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(Citation)
Economics Letters,202:109824

(Issue Date)
2021-05

(Resource Type)
journal article

(Version)
Accepted Manuscript

Group Incentive Compatibility and Welfare for Matching with Contracts

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March, 2021

Abstract

In the matching with contracts framework, under the assumption that each hospital has a choice function that satisfies the observable substitutability condition, we show that the strategy-proof cumulative offer mechanism is no longer guaranteed to be group strategy-proof, yet it nevertheless outputs a weakly Pareto efficient outcome for any input. We also discuss why, unlike the previous observations in the literature, the equivalence of strategy-proofness and group strategy-proofness does not carry over to our context.

JEL Classification: C78, D44, D47

Keywords: matching with contracts; observable substitutability; strategy-proofness; group strategy-proofness; Pareto efficiency.

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1 Introduction

Except for stability, incentives and efficiency are the two most important issues in the field of matching market design. It is well known that when we regard both issues as relevant only to some part of agents (e.g., doctors in the medical match and students in school choice), the so-called cumulative offer mechanism (for short, COM)[1] is stable and strategy-proof in various situations, whereas it fails to be Pareto efficient in general.

Meanwhile, it is a common conclusion that the strategy-proof COM is in fact *group* strategy-proof (in a weak sense) and, as a result, *weakly* Pareto efficient.[2] The two properties provide added justification for the use of the COM: The lack of group strategy-proofness may invite a (small) group of agents to coordinate and manipulate the social outcome, while the lack of weak Pareto efficiency implies unlimited inefficiency.

Despite their importance, however, it seems to have been unknown whether the same conclusion holds in the matching with contracts framework (Hatfield and Milgrom, 2005) when the hospitals' choice functions are *observably substitutabile* (Hatfield et al., 2020). This condition is weaker than most of the previously known substitutability conditions, while allowing us to broaden the area of application (Hatfield et al., 2017). Hatfield et al. (2020) discussed the existence of the strategy-proofness COM, yet left it open whether it is then group strategy-proofness and/or weakly Pareto efficient.

The current paper fills this gap in the literature by showing that under the assumption of observable substitutability, the strategy-proof COM is no longer guaranteed to be group strategy-proof, yet it is nevertheless weakly Pareto efficient in general.[3] We also

[1] In the current context, the COM is outcome-equivalent to the *deferred-acceptance* mechanism (Gale and Shapley, 1962). Hence, all the arguments go through unaltered with the latter mechanism.

[2] This conclusion can be found in different contexts with different levels of generality. See Dubins and Freedman (1981), Roth (1982), Demange and Gale (1985), Martínez et al. (2004), Kojima (2008), Hatfield and Kojima (2009, 2010), Hatfield and Kominers (2012, 2019), Jiao and Tian (2017), Fleiner et al. (2018), and Schlegel (2018, 2020). See also Barberà et al. (2016) who pointed out that the same kind of conclusion holds in contexts other than matching as well, and provided some explanation for this coincidence.

[3] Schlegel (2020) took a different approach from ours and showed that if we relax the observable sub-

explain why, unlike the previous observations, the equivalence of strategy-proofness and group strategy-proofness does not carry over to the current context.

The conclusions of the current paper are especially interesting in the context of school choice (Abdulkadiroğlu and Sönmez, 2003), where incentives and welfare of students are of primary importance. In this literature, a search for weakened substitutability conditions in the matching with contracts framework has advanced the frontier (e.g., Kominers and Sönmez 2016; Hafalir et al. 2019). Yet, the current paper suggests that while weak Pareto efficiency is generally satisfied, caution is needed as one cannot preclude group manipulations when substitutability is weakened too much.

2 Preliminaries

There are finite sets D and H of doctors and hospitals. The finite set of bilateral contracts is denoted by X; they are bilateral in that each contract $x \in X$ involves only one doctor $\mathrm{d}(x) \in D$ and one hospital $\mathrm{h}(x) \in H$, along with some unspecified contract term. A subset $X' \subset X$ of contracts is an *allocation* or a *feasible* set if it includes at most one contract for each doctor. The set of all possible allocations is denoted by $\mathscr{X} \subset 2^X$.

