$

where, without loss of generality, we made a choice of time so that the phase in (23) is zero. In this case, the no-ghost condition gives:

$$k < k_{\rm g} := \frac{1}{\alpha m\Phi_0/\kappa} = \frac{\kappa}{\alpha\sqrt{2\rho}} \tag{24}$$

where:

$$\rho := \frac{1}{2}m^2\Phi_0^2 \tag{25}$$

is the energy density of the dark matter field. Once we have used the observed energy density, the amplitude can be determined as:

$$\Phi_0 \simeq 2.1 \times 10^7\,\mathrm{eV} \left(\frac{10^{-10}\,\mathrm{eV}}{m}\right) \sqrt{\frac{\rho}{0.3\,\mathrm{GeV/cm^3}}} \,. \tag{26}$$

Finally, we can deduce the cutoff frequency f_g as:

$$f_\mathrm{g} \equiv \frac{k_\mathrm{g}}{2\pi} = 1.1 \times 10^9\,\mathrm{Hz} \left(\frac{10^8\,\mathrm{km}}{\ell}\right)^2 \sqrt{\frac{0.3\,\mathrm{GeV/cm^3}}{\rho}} \,. \tag{27}$$

Above this scale, we cannot use the Chern–Simons gravity to describe the propagation of gravitational waves in axion dark matter.

On energy scales below this, the equation of motion of gravitational waves in the Chern–Simons gravity is given by:

$$\ddot{h}_A + D_A \dot{h}_A + \frac{k^2}{a^2} h_A = 0 \tag{28}$$

where the function D_A is defined by:

$$D_A(t,k) := 3H + \frac{\dot{B}_A}{B_A} \,. \tag{29}$$

The first term is due to cosmological expansion, which we can ignore. The second term is due to the dark matter background (23). It is convenient to introduce the dimensionless parameter:

$$\delta := m^2 \alpha \Phi_0 / \kappa \tag{30}$$

in terms of which we have:

$$B_A = 1 + \frac{\rho_A \delta}{m} k \sin(mt). \tag{31}$$

The parameter can be estimated as:

$$\delta \simeq 2.3 \times 10^{-5} \left(\frac{m}{10^{-10}\,\mathrm{eV}}\right) \left(\frac{\ell}{10^8\,\mathrm{km}}\right)^2 \sqrt{\frac{\rho}{0.3\,\mathrm{GeV/cm^3}}} \,. \tag{32}$$

In the following sections, we analyze the features of gravitational waves in the axion dark matter background.

4. Parametric Resonances

Apparently, the dynamical Chern–Simons coupling induces the parity violation in the presence of axion dark matter. In fact, the equations for each circular polarization mode are different. Generically, the polarization dependent effect in gravitational waves characterized by δ is small for typical values of the model parameters. However, the effect can be exponentially enhanced due to the resonance.

To see this, let us introduce a new variable $\Psi_A(t,k)$ defined by:

$$h_A(t,k) = \exp\left(-\frac{1}{2}\int^t D_A(t',k)dt'\right) \Psi_A(t,k). \tag{33}$$

Then, the equation of motion for the gravitational wave becomes:

$$\ddot{\Psi}_A + \omega_A^2 \Psi_A = 0, \tag{34}$$

where:

$$\omega_A^2 := \frac{k^2}{a^2} - \frac{1}{2}\dot{D}_A - \frac{1}{4}D_A^2. \tag{35}$$

To the leading order of δ, the angular frequency is given by:

$$\omega_A^2 = k^2\left(1 + \rho_A f_0 \sin(mt)\right) \tag{36}$$

and Equation (34) takes the form of the Mathieu equation:

$$\ddot{\Psi}_A + k^2\left(1 + \rho_A f_0 \sin(mt)\right)\Psi_A = 0, \quad \text{where} \quad f_0 := \frac{1}{2}\frac{m}{k}\delta. \tag{37}$$

This describes an oscillator with a frequency k pumped by the polarization dependent periodic force with a magnitude f_0 and a frequency m. As is well known, the resonance occurs when:

$$k \sim m/2\,. \tag{38}$$

In this case, we obtain:

$$f_0 \sim \delta\,. \tag{39}$$

The amplitude of gravitational waves h_A grows exponentially $|h_A| \sim e^{\Gamma t}$ with the growth rate given by:

$$\begin{aligned}\Gamma &= \frac{m\delta}{8}\\ &= 2.8 \times 10^{-16}\,\text{eV}\left(\frac{m}{10^{-10}\,\text{eV}}\right)^2\left(\frac{\ell}{10^8\,\text{km}}\right)^2\sqrt{\frac{\rho}{0.3\,\text{GeV}/\text{cm}^3}}.\end{aligned} \tag{40}$$

