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Abstract

We consider the multiple products of relevant and marginal scalar composite operators at the Gaussian fixed-point in $D = 4$ dimensions. This amounts to perturbative construction of the ϕ^4 theory where the parameters of the theory are momentum dependent sources. Using the exact renormalization group (ERG) formalism, we show how the scaling properties of the sources are given by the short-distance singularities of the multiple products.

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I. INTRODUCTION

For a quantum field theory to have a well defined continuum limit, it must have an ultraviolet (UV) fixed-point of the renormalization group. Conversely, given a fixed-point, we can build a continuum limit with as many parameters as the number of relevant operators. Modulo the important question of convergence, the continuum limit with parameters can be constructed out of the multiple products of the relevant operators at the fixed-point. [1]

Following this idea, Cardy has considered a generic fixed-point action S^* with relevant operators $\mathcal{O}_i$. [2] Let the first order action of the theory be

$$S^* + \sum_i g_i \int d^D x \, \mathcal{O}_i(x) \, . \tag{1}$$

The short-distance singularity of the operator products are given as

$$\mathcal{O}_i(x)\mathcal{O}_j(y) \xrightarrow{x \to y} \sum_k \frac{c_{ij,k}}{|x-y|^{x_i+x_j-x_k}}\mathcal{O}_k(x) \, , \tag{2}$$

where x_i is the scale dimension of $\mathcal{O}_i$ in D-dimensional coordinate space. When $x_i+x_j-x_k \geq D$, the product is unintegrable, and the second order perturbation needs regularization. Consequently, the parameter g_i acquires a mixing term proportional to $\sum_{j,k} c_{jk,i}g_jg_k$ under the renormalization group. (This result has been reviewed in Appendix C of [3] using the exact renormalization group formalism.)

Even before this observation, we all had been familiar with the idea that the short-distance singularities are the source of renormalization and the associated renormalization group. Consider the perturbative construction of ϕ^4 theory in $D = 4 - \epsilon$ dimensions. The bare parameters and field are given by

$$m_0^2 = Z_m(\epsilon; \lambda)m^2 \, , \tag{3a}$$

$$\lambda_0 = Z_\lambda(\epsilon; \lambda)\lambda\mu^\epsilon \, , \tag{3b}$$

$$\phi_0 = \sqrt{Z(\epsilon; \lambda)} \, \phi \, , \tag{3c}$$

where the renormalization constants are given in the MS prescription. The renormalization constants, which cancel the UV divergences, are determined by the beta functions and the anomalous dimension as [4]

$$Z_\lambda(\epsilon; \lambda) = \exp\left(\int_0^\lambda dx \left(\frac{\epsilon}{\epsilon x + \beta(x)} - \frac{1}{x}\right)\right) \, , \tag{4a}$$

$$Z_m(\epsilon; \lambda) = \exp\left(-\int_0^\lambda dx\, \frac{\beta_m(x)}{\epsilon x + \beta(x)}\right), \tag{4b}$$

$$Z(\epsilon; \lambda) = \exp\left(-\int_0^\lambda dx\, \frac{2\gamma(x)}{\epsilon x + \beta(x)}\right). \tag{4c}$$

(Note that we have chosen the sign of beta functions to give the change of parameters toward the IR.)

In this paper we take the Gaussian fixed-point in $D = 4$ dimensions as the simplest example, and study how to construct the multiple products of three scalar composite operators $\phi^2, \phi^4, \partial_\mu\phi\partial_\mu\phi$ which are either relevant or marginal. (These operators will be called $\mathcal{O}_2, \mathcal{O}_4, \mathcal{N}$ in the main text.) Using the exact renormalization group (ERG) formalism, we construct a theory with momentum dependent sources coupled to the three composite operators. (There are many reviews of ERG. See [5] and references therein. We mostly follow the conventions of [6] and [7] out of familiarity.) We will obtain a precise relation between the short-distance singularities of the operator products and the mixing coefficients under scaling of the sources.

Momentum dependent sources for composite operators, equivalent to space dependent parameters, have been considered before. In generalizing Zamolodchikov's c-theorem, Jack and Osborn have introduced space dependent parameters using the dimensional regularization. [8–10] (See also [11] for more recent developments.) The correlation functions of the bare composite operators have poles in $\epsilon = 4 - D$, which they have related to the beta functions of the space dependent parameters, just as in (4). The poles result from the short-distance singularities we discuss here.

This paper is organized as follows. In Sec. II we briefly review the composite operators in the ERG formalism. We introduce the three scalar composite operators the multiple products of which constitute the main subject of this paper. In Sec. III we sketch how to construct multiple products of composite operators in the ERG formalism. This section is based upon the results of [12]. In Sec. IV we give concrete examples of the products of two composite operators. We then give a general discussion of the number operator in Sec. V. The number operator is an equation-of-motion operator, and the precise control we gain over their products greatly simplifies our study of the multiple products. In Sec. VI we introduce the generating functional W of the multiple products. The sources for W are momentum dependent parameters of the ϕ^4 theory. We show that the scaling property of

e^W is given by the coefficients of short-distance singularities in the multiple products of composite operators. The paper is concluded in Sec. VII. We sketch the calculations of some integrals of cutoff functions in Appendix A, derive the asymptotic behavior of two functions (F, G) in Appendix B, review quickly the equation-of-motion operators in the ERG formalism in Appendix C, and construct some products of three composite operators in Appendix D. In Appendix E we switch from momentum space to coordinate space to discuss the scaling properties of the space dependent parameters.

Throughout the paper we work in $D = 4$ dimensional Euclidean space, and use the following short-hand notation for the momentum space:

$$\int_p \equiv \int \frac{d^D p}{(2\pi)^D}, \quad \delta(p) \equiv (2\pi)^D \delta^{(D)}(p), \quad p \cdot \partial_p \equiv \sum_{\mu=1}^{D} p_\mu \frac{\partial}{\partial p_\mu}.$$

II.   COMPOSITE OPERATORS IN THE ERG FORMALISM

The Wilson action of the free massless scalar theory is given by

$$S_\Lambda[\phi] = -\frac{1}{2} \int_p \phi(-p) \frac{p^2}{K(p/\Lambda)} \phi(p) \tag{5}$$

in the momentum space, where Λ is a momentum cutoff. The smooth and positive cutoff function $K(p/\Lambda)$ is 1 at $p = 0$, is of order 1 for $p \sim \Lambda$, and approaches zero rapidly as $p/\Lambda \to \infty$. An example is given by $\exp(-p^2/\Lambda^2)$. The corresponding propagator $K(p/\Lambda)/p^2$ suppresses the high momentum modes. The cutoff dependence of the Wilson action is given by the ERG differential equation[1]

$$-\Lambda \frac{\partial}{\partial \Lambda} e^{S_\Lambda[\phi]} = \int_p \frac{\Delta(p/\Lambda)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} e^{S_\Lambda[\phi]}, \tag{6}$$

where we define

$$\Delta(p/\Lambda) \equiv \Lambda \frac{\partial}{\partial \Lambda} K(p/\Lambda) > 0. \tag{7}$$

To obtain the Gaussian fixed-point from S_Λ, we must rescale the momentum and field using the cutoff Λ. We introduce dimensionless momentum $\bar{p}$ and field $\bar{\phi}(\bar{p})$ by

$$p = \Lambda \bar{p}, \tag{8a}$$

$$\phi(p) = \Lambda^{-\frac{D+2}{2}} \bar{\phi}(\bar{p}). \tag{8b}$$

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[1] Strictly speaking, we must ignore the field independent part of the action for the validity of Eq. (6).

Please note that the mass dimension of the scalar field $\phi(x)$ in coordinate space is $\frac{D-2}{2}$, and that of its Fourier transform $\phi(p) = \int d^D x \, e^{-ipx} \phi(x)$ is $-\frac{D+2}{2}$. Writing (5) using the dimensionless momentum and field, we obtain the Gaussian fixed-point action

$$\bar{S}[\bar{\phi}] = -\frac{1}{2} \int_{\bar{p}} \bar{\phi}(-\bar{p}) \frac{\bar{p}^2}{K(\bar{p})} \bar{\phi}(\bar{p}) \,. \tag{9}$$

This satisfies the fixed-point equation

$$0 = \int_{\bar{p}} \left[\left(\frac{D+2}{2} + \bar{p} \cdot \partial_{\bar{p}} \right) \bar{\phi}(\bar{p}) \cdot \frac{\delta}{\delta \bar{\phi}(\bar{p})} + \frac{\Delta(\bar{p})}{\bar{p}^2} \frac{1}{2} \frac{\delta^2}{\delta \bar{\phi}(\bar{p}) \delta \bar{\phi}(-\bar{p})} \right] e^{\bar{S}[\bar{\phi}]} \,. \tag{10}$$

From now on we use this dimensionless convention by measuring all the dimensionful quantities in units of appropriate powers of the cutoff Λ. We omit the bars above momenta and fields entirely for the sake of simplicity. Hence, we write the Gaussian fixed-point action as

$$S[\phi] = -\frac{1}{2} \int_p \phi(-p) \frac{p^2}{K(p)} \phi(p) \,. \tag{11}$$

The correlation functions defined by

$$\langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle$$
$$\equiv \prod_{i=1}^n \frac{1}{K(p_i)} \cdot \left\langle \exp\left(-\int_p \frac{K(p)(1-K(p))}{p^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right) \phi(p_1) \cdots \phi(p_n) \right\rangle_S \tag{12}$$

satisfy the scaling relation

$$\langle\!\langle \phi(p_1 e^t) \cdots \phi(p_n e^t) \rangle\!\rangle = e^{-n\frac{D+2}{2}t} \langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle \,. \tag{13}$$

Here,

$$\langle \cdots \rangle_S \equiv \int [d\phi] e^{S[\phi]} \cdots \tag{14}$$

is the standard correlation function with the Boltzmann weight $e^{S[\phi]}$ given by the Wilson action. Let us explain the role of the exponentiated second order differential by considering the example of $n = 2$. We obtain

$$\langle\!\langle \phi(p)\phi(q) \rangle\!\rangle = \frac{1}{K(p)K(q)} \left(\langle \phi(p)\phi(q) \rangle - \frac{K(p)(1-K(p))}{p^2} \delta(p+q) \right)$$
$$= \frac{1}{K(p)K(q)} \left(\frac{K(p)}{p^2} - \frac{K(p)(1-K(p))}{p^2} \right) \delta(p+q)$$
$$= \frac{1}{p^2} \delta(p+q) \,. \tag{15}$$

The exponentiated differential modifies the high momentum behavior a little so that the correct propagator is recovered from the Wilson action. This simple modification works not only for the free theory but also for the interacting theory.[7]

A composite operator $\mathcal{O}(p)$ with scale dimension $-y$ satisfies the ERG differential equation

$$(p \cdot \partial_p + y - \mathcal{D}) \mathcal{O}(p) = 0, \tag{16}$$

where

$$\mathcal{D} \equiv \int_p \left[\left(p \cdot \partial_p + \frac{D+2}{2} \right) \phi(p) \cdot \frac{\delta}{\delta\phi(p)} + f(p)\frac{1}{2}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right]. \tag{17}$$

$f(p)$ is the derivative of the high-momentum propagator $h(p)$:

$$f(p) \equiv (p \cdot \partial_p + 2) \, h(p) = \frac{\Delta(p)}{p^2}, \tag{18a}$$

$$h(p) \equiv \frac{1 - K(p)}{p^2}. \tag{18b}$$

Please note that $-y$ is the scale dimension of the operator in momentum space. The scale dimension of the corresponding operator in coordinate space is $D - y$, often denoted as x (not to be confused with space coordinate). Eq. (16) implies that the correlation function of the composite operator

$$\langle\!\langle \mathcal{O}(p)\phi(p_1) \cdots \phi(p_n) \rangle\!\rangle$$
$$\equiv \prod_{i=1}^{n} \frac{1}{K(p_i)} \left\langle \mathcal{O}(p) \exp\left(-\int_q \frac{K(q)\,(1 - K(q))}{q^2}\frac{1}{2}\frac{\delta^2}{\delta\phi(q)\delta\phi(-q)} \right) \phi(p_1) \cdots \phi(p_n) \right\rangle_S \tag{19}$$

satisfies the scaling relation

$$\langle\!\langle \mathcal{O}(pe^t)\phi(p_1 e^t) \cdots \phi(p_n e^t) \rangle\!\rangle = e^{-\left(y + n\frac{D+2}{2}\right)t} \langle\!\langle \mathcal{O}(p)\phi(p_1) \cdots \phi(p_n) \rangle\!\rangle. \tag{20}$$

Note the absence of the factor $1/K(p)$ for the composite operator in Eq. (19). A composite operator acts like an infinitesimal variation of the action S, and Eq. (19) can be regarded as the corresponding infinitesimal variation of Eq. (12).

We are particularly interested in the scalar composite operators which are even in ϕ and relevant, i.e., the scale dimensions $-y$ are either zero or negative. There are only three independent operators: $\mathcal{O}_2$ corresponding to ϕ^2, $\mathcal{O}_4$ corresponding to ϕ^4, and $\mathcal{N}$ corresponding to $\partial_\mu\phi\partial_\mu\phi$. (We do not count $\partial^2\phi^2$ separately.) The purpose of this paper is to understand the multiple products of the three composite operators. Let us end this section by constructing these composite operators by solving the respective ERG differential equations.