Each doctor $d \in D$ has a strict preference relation $\succ_d$ over $\mathrm{Ac}(\succ_d) \cup \{\varnothing\}$ where $x \succ_d \varnothing$ for all $x \neq \varnothing$. Here, $\mathrm{Ac}(\succ_d) \subset \{x \in X \mid \mathrm{d}(x) = d\}$ is the set of "acceptable" contracts for $\succ_d$, and $\varnothing$ is the null contract which means he has no contract.[4] For each $\succ_d$ satisfying $\mathrm{Ac}(\succ_d) \neq \emptyset$, let $\mathrm{t}(\succ_d)$ denote the most preferred contract. In the same spirit, for each preference profile $\succ_D = (\succ_d)_{d \in D}$, let $\mathrm{T}(\succ_D) := \{\mathrm{t}(\succ_d) \mid d \in D \text{ and } \mathrm{Ac}(\succ_d) \neq \emptyset\}$. For each $d \in D$ and $Y \subset X$, let $\succ_d^{-Y}$ be such that (i) $\mathrm{Ac}(\succ_d^{-Y}) = \mathrm{Ac}(\succ_d) \setminus Y$, and (ii) $x \succeq_d^{-Y} x' \Leftrightarrow x \succeq_d x'$ for all $x, x' \in \mathrm{Ac}(\succ_d^{-Y})$. For each $\succ_d$ and $x \in X \cup \{\varnothing\}$, let $\mathrm{U}(\succ_d, x) := \{y \in$

stitutability condition as well as the incentive compatibility notions, the equivalence of strategy-proofness and group strategy-proofness is restored. See footnote 11 for more details.

[4]Since we assume that $\succ_d$ is defined only over $\mathrm{Ac}(\succ_d) \cup \{\varnothing\}$, when we write $x \succ_d y$ for some non-null contracts x and y, this automatically implies that both x and y are in $\mathrm{Ac}(\succ_d)$.

$\mathrm{Ac}(\succ_d) \cup \{\varnothing\} \mid x \not\succ_d y\}$ denote an *upper contour set* of $\succ_d$ at x.[5]

Each hospital $h \in H$ has a choice function $C_h : 2^X \to 2^X$ such that for each possible $X' \subset X$, (i) $C_h(X') \subset \{x \in X' \mid \mathrm{h}(x) = h\}$, (ii) $C_h(X') \in \mathscr{X}$, and (iii) $x \notin C_h(X' \cup \{x\})$ implies $C_h(X') = C_h(X' \cup \{x\})$ for all $x \in X$.[6]

We say that C_h is *induced* by a strict relation $\succ_h$ over 2^X if for all $Y \subset X$, $C_h(Y) = \max_{\succ_h}\{Z \in \mathscr{X} \mid Z \subset Y \text{ and } \mathrm{h}(x) = h \text{ for all } x \in Z\}$. Note that when C_h is induced by some strict relation, it satisfies the three conditions (i)–(iii) for it to be a legitimate choice function. Slightly abusing the notation, for each pair of disjoint sets $\Sigma, \Sigma' \in 2^{\mathscr{X}}$ of allocations, we write $\Sigma \succ_h \Sigma'$ whenever $Y \succ_h Y'$ holds for any $(Y, Y') \in \Sigma \times \Sigma'$.

Given $(\succ_D, C_H)$, an allocation $X' \in \mathscr{X}$ is *individually rational* (for short, IR) if (i) $x \in \mathrm{Ac}(\succ_{\mathrm{d}(x)})$ for all $x \in X'$, and (ii) $C_H(X') = X'$. A pair of a hospital h and a set $X'' \in \mathscr{X}$ of contracts *block* an IR allocation X' if (i) $C_h(X' \cup X'') = X'' \neq C_h(X')$ and (ii) $X'' \succeq_{\mathrm{d}(x)} X'$ for all $x \in X''$. An IR allocation X' is *stable* if it is not blocked by any $(h, X'') \in H \times \mathscr{X}$. An IR allocation is *weakly Pareto efficient (for doctors)* if it is not *strongly dominated* by other IR allocation; i.e., there is no IR allocation that makes *all* the doctors strictly better off.

Given $(\succ_D, C_H)$, the *cumulative offer process* (for short, COP) proceeds as follows:

- Step 1: Let some doctor d_1 propose his best contract x_1 in $\mathrm{Ac}(\succ_{d_1}) \neq \emptyset$.
- Step $t \geq 2$: Let some doctor d_t who signs no contract at $C_H(\{x_1, \ldots, x_{t-1}\})$ propose his best contract x_t in $\mathrm{Ac}(\succ_{d_t}) \setminus \{x_1, \ldots, x_{t-1}\} \neq \emptyset$.

The algorithm terminates at the end of step T when there remains no doctor who can propose a contract, and it outputs $C_H(\{x_1, \ldots, x_T\})$. The above description of the COP does not specify how d_t is chosen at each step. But this is inconsequential in the current paper as we will explain below.

