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Stability analysis of heat exchangers with delayed boundary feedback ⋆

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Abstract

In this paper, we analyze the stability of heat exchangers with delayed boundary feedback. Heat exchangers of both counter-flow type and parallel-flow type are treated, where they are respectively described by a coupled hyperbolic equation with time lags in the boundary feedback loop. In our previous work [Sano, H., IFAC-PapersOnLine, 49-8, pp. 43–47, 2016], a condition for the system with time lags to be exponentially stable was derived for each heat exchanger, after the time lag was expressed by a transport equation. However, the case with small time lags was excluded. In this paper, it is shown that, under the presence of any time lag, the upper bound of the feedback gain to assure the closed-loop stability is solved by computing the H_∞-norm of a complex function. That is, each closed-loop system with admissible feedback gain is robust with respect to any time lag.

Key words: Hyperbolic equation; boundary feedback; time lag; stability; semigroup.

1 Introduction

In recent years, the control theory of coupled first-order hyperbolic equations has been widely studied (e.g. [1–6,8–10,12,14,17,21,23–30] and the references therein). Many of the works are related to the counter-flow heat exchanger equation, and some of them the parallel-flow heat exchanger equation. We first review the works having relevance to these two kinds to heat exchanger equations.

i) Counter-flow heat exchanger: The following heterodirectional coupled hyperbolic equation with $\nu_1, \nu_2, h_1, h_2 > 0$ is called a counter-flow heat exchanger equation:

$$\begin{cases} \dfrac{\partial \theta_1}{\partial t}(t,x) = -\nu_1 \dfrac{\partial \theta_1}{\partial x}(t,x) + h_1(\theta_2(t,x) - \theta_1(t,x)), \\ \dfrac{\partial \theta_2}{\partial t}(t,x) = \quad \nu_2 \dfrac{\partial \theta_2}{\partial x}(t,x) + h_2(\theta_1(t,x) - \theta_2(t,x)), \\ \theta_1(t,0) = -k_1\theta_2(t,0),\ \theta_2(t,l) = -k_2\theta_1(t,l) + u(t). \end{cases} \quad (1)$$

For general heterodirectional coupled hyperbolic equations, Auriol and Di Meglio have constructed a control law that achieves the finite-time stabilization in minimum time using the backstepping method [1]. Coron *et al.* have proposed a design method to achieve the same object using Fredholm backstepping transformation [6]. Also, Di Meglio *et al.* have solved the stabilization problem by state feedback/output feedback using only one boundary control [9], where the backstepping method is applied as well. Hu *et al.* have solved the same problem and trajectory tracking problem for more general systems [12]. Further, Deutscher has treated the finite-time output regulation problem [8]. Besides, Villegas *et al.* have developed a method of stability analysis using the port Hamiltonian approach [27]. C.Z. Xu and Sallet have analyzed the stability of a network composed of two counter-flow heat exchanger equations [29]. Recently, an explicit sufficient stability condition was given for systems related to (1) using a frequency domain method [4].

ii) Parallel-flow heat exchanger: The following homodirectional coupled hyperbolic equation with $\nu_1, \nu_2, h_1, h_2 > 0$ is called a parallel-flow heat exchanger equation:

$$\begin{cases} \dfrac{\partial \theta_1}{\partial t}(t,x) = -\nu_1 \dfrac{\partial \theta_1}{\partial x}(t,x) + h_1(\theta_2(t,x) - \theta_1(t,x)), \\ \dfrac{\partial \theta_2}{\partial t}(t,x) = -\nu_2 \dfrac{\partial \theta_2}{\partial x}(t,x) + h_2(\theta_1(t,x) - \theta_2(t,x)), \\ \theta_1(t,0) = -k_1\theta_2(t,l),\ \theta_2(t,0) = -k_2\theta_1(t,l) + u(t). \end{cases} \quad (2)$$

For general homodirectional coupled hyperbolic equations, X. Xu and Dubljevic have solved the output regulation problem [30]. Also, the reachability problem and output tracking problem for the parallel-flow heat exchanger equation were studied in [21], [23], respectively.

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In this paper, we consider systems with $u(t) \equiv 0$ and $k_1 = 0$ in (1) and (2) [1]. Here, we assume that there is a time lag $\tau > 0$ in the boundary feedback loop. The goal of this work is to analyze the stability of such heat exchangers with delay. Thus, the problem setting is simple since we do not design $u(t)$ for some kind of purpose. In our previous work [24], the same problem was studied, where the time lag was expressed by using a transport equation and the port Hamiltonian approach [27] was applied to the whole system. Then, the same condition was obtained for assuring exponential stability for two kinds of heat exchangers. However, the condition in Propositions 2.2 and 3.2 of [24] excludes the interval $0 < \tau \le \frac{h_1 l}{\nu_1}$, that is, the case with small time lags. In other words, the problem of whether or not the system with small time lags is stable remains unsolved. Motivated by the work of Auriol *et al.* [2], we apply, in this paper, a frequency domain approach to the characteristic equation. For each heat exchanger, it is shown that, under the presence of any time lag, the upper bound of the feedback gain to assure the closed-loop stability is solved by computing the H_∞-norm of a complex function. That is, each closed-loop system with admissible feedback gain is robust with respect to any time lag. Numerical examples are also given to demonstrate the validity of the proposed method.

The contribution of this paper lies in:

- giving the characterization of the spectrum of each closed-loop operator, and
- giving the upper bound of feedback gain whose closed-loop system is robust with respect to any time lag.

Thus, this paper has the nature of compensating for the results [24] derived by the port Hamiltonian approach.

2 Counter-flow heat exchanger

2.1 System description and existing result

We shall consider the following counter-flow heat exchanger equation with delayed boundary feedback:

$$\begin{cases} \dfrac{\partial \theta_1}{\partial t}(t,x) = -\nu_1 \dfrac{\partial \theta_1}{\partial x}(t,x) + h_1(\theta_2(t,x) - \theta_1(t,x)), \\ \dfrac{\partial \theta_2}{\partial t}(t,x) = \quad \nu_2 \dfrac{\partial \theta_2}{\partial x}(t,x) + h_2(\theta_1(t,x) - \theta_2(t,x)), \\ \qquad\qquad\qquad\qquad\qquad (t,x) \in (0,\infty) \times (0,l), \\ \theta_1(t,0) = 0, \quad \theta_2(t,l) = -k\theta_1(t-\tau,l), \quad t > 0, \\ \theta_1(0,x) = \theta_{10}(x), \quad \theta_2(0,x) = \theta_{20}(x), \quad x \in [0,l], \\ \theta_1(s,l) = \phi(s), \quad s \in (-\tau, 0), \end{cases} \tag{3}$$

where $l > 0$ is the length of two tubes, $\theta_1(t,x), \theta_2(t,x) \in \mathbf{R}$ are the temperature of two fluids at time t and at the point $x \in [0,l]$, $\nu_1, \nu_2 > 0$ denote the fluid velocity, $h_1, h_2 > 0$ the heat exchange rate, k the feedback gain, $\tau > 0$ a time lag. In the engineering point of view, it is considered that the time lag is large to some extent, because the quick adjustment of heat is generally difficult.

Replacing the term $\theta_1(t-\tau,l)$ by a transport equation with velocity $\mu := \frac{l}{\tau}$ (e.g. Krstic and Smyshlyaev [16]), system (3) is equivalently expressed as

$$\begin{cases} \dfrac{\partial \theta_1}{\partial t}(t,x) = -\nu_1 \dfrac{\partial \theta_1}{\partial x}(t,x) + h_1(\theta_2(t,x) - \theta_1(t,x)), \\ \dfrac{\partial \theta_2}{\partial t}(t,x) = \quad \nu_2 \dfrac{\partial \theta_2}{\partial x}(t,x) + h_2(\theta_1(t,x) - \theta_2(t,x)), \\ \qquad\qquad\qquad\qquad\qquad (t,x) \in (0,\infty) \times (0,l), \\ \theta_1(t,0) = 0, \quad \theta_2(t,l) = -kw(t,l), \quad t > 0, \\ \theta_1(0,x) = \theta_{10}(x), \quad \theta_2(0,x) = \theta_{20}(x), \quad x \in [0,l], \\ \dfrac{\partial w}{\partial t}(t,x) = -\mu \dfrac{\partial w}{\partial x}(t,x), \quad (t,x) \in (0,\infty) \times (0,l), \\ w(t,0) = \theta_1(t,l), \quad t > 0, \\ w(0,x) = w_0(x) := \phi\big(\frac{1}{\mu}(l-x)\big), \quad x \in (0,l). \end{cases} \tag{4}$$

By conducting the variable transformation $[\theta_1(t,x), \theta_2(t,x), w(t,x)]^T = T[\varphi_1(t,x), \varphi_2(t,x), u(t,x)]^T$, where $T := \mathrm{diag}(\sqrt{h_1}, \sqrt{h_2}, 1)$, system (4) is written as

$$\frac{\partial}{\partial t}\begin{bmatrix} \varphi_1(t,x) \\ \varphi_2(t,x) \\ u(t,x) \end{bmatrix} = P_1' \frac{\partial}{\partial x}\begin{bmatrix} \varphi_1(t,x) \\ \varphi_2(t,x) \\ u(t,x) \end{bmatrix} - G_0 \begin{bmatrix} \varphi_1(t,x) \\ \varphi_2(t,x) \\ u(t,x) \end{bmatrix}, \tag{5}$$

where $\varphi_1(t,0) = 0$, $\varphi_2(t,l) = -\frac{k}{\sqrt{h_2}}u(t,l)$, $u(t,0) = \sqrt{h_1}\varphi_1(t,l)$, and

$$P_1' := \begin{bmatrix} -\nu_1 & 0 & 0 \\ 0 & \nu_2 & 0 \\ 0 & 0 & -\mu \end{bmatrix}, \quad G_0 := \begin{bmatrix} h_1 & -\sqrt{h_1 h_2} & 0 \\ -\sqrt{h_1 h_2} & h_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In [24], the following result is obtained:

