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Algebraic approach towards the exploitation of "softness": the input–output equation for morphological computation

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Abstract

Recently, soft robots that consist of soft and deformable materials have received much attention for their adaptability to uncertain environments. Although these robots are difficult to control with a conventional control theory due to their complex body dynamics, research from different perspectives attempts to actively exploit these body dynamics as an asset rather than a drawback. This approach is called morphological computation, in which, the soft materials are used for computation that includes a new kind of control strategy. In this paper, we propose a novel approach to analyze the computational properties of soft materials based on an algebraic method, called the *input–output equation* used in systems analysis, particularly in systems biology. We mainly focus on the two scenarios relevant to soft robotics, that is, analysis of the computational capabilities of soft materials and design of the input force to soft devices to generate the target behaviors. The input–output equation directly describes the relationship between inputs and outputs of a system, and hence by using this equation, important properties, such as the echo state property that guarantees reproducible responses against the same input stream, can be investigated for soft structures. Several application scenarios of our proposed method are demonstrated using typical soft robotic settings in detail, including linear/nonlinear models and hydrogels driven by chemical reactions.

Keywords

Soft robot, Morphological computation, Echo state property, Input-output equation, Gröbner basis

Introduction

Much attention has been given lately to soft robots, which are made of soft and deformable materials (see Rus and Tolley (2015); Laschi, Mazzolai and Cianchetti (2016); Bao, Fang, Chen, Wan, Xu, Yang and Zhang (2018).) Soft robots can often deal with tasks that are difficult for conventional rigid robots, such as locomotion in rough terrain (e.g., Shepherd, Ilievski, Choi, Morin, Stokes, Mazzeo, Chen, Wang and Whitesides (2011)) and manipulation of objects whose structures are unknown beforehand (e.g., Brown, Rodenberg, Amend, Mozeika, Steltz, Zakin, Lipson and Jaeger (2010)). Their application domains are still developing (Pfeifer, Lungarella and Iida (2012)).

In spite of the increasing popularity and success of this research stream, soft robots are difficult to control in general with the conventional control theory, and hence a new approach is required (George Thuruthel, Ansari, Falotico and Laschi (2018)). In this regard, a framework called *morphological computation* has been attracting attention (see Paul, Valero-Cuevas and Lipson (2006); Pfeifer, Lungarella and Iida (2007)), in which the softness is directly exploited as a computational resource. In fact, morphological computation was motivated by some observations of living creatures in nature. For example, creatures like octopuses have soft bodies and can control them without any difficulty (Kim, Laschi and Trimmer (2013); Li, Nakajima, Kuba, Gutnick, Hochner and Pfeifer (2012); Hochner (2012)). In Müller and Hoffmann (2017), a number of real-world examples are provided and then exploitation of morphology is suggested to be categorized into the three groups: (a) morphology facilitating control, (b) morphology facilitating perception and (c) morphological (in a sense, purely) computation. Research in the first direction includes Caluwaerts, D'Haene, Verstraeten and Schrauwen (2013); Füchslin, Dzyakanchuk, Flumini, Hauser, Hunt, Luchsinger, Reller, Scheidegger and Walker (2013); Moore, Ober-Blöbaum and Marsden (2012); Rückert and Neumann (2013), in which, for example, asymptotic behaviors of the systems are controlled by appropriately choosing certain parameters of the systems. Examples of (b) include artificial compound eyes shown in Floreano, Pericet-Camara, Viollet, Ruffier, Brückner, Leitel, Buss, Menouni, Expert, Juston, Dobrzynski, L'Eplattenier, Recktenwald, Mallot and Franceschini (2013), and of (c) the octopus-like device that can predict a certain class of time series (Kim, Laschi and Trimmer (2013); Li, Nakajima, Kuba, Gutnick, Hochner and Pfeifer (2012); Hochner (2012)). In this paper, we propose a novel approach to analyze the computational capabilities of soft structures used in morphological computation, mainly

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of the type (c); however, we also show an example where the input force is designed for the position control of a model of elastic rods, which is relevant to (a) to a certain extent.

The theoretical background of morphological computation of the type (c) (which is abbreviated as morphological computation in the following for simplicity) is given in Hauser, Ijspeert, Füchslin, Pfeifer and Maass (2011). Their approach is based on the theory of approximations of nonlinear filters given by Boyd and Chua (1985). A filter is a map that maps an input stream (e.g., information from sensors) to an output stream (e.g., control signal), or mathematically, a map from a space of functions to the set of real numbers $\mathbb{R}$. For example, approximation of Volterra series operators with the parameters $\sigma_1, \sigma_2, \mu_1, \mu_2$

$$y(t) = \mathcal{V}u(t) = \int_0^\infty \int_0^\infty h(r,s)u(t-s)u(t-r)\mathrm{d}s\mathrm{d}r,$$
$$h(r,s) = \exp((r-\mu_1)^2/2\sigma_1^2 + (s-\mu_2)^2/2\sigma_2^2)$$

is considered in Hauser, Ijspeert, Füchslin, Pfeifer and Maass (2011). Many target tasks in morphological computation can be considered as approximations of filters; however, morphological computation of other two categories (a) and (b) cannot always be implemented as filters. The paper by Boyd and Chua shows that a causal time-invariant filter with fading memory can be approximated by polynomials of outputs of huge linear systems. It is also shown that the approximate polynomials can be rewritten as a linear combination of outputs of certain nonlinear systems. This result can be a theoretical basis of the feasibility of morphological computation in that this shows the possibility of approximation of various causal time-invariant filters with fading memory by combining the outputs of nonlinear dynamical systems with complicated motions. It has been confirmed experimentally, such as by using octopus-inspired soft robotic arms, that many temporal machine learning tasks can be indeed implemented via morphological computation; for example, soft body dynamics have been shown to be capable of emulating nonlinear dynamical systems (Nakajima, Li, Hauser and Pfeifer 2014; Nakajima, Hauser, Li and Pfeifer 2015, 2018) and can be used to embed robust closed-loop control into the body (Nakajima, Li, Hauser and Pfeifer 2014; Zhao, Nakajima, Sumioka, Hauser and Pfeifer 2013).

This view of morphological computation is also compatible with the framework of reservoir computing, in which soft materials are considered as physical reservoirs. In fact, the theory of Boyd and Chua is also the basis of the theory of reservoir computing, which is a method for training recurrent neural networks (see, e.g., Jaeger (2001); Maass, Natschläger and Markram (2002); Jaeger and Haas (2004); Jaeger, Lukoševičius, Popovici and Siewert (2007); Lukoševičius and Jaeger (2009).) Recurrent neural networks are neural networks with a characteristic property, specifically, the presence of loops inside the network. Due to this property, these neural networks can learn time series, such as natural languages; however, designing a learning algorithm for such networks is difficult compared with algorithms for forward neural networks. Reservoir computing has received much attention as a simple but reliable approach to achieve this.

In reservoir computing, first a complex and huge neural network, which is called a reservoir, is prepared in advance. Then the target time series are predicted by a linear combination of outputs of the reservoir. In particular, in the learning stage only the output weights are learned, and the structure of the reservoir including the connection weights is unchanged. In the theory of reservoir computing, for example, a property called the *echo state property* (Jaeger (2001); Yildiz, Jaeger and Kiebel (2012)) is considered to be essential for a reservoir to have sufficient computational ability. Therefore, it is beneficial to investigate whether and when nonlinear systems used in morphological computation have the echo state property.

The theorem by Boyd and Chua is proved by a method of functional analysis and in this sense the existing approach described above is analytical. In this paper, we propose an alternative *algebraic* approach to morphological computation. The proposed approach is based on the technique of model identifiability analysis, which has been studied in the field of mathematical modeling, in particular, in systems biology (see Meshkat and Sullivant (2014); Meshkat, Rosen and Sullivant (2018)). In the model identifiability problem, the main interest is in whether the values of parameters included in mathematical models can be identified from observable data. Among the various existing methods for identifiability analysis, we focus on the method using computational differential algebra (Ritt (1950); Kolchin (1973)). The main tool of this method is the *input–output equation*, which is an equation that describes the hidden relations in a given mathematical model between the input variables and the output variables. This equation is obtained by eliminating the internal variables from the system of model equations while retaining the input and output variables in a given mathematical model. Substitution of the observed data into the input–output equation yields a system of equations with the parameters as the unknown variables. The identifiability of the parameters can be determined by analyzing the uniqueness of the solutions for this system of equations.

In this paper, we employ this technique to analyze the computational ability of systems used in morphological computation. In morphological computation, it is difficult to determine what relationship exists between the solvable tasks by a given system and the values of the physical parameters, for example, the spring constants and damping coefficients in mass–spring–damper systems.

However, the theoretical analysis of morphological computation and model identifiability analysis share the following common objectives. If the target tasks of morphological computation are restricted to approximation of time-invariant filters, the relationship between the computational ability of the systems and the values of the parameters is essentially revealed by determining the physical parameters so that the outputs of the systems can well approximate the target filters. Hence, the objective of the theoretical analysis of the time-series prediction by morphological computation is almost the same as determining the parameters of mathematical models so as to approximate the given observation data, which is the aim of model identifiability analysis. Therefore, the various existing techniques in model identifiability analysis must be useful for the theoretical analysis of morphological computation. Besides, the input–output equation can be used for designing the input function so that the output function becomes an well approximation of the target function. With this feature, the proposed approach has a potential application to control of soft devices.

In this paper, we show that the techniques, particularly, the input–output equations in model identifiability analysis, are in fact useful for analysis of morphological computation.

In the next section, we will explain how the input–output equation is used for analysis in morphological computation. The equation is derived by an algebraic method in which the *ideal*, which is a certain set of equations, is investigated using the *Gröbner basis*. We summarize the mathematical topics needed for understanding the method in Appendix 1 and 5. See also Hibi (2014) for details of the algebraic theory. Some applications to analysis of morphological computations of soft materials are shown, followed by the application to designing the input force for a model of elastic rods.

Methodology

In this section, we explain the input–output equation and why this equation is useful for analyzing the properties of the system. The detailed algebraic theory is summarized in Appendix 1 along with a list of mathematical terms in Appendix 5.

The input–output equation has been extensively studied especially in systems biology. This equation has been used for the parameter identifiability problem, in particular, to determine whether the parameters in a model are identified from the observed data; however, its application to other research area is limited. In this paper, we use this equation for the analysis of morphological computation. In addition, although here we focus on the morphological computation for approximation of time-invariant filters, an application to designing the input function to achieve the target outputs will be shown in the later section.

As a toy example, let us consider the following state-space model with an input and an output:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = a_{11}x_1 + a_{12}x_2 + u, \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = a_{21}x_1 + a_{22}x_2, \\ y = w_1 x_1 + w_2 x_2. \tag{1}$$

$x_1(t), x_2(t)$ are the state variables. $u(t)$ and $y(t)$ are the input and the output, respectively. $a_{11}, a_{12}, a_{21}, a_{22}$ are parameters of the system. w_1 and w_2 are the weights which are assumed to be user-specified. We may regard this system as a filter that maps the input function u to the output function y. Fundamental questions regarding filters introduced in this way are the *echo state property* (Jaeger (2001); Yildiz, Jaeger and Kiebel (2012)), the *point-wise separation property* (Boyd and Chua (1985); Maass, Natschläger and Markram (2002); Maass and Markram (2004)) and the *fading memory*. The echo state property refers to the asymptotic uniqueness of the output, that is, a system has the echo state property if the state of the system is uniquely determined by the input asymptotically. A set of filters is said to have the point-wise separation property if, for any pair of inputs u_1 and u_2 there exists a filter $\mathcal{F}$ in this set such that $\mathcal{F}(u_1) \neq \mathcal{F}(u_2)$. A filter that has weak dependence on the past data is said to have a fading memory. More precisely, a filter is said to have a fading memory if it is continuous with respect to a weighted norm of which weight is exponentially decay as the time goes back to the past. If a set of filters with both the point-wise separation property and the fading memory is combined with a system with the universal approximation property (e.g., artificial neural networks), then any nonlinear filters with fading memory can be approximated (Maass, Natschläger and Markram (2002); Maass and Markram (2004)).

In this paper, we propose a method to analyze the filters introduced by using physical systems from a slightly different point of view. While we consider the echo state property, which is certainly important because systems without this property do not yield a unique output from a given input, the above theoretical framework may not suitable for situations where the system under consideration can approximate only a certain limited class of filters. In the following, we propose an algebraic method to directly describe the filter defined by a given system, which may have a limited computational capability, thereby investigating the properties of the filters that can be approximated by the system. In other words, if a reservoir is considered as a bank of filters, the proposed method is useful to see what filters are in the bank.

To investigate the properties of the filter, it is useful to rewrite the equation as

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - (a_{11}+a_{22})\frac{\mathrm{d}y}{\mathrm{d}t} + (a_{11}a_{22} - a_{12}a_{21})y \\ = w_1\frac{\mathrm{d}u}{\mathrm{d}t} + (-w_1a_{22} + w_2a_{21})u \tag{2}$$

by eliminating the internal variables x_1 and x_2 and their derivatives. This equation is called the *input–output equation* in the studies of the parameter identifiability problem. Indeed, the filter $\mathcal{F}$ defined by the system (1) is essentially the map $\mathcal{F}$ that maps the input function $u(t)$ to the solution $y(t)$ of (2). Hence the echo state property is related to the asymptotic uniqueness of the solution y to the input–output equation. Besides, the set of filters that can be introduced by using this system can be investigated by analyzing the space (or the manifold if the equation is nonlinear) of the solutions y, that is, the dependence of y on the weights w_1 and w_2.

