Low order stabilizing controllers for a class of distributed parameter systems

Sano, Hideki

(Citation)
Automatica,92:49-55

(Issue Date)
2018-06

(Resource Type)
journal article

(Version)
Accepted Manuscript

Low order stabilizing controllers for a class of distributed parameter systems ⋆

Hideki Sano [a]

[a] *Department of Applied Mathematics, Graduate School of System Informatics, Kobe University, 1-1 Rokkodai, Nada, Kobe 657-8501, Japan*

Abstract

This paper is concerned with reduction of the order of finite-dimensional stabilizing controllers for a class of distributed parameter systems. Since the middle of the 1980's, the design method of finite-dimensional stabilizing controllers of Sakawa type has been generalized for a wider class of parabolic distributed parameter systems with boundary control and/or boundary observation. The controller of Sakawa type consists of two kinds of observers: one is an observer of Luenberger type and the other is an estimator for residual modes. Especially, the latter is called residual mode filter (RMF), and it plays an essential role in the design of finite-dimensional stabilizing controllers when the order of RMF is "sufficiently large". The purpose of this paper is to propose the design method containing low order RMF. An approach based on stability radius is employed.

Key words: Distributed parameter system; finite-dimensional controller; residual mode filter; stability radius; semigroup.

1 Introduction

In the control theory of distributed parameter systems, the system described by the following evolution equation with output equation has been used for a long time.

$$\dot{z}(t) = -Az(t) + Bu(t), \quad t > 0, \quad z(0) = z_0, \tag{1}$$

$$y(t) = Cz(t), \quad t > 0, \tag{2}$$

where $-A$ is the infinitesimal generator of a C_0-semigroup on a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. $B : \mathbf{R}^m \to H$ and $C : H \to \mathbf{R}^p$ are bounded linear operators. $z(t) \in H$ is the state variable, $u(t) \in \mathbf{R}^m$ the input variable, and $y(t) \in \mathbf{R}^p$ the output variable. For systems (1)–(2), the stabilization problem by finite-dimensional controllers have been investigated by many researchers (see e.g. [17,6,22,15,1,9,12,18,7,11,8] and the references therein). Generally, when one constructs a finite-dimensional model for an infinite-dimensional system and applies a finite-dimensional controller to the original infinite-dimensional system, spillover phenomenon may occur due to the influence of unmodeled modes. Sakawa first introduced two kinds of finite-dimensional observers for linear diffusion systems to reduce the influence of unmodeled modes for the closed-loop system with the finite-dimensional controller [17]. Then, Balas called one of them the residual mode filter (RMF), and clarified that the RMF plays an essential role for the construction of finite-dimensional stabilizing controllers [1] [1]. Furthermore, Sano and Kunimatsu showed that the method could be extended to infinite-dimensional systems with A^γ-bounded output operators [18]. In those papers, by choosing the order of the RMF "sufficiently large", the closed-loop stability was assured. Independently of Sakawa's work [17], Curtain gave a design method for finite-dimensional stabilizing controllers for linear parabolic systems with unbounded control and observation [6], in which Schumacher's design method [22] for the case with bounded control and observation was extended to the unbounded case. Since there was no upper bound on the order of controller in both works [22,6], they used the perturbation result of Weinstein-Aronszajn determinant [13] to make the controller design feasible. After that, Fuentes and Balas applied the perturbation theory of operators to obtain the lowest order of RMF [11]. Also, in [8], the method

⋆ This research is supported by KAKENHI (Grant-in-Aid for Scientific Research (C), No. 15K04999), Japan Society for the Promotion of Science. The material in this paper was not presented at any conference. Corresponding author H. Sano.

Email address: sano@crystal.kobe-u.ac.jp (Hideki Sano).

[1] For nonlinear distributed parameter systems, Balas also introduced nonlinear RMFs to construct finite-dimensional stabilizing controllers [2].

of LQG-balancing was developed for model reduction of a class of infinite-dimensional systems, and the method was successfully applied to construct robust controllers.

In this paper, we consider the problem of reducing the order of RMFs in finite-dimensional controllers of Sakawa type, under the assumption that the eigenvalues and eigenfunctions of the state operator are completely known. As a technical tool, we use stability radius theory [16,4], and the approach is different from that of [11]. First of all, we survey the Sakawa's design method [17] and then give the modified version using the stability radius. But, to calculate stability radius, we need the value of H_∞-norm of a transfer function whose realization is described by infinite-dimensional operators in a Hilbert space. From the computational point of view, we need to prepare a family of approximate finite-dimensional operators and then to calculate the H_∞-norm of their transfer functions. However, it is not assured that they converge to the value of H_∞-norm of the original transfer function. The purpose of this paper is to justify the convergence and to propose an algorithm to reduce the order of RMFs. In addition, the case where the bounded output operator is replaced by an A^γ-bounded output operator is discussed. Finally, we give a numerical example to demonstrate the validity of the theory.

2 Sakawa's design method and its modification

2.1 System description

To explain the existing result [17,1] briefly for system (1)–(2), we consider the case where the operator A is defined by

$$Af = \sum_{i=1}^{\infty} \lambda_i \langle f, \phi_i \rangle \phi_i, \quad f \in D(A),$$
$$D(A) = \left\{ f \in H \,;\, \sum_{i=1}^{\infty} \lambda_i^2 \langle f, \phi_i \rangle^2 < +\infty \right\}, \tag{3}$$

where $\{\lambda_i, i \geq 1\}$ is a sequence of real numbers such that $\lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots$, $\lim_{i\to\infty} \lambda_i = \infty$, and $\{\phi_i, i \geq 1\}$ is a complete orthogonal system in H. From the definition, it is clear that the operator A is self-adjoint on H. By using Hille-Yosida's theorem [24,10], we see that $-A$ generates the C_0-semigroup e^{-tA} whose expression is given by $e^{-tA} f = \sum_{i=1}^{\infty} e^{-\lambda_i t} \langle f, \phi_i \rangle \phi_i$, $t \geq 0, f \in H$.