Proposition 1 *Let $k^2 < \frac{h_2 \nu_1}{\nu_2 h_1}$ be satisfied. If $\frac{h_1 l}{\nu_1} < \tau < \frac{h_2 l}{\nu_2 k^2}$, then the operator A_T defined by*

$$\begin{aligned} &A_T z = P_1' \frac{\partial}{\partial x} z - G_0 z, \ z = [\varphi_1, \varphi_2, u]^T \in D(A_T), \\ &D(A_T) = \{\, z = [\varphi_1, \varphi_2, u]^T \in [H^1(0,l)]^3\,; \\ &\qquad \varphi_1(0) = 0,\ \varphi_2(l) = -\tfrac{k}{\sqrt{h_2}}u(l),\ u(0) = \sqrt{h_1}\varphi_1(l) \,\} \end{aligned} \tag{6}$$

generates an exponentially stable C_0-semigroup e^{tA_T} on $[L^2(0,l)]^3$. That is, system (3) is exponentially stable.

Remark 2 *When $\tau = 0$, the following facts are known:*
(i) *System (3) with $k = 0$ is exponentially stable [28].*
(ii) *System (3) is exponentially stable if $k^2 < \frac{h_2 \nu_1}{\nu_2 h_1}$ [27].*

[1] For the parallel-flow type, the system with zero boundary condition does not have finite poles. In (2), $\theta_2(t,0) = -k_2\theta_1(t,l)$, $k_2 \neq 0$ is needed to produce finite poles, whereas $u(t)$ is major control and is used for modal control etc. [15]. This paper focuses on the systems without major control.

2.2 Frequency domain approach

First, let us formulate system (4) in a Hilbert space $X := [L^2(0,l)]^3$ with inner product defined by $\langle f, g\rangle_X := a\langle f_1, g_1\rangle + b\langle f_2, g_2\rangle + \langle f_3, g_3\rangle$ for $f = [f_1, f_2, f_3]^T \in X$, $g = [g_1, g_2, g_3]^T \in X$, where $a := \frac{\mu h_1}{\nu_1}$, $b := \frac{\mu h_2}{\nu_2 k^2}$, and $\langle \varphi, \psi\rangle := \int_0^l \varphi(x)\overline{\psi(x)}dx$ for $\varphi, \psi \in L^2(0,l)$. Define the linear operator $A : D(A) \subset X \to X$ as

$$Af = \begin{bmatrix} -\nu_1 \frac{d}{dx} - h_1 & h_1 & 0 \\ h_2 & \nu_2 \frac{d}{dx} - h_2 & 0 \\ 0 & 0 & -\mu \frac{d}{dx} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix},$$

$$D(A) = \{\, f = [f_1, f_2, f_3]^T \in [H^1(0,l)]^3\,; \; f_1(0) = 0,\ f_2(l) = -kf_3(l),\ f_3(0) = f_1(l)\,\}. \quad (7)$$

Then, system (4) can be written as

$$\frac{d}{dt}\begin{bmatrix} \theta_1(t,\cdot) \\ \theta_2(t,\cdot) \\ w(t,\cdot) \end{bmatrix} = A \begin{bmatrix} \theta_1(t,\cdot) \\ \theta_2(t,\cdot) \\ w(t,\cdot) \end{bmatrix}, \quad \begin{bmatrix} \theta_1(0,\cdot) \\ \theta_2(0,\cdot) \\ w(0,\cdot) \end{bmatrix} = \begin{bmatrix} \theta_{10} \\ \theta_{20} \\ w_0 \end{bmatrix}, \quad (8)$$

where $\theta_{10}, \theta_{20}, w_0 \in L^2(0,l)$ is assumed. Now, we prove that the operator A generates a C_0-semigroup on X. For more general class of hyperbolic systems, the proof is given in Bastin and Coron [3]. But, the inner product $\langle \cdot, \cdot \rangle_X$ mentioned above is different from theirs. It is also possible to apply Le Gorrec *et al.* [10] to the proof.

Theorem 3 *For any $k \in \mathbf{R}$, the operator A defined by (7) generates a C_0-semigroup e^{tA} on X.*

Proof. First, we define the operator $T \in \mathcal{L}(X)$ as $T = \mathrm{diag}(\sqrt{h_1}, \sqrt{h_2}, 1)$. Then, the operator $T^{-1}AT$ is expressed as

$$T^{-1}ATf = \begin{bmatrix} -\nu_1 \frac{d}{dx} - h_1 & \sqrt{h_1 h_2} & 0 \\ \sqrt{h_1 h_2} & \nu_2 \frac{d}{dx} - h_2 & 0 \\ 0 & 0 & -\mu \frac{d}{dx} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix},$$

$$D(T^{-1}AT) = \{\, f = [f_1, f_2, f_3]^T \in [H^1(0,l)]^3\,; \; f_1(0) = 0,\ f_2(l) = -\tfrac{k}{\sqrt{h_2}} f_3(l),\ f_3(0) = \sqrt{h_1} f_1(l)\,\}.$$

Here, we see that

$$\mathrm{Re}\langle T^{-1}ATf, f\rangle_X \le \omega \|f\|_X^2, \quad \forall f \in D(T^{-1}AT), \quad (9)$$

where $\omega := \frac{1}{2}\max\{\frac{b}{a}h_1, \frac{a}{b}h_2\}$ (> 0). Also, using a method similar to [25], it can be shown that there exists a number $\lambda > 0$ such that the range of $\lambda I - T^{-1}AT$ is equal to X, i.e.,

$$R(\lambda I - T^{-1}AT) = X. \quad (10)$$

It follows from (9)–(10) that the operator $T^{-1}AT - \omega I$ generates a contractive C_0-semigroup $e^{t(T^{-1}AT - \omega I)}$ on X, by using the Lumer-Phillips' Theorem [20, Theorem 1.4.3] (see also the proof of [7, Corollary 2.2.3]). That is, the operator A generates a C_0-semigroup e^{tA} on X for any $k \in \mathbf{R}$. □

In this section, we set $\alpha_1 := \frac{h_1}{\nu_1}$, $\beta_1 := \frac{h_2}{\nu_2}$, $\alpha_2 := \frac{1}{\nu_1}$, $\beta_2 := \frac{1}{\nu_2}$, and define $\lambda_\pm := -\frac{(\sqrt{\alpha_1} \pm \sqrt{\beta_1})^2}{\alpha_2 + \beta_2}$, $k_\pm := \pm\sqrt{\frac{\beta_1}{\alpha_1}} e^{\lambda_\pm \tau}(1 \mp \frac{1}{l\sqrt{\alpha_1 \beta_1}})$ (double sign in same order).

Lemma 4 *Let $S = \{\lambda \in \mathbf{C}\,; [C(\lambda) + 2k\alpha_1 e^{-\tau\lambda}]\sinh z + \frac{2z}{l}\cosh z = 0\}$, where $z := \frac{l}{2}\sqrt{\Delta(\lambda)}$, $\Delta(\lambda) := C(\lambda)^2 - 4\alpha_1\beta_1$, $C(\lambda) := \alpha_1 + \beta_1 + (\alpha_2 + \beta_2)\lambda$, and $\sqrt{\Delta(\lambda)}$ is taken of non-negative real part. Then, the spectrum of A is given by $\sigma(A) = S$ if $k \neq k_\pm$, $\sigma(A) = S \cup \{\lambda_+\}$ if $k = k_+$, and $\sigma(A) = S \cup \{\lambda_-\}$ if $k = k_-$.*

Proof. For counter-flow type, we can obtain the spectrum of A by using a method similar to [28,17]. So, we omit the detail here. □

Based on Lemma 4, we have the following theorem:

Theorem 5 *If k satisfies $|k| < \frac{\gamma}{2\alpha_1}$, system (3) is exponentially stable for any time lag $\tau > 0$, where*

$$\gamma := \left\| \left(C(\lambda) + \frac{2z}{l}\frac{\cosh z}{\sinh z} \right)^{-1} \right\|_\infty^{-1}. \quad (11)$$

Proof. Set $\Phi(\lambda) := [C(\lambda) + 2k\alpha_1 e^{-\tau\lambda}]\sinh z + \frac{2z}{l}\cosh z$ and $\Phi_0(\lambda) := C(\lambda)\sinh z + \frac{2z}{l}\cosh z$. When $k = 0$, it is known that the open-loop system is exponentially stable [28]. This means that $|\Phi_0(\lambda)| > 0$ for all $\lambda \in \mathbf{C}_0^+ := \{\lambda \in \mathbf{C}\,; \mathrm{Re}(\lambda) \ge 0\}$.