Moreover, (2) is also used to design the input function so that the output of the system behaves in a desired way. As an example, in the later section we use the input–output equation for position control of a model of elastic rods. Another example is designing the parameters of the system so that the system have, for instance, an attractor suitable for the situation under consideration. In particular, for those purpose, one must choose the only 4 parameters $c_1 = a_{11} + a_{22}, c_2 = a_{11}a_{22} - a_{12}a_{21}, c_3 = w_1, c_4 = -w_1a_{22} + w_2a_{21}$ instead of each $a_{11}, a_{12}, a_{21}, a_{22}, w_1, w_2$, because the same c_1, c_2, c_3, c_4 results in the same dynamics of the output y.

In the following section of this paper, we consider the computational capabilities of systems described by the state-space model

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x;\theta) + u, \quad y = g(x;\theta), \tag{3}$$

where $x : \mathbb{R} \to \mathbb{R}^n$ is a function that depends on time t, $u : \mathbb{R} \to \mathbb{R}^m$ and $y : \mathbb{R} \to \mathbb{R}$ are the input and the output of the system respectively, both of which are function of time t. f and g are nonlinear functions that depend on the parameter $\theta \in \mathbb{R}^l$. Here we assume that f and g are polynomials of x. Note that this assumption is essential because we employ an algebraic approach and algebra is basically a mathematical theory for polynomials. If they are not polynomials, the model equations should be reformulated or approximated to be polynomials or piecewise polynomials by using, for example, the Taylor series expansion. There is a tradeoff between the number of terms used in the approximation and the computational time; however, the increase of the computational time is actually highly dependent on each specific state-space model. Finding an effective reformulation is an important future work.

Under the above assumption, the input–output equation for (3) is obtained by using an algebraic method: *the Gröbner basis in differential algebra* as summarized in Appendix 1. It is known as the elimination theorem that specified variables (e.g., state variables x_1 and x_2 in the above example) can be eliminated from a given system of polynomial equations by computing the Gröbner basis. This procedure is regarded as an extension of the Gaussian elimination for solving systems of linear equations to systems of polynomial equations. In particular, differential algebra is an algebraic framework in which the differentiation is allowed as an algebraic operation. Hence by using the Gröbner basis in differential algebra, the specified variables can be eliminated from the given system of differential equations. The input–output equation is obtained by eliminating the state variables from the model of the system under consideration.

This procedure of eliminating variables can be performed by computer algebra software, such as Singular, Magma, Mathematica, Maple. An example of a series of commands of Singular for computation of the Gröbner basis is shown in Remark 4 in Appendix 1. For details, see the summary of the algebraic theory and Example 1 in Appendix 1.

In the following sections, we illustrate this algebraic approach using some examples.

Illustration of Proposed Approach for Linear Systems

In the following sections, applications of the proposed method are shown. In this section, a general method to investigate computational abilities of linear systems are illustrated. Because the examples considered here are well-examined, some of the results (e.g., the echo state property of mass-spring systems) have already been established; however we use these examples for ease of understanding the proposed approach. As an example, a system of equations which describes two connected springs one of which is fixed to the wall (see Figure 1) is considered:

$$\begin{aligned} m\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} &= -k(x_1 - l) + k(x_2 - x_1 - l) - \gamma\frac{\mathrm{d}x_1}{\mathrm{d}t} + u, \\ m\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} &= -k(x_2 - x_1 - l) - \gamma\frac{\mathrm{d}x_2}{\mathrm{d}t}. \end{aligned} \tag{4}$$

We assume that the output from the system is given as

$$y(t) = w_1 x_1(t) + w_2 x_2(t).$$

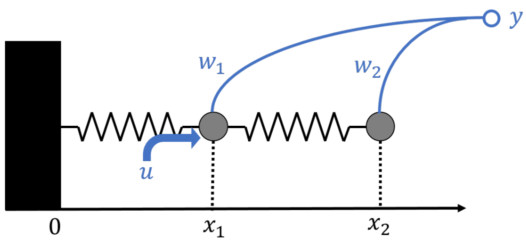


Figure 1. Two connected springs investigated in the 3rd section. One of the springs is fixed to a wall.

In this section, we consider the task of predicting time series $s(t)$ using the output of the system. Basically, this task is reduced to the problem of finding the values of the parameters w_1, w_2 so that $\|s(t) - y(t)\|$ is as small as possible. Here we consider what kind of functions can be well approximated by using this system, in other words, how many filters are in the bank of filters of this system as a reservoir.

The input–output equation of the system is obtained as follows:

$$\begin{aligned} & m^2\frac{\mathrm{d}^4 y}{\mathrm{d}t^4} + 2m\gamma\frac{\mathrm{d}^3 y}{\mathrm{d}t^3} + (3km + \gamma^2)\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 3k\gamma\frac{\mathrm{d}y}{\mathrm{d}t} + k^2 y \\ & = w_1((m\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}u}{\mathrm{d}t} + ku) + k^2 l) + w_2(ku + 2k^2 l). \end{aligned} \tag{5}$$

In the following, applications of the input–output equation are described using two types of systems: systems with and without input. Firstly, for a system without an input, (5) is rearranged to

$$\begin{aligned} & m^2\frac{\mathrm{d}^4 y}{\mathrm{d}t^4} + 2m\gamma\frac{\mathrm{d}^3 y}{\mathrm{d}t^3} + (3km + \gamma^2)\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 3k\gamma\frac{\mathrm{d}y}{\mathrm{d}t} + k^2 y \\ & = (w_1 + 2w_2)k^2 l. \end{aligned} \tag{6}$$

Remark 1. *Because the derived input–output equation often contains high-order derivatives, it is hard to interpret the physical meaning of the equation unlike the given model itself. However, the input–output equation is derived from the given model and hence it is equivalent to the model. Therefore, in fact, the input–output equation makes sense under the assumption that the given model is physically plausible.*

The functions that can be approximated by this system are the particular solution of (6) and the functions in

$$\mathrm{Ker}\left(m^2\frac{\mathrm{d}^4}{\mathrm{d}t^4} + 2m\gamma\frac{\mathrm{d}^3}{\mathrm{d}t^3} + (3km + \gamma^2)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 3k\gamma\frac{\mathrm{d}}{\mathrm{d}t} + k^2\mathcal{I}\right) \tag{7}$$

where I is the identity mapping.

The operator in (7) is a 4th order differential operator and hence the dimension of general solutions is 4. This is considered to be reasonable because the system consists of two point masses having two degrees of freedom, the position and the velocity, for each mass point.

Furthermore, changes in the values of the output weights w_1, w_2 do not affect (7), which accounts for a large part of possible approximated functions, but only affect the particular solution. In addition, the particular solution is affected by w_1, w_2 only through the term $(w_1+2w_2)k^2l$, which means that the system essentially has just one parameter (w_1+2w_2), that is, w_1 and w_2 with the same value of w_1+2w_2 yield the system with the same computational capability. In terms of reservoir computing, this means that although the system has the two parameters w_1 and w_2, the dimension of the corresponding bank of filters is reduced to 1. This result can be generalized to other linear systems.

Theorem 1. *Consider systems described by a linear state-space model*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + u, \quad y = w \cdot x, \tag{8}$$

where $x = (x_1, \ldots, x_n)$, $u = (u_1, \ldots, u_n)$, $w = (w_1, \ldots w_n)$ *and* A *is an* $n \times n$ *matrix. Suppose that the matrix* A *is irreducible. Then the weights* w *do not affect the general solutions of the input–output equation. In addition, the particular solution to the system is determined by*

$$\begin{aligned} &w_1 \det(\frac{\mathrm{d}}{\mathrm{d}t}I - A_1) + \cdots + w_n \det(\frac{\mathrm{d}}{\mathrm{d}t}I - A_n) \\ &= C_n \frac{\mathrm{d}^n u}{\mathrm{d}t^n} + \cdots + C_1 \frac{\mathrm{d}u}{\mathrm{d}t} + C_0 u, \end{aligned}$$

where A_j *is the matrix obtained by replacing the* j*th column with the vector* u *and the matrices* $C_n, C_{n-1}, \ldots, C_0$ *depend on* $w_1, \ldots, w_n$ *and the matrix* A.

A proof of this theorem is provided in Appendix 2. Besides, the explicit expression of the essential polynomials in terms of the weight parameters for the dynamics of the output, e.g., w_1+2w_2 in the above example, is obtained from $C_0, \ldots, C_n$. The number of independent conponents of C_j's implies the dimension of the bank of filters. In addition, the explicit expression $C_0, \ldots, C_n$ will be useful for implementations of morphological filters for real-world applications, for example, as shown in Remark 2 below.

Remark 2. *In Section 3.3.2 of Füchslin, Dzyakanchuk, Flumini, Hauser, Hunt, Luchsinger, Reller, Scheidegger and Walker (2013), an application of morphological control for radio-oncology is shown. They considered a model of tumor dynamics described by a system of differential equations with parameters called the dose equivalent. In the paper, they suggested to use the dose equivalent as a true parameter of the system. The input–output equation can be used for deriving these true parameters of systems. In fact, the essential polynomials of the weight parameters, such as* w_1+2w_2 *in the example, are regarded as the true parameters of the systems.*

Secondly, for the system with an input, (5) can be regarded as an equation that can find input u given a target function y so that the equation has the desired output y as its particular solution. In this case, if all of the functions in (7) decay over $t \to \infty$, the behavior of the solution after a sufficiently long time is determined only by the input, which implies that the system has the echo state property. Actually, in this case, the system has the particular solution alone as its solution after a long time, and thereby the target function can be approximated by the output by giving the input determined by (5). As mentioned above, the echo state property can be confirmed by analyzing the dissipation property of the system over $t \to \infty$. For that, the characteristic polynomial of the differential equation that is defined by the left hand side of (5) needs to have zeros in the left half plane. This can be checked by the Routh–Hurwitz stability criterion. The characteristic equation of the system is as follows:

$$m^2s^4 + 2m\gamma s^3 + (3km+\gamma^2)s^2 + 3k\gamma s + k^2 = 0. \tag{9}$$

To guarantee the existence of roots of the equation in the left half plane, the followings need to be satisfied:

- the coefficient of each term in the equation is not 0,
- all of the coefficients have the same sign, and
- all of the following determinants are positive:

$$D_1 = \left|2m\gamma\right|, \quad D_2 = \begin{vmatrix} 2m\gamma & 3k\gamma \\ m^2 & 3km+\gamma^2 \end{vmatrix},$$

$$D_3 = \begin{vmatrix} 2m\gamma & 3k\gamma & 0 \\ m^2 & 3km+\gamma^2 & k^2 \\ 0 & 2m\gamma & 3k\gamma \end{vmatrix},$$

$$D_4 = \begin{vmatrix} 2m\gamma & 3k\gamma & 0 & 0 \\ m^2 & 3km+\gamma^2 & k^2 & 0 \\ 0 & 2m\gamma & 3k\gamma & 0 \\ 0 & m^2 & 3km+\gamma^2 & k^2 \end{vmatrix}.$$

Because the first two conditions are clearly satisfied, we only need to check the third one. D_1 is obviously positive and because $D_4 = k^2 D_3$ the positivities of D_2, D_3 are the remaining concerns. D_2 is simplified as follows:

$$D_2 = 2m\gamma(3km+\gamma^2) - 3km^2\gamma = m\gamma(2\gamma^2+3km).$$

Therefore to make D_2 positive, $\gamma > 0$ must be satisfied. Similarly, for D_3 from

$$\begin{aligned} D_3 &= 2m\gamma(3k\gamma(3km+\gamma^2) - 2m\gamma k^2) - 9m^2k^2\gamma^2 \\ &= mk\gamma^2(5km+6\gamma^2) \end{aligned}$$

the same condition $\gamma > 0$ is sufficient as well. In other words, if $\gamma > 0$ is satisfied, the system always admits the echo state property.

As an illustration we provide a numerical example. We used the system with $m=1$, $k=1.5$, and $\gamma=1.0$ for predicting the function $11+\sin t$. The input to the system is derived by the input–output equation (5). The equation of motion (4) is solved by the Python function odeint with the initial condition $x_1(0)=5.5, v_1(0)=0, x_2(0)=5.5, v_2(0)=0$. Note that the frequencies of the springs in the system and the target function are different. Hence, to approximate the target function, a carefully designed input is required.

The result is shown in Figure 2. The target function $11+\sin t$ and the output of the system are shown by the red and blue curves, respectively. As shown in the figure, at the early stage of the computation ($t<8$), we can see a gap between the two curves. However, this gap becomes smaller as time proceeds and the target function is successfully approximated. This result implies that the system acquires the echo state property.