6

A. $\mathcal{O}_2$ with $y = 2$

We introduce $\mathcal{O}_2$ by the ERG equation

$$\left(p \cdot \partial_p + 2 - \mathcal{D}\right) \mathcal{O}_2(p) = 0 \,. \tag{21}$$

This gives

$$\mathcal{O}_2(p) = \frac{1}{2} \int_{p_1, p_2} \phi(p_1) \phi(p_2) \, \delta\left(p_1 + p_2 - p\right) + v_2 \delta(p) \,, \tag{22}$$

where

$$v_2 \equiv -\frac{1}{4} \int_q f(q) \,. \tag{23}$$

(f is defined by Eq. (18a).) We may graphically express $\mathcal{O}_2$ as

$$\mathcal{O}_2(p) = \qquad + \qquad , \tag{24}$$

where each open solid line corresponds to a factor of ϕ, and the loop denotes v_2, corresponding to an integral over the high-momentum propagator.[2] We give graphs merely to aid the intuitive understanding of the results, and they should not be taken too seriously.

B. $\mathcal{O}_4$ with $y = 0$

We introduce $\mathcal{O}_4$ by

$$\left(p \cdot \partial_p - \mathcal{D}\right) \mathcal{O}_4(p) = 0 \,. \tag{25}$$

This gives

$$\mathcal{O}_4(p) = \frac{1}{4!} \int_{p_1, \cdots, p_4} \prod_{i=1}^{4} \phi(p_i) \, \delta\left(\sum_{i=1}^{4} p_i - p\right)$$
$$+ v_2 \frac{1}{2} \int_{p_1, p_2} \phi(p_1) \phi(p_2) \, \delta(p_1 + p_2 - p) + \frac{1}{2} v_2^2 \delta(p) \tag{26}$$

$$= \qquad + \qquad + \qquad . \tag{27}$$

In fact

$$p^2 \mathcal{O}_2(p) = p^2 \frac{1}{2} \int_{p_1, p_2} \phi(p_1) \phi(p_2) \, \delta(p_1 + p_2 - p) \tag{28}$$

satisfies the same ERG differential equation as $\mathcal{O}_4(p)$.

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[2] The naive expression for v_2 is $\frac{1}{2} \int_p h(p)$ which is quadratically divergent, but the ERG gives a perfectly finite expression (23). A hand-waving derivation of this result is integration by parts, ignoring the divergent surface integral: $0 = \int_p (p \cdot \partial_p + 4) h(p) = \int_p f(p) + 2 \int_p h(p)$ gives $v_2 = \frac{1}{2} \int_p h(p) = -\frac{1}{4} \int_p f(p)$.

7

C. $\mathcal{N}$ with $y = 0$

The number operator $\mathcal{N}$ is an equation-of-motion operator, defined by

$$\mathcal{N}(p) \equiv -e^{-S} \int_q K(q) \frac{\delta}{\delta\phi(q)} \left(\phi(q+p)e^S\right).$$

(29)

This is computed as

$$\mathcal{N}(p) = \frac{1}{2} \int_{p_1,p_2} \phi(p_1)\phi(p_2) \left(p_1^2 + p_2^2\right) \delta(p_1 + p_2 - p) - \int_q K(q)\,\delta(p).$$

(30)

We will give a general discussion of the number operator and its multiple products in Sec. V.

III.   MULTIPLE PRODUCTS OF COMPOSITE OPERATORS

Multiple products of composite operators have been considered in [12]. Here we review what is necessary for this paper. For more details we refer the reader to [12]. See [13] for a recent application of the formalism.

Consider the product of a composite operator $\mathcal{O}_i(p)$ of scale dimension $-y_i$ and another $\mathcal{O}_j(p)$ of scale dimension $-y_j$.[3] Their product is given in the form

$$[\mathcal{O}_i(p)\mathcal{O}_j(q)] = \mathcal{O}_i(p)\mathcal{O}_j(q) + \mathcal{P}_{ij}(p,q).$$

(31)

The counterterm $\mathcal{P}_{ij}$ is necessary for the product to be a composite operator that satisfies the ERG equation

$$(p \cdot \partial_p + q \cdot \partial_q + y_i + y_j - \mathcal{D}) [\mathcal{O}_i(p)\mathcal{O}_j(q)] = 0.$$

(32)

This is equivalent to the scaling law

$$\langle\!\langle \left[\mathcal{O}_i(pe^t)\mathcal{O}_j(qe^t)\right] \phi(p_1 e^t) \cdots \phi(p_n e^t) \rangle\!\rangle = e^{-(y_i+y_j)t - n\frac{D+2}{2}t} \langle\!\langle \left[\mathcal{O}_i(p)\mathcal{O}_j(q)\right] \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle.$$

(33)

The ERG equation for the counterterm is obtained from (32) as

$$(p \cdot \partial_p + q \cdot \partial_q + y_i + y_j - \mathcal{D}) \mathcal{P}_{ij}(p,q) = \int_r f(r) \frac{\delta\mathcal{O}_i(p)}{\delta\phi(r)} \frac{\delta\mathcal{O}_j(q)}{\delta\phi(-r)}.$$

(34)

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[3] We use an abstract notation in this section. $\mathcal{O}_2, \mathcal{O}_4$, and $\mathcal{N}$ introduced in the previous section are examples of $\mathcal{O}_i$ discussed here. We will return to the concrete discussion of $\mathcal{O}_2, \mathcal{O}_4, \mathcal{N}$ in the next section.

The counterterm is induced by the short-distance singularity of the product of two composite operators. The singularity may induce a mixing of the product with a local composite operator $\mathcal{O}_k$ with scale dimension $-y_k$ if

$$-y_k + 2d = -y_i - y_j \iff y_k = y_i + y_j + 2d \tag{35}$$

where $d = 0, 1, 2, \cdots$. The ERG equation is then modified to

$$(p \cdot \partial_p + q \cdot \partial_q + y_i + y_j - \mathcal{D}) \left[\mathcal{O}_i(p)\mathcal{O}_j(q)\right] = \gamma_{ij,k}(p, q)\mathcal{O}_k(p + q), \tag{36}$$

where $\gamma_{ij,k}(p, q)$ is a degree $2d$ polynomial of p, q. Unless there is such y_k, there is no mixing. We will give examples in the next section. Since we can rewrite (36) as

$$(p \cdot \partial_p + q \cdot \partial_q + y_i + y_j - \mathcal{D}) \left(\left[\mathcal{O}_i(p)\mathcal{O}_j(q)\right] - \ln p \cdot \gamma_{ij,k}(p, q)\mathcal{O}_k(p + q)\right) = 0, \tag{37}$$

we find that the mixing causes the short-distance singularity

$$\left[\mathcal{O}_i(p)\mathcal{O}_j(q)\right] \overset{p,q \to \infty}{\longrightarrow} \ln p \cdot \gamma_{ij,k}(p, q)\mathcal{O}_k(p + q), \tag{38}$$

where $p + q$ is fixed. Note that the solution of (36) is ambiguous: $[\mathcal{O}_i(p)\mathcal{O}_j(q)]$ can be redefined with an addition of

$$\sigma_{ij,k}(p, q)\, \mathcal{O}_k(p + q) \tag{39}$$

where $\sigma_{ij,k}(p, q)$ is a polynomial of degree $2d$. But this redefinition does not affect the logarithmic singularity (38).

We can consider higher order products. The product of three composite operators is given as

$$\begin{aligned}
\left[\mathcal{O}_i(p)\mathcal{O}_j(q)\mathcal{O}_k(r)\right] = {} & \mathcal{O}_i(p)\mathcal{O}_j(q)\mathcal{O}_k(r) + \mathcal{P}_{ij}(p, q)\mathcal{O}_k(r) \\
& + \mathcal{P}_{jk}(q, r)\mathcal{O}_i(p) + \mathcal{P}_{ki}(r, p)\mathcal{O}_j(q) \\
& + \mathcal{P}_{ijk}(p, q, r),
\end{aligned} \tag{40}$$

where $\mathcal{P}_{ijk}$ takes care of the short-distance singularity that occurs when all the three operators come close together in space. Allowing for the possibility of mixing, we obtain the ERG equation for the product as

$$(p \cdot \partial_p + q \cdot \partial_q + r \cdot \partial_r + y_i + y_j + y_k - \mathcal{D}) \left[\mathcal{O}_i(p)\mathcal{O}_j(q)\mathcal{O}_k(r)\right]$$

$$= \gamma_{ij,l}(p,q)\left[\mathcal{O}_l(p+q)\mathcal{O}_k(r)\right] + \gamma_{jk,l}(q,r)\left[\mathcal{O}_l(q+r)\mathcal{O}_i(p)\right]$$

$$+ \gamma_{ki,l}(r,p)\left[\mathcal{O}_l(r+p)\mathcal{O}_j(q)\right] + \gamma_{ijk,l}(p,q,r)\mathcal{O}_l(p+q+r)\,, \tag{41}$$

where the mixing coefficient $\gamma_{ijk,l}(p,q,r)$ can be nonvanishing only if

$$-y_i - y_j - y_k + y_l = 2d' = 0, 2, 4, \cdots. \tag{42}$$

Then it is a polynomial of degree $2d'$. When all the three operators have large momenta, we find

$$\left[\mathcal{O}_i(p)\mathcal{O}_j(q)\mathcal{O}_k(r)\right]$$

$$\xrightarrow{p,q,r\to\infty} \frac{1}{2}(\ln p)^2 \cdot \left\{\gamma_{ij,l}(p,q)\gamma_{lk,m}(p+q,r) + \gamma_{jk,l}(q,r)\gamma_{li,m}(q+r,p)\right.$$

$$\left. + \gamma_{ki,l}(r,p)\gamma_{lj,m}(r+p,q)\right\}\mathcal{O}_m(p+q+r)$$

$$+ \ln p \cdot (\gamma_{ijk,l}(p,q,r) + \cdots)\,\mathcal{O}_l(p+q+r)\,, \tag{43}$$

where $p+q+r$ is fixed. Note that $\gamma_{ijk,l}$ is not the only source of the term proportional to $\ln p$. For example, the short-distance expansion of $\left[\mathcal{O}_l(p+q)\mathcal{O}_k(r)\right]$ contains a term $\tau_{lk,m}(p+q,r)\mathcal{O}_m(p+q+r)$ free of $\ln p$, which contributes a log term

$$\ln p \cdot \gamma_{ij,m}(p,q)\tau_{mk,l}(p+q,r) \tag{44}$$

as part of the dots in (43). ($\tau_{mk,l}$ is a polynomial of degree $-y_m - y_k + y_l$.) It is important to note that the mixing coefficient $\gamma_{ijk,l}$ depends on the definition of the products of two composite operators. If we redefine the products of two composite operators as (39), $\gamma_{ijk,l}(p,q,r)$ changes accordingly. We will derive the changes shortly.

The ERG equation for $\mathcal{P}_{ijk}$, corresponding to (41), is given by

$$\left(p\cdot\partial_p + q\cdot\partial_q + r\cdot\partial_r + y_i + y_j + y_k - \mathcal{D}\right)\mathcal{P}_{ijk}(p,q,r)$$

$$= \int_s f(s)\left(\frac{\delta\mathcal{P}_{ij}(p,q)}{\delta\phi(s)}\frac{\delta\mathcal{O}_k(r)}{\delta\phi(-s)} + \frac{\delta\mathcal{P}_{jk}(q,r)}{\delta\phi(s)}\frac{\delta\mathcal{O}_i(p)}{\delta\phi(-s)} + \frac{\delta\mathcal{P}_{ki}(r,p)}{\delta\phi(s)}\frac{\delta\mathcal{O}_j(q)}{\delta\phi(-s)}\right)$$

$$+ \gamma_{ij,l}(p,q)\mathcal{P}_{lk}(p+q,r) + \gamma_{jk,l}(q,r)\mathcal{P}_{li}(q+r,p)$$

$$+ \gamma_{ki,l}(r,p)\mathcal{P}_{lj}(r+p,q) + \gamma_{ijk,l}(p,q,r)\mathcal{O}_l(p+q+r)\,. \tag{45}$$

We will calculate products of three composite operators in Appendix D.