We can estimate the length $R_{\times 10}$ by which the gravitational wave grows ten times bigger from the growth rate as follows,

$$\begin{aligned}R_{\times 10} &= \frac{8\ln 10}{m\delta}\\ &= 5.2 \times 10^{-8}\,\text{pc}\left(\frac{10^{-10}\,\text{eV}}{m}\right)^2\left(\frac{10^8\,\text{km}}{\ell}\right)^2\sqrt{\frac{0.3\,\text{GeV}/\text{cm}^3}{\rho}}.\end{aligned}$$

The range of the wave number for the resonance is given by:

$$\frac{m}{2} - \frac{m}{8}\delta \le k \le \frac{m}{2} + \frac{m}{8}\delta.$$

This corresponds to a resonance width Δk_{res}:

$$\Delta k_{\text{res}} = \frac{1}{4}m\delta. \tag{41}$$

The phenomenological consequence of this result was discussed in a previous paper [13]. Recently, a more serious comparison with gravitational observation was made in [25].

4.1. Coherence Length

In the above analysis, we ignored the coherence issue of the dark matter background. In principle, if the length scale of the gravitational perturbation becomes comparable to the Jeans length scale, the dark matter cloud can no longer remain as homogeneous, and gravitational collapse will occur.

In other words, the coherence can be sustained only within the Jeans scale. As is known, the Jeans length r_J of axion dark matter can be deduced as:

$$\begin{aligned} r_J &= 6.7\times 10^{20}\,\mathrm{eV}^{-1}\left(\frac{m}{10^{-10}\,\mathrm{eV}}\right)^{-\frac{1}{2}}\left(\frac{\rho}{0.3\,\mathrm{GeV/cm^3}}\right)^{-\frac{1}{4}} \\ &= 4.3\times 10^{-3}\,\mathrm{pc}\left(\frac{m}{10^{-10}\,\mathrm{eV}}\right)^{-\frac{1}{2}}\left(\frac{\rho}{0.3\,\mathrm{GeV/cm^3}}\right)^{-\frac{1}{4}}. \end{aligned} \tag{42}$$

Thus, the condition $R_{\times 10} \leq r_J$ is necessary for the resonance to occur. This is satisfied for $m \geq m_c$ where the critical mass m_c is given by:

$$m_c = 5.3\times 10^{-14}\,\mathrm{eV}\left(\frac{10^8\,\mathrm{km}}{\ell}\right)^{\frac{4}{3}}\left(\frac{0.3\,\mathrm{GeV/cm^3}}{\rho}\right)^{\frac{1}{6}}. \tag{43}$$

4.2. One More Consistency Check

For having the resonance, we need one more condition. Axion dark matter within the dark matter halo has a virial velocity. From the simple dimensional analysis, one can estimate the virial velocity as:

$$v_{\mathrm{vir}} = 1.5\times 10^{-11}\left(\frac{m}{10^{-10}\,\mathrm{eV}}\right)^{-\frac{1}{2}}\left(\frac{\rho}{0.3\,\mathrm{GeV/cm^3}}\right)^{\frac{1}{4}}.$$

This velocity induces fluctuations in the frequency of axion dark matter given by:

$$\Delta k_{\mathrm{vir}} = \frac{1}{2}m v_{\mathrm{vir}}^2 \tag{44}$$

If the fluctuations are larger than the bandwidth Δk_{res} of the parametric resonance, the amplitude cannot grow efficiently. This is characterized by the ratio:

$$\gamma := \frac{\Delta k_{\mathrm{vir}}}{\Delta k_{\mathrm{res}}}. \tag{45}$$

If γ satisfies:

$$\gamma \ll 1, \tag{46}$$

the amplitude of the gravitational waves grows. On the other hand, if γ satisfies:

$$\gamma \geq 1, \tag{47}$$

then the frequency of axion dark matter easily escapes from the resonance band, and the gravitational wave never grows. Since it is calculated as:

$$\gamma = 2.0\times 10^{-17}\left(\frac{m}{10^{-10}\,\mathrm{eV}}\right)^{-2}\left(\frac{l}{10^8\,\mathrm{km}}\right)^{-2}, \tag{48}$$

we see that the amplitude of gravitational waves grows.

