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Identification method for polynomially parametrized LTI systems based on exhaustive modelling with algebraic elimination

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Abstract: Realization problems of impulse responses for linear time-invariant (LTI) systems are well-studied. In particular, the realizations with the least order of such systems are said to be minimal. Apart from these problems, parameter estimations of pre-defined LTI models with a specific parametrization of the system matrices are also important. In this paper, we propose a parameter estimation method for such LTI models by transforming a minimal realization obtained through black-box identifications. Our approach is based on the minimal realization theory, exhaustive modelling, and algebraic elimination. Contrary to the existing methods, the proposed method allows polynomial parametrizations of the system.

Key Words: minimal realization theory, parameter estimation, linear time-invariant system, exhaustive modelling, elimination theorem

1. Introduction

In this paper, a state-space realization method of impulse responses for linear time-invariant systems based on exhaustive modelling [1] is proposed. More precisely, under the assumptions that the input and output matrices are in standard forms and the elements of the state matrices are represented through polynomials of unknown parameters, we provide an estimation method for such parameters by combining exhaustive modelling and algebraic eliminations.

System realization is the process of constructing a state space representation from experimental data. Throughout this paper, we consider the realization of impulse responses for linear time-invariant systems with zero initial states. In general, although such representations are not unique in nature,

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ideally they should be small. Realizations with state variables of the least order are called minimal realizations, and which are also not unique. In particular, an infinite number of different minimal realizations can be obtained from a minimal realization through similarity transformations [2]. In the general context of system realizations, such a non-uniqueness has little significance because the main objective of realization is not finding a specific representation.

However, in such applications as systems biology and engineering, the realization problems that arise are sometimes different from those considered above. In such fields, models that describe the systems of interest are typically constructed based on domain knowledge on phenomena regarding the systems. Throughout this paper, we consider these models and provide a realization method for them.

We now define the realization problems considered in this paper. First, the assumptions of the models are explained. We assume that models constructed in advance are either a discrete or continuous linear time-invariant system. A discrete-time model with unknown parameter vector $p_{\mathrm{d}} \in \mathbb{R}^n$ with zero initial states considered is as follows:

$$\begin{aligned} &x(k+1) = A_{\mathrm{d}}(p_{\mathrm{d}})x(k) + B_{\mathrm{d}}(p_{\mathrm{d}})u(k), \quad y = C_{\mathrm{d}}(p_{\mathrm{d}})x(k), \\ &x(0) = (0,\dots,0)^T, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^N$ is a state vector, $u \in \mathbb{R}^L$ is an input vector, $y \in \mathbb{R}^M$ is an output vector, and k is a non-negative integer that represents time. A continuous-time model with unknown parameter vector $p_{\mathrm{c}} \in \mathbb{R}^n$ with zero initial states is as follows:

$$\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}t} = A_{\mathrm{c}}(p_{\mathrm{c}})x + B_{\mathrm{c}}(p_{\mathrm{c}})u, \quad y = C_{\mathrm{c}}(p_{\mathrm{c}})x, \\ &x(0) = (0,\dots,0)^T, \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^N$ is a state vector, $u \in \mathbb{R}^L$ is an input vector, and $y \in \mathbb{R}^M$ is an output vector, each of which is time t dependent. Furthermore, we assume the followings:

1. The elements of $A_{\mathrm{d}}(p_{\mathrm{d}}), A_{\mathrm{c}}(p_{\mathrm{c}})$ are assumed to be polynomial functions of $p_{\mathrm{d}}, p_{\mathrm{c}}$, i.e., elements in $\mathbb{R}[p_{\mathrm{d}}]$ and $\mathbb{R}[p_{\mathrm{c}}]$, where $\mathbb{R}[p_{\mathrm{d}}]$ and $\mathbb{R}[p_{\mathrm{c}}]$ represent sets of polynomials of p_{d} and p_{c} with real coefficients, respectively. We call this a polynomial parametrization of linear systems.

2. Here, x is divided into four subsets of state variables: $x = (x_1, x_2, x_3, x_4)^T$, where $x_1 \in \mathbb{R}^Q, x_2 \in \mathbb{R}^{M-Q}, x_3 \in \mathbb{R}^{L-Q}$, and $x_4 \in \mathbb{R}^{N-L-M+Q}$. In addition, x_1 and x_2 are observed but the rest are not observed, whereas x_1 and x_3 are variables having an input and the rest are without inputs. Thus, Q denotes the number of variables that have inputs and are observed. This assumption reduces $B_{\mathrm{d}}, B_{\mathrm{c}}, C_{\mathrm{d}}$, and C_{c} into standard forms, which are denoted as follows [1]:

$$B_{\mathrm{d}} = B_{\mathrm{c}} = \begin{pmatrix} I_{Q,Q} & O_{Q,L-Q} \\ O_{M-Q,Q} & O_{M-Q,L-Q} \\ O_{L-Q,Q} & I_{L-Q,L-Q} \\ O_{N-L-M+Q,Q} & O_{N-L-M+Q,L-Q} \end{pmatrix}, \quad C_{\mathrm{d}} = C_{\mathrm{c}} = \begin{pmatrix} I_{M,M} & O_{M,N-M} \end{pmatrix}$$

where $I_{Z,Z}$ with $Z \in \mathbb{Z}$ denotes a $Z \times Z$ identity matrix and O_{Z_1,Z_2} with $Z_1, Z_2 \in \mathbb{Z}$ denotes a $Z_1 \times Z_2$ zero matrix. Note that $B_{\mathrm{d}}, B_{\mathrm{c}}, C_{\mathrm{d}}, C_{\mathrm{c}}$ are independent of $p_{\mathrm{d}}, p_{\mathrm{c}}$.

3. Here, (1) and (2) are identifiable, that is, each of p_{d} and p_{c} are determined uniquely from the given impulse responses, which are clearly defined below.

Next to define the impulse responses, we define inputs for (1) and (2). The impulse inputs for (1) are as follows:

$$U(k) = (u_1(k), \dots, u_L(k))^T \in \mathbb{R}^{L\times L}, \quad u_i(k) = \begin{cases} e_i & (k=0,) \\ (0,\dots,0) & (k=1,2,\dots,) \end{cases} \in \mathbb{R}^L, \quad i = 1,\dots,L, \tag{3}$$

where $e_i \in \mathbb{R}^L$ is a vector whose i th element is 1, whereas all other values are zero. In other words, $U(0)$ is an idenftity matrix, and $U(k)$ $(k = 1, \ldots, L)$ are zero matrices. The impulse functions for (2) are as follows:

$$U(t) = (u_1(t), \ldots, u_L(t))^T \in \mathbb{R}^{L\times L}, \quad u_i(t) \in \mathbb{R}^L, \quad i = 1, \ldots, L, \tag{4}$$

where $u_i(t)$ is a vector whose i th element is a dirac delta function and all other elements are all zero. Finally, our problems are obtaining realizations that satisfy the model assumptions mentioned.

