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Original articles

A reaction–diffusion Susceptible–Vaccinated–Infected–Recovered model in a spatially heterogeneous environment with Dirichlet boundary condition

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Abstract

In this paper, we study a Susceptible–Vaccinated–Infected–Recovered (SVIR) epidemic model in a spatially heterogeneous environment under the Dirichlet boundary condition. We define the basic reproduction number $\Re_0$ by the spectral radius of the next generation operator, and show that it is a threshold parameter. The disease extinction and persistence in the case of a bounded domain are considered. More precisely, we show that the disease-free equilibrium is globally asymptotically stable if $\Re_0 < 1$; the system is uniformly persistent and an endemic equilibrium exists if $\Re_0 > 1$. To verify our theoretical results, we perform some numerical simulations, using the Fredholm discretization method to identify $\Re_0$.

Keywords: SVIR epidemic model; Diffusion; Basic reproduction number; Dirichlet boundary condition; Vaccination

1. Introduction

Since the earliest work on Susceptible–Infected–Recovered (SIR) epidemic model [18], there have been a lot of works on this subject, and all without exception derived the important role of mathematical modeling in investigating the mechanism of infectious disease. The Susceptible–Vaccinated–Infected–Recovered (SVIR) epidemic model stems from classical SIR epidemic model, where a vaccinated compartment is included. In recent years, the explorations of such models have received much attentions from researchers, which provide better understanding of the role of vaccination on disease control. These models have been studied in various model formulations in terms of ordinary differential equations, delay differential equations, multi-group models and partial differential equations.

Vaccination is especially important for people at higher risk of disease. For different infectious diseases, the vaccination strategies are allowed to vary from one to one. For example, the vaccination process for hepatitis B (HBV) is divided into several stages, and there is a fixed period between each doses [26]. Another example is the RTS,S/AS01 malaria vaccine, such vaccine is devoted against the most dangerous species of the malaria

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parasite called *Plasmodium falciparum*. Clinical trial shows that the efficacy of RTS,S/AS01 is decreasing since vaccination [30], and children should be targeted for special treatment as who do not receive booster doses has a higher risk of severe malaria [31]. It should be noted that susceptible individual needs time to acquire immunity after vaccination. The vaccinees who contact with an infected person are still considered to be at risk of infection before obtaining immunity.

At time t, let $S(t)$, $V(t)$, $I(t)$ and $R(t)$ be respectively the densities of susceptible, vaccinated, infected and removed individuals. The standard ODEs SVIR model studied in [22] takes the following form:

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - \beta_1 SI - \alpha S - \mu S, \\ \dfrac{dV}{dt} = \alpha S - \beta_2 VI - (\gamma + \mu)V, \\ \dfrac{dI}{dt} = \beta_1 SI + \beta_2 VI - \delta I - \mu I, \\ \dfrac{dR}{dt} = \gamma V + \delta I - \mu R. \end{cases} \tag{1.1}$$

Here α is the vaccination rate standing for the continuous vaccination strategy. The parameter Λ denotes the birth rate of populations. μ represents the mortality rate of populations. The parameter β_1 (resp. β_2) represents the transmission rate between susceptible and infected individuals (resp. between vaccinated and infected individuals). The parameters δ and γ denote respectively the recovery rate of infected and vaccinated individuals. In (1.1), the basic reproduction number (BRN) is identified as

$$\Re_0^{(1.1)} = \frac{\beta_1 \Lambda}{(\alpha + \mu)(\delta + \mu)} + \frac{\beta_2 \alpha \Lambda}{(\gamma + \mu)(\alpha + \mu)(\delta + \mu)}.$$

Based on $\Re_0^{(1.1)}$, the authors explored the sharp threshold dynamics of (1.1), i.e., the sign of $\Re_0^{(1.1)} - 1$ determines the global asymptotic stability of each equilibrium, which involves the disease extinction and persistence. Moreover, the authors also studied the effect of vaccination on disease control with continuous and pulse vaccination strategy, respectively, and observed that vaccination plays an important role in decreasing the $\Re_0^{(1.1)}$. It is also noted in [22] that if $\beta_2 = 0$ or $\gamma \to \infty$ (i.e., before obtaining immunity for vaccinees, neglecting the risk of infection or neglecting the time to obtain immunity), one leads to over-evaluating the vaccination effect.

Hereafter, in [19], the model (1.1) was extended to a multi-group formulation and the stability results for it were obtained by using a method of combining graph-theoretic approach with Lyapunov functional. One can also see [36,43] on the complete analysis for multi-group SVIR models with distributed delay. In [13], Henshaw and McCluskey proposed a SVIR model with immigration of individuals and general incidence, such model has no disease-free equilibrium since the introduction of immigration. By applying the Brouwer fixed point theorem, the Poincaré-Hopf theorem and the Lyapunov direct method, the authors proved that the model possesses a unique globally stable endemic equilibrium. Further, a number of studies [10,35,39] focused on the SVIR epidemic model with age-structure and investigated their global dynamics.

Recently, there have been growing interests in PDEs formulations of epidemic models with reaction–diffusion equations, as exploration of such models can reflect some reasonable factors in disease transmission, which also can potentially be applied to biological models such as ecology [40] and epidemiology [2,14,25]. Due to the complexity of models (specially in PDEs formulations), there have been mathematical challenges in analyzing the model, and even in practical applications. For models involving space variable, one point comes first is the spatial domain that population habitats, which should be subjected to some suitable boundary conditions (for example, Neumann boundary condition, Dirichlet boundary condition, etc.). In general, the domain where individuals live is spatially bounded. The reaction–diffusion disease models greatly have enriched the investigation of the disease transmission dynamics. On the one hand, individuals are allowed to randomly disperse on the habitats. On the other hand, various heterogeneities for disease transmission can be modeled. Some abstract results in dynamical systems theory are applied in analyzing the model dynamics, and some main mathematical concerns, such as existence of traveling wave [3], nonlocal infection [11,23], asymptotical profiles and effect of environmental heterogeneity [2,8,38,41,42] have been well investigated.

