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Inequality and Catching-Up Under Decreasing Marginal Impatience

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Abstract

This paper examines how endogenous time preference interacts with inequalities in economic development. We consider two distinct groups of households with intrinsic inequality (e.g., capitalists and workers), and show that (i) under *decreasing* marginal impatience (DMI), an unequal society may be preferable for poor households than an egalitarian one in which every household owns an equal share of asset; (ii) poor households tend to benefit more under DMI than CMI (*constant* marginal impatience) from positive shocks; (iii) inequality exhibits a sharp inverted-U shape as more people become rich, which should be good news for developing countries in catching up; and (iv) a tax on capital income reduces poor households' income when the fraction of the rich is sufficiently small. We also examine immigration and discuss capital mobility.

Keywords: Endogenous Time Preference; China; India; Inequality; Catching up; Marginal Impatience; Globalization; Investment

JEL codes: D1; D9; O1

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1 Introduction

Piketty's popular book (2014, English edition) has revived wide interest in the relationship between social inequality and capital-asset ownership. He argues that the world today is returning toward "patrimonial capitalism," in which much of the economy is dominated by inherited wealth: their power is increasing, creating an oligarchy. He thus proposes a global system of progressive wealth taxes, to avoid the vast majority of wealth coming under the control of a tiny minority.

However, some policy makers have also put forth the idea that tax cuts for the wealthy are stimulative, especially when the fraction of the rich is small, such as pre-takeoff in poor economies. Indeed, when China started its open-door policy about 40 years ago, the then-leader, Deng Xiaoping, in particular stressed to "allow a small fraction of the population to become rich first."[1]This leads one to ask, is inequality necessarily bad, in particular, for the poor?

In the present paper, we offer an alternative explanation. As assets are accumulated and reinvested, diminishing returns kick in on the one hand and the demand for labor increases on the other hand, raising the wage rate. These effects are especially strong when we incorporate *decreasing marginal impatience (DMI)*, under which households become more patient and thus invest more when they become richer, thereby increasing capital accumulation and eventually generating a trickle-down effect to the poor in the long run. The mechanism increases the capital stock, the productivity and the welfare of all households including the poor when the rich becomes richer. Such investment and production linkages between the rich and the poor are absent under constant marginal impatience (CMI). Hence, in contrast to the alarm caused by Piketty, we find that inequality may not be so bad after all; on the contrary, it might just be a "growing pain" or even a "necessary evil" on a country's catching-up path, especially pre take-off when the fraction of rich people is small, and as a means to increasing the incentives

[1]Deng Xiaoping made such remarks when meeting with a group of U.S. visitors in September 1986.

for investment and eventually enlarging the total pie. However, the result could be reversed when the fraction of the rich becomes sufficiently large, as we shall demonstrate later.

Under endogenous time preference with DMI, a poorer household consumes a higher fraction of its income than a richer household. The assumption is motivated by a number of empirical studies which find strong evidence that households discount the future at different rates, which is important in explaining income inequality, see for instance, Hausman (1979), Becker and Mulligan (1997), Samwick (1997), and Barsky et al. (1997). Also, Lawrance (1991) and Warner and Pleeter (2001) find that more-educated households tend to have lower discount rates than less-educated ones, thus heterogenous time preference may lead to inequality through long-term investment and human capital accumulation. In fact, some studies have found that the marginal propensity to save is considerably higher among wealthier people (Frederick et al., 2002). Recently, Dohmen et al. (2016) find a significant relationship between patience and development, with patience explaining a substantial fraction of development differences across countries. As for the microeconomic foundations of DMI, it may arise for the following reasons: the rich may invest more on health, beauty and education, enabling them to live longer and healthier, making them more optimistic for the future.

Also, the experiences of many developing countries provide good support for our study. In their *early years* of economic development, widespread subsidies are provided to the rich–those fortunate enough to be business owners, such as the policies applied in the *special economic zones* where tax holidays, export, import and land subsidies are common, as a means to jump-starting economic development. Some of these countries have achieved great success with such policies; yet, their income inequality has also been rising rapidly. For instance, the Chinese wealth Gini coefficient remains well above the warning level of 0.4 set by the United Nations, peaking at 0.55 in 2002 (Knight, 2014).[2] Further, among the so-called BRICS countries (Brazil, Russia, India, China and South Africa), the Gini coefficient in South Africa was 0.67 in 2008,

[2] Even official figures show the Gini to be 0.491 in 2008 (National Bureau of Statics of China).

followed by Brazil (0.53 in 2015), India (0.51 in 2013), and Russia (0.483 in 1993).[3]

Given these stylized facts, one naturally asks the following question: could inequality be responsible for the high growth rates in these economies? A similar question was asked by Kuznets (1955) that led to the discovery of the Kuznets curve. Recent studies by Chang et al. (2015) and Gu et al. (2015) find that income inequality is a significant contributor to China's savings glut, which has enabled the recent Chinese growth that is heavily dependent on investment (Song, Storesletten and Zilibotti, 2011). Earlier, Banerjee and Duflo (2003) find that with cross-country data, changes in inequality in any direction are associated with reduced growth in the next period.

Based on the above empirical evidence and stylized facts, we consider a society without "equal opportunity" to begin with, as is a fact in many developing countries with strong traditional institutions (e.g., some Latin American countries, China and India, etc.), especially before their reform and take-off periods, when the fraction of rich households is very small.[4] To be specific, there exist two types of households that are symmetric in all aspects except that one type owns asset and can invest (e.g., "the rich", "lenders", "capitalists"), while the other type (e.g., "the poor", "borrowers", "workers") is unable to own asset or does not have the technology to operate in the asset market and hence consumes all income at each point in time (i.e., "hand to mouth"). In such economies, the fraction of rich households is small and the financial market is inefficient so poor households can hardly obtain the skills to save and invest (Dupas and Robinson, 2013). And thus, in the present paper we simply assume poor

[3]See the World Bank GINI Index, and "IMF warns of growing inequality in China and India": http://www.livemint.com/Politics/mTf8d5oOqzMwavzaGy4yMN/IMF-warns-of-growing-inequality-in-India-and-China.html

[4]For instance, although slowly being relaxed, the infamous family registration (*hukou*) system in China determines whether one has rural status or urban status at birth (following the mother), the latter of which carries many benefits such as retirement pension, medical insurance, housing subsidies and better chances to attend good schools while the former has none of them. And India is wellknown for the Caste system, which has largely broken down in cities, but persists in rural areas where 72% of India's population resides. Several Latin American countries have the highest Gini index in the world, and such inequalities have been carried over from generation to generation, some even from the colonial period (Ferreira et al., 2004). As such the top 10% population is estimated to own 50% of the total income while the bottom 10% owns only 1.5%, compared to 30% and 2.5% for corresponding groups in developed countries.

households save nothing, and focus on examining how inequality evolves under globalization.

We find that *poor* households tend to benefit more under DMI than CMI, because DMI generates a trickle down effect that is *absent under CMI*. Specifically, in standard models with CMI, a positive productivity shock always raises the income gap, since rich households benefit in more ways or more directly from such shocks while poor households only benefit through changes in the wage rate. This result is consistent with Acemoglu (2002), who studies the impacts of skilled labor-biased technology improvement and finds it to be a major cause for income inequality in the 20th century. In contrast, under DMI, an increase in productivity *reduces the Gini-coefficient* due to the trickle down effect stemming from the rich investing more, and the level income-gap may also fall. These are exactly opposite to Acemoglu (2002).

Further, we demonstrate that under DMI, the welfare of an economy where everyone owns an equal share of capital is *lower* than if only some people own capital. More surprisingly and perhaps politically incorrect, when the fraction of rich households is sufficiently small, *subsidizing the rich and taxing the poor* raises the poor households' welfare in the steady state. The logic is, the lower the fraction of capitalists, the more they invest and the more capital the country accumulates under DMI. Hence, a country with higher inequality accumulates higher capital stock and enjoys higher levels of welfare per capita, ceteris paribus. On the contrary, if the same amount of capital stock is spread over more owners, each capitalist invests less and the steady-state welfare becomes lower. As such, the poor may accept inequality to a certain extent, as long as their income level rises.

In addition, we can clearly compare the welfare levels with even and uneven distributions of assets. Under CMI, the welfare per capita with even distribution is lower than the welfare of the rich but higher than that of the poor with uneven distribution; in contrast, under DMI, it can be lower than the poor's welfare with uneven distribution. That is to say, under DMI, if society were forced to be egalitarian such that every household owned an equal share of asset, the steady-state welfare could become the lowest, i.e., lower than the poor's welfare in an uneven-distribution steady-state. Hence, it may seem that we have assumed asymmetry of households to begin with, but given DMI, inequality of households turns out to be a natural consequence of endogenous time preference, and an even-distribution of assets would not be preferred by any group. Therefore, it is socially inefficient to eradicate inequalities if they are not out of reasonable bounds. The good news is though, we find that inequality exhibits an inverted-U shape under DMI, i.e., the Gini-coefficient first rises then falls as the share of population owning asset increases, in a similar shape to the Kuznets curve, albeit due to a different mechanism.