For a clearer understanding, we provide an example of the models considered. The following is a type of compartment models, which are widely used in various applications, e.g., infectious disease modelling and biochemical networks, along with an observation model [3]. In Section 3, the proposed method will be demonstrated using this model.

Example 1.

$$x(k+1) = \begin{pmatrix} 0.1 & p_1p_2 & 0.0 & p_3+p_2 \\ 0.0 & 0.0 & -0.1 & 0.2 \\ p_3-p_2 & -0.2 & -0.1 & p_1p_2p_3 \\ 0.1 & 0.0 & 0.5 & 0.3 \end{pmatrix} x(k) + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} u(k), \tag{5}$$

$$y(k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x(k), \tag{6}$$

where $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$ is a state vector, $u = (u_1, u_2)^T \in \mathbb{R}^2$ is an input vector, $y = (y_1, y_2)^T \in \mathbb{R}^2$ is an output vector, and k is a non-negative integer that represents time. A schematic representation of (5) and (6) is shown in Fig. 1. As shown here, the matrix multiplied with $x(k)$ in (5), which is called a state matrix, has certain constraints, i.e., some elements are known constants and others are functions of unknown parameters p_1, p_2, and p_3. Let us suppose that (5) is constructed as a compartment model that represents four types of interactions of chemical substances, each amount of which is represented by a state variable. The scale of the interactions between substances 2 and 4 are represented by constants based on the domain knowledge. In addition, substance 4 affects substance 2 at the scale of 0.2, whereas there is no effect from substance 2 on substance 4. The scale of the interaction from substance 2 to substance 1 is denoted as p_1p_2, in addition, there are two factors p_1 and p_2 that determines the interactions, each of which is proportional to x_2, where the values of p_1

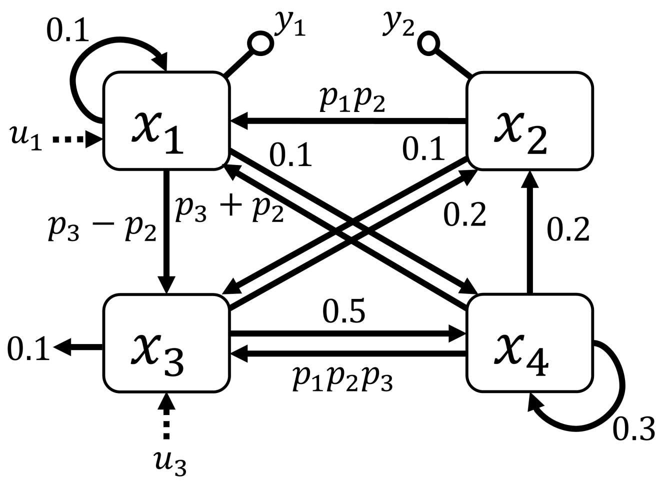

Fig. 1. A schematic representation of (5) and (6). State variables are represented by boxes labeled with the corresponding variables. These boxes are called compartments, when (5) is regarded as a compartment model. The inputs and outputs are represented by the dashed arrows and bars with circles, respectively. The interactions between the state variables and the self interactions are represented by the arrows. The values or functions of the parameters (p_1, p_2, p_3) near those arrows denote the scale of such interactions.

and p_2 are unknown. The matrices multiplied with $u(k)$ in (5) and $x(k)$ in (6) represents the impulsive stimuli applied to substances 1 and 3, and the responses of which are observed through x_1 and x_2. In this way, the matrices appeared in (5) and (6) are constructed based on the domain knowledge with unknown parameters to be estimated and known constants. In this way, linear systems considered in this type of situation differ from those in a conventional system realization, which assumes that all entries of the system matrices are unknown constants.

The problems defined above are usually tackled as numerical parameter estimations, one of which explores the parameters that fit the model outputs to the given impulse responses by numerically computing their trajectories with various parameters. However, if we change the viewpoint, the non-uniqueness of the minimal realization problems can be then used effectively to estimate the parameters of models having particular structures. In other words, it may be possible to find specific models that have the desired structures from minimal representations without structures. In fact, such identification methods have been studied in [4–6]; Parrilo et al. proposed an identification method based on the sum of squares method [5], whereas the methods proposed by Mercére et al. [4] are based on vectorizations of the system matrices. In the former approach, the parameters of the model are estimated along with a matrix that transforms a realized system into a model with the desired structure. The method proposed in [6] also estimates such parameters through two-step iterative methods. By contrast, the latter approach [4] estimates the matrix, which implicitly determines the values of the parameters, thereby simplifying the estimation procedure; however, it restricts the system matrices under consideration as those with affine parametrizations, which does not allow polynomial parametrizations.

In this study, under the assumptions that models (1) and (2) constructed in advance satisfy the above three assumptions and can be minimal realizations, that is, are both observable and controllable with true parameter values, we show that the parameters can be obtained by finding transformations from a minimal realization of the impulse responses. Our method is based on an idea of exhaustive modelling followed by an algebraic elimination. Exhaustive modelling, which is a concept in the context of a structural identifiability analysis [1], is applied to find matrices that transforms systems obtained through a black-box identification into the desired model. With our method in particular, an objective function that must be minimized to estimate the matrices is specified through algebraic eliminations, which avoids direct estimations of the model parameters.

The remainder of this paper is organized as follows. In Section 2, a method for a parameter estimation combining black-box identification methods and exhaustive modelling is proposed. This section is divided into four parts. In the first subsection, the definitions and basics of minimal realization theory are introduced. In the second subsection, two main theorems that support the application of a black-box identification method of linear systems into our problems are described. In the third subsections, an estimation method of a transformation matrix for transforming the models obtained through a black-box identification into the desired models, based on exhaustive modelling, is described. In the fourth subsection, we provide a short explanation of algebraic elimination, particularly for the elimination property of the Gröbner basis. Through algebraic eliminations, the choices of the objective functions to be minimized in the estimations of the transformation matrix are induced. Numerical examples are shown in Section 3.

2. Outline of proposed method

In this section, we propose a method for estimating p_d and p_c of linear time-invariant systems (1) and (2) such that their outputs are coincident with the given impulse responses. To provide the theorems supporting our method, we first introduce the definitions and basics of minimal realization theory. We then provide the theorems that clarify the conditions needed for our method and how to apply black-box identification methods. Finally, an estimation method of a matrix that transforms a minimal realization obtained by the black-box identifications into (1) and (2), which estimates p_d and p_c at the same time, is described.