An *observable offer process for h* is a sequence $(x_1, \ldots, x_n)$ of distinct contracts such that

[5]Note that when $x \notin \mathrm{Ac}(\succ_d)$, $\mathrm{U}(\succ_d, x) = \mathrm{Ac}(\succ_d) \cup \{\varnothing\}$ holds in our definition.

[6]The last condition (iii) is called the *irrelevance of rejected contracts* condition (Aygün and Sönmez, 2013), which is a mild rationality condition and is assumed almost universally in the literature.

$\mathrm{h}(x_t) = h$ and $\mathrm{d}(x_t)$ signs no contract at $C_h(\{x_1, \ldots, x_{t-1}\})$ for all $t \leq n$. A choice function C_h is *observably substitutable* (for short, OS) if $R_h(\{x_1, \ldots, x_{n-1}\}) \subset R_h(\{x_1, \ldots, x_n\})$ for any observable offer process $(x_1, \ldots, x_n)$ for h, where $R_h(X') := X' \setminus C_h(X')$ denotes the set of rejected contracts from any $X' \subset X$. Hatfield et al. (2020, Theorem 1b) showed that when each C_h is OS, for any $\succ_D$, the outcome of the COP is independent of which doctor is chosen at each step, and its outcome $C_H(\{x_1, \ldots, x_T\})$ is stable with respect to $(\succ_D, C_H)$. Thus, when each C_h is OS, we can define the *cumulative offer mechanism* denoted by ψ^{C} that outputs for any input $\succ_D$ the set of contracts $\psi^{\mathrm{C}}(\succ_D) := C_H(\{x_1, \ldots, x_T\})$ generated by the COP.

For the matter of incentives, ψ^{C} is *strategy-proof* if for any $\succ_D$ and $d \in D$, there is no $\succ'_d$ such that $\psi^{\mathrm{C}}_d(\succ'_d, \succ_{-d}) \succ_d \psi^{\mathrm{C}}_d(\succ_D)$. Here, $\psi^{\mathrm{C}}_d(\succ_D)$ refers to a (possibly null) contract that doctor d signs at $\psi^{\mathrm{C}}(\succ_D)$. Moreover, ψ^{C} is *group strategy-proof* if for any $\succ_D$ and $D' \subset D$, there is no $\succ'_{D'}$ such that $\psi^{\mathrm{C}}_d(\succ'_{D'}, \succ_{-D'}) \succ_d \psi^{\mathrm{C}}_d(\succ_D)$ for all $d \in D'$.[7]

For the strategy-proofness of ψ^{C}, we need further conditions: A choice function C_h is *observably size-monotonic* (for short, OSM) if $|C_h(\{x_1, \ldots, x_{n-1}\})| \leq |C_h(\{x_1, \ldots, x_n\})|$ for any observable offer process $(x_1, \ldots, x_n)$ for h. Under the assumption that C_h is OS, it is *non-manipulable via contractual terms* (for short, nMvCT) if ψ^{C} is strategy-proof in a situation $(D, \{h\}, \{x \in X \mid \mathrm{h}(x) = h\}, C_{\{h\}})$.[8] Hatfield et al. (2020, Theorem 4) showed that ψ^{C} is strategy-proof if each C_h is OS, OSM, and nMvCT.

3 Results

We first show that there is a situation (D, H, X, C_H) with only one hospital h for whom C_h is OS, OSM, and nMvCT, whereas ψ^{C} fails to be group strategy-proof.

[7]Notice that if ψ^{C} is group strategy-proof, it outputs a weakly Pareto efficient allocation for any input.

[8]This is slightly different from the original definition by Hatfield et al. (2020) which does not necessitate the assumption that C_h is OS. Our definition is equivalent to the original definition under this assumption, which we will impose throughout the paper.

Example 1. Let $H = \{h\}$, $D = \{d_x, d_y, d_z, d_w\}$, $X = \{x_1, x_2, y_1, y_2, z_1, z_2, w_1, w_2\}$ where $\mathrm{d}(v_1) = \mathrm{d}(v_2) = d_v$ for each $v \in \{x, y, z, w\}$. Let

$$\Sigma_1 = \{\{x_2, y_2, z_2, w_1\}, \{x_1, y_1, z_1, w_2\}\} \cup \{\{x_i, y_j, z_2, w_2\}\}_{i,j \in \{1,2\}},$$

$$\Sigma_2 = \{\{x_2, y_1, z_1, w_2\}, \{x_1, y_2, z_2, w_1\}\},$$

$$\Sigma_3 = \{\{x_2, z_2, w_1\}, \{x_1, z_1, w_2\}\},$$

$$\Sigma_4 = \{\{y_1, z_2, w_1\}, \{y_2, z_1, w_2\}\}, \text{and}$$

$$\Sigma_5 = \{\{x_1, y_1, w_1\}, \{x_2, y_2, z_1\}, \{x_1, y_2, w_1\}, \{x_2, y_1, z_1\}\}.$$