2.2 Partitioned system

In order to derive a finite-dimensional model for system (1)–(2), we use the orthgonal projection P_k defined by $P_k f = \sum_{i=1}^{k} \langle f, \phi_i \rangle \phi_i$. Here, using the operators P_l and P_n ($n > l$), we decompose system (1)–(2) as follows: First, we decompose the state variable $z(t)$ as $z(t) = z_1(t) + z_2(t) + z_3(t)$, where $z_1(t) := P_l z(t)$, $z_2(t) := (P_n - P_l) z(t)$, $z_3(t) := (I - P_n) z(t)$. Then, the state space H has the expression

$$H = \overbrace{P_l H}^{\dim = l} \oplus \overbrace{(P_n - P_l) H}^{\dim = n-l} \oplus \overbrace{(I - P_n) H}^{\dim = \infty}.$$

Accordingly, system (1)–(2) is expressed as follows (e.g. [1]):

$$\begin{cases} \dot{z}_1(t) = -A_1 z_1(t) + B_1 u(t), \;\; z_1(0) = P_l z_0, \\ \dot{z}_2(t) = -A_2 z_2(t) + B_2 u(t), \;\; z_2(0) = (P_n - P_l) z_0, \\ \dot{z}_3(t) = -A_3 z_3(t) + B_3 u(t), \;\; z_3(0) = (I - P_n) z_0, \\ y(t) = C_1 z_1(t) + C_2 z_2(t) + C_3 z_3(t), \end{cases} \tag{4}$$

where $A_1 := P_l A P_l$, $B_1 := P_l B$, $C_1 := C P_l$, $A_2 := (P_n - P_l) A (P_n - P_l)$, $B_2 := (P_n - P_l) B$, $C_2 := C(P_n - P_l)$, $A_3 := (I - P_n) A (I - P_n)$, $B_3 := (I - P_n) B$, $C_3 := C(I - P_n)$. In the above, the operator A_3 is unbounded, whereas all the other operators are bounded [2].

Hereafter, we identify the finite-dimensional Hilbert space $P_l H$ with the Euclidean space $\mathbf{R}^l$ with respect to the basis $\{\phi_1, \phi_2, \ldots, \phi_l\}$. In this way, each element in $P_l H$ is identified with an l-dimensional vector, and the operators A_1, B_1, and C_1 are identified with matrices with appropriate size. Similarly, each element in $(P_n - P_l) H$ is identified with an $(n - l)$-dimensional vector, and the operators A_2, B_2, and C_2 are identified with matrices with appropriate size.

2.3 Finite-dimensional controllers with RMFs

For the decomposed system (4), we consider the finite-dimensional system

$$\begin{cases} \dot{z}_1(t) = -A_1 z_1(t) + B_1 u(t), \\ \eta(t) = C_1 z_1(t), \end{cases} \tag{5}$$

as a finite-dimensional model of system (1)–(2). For the model, we set the following assumption.

Assumption 1 *(i) The integer $l\,(\geq 1)$ is chosen such that the eigenvalues of the matrix $-A_1$, $\sigma(-A_1)$ contains all unstable eigenvalues of the operator $-A$. (ii) The pair*

[2] The projections have been widely used. For example, Byrnes *et al.* solved the output regulation problem for a class of infinite-dimensional systems [3]. Christofides and Daoutidis applied approximate inertial manifolds to the stabilization problem of semilinear distributed parameter systems [5].

$(-A_1, B_1)$ *is controllable and the pair* $(C_1, -A_1)$ *is observable (see e.g. [25] for the definitions and the related theorems).*

Remark 1 *The second assumption (ii) can be relaxed as (ii') The pair* $(-A_1, B_1)$ *is stabilizable and the pair* $(C_1, -A_1)$ *is detectable.*

Under (ii) of Assumption 1 (or (ii') of Remark 1), we can choose a matrix F_1 such that $-A_1 - B_1F_1$ is Hurwitz, and we can choose a matrix G_1 such that $-A_1 - G_1C_1$ is Hurwitz (e.g. [25]). Here, we consider the observer-based controller

$$\begin{cases} \dot{w}_1(t) = (-A_1 - G_1C_1)w_1(t) + G_1y(t) + B_1u(t), \\ w_1(0) = w_{10}, \\ u(t) = -F_1w_1(t). \end{cases} \tag{6}$$

The control law (6) works as a stabilizing controller for the finite-dimensional model (5), however, it is not assured for the original system (1)–(2). For that reason, we use an RMF (7) together with the control law (6). Then, the whole controller is described as follows (see Fig. 1):

$$\begin{cases} \dot{w}_2(t) = -A_2w_2(t) + B_2u(t), \quad w_2(0) = w_{20}, \\ \hat{y}_2(t) = C_2w_2(t), \end{cases} \tag{7}$$

$$\begin{cases} \dot{w}_1(t) = (-A_1 - G_1C_1)w_1(t) + G_1(y(t) - \hat{y}_2(t)) \\ \qquad\qquad +B_1u(t), \quad w_1(0) = w_{10}, \\ u(t) = -F_1w_1(t). \end{cases} \tag{8}$$

Then, the following result is well-known.

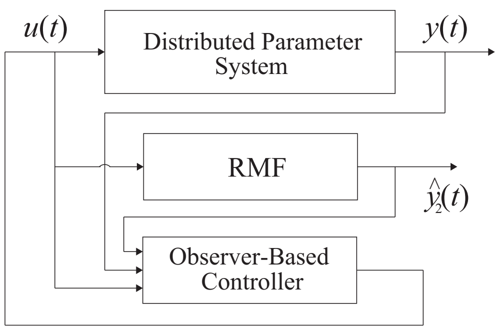


Fig. 1. Closed-loop system.