To address the effect of diffusion and the vaccination, Xu et al. [44] proposed the following SVIR model with diffusion:

$$\begin{cases} S_t - d_1 \Delta S = \ \Lambda - \beta_1 S f(I) - \alpha S - \mu S, \\ V_t - d_2 \Delta V = \ \alpha S - \beta_2 V f(I) - (\gamma + \mu) V, \\ I_t - d_3 \Delta I = \ \beta_1 S f(I) + \beta_2 V f(I) - \delta I - \mu I, \\ R_t - d_4 \Delta R = \ \gamma V + \delta I - \mu R, \end{cases} \tag{1.2}$$

where $S(x,t)$, $V(x,t)$, $I(x,t)$ and $R(x,t)$, $x \in \Omega \subset \mathbb{R}^n$, $t > 0$ represent the densities of susceptible, vaccinated, infected, and recovered individuals, respectively. $f(I) = I/(1+I)$, and d_i $(i = 1, 2, 3, 4)$ stand for the diffusion rates of individuals in each class. The epidemiological meanings of other coefficients are the same as in model (1.1). The global stability of the model (1.2) on bounded spatial habitat and the existence of traveling waves on unbounded spatial habitat were studied in [44]. In the recent works [46,47], the traveling waves for two different nonlocal diffusive SVIR models were also discussed.

Considering the spatial heterogeneity (i.e. space-dependent parameters) on a bounded domain Ω with smooth boundary $\partial\Omega$, Zhang et al. [45] derived and analyzed the following model, for $(x,t) \in \Omega \times (0,\infty)$,

$$\begin{cases} S_t - \nabla \cdot [d(x)\nabla S] = \Lambda(x) - \beta_1(x) S I - \alpha(x) S - \mu(x) S, \\ V_t - \nabla \cdot [d(x)\nabla V] = \alpha(x) S - \beta_2(x) V I - [\gamma(x) + \mu(x)] V, \\ I_t - \nabla \cdot [d(x)\nabla I] = \beta_1(x) S I + \beta_2(x) V I - [\mu(x) + \delta(x)] I, \\ R_t - \nabla \cdot [d(x)\nabla R] = \gamma(x) V + \delta(x) I - \mu(x) R, \end{cases} \tag{1.3}$$

with initial and boundary conditions,

$$S(\cdot,0) = S_0(\cdot) \geq 0, \quad V(\cdot,0) = V_0(\cdot) \geq 0, \quad I(\cdot,0) = I_0(\cdot) \geq 0, \quad R(\cdot,0) = R_0(\cdot) \geq 0. \tag{1.4}$$

$$[d(\cdot)\nabla S] \cdot \mathbf{n} = [d(\cdot)\nabla V] \cdot \mathbf{n} = [d(\cdot)\nabla I] \cdot \mathbf{n} = [d(\cdot)\nabla R] \cdot \mathbf{n} = 0, \quad (x,t) \in \partial\Omega \times (0,\infty). \tag{1.5}$$

Here $d(x)$ is a space-dependent function for diffusion rate of population individuals. The epidemiological meanings of other coefficients are the same as in model (1.1), but they depend on space variable x. The global dynamics of model and the effects of spatial heterogeneity were investigated in [45]. Their results also revealed that the fast and slow diffusion rates strongly affect the spatial distribution of disease.

Our current work is also inspired by a series of works on diffusive Susceptible–Infective–Susceptible (SIS) epidemic models. Allen et al. [2] derived and analyzed a diffusive SIS model without birth and death effects for the population, and the asymptotic behavior was investigated when the dispersal rate of susceptible individuals approaches to zero. In [8,41,42], by using mass action incidence instead of standard incidence, the global dynamics and asymptotical profiles of diffusive SIS epidemic models were investigated. By introducing exposed class into model in [2], Song et al. [29] proposed an SEIRS model in heterogeneous environment. They build up the connection between $\Re_0$ and the principal eigenvalue of associated elliptic eigenvalue problem. Further, the asymptotic profile was investigated as the dispersal rate of the susceptible individuals approaches to zero. For more recent epidemic models in heterogeneous environment, we refer readers to the papers [33,34,37] and the references therein.

Note that the model (1.3) possesses the same diffusion rate, and the above mentioned works were all subject to the homogeneous Neumann boundary condition. On the other hand, much less studies have been paid on the diffusive SVIR model subject to the homogeneous Dirichlet boundary condition in heterogeneous environment. The Dirichlet boundary condition is usually taken for considering an environment which boundary is hostile for the survival of individuals (see, e.g., [15]). Mathematically, the Dirichlet boundary condition would induce a difficulty in the analysis of the global asymptotic behavior because typical Lyapunov functionals for epidemic models include terms such as $\ln(I/I^*)$ (see, e.g., [21]), which could not be defined on the boundary in the case of the homogeneous Dirichlet boundary condition, where I^* denotes the infected population in the endemic equilibrium. This difficulty does not arise in the case of the homogeneous Neumann boundary condition [21]. Moreover, it was shown in [7,20] that the BRN can be changed and the stability of the system can be affected by the shape of the boundary in the case of the Dirichlet boundary condition, in contrast to the case of the homogeneous Neumann boundary condition in which BRN is unchanged by the shape of the boundary. Therefore, the study of epidemic models under the Dirichlet boundary condition would include not only application but also mathematical interests. Considering the population individuals disperse at non-uniformness of diffusion rates, in this paper, we shall study the following diffusive SVIR model on $\Omega \times (0, \infty)$,

$$\begin{cases} S_t - d_1\Delta S = \Lambda(x) - \beta_1(x)SI - \alpha(x)S - \mu(x)S, \\ V_t - d_2\Delta V = \alpha(x)S - \beta_2(x)VI - [\mu(x) + \gamma(x)]V, \\ I_t - d_3\Delta I = \beta_1(x)SI + \beta_2(x)VI - [\mu(x) + \delta(x)]I, \\ R_t - d_4\Delta R = \gamma(x)V + \delta(x)I - \mu(x)R, \end{cases} \tag{1.6}$$

with initial condition and homogeneous Dirichlet boundary condition:

$$\begin{cases} S(\cdot,0) = \phi_1(\cdot), \quad V(\cdot,0) = \phi_2(\cdot), \quad I(\cdot,0) = \phi_3(\cdot), \quad R(\cdot,0) = \phi_4(\cdot), & x \in \Omega, \\ S(\cdot,t) = V(\cdot,t) = I(\cdot,t) = R(\cdot,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \end{cases} \tag{1.7}$$

where $\Omega \subset \mathbb{R}^n$ ($n \in \{1, 2, \ldots\}$) is a bounded domain with smooth boundary. For each coefficient, we assume:

(A1) $d_i > 0$ and $\phi_i \geq 0$, $i = 1, 2, 3, 4$.