Remark 3. *The above procedure is also applied to designing the input for a kind of control problems, which will be illustrated in the later section.*

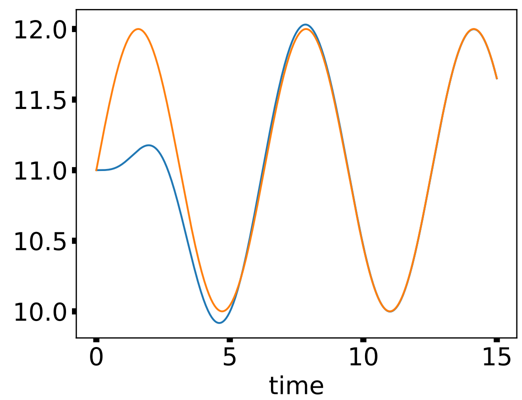


Figure 2. The target function(red curve) and system output with two linear springs along with the input determined by the input–output equation (5) (blue curve.)

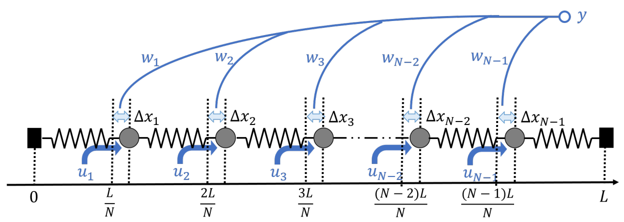


Figure 3. One dimensional mass-spring-damper array, as analyzed in this paper.

Application to Analysis of Linear Mass–Spring Array Models

In this section, the behaviors of 1-dimensional soft materials are analyzed as another application of the proposed method. The softness of the material, which has a total length of L, is modeled by the 1-dimensional mass-spring-damper array, which consists of $N+1$ point masses, of which the two masses at both ends are fixed (see Figure 3). The physical parameters are the spring constant k, the natural length l_0 and damper coefficient γ of the springs, and the point masses m. This model corresponds to an object such as a 1-dimensional tensegrity that consists of flexible and elastic materials with a fixed length. We assume that the natural length of springs is small enough and the distances between point masses $l = L/N$ are larger than l_0. In this situation, as N becomes smaller, the springs become less flexible and hence the system becomes firmer. That is to say, the larger N is, the softer the system becomes and hence we may regard N as the measure of the softness of the materials.

The position of the point mass is given as $x_j(t) = jL/N + \Delta x_j$. Note that both ends are fixed, as mentioned before: $x_0 = 0, x_N = L$. We assume that an external force u_j is applied to each point mass as the input to the system and the output is given as a linear combination of displacements of the point mass: $y = w_j \Delta x_j$.

The input–output equation in the case of $N = 6$ is derived in Appendix 3. The solution $y(t)$ to the equation is a linear combination of $\exp(st)$ with

$$s = \frac{-\tilde{\gamma} \pm \sqrt{\tilde{\gamma}^2 - 12\tilde{k}}}{2}, \frac{-\tilde{\gamma} \pm \sqrt{\tilde{\gamma}^2 - 8\tilde{k}}}{2},$$

$$\frac{-\tilde{\gamma} \pm \sqrt{\tilde{\gamma}^2 - 4\tilde{k}}}{2}, \frac{-\tilde{\gamma} \pm \sqrt{\tilde{\gamma}^2 - 8\tilde{k} \pm 4\sqrt{3}\tilde{k}}}{2}.$$

See Appendix 3 for details. From these, it is seen that s can be imaginary numbers or negative real numbers. In particular, when the damping term is strong so that $\tilde{\gamma} > 12\tilde{k}$, all of the values of s become negative real numbers. In this case, as for applications to morphological computation, such systems are able to approximate only monotonically decreasing systems. When $\tilde{\gamma}$ is small, s can be approximated as

$$s \simeq \pm\frac{\sqrt{-12\tilde{k}}}{2}, \frac{\pm\sqrt{-8\tilde{k}}}{2}, \frac{\pm\sqrt{-4\tilde{k}}}{2}, \frac{\pm\sqrt{-8\tilde{k} \pm 4\sqrt{3}\tilde{k}}}{2}.$$

By noting that $4\sqrt{3} \simeq 7.0$, the expressions can be rearranged as

$$s \simeq \pm\frac{\sqrt{-15\tilde{k}}}{2}, \frac{\sqrt{-12\tilde{k}}}{2}, \frac{\pm\sqrt{-8\tilde{k}}}{2}, \frac{\pm\sqrt{-4\tilde{k}}}{2}, \frac{\pm\sqrt{-\tilde{k}}}{2},$$

which means that the solution s uniformly cover a wide range of frequency. To satisfy the assumption, however, $\tilde{\gamma}$ needs to be considerably small. If not, the modes that correspond to $\pm\sqrt{-\tilde{k}}/2$ are negative real numbers. According to this, it is considered that the system with large damping coefficients is able to approximate high-frequency modes easily, but it has difficulties in approximating low frequencies.

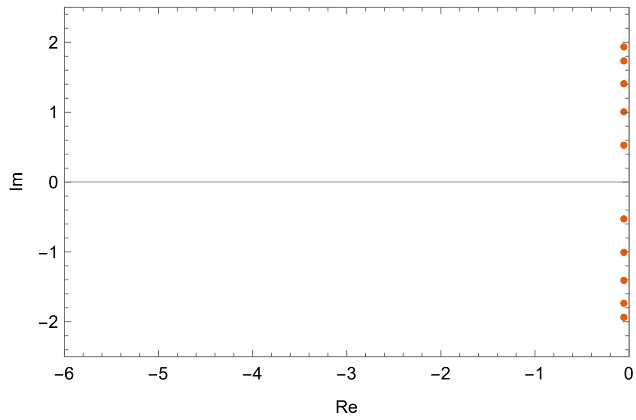
To confirm this, with $\tilde{k}$ fixed to 1. we derive the characteristic equations of the input–output equations with different N and $\tilde{\gamma}$ values. The solution of the characteristic equations are plotted on the complex plane and shown in Figure 4–6. As expected, for each N, if $\tilde{\gamma}$ is large, solutions in low frequency regions are sparse. In addition, the larger N is, which means that the softer the system is, the more the number of general solutions increases, and in particular, there tend to exist denser solutions in the high frequency region. Considering that the system is a model of a tensegrity, for approximating filters with high frequencies, N is preferred to be large, which means that one needs to increase the number of elastic rods and thus the degree of freedom of the system. However, this implies that weakening the tension is expected to work more effectively to approximate filters with low frequency, rather than increasing the number of elastic rods.

Application to Analysis of Nonlinear Models

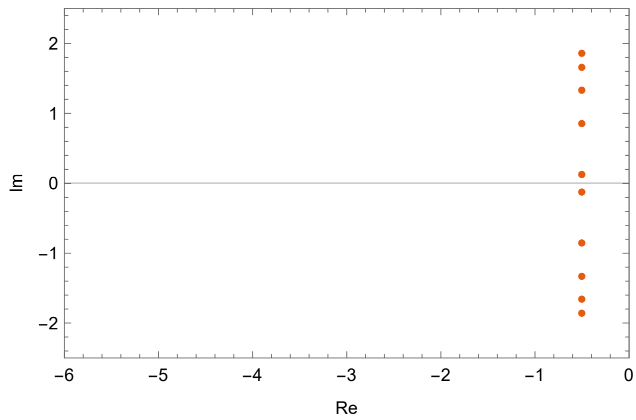
In this section, we analyze a system with nonlinear terms that are described by polynomials. For this sort of system, the input–output equations are derived in the same way as before. As an example, we consider hydrogels with chemical reactions, which is one of the expected materials for soft robots (Maeda, Hara, Sakai, Yoshida and Hashimoto (2007)). In particular, the materials are modeled as reaction-diffusion systems

$$\frac{\partial u}{\partial t} = d\Delta u + f(u).$$

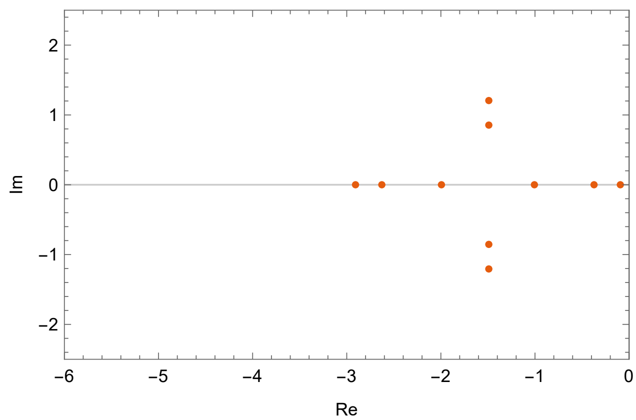
where d is a diffusion coefficient and f is assumed to be a polynomial (Maeda, Hara, Sakai, Yoshida and



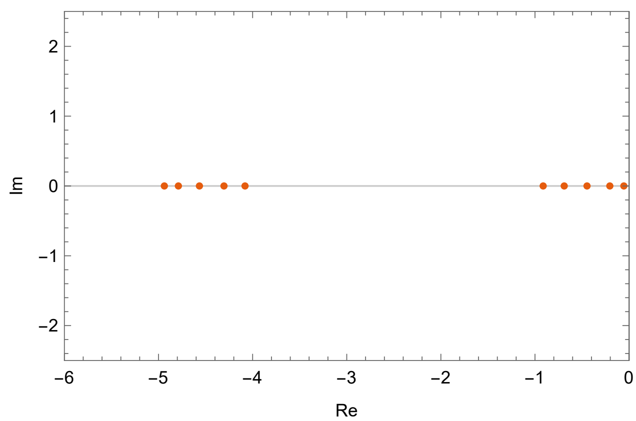
(a) $\tilde{k} = 1, \tilde{\gamma} = 0.1, N = 6.$



(b) $\tilde{k} = 1, \tilde{\gamma} = 1, N = 6.$

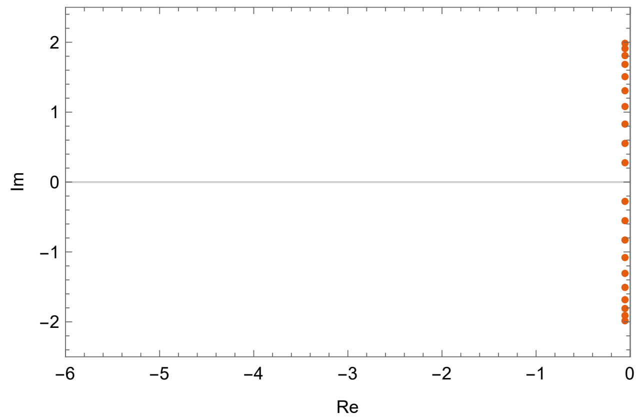


(c) $\tilde{k} = 1, \tilde{\gamma} = 3, N = 6.$

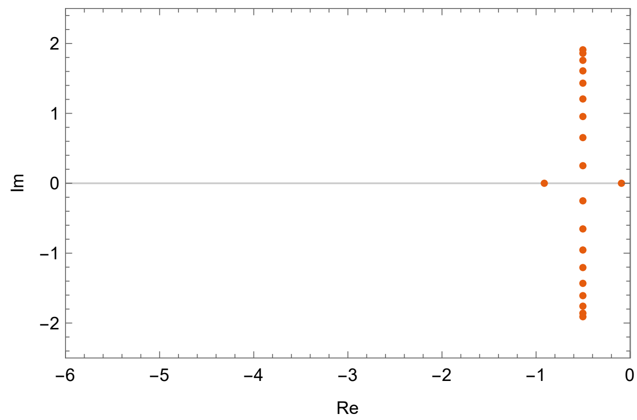


(d) $\tilde{k} = 1, \tilde{\gamma} = 5, N = 6.$

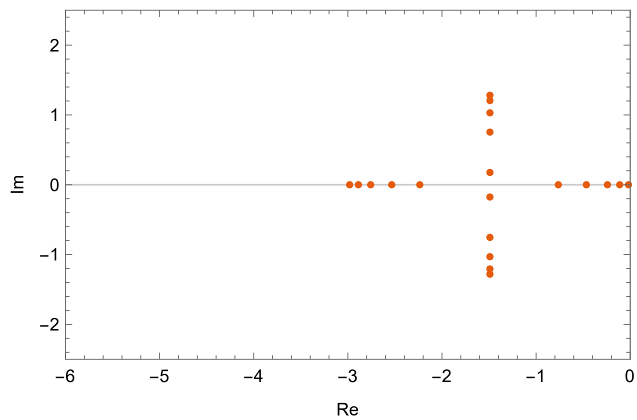
Figure 4. Distribution of the solutions of the characteristic equation of the input–output equation in the case of $N = 6$ and $\tilde{\gamma} = 0.1, 1, 3, 5$.