Changes of $\gamma_{ijk,l}$ induced by (39)

Let us redefine the products of two composite operators as

$$[\mathcal{O}_i(p)\mathcal{O}_j(q)]' \equiv \mathcal{O}_i(p)\mathcal{O}_j(q) + \mathcal{P}'_{ij}(p,q), \tag{46}$$

where

$$\mathcal{P}'_{ij}(p,q) = \mathcal{P}_{ij}(p,q) + \sigma_{ij,k}(p,q)\mathcal{O}_k(p+q), \tag{47}$$

as given by (39). We must then define the products of three composite operators as

$$
\begin{aligned}
[\mathcal{O}_i(p)\mathcal{O}_j(q)\mathcal{O}_k(r)]' \equiv{}& \mathcal{O}_i(p)\mathcal{O}_j(q)\mathcal{O}_k(r) + \mathcal{P}'_{ij}(p,q)\mathcal{O}_k(r) \\
& + \mathcal{P}'_{jk}(q,r)\mathcal{O}_i(p) + \mathcal{P}'_{ki}(r,p)\mathcal{O}_j(q) \\
& + \mathcal{P}'_{ijk}(p,q,r),
\end{aligned} \tag{48}
$$

where $\mathcal{P}'_{ijk}$ must be defined so that the product is a composite operator of scale dimension $-y_i - y_j - y_k$. To find $\mathcal{P}'_{ijk}$, we write the original product of three as

$$
\begin{aligned}
[\mathcal{O}_i(p)\mathcal{O}_j(q)\mathcal{O}_k(r)] ={}& \mathcal{O}_i(p)\mathcal{O}_j(q)\mathcal{O}_k(r) + \mathcal{P}'_{ij}(p,q)\mathcal{O}_k(r) \\
& + \mathcal{P}'_{jk}(q,r)\mathcal{O}_i(p) + \mathcal{P}'_{ki}(r,p)\mathcal{O}_j(q) \\
& + \mathcal{P}_{ijk}(p,q,r) - \sigma_{ij,m}(p,q)\mathcal{O}_m(p+q)\mathcal{O}_k(r) \\
& - \sigma_{jk,m}(q,r)\mathcal{O}_m(q+r)\mathcal{O}_i(p) - \sigma_{ki,m}(r,p)\mathcal{O}_m(r+p)\mathcal{O}_j(q).
\end{aligned} \tag{49}
$$

The last two lines cannot be the desired $\mathcal{P}'_{ijk}(p,q,r)$ since it is nonlocal, involving products of independent pieces. To make it local, we define the new product as

$$
\begin{aligned}
[\mathcal{O}_i(p)\mathcal{O}_j(q)\mathcal{O}_k(r)]' \equiv{}& [\mathcal{O}_i(p)\mathcal{O}_j(q)\mathcal{O}_k(r)] + \sigma_{ij,m}(p,q)\left[\mathcal{O}_m(p+q)\mathcal{O}_k(r)\right] \\
& + \sigma_{jk,m}(q,r)\left[\mathcal{O}_m(q+r)\mathcal{O}_i(p)\right] + \sigma_{ki,m}(r,p)\left[\mathcal{O}_m(r+p)\mathcal{O}_j(q)\right] \\
={}& \mathcal{O}_i(p)\mathcal{O}_j(q)\mathcal{O}_k(r) + \mathcal{P}'_{ij}(p,q)\mathcal{O}_k(r) \\
& + \mathcal{P}'_{jk}(q,r)\mathcal{O}_i(p) + \mathcal{P}'_{ki}(r,p)\mathcal{O}_j(q) + \mathcal{P}'_{ijk}(p,q,r),
\end{aligned} \tag{50}
$$

where

$$
\begin{aligned}
\mathcal{P}'_{ijk}(p,q,r) \equiv{}& \mathcal{P}_{ijk}(p,q,r) + \sigma_{ij,m}(p,q)\mathcal{P}_{mk}(p+q,r) \\
& + \sigma_{jk,m}(q,r)\mathcal{P}_{mi}(q+r,p) + \sigma_{ki,m}(r,p)\mathcal{P}_{mj}(r+p,q)
\end{aligned} \tag{51}
$$

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is local as desired. The new product, being the sum of composite operators, is obviously a composite operator of scale dimension $-y_i - y_j - y_k$.

Now, the new product satisfies the ERG equation

$$
(p \cdot \partial_p + q \cdot \partial_q + r \cdot \partial_r + y_i + y_j + y_k - \mathcal{D}) \left[\mathcal{O}_i(p)\mathcal{O}_j(q)\mathcal{O}_k(r) \right]'
$$
$$
= \gamma_{ij,l}(p,q) \left[\mathcal{O}_l(p+q)\mathcal{O}_k \right]' + \gamma_{jk,l}(q,r) \left[\mathcal{O}_l(q+r)\mathcal{O}_i(p) \right]'
$$
$$
+ \gamma_{ki,l}(r,p) \left[\mathcal{O}_l(r+p)\mathcal{O}_j(q) \right]' + \gamma'_{ijk,l}(p,q,r)\mathcal{O}_l(p+q+r) \,, \tag{52}
$$

where the new mixing coefficient is given by

$$
\gamma'_{ijk,l}(p,q,r) = \gamma_{ijk,l}(p,q,r)
$$
$$
+ \sigma_{ij,m}(p,q)\gamma_{mk,l}(p+q,r) - \gamma_{ij,m}(p,q)\sigma_{mk,l}(p+q,r)
$$
$$
+ \sigma_{jk,m}(q,r)\gamma_{mi,l}(q+r,p) - \gamma_{jk,m}(q,r)\sigma_{mi,l}(q+r,p)
$$
$$
+ \sigma_{ki,m}(r,p)\gamma_{mj,l}(r+p,q) - \gamma_{ki,m}(r,p)\sigma_{mj,l}(r+p,q) \,. \tag{53}
$$

If there is an operator with scale dimension

$$
-y_l = -y_i - y_j - y_k - 2d' \quad (d' = 0, 1, 2, \cdots) \,, \tag{54}
$$

we can redefine the three-operator product further by adding

$$
\sigma_{ijk,l}(p,q,r) \, \mathcal{O}_l(p+q+r) \,, \tag{55}
$$

where $\sigma_{ijk,l}$ is a polynomial of degree $2d'$. But this redefinition does not affect $\gamma'_{ijk,l}$.

IV. PRODUCTS OF TWO COMPOSITE OPERATORS

We consider the products of two composite operators, either $\mathcal{O}_2$ or $\mathcal{O}_4$ defined in Sec. II, in this section. The first two examples have been obtained in [12]. We include them here for completeness and the convenience of the reader. Through these examples we wish to verify the claim that the short-distance singularities are determined by the coefficients of mixing, or vice versa.

A. $[\mathcal{O}_2\mathcal{O}_2]$

We compute

$$
[\mathcal{O}_2(p)\mathcal{O}_2(q)] = \mathcal{O}_2(p)\mathcal{O}_2(q) + \mathcal{P}_{22}(p,q) \,, \tag{56}
$$

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where $\mathcal{P}_{22}$ satisfies

$$(p \cdot \partial_p + q \cdot \partial_q + 4 - D)\, \mathcal{P}_{22}(p,q) = \int_r f(r) \frac{\delta \mathcal{O}_2(p)}{\delta \phi(r)} \frac{\delta \mathcal{O}_2(q)}{\delta \phi(-r)} + \gamma_{22,0}\, \delta(p+q)\,. \tag{57}$$

Let

$$\mathcal{P}_{22}(p,q) = \frac{1}{2} \int_{p_1,p_2} \phi(p_1)\phi(p_2)\, \delta(p_1 + p_2 - p - q)\, c_{22,2}(p,q;p_1,p_2) + c_{22,0}(p)\delta(p+q)\,. \tag{58}$$

$[\mathcal{O}_2(p)\mathcal{O}_2(q)]$ being a composite operator of scale dimension -4, it is ambiguous by a constant multiple of $\delta(p+q)$. We remove this ambiguity by adopting a convention

$$c_{22,0}(0) = 0\,. \tag{59}$$

$c_{22,2}$ satisfies the ERG equation

$$\left(p \cdot \partial_p + q \cdot \partial_q + \sum_{i=1}^{2} p_i \cdot \partial_{p_i} + 2 \right) c_{22,2}(p,q;p_1,p_2) = f(p_1 - p) + f(p_2 - p)\,. \tag{60}$$

The solution is given by

$$c_{22,2}(p,q;p_1,p_2) = h(p_1 - p) + h(p_2 - p) = \quad\overset{\displaystyle p \quad\quad q}{\rule{0pt}{1.5em}}\!\!\!\bullet\!\!\!-\!\!\!-\!\!\!\bullet\; . \tag{61}$$

where the solid line connecting two vertices corresponds to a high-momentum propagator.

$c_{22,0}$ satisfies the ERG equation

$$p \cdot \partial_p c_{22,0}(p) = \frac{1}{2} \int_q f(q)h(q+p) + \gamma_{22,0}\,. \tag{62}$$

Now, $c_{22,0}(p)$ must be analytic at $p = 0$ since the fluctuations of momentum below the momentum cutoff 1 have not been integrated. For the analyticity at $p = 0$, we must choose

$$\gamma_{22,0} = - \int_q f(q)h(q) = -\frac{1}{(4\pi)^2}\,. \tag{63}$$

(This is calculated in Appendix A.) Using the convention (59), we obtain

$$c_{22,0}(p) = F(p) = \quad\overset{\displaystyle p \quad\quad\quad q}{\rule{0pt}{1.5em}}\!\!\leftarrow\!\cdots\!\bullet\!\!\bigcirc\!\!\bullet\!\cdots\!\rightarrow\; , \tag{64}$$

where $F(p)$ is defined by

$$F(p) \equiv \frac{1}{2} \int_q h(q)\, (h(q+p) - h(q))\,. \tag{65}$$

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Note that $\gamma_{22,0}$ determines the asymptotic behavior of $F(p)$ for large p. Since $F(p)$ satisfies

$$p \cdot \partial_p F(p) \xrightarrow{p \to \infty} \gamma_{22,0} \,, \tag{66}$$

we obtain

$$F(p) \xrightarrow{p \to \infty} \gamma_{22,0} \ln p \tag{67}$$

up to an additive constant. Hence, as p, q become large with $p + q$ fixed, we find

$$[\mathcal{O}_2(p)\mathcal{O}_2(q)] \xrightarrow{p,q \to \infty} \gamma_{22,0} \ln p \, \delta(p + q) \,. \tag{68}$$

B. $[\mathcal{O}_4\mathcal{O}_2]$

$[\mathcal{O}_4(p)\mathcal{O}_2(q)]$ satisfies the ERG equation

$$(p \cdot \partial_p + q \cdot \partial_q + 2 - \mathcal{D})\,[\mathcal{O}_4(p)\mathcal{O}_2(q)] = \gamma_{42,2}\mathcal{O}_2(p + q) + \gamma_{42,0}(p)\delta(p + q) \,. \tag{69}$$

Let

$$[\mathcal{O}_4(p)\mathcal{O}_2(q)] = \mathcal{O}_4(p)\mathcal{O}_2(q) + \mathcal{P}_{42}(p, q) \,. \tag{70}$$

$\mathcal{P}_{42}$ satisfies

$$(p \cdot \partial_p + q \cdot \partial_q + 2 - \mathcal{D})\,\mathcal{P}_{42}(p, q) = \int_r f(r)\frac{\delta \mathcal{O}_4(p)}{\delta \phi(r)}\frac{\delta \mathcal{O}_2(q)}{\delta \phi(-r)} + \gamma_{42,2}\mathcal{O}_2(p+q) + \gamma_{42,0}(p)\delta(p+q) \,. \tag{71}$$

To solve this, we expand

$$\mathcal{P}_{42}(p, q) = \frac{1}{4!}\int_{p_1,p_2,p_3,p_4} \prod_{i=1}^{4}\phi(p_i)\,\delta\left(\sum_{i=1}^{4}p_i - p - q\right) c_{42,4}(p, q; p_1, \cdots, p_4)$$

$$+ \frac{1}{2}\int_{p_1,p_2}\phi(p_1)\phi(p_2)\,\delta(p_1 + p_2 - p - q)\,c_{42,2}(p, q; p_1, p_2) + c_{42,0}(p)\delta(p + q) \,. \tag{72}$$

We first obtain

$$\left(p \cdot \partial_p + q \cdot \partial_q + \sum_{i=1}^{4}p_i \cdot \partial_{p_i} + 2\right) c_{42,4}(p, q; p_1, \cdots, p_4) = \sum_{i=1}^{4}f(p_i - q) \,, \tag{73}$$

which is solved by

$$c_{42,4}(p, q; p_1, \cdots, p_4) = \sum_{i=1}^{4}h(p_i - q) = \qquad \qquad . \tag{74}$$

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Let

$$c_{42,2}(p, q; p_1, p_2) = v_2 \left(h(p_1 - q) + h(p_2 - q) \right) + c_{42,2}^{1PI}(p, q)$$

where $c_{42,2}^{1PI}(p, q)$ satisfies

$$(p \cdot \partial_p + q \cdot \partial_q) \, c_{42,2}^{1PI}(p, q) = \int_r f(r) h(r + q) + \gamma_{42,2} \,. \tag{75}$$

For this to have a solution analytic at zero momenta, we must choose

$$\gamma_{42,2} = - \int_r f(r) h(r) = \gamma_{22,0} = - \frac{1}{(4\pi)^2} \,. \tag{76}$$

The solution is ambiguous by a constant, i.e., we can change $[\mathcal{O}_4(p)\mathcal{O}_2(q)]$ by a constant multiple of $\mathcal{O}_2(p+q)$. Note Eq. (75) is the same as Eq. (62). Hence, by adopting a convention

$$c_{42,2}^{1PI}(0, 0) = 0 \,, \tag{77}$$

we obtain

$$c_{42,2}^{1PI}(p, q) = F(q) \,, \tag{78}$$

where F is defined by (65).