2.1 Definitions and the basics of minimal realization theory

In this section, we provide the definitions and basics of minimal realization theory. The responses of (1) to impulse inputs (3) are denoted as follows:

$$Y(k) = (y_1(k), \dots, y_L(k)) \in \mathbb{R}^{M\times L}, \tag{7}$$

where $y_i \in \mathbb{R}^M (i = 1, \dots, L)$ is the response of discrete model (1) with u_i. In the same way, the set of state vectors with $U(k)$ are denoted as follows:

$$X(k) = (x_1(k), \dots, x_L(k)) \in \mathbb{R}^{N\times L},$$

where $x_i \in \mathbb{R}^N (i = 1, \dots, L)$ are the state vectors with u_i. Using these notations, the responses of (1) with $U(k)$ are as follows:

$$\begin{aligned}
Y(0) &= C_\mathrm{d}X(0) = O \in \mathbb{R}^{M\times N},\\
Y(1) &= C_\mathrm{d}(A_\mathrm{d}X(0) + B_\mathrm{d}U(0)) = C_\mathrm{d}B_\mathrm{d},\\
Y(2) &= C_\mathrm{d}(A_\mathrm{d}X(1) + B_\mathrm{d}U(1)) = C_\mathrm{d}A_\mathrm{d}B_\mathrm{d},\\
&\ \vdots\\
Y(k) &= C_\mathrm{d}(A_\mathrm{d}X(k-1) + B_\mathrm{d}U(k-1)) = C_\mathrm{d}{A_\mathrm{d}}^{k-1}B_\mathrm{d},
\end{aligned}$$

where $A_\mathrm{d}(p_\mathrm{d}), B_\mathrm{d}(p_\mathrm{d})$ and $C_\mathrm{d}(p_\mathrm{d})$ are written as $A_\mathrm{d}, B_\mathrm{d}$ and C_d for short. Based on the above, the realization of the impulse responses for the discrete model (1) is defined as follows:

Definition 1. $(A_\mathrm{d}, B_\mathrm{d}, C_\mathrm{d})$ is a realization of the given impulse responses to (3) for (1) if $Y(k) = C_\mathrm{d}{A_\mathrm{d}}^{k-1}B_\mathrm{d}$ $(k = 1, 2, \dots)$ is satisfied.

Note that this realization can be essentially regarded as a parameter estimation of (1) given the impulse responses because the model structure is restricted. A realization of the impulse responses is minimal if it is of least order, i.e., if the rank of A_d is N. Regarding the minimal realizations, the following are well-known [2].

Proposition 1. Let A, B and C be $N \times N, N \times L$ and $M \times N$ matrices, respectively. A realization (A, B, C) of the given discrete-time impulse responses is minimal, i.e., its order is N, if and only if it is both controllable and observable.

Proposition 2. Suppose that the following system with $(\tilde{A}, \tilde{B}, \tilde{C})$ and that with (A, B, C) yield the same y for the same u:

$$\begin{aligned}
\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{B}u(k), \quad y = \tilde{C}\tilde{x}(k),\\
\tilde{x}(0) = (0, \dots, 0)^T,
\end{aligned} \tag{8}$$

where $\tilde{x} \in \mathbb{R}^{\tilde{N}}$ is a state vector and $\tilde{A} \in \mathbb{R}^{\tilde{N}\times\tilde{N}}, \tilde{B} \in \mathbb{R}^{\tilde{N}\times L}$, and $\tilde{C} \in \mathbb{R}^{M\times\tilde{N}}$. In addition, suppose that $(\tilde{A}, \tilde{B}, \tilde{C})$ is a minimal realization. Then, (A, B, C) is also a minimal realization, that is, $\tilde{N} = N$, if and only if there exists a nonsingular matrix T such that $A = T\tilde{A}T^{-1},\ B = T\tilde{B}$ and $\ C = \tilde{C}T^{-1}$.

Now, we consider the continuous counterpart of the above. In the same way as before, the responses of (2) applied $U(t)$ are expressed as follows [2]:

$$Y(t) = C_\mathrm{c}\mathrm{e}^{A_\mathrm{c}t}B_\mathrm{c}. \tag{9}$$

Note that the (i, j) element of $Y(t)$ contains the ith element of $y(t)$ owing to an impulse function assigned to only the jth element of $u(t)$. This matrix is generally called the impulse response matrix for (2). Based on this, the realization of impulse responses is defined as follows:

Definition 2. $(A_\mathrm{c}, B_\mathrm{c}, C_\mathrm{c})$ is a realization of the given impulse responses to (4) for (2) if $Y(t) = C_\mathrm{c}\mathrm{e}^{A_\mathrm{c}t}B_\mathrm{c}$ is satisfied.

A realization of impulse responses for (2) is minimal if it is of least order, i.e., if the rank of A_c is N. The continuous counterparts of Propositions 1 and 2 are as follows [2]:

Proposition 3. Let A, B and C be $N \times N, N \times L$ and $M \times N$ matrices, respectively. A realization (A, B, C) of the given continuous-time impulse responses is minimal, i.e., its order is N, if and only if it is both controllable and observable.

Proposition 4. Suppose that the following system with $(\tilde{A}, \tilde{B}, \tilde{C})$ and that with (A, B, C) yield the same y for the same u:

$$\begin{aligned} \frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} &= \tilde{A}\tilde{x} + \tilde{B}u, \quad y = \tilde{C}\tilde{x}, \\ \tilde{x}(0) &= (0, \dots, 0)^T, \end{aligned} \tag{10}$$

where $\tilde{x} \in \mathbb{R}^{\tilde{N}}$ is a state vector and $\tilde{A} \in \mathbb{R}^{\tilde{N}\times\tilde{N}}, \tilde{B} \in \mathbb{R}^{\tilde{N}\times L}$, and $\tilde{C} \in \mathbb{R}^{M\times\tilde{N}}$. In addition, suppose also that $(\tilde{A}, \tilde{B}, \tilde{C})$ is a minimal realization. Then, (A, B, C) is also a minimal realization, that is, $\tilde{N} = N$, if and only if there exists a nonsingular matrix T such that $A = T\tilde{A}T^{-1}$, $B = T\tilde{B}$ and $C = \tilde{C}T^{-1}$.

2.2 Black-box identifications of $(\tilde{A}, \tilde{B}, \tilde{C})$

As mentioned in [4], as a naive choice to estimate the parameters of (1) and (2), the prediction error method [7] is occasionally chosen. The method often struggles with convexity problems, which tend to cause local minima issues. In other words, they require an appropriate initial estimate of the parameters. In comparison, black-box identification methods, which we consider in the following, are non-iterative, and hence do not suffer from local minima issues. Realizations obtained through black-box identification methods typically do not satisfy the model assumptions; however, there are similarity transformations that transform those realizations into the desired versions under certain assumptions, as shown in the following theorems:

Theorem 1. Suppose that $(A_\mathrm{d}, B_\mathrm{d}, C_\mathrm{d})$ is a realization of the given impulse responses for (1). In addition, suppose that (1) is both observable and controllable. If $(\tilde{A}, \tilde{B}, \tilde{C})$ is a minimal realization of the impulse responses for (8), then a nonsingular matrix T exists such that

$$A_\mathrm{d} = T\tilde{A}T^{-1}, \quad B_\mathrm{d} = T\tilde{B}, \quad C_\mathrm{d} = \tilde{C}T^{-1}. \tag{11}$$