Let C_h be a choice function induced by an arbitrary strict preference relation $\succ_h$ satisfying the following condition:

$$\Sigma_1 \succ_h \Sigma_2 \succ_h \Sigma_3 \succ_h \Sigma_4 \succ_h \Sigma_5 \succ_h$$
$$\{\text{all other feasible tripletons}\} \succ_h \{\text{all feasible doubletons}\} \succ_h \{\text{all singletons}\} \succ_h \emptyset,$$

where the unlisted sets are all unacceptable; i.e., not regarded as better than rejecting all the contracts. By construction, C_h is OSM.[9]

Claim 1. C_h is OS.

Proof. We only need to check observable offer processes $\boldsymbol{v} = (v_1, \ldots, v_4, \ldots)$ satisfying $\{v_1, \ldots, v_4\} \notin \Sigma_1 \cup \Sigma_2$. Table 1 lists all the possible observable offer processes satisfying the condition where the first four contracts are lined up so that d_x proposes first, followed by d_y, d_z, and d_w.[10] By checking the rows of Table 1, we can confirm that C_h is OS. ∎

[9] Indeed, it is *size-monotonic* (or satisfies the *law of aggregate demand*) (Hatfield and Milgrom, 2005) which requires that $|C_h(X')| \leq |C_h(X'')|$ should hold for *all* $X' \subset X''$.

[10] By the construction of C_h, for any observable offer process, the first three contracts should all be accepted by h, and hence the first four contracts should be the collection of the best acceptable contracts of the four doctors. Thus, there is no loss of generality to restrict attention to the possible observable offer processes where the first four contracts are lined up in this way.

	$\boldsymbol{v}$	C_h	R_h	cf.
[1]	(x_1, y_1, z_1, w_1)	$\{x_1, y_1, w_1\}$	$\{z_1\}$	
[2]	$(x_1, y_1, z_1, w_1, z_2)$	$\{y_1, z_2, w_1\}$	$\{x_1, z_1\}$	
[3]	$(x_1, y_1, z_1, w_1, z_2, x_2)$	$\{x_2, z_2, w_1\}$	$\{x_1, y_1, z_1\}$	
[4]	$(x_1, y_1, z_1, w_1, z_2, x_2, y_2)$	$\{x_2, y_2, z_2, w_1\}$	$\{x_1, y_1, z_1\}$	
[5]	(x_1, y_2, z_1, w_1)	$\{x_1, y_2, w_1\}$	$\{z_1\}$	
[6]	$(x_1, y_2, z_1, w_1, z_2)$	$\{x_1, y_2, z_2, w_1\}$	$\{z_1\}$	
[7]	(x_2, y_1, z_1, w_1)	$\{x_2, y_1, z_1\}$	$\{w_1\}$	
[8]	$(x_2, y_1, z_1, w_1, w_2)$	$\{x_2, y_1, z_1, w_2\}$	$\{w_1\}$	
[9]	(x_2, y_2, z_1, w_1)	$\{x_2, y_2, z_1\}$	$\{w_1\}$	
[10]	$(x_2, y_2, z_1, w_1, w_2)$	$\{y_2, z_1, w_2\}$	$\{x_2, w_1\}$	
[11]	$(x_2, y_2, z_1, w_1, w_2, x_1)$	$\{x_1, z_1, w_2\}$	$\{x_2, y_2, w_1\}$	
[12]	$(x_2, y_2, z_1, w_1, w_2, x_1, y_1)$	$\{x_1, y_1, z_1, w_2\}$	$\{x_2, y_2, w_1\}$	
[13]	(x_1, y_2, z_1, w_2)	$\{x_1, z_1, w_2\}$	$\{y_2\}$	[11]
[14]	$(x_1, y_2, z_1, w_2, y_1)$	$\{x_1, y_1, z_1, w_2\}$	$\{y_2\}$	[12]
[15]	(x_2, y_2, z_1, w_2)	$\{y_2, z_1, w_2\}$	$\{x_2\}$	[10]
[16]	$(x_2, y_2, z_1, w_2, x_1)$	$\{x_1, z_1, w_2\}$	$\{x_2, y_2\}$	[11]
[17]	$(x_2, y_2, z_1, w_2, x_1, y_1)$	$\{x_1, y_1, z_1, w_2\}$	$\{x_2, y_2\}$	[12]
[18]	(x_1, y_1, z_2, w_1)	$\{y_1, z_2, w_1\}$	$\{x_1\}$	[2]
[19]	$(x_1, y_1, z_2, w_1, x_2)$	$\{x_2, z_2, w_1\}$	$\{x_1, y_1\}$	[3]
[20]	$(x_1, y_1, z_2, w_1, x_2, y_2)$	$\{x_2, y_2, z_2, w_1\}$	$\{x_1, y_1\}$	[4]
[21]	(x_2, y_1, z_2, w_1)	$\{x_2, z_2, w_1\}$	$\{y_1\}$	[3]
[22]	$(x_2, y_1, z_2, w_1, y_2)$	$\{x_2, y_2, z_2, w_1\}$	$\{y_1\}$	[4]