Finally, we consider $c_{42,0}(p)$ that satisfies

$$(p \cdot \partial_p - 2) \, c_{42,0}(p) = \frac{1}{2} \int_r f(r) \cdot F(p) + v_2 \left(\int_r f(r) h(r + p) + \gamma_{42,2} \right) + \gamma_{42,0}(p) \,, \tag{79}$$

where $\gamma_{42,0}(p)$ is a constant multiple of p^2. For $c_{42,0}(p)$ to be analytic at $p = 0$, the rhs should have no term proportional to p^2. This determines

$$\gamma_{42,0}(p) = 0 \,. \tag{80}$$

The solution is ambiguous by a constant multiple of p^2. A particular solution is given by

$$c_{42,0}(p) = v_2 F(p) = \quad$$ $$. \tag{81}$$

For large p with $p + q$ fixed, we find the asymptotic behavior

$$[\mathcal{O}_4(p)\mathcal{O}_2(q)] \overset{p, q \to \infty}{\longrightarrow} \gamma_{42,2} \ln p \cdot \mathcal{O}_2(p + q) \,. \tag{82}$$

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C. $[\mathcal{O}_4\mathcal{O}_4]$

This example is not given in [12], and we would like to give more details than for the preceding examples. The ERG equation for the product is

$$
(p \cdot \partial_p + q \cdot \partial_q - \mathcal{D})\, [\mathcal{O}_4(p)\mathcal{O}_4(q)]
$$
$$
= \gamma_{44,4}\mathcal{O}_4(p+q) + \gamma_{44,\mathcal{N}}\mathcal{N}(p+q) + \gamma_{44,2}(p,q)\mathcal{O}_2(p+q) + \gamma_{44,0}(p)\delta(p+q)\,, \qquad (83)
$$

where $\gamma_{44,2}(p,q)$ is a degree 2 polynomial, and $\gamma_{44,0}(p)$ is proportional to $(p^2)^2$. The corresponding ERG equation for the counterterm is

$$
(p \cdot \partial_p + q \cdot \partial_q - \mathcal{D})\,\mathcal{P}_{44}(p,q)
$$
$$
= \int_r f(r)\frac{\delta\mathcal{O}_4(p)}{\delta\phi(r)}\frac{\delta\mathcal{O}_4(q)}{\delta\phi(-r)} + \gamma_{44,4}\mathcal{O}_4(p+q) + \gamma_{44,\mathcal{N}}\mathcal{N}(p+q)
$$
$$
+ \gamma_{44,2}(p,q)\mathcal{O}_2(p+q) + \gamma_{44,0}(p)\delta(p+q)\,. \qquad (84)
$$

The solution is ambiguous by a linear combination of $\mathcal{O}_4(p+q), \mathcal{N}(p+q), (p \cdot q)\mathcal{O}_2(p+q)$, and $p^4\delta(p+q)$ with constant coefficients, and we will introduce particular conventions to remove the ambiguities. We expand

$$
\mathcal{P}_{44}(p,q) = \sum_{n=1}^{3} \frac{1}{(2n)!} \int_{p_1,\cdots,p_{2n}} \prod_{i=1}^{2n} \phi(p_i)\, \delta\left(\sum_{i=1}^{2n} p_i - p - q\right) c_{44,2n}(p,q;p_1,\cdots,p_{2n})
$$
$$
+ \delta(p+q)\, c_{44,0}(p)\,. \qquad (85)
$$

The calculations of $c_{44,6}$ and $c_{44,4}$ are similar to what we have shown for the previous two examples. We simply state the results:

$$
c_{44,6}(p,q;p_1,\cdots,p_6) =
$$

$$
= h(p_1 + p_2 + p_3 - p) + \cdots\,. \qquad (86)
$$

$$
c_{44,4}(p,q;p_1,\cdots,p_4) =
$$

$$
= v_2 \sum_{i=1}^{4} \left(h(p_i - p) + h(p_i - q)\right) + c_{44,4}^{\mathrm{1PI}}(p,q;p_1,\cdots,p_4)\,, \qquad (87)
$$

where

$$
c_{44,4}^{\mathrm{1PI}}(p,q;p_1,\cdots,p_4) = F(p_1 + p_2 - p) + F(p_1 + p_2 - q) + (\mathrm{t, u})\text{-channels}\,. \qquad (88)
$$

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We have chosen

$$\gamma_{44,4} = -6 \int_q f(q)h(q) = 6\gamma_{42,2} = -\frac{6}{(4\pi)^2} \tag{89}$$

for the analyticity of $c_{44,4}^{\mathrm{1PI}}$ at zero momenta, and we have chosen a particular convention

$$c_{44,4}^{\mathrm{1PI}}(0,0;0,0,0,0) = 0 \tag{90}$$

to remove the ambiguity of $c_{44,4}^{\mathrm{1PI}}$ by a constant. For large p, q with $p + q$ fixed, we find

$$c_{44,4}(p,q;p_1,p_2,p_3,p_4) \overset{p,q\to\infty}{\longrightarrow} \gamma_{44,4} \ln p \,. \tag{91}$$

We can write $c_{44,2}$ as

$$c_{44,2}(p,q;p_1,p_2) =$$

$$= v_2^2 \left(h(p_1 - p) + h(p_2 - p) \right) + v_2 \left(F(p) + F(q) \right) + c_{44,2}^{\mathrm{1PI}}(p,q;p_1,p_2) \,, \tag{92}$$

where $c_{44,2}^{\mathrm{1PI}}$ satisfies the ERG equation

$$\left(p \cdot \partial_p + q \cdot \partial_q + \sum_{i=1}^{2} p_i \cdot \partial_{p_i} - 2 \right) c_{44,2}^{\mathrm{1PI}}(p,q;p_1,p_2)$$

$$= \int_r f(r) \left(F(r + p_1 - p) + F(r + p_2 - p) \right) + \frac{2}{3}\gamma_{44,4}v_2 + \gamma_{44,\mathcal{N}}(p_1^2 + p_2^2) + \gamma_{44,2}(p,q) \,. \tag{93}$$

$\gamma_{44,\mathcal{N}}$ and the quadratic polynomial $\gamma_{44,2}(p,q)$ are to be determined so that the rhs has no term of quadratic order at zero momenta. This is required by the analyticity of $c_{44,2}^{\mathrm{1PI}}(p,q;p_1,p_2)$.

To solve this we define $G(p)$ by

$$(p \cdot \partial_p - 2) G(p) = \int_q f(q)F(q + p) + \frac{1}{3}\gamma_{44,4}v_2 + \eta \, p^2 \,, \tag{94}$$

where

$$\eta \equiv -\frac{\partial}{\partial p^2} \int_q f(q)F(q + p) \Big|_{p=0} = \frac{1}{(4\pi)^2}\frac{1}{6} \tag{95}$$

removes the p^2 term from the rhs. See Appendix A for the calculation of η. We also remove the ambiguity of $G(p)$ by the convention

$$\frac{d}{dp^2}G(p) \Big|_{p=0} = 0 \,. \tag{96}$$

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Since

$$\left(p \cdot \partial_p + q \cdot \partial_q + \sum_{i=1}^{2} p_i \cdot \partial_{p_i} - 2 \right) (G(p_1 - p) + G(p_1 - q))$$

$$= \int_r f(r) \left(F(r + p_1 - p) + F(r + p_1 - q) \right) + \frac{2}{3} \gamma_{44,4} v_2 + \eta \left\{ (p_1 - p)^2 + (p_2 - p)^2 \right\}$$

$$= \int_r f(r) \left(F(r + p_1 - p) + F(r + p_1 - q) \right) + \frac{2}{3} \gamma_{44,4} v_2 + \eta \left(p_1^2 + p_2^2 \right) - 2\eta(p \cdot q) , \qquad (97)$$

we obtain

$$c_{44,2}^{1PI}(p, q; p_1, p_2) = G(p_1 - p) + G(p_2 - p) , \qquad (98)$$

and

$$\gamma_{44,\mathcal{N}} = \eta , \qquad (99a)$$

$$\gamma_{44,2}(p, q) = -2\eta \, p \cdot q . \qquad (99b)$$

To summarize, we have obtained

$$c_{44,2}(p, q; p_1, p_2) = v_2^2 \left(h(p_1 - p) + h(p_2 - p) \right) + v_2 \left(F(p) + F(q) \right)$$

$$+ G(p_1 - p) + G(p_2 - p) . \qquad (100)$$

Before computing $c_{44,0}(p)$, let us find the asymptotic behavior of $c_{44,2}(p, q; p_1, p_2)$. The asymptotic behavior of $G(p)$ is obtained in Appendix B as (B4):

$$G(p) \overset{p \to \infty}{\longrightarrow} \eta p^2 \ln p + 2\gamma_{42,2} v_2 \ln p , \qquad (101)$$

where we have dropped a constant multiple of p^2 and a constant. Hence, for large p, q with fixed $p + q$, we obtain the asymptotic behavior

$$c_{44,2}(p, q; p_1, p_2) \overset{p \to \infty}{\longrightarrow} \ln p \cdot \left[\eta \left(-2p \cdot q + p_1^2 + p_2^2 \right) + \gamma_{44,4} v_2 \right] . \qquad (102)$$

Finally, we consider

$$= v_2^2 F(p) + H(p) , \qquad (103)$$

where $H(p)$ is defined by

$$(p \cdot \partial_p - 4) H(p) = \int_q f(q) G(q + p) + v_2^2 \frac{1}{3} \gamma_{44,4} - \eta \int_q K(q) + 2\eta \, p^2 v_2 + \gamma_{44,0}(p) , \qquad (104)$$

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and the convention

$$\frac{d^2}{(dp^2)^2} H(p)\Big|_{p=0} = 0 \,. \tag{105}$$

$\gamma_{44,0}(p)$ is determined so that the rhs has no term of order p^4 at $p = 0$:

$$\gamma_{44,0}(p) = \gamma_{44,0} \cdot p^4 \,, \tag{106}$$

where

$$\gamma_{44,0} = -\frac{1}{2} \frac{d^2}{(dp^2)^2} \int_q f(q) G(q+p)\Big|_{p=0} = -\frac{1}{(4\pi)^6} \frac{1}{144} \,. \tag{107}$$

This is calculated in Appendix A. In Appendix B we derive the asymptotic behavior of $H(p)$ as

$$H(p) \stackrel{p\to\infty}{\longrightarrow} \ln p \left[\gamma_{44,0}\, p^4 + 2\eta\, v_2 p^2 + 2\gamma_{42,2} v_2^2 \right] \,, \tag{108}$$

where we have ignored constant multiples of $p^4, p^2, 1$.

To summarize, the asymptotic behavior of the operator product is given by

$$[\mathcal{O}_4(p)\mathcal{O}_4(q)]$$
$$\stackrel{p,q\to\infty}{\longrightarrow} \ln p \left(\gamma_{44,4}\mathcal{O}_4(p+q) + \gamma_{44,\mathcal{N}}\mathcal{N}(p+q) + \gamma_{44,2}(p,q)\mathcal{O}_2(p+q) + \gamma_{44,0}\, p^4\, \delta(p+q) \right) \,. \tag{109}$$

V.   NUMBER OPERATOR AND ITS MULTIPLE PRODUCTS

The number operator defined by (29) is an equation-of-motion operator, and it has the correlation function

$$\langle\!\langle \mathcal{N}(p)\, \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle = \sum_{i=1}^{n} \langle\!\langle \phi(p_1) \cdots \phi(p_i + p) \cdots \phi(p_n) \rangle\!\rangle \,. \tag{110}$$

See Appendix C for a quick review of the equation-of-motion composite operators in the ERG formalism.

Given a composite operator $\mathcal{O}$, we can create an equation-of-motion operator by

$$\mathcal{N}(p) \star \mathcal{O} \equiv -e^{-S} \int_q K(q) \frac{\delta}{\delta\phi(q)} \left([\phi(q+p)\mathcal{O}]\, e^S \right) \,, \tag{111}$$

where

$$[\phi(q+p)\mathcal{O}] = [\mathcal{O}\phi(q+p)] \equiv \phi(q+p)\mathcal{O} + h(q+p) \frac{\delta\mathcal{O}}{\delta\phi(-q-p)} \tag{112}$$

is the product of ϕ and $\mathcal{O}$ with the correlation function

$$\langle\!\langle [\mathcal{O}\phi(q)]\, \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle = \langle\!\langle \mathcal{O}\, \phi(q)\phi(p_1) \cdots \phi(p_n) \rangle\!\rangle \,. \tag{113}$$

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$\mathcal{N}(p) \star \mathcal{O}$ has the correlation function

$$\langle\!\langle \mathcal{N}(p) \star \mathcal{O} \, \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle = \sum_{i=1}^{n} \langle\!\langle \mathcal{O} \, \phi(p_1) \cdots \phi(p_i + p) \cdots \phi(p_n) \rangle\!\rangle \,, \tag{114}$$

where $\mathcal{O}$ is untouched by the number operator.