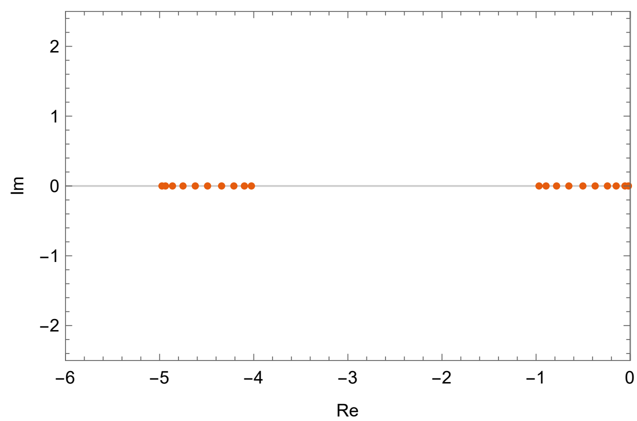
(a) $\tilde{k} = 1, \tilde{\gamma} = 0.1, N = 11.$



(b) $\tilde{k} = 1, \tilde{\gamma} = 1, N = 11.$

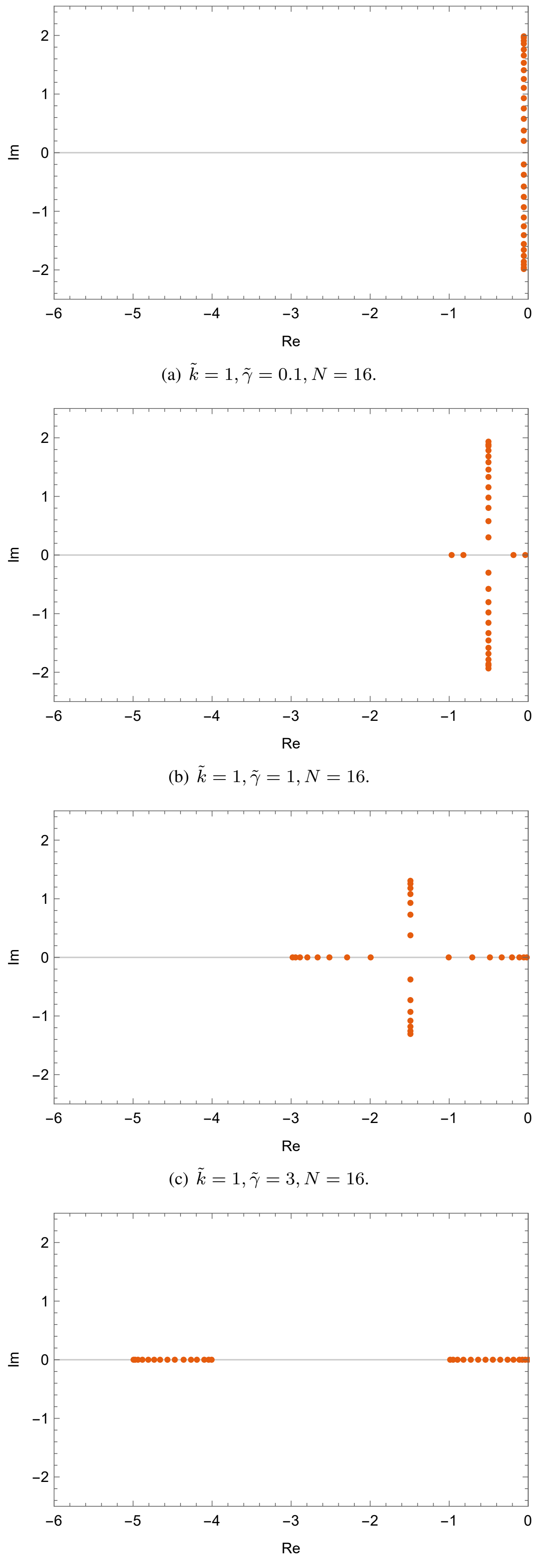


(c) $\tilde{k} = 1, \tilde{\gamma} = 3, N = 11.$



(d) $\tilde{k} = 1, \tilde{\gamma} = 5, N = 11.$

Figure 5. Distribution of the solutions of the characteristic equation of the input–output equation in the case of $N = 11$ and $\tilde{\gamma} = 0.1, 1, 3, 5$.



(a) $\tilde{k} = 1, \tilde{\gamma} = 0.1, N = 16.$

(b) $\tilde{k} = 1, \tilde{\gamma} = 1, N = 16.$

(c) $\tilde{k} = 1, \tilde{\gamma} = 3, N = 16.$

(d) $\tilde{k} = 1, \tilde{\gamma} = 5, N = 16.$

Figure 6. Distribution of the solutions of the characteristic equation of the input–output equation in the case of $N = 16$ and $\tilde{\gamma} = 0.1, 1, 3, 5$.

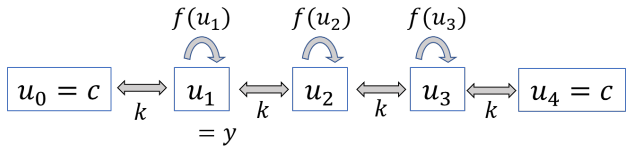


Figure 7. Discrete compartment model of hydrogels with chemical reactions.

Hashimoto (2007)). In addition, the domain is assumed to be 1-dimensional and the boundary condition is given as the Dirichlet boundary condition: $u(t, 0) = u(L, 0) = c$. As with the models described above, first the equations are discretized, and the properties of the system are analyzed by finding the input–output equation of the discretized equations. Specifically, the domain L is divided into N sections, and the equation is discretized in space by the central difference method, yielding

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = d\frac{u_{j+1} - 2u_j + u_j}{\Delta x^2} + f(u_j)$$

where $\Delta x = L/N$. $u_0 = u_N = c$ is given as a boundary condition. By replacing $d/\Delta x^2$ with k, we get

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = k(u_{j+1} - 2u_j + u_j) + f(u_j).$$

Generally, for nonlinear systems, input–output equations are likely to be considerably complicated. Even for a small system with fewer than 10 degrees of freedom, its input–output equation may consist of more than hundreds of terms. Based on this, for brevity, the input–output equation of the system mentioned above is derived where $N = 4$ with simple input and output as constant c and $y = u_1$, respectively. Note that this system can be regarded as the compartment model shown in Figure 7. As for f, we consider $f(u) = a(2u - 3u^2)$ where a is a constant. In this situation, the above partial differential equation forms the Allen–Cahn equation, in which free energy is defined as $au^2(1 - u)$. In the case of $N = 4$, the discretized model becomes

$$\begin{aligned}
\frac{\mathrm{d}u_1}{\mathrm{d}t} &= k(u_2 - 2u_1 + c) + a(2u_1 - 3u_1^2) \\
\frac{\mathrm{d}u_2}{\mathrm{d}t} &= k(u_3 - 2u_2 + u_1) + a(2u_2 - 3u_2^2) \\
\frac{\mathrm{d}u_3}{\mathrm{d}t} &= k(c - 2u_3 + u_2) + a(2u_3 - 3u_3^2) \\
y &= u_1.
\end{aligned}$$

Next, we explain the lack of the echo state property of this system for general parameter k and a. The input–output equation of the above equation is derived as follows:

$$\begin{aligned}
&4k^4c + 9k^3ac^2 + 8k^3ac + 6k^2a^2c^2 + 4k^2a^2c \\
&+ (-4k^4 - 36k^3ac - 20k^3a - 60k^2a^2c - 24k^2a^2 \\
&\qquad - 24ka^3c - 8ka^3)y \\
&+ (-10k^3 - 30k^2ac - 24k^2a - 24ka^2c - 12ka^2)\frac{\mathrm{d}y}{\mathrm{d}t} \\
&+ (-6k^2 - 6kac - 6ka)\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + (-k)\frac{\mathrm{d}^3y}{\mathrm{d}t^3} \\
&+ (45k^3a + 54k^2a^2c + 120k^2a^2 + 36ka^3c + 96ka^3
\end{aligned}$$

$$
\begin{aligned}
&\quad + 24a^4)y^2 \\
&+ (84k^2a + 36ka^2c + 132ka^2 + 48a^3)y\frac{\mathrm{d}y}{\mathrm{d}t} \\
&+ (18ka + 12a^2)y\frac{\mathrm{d}^2y}{\mathrm{d}t^2} \\
&+ (27ka + 18a^2)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + 6a\frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} \\
&+ (-108k^2a^2 - 180ka^3 - 72a^4)y^3 \\
&+ (-162ka^2 - 144a^3)y^2\frac{\mathrm{d}y}{\mathrm{d}t} \\
&- 18a^2y^2\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + (-36a^2)y\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 \\
&+ (81ka^3 + 54a^4)y^4 + 108a^3y^3\frac{\mathrm{d}y}{\mathrm{d}t} = 0.
\end{aligned}
$$

To investigate the echo state property of this system, following Yildiz, Jaeger and Kiebel (2012), we check the fixed points of the system. To this end, we establish $y = y_0 =$ constant to obtain

$$
\begin{aligned}
&4k^4c + 9k^3ac^2 + 8k^3ac + 6k^2a^2c^2 + 4k^2a^2c \\
&+ (-4k^4 - 36k^3ac - 20k^3a - 60k^2a^2c \\
&\quad - 24k^2a^2 - 24ka^3c - 8ka^3)y_0 \\
&+ (45k^3a + 54k^2a^2c + 120k^2a^2 + 36ka^3c + 96ka^3 \\
&\quad + 24a^4)y_0^2 \\
&+ (81ka^3 + 54a^4)y_0^4 = 0, \qquad (10)
\end{aligned}
$$

which is a 4th-order equation of y_0. Generally, there are 4 complex solutions for 4th-order equations, and if they are imaginary numbers, they must be conjugate. Thus, the possible solutions are 4 real solutions permitting multiple roots, 2 each of real and imaginary solutions, or 4 imaginary solutions. Therefore, except for the case that the solutions are 2 or 4 multiple roots, they are not unique. As a result, this implies that the system with general parameters does not have the echo state property.

Although this lack of the echo state property may make morphological computation using hydrogels with chemical reactions difficult, the above analysis also implies that the echo state property may hold locally around each of the fixed points. If carefully designed so that this local echo state property is maintained, morphological computation using the systems may be possible.

In addition, (10) is also useful from the perspective of morphological control. In morphological control, the asymptotic behaviors of a given system is often controlled while the parameters of the system and the initial condition are used as the control input (see, e.g., Füchslin, Dzyakanchuk, Flumini, Hauser, Hunt, Luchsinger, Reller, Scheidegger and Walker (2013)). (10) reveals how the steady state changes along with the parameters of the system, that is, the control input. This usage of the input–output equation is generalized for other systems to analyze the dependence of the asymptotic behaviors of the systems on the choice of the control inputs.

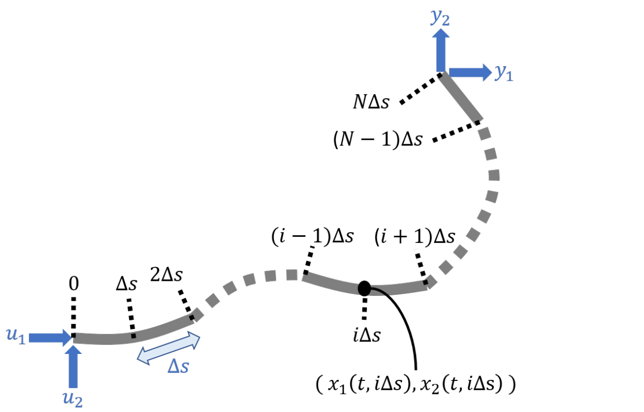


Figure 8. The discretized model of an elastic rod. The inputs are given as the external forces to the terminal portion of the rod and the outputs are the position of the head portion.

Application to an Inverse Problem Regarding the Position Control of Elastic Rods

In this section, as a potential application of the proposed method, an inverse problem related to the position control of elastic rods, which are used for, e.g., snake-like robots, is considered. For simplicity, let us consider an elastic rod on a 2-dimensional space. Typically, elastic rods in snakelike robotic devices are described by a backbone curve (see Chirikjian (2013))

$$
\begin{pmatrix} x_1(t,s) \\ x_2(t,s) \end{pmatrix} = \int_0^s \begin{pmatrix} \xi_1(t,s) \\ \xi_2(t,s) \end{pmatrix} \mathrm{d}s,
$$

where s is a curve parameter. ξ_1, ξ_2 are vectors representing the deformation of the rod. We suppose that $\xi_1 = \bar{\xi}_1, \xi_2 = \bar{\xi}_2$ when the rod is in its natural shape. Assuming that the density of the rod is ρ, and the elastic coefficient is k, the Lagrangian of this system is defined as follows:

$$
\rho \int_0^1 \left| \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x_1(t,s) \\ x_2(t,s) \end{pmatrix} \right|^2 \mathrm{d}s
- \frac{k}{2} \int_0^1 \left| \begin{pmatrix} \xi_1(t,s) \\ \xi_2(t,s) \end{pmatrix} - \begin{pmatrix} \bar{\xi}_1(t,s) \\ \bar{\xi}_2(t,s) \end{pmatrix} \right|^2 \mathrm{d}s.
$$

Dividing the rod into N segments (see Figure 8) and adding a damping term with the damping coefficient γ, we obtained the equation of motion:

$$
\rho \mathcal{I}^* \mathcal{I} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \begin{pmatrix} \xi_{1,1}(t) \\ \vdots \\ \xi_{1,N}(t) \\ \xi_{2,1}(t) \\ \vdots \\ \xi_{2,N}(t) \end{pmatrix} = -k \left(\begin{pmatrix} \xi_{1,1}(t) \\ \vdots \\ \xi_{1,N}(t) \\ \xi_{2,1}(t) \\ \vdots \\ \xi_{2,N}(t) \end{pmatrix} - \begin{pmatrix} \bar{\xi}_{1,1}(t) \\ \vdots \\ \bar{\xi}_{1,N}(t) \\ \bar{\xi}_{2,1}(t) \\ \vdots \\ \bar{\xi}_{2,N}(t) \end{pmatrix} \right)
- \gamma \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \xi_{1,1}(t) \\ \vdots \\ \xi_{1,N}(t) \\ \xi_{2,1}(t) \\ \vdots \\ \xi_{2,N}(t) \end{pmatrix} + f(t), \qquad (11)
$$

where $\xi_{1,i}(t)$, $\xi_{2,i}(t)$ are approximations of $\xi_1(t, i\Delta s)$ and $\xi_2(t, i\Delta s)$, $\Delta s = 1/N$, respectively. f denotes the external force. $\mathcal{I}$ is an approximation of the integral operator with respect to s and $\mathcal{I}^*$ is the adjoint operator of $\mathcal{I}$, of which matrix representations are supposed to be given as

$$\mathcal{I} = \begin{pmatrix} \tilde{\mathcal{I}} & O \\ O & \tilde{\mathcal{I}} \end{pmatrix}, \quad \tilde{\mathcal{I}} = \Delta s \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \ddots & 1 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$
$$\mathcal{I}^* = \mathcal{I}^\top.$$

Suppose that the inputs u_1 and u_2 are applied to the terminal portion of the elastic rod as external forces f, and the outputs y_1 and y_2 are the positions of the head portion:

$$f = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ u_1(t) \\ 0 \\ \vdots \\ 0 \\ u_2(t) \end{pmatrix},$$
$$\begin{pmatrix} y_1(t) & y_2(t) \end{pmatrix} =$$
$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \mathcal{I} \begin{pmatrix} \xi_{1,1}(t) \\ \vdots \\ \xi_{1,N}(t) \\ \xi_{2,1}(t) \\ \vdots \\ \xi_{2,N}(t) \end{pmatrix}.$$

We design the inputs $u_1(t), u_2(t)$ so that $y_1(t), y_2(t)$ approach asymptotically given functions $g_1(t), g_2(t)$ respectively.