(A2) $\Lambda(\cdot)$, $\alpha(\cdot) \in C^2(\Omega)$ are strictly positive and uniformly bounded on Ω. Moreover, $\Lambda(x) = \alpha(x) = 0$ for all $x \in \partial\Omega$.

(A3) $\mu(\cdot)$, $\beta_1(\cdot)$, $\beta_2(\cdot)$, $\gamma(\cdot)$ and $\delta(\cdot)$ are strictly positive and twice continuously differentiable on $\overline{\Omega}$.

It should be pointed out that spatially heterogeneous SVIR model with homogeneous Neumann boundary condition was studied in [45,48], the long-term dynamic behavior of each model was obtained based on BRN. In contrast, we focus on the model subject to (1.7). Since $R(\cdot,t)$ does not appear in the first three equations in (1.6), we can omit it and consider the following reduced system: for $(x,t) \in \Omega \times (0,\infty)$,

$$\begin{cases} S_t - d_1\Delta S = \Lambda(\cdot) - \beta_1(\cdot)SI - \alpha(\cdot)S - \mu(\cdot)S, \\ V_t - d_2\Delta V = \alpha(\cdot)S - \beta_2(\cdot)VI - [\mu(\cdot) + \gamma(\cdot)]V, \\ I_t - d_3\Delta I = \beta_1(\cdot)SI + \beta_2(\cdot)VI - [\mu(\cdot) + \delta(\cdot)]I, \end{cases} \tag{1.8}$$

with

$$\begin{cases} S(\cdot,0) = \phi_1(\cdot), \quad V(\cdot,0) = \phi_2(\cdot), \quad I(\cdot,0) = \phi_3(\cdot), & x \in \Omega, \\ S(\cdot,t) = V(\cdot,t) = I(\cdot,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty). \end{cases} \tag{1.9}$$

The rest of this paper is organized as follows. In Section 2, we show the well-posedness of the problem (1.8). In Section 3, we identify the BRN, $\Re_0$, and prove the extinction of disease when $\Re_0 < 1$. In Section 4, we show the persistence of disease and the existence of endemic equilibrium when $\Re_0 > 1$. Some numerical simulations are performed to validate theoretic assertions in Section 5.

2. Well-posedness of the problem

Before going into details, we first present some notations:

- Let $\mathbb{X}_0 := \left\{\varphi \in C(\overline{\Omega}, \mathbb{R}) : \varphi(x) = 0, \ \forall x \in \partial\Omega\right\}$, equipped with the supremum norm $\|\varphi\|_\infty := \sup_{x\in\Omega} |\varphi(x)|$, $\varphi \in C(\overline{\Omega}, \mathbb{R})$.
- Let $\mathbb{Y}_0 := \mathbb{X}_0 \times \mathbb{X}_0 \times \mathbb{X}_0$, equipped with the norm

$$\|\psi\|_{\mathbb{Y}_0} := \sum_{i=1}^{3} \|\psi_i\|_\infty, \quad \psi = (\psi_1, \psi_2, \psi_3) \in \mathbb{Y}_0.$$

- Let $A_i^0 := d_i\Delta$ ($i = 1, 2, 3$) be the differential operators defined on $Dom(A_i^0)$, where

$$Dom(A_i^0) := \left\{\varphi \in \mathbb{X}_0 \cap C^2(\Omega, \mathbb{R}) : \ A_i^0\varphi \in \mathbb{X}_0\right\}.$$

- Let A_i ($i = 1, 2, 3$) be the closure of A_i^0 in $\mathbb{X}_0$.
- Let $\mathbb{X}_0^+ := \left\{\varphi \in \mathbb{X}_0 : \varphi(x) \geq 0, \ \forall x \in \overline{\Omega}\right\}$ and $\mathbb{Y}_0^+ := \mathbb{X}_0^+ \times \mathbb{X}_0^+ \times \mathbb{X}_0^+$.

For $i = 1, 2, 3$, according to [27, Section 7.1], A_i generates the semigroup $\{T_i(t)\}_{t\geq 0}$, and $u_i(\cdot,t) := [T_i(t)\varphi_i](\cdot)$ is the classical solution of

$$\begin{cases} [u_i]_t = d_i\Delta u_i, & (x,t) \in \Omega \times (0,\infty), \\ u_i(\cdot,0) = \varphi_i(\cdot), & x \in \Omega, \\ u_i(\cdot,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty). \end{cases}$$

Let us define $\{T(t)\}_{t\geq 0} : \mathbb{Y}_0 \to \mathbb{Y}_0$, $A : Dom(A) \subset \mathbb{Y}_0 \to \mathbb{Y}_0$ and $F : \overline{\Omega} \times \mathbb{R}^3_+ \to \mathbb{R}^3$ by

$$\{T(t)\}_{t\geq 0} := \left\{\prod_{i=1}^{3} T_i(t)_{t\geq 0}\right\}, \quad A := \prod_{i=1}^{3} A_i, \quad F := \prod_{i=1}^{3} F_i,$$
$$Dom(A) := \prod_{i=1}^{3} Dom(A_i), \quad Dom(A_i) := \left\{\varphi \in \mathbb{X}_0 : \lim_{t\to+0} \frac{[T_i(t) - I_d]\varphi}{t} \text{ exists}\right\}, \quad i = 1, 2, 3,$$

where I_d is the identity operator and

$$\begin{cases} F_1(\cdot, \rho) := \Lambda(\cdot) - \beta_1(\cdot)\rho_1\rho_3 - [\mu(\cdot) + \alpha(\cdot)]\rho_1, \\ F_2(\cdot, \rho) := \alpha(\cdot)\rho_1 - \beta_2(\cdot)\rho_2\rho_3 - [\mu(\cdot) + \gamma(\cdot)]\rho_2, \\ F_3(\cdot, \rho) := \beta_1(\cdot)\rho_1\rho_3 + \beta_2(\cdot)\rho_2\rho_3 - [\mu(\cdot) + \delta(\cdot)]\rho_3 \end{cases} \quad x \in \overline{\Omega}, \quad \rho = (\rho_1, \rho_2, \rho_3) \in \mathbb{R}^3_+. \tag{2.1}$$

We can then rewrite the problem (1.8)–(1.9) as:

$$u_t = Au + F(\cdot, u), \quad t > 0, \quad u(0) = u_0, \tag{2.2}$$

where $u := (S, V, I)$ and $u_0 := (\phi_1, \phi_2, \phi_3)$. The following proposition could be obtained by applying Theorem 3.1 and Corollary 3.2 in [27, Chapter 7].