By definition the star product is commuting:

$$\mathcal{N}(p) \star \mathcal{N}(q) \star \mathcal{O} = \mathcal{N}(q) \star \mathcal{N}(p) \star \mathcal{O} \,. \tag{115}$$

This is easy to prove from the correlation functions:

$$\langle\!\langle \mathcal{N}(p) \star \mathcal{N}(q) \star \mathcal{O} \, \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle = \sum_{i=1}^{n} \langle\!\langle \mathcal{N}(q) \star \mathcal{O} \cdots \phi(p_i + p) \cdots \rangle\!\rangle$$

$$= \sum_{i=1}^{n} \langle\!\langle \mathcal{O} \cdots \phi(p_i + p + q) \cdots \rangle\!\rangle + \sum_{i \neq j} \langle\!\langle \mathcal{O} \cdots \phi(p_i + p) \cdots \phi(p_j + q) \cdots \rangle\!\rangle \,. \tag{116}$$

This is obviously symmetric under the interchange of p and q.

Given

$$\mathcal{O} = [\mathcal{O}_2(p_1) \cdots \mathcal{O}_2(p_k) \mathcal{O}_4(q_1) \cdots \mathcal{O}_4(q_l)] \,,$$

we define its product with a multiple number of $\mathcal{N}$'s as

$$[\mathcal{N}(p_1) \cdots \mathcal{N}(p_k) \, \mathcal{O}_2(p_1) \cdots \mathcal{O}_2(p_k) \mathcal{O}_4(q_1) \cdots \mathcal{O}_4(q_l)]$$

$$\equiv \mathcal{N}(p_1) \star \cdots \star \mathcal{N}(p_k) \star [\mathcal{O}_2(p_1) \cdots \mathcal{O}_2(p_k) \mathcal{O}_4(q_1) \cdots \mathcal{O}_4(q_l)] \,. \tag{117}$$

According to this definition, we find that the product with $\mathcal{N}$ does not produce any new short-distance singularity. Suppose the product $[\mathcal{O}(p)\mathcal{O}'(q)]$ satisfies the ERG equation

$$(p \cdot \partial_p + q \cdot \partial_q + y_i + y_j - \mathcal{D}) \, [\mathcal{O}_i(p)\mathcal{O}_j(q)] = \gamma_{ij,k}(p, q)\mathcal{O}_k(p + q) \,. \tag{118}$$

The rhs is due to the short-distance singularity. Taking the star product with $\mathcal{N}(r)$, we obtain

$$(p \cdot \partial_p + q \cdot \partial_q + r \cdot \partial_r + y_i + y_j - \mathcal{D}) \, [\mathcal{N}(r)\mathcal{O}_i(p)\mathcal{O}_j(q)] = \gamma_{ij,k}(p, q) \, [\mathcal{N}(r)\mathcal{O}_k(p + q)] \,. \tag{119}$$

$\mathcal{N}$ itself has no short-distance singularity with any composite operator. Thus, the only mixing coefficient involving $\mathcal{N}$ is of the type

$$\gamma_{4\cdots4,\mathcal{N}} \,. \tag{120}$$

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Let us consider two examples. The first example is

$$[\mathcal{N}(p)\mathcal{O}_2(q)] = \mathcal{N}(p) \star \mathcal{O}_2(q)$$

$$= -e^{-S} \int_r K(r) \frac{\delta}{\delta\phi(r)} \left[\left(\phi(r+p)\mathcal{O}_2(q) + h(r+p)\frac{\delta\mathcal{O}_2(q)}{\delta\phi(-r-p)} \right) e^S \right]$$

$$= \mathcal{N}(p)\mathcal{O}_2(q) + \mathcal{P}_{\mathcal{N}2}(p,q) , \tag{121}$$

where

$$\mathcal{P}_{\mathcal{N}2}(p,q) = \int_r \frac{\delta\mathcal{N}(p)}{\delta\phi(r)} h(r) \frac{\delta\mathcal{O}_2(q)}{\delta\phi(-r)}$$

$$- 2\mathcal{O}_2(p+q) + \left(2v_2 - \int_r K(r)h(r+p) \right) \delta(p+q) . \tag{122}$$

The second example is

$$[\mathcal{N}(p)\mathcal{O}_4(q)] = \mathcal{N}(p) \star \mathcal{O}_4(q)$$

$$= -e^{-S} \int_r K(r) \frac{\delta}{\delta\phi(r)} \left[\left(\phi(r+p)\mathcal{O}_4(q) + h(r+p)\frac{\delta\mathcal{O}_4(q)}{\delta\phi(-r-p)} \right) e^S \right]$$

$$= \mathcal{N}(p)\mathcal{O}_4(q) + \mathcal{P}_{\mathcal{N}4}(p,q) , \tag{123}$$

where

$$\mathcal{P}_{\mathcal{N}4}(p,q) = \int_r \frac{\delta\mathcal{N}(p)}{\delta\phi(r)} h(r) \frac{\delta\mathcal{O}_4(q)}{\delta\phi(-r)}$$

$$- 4\mathcal{O}_4(p+q) + \left(4v_2 - \int_r K(r)h(r+p) \right) \mathcal{O}_2(p+q)$$

$$- v_2 \frac{1}{2} \int_{p_1,p_2} \phi(p_1)\phi(p_2)\delta(p_1+p_2-p-q) \left(1 - K(p_1-p) + 1 - K(p_2-p) \right)$$

$$- 2v_2^2 \delta(p+q) . \tag{124}$$

VI.   GENERATING FUNCTIONAL

So far we have shown how to construct the multiple products of $\mathcal{O}_2$ and $\mathcal{O}_4$ by solving the respective ERG differential equations. Their products with the number operator $\mathcal{N}$ are given by star products as discussed in the previous section.

To better understand the relation among the multiple products, we couple momentum dependent sources to $\mathcal{O}_2, \mathcal{O}_4, \mathcal{N}$ and introduce a generating functional $W[J_2, J_4, J_{\mathcal{N}}]$ by

$$W[J_2, J_4, J_{\mathcal{N}}] \equiv \int_p \left(J_2(-p)\mathcal{O}_2(p) + J_4(-p)\mathcal{O}_4(p) + J_{\mathcal{N}}(-p)\mathcal{N}(p) \right)$$

$$
+ \sum_{k+l+m \geq 2} \frac{1}{k!\, l!\, m!} \int_{p_1,\cdots,p_k;q_1,\cdots,q_l;r_1,\cdots,r_m} J_2(-p_1) \cdots J_2(-p_k)
$$
$$
\times \, J_4(-q_1) \cdots J_4(-q_l) J_{\mathcal{N}}(-r_1) \cdots J_{\mathcal{N}}(-r_m)
$$
$$
\times \, \mathcal{P}_{2\cdots4\cdots\mathcal{N}\cdots}(p_1,\cdots,p_k,q_1,\cdots,q_l,r_1\cdots,r_m) . \tag{125}
$$

Its exponential $e^{W[J_2,J_4,J_{\mathcal{N}}]}$ is the sum of composite operators:

$$
e^{W[J_2,J_4,J_{\mathcal{N}}]} = 1 + \sum_{k+l+m \geq 1} \frac{1}{k!\, l!\, m!} \int_{p_1,\cdots,p_k;q_1,\cdots,q_l;r_1,\cdots,r_m} J_2(-p_1) \cdots J_2(-p_k)
$$
$$
\times \, J_4(-q_1) \cdots J_4(-q_l) J_{\mathcal{N}}(-r_1) \cdots J_{\mathcal{N}}(-r_m)
$$
$$
\times \, [\mathcal{O}_2(p_1) \cdots \mathcal{O}_2(p_k) \mathcal{O}_4(q_1) \cdots \mathcal{O}_4(q_l) \mathcal{N}(r_1) \cdots \mathcal{N}(r_m)] . \tag{126}
$$

We call W the generating functional because the multiple products are obtained as its differentials:

$$
[\mathcal{O}_2(p_1) \cdots \mathcal{O}_2(p_k) \mathcal{O}_4(q_1) \cdots \mathcal{O}_4(q_l) \mathcal{N}(r_1) \cdots \mathcal{N}(r_m)]
$$
$$
= \frac{\delta^{k+l+m}}{\delta J_2(-p_1) \cdots \delta J_2(-p_k) \delta J_4(-q_1) \cdots \delta J_4(-q_l) J_{\mathcal{N}}(-r_1) \cdots J_{\mathcal{N}}(-r_m)} e^{W[J_2,J_4,J_{\mathcal{N}}]} \Big|_{J_2 = J_4 = J_{\mathcal{N}} = 0} . \tag{127}
$$

By giving the scale dimensions $-2, -4, -4$ to $J_2(-p), J_4(-p), J_{\mathcal{N}}(-p)$, respectively, we can make $e^{W[J_2,J_4,J_{\mathcal{N}}]}$ a composite operator of scale dimension 0. We can then write the ERG differential equation compactly as

$$
\int_p \Big[\{(-p \cdot \partial_p - 2) J_2(-p) - B_2(-p)\} \frac{\delta}{\delta J_2(-p)}
$$
$$
+ \{(-p \cdot \partial_p - 4) J_4(-p) - B_4(-p)\} \frac{\delta}{\delta J_4(-p)} \tag{128}
$$
$$
+ \{(-p \cdot \partial_p - 4) J_{\mathcal{N}}(-p) - B_{\mathcal{N}}(-p)\} \frac{\delta}{\delta J_{\mathcal{N}}(-p)} - \mathcal{D} \Big] e^{W[J_2,J_4,J_{\mathcal{N}}]} = B_0[J_2, J_4]\, e^{W[J_2,J_4,J_{\mathcal{N}}]} ,
$$

where

$$
B_0[J_2, J_4] \equiv \sum_{k=0}^{\infty} \frac{1}{k!} \gamma_{4\cdots422,0} \int_{p_1,\cdots,p_k} J_4(-p_1) \cdots J_4(-p_k)
$$
$$
\times \frac{1}{2} \int_{q_1,q_2} J_2(-q_1) J_2(-q_2)\, \delta\, (p_1 + \cdots + p_k + q_1 + q_2)
$$
$$
+ \sum_{k=1}^{\infty} \frac{1}{k!} \int_{p_1,\cdots,p_k} \gamma_{4\cdots42,0}(p_1,\cdots,p_k,q) J_4(-p_1) \cdots J_4(-p_k) J_2(-q) \delta(p_1 + \cdots + p_k + q)
$$

$$+ \sum_{k=2}^{\infty} \frac{1}{k!} \int_{p_1,\cdots,p_k} \gamma_{4\cdots4,0}(p_1,\cdots,p_k)\, J_4(-p_1)\cdots J_4(-p_k)\, \delta(p_1+\cdots+p_k)\,, \quad (129)$$

$$B_2(-p) \equiv \sum_{k=1}^{\infty} \frac{1}{k!}\gamma_{4\cdots42,2} \int_{p_1,\cdots,p_k,q} J_4(-p_1)\cdots J_4(-p_k) J_2(-q)\, \delta(p_1+\cdots+p_k+q-p)$$

$$+ \sum_{k=2}^{\infty} \frac{1}{k!} \int_{p_1,\cdots,p_k} \gamma_{4\cdots4,2}(p_1,\cdots,p_k) J_4(-p_1)\cdots J_4(-p_k)\, \delta(p_1+\cdots p_k-p)\,, \quad (130)$$

$$B_4(-p) \equiv \sum_{k=2}^{\infty} \frac{1}{k!}\gamma_{4\cdots4,4} \int_{p_1,\cdots,p_k} J_4(-p_1)\cdots J_4(-p_k)\, \delta(p_1+\cdots+p_k-p)\,, \quad (131)$$

$$B_{\mathcal{N}}(-p) \equiv \sum_{k=2}^{\infty} \frac{1}{k!}\gamma_{4\cdots4,\mathcal{N}} \int_{p_1,\cdots,p_k} J_4(-p_1)\cdots J_4(-p_k)\, \delta(p_1+\cdots+p_k-p)\,. \quad (132)$$

Our notation for the mixing coefficients may be clear. For example, the product of k number of $\mathcal{O}_4$ satisfies the ERG equation

$$\left(\sum_{i=1}^{k} p_i \cdot \partial_{p_i} - \mathcal{D}\right)[\mathcal{O}_4(p_1)\cdots\mathcal{O}_4(p_k)]$$

$$= \gamma_{4\cdots4,4}\mathcal{O}_4(p_1+\cdots+p_k) + \gamma_{4\cdots4,\mathcal{N}}\mathcal{N}(p_1+\cdots+p_k)$$

$$+ \gamma_{4\cdots4,2}(p_1,\cdots,p_k)\mathcal{O}_2(p_1+\cdots+p_k) + \gamma_{4\cdots4,0}(p_1,\cdots,p_k)\,\delta(p_1+\cdots+p_k) + \cdots\,, \quad (133)$$

where the mixing of less than k number of $\mathcal{O}_4$'s is suppressed. As we have discussed in Sec. III, the mixing coefficients determine the short-distance singularity of the operator product. When all the momenta become large with the sum $\sum_1^k p_i$ fixed, the short-distance singularity is given by

$$[\mathcal{O}_4(p_1)\cdots\mathcal{O}_4(p_k)]$$

$$\overset{p_1,\cdots,p_k\to\infty}{\longrightarrow} \frac{k!}{2^{k-1}}(\ln p)^{k-1}\gamma_{44,4}^{k-2}\left[\gamma_{44,4}\mathcal{O}_4\left(\sum p_i\right) + \gamma_{44,\mathcal{N}}\mathcal{N}\left(\sum p_i\right) + \right.$$

$$+ \frac{1}{k}\sum_{n=1}^{k}\gamma_{44,2}\left(\sum p_i - p_n, p_n\right)\mathcal{O}_2\left(\sum p_i\right)$$

$$\left. + \frac{1}{k}\sum_{n=1}^{k}\gamma_{44,0}\left(\sum p_i - p_n, p_n\right)\delta\left(\sum p_i\right)\right]$$

$$+ \cdots$$

$$+ \ln p\left[\gamma_{4\cdots4,4}\mathcal{O}_4\left(\sum p_i\right) + \gamma_{4\cdots4,\mathcal{N}}\mathcal{N}\left(\sum p_i\right)\right.$$

$$\left. + \gamma_{4\cdots4,2}(p_1,\cdots,p_k)\mathcal{O}_2\left(\sum p_i\right) + \gamma_{4\cdots4,0}(p_1,\cdots,p_k)\,\delta\left(\sum p_i\right) + \cdots\right]\,, \quad (134)$$

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where p is any of the momenta p_i. The dotted parts, proportional to the powers of $\ln p$ less than $k - 1$, are determined by the lower order mixing coefficients.