(a) In the case of $\alpha_1 \neq \beta_1$, for all $\lambda \in \mathbf{C}_0^+$ we have $z \neq n\pi i$ $(n = 0, \pm1, \pm2, \ldots)$ [2], that is, $\sinh z \neq 0$. For any fixed $\lambda \in \mathbf{C}_0^+$, we can estimate $|\Phi(\lambda)|$ as follows:

$$\begin{aligned} |\Phi(\lambda)| &= |\Phi_0(\lambda) + 2k\alpha_1 e^{-\tau\lambda}\sinh z| \quad (12) \\ &\ge |\Phi_0(\lambda)| - |2k\alpha_1 e^{-\tau\lambda}\sinh z| \ge |\Phi_0(\lambda)| - 2\alpha_1|k||\sinh z| \\ &= |\sinh z| \left\{ \left| \frac{\Phi_0(\lambda)}{\sinh z} \right| - 2\alpha_1|k| \right\} \ge |\sinh z|(\gamma - 2\alpha_1|k|), \end{aligned}$$

where $\gamma := \inf_{\mathrm{Re}(\lambda)\ge 0} |C(\lambda) + \frac{2z}{l}\frac{\cosh z}{\sinh z}| = \|(C(\lambda) + \frac{2z}{l}\frac{\cosh z}{\sinh z})^{-1}\|_\infty^{-1}$. The notation $\|\cdot\|_\infty$ denotes the H_∞-norm and it is defined by $\|G\|_\infty := \sup_{\mathrm{Re}(\lambda)>0}|G(\lambda)|$ for G which is holomorphic on the open right-half plane on $\mathbf{C}$ and $\sup_{\mathrm{Re}(\lambda)>0}|G(\lambda)| < \infty$. Therefore, we have $|\Phi(\lambda)| > 0$ if $|k| < \frac{\gamma}{2\alpha_1}$. In other words, $\Phi(\lambda)$ does not have any root with non-negative real part. In the case

[2] Note that, if $z = n\pi i$ $(n = 0, \pm1, \pm2, \ldots)$, there holds $\mathrm{Re}(\lambda) = \mathrm{Re}(\frac{\pm\sqrt{4\alpha_1\beta_1 - 4n^2\pi^2/l^2} - \alpha_1 - \beta_1}{\alpha_2 + \beta_2}) < 0$ under $\alpha_1 \neq \beta_1$.

of $\alpha_1 \neq \beta_1$, since $\lambda_- < 0$, we see that A does not have any spectrum with non-negative real part for any time lag $\tau > 0$ under the condition $|k| < \frac{\gamma}{2\alpha_1}$.

(b) Next, in the case of $\alpha_1 = \beta_1$, for all $\lambda \in \mathbf{C}_0^+$ we have $z \neq n\pi i$ ($n = \pm 1, \pm 2, \ldots$). Especially, note that $\lambda = 0 \in \mathbf{C}_0^+$ corresponds to $z = 0$. For any fixed $\lambda (\neq 0) \in \mathbf{C}_0^+$, we can estimate $|\Phi(\lambda)|$ as follows:

$$\begin{aligned} |\Phi(\lambda)| &= |\Phi_0(\lambda) + 2k\alpha_1 e^{-\tau\lambda} \sinh z| \qquad (13) \\ &\geq |\Phi_0(\lambda)| - |2k\alpha_1 e^{-\tau\lambda} \sinh z| \geq |\Phi_0(\lambda)| - 2\alpha_1|k||\sinh z| \\ &= |\sinh z| \left\{ \left| \frac{\Phi_0(\lambda)}{\sinh z} \right| - 2\alpha_1|k| \right\} \geq |\sinh z|(\gamma' - 2\alpha_1|k|), \end{aligned}$$

where $\gamma' := \inf_{\substack{\mathrm{Re}(\lambda)\geq 0 \\ \lambda\neq 0}} |C(\lambda) + \frac{2z}{l}\frac{\cosh z}{\sinh z}|$. Note that $\gamma' = \gamma$. Therefore, we have $|\Phi(\lambda)| > 0$ if $|k| < \frac{\gamma}{2\alpha_1}$. Also, when $\lambda = 0 \in \mathbf{C}_0^+$, that is, $z = 0$, it follows from (13) that $|\Phi(\lambda)| = |\Phi_0(\lambda)| > 0$. Hence, for any $\lambda \in \mathbf{C}_0^+$ we obtain $|\Phi(\lambda)| > 0$ if $|k| < \frac{\gamma}{2\alpha_1}$. In the case where $\alpha_1 = \beta_1$ and $|k| < \frac{\gamma}{2\alpha_1}$, note that the spectrum $\lambda_- = 0$ does not appear, since $k_- = -(1 + \frac{1}{l\alpha_1})$ and $\frac{\gamma}{2\alpha_1} = 1 + \frac{1}{l\alpha_1} = |k_-|$. Accordingly, in the case of $\alpha_1 = \beta_1$, A does not have any spectrum with non-negative real part for any time lag $\tau > 0$ under the condition $|k| < \frac{\gamma}{2\alpha_1}$.

(c) Finally, we show that, under the condition $|k| < \frac{\gamma}{2\alpha_1}$, the spectrum of A does not approach asymptotically to the imaginary axis. Note that, since the operator A has compact resolvent, the spectrum $\sigma(A)$ entirely consists of isolated eigenvalues with finite multiplicities and has no accumulation point different from ∞. Assume that there exists a spectrum $\lambda_n = \xi_n + i\eta_n \in \mathbf{C} - \mathbf{C}_0^+$ such that $\xi_n \to 0$ and $|\eta_n| \to \infty$. Then, from Lemma 4, the spectrum λ_n satisfies

$$\begin{aligned} \Phi(\lambda_n) &= [C(\lambda_n) + 2k\alpha_1 e^{-\tau\lambda_n}] \sinh z_n + \frac{2z_n}{l} \cosh z_n \\ &= 0, \end{aligned} \qquad (14)$$

where $z_n := \frac{l}{2}\sqrt{\Delta(\lambda_n)}$. On the other hand, we have

$$\sup\{\, \mathrm{Re}(z)\,;\, \lambda = j\omega,\ \omega \in \mathbf{R} \,\} = \frac{l}{2}(\alpha_1 + \beta_1). \qquad (15)$$

From (15), it is not difficult to see that

$$|\sinh z| \geq \frac{1}{2}\left(e^{\frac{l}{2}(\alpha_1+\beta_1)} - e^{-\frac{l}{2}(\alpha_1+\beta_1)}\right) =: \delta, \qquad (16)$$

as $\lambda = j\omega$ and $|\omega| \to \infty$. Therefore, under the condition $|k| < \frac{\gamma}{2\alpha_1}$, it follows from (12) (or (13)) and (16) that

$$|\Phi(j\omega)| \geq \delta(\gamma - 2\alpha_1|k|) > 0 \quad \text{as} \quad |\omega| \to \infty. \qquad (17)$$

This contradicts (14). Hence, the spectrum of A does not approach asymptotically to the imaginary axis.

From (a)–(c), we see that, if k satisfies $|k| < \frac{\gamma}{2\alpha_1}$, the spectrum bound of A is strictly negative and further system (3) is exponentially stable for any time lag $\tau > 0$, by using the result by F.L. Huang [13] on the spectrum determined growth condition [3]. □

This proof shows that the closed-loop system with the same feedback gain is exponentially stable for the case of $\tau = 0$.

3 Parallel-flow heat exchanger

3.1 System description and existing result

In this section, we shall consider the following parallel-flow heat exchanger equation with delayed boundary feedback:

$$\begin{cases} \dfrac{\partial\theta_1}{\partial t}(t,x) = -\nu_1 \dfrac{\partial\theta_1}{\partial x}(t,x) + h_1(\theta_2(t,x) - \theta_1(t,x)), \\ \dfrac{\partial\theta_2}{\partial t}(t,x) = -\nu_2 \dfrac{\partial\theta_2}{\partial x}(t,x) + h_2(\theta_1(t,x) - \theta_2(t,x)), \\ \qquad\qquad (t,x) \in (0,\infty) \times (0,l), \\ \theta_1(t,0) = 0, \quad \theta_2(t,0) = -k\theta_1(t-\tau,l), \quad t > 0, \\ \theta_1(0,x) = \theta_{10}(x), \quad \theta_2(0,x) = \theta_{20}(x), \quad x \in [0,l], \\ \theta_1(s,l) = \phi(s), \quad s \in (-\tau, 0). \end{cases} \qquad (18)$$

By defining $\mu := \frac{l}{\tau}$, system (18) becomes

$$\begin{cases} \dfrac{\partial\theta_1}{\partial t}(t,x) = -\nu_1 \dfrac{\partial\theta_1}{\partial x}(t,x) + h_1(\theta_2(t,x) - \theta_1(t,x)), \\ \dfrac{\partial\theta_2}{\partial t}(t,x) = -\nu_2 \dfrac{\partial\theta_2}{\partial x}(t,x) + h_2(\theta_1(t,x) - \theta_2(t,x)), \\ \qquad\qquad (t,x) \in (0,\infty) \times (0,l), \\ \theta_1(t,0) = 0, \quad \theta_2(t,0) = -kw(t,l), \quad t > 0, \\ \theta_1(0,x) = \theta_{10}(x), \quad \theta_2(0,x) = \theta_{20}(x), \quad x \in [0,l], \\ \dfrac{\partial w}{\partial t}(t,x) = -\mu \dfrac{\partial w}{\partial x}(t,x), \quad (t,x) \in (0,\infty) \times (0,l), \\ w(t,0) = \theta_1(t,l), \quad t > 0, \\ w(0,x) = \phi\left(\frac{1}{\mu}(l-x)\right), \quad x \in (0,l). \end{cases} \qquad (19)$$