Finally, we examine policies such as a tax on capital earnings, to redistribute income from the rich to the poor, to keep inequality within boundaries.[5] The government could also increase expenditure on education such as scholarships and loans for low income families, and subsidies and loans for young entrepreneurs, etc. Nevertheless, we find that the effect of the tax can be reversed as the fraction of rich households rises: it reduces (raises) poor households' income when the fraction is sufficiently low (high). As might be the case of present-day China (about 40 years after opening up to the world), the fraction of the rich has exceeded a certain level, especially in big cities and along the coast, and it might be time to impose a wealth or property tax as suggested by Piketty.[6]

In the theoretical literature, Bourguignon (1981) extends Stiglitz (1969) and shows that under convex saving functions, locally stable unegalitarian stationary distributions are Pareto superior to the egalitarian one. It follows that, the optimal asymptotic distribution of income and wealth is unegalitarian even though all individuals in the population are assumed to be identical. However, he admits that his result applies only to equilibria where all individuals

[5] In China, on the surface, it seems to be GDP growth at all costs: the sudden surge in inequality puts pressure on everybody, to try to become richer, as faster as possible, resulting in landslides of public morality. Many newly constructed roads, bridges, buildings and even food products are of poor quality, causing fatal accidents; Air and water pollution has soared to hazardous levels. These phenomena have generated heated debates in the media and among researchers. There are soul searching cries in the popular press that China in its rush to modernity should slow down the pace, in order to reduce ever-increasing pollution, to decrease man-made errors and potential disasters, and to make more efficient use of its depleting resources.

[6] It is especially interesting to note that the Chinese Communist Party's 19th Congress held in October, 2017, stresses that the party's main task for the next five years is to reduce inequality.

have a positive wealth, which might not be the general case, especially in developing countries. In contrast, the present has a focus on developing countries and assumes poor households have zero savings (i.e., "hand-to-mouth"), especially in the early stages of catching up.

Elsewhere, Krusell and Smith (1998) demonstrate that introducing time preference heterogeneity can significantly improve the Aiyagari (1994) model in explaining income inequality, and Hendricks (2007) incorporates preference heterogeneity into the life-cycle model of Huggett (1996) to account for wealth inequality. Epstein (1987), Das (2003), Chang (2009) and Hirose and Ikeda (2012) investigate equilibrium stability and uniqueness issues under DMI. Different from the above, Uzawa (1968) and Kamihigashi (2000) examine cases of IMI (increasing marginal impatience) rather than DMI. In an economy with initial inequality, Ghiglino and Sorger (2002) show that redistribution of wealth may drive the economy from a steady state with strictly positive output to a poverty trap in which output converges asymptotically to zero. Benhabib et al. (2011) demonstrate that wealth follows a Pareto distribution in the right tail, driven by capital income risk rather than labor income, and subsequently, Benhabib et al. (2016) show that redistributive fiscal policy with idiosyncratic investment risk and uncertain lifetimes can generate a double Pareto wealth distribution. De Nardi and Fella (2017) survey numerous extensions and applications of the Bewley model (1977) on inequality due to various reasons. Closely related to us, Aghion and Bolton (1997) and Matsuyama (2000) examine inequality and credit market imperfections, generating a trickle-down effect of capital accumulation to the poor, which is in turn caused by changes in the interest rate. In contrast, in the present paper, we model the patience of asset owners. The poor is unable to save and borrow (impatient), and the trickle-down arises due to an increase in the wage rate. We focus on the effects of DMI preference, patience and total factor productivity. The interplays of the two types of households bring interesting results that are novel in the literature.

2 The Basic Model

We focus on an economy with two types of households, which are symmetric in all aspects except that one type owns asset (i.e., rich households), while the other type is unable to own asset for some reason (i.e., poor households).[7] That is, there exists intrinsic social inequality to begin with, as in many developing countries with strong traditional institutions and customs. In such economies, the fraction of rich households is small and the financial market is inefficient so poor households can hardly save or borrow. Later we shall extend the basic model to examine cases of all households owning an equal share of asset.

Our goal is to examine the relationship between inequality and economic growth and how social inequality evolves in transitional economies such as the so-called BRICS countries (Brazil, Russia, India, China, and South Africa). In the process of catching up, such economies inevitably face technology upgrading and other shocks. We analyze the impacts of technological improvement on the income gaps, and government policies including tax reform that might be used to mitigate the existing and possibly widening inequality.

Consider one good that is consumed and saved as capital, whose output is given by

$$Y = F(K, L),$$

where K and L are respectively capital stock and labor supply. Production exhibits constant returns to scale technology,

$$k \equiv \frac{K}{L} \quad \text{and} \quad f(k) \equiv F\left(\frac{K}{L}, 1\right).$$

[7] Of course, we can relax this assumption, to let all households own assets. However, under DMI, the economy will not converge to the steady state where everyone owns capital, if their initial asset holdings differ, because the rich will become richer through asset re-investment. Indeed, the steady state we shall examine in this paper is the same as the one when the borrowing constraint is binding only for one type of households. Note that this is a direct derivation from the findings in Epstein (1987).

Then, the capital rental rate R and the wage rate w are respectively

$$R = f'(k) \text{ and } w = f(k) - kf'(k).$$

The household's inelastic labor supply is normalized to 1, so that the number of households is denoted by L, and k is the capital stock per household or simply capital stock.

2.1 The Capital Market and Households' Distinct Incomes

In our model, poor households consume all current income so they do not participate in the capital market. Since their income comes from wages only, $i^*(= w)$, it increases in capital accumulation.

In contrast, rich households save a portion of their income as assets. The income for rich households with asset a is given by $w + ra$, while that for poor households (without asset) is only w, where $r = R - \delta$ is the interest rate and δ is the depreciation rate on capital.

Let $\theta \in (0, 1]$ denote the fraction of households owning asset in the whole economy. Then $k = a\theta$ from the market clearing condition for asset, $a\theta L = K$.[8] It follows that the income of rich households, i, becomes a function of k and θ as follows:

$$i(k, \theta) \equiv w(k) + \frac{r(k)k}{\theta},$$

where $w(k) \equiv f(k) - kf'(k)$ and $r(k) \equiv f'(k) - \delta$.

Notice that $w'(k) = -kf''(k) > 0$, and $r'(k) = f''(k) < 0$;[9] that is, when the capital stock rises, the wage rate increases but the interest rate decreases. In standard models with (one type of) representative households ($\theta = 1$), the two effects are exactly cancelled out ($w'(k)+r'(k)k = 0$), and hence household income, $f(k) - \delta k$, always increases with capital accumulation for any

[8] Poor households consume all income at each point in time, and hence consumption is $c^* = w$. Then, the goods market clears when: $(c + \dot{a})\theta L + c^*(1 - \theta)L + \delta K = wL + RK$.

[9] In what follows, we assume $w''(k) = -f''(k) - kf'''(k) < 0$ as is the case with Cobb-Douglas technology.

positive interest rate, and a representative household's income is maximized at the *golden-rule* level of capital stock where $R = \delta$, which yields $r = 0$.

In contrast, under $\theta < 1$ as in the present model, we have two distinct types of households, and these two opposite effects do *not* cancel out. In fact we have

Lemma 1 *If $\theta < 1$, capital accumulation can reduce the income of households holding assets, even when the interest rate is positive, and hence the income of rich households is maximized at a certain capital stock lower than the level given by the golden rule.*

Some explanations are in order. From the definition of $i(k, \theta)$ above, when θ is small, $r(k)k/\theta$ becomes large and rich households' income mainly consists of asset income, which declines when the interest rate falls due to capital accumulation.

Note that in standard models like the Ramsey model, households accumulate capital to a level lower than that given by the golden rule, and there is no case for capital accumulation to lower households' income along the optimal path to the steady state. However, in the present model with $\theta < 1$, it could happen, since capital accumulation lowers the interest rate and the rich households' asset income.

2.2 Optimization under DMI for Rich Households

So far all results are obtained without considering DMI. Next we formally incorporate DMI: each of the rich households maximizes the discounted sum of utility

$$\int_0^\infty u(c)X dt, \tag{1a}$$

subject to

$$\dot{a} = w + ra - c, \tag{1b}$$

$$\dot{X} = -\rho(c)X, \tag{1c}$$

where for $^\forall c > 0$, $u(c) > 0$; $u'(c) > 0$; $u''(c) < 0$. Also, $X \equiv \exp[-\int_0^t \rho(c)ds]$ is the discount factor which depends on the past and present levels of consumption through function $\rho(\cdot)$. Expression (1c) implies the rate at which X decreases is $\rho(c)$; to be more precise, X decreases at a constant speed under CMI, but at a decreasing speed under DMI.

Following Das (2003) and Chang (2009), household preference exhibits DMI,[10]

$$\rho'(c) < 0 < \rho''(c) \text{ for } ^\forall c > 0,\ \rho(0) < \infty,\ \text{and } \lim_{c\to\infty} \rho(c) = 0. \tag{2}$$

Intuitively, it says households with a higher consumption (income) discount future less, since they can afford to defer consumption of additional income and wealth.[11] This assumption is supported by a number of empirical studies, such as Lawrance (1991), Barsky et al. (1997) and Samwick (1997), as mentioned in the Introduction.

The Hamiltonian associated with our optimization problem is

$$H \equiv u(c)X + \lambda(w + ra - c) - \mu\rho(c)X,$$

where λ and μ are the co-state variables. The necessary conditions for optimality are[12]

$$\frac{\partial H}{\partial c} = u'(c)\,X - \lambda - \mu\rho'(c)X = 0, \tag{3a}$$
$$\frac{\partial H}{\partial a} = \lambda r = -\dot{\lambda}, \tag{3b}$$
$$\frac{\partial H}{\partial X} = u(c) - \mu\rho(c) = -\dot{\mu}. \tag{3c}$$

[10] In standard models with CMI, $\rho' = 0$, so ρ is constant. See Hirose and Ikeda (2008) for assumptions on u and ρ that can be applied to both DMI and IMI.