Proof. Based on Proposition 1, $(A_\mathrm{d}, B_\mathrm{d}, C_\mathrm{d})$ is a minimal realization of the given impulse responses for (1) because it is assumed to be both observable and controllable. According to Proposition 2, letting A, B and C be $N \times N, N \times L$ and $M \times N$ matrices, there exists a nonsingular matrix that satisfies the following:

$$A = T\tilde{A}T^{-1}, \quad B = T\tilde{B}, \quad C = \tilde{C}T^{-1},$$

where both (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$ are minimal realizations of the same impulse responses. Because $(A_\mathrm{d}, B_\mathrm{d}, C_\mathrm{d})$ is a minimal realization of the impulse responses, (A, B, C) can be used as an replacement. Hence, there exists a nonsingular matrix T such that (11) holds. □

Theorem 2. Suppose that $(A_\mathrm{c}, B_\mathrm{c}, C_\mathrm{c})$ is a realization of the given impulse responses for (2). If $(\tilde{A}, \tilde{B}, \tilde{C})$ is a minimal realization for (10), then a nonsingular matrix T exists such that

$$A_\mathrm{c} = T\tilde{A}T^{-1}, \quad B_\mathrm{c} = T\tilde{B}, \quad C_\mathrm{c} = \tilde{C}T^{-1}. \tag{12}$$

Proof. Based on Proposition 3, $(A_\mathrm{c}, B_\mathrm{c}, C_\mathrm{c})$ is a minimal realization of the given impulse responses for (2) since it is assumed to be both observable and controllable. According to Proposition 4, letting A, B and C be $N \times N, N \times L$ and $M \times N$ matrices, there exists a nonsingular matrix that satisfies

$$A = T\tilde{A}T^{-1}, \quad B = T\tilde{B}, \quad C = \tilde{C}T^{-1}, \tag{13}$$

where both of (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$ are minimal realizations of the same impulse responses. Because $(A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}})$ is a minimal realization of the impulse responses, (A, B, C) can be used as an replacement. Hence, a nonsingular matrix T exists such that (12) holds. □

The transformation of the systems through T in Theorems 1 and 2 is achieved by the coordinate transformations between the state variables $x = T\tilde{x}$.

Remark 1. In practice, the impulse responses observed from the systems might not be precisely coincident with the output of the models. However, in the context of modelling, it is natural to assume that (1) and (2) with the true parameters are realizations of the impulse responses. Therefore, in this sense, the assumptions regarding the model constructed to make use of Theorems 1 and 2 are simply the observability and controllability of (1) and (2).

It should be pointed out that a minimal realization of the given impulse responses $(\tilde{A}, \tilde{B}, \tilde{C})$ for (1) and (2) can actually be found using appropriate minimal realization algorithms. This is because the models constructed in advance are assumed to be realizations of the impulse responses provided true parameter values. Thus, the existence of a realization is guaranteed, which allows minimal realization algorithms to operate properly. Hence, the above theorems, each of which connects a minimal realization $(\tilde{A}, \tilde{B}, \tilde{C})$ to the desired structurally constrained realization $(A_{\mathrm{d}}, B_{\mathrm{d}}, C_{\mathrm{d}})$ and $(A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}})$, respectively, indicate that a set of constraints on p_{d} and p_{c} are obtained by finding T if (1) and (2) are both observable and controllable.

Because (11) and (12) are essentially the same, we denote system matrices and parameters as (A_p, B_p, C_p) and p in a unified manner for both discrete and continuous models in the following, meaning that our method is applicable for both. Based on this, in the first step of our method, a minimal realization of the given impulse responses $(\tilde{A}, \tilde{B}, \tilde{C})$ that may not have the same structure as (A_p, B_p, C_p) is obtained through black-box identification methods. If it has the desired structure, the estimation then ends at this step by comparing (A_p, B_p, C_p) and $(\tilde{A}, \tilde{B}, \tilde{C})$. Here, $(\tilde{A}, \tilde{B}, \tilde{C})$ is obtained by, for example, the eigensystem realization algorithm, ERA [8], for discrete time models and the subspace methods [7] that achieve minimal realizations for continuous-time models, respectively.

2.3 Estimations of T

In the second step, T that transforms $(\tilde{A}, \tilde{B}, \tilde{C})$ into(A_p, B_p, C_p) is estimated under the assumptions that (1) and (2) are both observable and controllable. As is evident from Theorems 1 and 2, T depends on both (A_p, B_p, C_p) and $(\tilde{A}, \tilde{B}, \tilde{C})$. As mentioned in Section 1, B_p and C_p are assumed to be in the standard forms as follows:

$$B_p = B = \begin{pmatrix} I_{Q,Q} & O_{Q,L-Q} \\ O_{M-Q,Q} & O_{M-Q,L-Q} \\ O_{L-Q,Q} & I_{L-Q,L-Q} \\ O_{N-L-M+Q,Q} & O_{N-L-M+Q,L-Q} \end{pmatrix}, \quad C_p = C = \begin{pmatrix} I_{M,M} & O_{M,N-M} \end{pmatrix}. \tag{14}$$

To clarify the independency of B_p and C_p from the parameters, we denote them as B and C. These are called the standard forms in the context of exhaustive modelling [1], which is one of the concepts for a structural identifiability analysis. Note that if B and C are known full rank matrices, they can be converted into the standard form through a singular value decomposition and similarity transformations, as described in [1]. These procedures do not effect A_p, meaning that if B and C are known full-rank matrices, our method can be utilized. These assumptions tend to be satisfied in common experimental settings.

We are now ready to explain the formulation of estimations for T. Our idea is based on exhaustive modelling [1], in which one considers all possible parameters that satisfy the given model structures. Assuming that $\tilde{B}$ and $\tilde{C}$ are already of standard forms, we estimate a nonsingular matrix, say T_h, that transforms $(\tilde{A}, \tilde{B}, \tilde{C}) = (\tilde{A}, B, C)$ into (A_p, B, C), which achieves the desired realization. Otherwise, when $(\tilde{B}, \tilde{C})$ obtained in the first step are not equal to (B, C), another nonsingular matrix, denoted as T_t, transforms $(\tilde{A}, \tilde{B}, \tilde{C})$ into $(\hat{A}, B, C)$, where $\hat{A}$ is an $N \times N$ matrix, which makes T_h applicable. In summary, we estimate T_h and T_t such that

$$\hat{A} = T_t \tilde{A} T_t^{-1}, \quad B = T_t \tilde{B}, \quad C = \tilde{C} T_t^{-1}, \tag{15}$$

$$A_p = T_h \hat{A} T_h^{-1}, \quad B = T_h B, \quad C = C T_h^{-1}, \tag{16}$$

where T is shown to be equal to $T_h T_t$ because $A_p = T_h \hat{A} T_h^{-1} = T_h T_t \tilde{A} T_t^{-1} T_h^{-1}$. As shown in Theorem 1, T and hence both T_h and T_t are guaranteed to be nonsingular.