Using the same variable transformation $[\theta_1(t,x), \theta_2(t,x), w(t,x)]^T = T[\varphi_1(t,x), \varphi_2(t,x), u(t,x)]^T$ as in Subsection 2.1, where $T := \mathrm{diag}(\sqrt{h_1}, \sqrt{h_2}, 1)$, system (19) is written as

$$\frac{\partial}{\partial t}\begin{bmatrix} \varphi_1(t,x) \\ \varphi_2(t,x) \\ u(t,x) \end{bmatrix} = \tilde{P}_1' \frac{\partial}{\partial x}\begin{bmatrix} \varphi_1(t,x) \\ \varphi_2(t,x) \\ u(t,x) \end{bmatrix} - G_0 \begin{bmatrix} \varphi_1(t,x) \\ \varphi_2(t,x) \\ u(t,x) \end{bmatrix}, \qquad (20)$$

[3] In Appendix B of this paper, it is explained about how to use the result by F.L. Huang for the parallel-flow type.

where $\varphi_1(t,0) = 0$, $\varphi_2(t,0) = -\frac{k}{\sqrt{h_2}}u(t,l)$, $u(t,0) = \sqrt{h_1}\varphi_1(t,l)$, and

$$\tilde{P}_1' := \begin{bmatrix} -\nu_1 & 0 & 0 \\ 0 & -\nu_2 & 0 \\ 0 & 0 & -\mu \end{bmatrix}, \ G_0 := \begin{bmatrix} h_1 & -\sqrt{h_1 h_2} & 0 \\ -\sqrt{h_1 h_2} & h_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In [24], the following result is obtained:

Proposition 6 *Let $k^2 < \frac{h_2\nu_1}{\nu_2 h_1}$ be satisfied. If $\frac{h_1 l}{\nu_1} < \tau < \frac{h_2 l}{\nu_2 k^2}$, then the operator $\tilde{A}_T$ defined by*

$$\begin{aligned} &\tilde{A}_T z = \tilde{P}_1' \frac{\partial}{\partial x} z - G_0 z, \ z = [\varphi_1, \varphi_2, u]^T \in D(\tilde{A}_T), \quad (21)\\ &D(\tilde{A}_T) = \{\, z = [\varphi_1, \varphi_2, u]^T \in [H^1(0,l)]^3\,; \\ &\quad \varphi_1(0) = 0,\ \varphi_2(0) = -\tfrac{k}{\sqrt{h_2}}u(l),\ u(0) = \sqrt{h_1}\varphi_1(l)\,\} \end{aligned}$$

generates an exponentially stable C_0-semigroup $e^{t\tilde{A}_T}$ on $[L^2(0,l)]^3$. That is, system (18) is exponentially stable.

3.2 Frequency domain approach

Similarly as in Subsection 2.2, we formulate system (19) in a Hilbert space $X := [L^2(0,l)]^3$ with inner product defined by $\langle f, g\rangle_X := a\langle f_1, g_1\rangle + b\langle f_2, g_2\rangle + \langle f_3, g_3\rangle$ for $f = [f_1, f_2, f_3]^T \in X$, $g = [g_1, g_2, g_3]^T \in X$, where $a := \frac{\mu h_1}{\nu_1}$, $b := \frac{\mu h_2}{\nu_2 k^2}$, and $\langle \varphi, \psi\rangle := \int_0^l \varphi(x)\overline{\psi(x)}dx$ for $\varphi, \psi \in L^2(0,l)$. Define the linear operator $\tilde{A} : D(\tilde{A}) \subset X \to X$ as

$$\begin{aligned} &\tilde{A}f = \begin{bmatrix} -\nu_1\frac{d}{dx} - h_1 & h_1 & 0 \\ h_2 & -\nu_2\frac{d}{dx} - h_2 & 0 \\ 0 & 0 & -\mu\frac{d}{dx} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, \\ &D(\tilde{A}) = \{\, f = [f_1, f_2, f_3]^T \in [H^1(0,l)]^3\,; \\ &\quad f_1(0) = 0,\ f_2(0) = -kf_3(l),\ f_3(0) = f_1(l)\,\}. \quad (22) \end{aligned}$$

Then, system (19) can be written as

$$\frac{d}{dt}\begin{bmatrix} \theta_1(t,\cdot) \\ \theta_2(t,\cdot) \\ w(t,\cdot) \end{bmatrix} = \tilde{A}\begin{bmatrix} \theta_1(t,\cdot) \\ \theta_2(t,\cdot) \\ w(t,\cdot) \end{bmatrix}, \ \begin{bmatrix} \theta_1(0,\cdot) \\ \theta_2(0,\cdot) \\ w(0,\cdot) \end{bmatrix} = \begin{bmatrix} \theta_{10} \\ \theta_{20} \\ w_0 \end{bmatrix}, \quad (23)$$

where $\theta_{10}, \theta_{20}, w_0 \in L^2(0,l)$ is assumed.

Theorem 7 *For any $k \in \mathbf{R}$, the operator $\tilde{A}$ defined by (22) generates a C_0-semigroup $e^{t\tilde{A}}$ on X.*

Proof. By using a method similar to the proof of Theorem 3, we can also give the proof of this theorem. So, we omit the detail here. □

In this section, we set $\alpha_1 := \frac{h_1}{\nu_1}$, $\beta_1 := \frac{h_2}{\nu_2}$, $\alpha_2 := \frac{1}{\nu_1}$, $\beta_2 := \frac{1}{\nu_2}$, $Q := \{n \in \mathbf{Z}\,;\ \frac{(2n-1)(\alpha_2-\beta_2)\pi}{2\sqrt{\alpha_1\beta_1}} > \frac{l}{2}(\alpha_2+\beta_2)\}$, and define $\tau_n := \frac{(2n-1)(\alpha_2-\beta_2)\pi}{2\sqrt{\alpha_1\beta_1}} - \frac{l}{2}(\alpha_2+\beta_2)$, $n \in Q$. For α_1, β_1, α_2, β_2 with $\alpha_2 \neq \beta_2$, define $\lambda_\pm := -\frac{\alpha_1-\beta_1}{\alpha_2-\beta_2} \pm \frac{2\sqrt{\alpha_1\beta_1}}{\alpha_2-\beta_2}i$, $p(\tau) := \frac{l}{2}[-(\alpha_1+\beta_1) + (\alpha_2+\beta_2+\frac{2\tau}{l})\frac{\alpha_1-\beta_1}{\alpha_2-\beta_2}]$.

Lemma 8 *Let $\tilde{S} = \{\lambda \in \mathbf{C}\,;\ \frac{z}{l}e^{\frac{l}{2}[\alpha_1+\beta_1+(\alpha_2+\beta_2)\lambda]} + k\alpha_1 e^{-\tau\lambda}\sinh z = 0\}$, where $z := \frac{l}{2}\sqrt{\tilde{\Delta}(\lambda)}$, $\tilde{\Delta}(\lambda) := \tilde{C}(\lambda)^2 + 4\alpha_1\beta_1$, $\tilde{C}(\lambda) := \alpha_1-\beta_1+(\alpha_2-\beta_2)\lambda$, and $\sqrt{\tilde{\Delta}(\lambda)}$ is taken of non-negative real part. In the case of $\alpha_2 \neq \beta_2$, the spectrum of $\tilde{A}$ is given by $\sigma(\tilde{A}) = \tilde{S} \cup \{\lambda_\pm\}$ if $\tau = \tau_n$ and $k = \frac{e^{-p(\tau_n)}}{\alpha_1 l}$ for some $n \in Q$, and $\sigma(\tilde{A}) = \tilde{S}$ if $\tau \neq \tau_n$ or $k \neq \frac{e^{-p(\tau_n)}}{\alpha_1 l}$ for any $n \in Q$. In the case of $\alpha_2 = \beta_2$, it is given by $\sigma(\tilde{A}) = \tilde{S}$.*

Proof. Since the operator $\tilde{A}$ has compact resolvent (see Appendix A), we have only to consider the eigenvalues of the operator $\tilde{A}$, that is, $\lambda \in \mathbf{C}$ such that $\tilde{A}f = \lambda f$, $f = [f_1, f_2, f_3]^T \neq 0$. This equation is equivalent to

$$\begin{cases} -\nu_1 f_1' - h_1 f_1 + h_1 f_2 = \lambda f_1, \quad f_1(0) = 0, \\ -\nu_2 f_2' + h_2 f_1 - h_2 f_2 = \lambda f_2, \quad f_2(0) = -kf_3(l), \\ -\mu f_3' = \lambda f_3, \quad f_3(0) = f_1(l), \end{cases} \quad (24)$$

which is further equivalent to the following equation:

$$\begin{cases} f_1'' + [\alpha_1+\beta_1+(\alpha_2+\beta_2)\lambda]f_1' \\ \qquad +[(\alpha_1\beta_2+\alpha_2\beta_1)\lambda + \alpha_2\beta_2\lambda^2]f_1 = 0, \\ f_3' = -\frac{\lambda}{\mu}f_3, \\ f_1(0) = 0,\ f_1'(0) + \alpha_1 k f_3(l) = 0,\ f_3(0) = f_1(l). \end{cases} \quad (25)$$

The characteristic equation of the first equation of (25) is

$$\begin{aligned} &r^2 + [\alpha_1+\beta_1+(\alpha_2+\beta_2)\lambda]r \\ &\qquad +[(\alpha_1\beta_2+\alpha_2\beta_1)\lambda + \alpha_2\beta_2\lambda^2] = 0. \end{aligned} \quad (26)$$

The solution of (26) is given by

$$r = -\frac{1}{2}[\alpha_1+\beta_1+(\alpha_2+\beta_2)\lambda] \pm \frac{1}{2}\sqrt{\tilde{\Delta}(\lambda)}.$$

Hereafter, we set

$$\begin{cases} r_1 := -\frac{1}{2}[\alpha_1+\beta_1+(\alpha_2+\beta_2)\lambda] + \frac{1}{2}\sqrt{\tilde{\Delta}(\lambda)}, \\ r_2 := -\frac{1}{2}[\alpha_1+\beta_1+(\alpha_2+\beta_2)\lambda] - \frac{1}{2}\sqrt{\tilde{\Delta}(\lambda)}. \end{cases} \quad (27)$$

Now, we consider two cases with respect to r_1 and r_2.