[11] While we assume rich households become more patient when their consumption rises in the model, later we shall show that in the steady state, their consumption necessarily rises along with income.

[12] These conditions are exactly the same as in Das (2003).

Let $Z \equiv \lambda/X$ to simplify notation. Then (3a) and (3b) can be rewritten as

$$Z = u'(c) - \mu\rho'(c),$$

$$\dot{Z} = Z[\rho(c) - r].$$

Using the above, our dynamic general equilibrium system can be described as

$$\dot{a} = w(k) + r(k)a - c, \tag{4a}$$

$$\dot{Z} = Z[\rho(c) - r(k)], \tag{4b}$$

$$\dot{\mu} = \mu\rho(c) - u(c), \tag{4c}$$

$$0 = u'(c) - \mu\rho'(c) - Z. \tag{4d}$$

2.3 The Steady State

We define the steady state of the model as when all variables for households with asset, i.e., a (or k), Z, μ and c, are constant, and the consumption of households without asset is also constant at $c^* = w(k)$. Then the steady state is a solution to the following system of equations[13]

$$0 = w(k) + \frac{r(k)k}{\theta} - c, \tag{5a}$$

$$0 = Z[\rho(c) - r(k)], \tag{5b}$$

$$0 = \mu\rho(c) - u(c), \tag{5c}$$

$$Z = u'(c) - \mu\rho'(c). \tag{5d}$$

These conditions say at the steady state, consumption must be equal to income (condition (5a)), the interest rate must be equal to the discount factor of rich households (5b), utility is constant ($\dot{\mu} = 0$, (5c)), and (5d) equates the current value of the shadow price to the marginal-utility

[13] Our model does not have a "satiated" steady state (Z being equal to zero) as discussed in Hirose and Ikeda (2008), since our assumption on u and ρ ensures that the steady state value of Z must be positive.

increase. In what follows, we use "$\sim$" to denote the steady state value of each variable.

More specifically, conditions (5a) and (5b) give the steady state solution pair $(\tilde{k}, \tilde{c})$, which can be rewritten as $c = i(k, \theta)$ and $k = \kappa(c)$, where

$$\kappa(c) \equiv r^{-1}(\rho(c)).$$

Due to DMI, the level of capital stock equating the interest rate to the discount factor is increasing in c:

$$\frac{d\kappa(c)}{dc} = \frac{\rho'(c)}{r'(k)} > 0. \tag{6}$$

Compared to under CMI, where $\rho'(c) = 0$, condition (6) then gives an important result.

Lemma 2 *Under DMI, an increase in the steady-state consumption of rich households raises the capital stock in the economy.*

Lemma 2 contains a very important mechanism that is key in deriving our results later. It basically says that by the definition of DMI, a rise in the rich's consumption makes them more patient, which raises their investment and the steady-state capital stock, leading to a trickle down effect via increasing the wage rate, and eventually benefiting the poor households. While under CMI, such a mechanism does not exist–rich households do not become more patient, and thus they do not invest more.

Once the steady state $\tilde{c}$ is determined, we see from (5c) and (5d) that

$$\tilde{\mu} = \frac{u(\tilde{c})}{\rho(\tilde{c})} \quad \text{and} \quad \tilde{Z} = \frac{u'(\tilde{c})\,\rho(\tilde{c}) - u(\tilde{c})\,\rho'(\tilde{c})}{\rho(\tilde{c})},$$

where $\tilde{\mu}$ (increasing in $\tilde{c}$) can be interpreted as the steady state level of welfare in the sense that the discounted sum of utility, $\int_0^\infty u(c)X dt$, is equal to $\tilde{\mu}$ when $c(t) = \tilde{c}$ for $^\forall t \geq 0$.

Finally we examine the stability and uniqueness of the steady state. Let k_1 and k_2 be the values of the capital stock that equate the interest rate to $\rho(0)$ and zero respectively:

$$k_1 \equiv r^{-1}(\rho(0)) \text{ and } k_2 \equiv r^{-1}(0),$$

where $k_1 < k_2$ holds.[14] Then, for any $\theta \in (0, 1]$,

$$i(k_1, \theta) = w(k_1) + \frac{\rho(0)k_1}{\theta} > 0,$$
$$i(k_2, \theta) = w(k_2) < \infty.$$

It is apparent from $d\kappa/dc > 0$, $k_1 = \kappa(0)$, and $k_2 = \lim_{c\to\infty} \kappa(c)$ that there necessarily exists an intersection of the two graphs, $c = i(k, \theta)$ and $k = \kappa(c)$, as in Figure 1.

In the rest of the paper, we assume[15]

Assumption 1: Both $c = i(k, \theta)$ and $k = \kappa(c)$ are strictly concave in k and c, respectively, for any $\theta \in (0, 1]$.

Remark 1 *The strict concavity of i is satisfied when technology takes the Cobb-Douglas form, while that of κ implies that we basically exclude the extreme case where rich households become much more patient when their income slightly rises.*[16] *An example of the pair of functions f and ρ that satisfies Assumption 1 is provided in Section 4 (see Assumptions 2 and 3 below).*

Under Assumption 1, at the intersection, the slope of $c = i(k, \theta)$ must be smaller than that

[14] The Inada conditions: $\lim_{k\to 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$ ensure the existence of k_1 and k_2.

[15] Strict concavity of i is not necessary for the results in the present paper, and that of κ is necessary only for the results about the income gap. However, we assume strict concavity to exclude the possibility of multiple steady states and the extreme case where the effect of DMI is very strong.

[16] If Assumption 1 is violated, there may be an odd number of steady states. Inequality (7) holds in the first steady state, but fails in the second one, ..., and it also holds in the last one. One can verify from the proof of Lemma 3 (Appendix 6.1) that the steady states with (7) are saddle points, while the others are unstable or equilibrium indeterminacy arises. Assuming the uniqueness of the steady state, we will focus on a saddle point.

of $c = \rho^{-1}(r(k))$, i.e., $\partial i/\partial k < (\partial\kappa/\partial c)^{-1}$, and hence we have:[17]

$$\frac{(\theta-1)w'(k)+\rho(c)}{\theta}\rho'(c) > f''(k). \tag{7}$$

Combining this with the stability analysis in Appendix 6.1, we obtain:[18]

Lemma 3 *Under Assumption 1, the steady state of the dynamic system is unique and a saddle point.*

The above completes the basic setup of the model. In the unique steady state under DMI, rich households save more when their income rises. More interestingly, poor households' welfare increases too, stemming from an increase in the wage rate by Lemma 2 (see also Figure 1).

In addition, we obtain a clear trickle down effect as follows[19]

Proposition 1 *(Trickle down of DMI): Under DMI, a decrease in the share of rich households, θ, raises poor households' income w and consumption $\tilde{c}^*$, which is absent under CMI.*

This Proposition follows straightforwardly from Lemma 2. It implies that the smaller is the share of rich households, the higher the income of the poor becomes. This is an effect that does not exist under CMI, where $k = \kappa(c)$ in Figure 1 becomes a vertical line passing through the steady-state capital stock, which is only a solution to $r(k) = \rho$, and independent of the steady-state consumption levels of the rich households. Therefore, even though a decrease in θ under CMI shifts up curve $c = i(k,\theta)$, raising $\tilde{c}$, it does not change $\tilde{c}^*$. In contrast, under DMI, $\rho(c) = r(k)$ yields the positively sloped curve $k = \kappa(c)$, and a higher $\tilde{c}$ requires a higher

[17] Condition (7) holds at any intersection and hence the steady state is unique, if the following is met:

$$\frac{1}{\rho(0)}\min_{k\in[k_1,k_2]}[-f''(k)] \geq -\frac{\rho'(0)}{\theta}\max_{k\in[k_1,k_2]}\left[1+(1-\theta)\frac{kf''(k)}{\rho(0)}\right].$$

This inequality degenerates to the bounded slope assumption in Chang (2009), when $\theta = 1$. Then the steady state of his model (of endogenous time preference with DMI) is unique and is a saddle point.

[18] Notice that with CMI, the intersection must uniquely exist, and (7) holds at the unique intersection. Also, one can easily verify that Lemma 3 remains valid under CMI.

[19] In the case where rich households obtain only capital income, their income is given by $(1-\theta)r(k)k/\theta$. Thus, the trickle down effect can arise in this case as well, which implies that other results in the paper will also follow.

$\tilde{k}$, raising $\tilde{c}^* = w(\tilde{k})$ with a trickle down: both E_1 and E_2 rise, along curves $k = \kappa(c)$ and $c^* = w(k)$ respectively. Intuitively, DMI connects the rich and the poor above what CMI does, through investment and production linkages, which benefits the poor in the long run.

3 Inequality under DMI

We now consider inequality under DMI, by changing the fraction of rich households, θ, and examine its impact on the steady state variables. As discussed above, lowering θ has a trickle down effect for the poor in the sense that it will raise the steady state capital stock and in turn the wage income of poor households. Again, this effect does not exist under CMI.

3.1 Income Gaps and the Gini Coefficient

In this subsection, we consider the income gap and the Gini coefficient under DMI. As shown below, although lowering θ widens the income gap, it does not necessarily imply expanding inequality estimated using the Gini coefficient. This sharply contrasts with the case of CMI, where the coefficient linearly decreases in θ.