Theorem 2 *[17,1] Suppose that Assumption 1 is satisfied and let another integer* n *be chosen such that* $n > l$*. Then, the control consisting of (7)–(8) becomes a finite-dimensional stabilizing controller for system (1)–(2), if the integer* n *is chosen sufficiently large.*

Remark 3 *In [18], Theorem 2 was extended to the system whose output operator was* A^γ*-bounded.*

2.4 Modification of Theorem 2

Let us introduce new variables $e_1(t) := z_1(t) - w_1(t)$ and $e_2(t) := z_2(t) - w_2(t)$. Then, the closed-loop system consisting of system (1)–(2) and the controller (7)–(8) is written as

$$\dot{\xi}(t) = (\mathcal{A} + \mathcal{BKC})\xi(t), \quad \xi(0) = \xi_0, \tag{9}$$

where the state $\xi(t) := [e_1(t)^T, e_2(t)^T, z_1(t)^T, z_2(t)^T, z_3(t)]^T$ belongs to the real Hilbert space $Z := \mathbf{R}^l \times \mathbf{R}^{n-l} \times \mathbf{R}^l \times \mathbf{R}^{n-l} \times (I - P_n)H$, and the operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{K}$ are defined by

$$\mathcal{A} = \begin{bmatrix} -A_1 - G_1C_1 & -G_1C_2 & 0 & 0 & 0 \\ 0 & -A_2 & 0 & 0 & 0 \\ B_1F_1 & 0 & -A_1 - B_1F_1 & 0 & 0 \\ B_2F_1 & 0 & -B_2F_1 & -A_2 & 0 \\ B_3F_1 & 0 & -B_3F_1 & 0 & -A_3 \end{bmatrix},$$

$$\mathcal{B} = \begin{bmatrix} -G_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & C_3 \end{bmatrix}, \quad \mathcal{K} = 1. \tag{10}$$

Then, we have the following modified result for Theorem 2:

Theorem 4 *Suppose that Assumption 1 is satisfied and let another integer* n *be chosen such that* $n > l$*. Then, the operator* $\mathcal{A}$ *defined by (10) generates an exponentially stable* C_0*-semigroup* $e^{t\mathcal{A}}$ *with norm bound* $\|e^{t\mathcal{A}}\|_{\mathcal{L}(Z)} \le Me^{-\nu t}$*,* $t \ge 0$ *on* Z*, where* $M \ge 1$ *and* $\nu > 0$ *are some constants independent of the integer* n*. If the integer* n *is chosen such that the inequality*

$$\|\mathcal{C}(\cdot I - \mathcal{A})^{-1}\mathcal{B}\|_\infty := \sup_{\omega \in \mathbf{R}} \|\mathcal{C}(j\omega I - \mathcal{A})^{-1}\mathcal{B}\| < 1 \tag{11}$$

is satisfied, the control law (7)–(8) becomes a finite-dimensional stabilizing controller for system (1)–(2). Especially, when the integer n *is chosen sufficiently large, the inequality (11) is always satisfied, that is, the control law (7)–(8) works as a finite-dimensional stabilizing controller for system (1)–(2).*

Proof. The proof of the first assertion follows by using techniques similar to those in [18]. The remainder of the proof is due to the result with respect to the stability radius [16,4]. As shown in [16,4], the stability radius $r_c(\mathcal{A}; \mathcal{B}, \mathcal{C})$ of the closed-loop system (9) is calculated as

$$r_c(\mathcal{A}; \mathcal{B}, \mathcal{C}) = \frac{1}{\sup_{\omega \in \mathbf{R}} \|G(j\omega)\|} = \frac{1}{\|G(\cdot)\|_\infty},$$

where $G(j\omega) := \mathcal{C}(j\omega I - \mathcal{A})^{-1}\mathcal{B}$. Therefore, when $r_c(\mathcal{A};\mathcal{B},\mathcal{C}) > \|\mathcal{K}\| = 1$, i.e., the condition (11) holds, the second conclusion of the theorem immediately follows.

In order to prove the third assertion, we use the Hille-Yosida's theorem [24,10]. Since the operator $\mathcal{A}$ generates the C_0-semigroup $e^{t\mathcal{A}}$ with norm bound $\|e^{t\mathcal{A}}\|_{\mathcal{L}(Z)} \leq Me^{-\nu t}$, $t \geq 0$ on Z, we have $\|(j\omega I - \mathcal{A})^{-1}\|_{\mathcal{L}(Z)} \leq \frac{M}{\nu}$ for all $\omega \in \mathbf{R}$, which implies that $\|(\cdot I - \mathcal{A})^{-1}\|_\infty \leq \frac{M}{\nu}$, where $M \geq 1$ and $\nu > 0$ are some constants independent of the integer n. Here, noting that $\|\mathcal{B}\| = \|G_1\|$ does not depend on n and that $\|\mathcal{C}\| = \|C_3\| \to 0$ as $n \to \infty$, we have $\|\mathcal{C}(\cdot I - \mathcal{A})^{-1}\mathcal{B}\|_\infty \leq \|\mathcal{C}\|\|(\cdot I - \mathcal{A})^{-1}\|_\infty\|\mathcal{B}\| \leq \|C_3\|\|G_1\|\frac{M}{\nu} \to 0$ as $n \to \infty$. Therefore, it follows that there exists an integer n_1 such that $\|\mathcal{C}(\cdot I - \mathcal{A})^{-1}\mathcal{B}\|_\infty < 1$ for all $n \geq n_1$. In other words, for all $n \geq n_1$, the control law (7)–(8) works as a finite-dimensional stabilizing controller for system (1)–(2). The proof is thus complete. □

Remark 5 *The version without RMF for Theorem 4 has been given in [9]. Note that in [9] finite-dimensional stabilizing controllers are not always obtained by increasing the integer l.*

In Theorem 4, we note that the algorithm needs iteration of infinite times to check the condition (11), since it contains the infinite-dimensional operators A_3, B_3, and C_3. In Section 3, we discuss whether or not it is possible to approximate the operators A_3, B_3, and C_3 of the theorem by finite-dimensional operators. Moreover, instead of (2), we discuss the case with unbounded output operator such as

$$y(t) = \tilde{C}(A + c)^\gamma z(t), \quad 0 < \gamma < 1, \tag{12}$$

where A is the unbounded operator defined by (3), $\tilde{C} : H \to \mathbf{R}^p$ is a bounded linear operator, and c is a constant chosen such that $\lambda_1 + c > 0$. Here, note that, by using the fractional power of the operator, parabolic distributed parameter systems with boundary observation are formulated as system (1) and (12) (see e.g. [15]). Such a formulation allows for the study of a much broader class of distributed parameter systems.