• The case of $r_1 \neq r_2$:
We set the solution f_1, f_3 of (25) as $f_1(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$, $f_3(x) = c_3 e^{-\frac{\lambda}{\mu} x}$. Then, from the boundary condition of (25), we have

$$\begin{bmatrix} r_1 & r_2 & \alpha_1 k e^{-\frac{\lambda}{\mu} l} \\ e^{r_1 l} & e^{r_2 l} & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{28}$$

Setting the determinant of the coefficient matrix Σ_1 of (28) to be equal to zero and noting that $\tau = \frac{l}{\mu}$, we obtain

$$\frac{z}{l} e^{\frac{l}{2}[\alpha_1+\beta_1+(\alpha_2+\beta_2)\lambda]} + k\alpha_1 e^{-\tau\lambda} \sinh z = 0. \tag{29}$$

The solution λ to this equation belongs to the spectrum of $\tilde{A}$, $\sigma(\tilde{A})$.

• The case of $r_1 = r_2 = -\frac{1}{2}[\alpha_1 + \beta_1 + (\alpha_2 + \beta_2)\lambda]$:
We set the solution f_1, f_3 of (25) as $f_1(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$, $f_3(x) = c_3 e^{-\frac{\lambda}{\mu} x}$. Then, from the boundary condition of (25), we have

$$\begin{bmatrix} r_1 & 1 & \alpha_1 k e^{-\frac{\lambda}{\mu} l} \\ e^{r_1 l} & l e^{r_1 l} & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{30}$$

Computing the determinant of the coefficient matrix Σ_2 of (30), we obtain

$$\det\Sigma_2 = -(1 + \alpha_1 k l e^{r_1 l - \lambda\tau}). \tag{31}$$

In addition to $r_1 = r_2$, if $\alpha_2 \neq \beta_2$, it follows that $\lambda = \lambda_\pm$ since $\tilde{\Delta}(\lambda) = 0$, and further that

$$\begin{aligned} r_1 l - \lambda\tau &= \frac{l}{2}[-(\alpha_1+\beta_1) - (\alpha_2+\beta_2+\frac{2}{\mu})\lambda_\pm] \\ &= \frac{l}{2}\Big[-(\alpha_1+\beta_1) + (\alpha_2+\beta_2+\frac{2\tau}{l})\frac{\alpha_1-\beta_1}{\alpha_2-\beta_2} \\ &\qquad \mp(\alpha_2+\beta_2+\frac{2\tau}{l})\frac{2\sqrt{\alpha_1\beta_1}}{\alpha_2-\beta_2} i\Big] \\ &= p(\tau) \mp (\alpha_2 l + \beta_2 l + 2\tau)\frac{\sqrt{\alpha_1\beta_1}}{\alpha_2-\beta_2} i. \end{aligned} \tag{32}$$

Especially, when $\tau = \tau_n$ and $k = \frac{e^{-p(\tau_n)}}{\alpha_1 l}$ for some $n \in Q$, it follows from (31)–(32) that $\det\Sigma_2 = 0$, which means that $\lambda_\pm \in \sigma(\tilde{A})$. On the other hand, when $\tau \neq \tau_n$ or $k \neq \frac{e^{-p(\tau_n)}}{\alpha_1 l}$ for any $n \in Q$, we have $\lambda_\pm \notin \sigma(\tilde{A})$.

If $\alpha_2 = \beta_2$, it is not difficult to see that $\tilde{\Delta}(\lambda) \neq 0$. Hence, the spectrum $\{\lambda_\pm\}$ does not appear. The proof is thus complete. □

Based on Lemma 8, we have the following theorem:

Theorem 9 *If k satisfies $|k| < \frac{\tilde{\gamma}}{\alpha_1}$, system (18) is exponentially stable for any time lag $\tau > 0$, where*

$$\tilde{\gamma} := \left\| \left(\frac{z}{l} \frac{e^{\frac{l}{2}[\alpha_1+\beta_1+(\alpha_2+\beta_2)\lambda]}}{\sinh z} \right)^{-1} \right\|_\infty^{-1}. \tag{33}$$

Proof. Set $\Psi(\lambda) := \frac{z}{l} e^{\frac{l}{2}[\alpha_1+\beta_1+(\alpha_2+\beta_2)\lambda]} + k\alpha_1 e^{-\tau\lambda} \sinh z$ and $\Psi_0(\lambda) := \frac{z}{l} e^{\frac{l}{2}[\alpha_1+\beta_1+(\alpha_2+\beta_2)\lambda]}$. When $k = 0$, it is known that the open-loop system is exponentially stable with any decay rate. This means that $|\Psi_0(\lambda)| > 0$ for all $\lambda \in \mathbf{C}_0^+ := \{\lambda \in \mathbf{C}\,;\, \mathrm{Re}(\lambda) \geq 0\}$.
(a') Under the condition $(\alpha_1 - \beta_1)(\alpha_2 - \beta_2) \neq 0$, it follows that, for all $\lambda \in \mathbf{C}_0^+$, $z \neq n\pi i$ $(n = 0, \pm1, \pm2, \ldots)$, that is, $\sinh z \neq 0$. Then, for any fixed $\lambda \in \mathbf{C}_0^+$ we can estimate $|\Phi(\lambda)|$ as follows:

$$\begin{aligned} &|\Psi(\lambda)| = |\Psi_0(\lambda) + k\alpha_1 e^{-\tau\lambda} \sinh z| \\ &\geq |\Psi_0(\lambda)| - |k\alpha_1 e^{-\tau\lambda} \sinh z| \geq |\Psi_0(\lambda)| - \alpha_1|k||\sinh z| \\ &= |\sinh z| \left\{ \left| \frac{\Psi_0(\lambda)}{\sinh z} \right| - \alpha_1|k| \right\} \geq |\sinh z|(\tilde{\gamma} - \alpha_1|k|), \end{aligned} \tag{34}$$

where $\tilde{\gamma}$ is defined by $\tilde{\gamma} := \inf_{\mathrm{Re}(\lambda)\geq 0} |\frac{z}{l} \frac{e^{\frac{l}{2}[\alpha_1+\beta_1+(\alpha_2+\beta_2)\lambda]}}{\sinh z}| = \|(\frac{z}{l}\frac{e^{\frac{l}{2}[\alpha_1+\beta_1+(\alpha_2+\beta_2)\lambda]}}{\sinh z})^{-1}\|_\infty^{-1}$. Therefore, it follows that $|\Psi(\lambda)| > 0$ if $|k| < \frac{\tilde{\gamma}}{\alpha_1}$. In other words, $\Psi(\lambda)$ does not have any root with non-negative real part. Here, in the case of $(\alpha_1 - \beta_1)(\alpha_2 - \beta_2) > 0$, it follows that $\mathrm{Re}(\lambda_\pm) = -\frac{\alpha_1-\beta_1}{\alpha_2-\beta_2} < 0$. Whereas, in the case where $(\alpha_1 - \beta_1)(\alpha_2 - \beta_2) < 0$ and $|k| < \frac{\tilde{\gamma}}{\alpha_1}$, the spectrum $\{\lambda_\pm\}$ does not appear, since $\frac{\tilde{\gamma}}{\alpha_1} \leq \frac{e^{-p(\tau_n)}}{\alpha_1 l}$ for any $n \in Q$. Hence, we see that, if $|k| < \frac{\tilde{\gamma}}{\alpha_1}$, $\tilde{A}$ does not have any spectrum with non-negative real part for any time lag $\tau > 0$.
(b') In the case of $\alpha_2 = \beta_2$, we have $z = \frac{l}{2}(\alpha_1 + \beta_1)$ for all $\lambda \in \mathbf{C}_0^+$. Therefore, under the condition $|k| < \frac{\tilde{\gamma}}{\alpha_1}$, we can estimate $|\Psi(\lambda)|$ as

$$\begin{aligned} |\Psi(\lambda)| &= |\Psi_0(\lambda) + k\alpha_1 e^{-\tau\lambda} \sinh z| \geq |\sinh z|(\tilde{\gamma} - \alpha_1|k|) \\ &= \sinh\{\tfrac{l}{2}(\alpha_1+\beta_1)\}(\tilde{\gamma} - \alpha_1|k|) > 0. \end{aligned}$$

Hence, if $|k| < \frac{\tilde{\gamma}}{\alpha_1}$, $\tilde{A}$ does not have any spectrum with non-negative real part for any time lag $\tau > 0$.
(c') Next, we consider the case of $\alpha_1 = \beta_1$ and $\alpha_2 \neq \beta_2$. Then, z takes values $n\pi i$ $(n = 0, \pm1, \pm2, \ldots)$ for some $\lambda \in \mathbf{C}_0^+$. Set $d := \inf_{\mathrm{Re}(\lambda)\geq 0} |\Psi_0(\lambda)| > 0$. Here, under the condition $|k| < \frac{\tilde{\gamma}}{\alpha_1}$, choose $\varepsilon > 0$ such that $d - \alpha_1\varepsilon|k| > 0$. Introduce the following two sets: $S_\varepsilon^1 := \{\lambda \in \mathbf{C}_0^+; |\sinh z| < \varepsilon\}$, $S_\varepsilon^2 := \{\lambda \in \mathbf{C}_0^+; |\sinh z| \geq \varepsilon\}$, where $\mathbf{C}_0^+ = S_\varepsilon^1 \cup S_\varepsilon^2$. Then, we can estimate $|\Psi(\lambda)|$ as follows:

- for all $\lambda \in S_\varepsilon^1$, $|\Psi(\lambda)| \geq |\Psi_0(\lambda)| - \alpha_1|k||\sinh z| \geq d - \alpha_1\varepsilon|k| > 0$,

- for all $\lambda \in S_\varepsilon^2$, $|\Psi(\lambda)| \geq |\sinh z|(\tilde{\gamma} - \alpha_1|k|) \geq \varepsilon(\tilde{\gamma} - \alpha_1|k|) > 0$.