Proposition 2.1. *If $u_0 \in \mathbb{Y}_0^+$, then (2.2) admits a unique positive solution $u(t) \in \mathbb{Y}_0^+$, $\forall t > 0$.*

Proof. From (2.1) and assumption (A2), we know that $x \in \overline{\Omega}$, and $\rho \in \mathbb{R}^3_+$, $F_i(x, \rho) \geq 0$ such that $\rho_i = 0$ for $i = 1, 2, 3$. Hence, by Theorem 3.1 and Corollary 3.2 in [27, Chapter 7], we can find a $\hat{t} = \hat{t}(u_0) \in (0, +\infty]$ such that (2.2) possesses a unique positive solution $u(t) \in \mathbb{Y}_0^+$, $t \in [0, \hat{t})$ with $\lim_{t\to\hat{t}-0} \|u(t)\|_{\mathbb{Y}_0} = +\infty$ if $\hat{t} < +\infty$.

We next prove that $\hat{t} = +\infty$. For $x \in \overline{\Omega}$, let $r_1(\cdot) := \mu(\cdot) + \alpha(\cdot)$, $r_2(\cdot) := \mu(\cdot) + \gamma(\cdot)$ and $r_3(\cdot) := \mu(\cdot) + \delta(\cdot)$. For $i = 1, 2, 3$, let $\Gamma_i(t, \cdot, y)$ be the fundamental solution of

$$\begin{cases} \varpi_t = d_i \Delta\varpi - r_i(\cdot)\varpi, & (x, t) \in \Omega \times (0, \infty), \\ \varpi(\cdot, 0) = \varpi_0(\cdot), & x \in \Omega, \\ \varpi(\cdot, t) = 0, & (x, t) \in \partial\Omega \times (0, \infty). \end{cases} \tag{2.3}$$

That is, $\varpi(\cdot, t) = \int_\Omega \Gamma_i(t, x, y)\varpi_0(y)\mathrm{d}y$ satisfies (2.3). By the arguments in [17, Chapter 2], we know that

(G1) $\Gamma_i(t, \cdot, y) \geq 0$, $\forall t > 0$, $\forall x, y \in \overline{\Omega}$.

(G2) $\Gamma_i(t, \cdot, y) > 0$, $\forall x, y \in \Omega$.

(G3) $\int_\Omega \Gamma_i(t, \cdot, y)\mathrm{d}y \leq \mathrm{e}^{-\underline{r}_i t}$, $\forall t > 0$, $\forall x \in \overline{\Omega}$, where $\underline{r}_i := \inf_{x\in\Omega} r_i(\cdot) > 0$.

(G4) $\lim_{t\to+0} \int_\Omega \Gamma_i(t, \cdot, y)\varphi(y)\mathrm{d}y = \varphi(\cdot)$, $\forall \varphi \in \mathbb{X}_0$, $\forall x \in \overline{\Omega}$.

(G5) $\int_\Omega \Gamma_i(t, x, z)\Gamma_i(s, z, y)\mathrm{d}z = \Gamma_i(t + s, x, y)$, $\forall t, s > 0$, $\forall x, y \in \overline{\Omega}$, .

Further, for $(x, t) \in \Omega \times (0, \infty)$, (1.8) can be rewritten as follows.

$$\begin{cases} S_t - d_1\Delta S = \Lambda(\cdot) - \beta_1(\cdot)SI - r_1(\cdot)S, \\ V_t - d_2\Delta V = \alpha(\cdot)S - \beta_2(\cdot)VI - r_2(\cdot)V, \\ I_t - d_3\Delta I = \beta_1(\cdot)SI + \beta_2(\cdot)VI - r_3(\cdot)I. \end{cases}$$

With the help of the fundamental solutions, we obtain, for $x \in \overline{\Omega}$ and $t > 0$,

$$S(\cdot,t) = \int_\Omega \Gamma_1(t,\cdot,y)\phi_1(y)\mathrm{d}y + \int_0^t \int_\Omega \Gamma_1(s,\cdot,y)\left[\Lambda(y) - \beta_1(y)S(y,t-s)I(y,t-s)\right]\mathrm{d}y\mathrm{d}s, \tag{2.4}$$

$$V(\cdot,t) = \int_\Omega \Gamma_2(t,\cdot,y)\phi_2(y)\mathrm{d}y + \int_0^t \int_\Omega \Gamma_2(s,\cdot,y)\left[\alpha(y)S(y,t-s) - \beta_2(y)V(y,t-s)I(y,t-s)\right]\mathrm{d}y\mathrm{d}s, \tag{2.5}$$

and

$$\begin{aligned} I(\cdot,t) = &\int_\Omega \Gamma_3(t,\cdot,y)\phi_3(y)\mathrm{d}y + \int_0^t \int_\Omega \Gamma_3(s,\cdot,y)\left[\beta_1(y)S(y,t-s) + \beta_2(y)V(y,t-s)\right] \\ &\times I(y,t-s)\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{2.6}$$

By the positivity of the solution, (G3) and (2.4),

$$\begin{aligned} S(\cdot,t) \le &\int_\Omega \Gamma_1(t,\cdot,y)\phi_1(y)\mathrm{d}y + \int_0^t \int_\Omega \Gamma_1(s,\cdot,y)\Lambda(y)\mathrm{d}y\mathrm{d}s \\ &< \mathrm{e}^{-\underline{r}_1 t}\|\phi_1\|_\infty + S^0(\cdot), \quad x \in \overline{\Omega}, \quad t > 0, \end{aligned} \tag{2.7}$$

where

$$S^0(\cdot) := \int_0^{+\infty} \int_\Omega \Gamma_1(s,\cdot,y)\Lambda(y)\mathrm{d}y\mathrm{d}s \le \frac{\overline{\Lambda}}{\underline{r}_1} < +\infty, \quad x \in \overline{\Omega}, \qquad \overline{\Lambda} := \sup_{x\in\Omega} \Lambda(\cdot). \tag{2.8}$$