Table 1: Observable offer processes $\boldsymbol{v} = (v_1, \ldots, v_4, \ldots)$ satisfying $\{v_1, \ldots, v_4\} \notin \Sigma_1 \cup \Sigma_2$.

$T(\succ_D)$	Table 1	d_x	d_y	d_z	d_w
$\{x_1, y_1, z_1, w_1\}$	[1]–[4]	B([7]–[8])	B([5]–[6])	B([18]–[20])	A
$\{x_1, y_2, z_1, w_1\}$	[5]–[6]	A	A	$B(\Sigma_2)$	A
$\{x_2, y_1, z_1, w_1\}$	[7]–[8]	A	A	A	$B(\Sigma_2)$
$\{x_2, y_2, z_1, w_1\}$	[9]–[12]	B([5]–[6])	B([7]–[8])	A	B([15]–[17])
$\{x_1, y_2, z_1, w_2\}$	[13]–[14]	A	$B(\Sigma_1)$	A	A
$\{x_2, y_2, z_1, w_2\}$	[15]–[17]	B([13]–[14])	$B(\Sigma_2)$	A	A
$\{x_1, y_1, z_2, w_1\}$	[18]–[20]	B([21]–[22])	$B(\Sigma_2)$	A	A
$\{x_2, y_1, z_2, w_1\}$	[21]–[22]	A	$B(\Sigma_1)$	A	A

Table 2: ψ^C is strategy-proof.

Claim 2. ψ^C is strategy-proof.

Proof. Note that no doctor has an incentive to misrepresent when $\psi^C(\succ_D) = T(\succ_D)$. By the construction of C_h, this is true when $|T(\succ_D)| < 4$ or $T(\succ_D) \in \Sigma_1 \cup \Sigma_2$. Hence there are eight remaining cases to consider, each corresponding to one of rows [1], [5], [7], [9], [13], [15], [18], and [21] of Table 1.

Table 2 summarizes the remaining argument. Each row corresponds to each remaining case of $T(\succ_D)$. The letter A indicates that $\psi^C_d(\succ_D) = t(\succ_d)$ holds for the corresponding doctor d, and thus he has no incentive to misrepresent. For example, when $T(\succ_D) = \{x_1, y_1, z_1, w_1\}$, d_w will receive w_1 no matter what preference relations other doctors report so long as $T(\succ_D) = \{x_1, y_1, z_1, w_1\}$ is satisfied (see rows [1]–[4] of Table 1).

On the other hand, the letter B([m]–[n]) (or $B(\Sigma)$) indicates that when the corresponding doctor d reports $\succ'_d$ instead of his true preference relation $\succ_d$ so that $Ac(\succ'_d) \neq \emptyset$ and $t(\succ'_d) \neq t(\succ_d)$ are satisfied, he will eventually obtain $\psi^C_d(\succ'_d, \succ_{-d}) = t(\succ'_d)$ as we can see from rows $[m]$–$[n]$ of Table 1 (or since $\mathrm{T}(\succ'_d, \succ_{-d}) \in \Sigma$). We claim that, in each case, the corresponding doctor has no incentive to misrepresent.