First, the incomes of households with and without asset in the steady state are respectively

$$\tilde{I} = w(\tilde{k}) + \frac{\rho(\tilde{c})\tilde{k}}{\theta} \quad \text{and} \quad \tilde{I}^* = w(\tilde{k})$$

We can define the income gap in terms of both *level* and *ratio* differences, respectively as:

$$\begin{aligned} g &\equiv \tilde{I} - \tilde{I}^* = \frac{\rho(\tilde{c})\tilde{k}}{\theta} \\ \text{and} \quad \hat{g} &\equiv \frac{\tilde{I}}{\tilde{I}^*} = 1 + \frac{\rho(\tilde{c})\tilde{k}}{\theta w(\tilde{k})}. \end{aligned}$$

Notice that when θ decreases, $\tilde{I}$ and $\tilde{I}^*$ rise along $k = \kappa(c)$ and $c^* = w(k)$ respectively.

Therefore, if κ and w are strictly concave in c and k respectively as in Figure 1,[20] then both g and $\hat{g}$ are decreasing in θ : Income gaps will rise, as the fraction of rich households decreases.[21]

The Gini coefficient can be calculated as,

$$G = 1 - \frac{\tilde{I}^*(1-\theta)^2/2 + \tilde{I}^*\theta(1-\theta) + \tilde{I}\theta^2/2}{[\tilde{I}^*(1-\theta) + \tilde{I}\theta]/2}$$
$$= \frac{\theta\hat{g}}{\theta\hat{g} + 1 - \theta} - \theta,$$

which gives:

Lemma 4 *The Gini coefficient G is increasing in the ratio income-gap $\hat{g}$.*

Also, we see that

$$G = \frac{\rho(\tilde{c})\tilde{k}(1-\theta)}{w(\tilde{k}) + \rho\tilde{k}}. \tag{8}$$

We are now in a position to state

Proposition 2 *Under DMI with Assumption 1, the concentration of wealth widens the income gap, but there exists a non-monotonic relationship between the share θ and the Gini coefficient G : As θ decreases from 1, G first rises but then falls and goes to zero as θ approaches zero; in contrast under CMI, G increases linearly as θ falls.*

Proof. It is apparent from Figure 1 that both g and $\hat{g}$ are decreasing in θ when κ is strictly concave in c, since we assume $w'' < 0$. And $G = 0$ holds when θ goes to 0, because

$$\lim_{\theta \to 0} \theta\tilde{I} = \lim_{\theta \to 0} \left[\theta w(\tilde{k}) + \rho(\tilde{c})\tilde{k}\right] = 0.$$

[20] Also, notice that $\kappa'' < 0$ corresponds to the case where the effect of DMI is not strong.

[21] g and $\hat{g}$ could be increasing in θ, if κ were convex in c. However, it would not happen, uncless rich households became much more patient when their income slightly rises, which yields a large increase in the capital stock and hence the wage rate.

Differentiation gives,

$$
\begin{aligned}
G_\theta &= \frac{\hat{g} + (1-\theta)\theta\hat{g}_\theta}{(\theta\hat{g} + 1 - \theta)^2} - 1 \\
&= \frac{(1-2\theta)(\hat{g}-1) + (1-\theta)\theta\hat{g}_\theta - [\theta(\hat{g}-1)]^2}{(\theta\hat{g} + 1 - \theta)^2},
\end{aligned}
$$

where $\hat{g}_\theta < 0$. Since $\hat{g} > 1$, G_θ must be negative for $\theta \geq 1/2$; but it can be positive for a sufficiently small θ,[22] because $G > 0$ for any $\theta \in (0,1)$ and $\lim_{\theta\to 0} G = 0$.

In the case of CMI, the linearity of G is apparent from the fact that $\tilde{k}$ and $\tilde{c}$ in (8) do not depend on θ. ■

Thus, with Assumption 1, as the share of rich households falls, both the level and the ratio income-gaps widen. But under DMI, the Gini coefficient G does not monotonically increase. Instead, it exhibits a sharp *inverted-U shape*, first increasing then decreasing, similar to the Kuznets curve (Kuznets, 1955), following a fall in the percentage owning assets, θ. It again stems from DMI and its trickle down effect, as discussed extensively before. The result stands in sharp contrast to the case of CMI, where G increases linearly as θ goes to zero.

Proposition 2 has important implications. In order to reduce inequality, Piketty (2014) proposes that a progressive annual global wealth tax of up to 2%, combined with a progressive income tax reaching as high as 80%. While in the present model, as the fraction of population owning assets decreases to below a certain level, inequality falls, even without any government intervention. Moreover, inequality also gradually falls as the fraction increases to above a certain level. Combined, these imply that inequality exhibits an inverted-U shape under *DMI*, i.e., the Gini-coefficient first rises then falls as the share of population owning assets increases.

[22]Indeed, one can verify that under Assumptions 2 and 3, $\lim_{\theta\to 0} \theta\hat{g}_\theta/(\hat{g}-1) = -(1+\beta)^{-1} > -1$.

3.2 Uneven but Preferred

With DMI, the decrease in θ raises the capital stock and the wage in the steady state, implying that a low θ may be beneficial. Indeed, we can show that an economy with an *uneven* distribution of income among households is preferable for *all* households to an economy with each household having an identical level of asset.

In contrast, with CMI, the even distribution may be preferred to an uneven one: the welfare of a typical household is higher than that of the poor but lower than that of the rich under an unequal society. Thus, with CMI, egalitarianism is good for the poor but bad for the rich; while with DMI, there is a possibility that egalitarianism is *not good for either* group.

3.2.1 Even distribution of income: $\theta = 1$

To see this clearly, suppose all households owned the same level of asset with $\theta = 1$. Then their income would be given by

$$\begin{aligned} i(k,1) &= w(k) + r(k)k \\ &= f(k) - \delta k, \end{aligned}$$

which is increasing in k if $k < k_2$.[23] In Figure 1, the $c^* = w(k)$ curve disappears, and the $c = i(k,\theta)$ curve flattens out to the blue curve, $c = i(k,1)$. It intersects with the curve $k = \kappa(c)$ at point E_0; that is, $\tilde{k}(1)$ and $\tilde{c}(1)$ denote the steady state levels of capital and consumption in the economy with $\theta = 1$ and DMI.

3.2.2 Uneven distribution of income: $\theta < 1$

Under DMI, a higher consumption makes households more patient and invest more, so the rich's consumption moves to point E_1, and the poor's consumption follows suit to point E_2. In Figure 1, the consumption of the typical household at point E_0 clearly lies below the consumption for

[23] Notice that k_2 corresponds to the Golden Rule level of capital (per household).

the poor, $\tilde{c}^*$, at point E_2. That is, the uneven distribution of wealth may provide a higher welfare level for everyone in the whole economy than the level under the even distribution.[24] And this situation necessarily arises when θ is sufficiently small, in which a tiny fraction of the population owns all assets, lowering their discount rate to accumulate capital close to the golden-rule level, and helping the whole country achieve sufficiently high wealth and consumption.

Formally, for $k < k_2$,

$$\frac{\partial i(k,\theta)}{\partial \theta} = -\frac{r(k)k}{\theta^2} < 0 \quad \text{and} \quad \lim_{\theta \to 0} i(k,\theta) = \infty,$$

which implies

$$\lim_{\theta \to 0} \tilde{k} = k_2 \quad \text{and} \quad \lim_{\theta \to 0} \tilde{c} = \infty. \tag{9}$$

Then, we see

$$\lim_{\theta \to 0} \tilde{c}^* = w(k_2) = i(k_2, 1),$$

and hence

$$\lim_{\theta \to 0} \tilde{c}^* > \tilde{c}(1).$$

We are now in a position to state

Proposition 3 *Under DMI, the steady-state welfare in the economy with the even distribution of wealth (i.e., everyone becoming a capitalist with an identical level of asset), is lower than the poor household's welfare in the steady state with an extremely small group of capitalists.*

The Proposition implies that under DMI, *uneven* distribution of income is preferable for everyone in the economy due to patience heterogeneity, essentially because rich households are more patient and invest more, which lowers the interest rate and in turn raises the wage income of poor households through production linkages. On the contrary, if the capital stock is spread

[24] Here, we define the steady state level of welfare for households without asset as $u(\tilde{c}^*)/\rho(\tilde{c}^*)$, analogous to that for households with asset.

over more owners (i.e., lowering inequality in income), then on average each capitalist saves and invests less, resulting in lower welfare in the long run. As such, the economy with an extremely small number of capitalists yields not only a lower value of the Gini coefficient, but also a higher welfare for all households than an economy with the even distribution of assets.

The generated consequences of trickle-down from this Proposition are similar to those in Aghion and Bolton (1997) and Matsuyama (2000), who examine capital market imperfections, while we analyze a different mechanism, namely, DMI.

3.3 Discussion and Application of Results: Immigration

Besides DMI, two assumptions are behind the surprising results in Propositions 2 and 3. One of which is $\lim_{c\to\infty}\rho(c)=0$,[25] and the other is that rich households do not coordinate with each other on the level of asset holdings, even when their population share is sufficiently small. On the contrary, the proposition may not hold, if rich households could strategically behave such that they optimize their discounted sum of utility subject to the following budget constraint,[26]

$$\dot{a} = w(a\theta) + r(a\theta)a - c.$$

Then, one of the steady state conditions on k and c would change as

$$\rho(c) = r(k) + (\theta - 1)w'(k).$$

In this case, the steady state capital stock would not converge to k_2 when θ goes to zero with $\lim_{\theta\to 0}\tilde{c}=\infty$ and $\lim_{c\to\infty}\rho(c)=0$.