Based on the above, we first explain how to estimate T_t. To this end, we introduce the following representation:

$$T_t = \begin{pmatrix} T_{t(1,1)} & T_{t(1,2)} & T_{t(1,3)} & T_{t(1,4)} \\ T_{t(2,1)} & T_{t(2,2)} & T_{t(2,3)} & T_{t(2,4)} \\ T_{t(3,1)} & T_{t(3,2)} & T_{t(3,3)} & T_{t(3,4)} \\ T_{t(4,1)} & T_{t(4,2)} & T_{t(4,3)} & T_{t(4,4)} \end{pmatrix},$$

where T_t is a $(Q + (M - Q) + (L - Q) + (N - L - M + Q))$ by $(Q + (M - Q) + (L - Q) + (N - L - M + Q))$ matrix. For instance, $T_{t(2,3)}$ is an $(M - Q) \times (L - Q)$ matrix. The constraints on C in (15) give $C = \tilde{C} T_t^{-1}$ and thus

$$\tilde{C} = \begin{pmatrix} I_{M,M} & O_{M,N-M} \end{pmatrix} \begin{pmatrix} T_{t(1,1)} & T_{t(1,2)} & T_{t(1,3)} & T_{t(1,4)} \\ T_{t(2,1)} & T_{t(2,2)} & T_{t(2,3)} & T_{t(2,4)} \\ T_{t(3,1)} & T_{t(3,2)} & T_{t(3,3)} & T_{t(3,4)} \\ T_{t(4,1)} & T_{t(4,2)} & T_{t(4,3)} & T_{t(4,4)} \end{pmatrix} \quad \Leftrightarrow \quad \tilde{C} = \begin{pmatrix} T_{t(1,1)} & T_{t(1,2)} & T_{t(1,3)} & T_{t(1,4)} \\ T_{t(2,1)} & T_{t(2,2)} & T_{t(2,3)} & T_{t(2,4)} \end{pmatrix}, \tag{17}$$

which means that the first L rows of T_t are equal to $\tilde{C}$. From the last $N - M$ rows of the constraints on B in (15),

$$\begin{pmatrix} O_{L-Q,Q} & I_{L-Q,L-Q} \\ O_{N-L-M+Q,Q} & O_{N-L-M+Q,L-Q} \end{pmatrix} = \begin{pmatrix} T_{t(3,1)} & T_{t(3,2)} & T_{t(3,3)} & T_{t(3,4)} \\ T_{t(4,1)} & T_{t(4,2)} & T_{t(4,3)} & T_{t(4,4)} \end{pmatrix} \tilde{B}.$$

Let $(\tilde{B}_1 \mid \tilde{B}_2)^T$ be $\tilde{B}$, where $\tilde{B}_1$ and $\tilde{B}_2$ are $M \times L$ and $(N - M) \times L$ matrices, and $(B_1 \mid B_2)^T$ be B with matrices of the same dimensions. The above equation can then be represented as follows:

$$B_2 = \begin{pmatrix} T_{t(3,1)} & T_{t(3,2)} \\ T_{t(4,1)} & T_{t(4,2)} \end{pmatrix} \tilde{B}_1 + \begin{pmatrix} T_{t(3,3)} & T_{t(3,4)} \\ T_{t(4,3)} & T_{t(4,4)} \end{pmatrix} \tilde{B}_2.$$

Assuming $\tilde{B}_2$ is nonsingular, it follows that

$$\begin{pmatrix} T_{t(3,3)} & T_{t(3,4)} \\ T_{t(4,3)} & T_{t(4,4)} \end{pmatrix} = \left(B_2 - \begin{pmatrix} T_{t(3,1)} & T_{t(3,2)} \\ T_{t(4,1)} & T_{t(4,2)} \end{pmatrix} \tilde{B}_1 \right) \tilde{B}_2^{-1}. \tag{18}$$

Thus, the indeterminants are $T_{t(3,1)}, T_{t(3,2)}, T_{t(4,1)}$ and $T_{t(4,2)}$. Before considering the constraints on $\hat{A}$ in (15), which are used for the determination of $T_{t(3,1)}, T_{t(3,2)}, T_{t(4,1)}$ and $T_{t(4,2)}$, we consider the formulation of estimations for T_h, assuming that an appropriate T_t is obtained.

In the same way as for T_t, regarding the constraints on C in (16), the first L rows of T_h are equal to C:

$$\begin{pmatrix} I_{M,M} & O_{M,N-M} \end{pmatrix} = \begin{pmatrix} T_{h(1,1)} & T_{h(1,2)} & T_{h(1,3)} & T_{h(1,4)} \\ T_{h(2,1)} & T_{h(2,2)} & T_{h(2,3)} & T_{h(2,4)} \end{pmatrix}.$$

The constraints on B in (16) indicate that B is equal to the following:

$$\begin{pmatrix} I_{Q,Q} & O_{Q,L-Q} \\ O_{M-Q,Q} & O_{M-Q,L-Q} \\ O_{L-Q,Q} & I_{L-Q,L-Q} \\ O_{N-L-M+Q,Q} & O_{N-L-M+Q,L-Q} \end{pmatrix} \begin{pmatrix} T_{h(1,1)} & T_{h(1,2)} & T_{h(1,3)} & T_{h(1,4)} \\ T_{h(2,1)} & T_{h(2,2)} & T_{h(2,3)} & T_{h(2,4)} \\ T_{h(3,1)} & T_{h(3,2)} & T_{h(3,3)} & T_{h(3,4)} \\ T_{h(4,1)} & T_{h(4,2)} & T_{h(4,3)} & T_{h(4,4)} \end{pmatrix} = \begin{pmatrix} T_{h(1,1)} & T_{h(1,3)} \\ T_{h(2,1)} & T_{h(2,3)} \\ T_{h(3,1)} & T_{h(3,3)} \\ T_{h(4,1)} & T_{h(4,3)} \end{pmatrix}.$$

Thus, $T_{h(3,2)}, T_{h(3,4)}, T_{h(4,2)}$ and $T_{h(4,4)}$ are the only undetermined blocks in T_h. More precisely, T_h is

$$\begin{pmatrix} I_{Q,Q} & O_{Q,M-Q} & O_{Q,L-Q} & O_{Q,N-L-M+Q} \\ O_{M-Q,Q} & I_{M-Q,M-Q} & O_{M-Q,L-Q} & O_{M-Q,N-L-M+Q} \\ O_{L-Q,Q} & T_{h(3,2)} & I_{L-Q,L-Q} & T_{h(3,4)} \\ O_{L-Q,Q} & T_{h(4,2)} & O_{L-Q,L-Q} & T_{h(4,2)} \end{pmatrix}, \tag{19}$$

which is the same as the result obtained in [1]. As shown in [1], T_h, which transforms $\hat{A}$ into A_p, does not change the $(1,1), (1,3), (2,1)$, and $(2,3)$ blocks. In other words, these blocks of $\hat{A}$ must be equal to those of A_p, which reveals the hidden constraints on $\hat{A}$.