To see this, note that since C_h is OS, once some doctor's contract is rejected at some step of the COP, it will never be chosen in later steps. Hence, no corresponding doctor has an incentive to misrepresent his preference relation by adding or removing the secondly-preferred contract. In other words, a corresponding doctor d can gain profits by reporting $\succ'_d$ instead of his true preference relation $\succ_d$ only if $\mathrm{t}(\succ_d) \neq \mathrm{t}(\succ'_d) = \psi^{\mathrm{C}}_d(\succ'_d, \succ_{-d})$, which in turn implies $|\mathrm{Ac}(\succ_d)| = 2$ and $\psi^{\mathrm{C}}_d(\succ_D) = \varnothing$. We complete the proof by showing that for each d, when $|\mathrm{Ac}(\succ_d)| = 2$, he will be assigned some non-null contract at the outcome of ψ^{C} irrespective of what preference relations other doctors report (as we can see from rows [4], [6], [8], [12], [14], [17], [20], and [22] of Table 1). ■

Claim 3. ψ^{C} is not group strategy-proof.

Proof. Consider the set of preferences for D:

$\succ_{d_x}$	$\succ_{d_y}$	$\succ_{d_z}$	$\succ_{d_w}$	$\succ'_{d_x}$	$\succ'_{d_y}$
x_1	y_1	z_1	w_1	x_2	y_2
x_2	y_2	z_2	w_2	x_1	y_1

which is read as $x_1 \succ_{d_x} x_2$, $y_2 \succ'_{d_y} y_1$, and so on. Then, $\psi^{\mathrm{C}}(\succ_D) = \{x_2, y_2, z_2, w_1\}$ (see rows [1]–[4] of Table 1) and $\psi^{\mathrm{C}}(\succ'_{\{d_x,d_y\}}, \succ_{\{d_z,d_w\}}) = \{x_1, y_1, z_1, w_2\}$ (resp. [9]–[12]) hold. Hence, under $\succ_D$, d_x and d_y can get strictly better off by jointly reporting $\succ'_{\{d_x,d_y\}}$. ■

Hence, unlike the previous literature and the previous settings considered therein, the group strategy-proofness of ψ^{C} does not follow from its strategy-proofness.[11] Nevertheless, we show that weak Pareto efficiency follows directly from strategy-proofness.

[11]This observation might seem at odds with Schlegel (2020) who established that ψ^{C} is group strategy-proof when each choice function is OS and OSM. Note, however, that Schlegel (2020) considered the setting where (i) there is a natural ordering over contract terms (e.g., salary), (ii) possible preference relations (in the message space) are restricted to those that are monotone with respect to this ordering, and (iii) both OS and OSM are defined with respect to observable offer processes that are generated from monotone preferences. Hence his OS, OSM, and (group) strategy-proof notions are all weaker than those of the current paper.

Theorem 1. *Suppose each C_h is OS. If ψ^{C} is strategy-proof, then it is weakly Pareto efficient; i.e., for any input $\succ_D$, its output $\psi^{\mathrm{C}}(\succ_D)$ is weakly Pareto efficient at $(\succ_D, C_H)$.*

Lemma 1. *Suppose each C_h is OS and ψ^{C} is strategy-proof. Then, for any $(\succ_D, \succ'_d)$ such that $\mathrm{U}(\succ_d, \psi^{\mathrm{C}}_d(\succ_D)) = \mathrm{U}(\succ'_d, \psi^{\mathrm{C}}_d(\succ_D))$, we have $\psi^{\mathrm{C}}(\succ_D) = \psi^{\mathrm{C}}(\succ'_d, \succ_{-d})$.*

Proof of Lemma 1. Toward a contradiction, let $(\succ_D, \succ'_d)$ be a *minimal counterexample*; i.e., we have both $\mathrm{U}(\succ_d, \psi^{\mathrm{C}}_d(\succ_D)) = \mathrm{U}(\succ'_d, \psi^{\mathrm{C}}_d(\succ_D))$ and $\psi^{\mathrm{C}}(\succ_D) \neq \psi^{\mathrm{C}}(\succ'_d, \succ_{-d})$, and if $(\hat{\succ}_D, \hat{\succ}'_{\hat{d}})$ is another counterexample, we have $\sum_{d' \neq d} |\mathrm{Ac}(\succ_{d'})| \leq \sum_{d' \neq \hat{d}} |\mathrm{Ac}(\hat{\succ}_{d'})|$. By the strategy-proofness of ψ^{C}, d signs the same (possibly null) contract at $\psi^{\mathrm{C}}(\succ_D)$ and at $\psi^{\mathrm{C}}(\succ'_d, \succ_{-d})$, which implies that (i) $((\succ'_d, \succ_{-d}), \succ_d)$ is also a minimal counterexample, and (ii) $\psi^{\mathrm{C}}(\succ_D)$ and $\psi^{\mathrm{C}}(\succ'_d, \succ_{-d})$ are distinct stable allocations at $(\succ_D, C_H)$. Then, the above observation (ii) and Lemma 2 of Hirata and Kasuya (2017)[12] imply that there exists some doctor $\mathrm{d}(y) \neq d$ such that

$$\varnothing \neq \underbrace{\psi^{\mathrm{C}}_{\mathrm{d}(y)}(\succ_D)}_{y} \neq \underbrace{\psi^{\mathrm{C}}_{\mathrm{d}(y)}(\succ'_d, \succ_{-d})}_{y'} \neq \varnothing.$$