Proposition 3 also contrasts with Piketty (2014), where inequality increases if the interest rate is higher than the growth rate, since the rich gets richer through investment. However, in

[25] If $\lim_{c\to\infty}\rho(c)>0$, $\lim_{\theta\to 0}\tilde{k}<k_2$, and hence $\lim_{\theta\to 0}G>0$ and Proposition 3 may not hold.

[26] This specification may be justified if rich households coordinate on asset holdings, such as when only one household owns all assets in the whole economy; then $k=\alpha\theta$. See also Sorger (2008), who considers strategic saving decisions in the Ramsey model.

our model with DMI, the process does not stop there, because investment by the rich lowers the interest rate and raises the wage income, benefiting the poor with a trickle down.

As an application, we now can examine the effects of immigration, since it can be viewed as a change in θ. International labor migration is an integral part of our globalized economy. Workers migrate from poor to rich countries, to seek better opportunities and earn higher income. According to the United Nations,[27] the world stock of migrants reached 243 million in 2015. Here we look into how international migration affects the economy in terms of inequality and welfare.

Consider a developed home country that has both rich and poor households. Poor households (without asset) from a foreign country immigrate into this country, which causes a decrease in the share of rich households θ. Thus, immigration necessarily brings a positive effect on the steady state *welfare* of all households, because $\partial\tilde{k}/\partial\theta < 0$. Applying Proposition 2, we also obtain the impact of immigration on the income gaps.

Proposition 4 *Let Assumption 1 hold. Immigration (lowering θ) increases all households' welfare in the long run, but it widens both the level and the ratio income-gaps under DMI as well as under CMI.*

This Proposition implies that immigration may cause polarization in the rich countries that host immigrants, which is arguably one of the most important reasons for Brexit in the EU and Donald J. Trump taking over the U.S. presidency in 2017. However, after a temporary fall in the wage rate due to the increase in the labor force, the income of poor households will rise above the level before immigration as the economy accumulates capital stock.

To gain an intuitive explanation for the Proposition, we now formally examine the economy that has reached a steady state such that

$$\tilde{k} = \frac{K}{L}, \quad \tilde{a} = \frac{K}{\theta L}, \quad \tilde{c} = w\left(\frac{K}{L}\right) + \rho\frac{K}{\theta L}, \quad \text{and} \quad \tilde{c}^{*} = w\left(\frac{K}{L}\right),$$

[27] U.N. Trends in International Migrant Stock: the 2015 Revision.

where $\rho = r(K/L)$.

As more immigrants move in, the labor input L rises to L' (the ratio θ falls to θ'), and hence the wage rate falls and the capital rental rises. The whole economy thus accumulates capital and will eventually reach a new steady state, which can be characterized by

$$\tilde{k} = \frac{K'}{L'}, \quad \tilde{a} = \frac{K'}{\theta L}, \quad \tilde{c} = w\left(\frac{K'}{L'}\right) + \rho\frac{K'}{\theta L}, \quad \text{and} \quad \tilde{c}^* = w\left(\frac{K'}{L'}\right),$$

where $\rho = r(K'/L')$, and the rich households own more asset than in the old steady state.

Under CMI, $K'/L' = K/L$ must hold, and hence rich households' steady state consumption will rise, while that of poor households does not change. As a consequence, existing poor households are harmed by immigration due to the short-term fall in the wage rate. Under DMI, however, an increase in the capital labor ratio ($K'/L' > K/L$) is accompanied by an increase in the consumption level of rich households since they become more patient, and thus poor households are made better off in the long run through the rich's added investment. In other words, DMI generates a further effect above CMI, which is a positive trickle-down.

Propositions 4 implies that immigration that adds poor people to the host country, increases the steady-state welfare of all households in the country due to capital accumulation stimulated by labor migration, but it also widens the income gap between the rich and the poor. These are consistent with observations in North America and Europe which have experienced increases in immigration and average income, but the income gap also rises noticeably.

Note that in practice, many countries adopt policies to attract skilled immigrants. In other words, such policies aim to attract the human capital or embodied skills in the immigrants, which are not the focus of the present model. Also, in the case where governments provide some income transfer from the rich to the poor, poor households may be made worse off by immigration due to the decrease in the amount of transfer per capita.

Finally, if capital can move across countries, the poor must become better off at least in the long run. This is because, i) if capital outflow occurs, the rental on capital in the rest of the world must be higher than the domestic rental rate, and the domestic capital labor ratio must temporally decline. But the higher capital income from other countries will make the rich richer and become more patient and hence accumulate more, resulting in a higher capital labor ratio in the country than before; ii) if capital inflow occurs, then it straightly implies that the capital labor ratio will rise.

4 Income Transfer

In this section, we examine how income transfer through a tax on capital return and technology advancement affect the steady state. Precisely due to the impact of DMI, any change that makes households with asset richer also has a positive effect on the steady state income of households without asset, as we shall clearly demonstrate below.

Some political economy models start with the premise that inequality leads to redistribution and then argue that redistribution may hurt growth, see for instance Alesina and Rodrik (1994), Persson and Tabellini (1991), Perotti (1996), and Benhabib and Rustichini (1998). Indeed, redistribution narrows the income gaps also in our model. However, an income transfer from the rich to the poor may lower not only the steady state capital stock, *but also the steady state welfare of poor households,* who may initially gain from the transfer nevertheless. This will happen if the share θ is small, when the positive direct effect of the transfer is dominated by the indirect effect through a fall in the steady state capital stock. Thus, θ is a key parameter that determines the two opposite effects of the transfer.

To see this, we introduce a 'capital tax' τ on the asset income of rich households, intended to reduce the income gap. The tax revenue is used as a direct lump-sum transfer, T, to poor households. The post-tax income for rich households is then

$$I = w + (1 - \tau)ra,$$

while that for poor households becomes

$$I^* = w + T,$$

for which the government budget constraint $\tau r a \theta = T(1-\theta)$ holds. Also, it is natural to assume the income of rich households to be higher than that of poor households, $I > I^*$, for which $\tau < 1 - \theta$ is required.

Next, the steady state is a solution to the following system of equations

$$0 = w(k) + \frac{1-\tau}{\theta} r(k)k - c, \tag{10a}$$

$$0 = Z[\rho(c) - (1-\tau)r(k)], \tag{10b}$$

$$0 = \mu\rho(c) - u(c), \tag{10c}$$

$$Z = u'(c) - \mu\rho'(c). \tag{10d}$$

Introducing τ dose not change the model, except that the interest rate is given by $(1-\tau)r$. And the steady state solution pair $(\tilde{k}, \tilde{c})$ is given by the intersection between $c = i(k, \theta, \tau)$ and $k = \kappa(c, \tau)$, where

$$i(k, \theta, \tau) \equiv w(k) + \frac{1-\tau}{\theta} r(k)k,$$
$$\kappa(c, \tau) \equiv r^{-1}\left(\frac{\rho(c)}{1-\tau}\right).$$

In the rest of the paper, we assume[28]

Assumption 1A: Both $c = i(k, \theta, \tau)$ and $k = \kappa(c, \tau)$ are strictly concave in k and c, respectively, for any pair (θ, τ) with $\theta \in (0, 1]$ and $\tau < 1 - \theta$.

[28] Assumptions 2 and 3 are sufficient for Assumption 1A also in this case. In what follows, we do not exclude the possibility of a negative τ to the extent that it yields a positive I^*.

At the unique intersection, we have

$$\frac{(\theta - 1 + \tau)w'(k) + \rho(c)}{\theta}\rho'(c) > (1-\tau)f''(k). \tag{11}$$

4.1 Impacts on Steady State Welfare

We next consider the effects of changes in A and τ on the steady state level of welfare for both types of households, where $A > 0$ is the total productivity with $f(k) = A\hat{f}(k)$ for ${}^{\forall}k$. Totally differentiating $c = i(k, \theta, \tau)$ and $k = \kappa(c, \tau)$ with respect to A and τ to give

$$\begin{aligned}
&\begin{pmatrix} -\frac{1-\tau-\theta}{\theta}\tilde{k}f'' - \frac{\rho}{\theta} & 1 \\ (1-\tau)f'' & -\rho' \end{pmatrix}\begin{pmatrix} dk \\ dc \end{pmatrix} \\
&= \begin{pmatrix} \frac{f}{A} + \frac{1-\tau-\theta}{A\theta}\tilde{k}f' & -\frac{\rho\tilde{k}}{(1-\tau)\theta} \\ -\frac{1-\tau}{A}f' & \frac{\rho}{1-\tau} \end{pmatrix}\begin{pmatrix} dA \\ d\tau \end{pmatrix},
\end{aligned}$$

where each element of the matrices is evaluated at the steady state. Then, we obtain:

Lemma 5 *Under Assumption 1A,*

$$\frac{\partial \tilde{k}}{\partial A} = \frac{-\theta\rho' f + [(1-\tau)\theta - (1-\tau-\theta)\rho'\tilde{k}]f'}{AD\theta} > 0, \tag{12a}$$

$$\frac{\partial \tilde{c}}{\partial A} = \frac{(1-\tau)(\rho f' - \theta f f'')}{AD\theta} > 0, \tag{12b}$$

$$\frac{\partial \tilde{k}}{\partial \tau} = -\frac{\rho(\theta - \rho'\tilde{k})}{D(1-\tau)\theta} < 0, \tag{12c}$$

$$\frac{\partial \tilde{c}}{\partial \tau} = -\frac{\rho(\rho - \theta\tilde{k}f'')}{D(1-\tau)\theta} < 0, \tag{12d}$$

where

$$D \equiv \left(\frac{1-\tau-\theta}{\theta}\tilde{k}f'' + \frac{\rho}{\theta}\right)\rho' - (1-\tau)f'' > 0 \quad \textit{and} \quad (1-\tau)\theta - (1-\tau-\theta)\rho'\tilde{k} > 0$$

hold from (11).