3 Main result

By using the orthogonal projection P_k defined in Section 2, we decompose the state variable $z(t)$ as $z(t) = z_1(t) + z_2(t) + z_{3a}(t) + z_{3b}(t)$, where $z_1(t) := P_l z(t)$, $z_2(t) := (P_n - P_l)z(t)$, $z_{3a}(t) := (P_N - P_n)z(t)$, $z_{3b}(t) := (I - P_N)z(t)$, $N > n > l$. Note that $z_{3a}(t) + z_{3b}(t) = z_3(t)$. Also, the space H is expressed as

$$H = \overbrace{P_l H}^{\dim = l} \oplus \overbrace{(P_n - P_l)H}^{\dim = n-l} \oplus \underbrace{\overbrace{(P_N - P_n)H}^{\dim = N-n} \oplus \overbrace{(I - P_N)H}^{\dim = \infty}}_{=(I-P_n)H}.$$

Then, the infinite-dimensional operators A_3, B_3, and C_3 are equivalently expressed as follows:

$$A_3 = \begin{bmatrix} A_{3a} & 0 \\ 0 & A_{3b} \end{bmatrix}, \quad B_3 = \begin{bmatrix} B_{3a} \\ B_{3b} \end{bmatrix}, \quad C_3 = \begin{bmatrix} C_{3a} & C_{3b} \end{bmatrix},$$

where $A_{3a} := (P_N - P_n)A(P_N - P_n)$, $B_{3a} := (P_N - P_n)B$, $C_{3a} := C(P_N - P_n)$, $A_{3b} := (I - P_N)A(I - P_N)$, $B_{3b} := (I - P_N)B$, $C_{3b} := C(I - P_N)$. Here, note that the operators A_{3a}, B_{3a}, and C_{3a} are identified with matrices with appropriate size. Then, the operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are expressed as

$$\mathcal{A} = \begin{bmatrix} -A_1 - G_1C_1 & -G_1C_2 & 0 & 0 & 0 & 0 \\ 0 & -A_2 & 0 & 0 & 0 & 0 \\ B_1F_1 & 0 & -A_1 - B_1F_1 & 0 & 0 & 0 \\ B_2F_1 & 0 & -B_2F_1 & -A_2 & 0 & 0 \\ B_{3a}F_1 & 0 & -B_{3a}F_1 & 0 & -A_{3a} & 0 \\ B_{3b}F_1 & 0 & -B_{3b}F_1 & 0 & 0 & -A_{3b} \end{bmatrix},$$

$$\mathcal{B} = \begin{bmatrix} -G_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & C_{3a} & C_{3b} \end{bmatrix}. \tag{13}$$

Further, we set the truncated operators as

$$\mathcal{A}_N = \begin{bmatrix} -A_1 - G_1C_1 & -G_1C_2 & 0 & 0 & 0 \\ 0 & -A_2 & 0 & 0 & 0 \\ B_1F_1 & 0 & -A_1 - B_1F_1 & 0 & 0 \\ B_2F_1 & 0 & -B_2F_1 & -A_2 & 0 \\ B_{3a}F_1 & 0 & -B_{3a}F_1 & 0 & -A_{3a} \end{bmatrix},$$

$$\mathcal{B}_N = \begin{bmatrix} -G_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{C}_N = \begin{bmatrix} 0 & 0 & 0 & 0 & C_{3a} \end{bmatrix}. \tag{14}$$

Now, let us define two transfer functions as follows:

$$G(j\omega) = \mathcal{C}(j\omega I - \mathcal{A})^{-1}\mathcal{B}, \tag{15}$$
$$G_N(j\omega) = \mathcal{C}_N(j\omega I - \mathcal{A}_N)^{-1}\mathcal{B}_N. \tag{16}$$

The following theorem, remarks, and algorithm are our main result in this paper.

Theorem 6 *Suppose that Assumption 1 is satisfied. Then, the operator $\mathcal{A}_N$ defined by (14) generates a C_0-semigroup $e^{t\mathcal{A}_N}$ with norm bound $\|e^{t\mathcal{A}_N}\| \le Me^{-\nu t}$, $t \ge 0$ on the Euclidean space $Z_N := \mathbf{R}^l \times \mathbf{R}^{n-l} \times \mathbf{R}^l \times \mathbf{R}^{n-l} \times \mathbf{R}^{N-n}$, where $M \ge 1$ and $\nu > 0$ are some constants independent of the integer N. Moreover, there holds*

$$\|G_N(\cdot)\|_\infty \to \|G(\cdot)\|_\infty \quad as \quad N \to \infty,$$

that is, $r_c(\mathcal{A}_N; \mathcal{B}_N, \mathcal{C}_N) \to r_c(\mathcal{A}; \mathcal{B}, \mathcal{C})$ as $N \to \infty$. Accordingly, if $\|G_N(\cdot)\|_\infty < 1$ is satisfied for sufficiently large N, the control law (7)–(8) works as a finite-dimensional stabilizing controller for system (1)–(2).