This problem, where the inputs are designed so that the outputs of the system approximate the target functions, is essentially the same as the problem considered in the 3rd section (see Remark 3) and hence can be solved in the same manner. For instance, when $g_1(t) = \cos(t)$, $g_2(t) = \sin(t)$, $\rho = 12, k = 1, \gamma = 1$, the inputs $u_1(t)$, $u_2(t)$ are obtained from the input–output equation as shown in Figure 9. Some relatively small coefficients were ignored when deriving the input–output equation. See Appendix 4 for details. The numerical results are shown in Fig. 10. In the numerical calculations, the equations of motion are discretized by using the variational integrator (see, e.g., Marsden and West (2001)). If t is large enough, it can be seen that the outputs y_1 and y_2 are close to the target functions g_1 and g_2. In terms of morphological computation, this indicates that the outputs are approaching the target functions over time due to the echo state property. Here, y_1, y_2 do not completely converge to g_1, g_2 because of the neglected terms and the numerical calculation errors. Actually, the neglected terms are terms including a derivative of the 9th order or higher. Hence although the coefficients are small, the effects on the result may be considerable. In addition, y_1 and y_2 once significantly departed from the target functions before asymptotically approaching them. This is not preferable in practical use, and it is necessary to combine the method with the techniques in the optimal control to design robust inputs so that

$$\int \left| \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} - \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \right|^2 \mathrm{d}t$$

is minimized in future work. Besides, the input functions shown in Figure 9 are oscillatory for large t. This is due to the numerical errors caused in the computation of the solutions to the input–output equation; stable numerical methods for the computation of the inputs must be developed for practical applications.

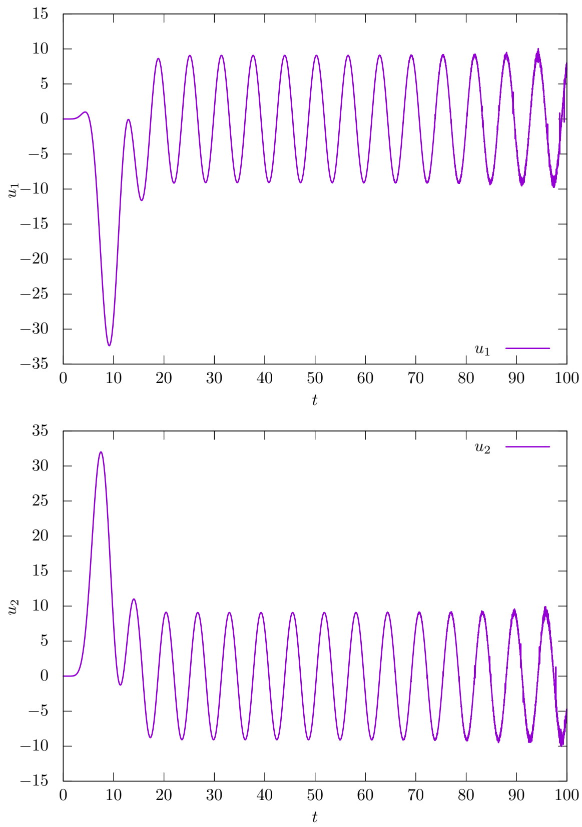


Figure 9. The inputs $u_1(t)$, $u_2(t)$ to the elastic rod that are designed so that the position of the head portion of the rod asymptotically approaches to $g_1(t) = \cos(t)$, $g_2(t) = \sin(t)$.

Conclusion and Future Work

In this paper, analysis using the input–output equations is illustrated as a new theoretical approach to the theoretical analysis of morphological computation. In addition, an application of designing the input functions to a model of elastic rods is also provided as a potential application of the method for control. The input–output equations specify the relationships between the input and output with model parameters explicitly. Thus, the equations provide much information on the existence of the echo state property of systems and changes of the possible approximate functional space for outputs of systems. In particular, even for nonlinear equations, theoretical analysis of the echo state property becomes possible to some extent.

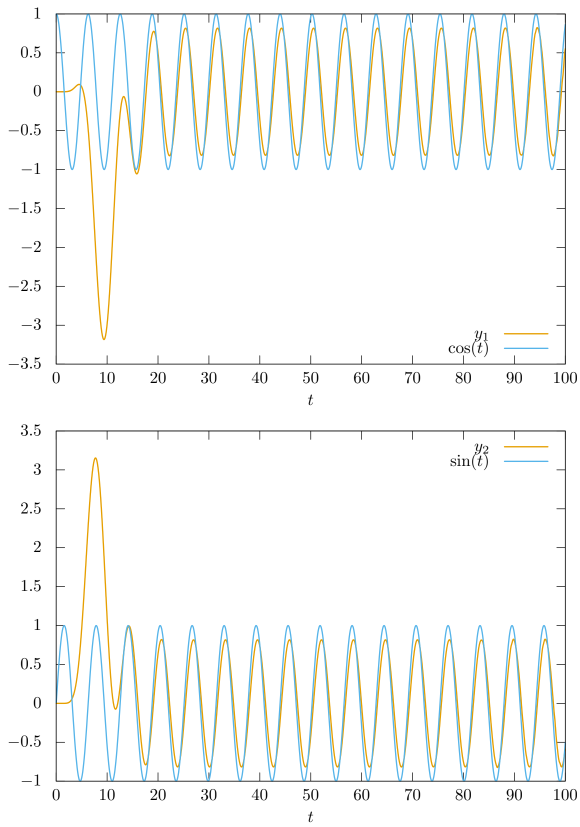


Figure 10. The outputs $y_1(t)$, $y_2(t)$ of the elastic rod when the input forces are designed so that the position of the head portion of the rod asymptotically approaches to $g_1(t) = \cos(t)$, $g_2(t) = \sin(t)$. When t is large enough ($t > 20$) $y_1(t)$, $y_2(t)$ are close to $g_1(t)$, $g_2(t)$. The differences are small and are possibly due to the numerical errors and the neglected term in the input–output equation.

To apply the proposed approach to controlling real soft robots, first, appropriate mathematical models for the robots need to be constructed. Then, if the state variables can be successfully removed from the ideal defined from the mathematical models, the input–output equation can be derived and hence the analyses described in the previous sections can be applied. In particular, when a linear or low-order approximation of the model is possible, the input–output equation can be computed for large systems by using simple matrix operations or the Gröbner basis. Generally, however, neither the linear nor the low-order approximation is appropriate, finding the input–output equations requires the Gröbner basis, which is often computationally expensive for large nonlinear systems. Although the parallel computation of the Gröbner basis has been partially explored (e.g., Sawada, Terasaki and Aiba (1994)), further development of computer algebra software is necessary to deal with such systems. Considering the above, applying the method to real soft robots at the present time can be performed by piecewise-linear approximation of the model; that is, the possible configurations of the robots are first divided into several regions, such that the input–output equation can be obtained by linear approximation of the model for each region. This approach should be considered in future work.

As for other future studies, firstly, the input–output equation is typically very complicated, even for small models; thus, to apply the proposed method to complicated physical models, the techniques of model reduction and sparse modeling must be incorporated. Secondly, introducing more advanced techniques of algebraic geometry should be considered. For instance, by first applying the finite difference method to a nonlinear differential equation and then analyzing the number of solutions of the discretized equation, the size of the set of solutions reportedly can be investigated (see Hao, Hu and Sommese (2014)). Moreover, the set of solutions of nonlinear equations is generally considered as a manifold in which the initial values are used as coordinate systems. For these types of manifolds, geometric considerations have been explored using a differential algebraic method. Thus, the differential algebraic approach is surely useful for the analysis of the computational abilities of nonlinear systems and must be integrated into the proposed method.

Declaration of conflicting interests

The authors declare that there is no conflict of interest.

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Appendix 1: Summary of the Algebraic Theory

The Parameter Identifiability Problem

The parameter identifiability problem is a problem of determining whether the parameter θ can be identified from the observable input data and the output data u and y. In this problem, the input–output equation is used to determine whether the parameter θ is identified from the values u and y. In fact, substitution of the values of u, y and their derivatives at some t into the input–output equation gives a system of equations for the unknown parameters. In general, the identifiability of the parameters is determined by investigating the rank of the system of equations. For example, for the equation of motion of a harmonic oscillator with an input and an output

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = -\theta x + u, \quad y = x,$$

suppose that $y(t_0) \neq 0$ at $t = t_0$. Then the parameter is given using the data at t_0 as

$$\theta = \frac{1}{y(t_0)} \left(u(t_0) - \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}(t_0) \right),$$

and hence the system is identifiable.

The Method to Derive the Input–Output Equation

We describe in detail the method to derive the input–output equations along with the algebraic theories. See also Appendix 5 for the list of definitions of mathematical terms. Two methods to derive the input–output equations are known: a method that can be used for nonlinear equations and an efficient method for linear systems. Here we describe the former method; for efficient methods for linear systems, see, for example, Meshkat and Sullivant (2014).

To derive the input–output equations, we use the Gröbner basis of an ideal, which is a set determined from the given model equations. More precisely, we assume that the model equations are given in the form

$$(\text{polynomial}) = 0$$

and identify the polynomials with the model equations. In general, the model equations are given as differential equations; actually, the differential equations can be treated as polynomials by using differential algebra, which will be explained later.

The main idea is that instead of concretely transforming the equations to find the input–output equation (2), the set of all possible equations obtained by transforming (2) is examined to see whether this set has an expression that does not contain the internal variables. Roughly speaking, the set of equations obtained by transforming the original equation is called an ideal. The Gröbner basis of an ideal is similar to the basis of a linear space, with which it is possible to determine whether a given polynomial belongs to the ideal.

Basically, deriving an equation by transformation of a given set of equations is made by successive applications of additions, subtractions and multiplications of polynomials. For example, the elimination of the variable x from the equations

$$y = x^2, \quad x = z$$

gives

$$y = z^2.$$

To obtain this result, first the terms in the right-hand side are moved to the left-hand side to obtain

$$y - x^2 = 0, \quad x - z = 0$$

where the left-hand side form the polynomials to manipulate. Then, the addition of the first equation, the second one multiplied by x, and the second one multiplied by z gives

$$\begin{aligned} y - x^2 = 0 \;&\Rightarrow\; y - x^2 + x(x-z) + z(x-z) = 0 \\ &\Rightarrow\; y - x^2 + x^2 - xz + zx - z^2 = 0 \\ &\Rightarrow\; y - z^2 = 0. \end{aligned}$$

As seen in this simple example, the set of equations derived by transformation of the given equations $y - x^2 = 0$ and $x - z = 0$ is essentially corresponding to the set of polynomials

$$p(x,y,z)(y - x^2) + q(x,y,z)(x - z) \tag{12}$$

for some polynomials $p(x,y,z)$ and $q(x,y,z)$. This set of polynomials (12) is called the ideal generated by $y - x^2$ and $x - z$ and is typically denoted by $\langle y - x^2, x - z\rangle$. The important point is that if a polynomial of y and z only is found in the ideal, it means that the polynomial can be derived from the given system of equations and hence it is an input–output equation. Thus, the derivation of the input–output equations is reduced to finding a polynomial of y and z in the ideal $\langle y - x^2, x - z\rangle$.