To convince ourselves that (128) is correct, it may suffice to check three terms. The term first order in J_2 gives

$$\int_p [(-p \cdot \partial_p - 2) J_2(-p) - J_2(-p)\mathcal{D}] \, \mathcal{O}_2(p) = \int_p J_2(-p) \, (p \cdot \partial_p + 2 - \mathcal{D}) \, \mathcal{O}_2(p) = 0 \,, \quad (135)$$

which is correct. Similarly, the term first order in J_4 gives

$$\int_p [(-p \cdot \partial_p - 4) J_4(-p) - J_4(-p)\mathcal{D}] \, \mathcal{O}_4(p) = \int_p J_4(-p) \, (p \cdot \partial_p - \mathcal{D}) \, \mathcal{O}_4(p) = 0 \,, \quad (136)$$

which is correct again. The term proportional to $J_4 J_2$ gives

$$\int_{p,q} \left[(-p \cdot \partial_p - 2) \, J_2(-p) \cdot J_4(-q) \, [\mathcal{O}_2(p)\mathcal{O}_4(q)] \right.$$

$$\left. + J_2(-p) \, (-q \cdot \partial_q - 4) \, J_4(-q) \cdot [\mathcal{O}_2(p)\mathcal{O}_4(q)] - \gamma_{42,2} \, J_2(-p) J_4(-q) \, \mathcal{O}_2(p+q) \right]$$

$$= \int_{p,q} \gamma_{42,0}(q,p) J_2(-p) J_4(-q) \, \delta(p+q) \,. \quad (137)$$

This gives the correct equation

$$(p \cdot \partial_p + q \cdot \partial_q + 2 - \mathcal{D}) \, [\mathcal{O}_2(p)\mathcal{O}_4(q)] = \gamma_{42,2}\mathcal{O}_2(p+q) + \gamma_{42,0}(q,p) \, \delta(p+q) \,. \quad (138)$$

We now recall that the multiple product of the number operator is given by

$$[\mathcal{O}_2(p_1) \cdots \mathcal{O}_4(q_1) \cdots \mathcal{N}(r_1) \cdots \mathcal{N}(r_m)]$$

$$= \mathcal{N}(r_1) \star \cdots \star \mathcal{N}(r_m) \star [\mathcal{O}_2(p_1) \cdots \mathcal{O}_4(q_1) \cdots] \,. \quad (139)$$

This is equivalent to

$$\frac{\delta}{\delta J_{\mathcal{N}}(-p)} e^{W[J_2,J_4,J_{\mathcal{N}}]} = \mathcal{N}(p) \star e^{W[J_2,J_4,J_{\mathcal{N}}]}$$

$$= -e^{-S} \int_q K(q) \frac{\delta}{\delta \phi(q)} \left(e^S \left(\phi(q+p) + h(q+p) \frac{\delta}{\delta \phi(-q-p)} \right) e^{W[J_2,J_4,J_{\mathcal{N}}]} \right) \,. \quad (140)$$

To show this, it suffices to derive (139) from (140). Using (140) m times, we obtain

$$\frac{\delta}{\delta J_{\mathcal{N}}(-p_m)} \cdots \frac{\delta}{\delta J_{\mathcal{N}}(-p_1)} e^{W[J_2,J_4,J_{\mathcal{N}}]} = \mathcal{N}(p_1) \star \cdots \mathcal{N}(p_m) \star e^{W[J_2,J_4,J_{\mathcal{N}}]} \,. \quad (141)$$

Differentiating this further with respect to J_2 and J_4 a number of times and setting the sources to zero, we obtain (139). We have thus verified (140).

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A. Constant parameters

Let us consider a special case where the sources are constants:

$$J_2(-p) = -m^2 \, \delta(p) \,, \tag{142a}$$

$$J_4(-p) = -\lambda \, \delta(p) \,, \tag{142b}$$

$$J_{\mathcal{N}}(-p) = z \, \delta(p) \,. \tag{142c}$$

We then write $W[J_2, J_4, J_{\mathcal{N}}]$ as $W(m^2, \lambda, z)$. $S + W(m^2, \lambda, z)$ is the Wilson action of the perturbative ϕ^4 theory. The parameter z changes the normalization of the field.

(128) gives the ERG equation

$$\left[(2 + \beta_m(\lambda)) \, m^2 \partial_{m^2} + \beta(\lambda) \partial_\lambda - \gamma(\lambda) \partial_z - \mathcal{D} \right] e^{W(m^2, \lambda, z)} = B_0(m^2, \lambda) \, e^{W(m^2, \lambda, z)} \,, \tag{143}$$

where the beta functions and the anomalous dimension are given by

$$\beta_m(\lambda) = -\sum_{k=1}^{\infty} \frac{1}{k!} \gamma_{4\cdots 42, 2}(-\lambda)^k = -\frac{\lambda}{(4\pi)^2} + \frac{\lambda^2}{(4\pi)^4} \frac{5}{6} + \cdots \,, \tag{144}$$

$$\beta(\lambda) = \sum_{k=2}^{\infty} \frac{1}{k!} \gamma_{4\cdots 4, 4}(-\lambda)^k = -\frac{3\lambda^2}{(4\pi)^2} + \frac{\lambda^3}{(4\pi)^3} \frac{17}{3} + \cdots \,, \tag{145}$$

$$\gamma(\lambda) = \sum_{k=2}^{\infty} \frac{1}{k!} \gamma_{4\cdots 4, \mathcal{N}}(-\lambda)^k = \frac{\lambda^2}{(4\pi)^2 \, 12} + \cdots \,. \tag{146}$$

The correlation functions are given by

$$\langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_{m^2, \lambda, z} \equiv \left\langle\!\!\left\langle e^{W(m^2, \lambda, z)} \phi(p_1) \cdots \phi(p_n) \right\rangle\!\!\right\rangle \,. \tag{147}$$

Differentiating this with respect to z, we obtain

$$\partial_z \langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_{m^2, \lambda, z} = \left\langle\!\!\left\langle \mathcal{N}(0) \star e^{W(m^2, \lambda, z)} \phi(p_1) \cdots \phi(p_n) \right\rangle\!\!\right\rangle$$

$$= n \, \langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_{m^2, \lambda, z} \,. \tag{148}$$

Hence, we obtain the z-dependence as

$$\langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_{m^2, \lambda, z} = e^{nz} \, \langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle_{m^2, \lambda, z=0} \,. \tag{149}$$

B. Source for the identity operator

We find it handy to introduce a source $J_0(p)$ that couples to the identity operator $\delta(p)$:

$$W[J_0, J_2, J_4, J_{\mathcal{N}}] = \int_p J_0(-p)\delta(p) + W[J_2, J_4, J_{\mathcal{N}}] = J_0(0) + W[J_2.J_4, J_{\mathcal{N}}] \,. \tag{150}$$

We can then rewrite the ERG equation (128) as

$$\int_p \left[\{-p \cdot \partial_p J_0(-p) - B_0[J_2, J_4]\} \frac{\delta}{\delta J_0(-p)} \right.$$

$$+ \{(-p \cdot \partial_p - 2) J_2(-p) - B_2(-p)\} \frac{\delta}{\delta J_2(-p)}$$

$$+ \{(-p \cdot \partial_p - 4) J_4(-p) - B_4(-p)\} \frac{\delta}{\delta J_4(-p)}$$

$$\left. + \{(-p \cdot \partial_p - 4) J_\mathcal{N}(-p) - B_\mathcal{N}(-p)\} \frac{\delta}{\delta J_\mathcal{N}(-p)} - \mathcal{D} \right] e^{W[J_0, J_2, J_4, J_\mathcal{N}]} = 0 . \tag{151}$$

C. Change of parameters

In Sec. IV we have constructed the products of two composite operators by solving the respective ERG differential equations. Some of the equations do not have unique solutions, and we have made arbitrary choices. For example, we have found $[\mathcal{O}_2(p)\mathcal{O}_2(q)]$ is ambiguous by a constant multiple of $\delta(p+q)$.

This ambiguity is related to our freedom of changing parameters (or sources) as long as we preserve scale dimensions and respect locality. In general we can introduce the following change of parameters:

$$J_0'(-p) = J_0(-p)$$

$$+ \sum_{k=1}^{\infty} \frac{1}{k!} C_{0,k} \frac{1}{2} \int_{p_1, \cdots, p_k, q, l} J_2(-q) J_2(-l) J_4(-p_1) \cdots J_4(-p_k) \, \delta\left(\sum_{i=1}^{k} p_i + q + l - p \right)$$

$$+ \sum_{k=0}^{\infty} \frac{1}{k!} \int_{p_1, \cdots, p_k, q} J_2(-q) J_4(-p_1) \cdots J_4(-p_k) \, C_{0,k}'(q; p_1, \cdots, p_k) \, \delta\left(\sum_{i=1}^{k} + q - p \right)$$

$$+ \sum_{k=1}^{\infty} \frac{1}{k!} \int_{p_1, \cdots, p_k} J_4(-p_1) \cdots J_4(-p_k) \, C_{0,k}''(p_1, \cdots, p_k) \, \delta\left(\sum_{i=1}^{k} p_i - p \right) , \tag{152}$$

where $C_{0,k}', C_{0,k}''$ are quadratic, quartic in momenta, respectively.

$$J_2'(-p) = J_2(-p) + \sum_{k=1}^{\infty} \frac{1}{k!} C_{2,k} \int_{p_1, \cdots, p_k, q} J_2(-q) J_4(-p_1) \cdots J_4(-p_k) \, \delta\left(q + \sum_{i=1}^{k} p_i - p \right)$$

$$+ \sum_{k-1}^{\infty} \frac{1}{k!} \int_{p_1, \cdots, p_k} C_{2,k}'(p_1, \cdots, p_k) \, \delta\left(q + \sum_{i=1}^{k} p_i - p \right) , \tag{153}$$

where $C'_{2,k}$ are quadratic in momenta, and

$$J'_4(-p) = J_4(-p) + \sum_{k=2}^{\infty} \frac{1}{k!} C_{4,k} \int_{p_1,\cdots,p_k} J_4(-p_1) \cdots J_4(-p_k)\, \delta\left(q + \sum_{i=2}^{k} p_i - p\right). \quad (154)$$

The above change of parameters should not change the generating functional in the sense that

$$e^{W[J_2,J_4,J_\mathcal{N}]} = e^{W'[J'_2,J'_4,J_\mathcal{N}]}. \quad (155)$$

The products of composite operators, defined as differentials of the generating functional, change accordingly. For example,

$$\begin{aligned}
[\mathcal{O}_2(p)\mathcal{O}_2(q)] &\equiv \frac{\delta^2}{\delta J_2(-p)\delta J_2(-q)} e^{W[J_2,J_4,J_\mathcal{N}]}\Big|_{J_2=J_4=J_\mathcal{N}=0} \\
&= \frac{\delta^2}{\delta J'_2(-p)\delta J'_2(-q)} e^{W'[J'_2,J'_4,J_\mathcal{N}]}\Big|_{J'_2=J'_4=J_\mathcal{N}=0} + \frac{\delta^2 J'_0(0)}{\delta J_2(-p)\delta J_2(-q)}\Big|_{J_2=J_4=J_\mathcal{N}=0} \\
&= [\mathcal{O}_2(p)\mathcal{O}_2(q)]' + C_{0,0}\delta(p+q). \quad (156)
\end{aligned}$$

As we saw in some details in Sec. III, redefined multi-operator products have different mixing coefficients. This is easy to understand since the mixing coefficients define the beta functions $B_0, \cdots, B_4$ of the parameters, and the redefined parameters obtain different beta functions B'_0, B'_2, and B'_4. These beta functions are easier to think of in coordinate space, and we give a general form of the beta functions and consider their simplest form in Appendix E.