Proof. By Assumption 1, the C_0-semigroup generated by the matrix

$$\mathcal{A}_1 := \begin{bmatrix} -A_1 - G_1C_1 & -G_1C_2 & 0 & 0 \\ 0 & -A_2 & 0 & 0 \\ B_1F_1 & 0 & -A_1 - B_1F_1 & 0 \\ B_2F_1 & 0 & -B_2F_1 & -A_2 \end{bmatrix}$$

has a norm bound $\|e^{t\mathcal{A}_1}\| \le M_1e^{-\nu_1 t}$, $t \ge 0$, where $M_1 \ge 1$ and $\nu_1 > 0$ are some constants independent of the integer N. Also, the C_0-semigroup generated by the matrix $-A_{3a}$ has a norm bound $\|e^{-tA_{3a}}\| \le e^{-\lambda_{n+1}t}$, $t \ge 0$. Here, noting that $\|B_{3a}F_1\| \le \|B\|\|F_1\|$, we see that the first assertion holds by using a technique similar to what was used in [18].

Next, we estimate the H_∞-norm of $G(j\omega) - G_N(j\omega)$. From (13)–(16), we have

$$\begin{aligned} G(j\omega) &= C_{3a}(j\omega I + A_{3a})^{-1}B_{3a}H(j\omega) \\ &\quad + C_{3b}(j\omega I + A_{3b})^{-1}B_{3b}H(j\omega), \\ G_N(j\omega) &= C_{3a}(j\omega I + A_{3a})^{-1}B_{3a}H(j\omega), \end{aligned}$$

by straightforward calculation, where

$$H(j\omega) := \begin{bmatrix} F_1 & 0 & -F_1 & 0 \end{bmatrix} H_1(j\omega) \begin{bmatrix} -G_1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$H_1(j\omega) := \begin{bmatrix} j\omega I + A_1 + G_1C_1 & G_1C_2 & 0 & 0 \\ 0 & j\omega I + A_2 & 0 & 0 \\ -B_1F_1 & 0 & j\omega I + A_1 + B_1F_1 & 0 \\ -B_2F_1 & 0 & B_2F_1 & j\omega I + A_2 \end{bmatrix}^{-1}.$$

From these, it follows that

$$G(j\omega) - G_N(j\omega) = C_{3b}(j\omega I + A_{3b})^{-1}B_{3b}H(j\omega).$$

By Assumption 1, it is easy to see that $\|H(\cdot)\|_\infty < +\infty$. Also, noting that

$$\|(\cdot I + A_{3b})^{-1}\|_\infty \le \frac{1}{\lambda_{N+1}} \to 0 \;\text{ as }\; N \to \infty, \tag{17}$$

and that $\|B_{3b}\|$, $\|C_{3b}\| \to 0$ as $N \to \infty$, we have $|\|G(\cdot)\|_\infty - \|G_N(\cdot)\|_\infty| \le \|G(\cdot) - G_N(\cdot)\|_\infty \to 0$ as $N \to \infty$, which implies that the second assertion holds.

If $\|G_N(\cdot)\|_\infty < 1$ is satisfied for sufficiently large N, it follows from the second assertion that $\|G(\cdot)\|_\infty < 1$. Therefore, from Theorem 4, the third assertion holds. □

Remark 7 *When the output equation (12) is used instead of (2), we obtain the same assertions as in Theorem 6, by replacing the control law (7)–(8) by*

$$\begin{cases} \dot{w}_2(t) = -A_2w_2(t) + B_2u(t), \quad w_2(0) = w_{20}, \\ \hat{y}_2(t) = \tilde{C}_2(A_2 + c)^\gamma w_2(t), \end{cases}$$

$$\begin{cases} \dot{w}_1(t) = (-A_1 - G_1\tilde{C}_1(A_1 + c)^\gamma)w_1(t) \\ \qquad\qquad + G_1(y(t) - \hat{y}_2(t)) + B_1u(t), \; w_1(0) = w_{10}, \\ u(t) = -F_1w_1(t). \end{cases}$$

In this case, since the operators C_1, C_2, C_{3a}, and C_{3b} are replaced as $\tilde{C}_1(A_1 + c)^\gamma$, $\tilde{C}_2(A_2 + c)^\gamma$, $\tilde{C}_{3a}(A_{3a} + c)^\gamma$, and $\tilde{C}_{3b}(A_{3b} + c)^\gamma$ in the operators (13)–(14), we need to use the following estimate instead of (17):

$$\begin{aligned} &\|(A_{3b} + c)^\gamma(\cdot I + A_{3b})^{-1}\|_\infty \\ &\le \frac{(\lambda_{N+1} + c)^\gamma + \lambda_{N+1}^\gamma\Gamma(1-\gamma)}{\lambda_{N+1}} \to 0 \;\; as \;\; N \to \infty, \end{aligned} \tag{18}$$

where $\Gamma(\cdot)$ is the gamma function. For the derivation of (18), see [19].

Remark 8 *Theorems 2, 4, and 6 also hold in the case where the operator A is replaced by a Riesz-spectral operator (see e.g. [7] for Riesz-spectral operators). The fact is shown based on the work in Curtain [6] and Sano [20].*

Lastly, by using Theorem 6, we can give an algorithm to reduce the order of RMFs in finite-dimensional controllers of Sakawa type.

<u>**Algorithm 1**</u>

Step 1. Derive a finite-dimensional model (5) that satisfies Assumption 1. Let the order of the model be l.

Step 2. Construct an observer-based controller (8) of the order l.

Step 3. Set $k = 1$.

Step 4. Construct an RMF (7) of the order $n - l = k$.

Step 5. For $G_N(\cdot)$ defined by (16), does $\|G_N(\cdot)\|_\infty < 1$ hold for sufficiently large N ? If yes, go to Step 6. If no, $k = k + 1$ and then go to Step 4.