From the Lemma, one sees that the steady state capital stock increases when technology A improves, which must in turn raise the steady state level of welfare for all households, since

$$c^* = i^*(k, \theta, \tau) \equiv w(k) + \frac{\tau r(k)k}{1-\theta}$$

and $\partial i^*/\partial k > 0$.

On the other hand, the effect of τ on poor households' welfare is ambiguous, because it has a negative effect on the steady state capital stock, and hence it may reduce poor households' income even counting the transfers.

As a particularly interesting finding, it is possible for the 'capital tax' on the rich to lower the steady state welfare of poor households, especially in an economy where almost all households are poor and unable to own asset (e.g., a low θ). In such a scenario, the government can increase the steady state welfare for all households by setting τ to be *negative*, in effect subsidizing the rich and taxing the poor![29]

To be more specific, partially differentiating $\tilde{c}^*$ with respect to τ yields

$$\begin{aligned}
\frac{\partial \tilde{c}^*}{\partial \tau} &= \frac{r(\tilde{k})\tilde{k}}{1-\theta} + \left\{ w'(\tilde{k}) + \frac{\tau}{1-\theta}\left[r'(\tilde{k})\tilde{k} + r(\tilde{k}) \right] \right\} \frac{\partial \tilde{k}}{\partial \tau} \\
&= \frac{r(\tilde{k})\tilde{k}}{1-\theta} + \frac{\tau r(\tilde{k}) + (1-\theta-\tau) w'(\tilde{k})}{1-\theta} \cdot \frac{\partial \tilde{k}}{\partial \tau},
\end{aligned}$$

where the first term on the RHS denotes a positive transfer effect, and the second is a negative distortion effect, which comes from the fact that levying a tax on capital reduces the capital stock in the economy.

[29] Hamada (1967) considers the effect of transfers between capitalists and workers on the latter's income on the equilibrium growth path, but with constant saving ratios. He finds that the optimal tax rate is zero, $\tau = 0$.

From Lemma 5, we see

$$\frac{\partial \tilde{c}^*}{\partial \tau} = \hat{D}\left\{\tilde{k}\left[(1-\tau-\theta)\rho'\tilde{k}f'' + \rho\rho' - (1-\tau)f''\theta\right] + (1-\theta)(1-\tau)\tilde{k}f''(\theta-\rho'\tilde{k}) \right.$$
$$\left. -\tau\rho(\theta-\rho'\tilde{k}) - \tau\tilde{k}\rho'(\rho-\theta\tilde{k}f'')\right\}$$
$$= \hat{D}\left[\rho\rho'\tilde{k} - \theta^2(1-\tau)\tilde{k}f'' - \tau\rho\theta\right], \tag{13}$$

where $\hat{D} \equiv \rho/D\theta(1-\theta)(1-\tau)^2$.

We can summarize the above results as

Proposition 5 *Given Assumption 1A, a positive capital-income tax on rich households raises the welfare of the poor under CMI, but may not do so under DMI: for a sufficiently small θ, subsidizing the rich and taxing the poor may raise all households' welfare in the steady state under DMI.*[30]

Proof. See Appendix 6.2. ■

Note that this result can be much more clearly demonstrated, if we specify as follows:

Assumption 2: The production technology takes the Cobb-Douglas form: $f(k) = Ak^{\alpha}$, $\alpha \in (0,1)$;

and

Assumption 3: $\rho(c) = B(c+1)^{-\beta}$, $\beta \in (0, 1-\alpha)$,[31]

where Assumption 3 implies that as capital accumulates, the *interest rate falls faster than the speed at which rich households become patient* (as their consumption levels rise), i.e.,

$$\left|\frac{d\rho}{\rho}\Big/\frac{d(c+1)}{c+1}\right| < \left|\frac{dR}{R}\Big/\frac{dk}{k}\right|.$$

[30] For a sufficiently high θ though, a positive tax raises the welfare of the poor under DMI, as discussed below.
[31] One can verify that Assumption 3 is consistent with (2).

Under Assumptions 2 and 3, which imply that the effect of DMI is weak, we find that for any $\tau \in (-\beta, 1-\theta)$, there exists an economy with some θ, where reducing the capital tax or raising the capital subsidy increases the steady state welfare of poor households, and for such a small value of θ, the welfare is maximized at some $\tau \in (-\beta, 0)$,[32] rather than by an income transfer to them.

The results under DMI contrast sharply with those under CMI. Poor households' steady state income is more likely to decrease under DMI than under CMI, when the tax on rich households rises (the first term in (13)). That is, a positive capital tax on rich households (with a transfer to the poor) is preferable for poor households under CMI by increasing their present and future incomes; But under DMI, it is possible that $\partial\tilde{c}^*/\partial\tau < 0$ for $\tau \geq 0$. Although the effect of DMI is not strong, subsidizing the rich and taxing the poor will raise poor households' welfare in the steady state if the population share of capitalists is sufficiently low in the economy, as might be the case in some least developing countries.

Thus, in our model with DMI, an uneven distribution of wealth may be good and it may not be necessary to reduce the income gaps across households, because for a sufficiently small θ, the Gini coefficient is low and the income transfer from the rich to the poor makes the poor worse off in the long run.

On the other hand, in a more mature economy with a higher θ (a higher fraction of population owning assets), then under Assumptions 2 and 3,[33]

$$\frac{\partial\tilde{c}^*}{\partial\tau} > 0 \text{ for any feasible } \tau \leq 0;$$

[32] See the proof of Proposition 5 in Appendix 6.2.

[33] Here, "feasible" means $i^*(\tilde{k}, \theta, \tau) = w(\tilde{k}) + \tau r(\tilde{k})\tilde{k}/(1-\theta) \geq 0$, i.e., $\tau \geq -(1-\theta)w(\tilde{k})/r(\tilde{k})\tilde{k}$, the right-hand side of which goes to zero when θ goes to 1.

that is, some positive capital tax will maximize poor households' steady state welfare. This arises because (13) yields

$$\frac{\partial \tilde{c}^*}{\partial \tau} = \hat{D}\theta \left\{ \left[\left(-\frac{\tau}{\theta}\tilde{k}f'' + \frac{\rho}{\theta} \right) \rho' - (1-\tau)f'' \right] \tilde{k} - \tau \left(\rho - \frac{\rho' \tilde{k}^2 f''}{\theta} \right) + (1-\theta)(1-\tau)\tilde{k}f'' \right\},$$

where, if θ is sufficiently close to 1, then (i). the term in [...] is positive from $D > 0$; (ii) $\rho > \rho' \tilde{k}^2 f''$ holds under Assumptions 2 and 3, and hence the second term is also non-negative given $\tau \leq 0$; and (iii) the first term in braces {...} dominates the last term when θ goes to 1 (see footnote 30). Thus, when θ is sufficiently high, the tax on capital income increases the welfare of the poor.

Intuitively, when θ is sufficiently high, there are enough rich people in the economy, and thus it is relatively easier to raise the welfare of the poor by taxing the rich on the one hand, and on the other hand, a large share of rich households implies a high capital stock, under which a capital tax does not lower investment incentives too much. As might be the case for China nowadays, when the fraction of the rich has reached a certain high level, especially in some big cities and along the coast, while the hinterland remains poor, and the government begins to shift its task focus to reduce inequality.

4.2 Effects on Income Gaps

We have considered the effect of a capital tax on welfare and shown that when positive productivity shock occurs, all households become better off in the steady state. A related issue is whether their income gap widens or not, which we investigate now.[34]

With the capital income tax, the incomes of households with and without assets in the

[34] Here we omit the effect of τ on the income gaps, which can be divided into the sum of a direct effect and an indirect effect. The indirect effect can be negative under strong DMI, when a tax hike on rich households reduces the capital stock by a large scale, and hence the wage rate falls sharply. Then it is possible for the income gap to widen, if the indirect effect dominates the direct one. However, the tax mostly narrows the gaps as in the case of CMI.

steady state become respectively

$$\tilde{I} = w(\tilde{k}) + \frac{\rho(\tilde{c})\tilde{k}}{\theta}$$
$$\text{and} \quad \tilde{I}^* = w(\tilde{k}) + \frac{\tau\rho(\tilde{c})\tilde{k}}{(1-\theta)(1-\tau)},$$

from which we have the new income gaps as

$$g = \frac{1-\theta-\tau}{\theta(1-\theta)(1-\tau)}\rho(\tilde{c})\tilde{k}$$
$$\text{or} \quad \hat{g} = \frac{\theta + \frac{\rho(\tilde{c})\tilde{k}}{w(\tilde{k})}}{\theta + \frac{\theta\tau}{(1-\theta)(1-\tau)} \cdot \frac{\rho(\tilde{c})\tilde{k}}{w(\tilde{k})}},$$

where g and $\hat{g}$ (and hence G) are increasing in $\rho(\tilde{c})\tilde{k}$ and $\rho(\tilde{c})\tilde{k}/w(\tilde{k})$ respectively.