To summarize, the estimations of T_t and T_h are formulated as follows:

$$\min_{T_{t(3,1)}, T_{t(3,2)}, T_{t(4,1)}, T_{t(4,2)}} ||\hat{A}(\lambda) - A_p(\lambda)||_2^2, \quad \text{s.t. } \hat{A} = T_t \tilde{A} T_t^{-1}, \tag{20}$$

$$\min_{T_{h(3,2)}, T_{h(3,4)}, T_{h(4,2)}, T_{h(4,4)}} ||A_f - A_p||_2^2, \quad \text{s.t. } A_f = T_h \hat{A} T_h^{-1}. \tag{21}$$

Here, λ denotes the $(1,1), (1,3), (2,1)$ and $(2,3)$ blocks, and T_t satisfies (17) and (18). Furthermore, A_f denotes an estimation of A_p, where (16) holds. Note that the objective function $||A_f - A_p||_2^2$ needs to be carefully designed, considering that the true parameters are unknown in practice. The details on specifications of $||A_f - A_p||_2^2$ are discussed in the next subsection. Here, p can be estimated by comparing A_f with A_p, where A_f is obtained as $T\tilde{A}T^{-1}$ such that $T = T_h T_t$ with T_h and T_t estimated above.

2.4 Specifications of the objective function in (21) based on algebraic elimination

To explain a difficulty that arises in designing $||A_f - A_p||_2^2$, we give simple examples in the following. They illustrate the model-dependencies of the existence of hidden constraints on the parameters to be considered in the specifications of $||A_f - A_p||_2^2$.

Example 2. Suppose we have the following:

$$A_p = \begin{pmatrix} p_1 & 0 \\ 0 & -p_1 \end{pmatrix}, \quad A_f = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} = T\tilde{A}T^{-1},$$

where p_1, p_2 are parameters and a_{11}, a_{22} are constants that are computed through a similarity transformation of $(\tilde{A}, \tilde{B}, \tilde{C})$. Clearly, $p_1 = a_{11} = -a_{22}$ must be satisfied so that $||A_f - A_p||_2^2$ is equal to zero. Thus, $(a_{11} + a_{22})^2$ should be a choice of the objective function to be minimized for a minimization of $||A_f - A_p||_2^2$.

Example 3. Suppose that we have the following:

$$A_p = \begin{pmatrix} p_1 p_2 & 0 \\ 0 & p_1 + p_2 \end{pmatrix}, \quad A_f = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} = T\tilde{A}T^{-1}$$

where p_1 and p_2 are the parameters and a_{11} and a_{22} are constants, as in the previous example. By solving the simultaneous equations $p_1 p_2 = a_{11}$ and $p_1 + p_2 = a_{22}$ in terms of p_1 and p_2, we obtain the following:

$$(p_1, p_2) = \left(\frac{1}{2} \left(a_{22} \mp \sqrt{a_{22}^2 - 4a_{11}} \right), \frac{1}{2} \left(a_{22} \pm \sqrt{a_{22}^2 - 4a_{11}} \right) \right).$$

This implies that by computing a_{11} and a_{22} through similarity transformations, the values of p_1 and p_2 are implicitly determined. Hence, $p_1 p_2 - a_{11} = 0$ and $p_1 + p_2 - a_{22} = 0$ hold for arbitrary values of a_{11} and a_{22} determined through the similarity transfomrations.

Remark 2. Although the elements of A_f except for a_{11}, a_{22} are assumed to be zeros in the examples for simplicity, their values depend on the similarity transformations applied to $(\tilde{A}, \tilde{B}, \tilde{C})$ in practice. Therefore, we must evaluate the norm of $A_f - A_p$ by considering these elements, each of which is a constant, along with the hidden constrains on the parameters.

As illustrated in the above examples, suitable specifications of $||A_f - A_p||_2^2$ depend on the parametrization of the models. To check whether there are constraints on the elements of A_f, that is, hidden constraints imposed to similarity transformations, we introduce algebraic techniques. We denote S as the set of pairs of indices that represent unknown elements of A_p, i.e., those containing polynomials of p. Suppose f_{ij} is the (i, j) element of $(A_f - A_p)$, where $(i, j) \in S$. Here, f_{ij} is a polynomial in $\mathbb{R}[p, a]$, where a is a vector in which each element is the (i, j) element of A_f. In addition, $\mathbb{R}[p, a]$, a polynomial ring, denotes a set of polynomials whose coefficients are real numbers, and where p and a are the variables. Let $I_p \subset \mathbb{R}[p, a]$ be the ideal generated by $\{f_{ij} \mid (i, j) \in S\}$ which is called the generator of I_p. In general, an ideal generated by a generator, which is a finite set of polynomials in a polynomial ring, is essentially, the set of polynomials in the ring obtained through algebraic manipulations of the generator. The constraints of the elements of A_f are in

$$I_p \cap \mathbb{R}[a]. \tag{22}$$

If there are no polynomials in (22), there are no hidden constraints imposed on the similarity transformations. By contrast, if such polynomials exist, to deal with such polynomials systemeatically, we utilize the elimination property of the Gröbner basis [9].

Definition 3. For an ideal I on a polynomial ring $K[x_1, \ldots, x_n]$ over a field K with a given monomial order, if its generator $G = \{g_1, \ldots, g_n\}$ satisfies

$$f \in I \Leftrightarrow f \text{ is divisible by } G,$$

G is called the Gröbner basis of I.

Remark 3. A monomial order of a polynomial ring determines the order between two arbitrary monomials in the polynomial ring. To divide a polynomial by a finite set of polynomials on a polynomial ring $K[x_1, \ldots, x_n]$ algorithmically, the monomial order needs to be specified on the ring.

Proposition 5. Suppose that lexicographic ordering is used as the monomial ordering in a ring of polynomials $K[x_1, \ldots, x_n]$ over field K such that $x_1 > x_2 > \ldots > x_n$. If G is reduced, the Gröbner basis of ideal $I \subset K[x_1, \ldots, x_n]$ for every $1 \le j \le n$, $G \cap K[x_j, \ldots, x_n]$ is the Gröbner basis of $I \cap K[x_j, \ldots, x_n]$.