This problem can be solved immediately using the Gröbner basis. In the above calculations, $y - x^2$ and $x - z$ are transformed to derive $y - z^2$. However if these calculations are reversed, $y - x^2$ is also obtained from $y - x^2$ and $x - z$. This shows that $\langle y - x^2, x - z\rangle$ is also generated by $y - z^2$ and $x - z$: $\langle y - x^2, x - z\rangle = \langle y - z^2, x - z\rangle$. In this way, both $y - x^2, x - z$ and $y - z^2, x - z$ form a kind of basis of the ideal $\langle y - x^2, x - z\rangle = \langle y - z^2, x - z\rangle$ in the sense that they both generate this ideal. Among these basis, there exists a special type of basis, the Gröbner basis. The Gröbner basis of an ideal I is a set of polynomials that generates I and also has an excellent property, specifically, the uniqueness of the remainder of division.

To define the Gröbner basis precisely, we prepare some algebraic terms. A set K is called a *field* if addition, subtraction, multiplication and division are defined and the results of these operations belong to K. For example, the sets of real numbers and rational numbers are both fields. The set of integers is not a field because, for example, 2/3 is not an integer. Similarly, a set in which addition, subtraction, and multiplication are defined is called a *ring*. Thus, the set of integers is a ring. The set of polynomials whose coefficients are rational numbers is also a ring, while the set of rational polynomials forms a field. For further algebraic terms, see Appendix 5.

Definition 1. *For an ideal I on a polynomial ring $K[x_1, \ldots, x_n]$ over a field K with a given monomial order, if its generator $G = \{g_1, \ldots, g_s\}$ satisfies*

$$f \in I \Leftrightarrow f \text{ is divisible by } G,$$

G is called the Gröbner basis of I.

Actually, the remainder of division by polynomials of multiple variables is generally not unique. The above property of the Gröbner basis ensures that a function is in the ideal if and only if the remainder of division by the polynomials in the Gröbner basis is zero. In other words, all polynomials in the ideal can be represented as a linear combination of the Gröbner basis. In this sense the Gröbner basis reveals all the members of the ideal and hence by computing the Gröbner basis of the ideal generated by polynomials corresponding to given equations, we can essentially enumerate all algebraic equations that can be derived from the given equations.

For example, $y - x^2$ and $x - z$ are in fact a Gröbner basis of the ideal $\langle y - x^2, x - z\rangle$. Similarly $y - z^2$ and $x - z$ are also a Gröbner basis and each polynomial that belongs to this ideal is written as

$$\tilde{p}(x, y, z)(y - z^2) + \tilde{q}(x, y, z)(x - z)$$

with other polynomials $\tilde{p}(x, y, z)$ and $\tilde{q}(x, y, z)$. As in this example, the Gröbner basis is not uniquely determined. A different basis is obtained according to the ordering of the variables. The ordering specifies priority between the variables and is important when the Gröbner basis is used to eliminate variables; as stated in the following theorem, the desired elimination can be achieved if a carefully designed ordering, for example a lexicographical or block ordering (see Appendix 5), is employed.

Theorem 2. (Elimination Theorem). *Suppose that the lexicographic ordering is used as the monomial ordering in a ring of polynomials $K[x_1, \ldots, x_n]$ over a field K so that $x_1 > x_2 > \ldots > x_n$. If G is a reduced Gröbner basis of an ideal $I \subset K[x_1, \ldots, x_n]$ then $G \cap K[x_1, \ldots, x_j]$ is a Gröbner basis of $I \cap K[x_1, \ldots, x_j]$.*

An ordering other than the lexicographic ordering can be used; see Hibi (2014) for details. The important point here is that, in the above example, the polynomial $y - z^2$ that is obtained by eliminating the variable x is a polynomial in the ideal $\langle y - x^2, x - z\rangle$ that has only y and z as its variables. The set of such polynomials also defines an ideal. The above theorem states that the Gröbner basis of this ideal is immediately obtained from the Gröbner basis of the original ideal associated with an appropriate ordering. For example, $G = \{y - z^2, x - z\}$ is the Gröbner basis of the ideal $\langle y - x^2, x - z\rangle \subset \mathbb{Q}[z, y, x]$ with the lexicographic ordering. Then $G \cap \mathbb{Q}[z, y] = \{y - z^2\}$ is the Gröbner basis of $\langle y - x^2, x - z\rangle \cap \mathbb{Q}[z, y]$. This shows that polynomials in $\langle y - x^2, x - z\rangle$ with only variables y and z are written as

$$r(y, z)(y - z^2)$$

where $r(y, z)$ is a polynomial. Hence, $y - z^2 = 0$ is essentially the only equation that can be obtained by eliminating x from the original equations.

Remark 4. *The Gröbner basis can be computed using computer algebra systems. For example, it can be computed in the following way using the free software Singular (Decker, Greuel, Pfister and Schönemann (2018)). The commands of Singular are shown after the colon of each the step.*

1. *Specify the ring and the monomial ordering: ring r = (0),(x,y,z),lp;*
2. *Define the ideal: ideal i = (y-x^2, x-z);*
3. *Compute the Gröbner basis: groebner(i);*

Note that the above commands are not efficient in general. Faster ways are found in, for example, Hibi (2014).

The above method, which is explained using polynomials as examples, is applied to ordinary differential equations as described below. There are mainly two ways to deal with differential operators. One is to specify the differentiated equations, which are calculated by differentiating the given equations by hand in advance, and the other is to equip differential operators with the operations that belong to the corresponding ring where the ideal is defined. In this paper we adopt the former method for simplicity. In fact, as shown in the following theorem, it is guaranteed that the elimination of variables that are neither input nor output is achieved if the given equations are differentiated N times, where N is the number of state variables in the system.

Theorem 3. (Meshkat, Rosen and Sullivant (2018)). *For the model described by polynomials f, g with N state variables x, R input variables, and an output variable y*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, u; \theta), \quad y = g(x; \theta),$$

if the ideal P is defined as

$$\begin{aligned}
&P = \\
&\langle \frac{\mathrm{d}x}{\mathrm{d}t} - f, \ldots, \frac{\mathrm{d}^N x}{\mathrm{d}t^N} - f^{(N-1)}, \ldots y - g, \ldots, y^{(N)} - g^{(N)}\rangle \\
&\subset \mathbb{Q}(\theta)[x, y, u, \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}u}{\mathrm{d}t}, \\
&\qquad \ldots, \frac{\mathrm{d}^{N-1}x}{\mathrm{d}t^{N-1}}, \frac{\mathrm{d}^{N-1}y}{\mathrm{d}t^{N-1}}, \frac{\mathrm{d}^{N-1}u}{\mathrm{d}t^{N-1}}, \frac{\mathrm{d}^N x}{\mathrm{d}t^N}, \frac{\mathrm{d}^N y}{\mathrm{d}t^N}\rangle
\end{aligned}$$

then $P \cap \mathbb{Q}(\theta)[y, u, \ldots \frac{\mathrm{d}^{N-1}y}{\mathrm{d}t^{N-1}}, \frac{\mathrm{d}^{N-1}u}{\mathrm{d}t^{N-1}}, \frac{\mathrm{d}^N y}{\mathrm{d}t^N}]$ is not the zero ideal.

Example 1. *Let us consider the harmonic oscillator*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = -\theta x + u, \quad y = x. \tag{13}$$

as an example. Given Theorem 3, to eliminate two state variables x, v from (13), up to the 2nd derivatives of the equations in the system need to be included as generators of the ideal. Specifically, first the equations

$$\begin{aligned}
&\frac{\mathrm{d}x}{\mathrm{d}t} = v, \quad \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t}, \\
&\frac{\mathrm{d}v}{\mathrm{d}t} = -\theta x + u, \quad \frac{\mathrm{d}^2 v}{\mathrm{d}t^2} = -\theta\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}u}{\mathrm{d}t}, \\
&y = x, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}
\end{aligned}$$

are specified in advance, and thus the Gröbner basis of the ideal

$$\begin{aligned}
&\langle \frac{\mathrm{d}x}{\mathrm{d}t} - v, \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - \frac{\mathrm{d}v}{\mathrm{d}t}, \frac{\mathrm{d}v}{\mathrm{d}t} + \theta x - u, \frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + \theta\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}u}{\mathrm{d}t}, \\
&y - x, \frac{\mathrm{d}y}{\mathrm{d}t} - \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\rangle
\end{aligned}$$

is calculated to eliminate state variables. In fact, by using the free software Singular, the Gröbner basis is obtained as

$$\theta y - u + \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}, \frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + \theta\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{\mathrm{d}u}{\mathrm{d}t}, \frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}y}{\mathrm{d}t},$$

$$v - \frac{\mathrm{d}x}{\mathrm{d}t},\ x - y,$$

where

$$\theta y - u + \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}$$

does not have the variables x and v. Hence the input–output equation (2) is obtained by setting this polynomial as 0:

$$\theta y - u + \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = 0.$$

Appendix 2: Proof of Theorem 1.

In this section, we show the proof of Theorem 1.

Proof. Because A is irreducible the input–output function of the system described by (8) becomes

$$\det(\frac{\mathrm{d}}{\mathrm{d}t}I - A)y = w_1 \det(\frac{\mathrm{d}}{\mathrm{d}t}I - A_1) \quad (14)$$

$$+ w_2 \det(\frac{\mathrm{d}}{\mathrm{d}t}I - A_2) + \cdots + w_n \det(\frac{\mathrm{d}}{\mathrm{d}t}I - A_n), \quad (15)$$

where A_j is the matrix that is obtained by replacing the jth column with the vector u (see Meshkat and Sullivant (2014)). The general solutions of the input–output equation are the solutions to

$$\det(\frac{\mathrm{d}}{\mathrm{d}t}I - A)y = 0,$$

and hence the weight does not affect the general solutions.

In addition, the particular solution to the system is determined by the right-hand side of the input–output equation (14). Because the size of the matrix A is n, the right-hand side contains the differential of the input function u up to the order n. Therefore we can write the right-hand side as

$$w_1 \det(\frac{\mathrm{d}}{\mathrm{d}t}I - A_1) + \cdots + w_n \det(\frac{\mathrm{d}}{\mathrm{d}t}I - A_n)$$
$$= C_n \frac{\mathrm{d}^n u}{\mathrm{d}t^n} + \cdots + C_1 \frac{\mathrm{d}u}{\mathrm{d}t} + C_0 u$$

with matrices $C_n, C_{n-1}, \ldots, C_0$ which depend on $w_1, \ldots, w_n$ and the matrix A of the parameters of the system. □

Because the same C_j's gives the same dynamics of the output y, C_j's are used as the true parameters of the system.

Appendix 3: The Equation of Motion and the Input–Output Equation of the Mass–Spring Array Model

The Lagrangian of the system shown in Figure 3 without the input and dampers is

$$\mathcal{L}(\Delta x_1, \ldots, \Delta x_{N-1}, \Delta\dot{x}_1, \ldots, \Delta\dot{x}_{N-1})$$
$$= \sum_{j=1}^{N-1} \frac{m}{2}(\dot{x}_j)^2 - \sum_{j=0}^{N-1} \frac{k}{2}(x_{j+1} - x_j - l_0)^2$$
$$= \sum_{j=1}^{N-1} \frac{m}{2}(\Delta\dot{x}_j)^2 - \sum_{j=1}^{N-2} \frac{k}{2}(\Delta x_{j+1} - \Delta x_j + l - l_0)^2$$
$$- \frac{k}{2}(\Delta x_1 + l - l_0)^2 - \frac{k}{2}(-\Delta x_{N-1} + l - l_0)^2$$

which gives the Euler–Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \Delta x_j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \Delta\dot{x}_j} = 0$$
$$\Leftrightarrow m\Delta\ddot{x}_j =$$
$$k(\Delta x_{j+1} - \Delta x_j + l - l_0) - k(\Delta x_j - \Delta x_{j-1} + l - l_0).$$

Therefore, the equation of motion of the system with the input and dampers is

$$m\Delta\ddot{x}_j = k(\Delta x_{j+1} - 2\Delta x_j + \Delta x_{j-1}) - \gamma\Delta\dot{x}_j + u_j.$$

By using $\tilde{k} = k/m$ and $\tilde{\gamma} = \gamma/m$, the number of the physical parameters are reduced to two:

$$\Delta\ddot{x}_j = \tilde{k}(\Delta x_{j+1} - 2\Delta x_j + \Delta x_{j-1}) - \tilde{\gamma}\Delta\dot{x}_j + u_j,$$
$$y = \sum_{j=1}^{N-1} w_j \Delta x_j.$$