Step 6. The order of RMF is k. ¶

Remark 9 *Theorem 4 assures that Algorithm 1 can be terminated after a finite number of iterations.*

Remark 10 *Based on the result by Balas [1], one could replace the observer-based controller (8) in Step 2 with a static output feedback controller* $u(t) = -Q_1(y(t) - \hat{y}_2(t))$*, if for the model (5) there exists a matrix* Q_1 *such that* $\sigma(-A_1 - B_1Q_1C_1) \subset \{\,\lambda \in \mathbf{C}\,;\, \mathrm{Re}(\lambda) < 0\,\}$*. Then,* $G_N(\cdot)$ *in Step 5, that is,* $\mathcal{A}_N$*,* $\mathcal{B}_N$*, and* $\mathcal{C}_N$ *in (14) are replaced by appropriate operators.*

4 Numerical example

We consider the following linear diffusion system:

$$\begin{cases} z_t(t,x) = \varepsilon z_{xx}(t,x) + \mu z(t,x) \\ \qquad\qquad +b(x)u(t), \quad t > 0,\ x \in (0,1), \\ z_x(t,0) = 0, \quad z(t,1) = 0, \quad t > 0, \\ z(0,x) = z_0(x), \quad x \in [0,1], \end{cases} \tag{19}$$

where $z(t,x) \in \mathbf{R}$ is the temperature at time t and at the point $x \in [0,1]$, $u(t) \in \mathbf{R}$ is the control input, and, $\varepsilon > 0$ and $\mu > 0$ are physical parameters. $b(x) := \frac{1}{r}\mathbf{1}_{[x_0-r/2,x_0+r/2]}(x)$ denotes the actuator influence function, where $\mathbf{1}_{[\,\cdot\,,\,\cdot\,]}(x)$ denotes the characteristic function. We first consider the following observation for system (19):

$$y(t) = \int_0^1 c(x)z(t,x)dx, \tag{20}$$

where $c(x) := \frac{1}{r}\mathbf{1}_{[x_1-r/2,x_1+r/2]}(x)$ is the sensor influence function.

We formulate system (19)–(20) in a Hilbert space $L^2(0,1)$, where $L^2(0,1)$ is the usual L^2-space with inner product $\langle\varphi,\psi\rangle := \int_0^1 \varphi(x)\psi(x)dx$, $\varphi,\psi \in L^2(0,1)$. Setting $\mathcal{L}\varphi = -\varepsilon\varphi'' - \mu\varphi$, we define the unbounded operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ as

$$A\varphi = \mathcal{L}\varphi, \quad \varphi \in D(A),$$
$$D(A) = \{\,\varphi \in H^2(0,1)\,;\, \varphi'(0) = 0,\ \varphi(1) = 0\,\}.$$

Then, A is a self-adjoint operator in $L^2(0,1)$ and it has the following eigenvalues and eigenfunctions: $\lambda_i = \varepsilon(i-\frac{1}{2})^2\pi^2 - \mu$, $\varphi_i(x) = \sqrt{2}\cos(i-\frac{1}{2})\pi x$, $i \geq 1$, where $\{\varphi_i\}_{i=1}^\infty$ forms a complete orthogonal system in $L^2(0,1)$. Note that the operator $-A$ generates an analytic semigroup e^{-tA} on $L^2(0,1)$ whose growth bound is equal to $-\lambda_1$. If $-\lambda_1 > 0$, it is clear that system (19)–(20) is unstable. Here, by defining the bounded operators $B : \mathbf{R} \to L^2(0,1)$ and $C : L^2(0,1) \to \mathbf{R}$ as

$$Bv = bv, \quad v \in \mathbf{R},$$
$$C\zeta = \langle c,\zeta\rangle, \quad \zeta \in L^2(0,1),$$

system (19)–(20) is expressed as in (1)–(2).

Next, we consider the following boundary observation for system (19):

$$y(t) = z_x(t,1). \tag{21}$$

In this case, by defining the unbounded operator $C : D(A) \to \mathbf{R}$ as $C\zeta = \zeta'(1)$, $\zeta \in D(A)$, we can express the observation equation (21) as

$$y(t) = Cz(t). \tag{22}$$

On the other hand, using techniques similar to those in [21], we can formulate the observation equation (21) as

$$y(t) = \tilde{C}(A+c)^\gamma z(t), \tag{23}$$

where $\gamma := \frac{3}{4} + \epsilon' \in (\frac{3}{4},1)$, $\tilde{C} : L^2(0,1) \to \mathbf{R}$ is the bounded operator defined by

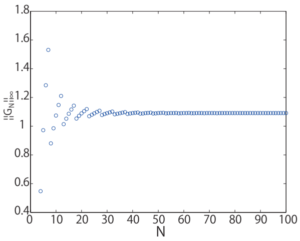
$$\tilde{C}\xi = \langle -\tfrac{1}{\varepsilon}(A+c)^{\frac{1}{4}-\epsilon'}h, \xi\rangle, \quad \xi \in L^2(0,1),$$

and c is a constant chosen such that $\lambda_1 + c > 0$. In the above, $h \in H^2(0,1)$ is the unique solution of the boundary value problem

$$(\mathcal{L}+c)h = 0 \ \text{ in } (0,1), \quad h'(0) = 0, \ \ h(1) = 1.$$

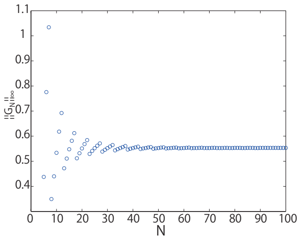
As for the derivation of (23), refer to [21] for details. Especially, when $c = \mu$, the solution is concretely given by $h(x) \equiv 1$. Here, note that the operator $\tilde{C}(A+c)^\gamma$ of (23) is the Λ-extension of the operator C of (22) (see e.g. [23] for the definition of Λ-extension).

Now, let $\varepsilon = 0.1$, $\mu = 1$, $x_0 = 0.8$, $x_1 = 0.4$, $r = 0.02$, and $\epsilon' = 0.15$. Then, we see that $-A$ has one unstable eigenvalue. Next, by setting $l = 2$, we can derive two models $(-A_1, B_1, C_1)$ and $(-A_1, B_1, \tilde{C}_1(A_1+\mu)^\gamma)$ that satisfy Assumption 1. These models correspond to the low order finite-dimensional models of system (19)–(20) and system (19), (21), respectively. For each model, we choose F_1 as an optimal regulator gain and choose G_1 as an optimal filter gain (e.g. [25]). For the model



$(l, n) = (2, 3)$.