Accordingly, if $|k| < \frac{\tilde{\gamma}}{\alpha_1}$, there holds $|\Psi(\lambda)| \geq \min\{d - \alpha_1\varepsilon|k|, \varepsilon(\tilde{\gamma} - \alpha_1|k|)\} > 0$ for all $\lambda \in \mathbf{C}_0^+$. Note that the spectrum $\{\lambda_\pm\}$ does not appear, since $\frac{\tilde{\gamma}}{\alpha_1} \leq \frac{e^{\frac{l}{2}(\alpha_1+\beta_1)}}{\alpha_1 l} = \frac{e^{-p(\tau_n)}}{\alpha_1 l}$ for any $n \in Q$. Hence, if $|k| < \frac{\tilde{\gamma}}{\alpha_1}$, $\tilde{A}$ does not have any spectrum with non-negative real part for any time lag $\tau > 0$.

(d') Finally, we show that, under the condition $|k| < \frac{\tilde{\gamma}}{\alpha_1}$, the spectrum of $\tilde{A}$ does not approach asymptotically to the imaginary axis. Note that the operator $\tilde{A}$ has compact resolvent. From the discussion of (b')–(c'), in the case of $\alpha_2 = \beta_2$ or in the case of $\alpha_1 = \beta_1$ and $\alpha_2 \neq \beta_2$, it follows easily that the asymptotic behaviour to the imaginary axis is negative, since $\tilde{A}$ has compact resolvent. So, we consider the case of $(\alpha_1 - \beta_1)(\alpha_2 - \beta_2) \neq 0$. Assume that there exists a spectrum $\lambda_n = \xi_n + i\eta_n \in \mathbf{C} - \mathbf{C}_0^+$ such that $\xi_n \to 0$ and $|\eta_n| \to \infty$. Then, from Lemma 8, the spectrum λ_n satisfies

$$\Psi(\lambda_n) := \frac{z_n}{l} e^{\frac{l}{2}[\alpha_1+\beta_1+(\alpha_2+\beta_2)\lambda_n]} + k\alpha_1 e^{-\tau\lambda_n} \sinh z_n = 0, \tag{35}$$

where $z_n := \frac{l}{2}\sqrt{\tilde{\Delta}(\lambda_n)}$. On the other hand, we have

$$\inf\{\,\mathrm{Re}(z)\,;\,\lambda = j\omega,\,\omega \in \mathbf{R}\,\} = \frac{l}{2}|\alpha_1 - \beta_1| > 0. \tag{36}$$

From (36), it is not difficult to see that

$$|\sinh z| \geq \frac{1}{2}(e^{\frac{l}{2}|\alpha_1-\beta_1|} - e^{-\frac{l}{2}|\alpha_1-\beta_1|}) =: \tilde{\delta}, \tag{37}$$

as $\lambda = j\omega$ and $|\omega| \to \infty$. Therefore, under the condition $|k| < \frac{\tilde{\gamma}}{\alpha_1}$, it follows from (34) and (37) that

$$|\Psi(j\omega)| \geq \tilde{\delta}(\tilde{\gamma} - \alpha_1|k|) > 0 \quad \text{as } |\omega| \to \infty. \tag{38}$$

This contradicts (35). Hence, the spectrum of $\tilde{A}$ does not approach asymptotically to the imaginary axis.

From (a')–(d'), we see that, if k satisfies $|k| < \frac{\tilde{\gamma}}{\alpha_1}$, the spectrum bound of $\tilde{A}$, $\sigma_0 := \sup\{\mathrm{Re}(\lambda); \lambda \in \sigma(\tilde{A})\}$, is strictly negative and further system (18) is exponentially stable for any time lag $\tau > 0$, by using the result by F.L. Huang [13] on the spectrum determined growth condition (see Appendix B for the proof of exponential stability). □

Similarly to the proof of Theorem 5, the above proof shows that the closed-loop system with the same feedback gain is exponentially stable for the case of $\tau = 0$.

4 Numerical examples

Let $l = 1$, $\nu_1 = 0.68$, $\nu_2 = 0.72$, $h_1 = 1.586$, $h_2 = 1.635$. From Propositions 1 and 6, the upper bound of $|k|$, i.e. $\sqrt{\frac{h_2\nu_1}{\nu_2 h_1}}$ is calculated as 0.9867. Then, under a gain k with $|k| < 0.9867$, the admissible time lag for both systems (3) and (18) becomes $2.3324 < \tau < \frac{2.2708}{k^2}$. On the other hand, it follows from Theorem 5 that $\gamma/(2\alpha_1) = \|G\|_\infty^{-1}/(2\alpha_1) = 1.4157$ and system (3) with $|k| < 1.4157$ is exponentially stable for any time lag $\tau > 0$, where $G(\lambda) := (C(\lambda) + \frac{2z}{l}\frac{\cosh z}{\sinh z})^{-1}$. Also, it follows from Theorem 9 that $\tilde{\gamma}/\alpha_1 = \|\tilde{G}\|_\infty^{-1}/\alpha_1 = 1.9936$ and system (18) with $|k| < 1.9936$ is exponentially stable for any time lag $\tau > 0$, where $\tilde{G}(\lambda) := (\frac{z}{l}\frac{e^{\frac{l}{2}[\alpha_1+\beta_1+(\alpha_2+\beta_2)\lambda]}}{\sinh z})^{-1}$. By [7, Lemma A.6.17], the computation of $\|G\|_\infty := \sup_{\mathrm{Re}(\lambda)>0}|G(\lambda)|$ is replaced by $\|G\|_\infty = \sup_{\omega\in\mathbf{R}}|G(j\omega)|$. In Figure 1, $|G(j\omega)|$ and $|\tilde{G}(j\omega)|$ are plotted on a graph using Matlab software. Thus, we see that the upper bounds of the feedback gains are not conservative compared with the ones obtained by Propositions 1 and 6. In addition, feedback gains less than the upper bounds make the closed-loop systems exponentially stable for any time lag. This shows that, by the proposed method, the condition for the exponential stability could be weakened.

Figures 2 and 3 are the numerical simulation results for systems of counter-flow type and parallel-flow type, in which the solution of the open-loop system ($k = 0$) is compared with that of the closed-loop system without delay, where $\|\theta_i(t,\cdot)\|$ denotes the L^2-norm of $\theta_i(t,x)$ $(i = 1,2)$. Especially, from Figure 3 we see that, for systems of parallel-flow type, the closed-loop system is stable but it cannot enhance the stability compared with the open-loop system. It is a natural result, since heat exchangers of parallel-flow type are finite-time stabilizable (see also the footnote of Section 1).

Figure 4 shows the numerical simulation results with time lag $\tau = 0.1$ and $\tau = 7.5$, where we set the same gain $k = -1.20$ satisfying $|k| < 1.4157$ and $|k| < 1.9936$. Figure 5 shows the simulation results with time lag $\tau = 0.1$ and $\tau = 7.5$, where $k = 1.20$. Comparing Figure 5 with Figure 4, we see that the system with negative feedback gain $(-k) = -1.20$ is more stable than the one with postitive feedback gain $(-k) = 1.20$ in the case $\tau = 0.1$ of the counter-flow type. From Figure 5, we further see that the system with time lag $\tau = 0.1$ is more stable than the one with time lag $\tau = 7.5$ in the case $(-k) = -1.20$ of the counter-flow type. Figure 6 shows the simulation results for the case where the time lag was fixed as $\tau = 0.1$ but the gain k varied. Figure 7 also shows the simulation results for the case with fixed time lag $\tau = 7.5$ but with different gains k. In these simulations (Figures 2–7), we used the same initial condition $\theta_{10}(x) = e^{-100(x-0.35)^2}$ (> 0), $\theta_{20}(x) =$

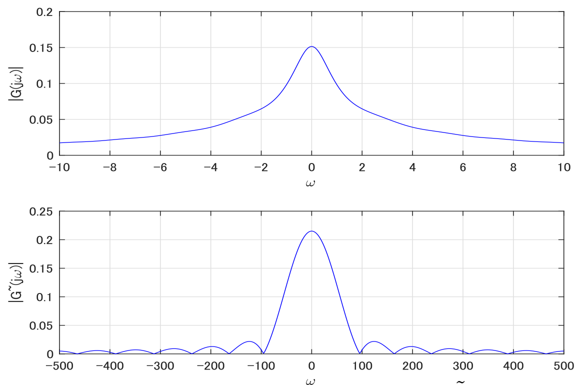

Fig. 1. Top: $|G(j\omega)|$. Bottom: $|\tilde{G}(j\omega)|$.