The above is called the elimination theorem. See [9] for details. Based on this, given a lexicographic ordering in $\mathrm{R}[p, a]$, $p > a$, we consider the Gröbner basis of ideal $I_p \subset \mathbb{R}[p, a]$. If $I_p \cap \mathbb{R}[a]$ contains non-zero polynomials, the Gröbner basis of I_p that does not contain p is that of (22) and is considered to be the specifications of the hidden constraints imposed on the similarity transformations. Because we assume the identifiability of the models, the values of p are assumed to be determined using those of a. In other words, polynomials in $I_p \backslash (I_p \cap \mathbb{R}[a])$ may be considered to be zeros by asserting constants to a. See [10] for details on the identifiability of the models. In conclusion, we propose choosing the following for evaluation of $||A_f - A_p||_2^2$:

$$\sum_{\bar{i},\bar{j}} (\text{the } (\bar{i}, \bar{j}) \text{ element of } (A_f - A_p))^2 + \sum_k {G_k}^2$$

where $(\bar{i}, \bar{j}) \in \bar{S}$, in which $\bar{S}$ denotes the complement of S, and G_k denotes the kth element of the Gröbner basis of (22) with respect to a lexicographic order $p > a$. To conclude this section, an example of the specifications of $||A_f - A_p||_2^2$ is provided below, which we utilize in the next section again.

Example 4. Suppose we have

$$A_p = \begin{pmatrix} 0.1 & p_1p_2 & 0.0 & p_3+p_2 \\ 0.0 & 0.0 & -0.1 & 0.2 \\ p_3-p_2 & -0.2 & -0.1 & p_1p_2p_3 \\ 0.1 & 0.0 & 0.5 & 0.3 \end{pmatrix}, \quad A_f = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = T\tilde{A}T^{-1} \tag{23}$$

where $S = \{(1,2),(1,4),(3,1),(3,4)\}$, and $I_p \subset \mathbb{R}[p_1,p_2,p_3,a_{12},a_{14},a_{31},a_{34}]$ is the ideal generated by f_{12}, f_{14}, f_{31} and f_{34}, i.e.,

$$I_p = \langle f_{12}, f_{14}, f_{31}, f_{34}\rangle = \langle a_{12} - p_1p_2, a_{14} - (p_3+p_2), a_{31} - (p_3-p_2), a_{34} - p_1p_2p_3\rangle.$$

We fix the monomial order of the ring $\mathbb{R}[p_1,p_2,p_3,a_{12},a_{14},a_{31},a_{34}]$ to the lexicograhic order $p_1 > p_2 > p_3 > a_{12} > a_{14} > a_{31} > a_{34}$ and compute the Gröbner basis of I_p. We then obtain the Gröbner basis G as follows:

$$G = \{a_{12}a_{14} + a_{12}a_{31} - 2a_{34}, 2p_3 - a_{14} - a_{31}, 2p_2 - a_{14} + a_{31}, \\ p_1a_{14} - p_1a_{31} - 2a_{12}, p_1a_{12}a_{31} - p_1a_{34} + {a_{12}}^2\}.$$

Hence, we obtain

$$a_{12}a_{14} + a_{12}a_{31} - 2a_{34} =: G_1. \tag{24}$$

as the Gröbner basis of $I_p \cap \mathbb{R}[a_1,a_2,a_3,a_4]$. Based on this, $||A_f - A_p||_2^2$ is minimized by minimizing

$$\sum_{\bar{i},\bar{j}} (\text{the } (\bar{i},\bar{j}) \text{ element of } (A_f - A_p))^2 + {G_1}^2, \tag{25}$$

where $\bar{S} = \{(1,1),(1,3),(2,1),(2,2),(2,3),(2,4),(3,2),(3,3),(4,1),(4,2),(4,3),(4,4)\}$.

3. Numerical example

To evaluate the numerical stability of the proposed method, an estimation of parameter $p_{\mathrm{d}} = (p_1,p_2,p_3) \in \mathbb{R}^3$ of a discrete-time linear time invariant model

$$x(k+1) = A_p x(k) + Bu(k), \quad y = Cx(k), \\ x(0) = (0,\ldots,0)^T, \tag{26}$$

with A_p as shown in (23) and the standard forms of B and C as shown in (14), is conducted. In this case $N = 4, L = 2, M = 2$. Here, (26) is schematically represented in Fig. 1. This model is observable, controllable and identifiable, which means p_{d} is uniquely determined provided the impulse responses. The impulse inputs (3) are applied to (26) with $(p_1,p_2,p_3) = (0.2,-0.5,-0.9)$, and their responses are generated by the model. Using these responses, p_{d} is estimated using the proposed method.

As the first step, we obtain the minimal realizations of the impulse responses for (26) using black-box identification methods. To do so, we apply ERA [8], which is a method for identifying a minimal realization of impulse responses for discrete linear time-invariant systems. We briefly explain the procedure of the ERA in the following. In general, to obtain a minimal realizations using ERA, a matrix called the Hankel matrix is first constructed using the given impulse responses. Using (7), the Hankel matrix used in the ERA is represented as follows:

$$\begin{pmatrix} Y(0) & Y(1) & \cdots & Y(K) \\ Y(1) & Y(2) & \cdots & Y(K+1) \\ \vdots & \vdots & \ddots & \vdots \\ Y(K) & Y(K+1) & \cdots & Y(2K) \end{pmatrix}$$

where K is a positive integer. The singular value decomposition is then applied to the matrix, $H = USV^*$, where $U,S,V \in \mathbb{R}^{(L\times(K+1))\times(M\times(K+1))}$; U and V are unitary matrices; and S is a diagonal matrix whose diagonal elements are the singular values of H. Because we need to obtain N-dimensional minimal realizations, H is approximated by regarding the smallest $L \times (K+1) - N$ singular values as zeros: $H \simeq \tilde{U}\tilde{S}\tilde{V}^*$, where $\tilde{U}\tilde{S}\tilde{V} \in \mathbb{R}^{N \times N}$. In general, H can be decomposed as $H \simeq (\tilde{U}\tilde{S}^{1/2})(\tilde{S}^{1/2}\tilde{V}) = M_{\mathrm{o}}M_{\mathrm{c}}$, where $M_{\mathrm{o}}, M_{\mathrm{c}}$ are the approximated observability and controllability matrices. The first L rows of M_{c} and the first M columns of M_{o} can be regarded as estimated $\tilde{C}$ and $\tilde{B}$, respectively. Then, $\tilde{A}$ is estimated using the shifting property of the Hankel matrix.