By the above observation (i), there is no loss of generality to assume $y \succ_{\mathrm{d}(y)} y'$. Let $Y := \mathrm{Ac}(\succ_{\mathrm{d}(y)}) \setminus \mathrm{U}(\succ_{\mathrm{d}(y)}, y)$, which is nonempty. Then, $\psi^{\mathrm{C}}(\succ_D) = \psi^{\mathrm{C}}(\succ^{-Y}_D)$ holds. In contrast, $\mathrm{d}(y)$ should sign the null contract at $\psi^{\mathrm{C}}(\succ'_d, \succ^{-Y}_{-d})$. Hence, $(\succ^{-Y}_D, \succ'_d)$ is also a counterexample, which contradicts the minimality assumption. ■

Proof of Theorem 1. Suppose there exists $\succ_D$ such that $\psi^{\mathrm{C}}(\succ_D)$ is strongly dominated by an IR allocation $Y = \{y_d\}_{d \in D}$. Pick some $d \in D$ and let $\hat{\succ}_d$ be a preference relation where y_d is ranked first by d and is otherwise unchanged from $\succ_d$. Then, by Lemma 1, we have $\psi^{\mathrm{C}}(\hat{\succ}_d, \succ_{-d}) = \psi^{\mathrm{C}}(\succ_D)$. Repeating the same argument (at most $|D|$ times), we

[12] It asserts the following: Fix $(\succ_D, C_H)$ (where each C_h is not necessarily OS) and suppose $X', X'' \in \mathscr{X}$ are two distinct stable allocations at $(\succ_D, C_H)$. Then, some doctor signs distinct non-null contracts at X' and X''; i.e., there exist d and x', x'' such that $\mathrm{d}(x') = \mathrm{d}(x'') = d$ and $X' \ni x' \neq x'' \in X''$.

will eventually obtain $\psi^{C}(\hat{\succ}_D) = \psi^{C}(\succ_D)$ where $T(\hat{\succ}_D) = Y$. However, if $T(\hat{\succ}_D) = Y$ is IR at $(\succ_D, C_H)$ and hence at $(\hat{\succ}_D, C_H)$, the COP with its input $\hat{\succ}_D$ terminates at the end of step $|D|$ and outputs $C_H(Y) = Y$. Hence, $\psi^{C}(\hat{\succ}_D) = Y$ should follow. Since $\psi^{C}(\succ_D)$ is different from Y, we have the desired contradiction. ■

Corollary 1. *Suppose each C_h is OS, OSM, and nMvCT. Then, ψ^{C} is weakly Pareto efficient.*

Let us note that, aside from OS, we cannot generalize the above claim by dropping any of the other two conditions. On the one hand, Hatfield et al. (2020, Example 4) demonstrated that the weak Pareto efficiency of ψ^{C} may fail when a choice function of some hospital satisfies all but the nMvCT condition. On the other hand, the importance of (observable) size-monotonicity for weak Pareto efficiency is well established: See Martínez et al. (2004, Example 2) and Kojima (2008, Corollary 1).

Finally, let us briefly discuss why, unlike the previous observations in the literature, the equivalence of strategy-proofess and group strategy-proofness for ψ^{C} does not carry over. Note that the equivalence results in Hatfield and Kojima (2010), Hatfield and Kominers (2012), Fleiner et al. (2018), Hatfield and Kominers (2019), and Schlegel (2020) are all established either directly or indirectly by an adaptation of the method of Hatfield and Kojima (2009) (or more generally, Barberà et al. 2016). Their method is built on the premise that the doctor-optimal stable mechanism[13] satisfies (i) the same kind of invariance condition as in Lemma 1 provided that it is strategy-proof, and (ii) a kind of monotonicity condition which requires that whenever some doctors move their contracts in $\psi^{C}(\succ_D)$ to the top of their preference rankings, these contracts must continue to be chosen.[14] For ψ^{C} in our setting, the first condition is assured as we saw in Lemma 1, which allows us to

[13] As demonstrated in Hatfield et al. (2020, Example 5), a doctor-optimal stable allocation (i.e., a stable allocation that is unanimously preferred by all doctors to other stable allocations) does not exist in general even when each C_h is OS, OSM, and nMvCT.