5. Gravitational Faraday Rotation

The fact that the angular frequency (36) is different for different circular polarization modes implies that the phase velocity:

$$v_A^{(p)} := \frac{\omega_A}{k} = \sqrt{1+\rho_A f_0 \sin mt}. \tag{49}$$

is different for different circular polarization modes. It also implies a phase shift between the R and L polarization arising as the wave propagates:

$$\Delta\phi = \int_0^t dt' (\omega_R - \omega_L) = \frac{\delta}{2}(1 - \cos mt) + \mathcal{O}(\delta^2). \tag{50}$$

This is characterized by a period of phase oscillation:

$$T = \frac{2\pi}{m} \sim (\frac{m}{10^{-10}\,\text{eV}})^{-1} \times 10^{-5}\,\text{s}. \tag{51}$$

The amplitude of the gravitational Faraday rotation is given by:

$$(\Delta\phi)_{\max} = \delta = 2.2 \times 10^{-5} \left(\frac{m}{10^{-10}\,\text{eV}}\right) \left(\frac{\ell}{10^8\,\text{km}}\right)^2 \sqrt{\frac{\rho}{0.3\,\text{GeV/cm}^3}}\,. \tag{52}$$

Hence, the gravitational Faraday rotation is sizable for the axion with $m \geq 10^{-5}$ eV.

The observability of this gravitational Faraday effect is not trivial. In the case that we know the initial polarization of gravitational waves, we can observe the effect. In fact, in the case of electromagnetic waves, there are such sources. For the gravitational waves, we leave the issue for future work.

6. Velocity of Gravitational Waves

The propagation of gravitational waves is characterized by the group velocity $v_A^{(g)} := \partial\omega_A/\partial k$. To the leading order of δ, we obtain:

$$v_A^{(g)} = 1 + \frac{1}{32}\frac{m^2}{k^2}\delta^2 \sin^2(mt) + \mathcal{O}(\delta^3)\,, \tag{53}$$

where we assumed $\delta \ll 1$ and dropped terms that are of the order of δ^2, but without the m^2/k^2 factor. The reason is because $k \sim 4.1 \times 10^{-13}\,\text{eV} \times (f_{gw}/100\,\text{Hz})$; hence, $m^2/k^2 \sim 10^5 \gg 1$ for the range of k, and we consider $m \sim 10^{-10}$ eV here. Note that (53) is independent of polarization up to the second order of δ. The group velocity is always greater than the speed of light and has a maximal deviation of:

$$\begin{aligned} \Delta c &= \frac{1}{32}\frac{m^2}{k^2}\delta^2 \\ &= 9.4 \times 10^{-7} \left(\frac{100\,\text{Hz}}{f_{gw}}\right)^2 \left(\frac{\ell}{10^8\,\text{km}}\right)^4 \left(\frac{m}{10^{-10}\,\text{eV}}\right)^4 \left(\frac{\rho}{0.3\,\text{GeV/cm}^3}\right). \end{aligned} \tag{54}$$

The current constraints on Δc coming from the observation of GW170817 and GRB170817A are given by:

$$-3 \times 10^{-15} < \Delta c < +7 \times 10^{-16}\,. \tag{55}$$

We take the value $+5 \times 10^{-16}$ in this range as a representative one:

$$\Delta c \leq 5 \times 10^{-16}. \tag{56}$$

If we observe gravitational waves oscillating in the low frequency and axion dark matter that has a heavier mass, we may give the stronger constraint to the coupling constant ℓ. Especially, if the gravitational waves are through the core of the Galaxy, the velocity of the gravitational waves will be modified strongly. For example, if we use gravitational waves that have a frequency of about 1 Hz and we assume the density of axion dark matter is about 0.3 GeV/cm^3 and the mass of axion dark matter is about 10^{-10} eV, the constraint on the Chern–Simons coupling constant reads:

$$\ell \leq 4.8 \times 10^4 \,\text{km} \,. \tag{57}$$

Once we obtain this constraint, by observing at a lower frequency, say 10^{-4} Hz, we can further constrain the Chern–Simons coupling constant as:

$$\ell \leq 4.8 \times 10^2 \,\text{km} \,. \tag{58}$$

For the extreme case of $f_{\text{gw}} = 10^{-9}$ Hz and $m = 10^3$ eV, we obtain the stringent constraint on the coupling constant:

$$\ell \leq 1.5 \times 10^{-13} \,\text{km} \,. \tag{59}$$

We note that the constraints on ℓ are actually not very sensitive to the axion density as it scales as $\ell \propto \rho^{-1/4}$. The average density of the universe is 10^{-5} GeV/cm^3. Hence, even if we take the most conservative value for the dark matter density, our constraints above on ℓ will change by just one order of magnitude.