In our situation, we let $K = 49$, which means a 100×100 Hankel matrix is constructed. The resulting minimal realization is as follows:

$$\begin{aligned}
\tilde{A} &= \begin{pmatrix} 0.2727 & -0.6099 & 0.4397 & -0.0319 \\ -0.2013 & 0.4615 & 0.5645 & -0.0429 \\ -0.2312 & -0.4998 & -0.4292 & -0.0465 \\ 0.0245 & 0.1612 & -0.2761 & -0.0050 \end{pmatrix}, \\
\tilde{B} &= \begin{pmatrix} 0.8730 & -0.2288 & 0.0333 & -0.1961 \\ 0.0653 & -0.6318 & -0.5308 & -0.0075 \end{pmatrix}^T, \\
\tilde{C} &= \begin{pmatrix} -1.0841 & -0.2070 & 0.1131 & -0.0123 \\ 0.0516 & 0.0802 & -0.0844 & -0.3376 \end{pmatrix}.
\end{aligned} \tag{27}$$

Next, we estimate $T = T_t, T_h$, where T_t and T_h satisfies (15) and (16). This estimation is formulated as (20) and (21). Finding the minimizers of (20) and (21) requires solving a nonlinear least-squares problem. We used the Levenberg–Marquardt algorithm as a solver with the initial estimates of

$$T_{t(3,1)}, T_{t(3,2)}, T_{t(4,1)}, T_{h(3,2)}, T_{h(3,4)}, T_{h(4,2)}, T_{h(4,4)}$$

randomly chosen within $[0, 1]$. The estimations are conducted 10 times from different initial estimates. The residual in the estimation is defined as (25). The last term of (25) expresses a hidden constraint on the similarity transformation T.

The results are as follows. The logarithms of the residuals were

$$-45.872, -43.618, -64.415, -69.683, -0.091089, -56.622, -0.091089, -57.955, -69.442, -47.817.$$

The fifth and seventh residuals were relatively high compared to the others, which implies that those estimates failed. These results suggests that a poor initial guess of T may result in a failed estimation. However, even for randomly chosen initial guesses, the estimation tends to be a success. By removing such results and renumbering others in order, the means of the estimated parameters are computed as follows:

$$\left(\frac{1}{7}\sum_{j=1}^{7} q_{1,j}, \frac{1}{7}\sum_{j=1}^{7} q_{2,j}, \frac{1}{7}\sum_{j=1}^{7} q_{3,j} \right) = (0.20000, -0.50000, -0.90000) \tag{28}$$

where $q_{i,j} (i = 1, \ldots, 3, j = 1, \ldots, 7)$ denotes the jth estimated value of p_i. The average of the relative errors of estimated parameters are as follows:

$$\frac{1}{3}\sum_{i=1}^{3} \left(\frac{1}{7}\sum_{j=1}^{7} \left(\frac{p_{i,j} - q_{i,j}}{p_{i,j}} \right)^2 \right) = 1.4588 \times 10^{-18}, \tag{29}$$

which shows that the estimations are extremely precise on average. The following is one of the estimated transformations:

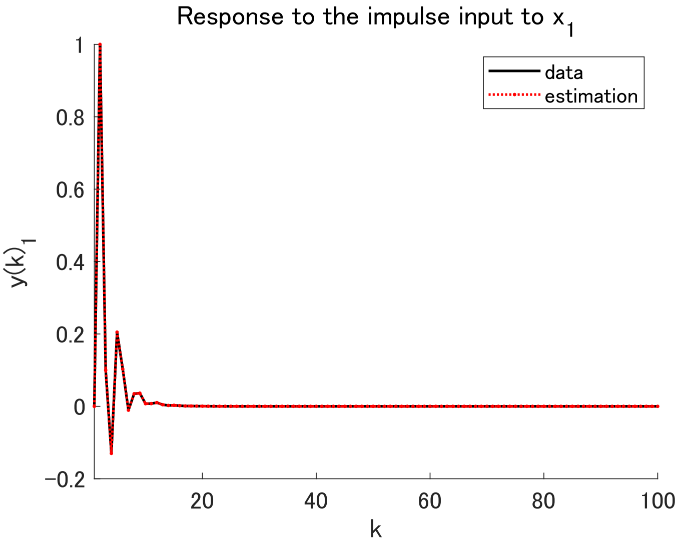

Fig. 2. A comparison of impulse response y_1 given impulse input to x_1. The data (shown by the solid line) and the estimated response (shown by the dotted line) were obtained by applying $T = T_h T_t$, where T_h and T_t are as shown in (30) and (31), to a minimal realization (27).

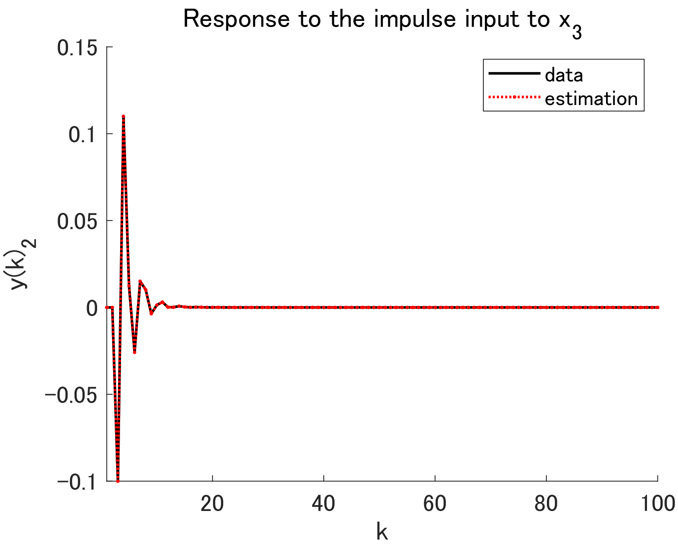

Fig. 3. A comparison of impulse response y_2 given the impulse input to x_3. The data (shown in the solid line) and the estimated response (shown in the dotted line) were obtained by applying $T = T_h T_t$, where T_h and T_t are as shown in (30) and (31), to a minimal realization (27).

$$T_t = \begin{pmatrix} -1.0841 & -0.20695 & 0.11314 & -0.012328 \\ 0.051574 & 0.080249 & -0.084413 & -0.33755 \\ 0.11607 & -0.57011 & -1.1903 & -0.053918 \\ 0.32813 & -1.3803 & 1.6773 & 0.43479 \end{pmatrix}, \tag{30}$$

$$T_h = \begin{pmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.00082823 & 1.0000 & 0.093483 \\ 0.0000 & 0.39921 & 0.0000 & 0.3005 \end{pmatrix}. \tag{31}$$

Figures 2 and 3 show examples of comparisons of the data and the estimated impulse responses associated to (30) and (31). Although it is obvious from the high accuracy of the estimations shown in (29), the impulse responses generated by (26) provided estimated parameters show that a good fit with the impulse responses of (26) provided that true parameters are applied.

4. Concluding Remarks

In this paper, a parameter estimation method for linear time-invariant systems from the given impulse responses through a realization method including techniques of exhaustive modelling is proposed. To estimate the similarity transformations from a system with B, C in a standard form to a system that has pre-defined structures, an appropriate definition of the objective function to be minimized is required. To specify the function, we proposed the application of algebraic elimination techniques. As such, because our method is based on algebra, polynomial parametrizations of A_p are allowed, which has not been achieved through existing methods. Although we restrict the systems to be realized to those with B, C in the standard form, extensions for other forms of B, C are worth investigating further. Investigations into the computational aspects along with the integration of our method into the null-space method proposed in [4] is also considered as a future study.

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