[14] Barberà et al. (2016) derived the first condition by strategy-proofness and their $\mathscr{H}$-respectfulness condition, whereas they assumed the second condition by their $\mathscr{H}$-joint monotonicity condition.

achieve weak Pareto efficiency. In contrast, the second condition is not satisfied in general as we saw in Claim 3 of Example 1. Hence, the lack of (some kind of) monotonicity seems to be responsible for the nonequivalence, which awaits further investigation.

Acknowledgment

We thank the anonymous referee for useful comments. We are also grateful for financial support from JSPS KAKENHI (#19K01542).

References

ABDULKADIROĞLU, A. AND T. SÖNMEZ (2003): "School Choice: A Mechanism Design Approach," *American Economic Review*, 93, 729–747.

AYGÜN, O. AND T. SÖNMEZ (2013): "Matching with contracts: Comment," *American Economic Review*, 103, 2050–51.

BARBERÀ, S., D. BERGA, AND B. MORENO (2016): "Group Strategy-Proofness in Private Good Economies," *American Economic Review*, 106, 1073–1099.

DEMANGE, G. AND D. GALE (1985): "The Strategy Structure of Two-Sided Matching Markets," *Econometrica*, 873–888.

DUBINS, L. E. AND D. A. FREEDMAN (1981): "Machiavelli and the Gale-Shapley Algorithm," *American Mathematical Monthly*, 88, 485–491.

FLEINER, T., Z. JANKÓ, A. TAMURA, AND A. TEYTELBOYM (2018): "Trading Networks with Bilateral Contracts," *mimeo*.

GALE, D. AND L. S. SHAPLEY (1962): "College Admissions and the Stability of Marriage," *American Mathematical Monthly*, 69, 9–15.

HAFALIR, I. E., F. KOJIMA, AND M. B. YENMEZ (2019): "Interdistrict school choice: A theory of student assignment," *mimeo*.

HATFIELD, J. W. AND F. KOJIMA (2009): "Group incentive compatibility for matching with contracts," *Games and Economic Behavior*, 67, 745–749.

——— (2010): "Substitutes and Stability for Matching with Contracts," *Journal of Economic Theory*, 145, 1704–1723.

HATFIELD, J. W. AND S. D. KOMINERS (2012): "Matching in Networks with Bilateral Contracts," *American Economic Journal: Microeconomics*, 4, 176–208.

——— (2019): "Hidden Substitutes," *mimeo*.

HATFIELD, J. W., S. D. KOMINERS, AND A. WESTKAMP (2017): "Stable and Strategy-Proof Matching with Flexible Allotments," *American Economic Review Papers and Proceedings*, 107, 214–219.

——— (2020): "Stability, Strategy-Proofness, and Cumulative Offer Mechanisms," *The Review of Economic Studies*, forthcoming.

HATFIELD, J. W. AND P. R. MILGROM (2005): "Matching with Contracts," *American Economic Review*, 95, 913–935.

HIRATA, D. AND Y. KASUYA (2017): "On Stable and Strategy-Proof Rules in Matching Markets with Contracts," *Journal of Economic Theory*, 168, 27–43.

JIAO, Z. AND G. TIAN (2017): "The Blocking Lemma and strategy-proofness in many-to-many matchings," *Games and Economic Behavior*, 102, 44–55.

KOJIMA, F. (2008): "The Law of Aggregate Demand and Welfare in the Two-Sided Matching Market," *Economics Letters*, 99, 581–584.

KOMINERS, S. D. AND T. SÖNMEZ (2016): "Matching with Slot-Specific Priorities: Theory," *Theoretical Economics*, 11, 683–710.

MARTÍNEZ, R., J. MASSÓ, A. NEME, AND J. OVIEDO (2004): "On group strategy-proof mechanisms for a many-to-one matching model," *International Journal of Game Theory*, 33, 115–128.

ROTH, A. E. (1982): "The Economics of Matching: Stability and Incentives," *Mathematics of Operations Research*, 7, 617–628.

SCHLEGEL, J. C. (2018): "Group-Strategy-Proof Mechanisms for Job Matching with Continuous Transfers," *mimeo*.

——— (2020): "Equivalent choice functions and stable mechanisms," *Games and Economic Behavior*, 123, 41–53.