Similarly, from (2.5) and (2.7),

$$\begin{aligned} V(x,t) \le &\int_\Omega \Gamma_2(t,\cdot,y)\phi_2(y)\mathrm{d}y + \int_0^t \int_\Omega \Gamma_2(s,\cdot,y)\alpha(y)\left[\mathrm{e}^{-\underline{r}_1(t-s)}\|\phi_1\|_\infty + S^0(y)\right]\mathrm{d}y\mathrm{d}s \\ &< \mathrm{e}^{-\underline{r}_2 t}\|\phi_2\|_\infty + \frac{\overline{\alpha}}{\underline{r}_2}\left(1 - \mathrm{e}^{-\underline{r}_2 t}\right)\|\phi_1\|_\infty + V^0(\cdot), \quad x \in \overline{\Omega}, \quad t > 0, \end{aligned} \tag{2.9}$$

where $\overline{\alpha} := \sup_{x\in\Omega} \alpha(\cdot)$ and

$$V^0(\cdot) := \int_0^{+\infty} \int_\Omega \Gamma_2(s,\cdot,y)\alpha(y)S^0(y)\mathrm{d}y\mathrm{d}s \le \frac{\overline{\alpha}}{\underline{r}_2}\|S^0\|_\infty < +\infty, \quad x \in \overline{\Omega}. \tag{2.10}$$

Moreover, from (2.6)–(2.9), we obtain

$$I(\cdot,t) \le \mathrm{e}^{-\underline{r}_3 t}\|\phi_3\|_\infty + K\int_0^t \int_\Omega \Gamma_3(t-\sigma,\cdot,y)I(y,\sigma)\mathrm{d}y\mathrm{d}\sigma, \quad x \in \overline{\Omega}, \quad t > 0, \tag{2.11}$$

where

$$K := \overline{\beta}_1(\|\phi_1\|_\infty + \|S^0\|_\infty) + \overline{\beta}_2\left(\|\phi_2\|_\infty + \frac{\overline{\alpha}}{\underline{r}_2}\|\phi_1\|_\infty + \|V^0\|_\infty\right), \quad \overline{\beta}_i := \sup_{x\in\Omega} \beta_i(\cdot), \quad i = 1, 2.$$

By (2.11), we obtain

$$\|I(\cdot,t)\|_\infty \le \mathrm{e}^{-\underline{r}_3 t}\|\phi_3\|_\infty + K\int_0^t \mathrm{e}^{-\underline{r}_3(t-\sigma)}\|I(\cdot,\sigma)\|_\infty\mathrm{d}\sigma, \quad t > 0.$$

Applying the Gronwall's inequality obtains

$$\|I(\cdot,t)\|_\infty \le \mathrm{e}^{(K-\underline{r}_3)t}\|\phi_3\|_\infty, \quad t > 0. \tag{2.12}$$

By (2.7), (2.9) and (2.12), $\|u(t)\|_{\mathbb{Y}_0} < +\infty$, $\forall t > 0$. Hence, $\hat{t} = +\infty$, and the unique positive solution $u(t) \in \mathbb{Y}_0^+$ exists. This proves Proposition 2.1. □

From Proposition 2.1, for any $u_0 \in \mathbb{Y}_0^+$, let us define $\Phi : \mathbb{R}_+ \times \mathbb{Y}_0^+ \to \mathbb{Y}_0^+$ by

$$\Phi(t, u_0) = u(t), \ \ t \ge 0,$$

which is the solution semiflow. We now prove the dissipativity of the semiflow Φ.

Proposition 2.2. *For all $u_0 \in \mathbb{Y}_0^+$, there exists a $K_0 > 0$ such that* $\limsup_{t\to+\infty} \|\Phi(t,u_0)\|_{\mathbb{Y}_0} < K_0$.

Proof. We can obtain the following relationship by using (2.7) and (2.9),

$$\limsup_{t\to+\infty} S(\cdot,t) \le S^0(\cdot) \quad \text{and} \quad \limsup_{t\to+\infty} V(\cdot,t) \le V^0(\cdot), \qquad x \in \overline{\Omega}, \tag{2.13}$$

respectively. Thus, we see that $\limsup_{t\to+\infty} \|S(\cdot,t)\|_\infty \le \|S^0\|_\infty =: K_1$ and $\limsup_{t\to+\infty} \|V(\cdot,t)\|_\infty \le \|V^0\|_\infty = K_2$. It remains to show that $\limsup_{t\to+\infty} \|I(\cdot,t)\|_\infty \le K_3$ for some $K_3 > 0$. In what follows, we apply the method of Alikakos [1] (see also Wu and Zou [42, Proof of Lemma 2.4]) to obtain the L^∞-estimate from an L^1-estimate. For $p \ge 1$, denote by $\|\cdot\|_p$ the L^p-norm, i.e., $\|\varphi\|_p := (\int_\Omega |\varphi(\cdot)|^p \mathrm{d}x)^{1/p}$, $\forall \varphi \in L^p(\Omega,\mathbb{R})$.

We first claim that $I(\cdot,t)$ satisfies L^1-estimate, that is, there exists a $M_1 > 0$ such that

$$\limsup_{t\to+\infty} \|I(\cdot,t)\|_1 \le M_1. \tag{2.14}$$

In fact, integrating and adding all equations of (1.8) directly gives,

$$\begin{aligned}\frac{\partial}{\partial t}\int_\Omega [S(\cdot,t)+V(\cdot,t)+I(\cdot,t)]\,\mathrm{d}x =& d_1 \int_{\partial\Omega} \frac{\partial S(\cdot,t)}{\partial \mathbf{n}}\mathrm{d}A + d_2 \int_{\partial\Omega} \frac{\partial V(\cdot,t)}{\partial \mathbf{n}}\mathrm{d}A + d_3 \int_{\partial\Omega} \frac{\partial I(\cdot,t)}{\partial \mathbf{n}}\mathrm{d}A \\ &+ \int_\Omega \Lambda(x)\mathrm{d}x \\ &- \int_\Omega \mu(\cdot)[S(\cdot,t)+V(\cdot,t)+I(\cdot,t)]\mathrm{d}x \\ &- \int_\Omega [\gamma(\cdot)V(\cdot,t)+\delta(\cdot)I(\cdot,t)]\mathrm{d}x,\end{aligned}$$