$\|G_{100}\|_\infty = 1.0908\ (> 1)$.



$(l, n) = (2, 4)$.

$\|G_{100}\|_\infty = 0.5531\ (< 1)$.

Fig. 2. The case of distributed observation (20).

$(-A_1, B_1, C_1)$, we set the weights as $Q = 750I_2$, $R = 1$. As a result, $-A_1 - B_1F_1$ and $-A_1 - G_1C_1$ become Hurwitz. Fig. 2 shows that Algorithm 1 can be terminated after 2 iterations and that the order of RMF is obtained as $n - l = 2$.

For the model $(-A_1, B_1, \tilde{C}_1(A_1+\mu)^\gamma)$, we set the weights as $Q = 5I_2$, $R = 1$. As well, $-A_1 - B_1F_1$ and $-A_1 - G_1\tilde{C}_1(A_1+\mu)^\gamma$ become Hurwitz. Fig. 3 shows that Algorithm 1 can be terminated after 6 iterations and that the order of RMF is $n - l = 6$.

As shown in the numerical example, the convergence speed of the case with boundary observation (21) is late compared with that of the case with distributed observation (20). The difference is caused by the estimates (17) and (18). In the numerical simulation, we used MATLAB Control System Toolbox.

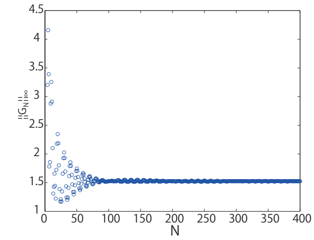
5 Concluding remarks

In this paper, we treated the problem of reducing the order of finite-dimensional stabilizing controllers for parabolic distributed parameter systems, using an approach based on stability radius. Here, we remark that it is possible to express the closed-loop system as

$$\dot{\xi}(t) = (\mathcal{A}' + \mathcal{B}'\mathcal{K}'\mathcal{C}')\xi(t), \quad \xi(0) = \xi_0, \tag{24}$$

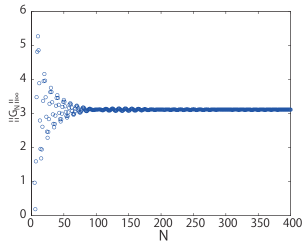
where the state $\xi(t)$ is in the real Hilbert space Z, and the operators $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{C}'$, and $\mathcal{K}'$ are defined by

$$\mathcal{A}' = \begin{bmatrix} -A_1 - G_1C_1 & -G_1C_2 & 0 & 0 & -G_1C_3 \\ 0 & -A_2 & 0 & 0 & 0 \\ B_1F_1 & 0 & -A_1 - B_1F_1 & 0 & 0 \\ B_2F_1 & 0 & -B_2F_1 & -A_2 & 0 \\ 0 & 0 & 0 & 0 & -A_3 \end{bmatrix},$$



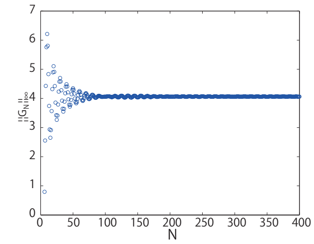
$(l, n) = (2, 3)$.

$\|G_{400}\|_\infty = 1.5241\ (> 1)$.



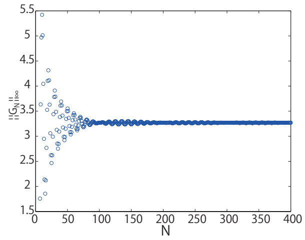
$(l, n) = (2, 4)$.

$\|G_{400}\|_\infty = 3.1131\ (> 1)$.



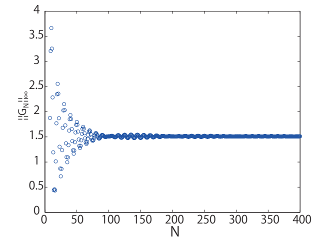
$(l, n) = (2, 5)$.

$\|G_{400}\|_\infty = 4.0633\ (> 1)$.



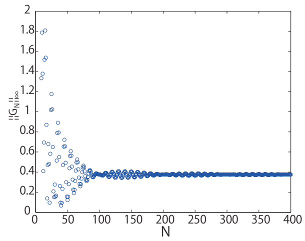
$(l, n) = (2, 6)$.

$\|G_{400}\|_\infty = 3.2691\ (> 1)$.



$(l, n) = (2, 7)$.

$\|G_{400}\|_\infty = 1.5109\ (> 1)$.



$(l, n) = (2, 8)$.

$\|G_{400}\|_\infty = 0.3755\ (< 1)$.

Fig. 3. The case of boundary observation (21).

$$\mathcal{B}' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ B_3 \end{bmatrix}, \quad \mathcal{C}' = \begin{bmatrix} F_1 & 0 & -F_1 & 0 & 0 \end{bmatrix}, \quad \mathcal{K}' = 1.$$

Then, we have the similar results as in Theorems 4 and 6 and Algorithm 1. Here, noting that the transfer function $G'(j\omega) := \mathcal{C}'(j\omega I - \mathcal{A}')^{-1}\mathcal{B}'$ is equal to the transfer function $G(j\omega)$ defined by (15) and that $\mathcal{K} = \mathcal{K}' = 1$, we see that the order of RMF obtained by the algorithm is the same for the both expression (9) and (24) of the closed-loop system.