VII. CONCLUSIONS

In this paper we have used the exact renormalization group (ERG) formalism to construct the multiple products of composite operators at the Gaussian fixed-point in $D = 4$ dimensions. We have considered only three scalar composite operators $\mathcal{O}_2, \mathcal{O}_4, \mathcal{N}$ whose scale dimensions in momentum space are $-2, 0, 0$, respectively. Their products do not generate any new operators, and the algebra is closed in this sense. We have shown, in Sec. III, that the unintegrable short-distance singularities of their multiple products determine the mixing coefficients under scaling. The mixing coefficients in turn determine the scaling properties of the sources coupled to the three operators as given in (128) to (132). Though these results are expected, we would like to emphasize the ease and clarity with which they are drawn naturally out of the ERG formalism.

On the technical side, our choice of the multiple products of the number operator (117) greatly simplifies the dependence of the theory on the source $J_\mathcal{N}$. The dependence is just as what we expect naively from the change of field normalization, and the products with the number operator do not generate any short-distance singularity.

It would be nice to apply the ERG formalism to study the operator algebra at a more nontrivial fixed-point such as the Wilson-Fisher fixed-point in $2 < D < 4$ dimensions.

Appendix A: Calculations of the mixing coefficients

For the calculations of the mixing coefficients, we need to compute the following integrals:

$$I_1 \equiv \int_q f(q)h(q) \,, \tag{A1}$$

$$I_2 \equiv \frac{d}{dp^2} \int_q f(q)F(q+p)\Big|_{p=0} \,, \tag{A2}$$

$$I_3 \equiv \frac{1}{2}\frac{d^2}{(dp^2)^2} \int_q f(q)G(q+p)\Big|_{p=0} \,, \tag{A3}$$

$$I_4 \equiv \int_p f(p) \int_q h(q)^2 h(p+q) + 2\int_p f(p)h(p)F(p) \,. \tag{A4}$$

All these integrals are universal, i.e., their values are independent of the choice of a cutoff function $K(p)$ or equivalently $h(p) \equiv \frac{1-K(p)}{p^2}$. Let us show the universality of the first two integrals. Under an infinitesimal change δh of h, we find

$$\begin{aligned}
\delta I_1 &= \int_q (\delta f(q) \cdot h(q) + f(q) \cdot \delta h(q)) \\
&= \int_q ((q \cdot \partial_q + 2)\,\delta h(q) \cdot h(q) + f(q)\delta h(q)) \\
&= \int_q \delta h(q)\,(-(q \cdot \partial_q + 2)\,h(q) + f(q)) = 0 \,.
\end{aligned} \tag{A5}$$

For I_2, we need

$$\begin{aligned}
\delta F(p) &= \delta\frac{1}{2}\int_q h(q)\,(h(q+p) - h(q)) \\
&= \int_q \delta h(q)\,(h(q+p)h(q)) \,.
\end{aligned} \tag{A6}$$

Hence, we obtain

$$\delta \int_q f(q)F(q+p) = \int_q (\delta f(q) \cdot F(q+p) + f(q)\delta F(q+p))$$

$$= \int_q \left[\delta h(q)(-q \cdot \partial_q - 2)F(q+p) + f(q) \int_r \delta h(r) \left(h(r+q+p) - h(r) \right) \right)$$

$$= \int_q \delta h(q) \left[(-q \cdot \partial_q - p \cdot \partial_p)F(q+p) + (p \cdot \partial_p - 2)F(q+p) \right.$$

$$\left. + \int_r f(r) \left(h(r+q+p) - h(q) \right) \right]$$

$$= (p \cdot \partial_p - 2) \int_q \delta h(q)F(q+p) + \int_q \delta h(q) \int_r f(r) \left(h(r) - h(q) \right), \qquad \text{(A7)}$$

where the first integral has no p^2 term, and the second integral is a constant. Hence,

$$\delta I_2 = 0. \qquad \text{(A8)}$$

Since the integrals are independent of the choice of cutoff function, we can make any appropriate choice such as

$$K(p) = e^{-p^2}, \qquad \text{(A9a)}$$

$$h(p) = \int_0^1 ds \, e^{-sp^2} = \frac{1 - e^{-p^2}}{p^2}, \qquad \text{(A9b)}$$

$$f(p) = (p \cdot \partial_p + 2) \, h(p) = 2e^{-p^2}. \qquad \text{(A9c)}$$

The advantage of this choice is that all the momentum integrals become Gaussian. The results are as follows:

$$I_1 = \frac{1}{(4\pi)^2} \qquad \text{(A10)}$$

$$I_2 = -\frac{1}{(4\pi)^4} \frac{1}{6} \qquad \text{(A11)}$$

$$I_3 = \frac{1}{(4\pi)^6} \frac{1}{144} \qquad \text{(A12)}$$

$$I_4 = \frac{1}{(4\pi)^4}. \qquad \text{(A13)}$$

We only calculate I_2 here. We first calculate

$$F(p) = \frac{1}{2} \int_q h(q) \left(h(q+p) - h(q) \right)$$

$$= \frac{1}{2} \int_q \int_0^1 ds \, e^{-sq^2} \int_0^1 dt \left(e^{-t(q+p)^2} - e^{-tq^2} \right)$$

$$= \frac{1}{2} \frac{1}{(4\pi)^2} \int_0^1 ds \int_0^1 dt \frac{1}{(s+t)^2} \left(e^{-\frac{st}{s+t}p^2} - 1 \right). \qquad \text{(A14)}$$

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This gives

$$\int_q f(q)F(q+p) = \frac{1}{(4\pi)^4} \int_0^1 ds \int_0^1 dt \, \frac{1}{(s+t+st)^2} \left(e^{-\frac{st}{s+t+st}p^2} - 1 \right). \tag{A15}$$

Hence, we obtain

$$\frac{d}{dp^2} \int_q f(q)F(q+p) \Big|_{p=0} = \frac{1}{(4\pi)^4} \int_0^1 ds \int_0^1 dt \, \frac{-st}{(s+t+st)^3}$$
$$= -\frac{1}{(4\pi)^4} \frac{1}{6}. \tag{A16}$$

Appendix B: Asymptotic behavior of $G(p), H(p)$

$G(p)$ satisfies the differential equation (94)

$$(p \cdot \partial_p - 2)\, G(p) = \int_q f(q)F(q+p) + 2\gamma_{42,2}v_2 + \eta\, p^2. \tag{B1}$$

Hence, using (62) satisfied by F, we obtain

$$(p \cdot \partial_p - 2)\, (G(p) - 2v_2 F(p)) = \int_q f(q)\, (F(q+p) - F(p)) + \eta\, p^2. \tag{B2}$$

This gives

$$(p \cdot \partial_p - 2)\, (G(p) - 2v_2 F(p)) \xrightarrow{p\to\infty} \eta\, p^2. \tag{B3}$$

Hence, we obtain the asymptotic behavior

$$G(p) \xrightarrow{p\to\infty} \eta\, p^2 \ln p + 2\gamma_{42,2}v_2 \ln p, \tag{B4}$$

where we have dropped constant multiples of $p^2, 1$.

$H(p)$ satisfies the differential equation (104):

$$(p \cdot \partial_p - 4)\, H(p) = \int_q f(q)G(q+p) + v_2^2 \frac{1}{3}\gamma_{44,4} - \eta \int_q K(q) + 2\eta p^2 v_2 + \gamma_{44,0} \cdot p^4. \tag{B5}$$

This gives

$$(p \cdot \partial_p - 4)\, \big(H(p) - 2v_2 G(p) + 2v_2^2 F(p)\big)$$
$$= \int_q f(q)\, (G(q+p) - G(p)) - 2v_2 \int_q f(q)\, (F(q+p) - F(p)) + 2v_2^2 \int_q f(q)h(q+p)$$
$$+ \gamma_{44,0}\, p^4 - \eta \int_q K(q). \tag{B6}$$

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Hence, we obtain

$$(p \cdot \partial_p - 4) \left(H(p) - 2v_2 G(p) + 2v_2^2 F(p) \right) \overset{p \to \infty}{\longrightarrow} \gamma_{44,0}\, p^4 - \eta \int_q K(q)\,. \tag{B7}$$

This gives the asymptotic behavior

$$H(p) \overset{p \to \infty}{\longrightarrow} \ln p \cdot \left(\gamma_{44,0}\, p^4 + 2\eta\, p^2 v_2 + 2\gamma_{42,2} v_2^2 \right) \,, \tag{B8}$$

where we have ignored constant multiples of $p^4, p^2, 1$.

Appendix C: Equation-of-motion composite operators

For a composite operator $\mathcal{O}(p)$ of momentum p, we define an equation-of-motion composite operator by

$$\mathcal{E}_{\mathcal{O}}(p) \equiv -e^{-S} \int_q K(q) \frac{\delta}{\delta\phi(q)} \left(\mathcal{O}(q+p)\, e^S \right) \,. \tag{C1}$$

By definition (19), we obtain, using functional integration by parts,

$$
\begin{aligned}
&\left\langle\!\!\left\langle \mathcal{E}_{\mathcal{O}}(p)\, \phi(p_1) \cdots \phi(p_n) \right\rangle\!\!\right\rangle \\
&= \prod_{i=1}^{n} \frac{1}{K(p_i)} \left\langle \mathcal{E}_{\mathcal{O}}(p) \exp\left(-\int_r \frac{K(r)\,(1-K(r))}{r^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(r)\delta\phi(-r)} \right) \phi(p_1) \cdots \phi(p_n) \right\rangle_S \\
&= \prod_{i=1}^{n} \frac{1}{K(p_i)} \int_q K(q) \left\langle \mathcal{O}(q+p) \exp\left(-\int_r \frac{K(r)\,(1-K(r))}{r^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(r)\delta\phi(-r)} \right) \right. \\
&\qquad\qquad\qquad \left. \times \frac{\delta}{\delta\phi(q)} \left(\phi(p_1) \cdots \phi(p_n) \right) \right\rangle_S \\
&= \prod_{i=1}^{n} \frac{1}{K(p_i)} \sum_{j=1}^{n} K(p_j) \left\langle \mathcal{O}(p_j+p) \exp\left(-\int_r \frac{K(r)\,(1-K(r))}{r^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(r)\delta\phi(-r)} \right) \right. \\
&\qquad\qquad\qquad \left. \times \phi(p_1) \cdots \widehat{\phi(p_j)} \cdots \phi(p_n) \right\rangle_S \\
&= \sum_{j=1}^{n} \left\langle\!\!\left\langle \mathcal{O}(p_j+p)\, \phi(p_1) \cdots \widehat{\phi(p_j)} \cdots \phi(p_n) \right\rangle\!\!\right\rangle \,, \tag{C2}
\end{aligned}
$$

where $\phi(p_j)$ under the hat is omitted.

Since the elementary field ϕ is also a composite operator for the free theory[4], we can choose $\mathcal{O}(p) = \phi(p)$ to obtain

$$\mathcal{E}_\phi(p) = \mathcal{N}(p)\,, \tag{C3}$$

---

[4] This is not the case for interacting theories.

which is Eq. (29), and

$$\langle\!\langle \mathcal{N}(p)\,\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle = \sum_{i=1}^{n}\langle\!\langle \phi(p_1)\cdots\phi(p_i+p)\cdots\phi(p_n)\rangle\!\rangle\,, \tag{C4}$$

which is Eq. (110).

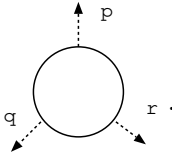**Appendix D: Products of three composite operators**

In this appendix we sketch the calculations of the products of three composite operators.

1. $[\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2]$

The counterterm $\mathcal{P}_{222}(p,q,r)$ satisfies the ERG equation with no mixing:

$$(p\cdot\partial_p + q\cdot\partial_q + r\cdot\partial_r + 6 - \mathcal{D})\,\mathcal{P}_{222}(p,q,r)$$
$$= \int_s f(s)\left(\frac{\delta\mathcal{P}_{22}(p,q)}{\delta\phi(s)}\frac{\delta\mathcal{O}_2(r)}{\delta\phi(-s)} + \frac{\delta\mathcal{P}_{22}(q,r)}{\delta\phi(s)}\frac{\delta\mathcal{O}_2(p)}{\delta\phi(-s)} + \frac{\delta\mathcal{P}_{22}(r,p)}{\delta\phi(s)}\frac{\delta\mathcal{O}_2(q)}{\delta\phi(-s)}\right). \tag{D1}$$

The solution is given by

$$\mathcal{P}_{222}(p,q,r) = \int_s h(s)h(s+q)h(s+q+r) = \quad \tag{D2}$$

2. $[\mathcal{O}_4\mathcal{O}_2\mathcal{O}_2]$

The ERG equation is given by

$$(p\cdot\partial_p + q\cdot\partial_q + r\cdot\partial_r + 4 - \mathcal{D})\,\mathcal{P}_{422}(p,q,r)$$
$$= \int_s f(s)\left(\frac{\delta\mathcal{P}_{42}(p,q)}{\delta\phi(s)}\frac{\delta\mathcal{O}_2(r)}{\delta\phi(-s)} + \frac{\delta\mathcal{P}_{42}(p,r)}{\delta\phi(s)}\frac{\delta\mathcal{O}_2(q)}{\delta\phi(-s)} + \frac{\delta\mathcal{P}_{22}(q,r)}{\delta\phi(s)}\frac{\delta\mathcal{O}_4(p)}{\delta\phi(-s)}\right)$$
$$+ \gamma_{42,2}\left(\mathcal{P}_{22}(p+q,r) + \mathcal{P}_{22}(p+r,q)\right) + \gamma_{422,0}\,\delta(p+q+r)\,. \tag{D3}$$

Let

$$\mathcal{P}_{422}(p,q,r) = \sum_{n=0,1,2}\frac{1}{(2n)!}\int_{p_1,\cdots,p_{2n}}\prod_{i=1}^{2n}\phi(p_i)\,\delta\left(\sum_i p_i - p - q - r\right)c_{422,2n}(p,q,r;p_1,\cdots,p_{2n})\,. \tag{D4}$$

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We obtain

$$c_{422,4}(p,q,r;p_1,p_2,p_3,p_4) = \qquad + \qquad + \qquad , \qquad (D5)$$

$$c_{422,2}(p,q,r;p_1,p_2) = \qquad + \qquad + \qquad (D6)$$

and

$$c_{422,0}(p,q,r) = v_2 \int_s h(s)h(s+q)h(s+q+p) + F(q)F(r) \qquad (D7)$$

$$= \qquad + \qquad (D8)$$

with

$$\gamma_{422,0} = 0 \,. \qquad (D9)$$

Note that the final result depends on our choice of $\mathcal{P}_{22}$; if we changed $\mathcal{P}_{22}(p,q)$ by $a\,\delta(p+q)$, where a is a constant, we would obtain $\gamma_{422,0} = -2\gamma_{42,2}\,a$.