where $\mathbf{n}$ is the outward unit normal vector and $\mathrm{d}A$ is the surface element on $\partial\Omega$. By [27, Corollary 2.3 in Chapter 7], we see that $\partial S/\partial \mathbf{n}$, $\partial V/\partial \mathbf{n}$ and $\partial I/\partial \mathbf{n}$ are non-positive on $\partial\Omega \times (0,+\infty)$. Hence,

$$\frac{\partial}{\partial t}\int_\Omega [S(\cdot,t)+V(\cdot,t)+I(\cdot,t)]\,\mathrm{d}x \le \|\Lambda\|_1 - \underline{\mu}\int_\Omega [S(\cdot,t)+V(\cdot,t)+I(\cdot,t)]\mathrm{d}x, \qquad t>0,$$

where $\underline{\mu} := \inf_{x\in\Omega} \mu(x) > 0$. One has

$$\limsup_{t\to+\infty} (\|S(\cdot,t)\|_1 + \|V(\cdot,t)\|_1 + \|I(\cdot,t)\|_1) \le \frac{\|\Lambda\|_1}{\underline{\mu}}.$$

Hence, (2.14) holds for $M_1 := \|\Lambda\|_1/\underline{\mu}$.

We next claim that, for any $p \in \{1,2,\ldots\}$, there exists $M_p > 0$ such that

$$\limsup_{t\to+\infty} \|I(\cdot,t)\|_p \le M_p \tag{2.15}$$

for all $u_0 \in \mathbb{Y}_0^+$. By (2.14), we see that (2.15) is true for $p=1$. Suppose that (2.15) is true for some $p = 2^{k-1}$, $k \in \{1,2,\ldots\}$. By multiplying I^{2^k-1} the third equation of (1.8) and then integrating over Ω,

$$\begin{aligned}\frac{1}{2^k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega I^{2^k}\mathrm{d}x =& d_3\left(\int_{\partial\Omega} I^{2^k-1}\frac{\partial I}{\partial \mathbf{n}}\mathrm{d}A - \int_\Omega \nabla I\cdot\nabla I^{2^k-1}\mathrm{d}x\right) + \int_\Omega [\beta_1(\cdot)S+\beta_2(\cdot)V]\,I^{2^k}\mathrm{d}x \\ &- \int_\Omega [\mu(\cdot)+\delta(\cdot)]\,I^{2^k}\mathrm{d}x \\ \le& -d_3\int_\Omega \nabla I\cdot\nabla I^{2^k-1}\mathrm{d}x + \int_\Omega [\beta_1(\cdot)S+\beta_2(\cdot)V]\,I^{2^k}\mathrm{d}x - \int_\Omega [\mu(\cdot)+\delta(\cdot)]\,I^{2^k}\mathrm{d}x, \quad t>0,\end{aligned}$$

where note that $\partial I/\partial \mathbf{n} \le 0$ on $\partial\Omega\times(0,+\infty)$. By (2.13), we obtain

$$\frac{1}{2^k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega I^{2^k}\mathrm{d}x \le -d_3\int_\Omega \nabla I\cdot\nabla I^{2^k-1}\mathrm{d}x + (\overline{\beta}_1+\overline{\beta}_2)(K_1+K_2)\int_\Omega I^{2^k}\mathrm{d}x - (\underline{\mu}+\underline{\delta})\int_\Omega I^{2^k}\mathrm{d}x, \quad t>t_0,$$

where $\underline{\delta} := \inf_{x\in\Omega} \delta(x) > 0$. Since

$$
\begin{aligned}
-d_3 \int_\Omega \nabla I \cdot \nabla I^{2^k-1} \mathrm{d}x =& - d_3(2^k-1) \int_\Omega I^{2^k-2} (\nabla I \cdot \nabla I)\, \mathrm{d}x \\
=& - d_3(2^k-1) \int_\Omega \left(I^{2^{k-1}-1} \nabla I \cdot I^{2^{k-1}-1} \nabla I \right) \mathrm{d}x \\
=& - d_3 \frac{2^k-1}{2^{2k-2}} \int_\Omega |\nabla I^{2^{k-1}}|^2 \mathrm{d}x,
\end{aligned}
$$

we obtain that

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega I^{2^k} \mathrm{d}x \le -D_k \int_\Omega |\nabla I^{2^{k-1}}|^2 \mathrm{d}x + E_k \int_\Omega I^{2^k} \mathrm{d}x - 2^k(\underline{\mu} + \underline{\delta}) \int_\Omega I^{2^k} \mathrm{d}x, \quad t > t_0, \tag{2.16}
$$

where

$$
D_k = d_3 \frac{2^k-1}{2^{k-2}}, \quad E_k = 2^k(\overline{\beta}_1 + \overline{\beta}_2)(K_1 + K_2).
$$

Let $\varepsilon := \min(D_k/(2E_k), 1/2)$ and $\xi_k := I^{2^{k-1}}$. It then follows from the Nirenberg–Gagliardo interpolation inequality and the Young inequality as in [1, Proof of Theorem 3.1] that there exists $C > 0$ such that

$$
\|\xi_k\|_2^2 \le \varepsilon \|\xi_k\|_{1,2}^2 + C\varepsilon^{-\frac{n}{2}} \|\xi\|_1^2,
$$

where $\|\cdot\|_{1,2}$ denotes the Sobolev norm, that is, $\|\varphi\|_{1,2} := (\|\varphi\|_2^2 + \|\nabla\varphi\|_2^2)^{1/2}$, $\forall \varphi \in W^{1,2}(\Omega, \mathbb{R})$. We then have that

$$
\|\xi_k\|_2^2 \le 2(1-\varepsilon)\|\xi_k\|_2^2 \le 2\left(\varepsilon \|\nabla\xi_k\|_2^2 + C\varepsilon^{-\frac{n}{2}} \|\xi_k\|_1^2\right) \le \frac{D_k}{E_k} \|\nabla \xi_k\|_2^2 + C_\varepsilon \|\xi_k\|_1^2,
$$

where $C_\varepsilon := 2C\varepsilon^{-n/2}$. Hence, we have

$$
\begin{aligned}
-D_k \int_\Omega |\nabla I^{2^{k-1}}|^2 \mathrm{d}x + E_k \int_\Omega I^{2^k} \mathrm{d}x =& - D_k \|\nabla \xi_k\|_2^2 + E_k \|\xi_k\|_2^2 \\
&\le E_k C_\varepsilon \|\xi_k\|_1^2 = E_k C_\varepsilon \|I^{2^{k-1}}\|_1^2 = E_k C_\varepsilon \|I\|_{2^{k-1}}^{2^k}, \quad t > t_0.
\end{aligned}
$$