Calculations give that under Assumptions 1A and 2,

$$\frac{\partial}{\partial A}\left[\rho(\tilde{c})\tilde{k}\right] = \frac{\rho[\rho+(1-\tau)\delta]}{AD}\left[1 + \frac{(1-\alpha)\delta}{\alpha\rho}\rho'\tilde{k}\right], \tag{14a}$$
$$\frac{\partial}{\partial A}\left[\frac{\rho(\tilde{c})\tilde{k}}{w(\tilde{k})}\right] = \frac{(1-\tau)\delta\{(1-\tau)\alpha\rho + \theta(1-\alpha)[\rho+(1-\tau)\delta]\}}{AD\theta(1-\alpha)[\rho+(1-\tau)\delta]}\rho' < 0, \tag{14b}$$

Therefore, we have

(i). under CMI,

$$g_A > 0, \quad \text{and} \quad G_A = 0; \tag{15a}$$

(ii). under DMI,

$$G_A < 0, \tag{15b}$$

where $g_A \equiv \partial g/\partial A$ and $G_A \equiv \partial G/\partial A$, respectively.

Some explanation is in order. In (i), under CMI, there is no trickle down effect, and the income gap will widen when technology advancement occurs; while in (ii) under DMI, poor households benefit more from positive shocks, which tends to reduce the Gini coefficient with a rise in productivity. Also, there is a possibility that technology improvement narrows the level-gap g under DMI. However, with Assumption 3, we have $g_A > 0$, as in the case of CMI.

Proposition 6 *Given Assumptions 1A and 2, (i). An increase in productivity A reduces the Gini coefficient G under DMI; (ii). It widens the level-gap g when DMI is not strong (i.e., when Assumption 3 holds).*

Proof. See Appendix 6.3. ■

Acemoglu (2002) argues that the widening of income gap is a result of technology improvement over the past century. Our results on the increase of A partly reconfirm his prediction: It is only true for the level gap ($g_A > 0$), but with regards to the Gini coefficient G, it is not true under DMI as shown in Proposition 6 ($G_A < 0$), due to the trickle down.

Also, Proposition 6 to some extent justifies the 'capital tax' on rich households, which is similar to what Piketty (2014) proposes. However, it has a negative effect on the steady state capital stock, and hence it may reduce poor households' income including transfers. Then it is necessary to look into the trade-off between the level of welfare and the inequality among households. From Proposition 5, the poor's welfare can be increased by the tax when θ is sufficiently high, but may be decreased when θ is sufficiently low, consistent with some of the findings in Saez and Zucman (2014) for the U.S.

5 Concluding Remarks

In an economy with intrinsic inequality to begin with, we have examined how endogenous time preference affects social inequality, with special focus on DMI. Our analysis has shown that (i) poor households tend to benefit more from positive shocks under DMI than under CMI; (ii) positive shocks widen the income gap between the rich and the poor when the effect of DMI is small; (iii) inequality may be a necessary evil, in order for a country to increase its welfare faster, especially under DMI and when the fraction of the rich is small.

In the model, the actions of the rich affect the poor but not the other way around. If the poor also becomes impatient, the whole effect in the economy is harder to predict, but as long as patience increases in wealth or consumption, then our qualitative predictions remain valid.

Our result that increasing inequality (a fall in θ) makes the poor households better off is derived in the absence of international credit market. If international lending and borrowing are available, this result may be altered, mainly because international borrowing may change the asset holdings ratio θ. For example, in some developing countries, due to the lack of capital stock, the interest rate could be higher than those in developed countries, causing capital inflow. In an extreme, if capital is completely freely mobile across countries, the rich in the developed countries will not accumulate capital and invest in their home countries but simply move capital to invest in the developing countries. Then poor households in the receiving countries will be made better off by such capital inflow due to wage increases.

Finally on an international scale, patience may impact rich and poor countries differently in terms of income inequality. Developing countries use relatively inferior technology and their capital markets are less complete, leading to lower income. An increase in patience may raise the income inequality because rich households tend to gain more, through higher interest rates when capital markets are less complete.

All these remain interesting issues to be examined in the future.

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6 Appendix

6.1 Proof of Lemma 3

We evaluate the elements of a Jacobian J in the system (Eqs. (4a)–(4d)) to study the local dynamics around the steady state. Here, we introduce the capital tax τ in the system as in

Section 4 to show that the stability of the steady state does not change. Differentiation gives,

$$
\begin{aligned}
J(x) &= \det[J - xI] \\
&= \det \begin{bmatrix} [\theta-(1-\tau)]w'+\rho-x & 0 & 0 & -1 \\ -\theta Z(1-\tau)r' & -x & 0 & Z\rho' \\ 0 & 0 & \rho-x & -Z \\ 0 & -1 & -\rho' & -M \end{bmatrix} \\
&= \Gamma(x,\theta,\tau),
\end{aligned}
$$

where $M \equiv -u'' + \mu\rho'' > 0$ and

$$
\begin{aligned}
\Gamma(x,\theta,\tau) \;\equiv\; & Mx^3 - M\{[\theta-(1-\tau)]w'+2\rho\}x^2 + (M\rho\{[\theta-(1-\tau)]w'+\rho\} \\
& -Z[\rho\rho' - \theta(1-\tau)r'])x + \rho Z\{[\theta-(1-\tau)]w'\rho' + \rho\rho' - \theta(1-\tau)r'\}.
\end{aligned}
$$

Notice that (11) implies $\Gamma(0,\theta,\tau) > 0$. This characteristic equation can be used to derive the local dynamics of the system in the neighborhood of the steady state.

It is clear from $M > 0$ and $J(0) = \Gamma(0,\theta,\tau) > 0$ that $J(x) = 0$ has at least one negative root, x_1. If $[\theta-(1-\tau)]w' + 2\rho = 0$, then

$$
\begin{aligned}
J(x) &= Mx^3 + J'(0)x + J(0) \\
&= M(x-x_1)\left[x^2 + x_1 x - \frac{J(0)}{Mx_1}\right].
\end{aligned}
$$

Thus the other two roots, x_2 and x_3, satisfy $x_2 + x_3 = -x_1 > 0$ and $x_2x_3 = -J(0)/Mx_1 > 0$, which implies that they have positive real parts.

Suppose that $[\theta-(1-\tau)]w' + 2\rho \neq 0$. Then, applying Routh's (1905) theorem, the number of the roots of $J(x) = 0$ with positive real parts equals the number of changes in signs in the following sequence:

$$M, \quad -M\{[\theta-(1-\tau)]w'+2\rho\}, \quad \frac{\Gamma([\theta-(1-\tau)]w'+2\rho,\theta,\tau)}{[\theta-(1-\tau)]w'+2\rho}, \quad \Gamma(0,\theta,\tau).$$

(i). $[\theta-(1-\tau)]w'+2\rho>0$. Then, the number of changes in signs is two irrespective of the sign of the third term.

(ii). $[\theta-(1-\tau)]w'+2\rho<0$. Then, $[\theta-(1-\tau)]w'+\rho<0$ and

$$\begin{aligned}&\frac{\Gamma([\theta-(1-\tau)]w'+2\rho,\theta,\tau)}{[\theta-(1-\tau)]w'+2\rho}\\ &=\frac{M\rho\{[\theta-(1-\tau)]w'+\rho\}\{[\theta-(1-\tau)]w'+2\rho\}+Z\theta(1-\tau)r'\{[\theta-(1-\tau)]w'+\rho\}-Z\rho^2\rho'}{[\theta-(1-\tau)]w'+2\rho}\\ &<0,\end{aligned}$$

which implies that the number of changes is two.

6.2 Proof of Proposition 5

Since

$$\frac{\partial\tilde{c}^*}{\partial\tau}=\frac{\rho\theta}{D(1-\theta)(1-\tau)^2}\left[\frac{\rho\rho'\tilde{k}}{\theta^2}-(1-\tau)\tilde{k}f''-\frac{\tau\rho}{\theta}\right], \tag{16}$$

for any $\tau\le 0$,

$$\frac{\partial\tilde{c}^*}{\partial\tau}>0 \tag{17}$$

holds in the case of CMI ($\rho'=0$).

We now show that if θ is sufficiently small, then the first term in square brackets of (16) dominates the second one, and hence we have

$$\frac{\partial\tilde{c}^*}{\partial\tau}<0 \text{ for } \tau\in[0,1-\theta) \tag{18}$$

under $\rho'<0$.

First, we see from (9) that

$$\lim_{\theta\to 0} \tilde{k} f''(\tilde{k}) = k_2 f''(k_2) \quad \text{and} \quad \lim_{\theta\to 0} \rho(\tilde{c}) = 0. \tag{19}$$

Also, we obtain $\partial\tilde{k}/\partial\theta$ and $\partial\tilde{c}/\partial\theta$ from totally differentiating $c = i(k, \theta, \tau)$ and $k = \kappa(c, \tau)$ with respect to θ as follows:

$$\frac{\partial\tilde{k}}{\partial\theta} = \frac{\rho\rho'\tilde{k}}{D\theta^2} < 0, \tag{20a}$$

$$\frac{\partial\tilde{c}}{\partial\theta} = \frac{(1-\tau)\rho\tilde{k}f''}{D\theta^2} < 0. \tag{20b}$$

Using L'Hôpital's rule, we have

$$\begin{aligned} \lim_{\theta\to 0} \frac{\rho(\tilde{c})}{\theta} &= \lim_{\theta\to 0} \rho' \frac{\partial\tilde{c}}{\partial\theta} \\ &= \lim_{\theta\to 0} \frac{(1-\tau)\tilde{k}f''}{\frac{\theta}{\rho}(1-\tau-\theta)\tilde{k}f'' + \theta - \frac{\theta^2}{\rho\rho'}(1-\tau)f''}. \end{aligned} \tag{21}$$

However, (9) and the fact that $\tilde{c} = w(\tilde{k}) + \rho(\tilde{c})\tilde{k}/\theta$ together imply

$$\lim_{\theta\to 0} \frac{\rho(\tilde{c})}{\theta} = \infty. \tag{22}$$

From (21) and (22), we have

$$\lim_{\theta\to 0} \frac{\rho(\tilde{c})\rho'(\tilde{c})}{\theta^2} = -\infty. \tag{23}$$

If θ is sufficiently small, then (18) holds, which proves Proposition 5.