The input–output equation of this system, for example, when $N = 6$, is as follows:

$$\frac{\mathrm{d}^{10}y}{\mathrm{d}t^{10}} + 5\tilde{\gamma}\frac{\mathrm{d}^9 y}{\mathrm{d}t^9} + 10(\tilde{\gamma}^2 + \tilde{k})\frac{\mathrm{d}^8 y}{\mathrm{d}t^8} + (10\tilde{\gamma}^3 + 40\tilde{\gamma}\tilde{k})\frac{\mathrm{d}^7 y}{\mathrm{d}t^7}$$
$$+ (5\tilde{\gamma}^4 + 60\tilde{\gamma}^2\tilde{k}m + 36\tilde{k}^2)\frac{\mathrm{d}^6 y}{\mathrm{d}t^6}$$
$$+ (\tilde{\gamma}^5 + 40\tilde{\gamma}^3\tilde{k} + 108\tilde{\gamma}\tilde{k}^2)\frac{\mathrm{d}^5 y}{\mathrm{d}t^5}$$
$$+ (10\tilde{\gamma}^4\tilde{k} + 108\tilde{\gamma}^2\tilde{k}^2 + 56\tilde{k}^3)\frac{\mathrm{d}^4 y}{\mathrm{d}t^4}$$
$$+ (36g^3k^2 + 112gk^3)\frac{\mathrm{d}^3 y}{\mathrm{d}t^3} + (56\tilde{\gamma}^2\tilde{k}^3 + 35\tilde{k}^4)\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}$$
$$+ 35\tilde{\gamma}\tilde{k}^4\frac{\mathrm{d}y}{\mathrm{d}t} + 6\tilde{k}^5 y$$
$$= (\tilde{k}^4 w_1 - 4\tilde{k}^3(2\tilde{k} + \tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2})w_1$$
$$- 2\tilde{\gamma}\tilde{k}^2\frac{\mathrm{d}}{\mathrm{d}t}(2\tilde{k} + \tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2})w_1$$
$$- 2\tilde{k}^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}(2\tilde{k} + \tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2})w_1$$
$$- \tilde{k}^2(2\tilde{k} + \tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2})^2 w_1 + 2\tilde{k}(2\tilde{k} + \tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2})^3 w_1$$
$$+ \tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}(2\tilde{k} + \tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2})^3 w_1$$
$$+ \frac{\mathrm{d}^2}{\mathrm{d}t^2}(2\tilde{k} + \tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2})^3 w_1 - 2\tilde{k}^3(2\tilde{k} + \tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2})w_2$$
$$+ \tilde{k}(2\tilde{k} + \tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2})^3 w_2 - \tilde{k}^4 w_3$$
$$+ \tilde{k}^2(2\tilde{k} + \tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2})^2 w_3$$
$$+ \tilde{k}^3(2\tilde{k} + \tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4 + \tilde{k}^4 w_5)u_1$$
$$+ (-2\tilde{k}^3(2\tilde{k} + \tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2})w_1 + \tilde{k}(2\tilde{k} + \tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2})^3 w_1$$
$$- 4\tilde{k}^3(2\tilde{k} + \tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2})w_2 - 2\tilde{\gamma}\tilde{k}^2\frac{\mathrm{d}}{\mathrm{d}t}(2\tilde{k} + \tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2})w_2$$
$$- 2\tilde{k}^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}(2\tilde{k} + \tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2})w_2 + 2\tilde{k}(2\tilde{k} + \tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2})^3 w_2$$

$$
\begin{aligned}
&+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})^3 w_2+\frac{\mathrm{d}^2}{\mathrm{d}t^2}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})^3 w_2\\
&-2\tilde{k}^4 w_3-\tilde{\gamma}\tilde{k}^3\frac{\mathrm{d}}{\mathrm{d}t}w_3-\tilde{k}^3\frac{\mathrm{d}^2}{\mathrm{d}t^2}w_3\\
&+2\tilde{k}^2(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})^2 w_3+\tilde{\gamma}\tilde{k}\frac{\mathrm{d}}{\mathrm{d}t}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})^2 w_3\\
&+\tilde{k}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})^2 w_3\\
&+2\tilde{k}^3(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4+\tilde{\gamma}\tilde{k}^2\frac{\mathrm{d}}{\mathrm{d}t}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4\\
&+\tilde{k}^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4\\
&+2\tilde{k}^4 w_5+\tilde{\gamma}\tilde{k}^3\frac{\mathrm{d}}{\mathrm{d}t}w_5+\tilde{k}^3\frac{\mathrm{d}^2}{\mathrm{d}t^2}w_5)u_2\\
&+(-\tilde{k}^4 w_1+\tilde{k}^2(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})^2 w_1\\
&+\tilde{k}^2(-2\tilde{k}^2-\tilde{\gamma}\tilde{k}\frac{\mathrm{d}}{\mathrm{d}t}-\tilde{k}\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_2\\
&-(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})^2(-2\tilde{k}^2-\tilde{\gamma}\tilde{k}\frac{\mathrm{d}}{\mathrm{d}t}-\tilde{k}\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_2\\
&-\tilde{k}^2(-\tilde{k}^2+(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})^2)w_3\\
&+(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})^2(-\tilde{k}^2+(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})^2)w_3\\
&+3\tilde{k}^3(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4+4\tilde{\gamma}\tilde{k}^2\frac{\mathrm{d}}{\mathrm{d}t}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4\\
&+\tilde{\gamma}^2\tilde{k}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4\\
&+4\tilde{k}^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4\\
&+2\tilde{\gamma}\tilde{k}\frac{\mathrm{d}^3}{\mathrm{d}t^3}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4\\
&+\tilde{k}\frac{\mathrm{d}^4}{\mathrm{d}t^4}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4+3\tilde{k}^4 w_5+4\tilde{\gamma}\tilde{k}^3\frac{\mathrm{d}}{\mathrm{d}t}w_5\\
&+\tilde{\gamma}^2\tilde{k}^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}w_5+4\tilde{k}^3\frac{\mathrm{d}^2}{\mathrm{d}t^2}w_5+2\tilde{\gamma}\tilde{k}^2\frac{\mathrm{d}^3}{\mathrm{d}t^3}w_5+\tilde{k}^2\frac{\mathrm{d}^4}{\mathrm{d}t^4}w_5)u_3\\
&+(\tilde{k}^3(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_1+2\tilde{k}^3(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_2\\
&+\tilde{\gamma}\tilde{k}^2\frac{\mathrm{d}}{\mathrm{d}t}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_2\\
&+\tilde{k}^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_2\\
&+\tilde{k}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})(-\tilde{k}^2+(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})^2)w_3\\
&+4\tilde{k}^3(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4\\
&+10\tilde{\gamma}\tilde{k}^2\frac{\mathrm{d}}{\mathrm{d}t}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4\\
&+6\tilde{\gamma}^2\tilde{k}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4\\
&+10\tilde{k}^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4\\
&+\tilde{\gamma}^3\frac{\mathrm{d}^3}{\mathrm{d}t^3}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4\\
&+12\tilde{\gamma}\tilde{k}\frac{\mathrm{d}^3}{\mathrm{d}t^3}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4\\
&+3\tilde{\gamma}^2\frac{\mathrm{d}^4}{\mathrm{d}t^4}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4\\
&+6\tilde{k}\frac{\mathrm{d}^4}{\mathrm{d}t^4}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4\\
&+3\tilde{\gamma}\frac{\mathrm{d}^5}{\mathrm{d}t^5}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4\\
&+\frac{\mathrm{d}^6}{\mathrm{d}t^6}(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_4+4\tilde{k}^4 w_5\\
&+10\tilde{\gamma}\tilde{k}^3\frac{\mathrm{d}}{\mathrm{d}t}w_5+6\tilde{\gamma}^2\tilde{k}^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}w_5+10\tilde{k}^3\frac{\mathrm{d}^2}{\mathrm{d}t^2}w_5\\
&+\tilde{\gamma}^3\tilde{k}\frac{\mathrm{d}^3}{\mathrm{d}t^3}w_5+12\tilde{\gamma}\tilde{k}^2\frac{\mathrm{d}^3}{\mathrm{d}t^3}w_5+3\tilde{\gamma}^2\tilde{k}\frac{\mathrm{d}^4}{\mathrm{d}t^4}w_5+6\tilde{k}^2\frac{\mathrm{d}^4}{\mathrm{d}t^4}w_5\\
&+3\tilde{\gamma}\tilde{k}\frac{\mathrm{d}^5}{\mathrm{d}t^5}w_5+\tilde{k}\frac{\mathrm{d}^6}{\mathrm{d}t^6}w_5)u_4\\
&+(\tilde{k}^4 w_1+\tilde{k}(2\tilde{k}^3+\tilde{\gamma}\tilde{k}^2\frac{\mathrm{d}}{\mathrm{d}t}+\tilde{k}^2\frac{\mathrm{d}^2}{\mathrm{d}t^2})w_2\\
&+\tilde{k}^2(-\tilde{k}^2+(2\tilde{k}+\tilde{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\mathrm{d}^2}{\mathrm{d}t^2})^2)w_3\\
&+(4\tilde{k}^4+10\tilde{\gamma}\tilde{k}^3\frac{\mathrm{d}}{\mathrm{d}t}+6\tilde{\gamma}^2\tilde{k}^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}+10\tilde{k}^3\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\tilde{\gamma}^3\tilde{k}\frac{\mathrm{d}^3}{\mathrm{d}t^3}\\
&+12\tilde{\gamma}\tilde{k}^2\frac{\mathrm{d}^3}{\mathrm{d}t^3}+3\tilde{\gamma}^2\tilde{k}\frac{\mathrm{d}^4}{\mathrm{d}t^4}+6\tilde{k}^2\frac{\mathrm{d}^4}{\mathrm{d}t^4}+3\tilde{\gamma}\tilde{k}\frac{\mathrm{d}^5}{\mathrm{d}t^5}+\tilde{k}\frac{\mathrm{d}^6}{\mathrm{d}t^6})w_4\\
&+5\tilde{k}^4 w_5+20\tilde{\gamma}\tilde{k}^3\frac{\mathrm{d}}{\mathrm{d}t}w_5+21\tilde{\gamma}^2\tilde{k}^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}w_5+20\tilde{k}^3\frac{\mathrm{d}^2}{\mathrm{d}t^2}w_5\\
&+8\tilde{\gamma}^3\tilde{k}\frac{\mathrm{d}^3}{\mathrm{d}t^3}w_5+42\tilde{\gamma}\tilde{k}^2\frac{\mathrm{d}^3}{\mathrm{d}t^3}w_5+\tilde{\gamma}^4\frac{\mathrm{d}^4}{\mathrm{d}t^4}w_5\\
&+24\tilde{\gamma}^2\tilde{k}\frac{\mathrm{d}^4}{\mathrm{d}t^4}w_5+21\tilde{k}^2\frac{\mathrm{d}^4}{\mathrm{d}t^4}w_5+4\tilde{\gamma}^3\frac{\mathrm{d}^5}{\mathrm{d}t^5}w_5+24\tilde{\gamma}\tilde{k}\frac{\mathrm{d}^5}{\mathrm{d}t^5}w_5\\
&+6\tilde{\gamma}^2\frac{\mathrm{d}^6}{\mathrm{d}t^6}w_5+8\tilde{k}\frac{\mathrm{d}^6}{\mathrm{d}t^6}w_5+4\tilde{\gamma}\frac{\mathrm{d}^7}{\mathrm{d}t^7}w_5+\frac{\mathrm{d}^8}{\mathrm{d}t^8}w_5)u_5=0.
\end{aligned}
$$

The general solutions of the equation are non-trivial solutions y which satisfy

$$
\begin{aligned}
&\frac{\mathrm{d}^{10}y}{\mathrm{d}t^{10}}+5\tilde{\gamma}\frac{\mathrm{d}^9 y}{\mathrm{d}t^9}+10(\tilde{\gamma}^2+\tilde{k})\frac{\mathrm{d}^8 y}{\mathrm{d}t^8}+(10\tilde{\gamma}^3+40\tilde{\gamma}\tilde{k})\frac{\mathrm{d}^7 y}{\mathrm{d}t^7}\\
&+(5\tilde{\gamma}^4+60\tilde{\gamma}^2\tilde{k}m+36\tilde{k}^2)\frac{\mathrm{d}^6 y}{\mathrm{d}t^6}\\
&+(\tilde{\gamma}^5+40\tilde{\gamma}^3\tilde{k}+108\tilde{\gamma}\tilde{k}^2)\frac{\mathrm{d}^5 y}{\mathrm{d}t^5}\\
&+(10\tilde{\gamma}^4\tilde{k}+108\tilde{\gamma}^2\tilde{k}^2+56\tilde{k}^3)\frac{\mathrm{d}^4 y}{\mathrm{d}t^4}\\
&+(36g^3k^2+112gk^3)\frac{\mathrm{d}^3 y}{\mathrm{d}t^3}+(56\tilde{\gamma}^2\tilde{k}^3+35\tilde{k}^4)\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\\
&+35\tilde{\gamma}\tilde{k}^4\frac{\mathrm{d}y}{\mathrm{d}t}+6\tilde{k}^5 y=0.
\end{aligned}
$$

The solution $y(t)$ is a linear combination of $\exp(st)$, where s is the solution to the characteristic equation of the above differential equation:

$$
\begin{aligned}
&s^5+5\tilde{\gamma}s^9+10(\tilde{\gamma}^2+\tilde{k})s^8+(10\tilde{\gamma}^3+40\tilde{\gamma}\tilde{k})s^7\\
&+(5\tilde{\gamma}^4+60\tilde{\gamma}^2\tilde{k}m+36\tilde{k}^2)s^6+(\tilde{\gamma}^5+40\tilde{\gamma}^3\tilde{k}+108\tilde{\gamma}\tilde{k}^2)s^5\\
&+(10\tilde{\gamma}^4\tilde{k}+108\tilde{\gamma}^2\tilde{k}^2+56\tilde{k}^3)s^4\\
&+(36g^3k^2+112gk^3)s^3+(56\tilde{\gamma}^2\tilde{k}^3+35\tilde{k}^4)s^2+35\tilde{\gamma}\tilde{k}^4 s\\
&+6\tilde{k}^5=0.
\end{aligned}
$$