Further from (2.16), one has that

$$
\frac{\mathrm{d}}{\mathrm{d}t} \|I(\cdot,t)\|_{2^k}^{2^k} \le E_k C_\varepsilon \|I(\cdot,t)\|_{2^{k-1}}^{2^k} - 2^k(\underline{\mu} + \underline{\delta}) \|I(\cdot,t)\|_{2^k}^{2^k}, \quad t > t_0.
$$

This implies that

$$
\limsup_{t\to+\infty} \|I(\cdot,t)\|_{2^k}^{2^k} \le \frac{E_k C_\varepsilon M_{2^{k-1}}^{2^k}}{2^k(\underline{\mu} + \underline{\delta})},
$$

and thus,

$$
\limsup_{t\to+\infty} \|I(\cdot,t)\|_{2^k} \le \sqrt[2^k]{\frac{(\overline{\beta}_1 + \overline{\beta}_2)(K_1+K_2)C_\varepsilon}{\underline{\mu} + \underline{\delta}}} M_{2^{k-1}} =: M_{2^k}.
$$

Hence, (2.15) is also true for $p = 2^k$. Thus, according to the continuous embedding $L^q(\Omega, \mathbb{R}) \subset L^p(\Omega, \mathbb{R})$, $q \ge p \ge 1$, (2.15) holds for any $p \in \{1, 2, \ldots\}$.

We finally claim that there exists a $K_3 > 0$ such that

$$
\limsup_{t\to+\infty} \|I(\cdot,t)\|_\infty \le K_3, \tag{2.17}
$$

for any $u_0 \in \mathbb{Y}_0^+$. Let $p > 0$ such that $p > \max(n/2, 1)$ and let $a \in (n/(2p), 1)$. Denote by $A_p := d_3\Delta - [\mu(\cdot) + \delta(\cdot)]$ the linear operator whose domain is $Dom(A_p) := \{\varphi \in W^{2,p}(\Omega,\mathbb{R}) \cap W_0^{1,p}(\Omega,\mathbb{R}) : \varphi(x) = 0,\ \forall x \in \partial\Omega\}$. Clearly, A_p is the infinitesimal generator of an analytic semigroup $\{T_p(t)\}_{t\ge0}$ on $L^p(\Omega,\mathbb{R})$ (see, e.g. [24, Theorem 3.5 in Chapter 7]). Let A_p^a be the fractional power of A_p defined by

$$
A_p^a := (A_p^{-a})^{-1}, \quad \text{where} \quad A_p^{-a} := \frac{\sin \pi a}{\pi} \int_0^{+\infty} t^{-a}(tI_d + A_p)^{-1} \mathrm{d}t.
$$

By [24, Theorem 6.8 in Chapter 2], we see that A_p^a is a closed linear operator with $Dom(A_p^a) = R(A_p^{-a})$ is dense in $L^p(\Omega, \mathbb{R})$. Let X^a be the Banach space obtained by endowing $Dom(A_p^a)$ with the graph norm $\|\cdot\|_a$ of A_p^a, that is, $\|\varphi\|_a := \|\varphi\|_p + \|A_p^a \varphi\|_p$ for $\varphi \in Dom(A_p^a)$. We then see by [24, Theorem 4.3 in Chapter 8] that $X^a \subset C(\overline{\Omega}, \mathbb{R})$ and the imbedding is continuous. Moreover, by [24, Theorem 6.13 in Chapter 2], there exists a $M_a > 0$ such that

$$\|A_p^a T_p(t)\| \le M_a t^{-a}, \ t > 0.$$

By (2.13) and (2.15), there exist $K_4 > 0$ and $t_1 > 0$ such that for any $u_0 \in \mathbb{Y}_0^+$ and $t > t_1$,

$$\|S(\cdot, t)\|_\infty < K_4, \quad \|V(\cdot, t)\|_\infty < K_4 \ \text{ and } \ \|I(\cdot, t)\|_p < K_4.$$

We then have that, for all $t > t_1$,

$$\begin{aligned} \|A_p^a I(\cdot, t)\|_p \le & \|A_p^a T_p(1) I(\cdot, t-1)\|_p + \int_{t_1}^t \|A_p^a T_p(t-s) [\beta_1(\cdot) S(\cdot, s) + \beta_2(\cdot) V(\cdot, s)] I(\cdot, s)\|_p \mathrm{d}s \\ \le & M_a K_4 + M_a(\overline{\beta}_1 + \overline{\beta}_2) K_4^2 \int_{t_1}^t \frac{1}{(t-s)^a} \mathrm{d}s \\ \le & M_a K_4 \left[1 + \frac{(\overline{\beta}_1 + \overline{\beta}_2) K_4}{1-a} \right]. \end{aligned}$$

Hence, according to the continuous embedding $X^a \subset C(\overline{\Omega}, \mathbb{R})$, (2.17) directly follows. This proves Proposition 2.2. □

We directly have the result on global attractor.