Finally, under Assumptions 2 and 3, we have

$$\begin{aligned}\lim_{\theta\to 0}\left[-\frac{\rho'(\tilde{c})\tilde{k}}{\theta}\right] &= \lim_{\theta\to 0}\frac{\beta\tilde{k}(1-\tau)r(\tilde{k})}{\theta(\tilde{c}+1)} \\ &= \lim_{\theta\to 0}\frac{\beta\tilde{k}(1-\tau)r(\tilde{k})}{\theta w(\tilde{k})+\tilde{k}(1-\tau)r(\tilde{k})+\theta} \\ &= \beta. \end{aligned} \tag{24}$$

Since (16) yields

$$\frac{\partial\tilde{c}^*}{\partial\tau} = -\frac{\rho\theta}{D(1-\theta)(1-\tau)^2}\left[(1-\tau)\tilde{k}f'' + \frac{\rho}{\theta}\left(\tau - \frac{\rho'\tilde{k}}{\theta}\right)\right],$$

we see from (22) and (24) that for any $\tau > -\beta$, there exists some θ such that

$$\frac{\partial\tilde{c}^*}{\partial\tau} < 0.$$

Also, if $\tau \leq -\beta$, then $\partial\tilde{c}^*/\partial\tau > 0$ holds for any θ : the value of τ that maximizes the steady state levels of welfare for poor households must be greater than $-\beta$. This is because

$$\begin{aligned}\frac{\partial}{\partial\theta}\left[\frac{\rho'(\tilde{c})\tilde{k}}{\theta}\right] &= \frac{(\rho''\frac{\partial\tilde{c}}{\partial\theta}\tilde{k}+\rho'\frac{\partial\tilde{k}}{\partial\theta})\theta-\rho'\tilde{k}}{\theta^2} \\ &= \frac{(\rho')^2\tilde{k}f''}{D\theta^3}\left[\frac{\rho\rho''}{(\rho')^2}(1-\tau)\tilde{k}-(1-\tau-\theta)\tilde{k}+\frac{\theta(1-\tau)}{\rho'}\right] \\ &= \frac{(\rho')^2\tilde{k}f''}{D\theta^3\beta\rho}\Big([(\beta+1)(1-\tau)-\beta(1-\tau-\theta)](1-\tau)(\alpha A\tilde{k}^\alpha-\delta\tilde{k}) \\ &\quad -(1-\tau)\{A\tilde{k}^\alpha[(1-\alpha)\theta+\alpha(1-\tau)]-(1-\tau)\delta\tilde{k}+\theta\}\Big) \\ &= -\frac{(1-\tau)(\rho')^2\tilde{k}f''}{D\theta^2\beta\rho}[(1-\alpha-\alpha\beta)A\tilde{k}^\alpha+\beta\delta\tilde{k}+1] \\ &> 0.\end{aligned}$$

6.3 Proof of Proposition 6

We show that under Assumptions 2 and 3,

$$\frac{\rho'(\tilde{c})\tilde{k}}{\rho(\tilde{c})} > -\frac{\alpha\beta}{(1-\alpha)\delta} \tag{25}$$

holds for any pair of parameters, and hence g_A must be positive due to (14a) and $\beta < 1-\alpha$. Under Assumption 3, we have

$$\frac{\rho'(\tilde{c})\tilde{k}}{\rho(\tilde{c})} = -\frac{\beta\tilde{k}}{\tilde{c}+1} \tag{26}$$

and from Lemma 5 and Assumption 2,

$$\frac{\partial}{\partial A}\left(-\frac{\beta\tilde{k}}{\tilde{c}+1}\right) = -\frac{\beta\tilde{k}^{\alpha-1}\{[\alpha(1-\tau)+(1-\alpha)\theta]\beta\rho(\tilde{c})\tilde{k}+\alpha(1-\tau)\theta\}}{D\theta(\tilde{c}+1)^2} < 0. \tag{27}$$

Next we show that

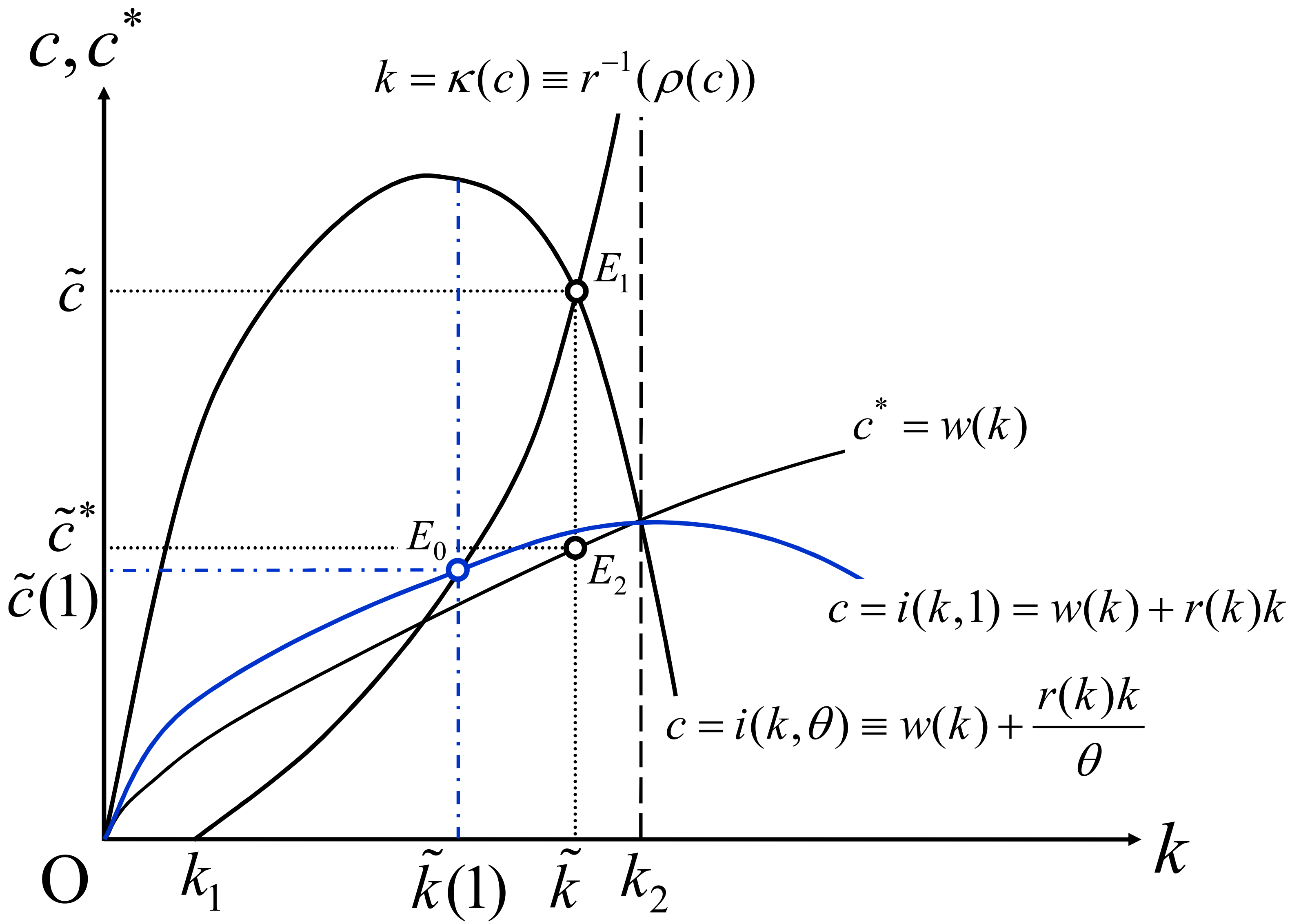
$$\lim_{A\to\infty}\left(-\frac{\beta\tilde{k}}{\tilde{c}+1}\right) = -\frac{\alpha\beta}{(1-\alpha)\delta}. \tag{28}$$

One can easily verify that $\lim_{A\to\infty}\tilde{k} = \lim_{A\to\infty}\tilde{c} = \infty$. This implies $\lim_{A\to\infty}A\hat{f}'(\tilde{k}) = \delta$, because $\lim_{c\to\infty}\rho(c) = 0$ and $\rho(c) = (1-\tau)[A\hat{f}'(k)-\delta]$ hold at any steady state. Then (28) holds since

$$\frac{\tilde{k}}{\tilde{c}+1} = \frac{1}{(1-\alpha)A\hat{f}'(\tilde{k})/\alpha+(1-\tau)[A\hat{f}'(\tilde{k})-\delta]/\theta+1/\tilde{k}}.$$

From (26)–(28), we may conclude that (25) holds for any pair of parameters.

Figure 1



c, c^*
$k = \kappa(c) \equiv r^{-1}(\rho(c))$
$\tilde{c}$
E_1
$c^* = w(k)$
$\tilde{c}^*$
E_0
E_2
$\tilde{c}(1)$
$c = i(k,1) = w(k) + r(k)k$
$c = i(k,\theta) \equiv w(k) + \frac{r(k)k}{\theta}$
O
k_1
$\tilde{k}(1)$
$\tilde{k}$
k_2
k