3. $[\mathcal{O}_4\mathcal{O}_4\mathcal{O}_2]$

The ERG equation is given by

$$
\begin{aligned}
&(p\cdot\partial_p + q\cdot\partial_q + r\cdot\partial_r + 2 - \mathcal{D})\,\mathcal{P}_{442}(p,q,r) \\
&= \int_s f(s)\left(\frac{\delta\mathcal{P}_{44}(p,q)}{\delta\phi(s)}\frac{\delta\mathcal{O}_2(r)}{\delta\phi(-s)} + \frac{\delta\mathcal{P}_{42}(p,r)}{\delta\phi(s)}\frac{\delta\mathcal{O}_4(q)}{\delta\phi(-s)} + \frac{\delta\mathcal{P}_{44}(q,r)}{\delta\phi(s)}\frac{\delta\mathcal{O}_4(p)}{\delta\phi(-s)}\right) \\
&\quad + \gamma_{44,4}\mathcal{P}_{42}(p+q,r) + \gamma_{44,\mathcal{N}}\mathcal{P}_{\mathcal{N}2}(p+q,r) + \gamma_{44,2}(p,q)\mathcal{P}_{22}(p+q,r) \\
&\quad + \gamma_{42,2}\left(\mathcal{P}_{42}(q,p+r) + \mathcal{P}_{42}(p,q+r)\right) \\
&\quad + \gamma_{442,2}\mathcal{O}_2(p+q+r) + \gamma_{442,0}(p,q,r)\delta(p+q+r)\,, \qquad (D10)
\end{aligned}
$$

where $\gamma_{442,0}(p,q,r)$ is quadratic. Before we calculate, let us remark that $\gamma_{442,2}$ depends on our choice of $\mathcal{P}_{42}$. If we changed $\mathcal{P}_{42}(p,q)$ by $a\mathcal{O}_2(p+q)$, where a is a constant, $\gamma_{442,2}$

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would change by $-\left(\gamma_{44,4}+\gamma_{42,2}\right)a$. Similarly, $\gamma_{442,0}$ would change if we changed $\mathcal{P}_{42},\mathcal{P}_{22}$ in proportion to the identity operator.

We only consider the 1PI part of $c_{442,4}, c_{442,2}$ here.

$$c_{442,4}^{1PI}(p,q,r;p_1,p_2,p_3,p_4) = \quad\text{}\quad,\tag{D11}$$

$$c_{442,2}^{1PI}(p,q,r;p_1,p_2) = \quad\text{}$$
$$+\quad\text{}\quad,\tag{D12}$$

where the last graph satisfies the ERG equation

$$\left(p\cdot\partial_p + q\cdot\partial_q + r\cdot\partial_r + \sum_{i=1,2} p_i\cdot\partial_{p_i}\right)\quad\text{}$$

$$= \int_s f(s)\left(\int_t h(t)h(p_1-p-s-t)h(p_1-p-r-s-t)\right.$$
$$\left.+ h(s-r)F(p_1-p-s) + h(s+r)F(p_1-p-s-r)\right)$$
$$+ \frac{1}{3}\gamma_{44,4}F(r) - \gamma_{44,\mathcal{N}} + \frac{1}{2}\gamma_{442,2}\,.\tag{D13}$$

$\gamma_{442,2}$ is determined so that rhs vanishes at zero momenta:

$$\int_s f(s)\int_t h(t)^2 h(s+t) + 2\int_s f(s)h(s)F(s) - \gamma_{44,\mathcal{N}} + \frac{1}{2}\gamma_{442,2} = 0\,.\tag{D14}$$

Using the integral I_4 in Appendix A and $\gamma_{44,\mathcal{N}} = \frac{1}{(4\pi)^4}\frac{1}{6}$, we obtain

$$\gamma_{442,2} = -\frac{1}{(4\pi)^4}\frac{5}{3}\,.\tag{D15}$$

4. $[\mathcal{O}_4\mathcal{O}_4\mathcal{O}_4]$

The ERG equation is

$$
(p \cdot \partial_p + q \cdot \partial_q + r \cdot \partial_r - \mathcal{D}) \, \mathcal{P}_{444}(p, q, r)
$$
$$
= \int_s f(s) \left(\frac{\delta \mathcal{P}_{44}(p,q)}{\delta \phi(s)} \frac{\delta \mathcal{O}_4(r)}{\delta \phi(-s)} + \frac{\delta \mathcal{P}_{44}(q,r)}{\delta \phi(s)} \frac{\delta \mathcal{O}_4(p)}{\delta \phi(-s)} + \frac{\delta \mathcal{P}_{44}(r,p)}{\delta \phi(s)} \frac{\delta \mathcal{O}_4(q)}{\delta \phi(-s)} \right)
$$
$$
+ \gamma_{44,4} \left(\mathcal{P}_{44}(p+q, r) + \mathcal{P}_{44}(q+r, p) + \mathcal{P}_{44}(r+p, q) \right)
$$
$$
+ \gamma_{44,\mathcal{N}} \left(\mathcal{P}_{\mathcal{N}4}(p+q, r) + \mathcal{P}_{\mathcal{N}4}(q+r, p) + \mathcal{P}_{\mathcal{N}4}(r+p, q) \right)
$$
$$
+ \gamma_{44,2}(p,q)\mathcal{P}_{24}(p+q, r) + \gamma_{44,2}(q,r)\mathcal{P}_{24}(q+r, p) + \gamma_{44,2}(r,p)\mathcal{P}_{24}(r+p, q)
$$
$$
+ \gamma_{444,4}\mathcal{O}_4(p+q+r) + \gamma_{444,\mathcal{N}}\mathcal{N}(p+q+r)
$$
$$
+ \gamma_{444,2}(p,q,r)\mathcal{O}_2(p+q+r) + \gamma_{444,0}(p,q,r) \,, \tag{D16}
$$

where $\gamma_{444,2}$ is quadratic, and $\gamma_{444,0}$ quartic in momenta.

Here we only compute part of the 1PI part $c_{444,4}^{\rm 1PI}(p,q,r;p_1,p_2,p_3,p_4)$ satisfying the ERG equation

$$
\left(p \cdot \partial_p + \cdots + \sum_{i=1}^{4} p_i \cdot \partial_{p_i} \right)
$$
$$
= \int_s f(s) \left(\int_t h(t)h(p_1 - p - s - t)h(p_1 + p_3 + p_4 - p - s - t - r) \right.
$$
$$
+ F(p_1 - p - s)h(s + p_3 + p_4 - r) + F(p_2 - q - s)h(s + p_3 + p + 4 - r) \Big)
$$
$$
+ \frac{1}{3}\gamma_{44,4}F(p_1 + p_2 - p - q) + \frac{1}{12}\left(-4\gamma_{44,\mathcal{N}} + \frac{1}{3}\gamma_{444,4} \right). \tag{D17}
$$

This is the same equation (and graph) as (D14). $\gamma_{444,4}$ is determined so that the rhs vanishes at zero momenta:

$$
\frac{1}{12}\left(4\gamma_{44,\mathcal{N}} - \frac{1}{3}\gamma_{444,4} \right) = \int_s f(s) \int_t h(t)h(s+t)^2 + 2\int_s f(s)h(s)F(s) = \frac{1}{(4\pi)^4}. \tag{D18}
$$

Hence, using $\gamma_{44,\mathcal{N}} = \frac{1}{(4\pi)^4}\frac{1}{6}$, we obtain

$$
\gamma_{444,4} = -\frac{34}{(4\pi)^4}. \tag{D19}
$$

It turns out that even if we changed $\mathcal{P}_{44}(p,q)$ by $a\,\mathcal{O}_4(p+q)$, $\gamma_{444,4}$ would remain intact. (If we substitute $\sigma_{44,4} = a$ into (53), we get zero.)

Appendix E: In coordinate space

Following [8–10], we consider space dependent parameters which are simply the Fourier transforms of the momentum dependent parameters:

$$\lambda(x) = -\int_p e^{-ipx} J_4(-p)\,, \tag{E1a}$$

$$m^2(x) = -\int_p e^{-ipx} J_2(-p)\,, \tag{E1b}$$

$$g(x) = -\int_p e^{-ipx} J_0(-p)\,. \tag{E1c}$$

The RG equations of the parameters are given by

$$\frac{d}{dt}\lambda(x) = x_\mu \cdot \partial_\mu \lambda(x) + B_4(x)\,, \tag{E2a}$$

$$\frac{d}{dt}m^2(x) = (x_\mu \cdot \partial_\mu + 2)\, m^2(x) + B_2(x)\,, \tag{E2b}$$

$$\frac{d}{dt}g(x) = (x_\mu \cdot \partial_\mu + 4)\, g(x) + B_0(x)\,, \tag{E2c}$$

where

$$B_4(x) \equiv \int_p e^{-ipx} B_4(-p) = \beta\left(\lambda(x)\right)\,, \tag{E3a}$$

$$B_2(x) \equiv \int_p e^{-ipx} B_2(-p)$$
$$= \beta_m(\lambda(x))m^2(x) + \beta_{m1}(\lambda(x))\partial^2\lambda(x) + \beta_{m2}(\lambda(x))\frac{1}{2}\partial_\mu\lambda(x)\partial_\mu\lambda(x)\,, \tag{E3b}$$

$$B_0(x) \equiv \int_p e^{-ipx} B_0(-p)$$
$$= \beta_{0m}(\lambda(x))\frac{1}{2}\left(m^2(x)\right)^2 + \beta_{0m1}(\lambda(x))\partial^2\lambda(x) + \beta_{0m2}(\lambda(x))\frac{1}{2}\partial_\mu\lambda(x)\partial_\mu\lambda(x)$$
$$+ \beta_{01}(\lambda(x))(\partial^2)^2\lambda(x) + \beta_{02}(\lambda(x))\frac{1}{2}\partial^2\lambda(x)\partial^2\lambda(x) + \beta_{03}(\lambda(x))\frac{1}{2}\partial_\mu\partial_\nu\lambda(x)\partial_\mu\partial_\nu\lambda(x)\,. \tag{E3c}$$

For $B_0(x)$, we have given the most general form up to total derivatives.

We can introduce a new parameter

$$\lambda'(x) = \lambda(x) + \mathrm{O}\left(\lambda(x)^2\right) \tag{E4}$$

so that its beta function is two-loop exact:

$$B_4'(x) = -\frac{3\lambda'(x)^2}{(4\pi)^2} + \frac{\lambda'(x)^3}{(4\pi)^3}\frac{17}{3}\,. \tag{E5}$$

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Similarly, we can introduce $m^{2\prime}(x)$ so that

$$B_2'(x) = -m^{2\prime}(x)\frac{\lambda'(x)}{(4\pi)^2} + \frac{1}{(4\pi)^4}\frac{1}{6}\partial_\mu\lambda'(x)\partial_\mu\lambda'(x) \tag{E6}$$

exactly. We should also be able to introduce $g'(x)$ with a simple $B_0'(x)$. With such a choice of parameters, we have no higher order mixing coefficients. This is possible only if we make a judicious choice of solutions to the ERG differential equations. Suppose we have constructed products of up to k operators. The ERG equations leave

$$[\mathcal{O}_{i_1}(p_1)\cdots\mathcal{O}_{i_k}(p_k)] \tag{E7}$$

ambiguous up to

$$\sigma_{i_1\cdots i_k,j}(p_1,\cdots,p_k)\mathcal{O}_j(p_1+\cdots+p_k) \tag{E8}$$

where σ is a polynomial of degree $-\sum_{l=1}^k y_{i_l} + y_j$. We choose $\sigma_{i_1\cdots i_k,j}$ so that the mixing coefficients of the $(k+1)$-operator products cancel. Only with this contrived choice of lower order products, we can eliminate the mixing coefficients.

ACKNOWLEDGMENTS

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