On the other hand, assuming the coupling constant $\ell = 10^2$ km and $f_{\text{gw}} = 10^{-9}$ Hz, we obtain the constraint on the abundance of the axion with $m = 10^{-10}$ eV as:

$$\Omega_{\text{axion}} < 3.4 \times 10^{-3}. \tag{60}$$

Thus, we see that the gravitational waves can provide useful constraints to the Chern–Simons coupling constant and the abundance of axion dark matter.

7. Conclusions

We studied gravitational waves propagating in axion dark matter. In the presence of the axion, it is natural to consider the coupling of the axion to the gravitational Chern–Simons term. Since the axion condensation violates the parity symmetry, there is a chance to observe parity violation effects in the gravity sector using gravitational waves. We found that the coherent oscillation of the axion field leads to the parametric amplification of gravitational waves with a specific frequency. We investigated the gravitational birefringence induced by the difference in the phase velocity of the different polarization modes. We also derived the group velocity of gravitational waves, which is independent of the polarization at the leading order. In particular, we gave a constraint on the Chern–Simons coupling constant and the abundance of axion dark matter from the observation of GW170817 and GRB170817A.

We considered a homogeneous axion background in this paper, and we found that the Faraday effect is sizable for the axion mass range $m \geq 10^{-5}$ eV. This corresponds to an oscillatory period $T \leq 10^{-10}$ s or an oscillatory length $L \leq 1$ cm. In [26], it was discussed that archioles could introduce a large-scale modulation of energy density in axionic dark matter. However, such a modulation is significant only at an astronomical length scale much greater than L. Thus, the inhomogeneity of the dark matter density due to archioles has no effect on the Faraday rotation we discussed in this paper.

The axion dark matter could also lead to the formation of axion stars [27]. The mass and size of the axion stars are model dependent. For an axion mas $m = 10^{-4}$ eV, the radius of the dilute axion star is of the order of $R_a = 10^6$–10^7 cm, and the radius of the dense axion star is of the order of $R_a = 10^{-1}$–10^6 cm, In the above, we found that a sizable amount of Faraday rotation would be generated after the gravitational wave has past through a distance $L \sim 1/m$ through axion matter. For $m = 10^{-4}$ eV, this corresponds to $L \sim 10^{-1}$ cm. Now, if $L \ll R_a$, there will be enough axion dark matter for the gravitational wave to travel through, and our result for Faraday rotation holds. Therefore, for dilute axion stars and for most of the range of the dense axion stars, our result remains unchanged. On the other hand, if $L \gg R_a$, then the axion star is too small in size to induce any Faraday rotation. As far as the Faraday effect is concerned, these axion stars are transparent to gravitational waves. Finally, for $L \sim R_a$, our analysis failed, and a more accurate analysis taking into account the effect of inhomogeneity on the gravitational wave propagation has to be carried out separately.

There are various ways to proceed. It is important to perform a comparison of our results with real data. It is also interesting to study gravitational waves propagating in ultralight vector dark matter [28,29]. We showed in our paper that the Chern–Simons coupling could translate the oscillatory behavior of the scalar field background into a resonant response of the gravitational waves. For a massive vector field, it can couple to the gravity through the Chern–Simons coupling in the same manner as the derivative of the scalar field,

$$S_{int} = \int d^4x \sqrt{-g} \epsilon^{\mu_1\mu_2\mu_3\mu_4} A_{\mu_1} \omega_{\mu_2\mu_3\mu_4}, \tag{61}$$

where ω is the gravitational Chern–Simons three-form: $R\tilde{R} = d\omega$. It has been shown that the massive vector field can display an oscillatory behavior, so we expect that a similar analysis of the present manuscript can be extended to the case of vector dark matter. This fact can also apply to higher spin dark matter. In string theory, a candidate for a consistent formulation of quantum gravity, there appears naturally not just the axion field, but also the presence of higher spin fields. Some of these have odd parity, and there may exist some consistent Chern–Simons-like coupling of them to gravity. The existence of such a coupling may be detected through string amplitude calculation, and it is interesting to explore this possibility.

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