Appendix 4: the Input–Output Equation for the Discretized Model of Elastic Rods

We derived the asymptotic input–output equation for the discretized model (11) of an elastic rod divided into 10 segments:

$$
\begin{aligned}
&k^{10}y_1 + 10gk^9y_1^{(1)} + (45g^2k^8 + 0.55k^9r)y_1^{(2)}\\
&+ (120g^3k^7 + 4.95gk^8r)y_1^{(3)}\\
&+ (210g^4k^6 + 19.8g^2k^7r + 0.0495k^8r^2)y_1^{(4)}\\
&+ (252g^5k^5 + 46.2g^3k^6r + 0.396gk^7r^2)y_1^{(5)}\\
&+ (210g^6k^4 + 69.3g^4k^5r\\
&\quad + 1.386g^2k^6r^2 + 0.001716k^7r^3)y_1^{(6)}\\
&+ (120g^7k^3 + 69.3g^5k^4r + 2.772g^3k^5r^2\\
&\quad + 0.012012gk^6r^3)y_1^{(7)}\\
&+ (45g^8k^2 + 46.2g^6k^3r + 3.465g^4k^4r^2 + 0.036036g^2k^5r^3\\
&\quad + 0.00003003k^6r^4)y_1^{(8)}\\
&+ (10g^9k + 19.8g^7k^2r + 2.772g^5k^3r^2 + 0.06006g^3k^4r^3\\
&\quad + 0.00018018gk^5r^4)y_1^{(9)}\\
&+ (g^{10} + 4.95g^8kr + 1.386g^6k^2r^2 + 0.06006g^4k^3r^3\\
&\quad + 0.00045045g^2k^4r^4 + 3.003\times10^{-7}k^5r^5)y_1^{(10)}\\
&+ (0.55g^9r + 0.396g^7kr^2 + 0.036036g^5k^2r^3\\
&\quad + 0.0006006g^3k^3r^4 + 1.5015\times10^{-6}gk^4r^5)y_1^{(11)}\\
&+ (0.0495g^8r^2 + 0.012012g^6kr^3 + 0.00045045g^4k^2r^4\\
&\quad + 3.003\times10^{-6}g^2k^3r^5 + 1.82\times10^{-9}k^4r^6)y_1^{(12)}\\
&+ (0.001716g^7r^3 + 0.00018018g^5kr^4\\
&\quad + 3.003\times10^{-6}g^3k^2r^5 + 7.28\times10^{-9}gk^3r^6)y_1^{(13)}\\
&+ (0.00003003g^6r^4 + 1.5015\times10^{-6}g^4kr^5\\
&\quad + 1.092\times10^{-8}g^2k^2r^6 + 6.8\times10^{-12}k^3r^7)y_1^{(14)}\\
&+ (3.003\times10^{-7}g^5r^5 + 7.28\times10^{-9}g^3kr^6\\
&\quad + 2.04\times10^{-11}gk^2r^7)y_1^{(15)}\\
&+ (1.82\times10^{-9}g^4r^6 + 2.04\times10^{-11}g^2kr^7\\
&\quad + 1.53\times10^{-14}k^2r^8)y_1^{(16)}\\
&+ (6.8\times10^{-12}g^3r^7 + 3.06\times10^{-14}gkr^8)y_1^{(17)}\\
&+ (1.53\times10^{-14}g^2r^8 + 1.9\times10^{-17}kr^9)y_1^{(18)}\\
&+ 1.9\times10^{-17}gr^9y_1^{(19)} + 10^{-20}r^{10}y_1^{(20)}\\
&= 0.1k^9u_1 + 0.9gk^8u_1^{(1)}\\
&+ 0.1(36g^2k^7 + 3.46945\times10^{-17}k^8r)u_1^{(2)}\\
&+ 0.1(84g^3k^6 + 2.77556\times10^{-16}gk^7r)u_1^{(3)}\\
&+ 0.1(126g^4k^5 + 5.55112\times10^{-16}g^2k^6r\\
&\quad + 8.13152\times10^{-18}k^7r^2)u_1^{(4)}\\
&+ 0.1(126g^5k^4 - 1.11022\times10^{-15}g^3k^5r\\
&\quad + 1.30104\times10^{-17}gk^6r^2)u_1^{(5)}\\
&+ 0.1(84g^6k^3 - 1.21431\times10^{-16}g^2k^5r^2\\
&\quad + 3.17637\times10^{-19}k^6r^3)u_1^{(6)}\\
&+ 0.1(36g^7k^2 - 1.11022\times10^{-15}g^5k^3r\\
&\quad + 1.56125\times10^{-16}g^3k^4r^2 + 1.0842\times10^{-18}gk^5r^3)u_1^{(7)}\\
&+ 0.1(9g^8k + 5.55112\times10^{-16}g^6k^2r\\
&\quad + 1.56125\times10^{-16}g^4k^3r^2\\
&\quad + 3.04932\times10^{-18}g^2k^4r^3 + 1.58819\times10^{-21}k^5r^4)u_1^{(8)}\\
&+ 0.1(g^9 + 2.77556\times10^{-16}g^7kr - 1.21431\times10^{-16}g^5k^2r^2\\
&\quad + 3.18484\times10^{-18}g^3k^3r^3\\
&\quad - 3.09696\times10^{-20}gk^4r^4)u_1^{(9)}\\
&+ 0.1(3.46945\times10^{-17}g^8r + 1.30104\times10^{-17}g^6kr^2\\
&\quad + 3.04932\times10^{-18}g^4k^2r^3 + 5.82335\times10^{-21}g^2k^3r^4\\
&\quad + 2.23274\times10^{-23}k^4r^5)u_1^{(10)}\\
&+ 0.1(8.13152\times10^{-18}g^7r^2 + 1.0842\times10^{-18}g^5kr^3\\
&\quad + 5.82335\times10^{-21}g^3k^2r^4 + 5.62224\times10^{-23}gk^3r^5)u_1^{(11)}\\
&+ 0.1(3.17637\times10^{-19}g^6r^3 - 3.09696\times10^{-20}g^4kr^4\\
&\quad - 3.44185\times10^{-22}g^2k^2r^5 - 2.26182\times10^{-25}k^3r^6)u_1^{(12)}\\
&+ 0.1(1.58819\times10^{-21}g^5r^4 + 5.62224\times10^{-23}g^3kr^5\\
&\quad - 1.29247\times10^{-24}gk^2r^6)u_1^{(13)}\\
&+ 0.1(2.23274\times10^{-23}g^4r^5\\
&\quad - 1.29247\times10^{-24}g^2kr^6 + 1.10441\times10^{-27}k^2r^7)u_1^{(14)}\\
&+ 0.1(-2.26182\times10^{-25}g^3r^6\\
&\quad + 2.20881\times10^{-27}gkr^7)u_1^{(15)}\\
&+ 0.1(1.10441\times10^{-27}g^2r^7\\
&\quad - 2.00297\times10^{-31}kr^8)u_1^{(16)}\\
&- 2.00297\times10^{-32}gr^8u_1^{(17)}.
\end{aligned}
$$

The equation for y_2 is omitted for simplicity. $y_1^{(n)}$, $u_1^{(n)}$ represents the nth order derivative of $y_1(t)$ and $u_1(t)$ with respect to t. Substituting the target functions g_1 into y_1 in this equation leads to a differential equation, of which unknown variable is u_1. The solution of this differential equation can be used as the input that makes y_1 asymptotically approach g_1; however, there are higher derivatives of u_1 with small coefficients on the right-hand side, which may make the solution unstable. Therefore we deleted the terms with negligible coefficients to get

$$
\begin{aligned}
&(\text{right-hand side}) \simeq 0.1k^9u_1 + 0.9gk^8u_1^{(1)}\\
&+ 0.1\times 36g^2k^7u_1^{(2)} + 0.1\times 84g^3k^6u_1^{(3)}\\
&+ 0.1\times 126g^4k^5u_1^{(4)} + 0.1\times 126g^5k^4u_1^{(5)}\\
&+ 0.1\times 84g^6k^3u_1^{(6)} + 0.1\times 36g^7k^2u_1^{(7)} + 0.1\times 9g^8ku_1^{(8)}\\
&+ 0.1\times g^9u_1^{(9)}.
\end{aligned}
$$

The input functions u_1 and u_2 shown in Figure 9 are obtained by solving this equation.

Appendix 5: The List of the Mathematical Terms

The mathematical terms used in the paper are listed below.

Explanation of the Terms Related to the Definition of the Gröbner Basis

- field: a set in which addition, subtraction, multiplication and division are defined.
(e.g.) the set of real numbers, the set of rational numbers, the set of complex numbers.

- ring: a set in which addition, subtraction and multiplication are defined.
(e.g.) the set of integers, the set of polynomials whose coefficients are rational numbers.

- polynomial ring $K[x_1, \ldots, x_n]$: a ring of polynomials in n variables $x_1, \ldots, x_n$ with coefficients in the field K.

- ideal (on a polynomial ring $K[x]$): a non-zero subset of $K[x]$ that satisfies

$$\text{if } f \in I, g \in I, \text{ then } f+g \in I,$$
$$\text{if } f \in I, g \in K[x], \text{ then } gf \in I$$

where $K[x]$ is a polynomial ring and f, g are polynomials. See “generator” for an example of ideals.

- order: $\leq$ is an order on a set Σ if the followings are satisfied for all $a, b, c \in \Sigma$:

$$a \leq a,$$
$$\text{if } a \leq b \text{ and } b \leq a, \text{ then } a = b,$$
$$\text{if } a \leq b \text{ and } b \leq c, \text{ then } a \leq c.$$

In this paper we only use the lexicographic order. This is also a total order and a monomial order, which are described below. Especially for lexicographic order, we show an example later.

- total order: an order $\leq$ is a total order on a set Σ if

$$a \leq b \text{ or } b \leq a, \text{ for all } a, b \in \Sigma.$$

- monomial order (of a ring $K[x]$): a total order $<$ on a set of monomials M_n of a ring $K[x] = K[x_1, \ldots, x_n]$ is a monomial order if the followings are satisfied:

$$1 < u \text{ for all } u \in M_n, u \neq 1,$$
$$\text{if } u < v, u, v \in M_n, \text{ then } uw < vw \text{ for all } w \in M_n.$$

- generator (of an ideal I): For an ideal I, if there exists a non-zero subset $\{f_\lambda \mid \lambda \in \Lambda\}$ of a ring $K[x]$ that satisfies

$$I = \left\{ \sum_{\lambda \in \Lambda} g_\lambda f_\lambda \,\middle|\, \text{for all } g_\lambda \in K[x] \right\},$$

$\{f_\lambda : \lambda \in \Lambda\}$ is called a generator of I. In that case, I is denoted by $I = \langle \{f_\lambda \mid \lambda \in \Lambda\} \rangle$.
(e.g.) the ideal I on a polynomial ring $K[x, y, z]$ of which generator is $y - x^2, x - z$ is

$$\begin{aligned} I &= \langle y - x^2, x - z \rangle \\ &= \{p(x,y,z)(y - x^2) + q(x,y,z)(x - z) \mid \\ &\qquad p(x,y,z),\ q(x,y,z) \in K[x,y,z]\}. \end{aligned}$$

Explanation of the Terms Related to “Elimination Theorem”

- lexicographic order: a monomial order $<$ is a lexicographic order if it satisfies the condition that for two monomials $u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $v = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$, $u < v$ if at least one of the followings holds:
  - the degree of v is larger than that of u
  - the degree of v is equal to that of u and the left most non-zero element of the vector $(b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n)$ is positive.

(e.g.) If $u = x_1 x_2 x_3^2$ (the degree is 4), $v = x_1^2 x_2^3 x_3$ (the degree is 6) then $u < v$ with respect to the lexicographic order since the first condition is satisfied.
(e.g.) If $u = x_1^3 x_2^3 x_3^2$ (the order is 8), $v = x_1^2 x_2^3 x_3^3$(the order is 8) then $v < u$ with respect to the lexicographic order. In fact, the first condition is not satisfied and the left most non-zero element of $(a_1 - b_1, a_2 - b_2, a_3 - b_3) = (1, 0, -1)$ is positive.

- support (of a polynomial f): a set of monomials $\{u_1, \ldots, u_t\}$ appears in a non-zero polynomial $f = a_1 u_1 + \cdots + a_t u_t$ of $K[x]$ where $0 \neq a_i \in K (i = 1, \ldots, t)$.

- leading monomial (of a polynomial f): the largest monomial in the support of a polynomial f with respect to a given monomial order $<$ of a ring $K[x]$, which is denoted by $\mathrm{in}_<(f)$.

- reduced Gröbner basis (of an ideal I with respect to a given monomial order $<$ on a polynomial ring $K[x]$: A Gröbner basis $\{g_1, \ldots, g_s\}$ of an ideal I with respect to a monomial order $<$ is the reduced Gröbner basis if the followings are satisfied:
  - the coefficient of $\mathrm{in}_<(g_i)$ of a polynomial g_i is 1 $(1 \leq i \leq s)$
  - if $i \neq j$, monomials in the support of g_i is not divisible by $\mathrm{in}_<(g_i)$.

The reduced Gröbner basis of an ideal I on a polynomial ring $K[x]$ with respect to a given monomial order is unique.