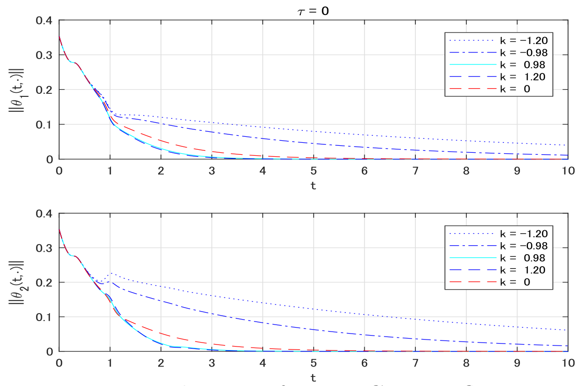

Fig. 2. The case of $\tau = 0$. Counter-flow type.

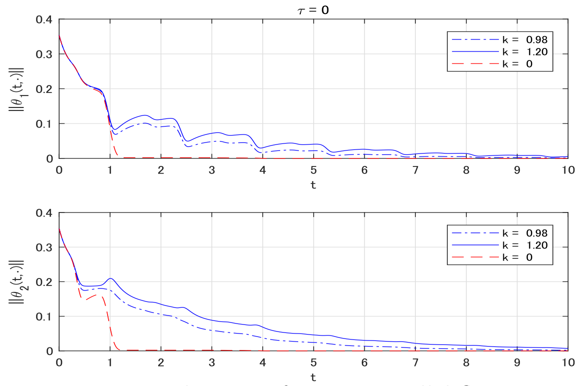

Fig. 3. The case of $\tau = 0$. Parallel-flow type.

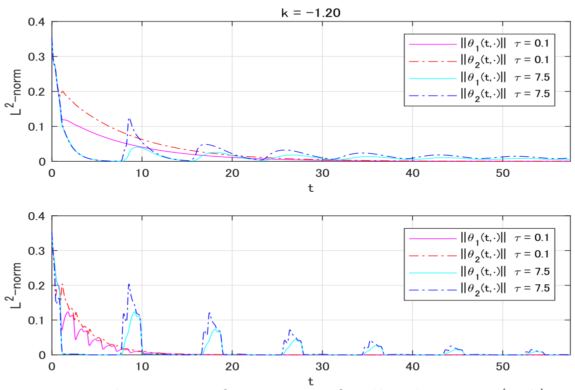

Fig. 4. The case of positive feedback gain $(-k) = 1.20$. Top: Counter-flow type. Bottom: Parallel-flow type.

$e^{-100(x-0.75)^2}$ (> 0), $\phi(s) \equiv 0$.

Remark 10 *When the initial data is given as* $\theta_{10}(x) = -e^{-100(x-0.35)^2}$ (< 0), $\theta_{20}(x) = -e^{-100(x-0.75)^2}$ (< 0), *we can obtain the similar simulation results for both cases. However, for the initial condition* $\theta_{10}(x) = e^{-100(x-0.35)^2}$ (> 0), $\theta_{20}(x) = -e^{-100(x-0.75)^2}$ (< 0), *it is difficult to enhance the stability for systems of counter-flow type, since the degree of the stability becomes higher from the first (note that* $\int_0^l \theta_{10}(x)\theta_{20}(x)dx$ *becomes negative in this case). Thus, it depends on the initial data*[4]*. On the other hand, for systems of parallel-flow type, one cannot enhance the stability by boundary control as mentioned above. Instead, we need to consider distributed control. In order to enhance the stability for such systems, we can consider distributed control via switching as an alternative approach. See e.g. [22] for switching control of parallel-flow heat exchangers without delay.*

Remark 11 *It might seem strange to consider parallel-flow systems with boundary control, but, we note that, in chemical process control, it is needed to cool chemical substances slowly, depending on the situation (e.g. [19]).*

Remark 12 *There is a possibility that the systems treated in this paper could be used as target systems in PDE backstepping designs.*

5 Conclusion

In this paper, we investigated the stability of two kinds of heat exchangers with delayed boundary feedback. By using feedback gains less than the upper bounds of Theorems 5 and 9, the closed-loop systems could be exponentially stable under the presence of any time lag. Further, as stated at the end of Sections 2 and 3, the closed-loop systems with the same feedback gains are exponentially stable for the case without delay. This is a contrasting result compared with the boundary control of a string studied by Gugat [11]. In that paper, the string is fixed at one end and stabilized by delayed boundary feedback at the other end, where the delay is the round-trip time of the wave in the string. It has been shown that the closed-loop system is exponentially stable for feedback gains chosen from a bounded interval, however, the corresponding boundary feedback without delay is not stabilizing the string.

Also, it has been reported about systems which are not robust to small delay in the feedback loop (e.g. [18,2] and the references therein). In this paper, we did not treat the case where both ends were governed by boundary feedback. Such a system may not be robust with respect to small delay, as shown for a class of boundary control hy-

[4] As for stability, the assertion of Theorem 5 holds for all initial data in L^2-space.

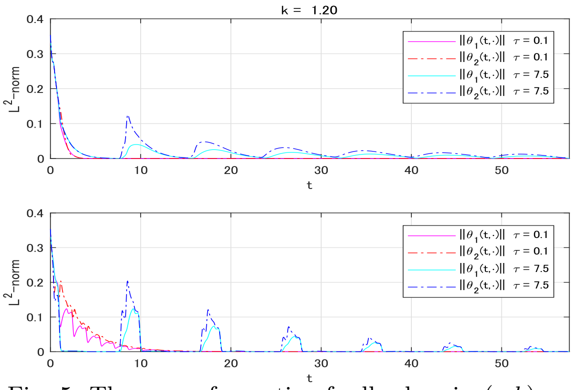

Fig. 5. The case of negative feedback gain $(-k) = -1.20$. Top: Counter-flow type. Bottom: Parallel-flow type.

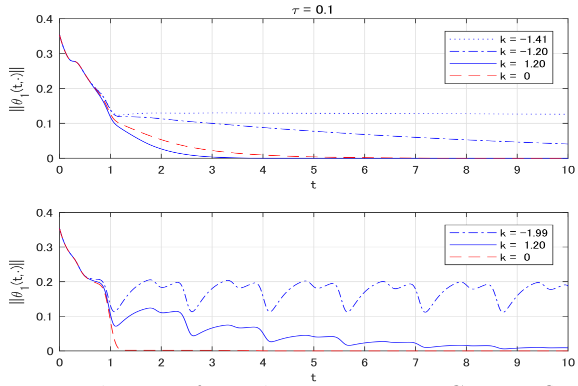

Fig. 6. The case of time lag $\tau = 0.1$. Top: Counter-flow type. Bottom: Parallel-flow type.

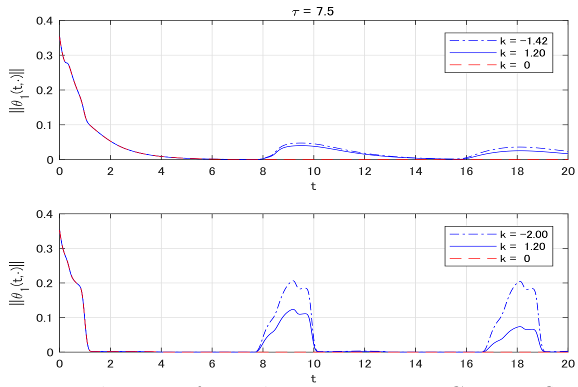

Fig. 7. The case of time lag $\tau = 7.5$. Top: Counter-flow type. Bottom: Parallel-flow type.

perbolic systems in [2]. As a further problem, the author plans to conduct the stability analysis of the heat exchangers with delayed boundary feedback at both ends.

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Appendix A. The resolvent operator of $\tilde{A}$

For a sufficiently large $\lambda > 0$, the operator $(\lambda I - \tilde{A})^{-1} : X \to X$ is expressed as

$$(\lambda I - \tilde{A})^{-1} g = (T_\lambda S_\lambda + U_\lambda) g, \quad g \in X, \tag{39}$$

where

$$S_\lambda g := \int_0^l e^{-(l-y)M^{-1}\Lambda} M^{-1} g(y) dy, \quad g \in X, \tag{40}$$

$$(T_\lambda h)(x) := e^{(l-x)M^{-1}\Lambda} \{(I - e^{-lM^{-1}\Lambda} K)^{-1} - I\} h, \quad h \in \mathbf{C}^3, \tag{41}$$

$$(U_\lambda g)(x) := \int_0^x e^{-(x-y)M^{-1}\Lambda} M^{-1} g(y) dy, \quad g \in X, \tag{42}$$

$$\Lambda := \begin{bmatrix} \lambda + h_1 & -h_1 & 0 \\ -h_2 & \lambda + h_2 & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \quad K := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -k \\ 1 & 0 & 0 \end{bmatrix},$$

$$M := \mathrm{diag}(\nu_1, \nu_2, \mu).$$

Here, it is easily verified that $S_\lambda : X \to \mathbf{C}^3$ is a bounded operator and $T_\lambda : \mathbf{C}^3 \to X$, $U_\lambda : X \to X$ are compact operators. Therefore, we see that the resolvent operator $(\lambda I - \tilde{A})^{-1} : X \to X$ with the λ is compact (e.g. [7, Lemma A.3.22 and Theorem A.3.24]).