While we demonstrated a design method of low order stabilizing controllers, it is not clear whether or not the order of RMF obtained in Algorithm 1 is minimal. For example, for the same closed-loop system, one could also consider the following alternative:

$$\dot{\xi}(t) = (\tilde{\mathcal{A}} + \tilde{\mathcal{B}}\tilde{\mathcal{K}}\tilde{\mathcal{C}})\xi(t), \quad \xi(0) = \xi_0, \tag{25}$$

where

$$\tilde{\mathcal{A}} = \begin{bmatrix} -A_1 - G_1C_1 & -G_1C_2 & 0 & 0 & 0 \\ 0 & -A_2 & 0 & 0 & 0 \\ B_1F_1 & 0 & -A_1 - B_1F_1 & 0 & 0 \\ B_2F_1 & 0 & -B_2F_1 & -A_2 & 0 \\ 0 & 0 & 0 & 0 & -A_3 \end{bmatrix},$$

$$\tilde{\mathcal{B}} = \begin{bmatrix} -G_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & B_3 \end{bmatrix}, \tilde{\mathcal{C}} = \begin{bmatrix} 0 & 0 & 0 & 0 & C_3 \\ F_1 & 0 & -F_1 & 0 & 0 \end{bmatrix}, \tilde{\mathcal{K}} = I_2.$$

On the other hand, Fuentes and Balas employed the perturbation theory of operators to get the lowest order of RMF [11] as stated in Section 1. The problem of comparing our method numerically with it remains as the future study.

Further, the author plans to study the design method of finite-dimension for distributed parameter systems with input delay such as treated in [14].

Acknowledgements

The author would like to thank the reviewers for their very helpful comments and suggestions.

References

[1] Balas, M.J. (1988). Finite-dimensional controllers for linear distributed parameter systems: exponential stability using residual mode filters. *J. Math. Anal. Appl.*, 133, 283–296.

[2] Balas, M.J. (1991). Nonlinear finite-dimensional control of a class of nonlinear distributed parameter systems using residual mode filters: a proof of local exponential stability. *J. Math. Anal. Appl.*, 162, 63–70.

[3] Byrnes, C.I., Gilliam, D.S., and Shubov, V.I. (2003). Geometric theory of output regulation for linear distributed parameter systems. *Research Directions in Distributed Parameter Systems. Edited by R.C. Smith and M.A. Demetriou. Chapter 6.* SIAM, Philadelphia.

[4] Chicone, C. and Latushkin, Y. (1999). *Evolution Semigroups in Dynamical Systems and Differential Equations. Mathematical Surveys and Monographs, Vol. 70.* American Mathematical Society.

[5] Christofides, P.D. and Daoutidis, P. (1997). Finite-dimensional control of parabolic PDE systems using approximate inertial manifolds. *J. Math. Anal. Appl.*, 216, 398–420.

[6] Curtain, R.F. (1984). Finite dimensional compensators for parabolic distributed systems with unbounded control and observation. *SIAM J. Control Optim.*, 22, 255–276.

[7] Curtain, R.F. and Zwart, H.J. (1995). *An Introduction to Infinite-Dimensional Linear Systems Theory. Texts in Applied Mathematics, Vol. 21.* Springer-Verlag, New York.

[8] Curtain, R.F. (2003). Model reduction for control design for distributed parameter systems. *Research Directions in Distributed Parameter Systems. Edited by R.C. Smith and M.A. Demetriou. Chapter 4.* SIAM, Philadelphia.

[9] El Jai, A. and Pritchard, A.J. (1988). *Sensors and Controls in the Analysis of Distributed Systems. English Language Edition.* Ellis Horwood Limited, Chichester.

[10] Engel, K.-J. and Nagel, R. (2000). *One-Parameter Semigroups for Linear Evolution Equations. Graduate Texts in Mathematics, Vol. 194.* Springer-Verlag, New York.

[11] Fuentes, R.J. and Balas, M.J. (1999). Stable controller design for deployable precision structures using perturbation theory. *Proc. AIAA Guidance, Navigation, and Control Conference and Exhibit 1999*, 1410–1415.

[12] Ito, K. (1990). Finite-dimensional compensators for infinite-dimensional systems via Galerkin approximation. *SIAM J. Control Optim.*, 28, 1251–1269.

[13] Kato, T. (1966). *Perturbation Theory for Linear Operators.* Springer-Verlag, Berlin.

[14] Krstic, M. (2009). *Delay Compensation for Nonlinear, Adaptive, and PDE Systems. Systems & Control: Foundations & Applications.* Birkhäuser, Boston.

[15] Nambu, T. (1985). On stabilization of partial differential equations of parabolic type: boundary observation and feedback. *Funkcialaj Ekvacioj, Serio Internacia*, 28, 267–298.

[16] Pritchard, A.J. and Townley, S. (1989). Robustness of linear systems. *J. Differential Equations*, 77, 254–286.

[17] Sakawa, Y. (1983). Feedback stabilization of linear diffusion systems. *SIAM J. Control Optim.*, 21, 667–676.

[18] Sano, H. and Kunimatsu, N. (1994). Feedback stabilization of infinite-dimensional systems with A^γ-bounded output operators. *Appl. Math. Lett.*, 7(5), 17–22.

[19] Sano, H. (1999). Finite-dimensional H_∞ control of linear parabolic systems with unbounded output operators. *Int. J. Control*, 72(16), 1466–1479.

[20] Sano, H. (2002). Finite-dimensional H_∞ control of flexible beam equation systems. *IMA J. Math. Control Inform.*, 19, 477–491.

[21] Sano, H. (2012). Stability-enhancing control of a coupled transport-diffusion system with Dirichlet actuation and Dirichlet measurement. *J. Math. Anal. Appl.*, 388, 1194–1204.

[22] Schumacher, J.M. (1983). A direct approach to compensator design for distributed parameter systems. *SIAM J. Control Optim.*, 21, 823–836.

[23] Weiss, G. and Curtain, R.F. (1997). Dynamic stabilization of regular linear systems. *IEEE Trans. Automatic Control*, 42(1), 4–21.

[24] Yosida, K. (1980). *Functional Analysis. Sixth Edition.* Springer-Verlag, Berlin.

[25] Zhou, K., Doyle, J., and Glover, K. (1997). *Robust and Optimal Control. Translated from the English by K.Z. Liu and Z.H. Luo.* Corona, Tokyo, (in Japanese).