Therefore, the spectrum $\sigma(\tilde{A})$ entirely consists of isolated eigenvalues with finite multiplicities and has no accumulation point different from ∞.

Appendix B. The proof of exponential stability

Theorem B ([13, Theorem 1 and Theorem 4]) *Let $e^{t\mathcal{A}}$ be a C_0-semigroup with the infinitesimal generator $\mathcal{A}$ in a Hilbert space $\mathcal{H}$ and set $\omega_0(\mathcal{A}) := \lim_{t\to\infty} \frac{\log \|e^{t\mathcal{A}}\|_{\mathcal{L}(\mathcal{H})}}{t}$, $\sigma_0(\mathcal{A}) := \sup\{\, \mathrm{Re}(\lambda) \,;\, \lambda \in \sigma(\mathcal{A}) \,\}$, where $\sigma(\mathcal{A})$ denotes the spectrum of $\mathcal{A}$. Then, the spectrum determined growth condition $\sigma_0(\mathcal{A}) = \omega_0(\mathcal{A})$ is satisfied if and only if $\sup\{\, \|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \,;\, \mathrm{Re}(\lambda) \geq \sigma \,\} < \infty$ holds for each $\sigma > \sigma_0(\mathcal{A})$. Furthermore, the computational formula relating to $\omega_0(\mathcal{A})$ is the following:*

$$\omega_0(\mathcal{A}) = \inf\{\, \sigma \,;\, \sigma > \sigma_0(\mathcal{A}) \text{ and } \sup\{\, \|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \,;\, \mathrm{Re}(\lambda) \geq \sigma \,\} < \infty \,\}.$$

Proof of exponential stability. In Theorem 7, it is shown that the operator $\tilde{A}$ generates a C_0-semigroup $e^{t\tilde{A}}$ on X. Therefore, the C_0-semigroup $e^{t\tilde{A}}$ has the operator norm bound $\|e^{t\tilde{A}}\|_{\mathcal{L}(X)} \leq M e^{\omega t}$, $t \geq 0$ for some $M > 0$ and $\omega \in \mathbf{R}$. Then, it follows from [20, Theorem 1.5.3 and

Remark 1.5.4] that, for each $\epsilon > 0$, $\|(\lambda I - \tilde{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{\mathrm{Re}(\lambda)-\omega} \leq \frac{M}{\epsilon}$ holds for all $\lambda \in \mathbf{C}$ satisfying $\mathrm{Re}(\lambda) \geq \omega + \epsilon$. Now, let the set E be defined by $E := \{\, \lambda \in \mathbf{C}\,;\ \sigma_0 + \epsilon \leq \mathrm{Re}(\lambda) \leq \omega + \epsilon,\ |\mathrm{Im}(\lambda)| \leq h \,\}$, where $\sigma_0 = \sup\{\mathrm{Re}(\lambda);\ \lambda \in \sigma(\tilde{A})\}\ (< 0)$. Then, since $E \subset \rho(\tilde{A})$ holds for any $h > 0$, it is easily seen that $\sup_{\lambda\in E} \|(\lambda I - \tilde{A})^{-1}\|_{\mathcal{L}(X)} < \infty$. Set $E_h := \{\, \lambda \in \mathbf{C}\,;\ \sigma_0 + \epsilon \leq \mathrm{Re}(\lambda) \leq \omega + \epsilon,\ |\mathrm{Im}(\lambda)| \geq h \,\}$. In order to apply Theorem B, it must be shown that there exists an $h > 0$ such that

$$\sup_{\lambda\in E_h} \|(\lambda I - \tilde{A})^{-1}\|_{\mathcal{L}(X)} < \infty. \tag{43}$$

For each $\lambda \in E_h$ and each $g = [g_1, g_2, g_3]^T \in X$, $(\lambda I - \tilde{A})^{-1} g$ is expressed as (39). The state-transition matrix of $-M^{-1}\Lambda$ is calculated as follows:

$$e^{-xM^{-1}\Lambda} = \begin{bmatrix} R_{11}(x) & R_{12}(x) & 0 \\ R_{21}(x) & R_{22}(x) & 0 \\ 0 & 0 & e^{-\lambda x/\mu} \end{bmatrix}, \tag{44}$$

$R_{11}(x) := \frac{\tilde{C}(\lambda)}{2\sqrt{\tilde{\Delta}(\lambda)}}(e^{r_2 x} - e^{r_1 x}) + \frac{e^{r_1 x} + e^{r_2 x}}{2}$, $R_{12}(x) := \frac{\alpha_1}{\sqrt{\tilde{\Delta}(\lambda)}}(e^{r_1 x} - e^{r_2 x})$, $R_{21}(x) := \frac{\beta_1}{\sqrt{\tilde{\Delta}(\lambda)}}(e^{r_1 x} - e^{r_2 x})$, $R_{22}(x) := \frac{\tilde{C}(\lambda)}{2\sqrt{\tilde{\Delta}(\lambda)}}(e^{r_1 x} - e^{r_2 x}) + \frac{e^{r_1 x} + e^{r_2 x}}{2}$, where r_1 and r_2 are the same ones as in the proof of Lemma 8. Note that $r_1 \neq r_2$ since $\lambda \in E_h$. Furthermore, $(I - e^{-lM^{-1}\Lambda}K)^{-1} - I$ as calculated as

$$\begin{aligned} &(I - e^{-lM^{-1}\Lambda}K)^{-1} - I \\ &= \frac{1}{\Theta(\lambda)} \begin{bmatrix} -ke^{-\tau\lambda}R_{12}(l) & 0 & -kR_{12}(l) \\ -ke^{-\tau\lambda}R_{22}(l) & 0 & -kR_{22}(l) \\ e^{-\tau\lambda} & 0 & -ke^{-\tau\lambda}R_{12}(l) \end{bmatrix}, \\ &\Theta(\lambda) := \det(I - e^{-lM^{-1}\Lambda}K). \end{aligned} \tag{45}$$

Now, we have the following estimates for the real part and the imaginary part of $\sqrt{\tilde{\Delta}(\lambda)}$:
If $h > 0$ is sufficiently large, there hold

- $|(\alpha_2 - \beta_2)\mathrm{Re}(\lambda) + \alpha_1 - \beta_1| \leq \mathrm{Re}\sqrt{\tilde{\Delta}(\lambda)}$ $\leq \sqrt{\{(\alpha_2 - \beta_2)\mathrm{Re}(\lambda) + \alpha_1 - \beta_1\}^2 + 4\alpha_1\beta_1}$,
- $\sqrt{(\alpha_2 - \beta_2)^2(\mathrm{Im}(\lambda))^2 - 4\alpha_1\beta_1} \leq |\mathrm{Im}\sqrt{\tilde{\Delta}(\lambda)}|$ $\leq |\alpha_2 - \beta_2||\mathrm{Im}(\lambda)|$,

for any $\lambda \in E_h$. Therefore, in the case of $\alpha_2 \neq \beta_2$, there exists a positive constant m such that, for any $\lambda \in E_h$,

$$\begin{aligned} &\sup_{0\leq x\leq l} \|e^{xM^{-1}\Lambda}\| \leq m, \quad \sup_{0\leq x\leq l} \|e^{-xM^{-1}\Lambda}\| \leq m, \\ &\|(I - e^{-lM^{-1}\Lambda}K)^{-1} - I\| \leq m, \quad h\text{: sufficiently large.} \end{aligned} \tag{46}$$

Here, we have used $\Theta(\lambda) = 1 + \frac{l}{z} e^{-\frac{l}{2}[\alpha_1 + \beta_1 + (\alpha_2 + \beta_2)\lambda]}\alpha_1 ke^{-\tau\lambda}\sinh z$. Hence, since $\|S_\lambda\|_{\mathcal{L}(X,\mathbf{C}^3)}$, $\|T_\lambda\|_{\mathcal{L}(\mathbf{C}^3,X)}$, and $\|U_\lambda\|_{\mathcal{L}(X)}$ are uniformly bounded with respect to $\lambda \in E_h$ (h: sufficiently large), it follows from (39) that

$$\|(\lambda I - \tilde{A})^{-1} g\|_X \leq \text{const.}\, \|g\|_X. \tag{47}$$

The inequality (47) holds for all $g \in X$ and for all $\lambda \in E_h$. That is, we have

$$\sup_{\lambda\in E_h} \|(\lambda I - \tilde{A})^{-1}\|_{\mathcal{L}(X)} \leq \text{const.} \tag{48}$$

This shows that (43) holds. On the other hand, in the case of $\alpha_2 = \beta_2$, it is not difficult to see that (48) holds, since $z = \frac{l}{2}(\alpha_1 + \beta_1)$ for all λ. Thus, we finally obtain $\sup\{\, \|(\lambda I - \tilde{A})^{-1}\|_{\mathcal{L}(X)}\,;\ \mathrm{Re}(\lambda) \geq \sigma_0 + \epsilon \,\} < \infty$. Accordingly, it follows from Lemma B that, for any $\epsilon > 0$, there exists a constant $M_\epsilon > 0$ such that $\|e^{t\tilde{A}}\|_{\mathcal{L}(X)} \leq M_\epsilon e^{(\sigma_0+\epsilon)t}$, $t \geq 0$, where σ_0 (< 0) is the spectral bound of the operator $\tilde{A}$.

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