Proposition 2.3. *For all $u_0 \in \mathbb{Y}_0^+$, $\Phi(t, u_0)$ possesses a global attractor $\mathscr{A} \subset \mathbb{Y}_0^+$, i.e., $\mathscr{A}$ is compact invariant and attracts all bounded sets in $\mathbb{Y}_0^+$.*

Proof. By [27, Theorem 3.1 (e) in Chapter 7], there exists a $t_0 > 0$ such that $\Phi(t, \cdot)$, $\forall t > t_0$ is compact. Since the dissipativity of Φ was proved in Proposition 2.2, the assertion follows from [12, Theorem 3.4.8]. □

3. Basic reproduction number and extinction of disease

Model (1.8)–(1.9) admits a disease-free equilibrium $E^0 = (S^0(\cdot), V^0(\cdot), 0) \in \mathbb{Y}_0^+ \cap C^2(\Omega, \mathbb{R}^3)$, where S^0, V^0 satisfy

$$\begin{cases} 0 = d_1 \Delta S^0 + \Lambda(\cdot) - [\mu(\cdot) + \alpha(\cdot)] S^0, & x \in \Omega, \\ 0 = d_2 \Delta V^0 + \alpha(\cdot) S^0 - [\mu(\cdot) + \gamma(\cdot)] V^0, & x \in \Omega, \\ S^0(\cdot) = V^0(\cdot) = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

In fact, since $\Gamma_i(t, \cdot, y)$, $i = 1, 2, 3$ satisfies (2.3), we have

$$\begin{aligned} d_1 \Delta S^0(\cdot) = & \int_0^{+\infty} \int_\Omega d_1 \Delta \Gamma_1(s, \cdot, y) \Lambda(y) \mathrm{d}y \mathrm{d}s = \int_0^{+\infty} \int_\Omega \left[\frac{\partial \Gamma_1(s, \cdot, y)}{\partial s} + r_1(\cdot) \Gamma_1(s, \cdot, y) \right] \Lambda(y) \mathrm{d}y \mathrm{d}s \\ = & -\Lambda(\cdot) + [\mu(\cdot) + \alpha(\cdot)] S^0(\cdot), \quad x \in \Omega, \\ d_2 \Delta V^0(\cdot) = & \int_0^{+\infty} \int_\Omega d_2 \Delta \Gamma_2(s, \cdot, y) \alpha(y) S^0(y) \mathrm{d}y \mathrm{d}s = \int_0^{+\infty} \int_\Omega \left[\frac{\partial \Gamma_2(s, \cdot, y)}{\partial s} + r_2(\cdot) \right] \alpha(y) S^0(y) \mathrm{d}y \mathrm{d}s \\ = & -\alpha(\cdot) S^0(\cdot) + [\mu(\cdot) + \gamma(\cdot)] V^0(\cdot), \quad x \in \Omega, \end{aligned}$$

and $S^0(\cdot) = V^0(\cdot) = 0$ for all $x \in \partial\Omega$. Thus, (3.1) holds.

Linearizing system (1.8)–(1.9) around E^0, yields,

$$\begin{cases} I_t = d_3 \Delta I + \beta_1(\cdot) S^0(\cdot) I + \beta_2(\cdot) V^0(\cdot) I - [\mu(\cdot) + \delta(\cdot)] I, & (x, t) \in \Omega \times (0, \infty), \\ I(\cdot, 0) = \phi_3(\cdot), & x \in \Omega, \\ I(\cdot, t) = 0, & (x, t) \in \partial\Omega \times (0, \infty). \end{cases}$$

Solving above equation gives

$$I(\cdot,t)=\int_\Omega \Gamma_3(t,\cdot,y)\phi_3(y)\mathrm{d}y+\int_0^t\int_\Omega \Gamma_3(t-s,\cdot,y)\left[\beta_1(y)S^0(y)+\beta_2(y)V^0(y)\right]I(y,s)\mathrm{d}y\mathrm{d}s,$$
$$x\in\overline{\Omega},\quad t>0.$$

Let $v(\cdot,t):=\left[\beta_1(\cdot)S^0(\cdot)+\beta_2(\cdot)V^0(\cdot)\right]I(\cdot,t)$ be the density of the newly infected individuals, the renewal equation reads as,

$$v(\cdot,t)=g(\cdot,t)+\left[\beta_1(\cdot)S^0(\cdot)+\beta_2(\cdot)V^0(\cdot)\right]\int_0^t\int_\Omega \Gamma_3(s,\cdot,y)v(y,t-s)\mathrm{d}y\mathrm{d}s,\quad (x,t)\in\overline{\Omega}\times(0,\infty),$$

where $g(\cdot,t)=\left[\beta_1(\cdot)S^0(\cdot)+\beta_2(\cdot)V^0(\cdot)\right]\int_\Omega \Gamma_3(t,\cdot,y)\phi_3(y)\mathrm{d}y$. Following the standard procedures in [9], we define the $\Re_0$, by the spectral radius of $\mathscr{L}:\mathbb{X}_0\to\mathbb{X}_0$,

$$\mathscr{L}\varphi(\cdot):=\left[\beta_1(\cdot)S^0(\cdot)+\beta_2(\cdot)V^0(\cdot)\right]\int_0^{+\infty}\int_\Omega \Gamma_3(s,\cdot,y)\varphi(y)\mathrm{d}y\mathrm{d}s,\quad \varphi\in\mathbb{X}_0,\quad x\in\overline{\Omega},\tag{3.2}$$

where $\mathscr{L}$ is called the next generation operator.

Lemma 3.1. *$\Re_0$ is a simple eigenvalue of $\mathscr{L}$ with a strictly positive eigenvector.*

Proof. By (A2), (G2), (2.8) and (2.10), we see that $S^0(\cdot)>0$ and $V^0(\cdot)>0$ on Ω. Hence, by (A3) and (G2), $\mathscr{L}$ is strongly positive, i.e., $\mathscr{L}\varphi(\cdot)>0$, $\forall x\in\Omega$, $\varphi\in\mathbb{X}_0^+\setminus\{0\}$. Moreover, $\mathscr{L}$ is compact since it has a continuous kernel $[\beta_1(\cdot)S^0(\cdot)+\beta_2(\cdot)V^0(\cdot)]\int_0^{+\infty}\Gamma_3(s,\cdot,y)\mathrm{d}s$, which is strictly positive for all $(x,y)\in\Omega\times\Omega$. Hence, the assertion holds from the Krein–Rutman theorem [16, Proposition 10.9]. This completes the proof. □

Let us define

$$\begin{cases}B^0\varphi(\cdot):=d_3\Delta\varphi(\cdot)-[\mu(\cdot)+\delta(\cdot)]\varphi(\cdot), & \varphi\in Dom(B^0),\quad x\in\Omega,\\ C\varphi(\cdot):=[\beta_1(\cdot)S^0(\cdot)+\beta_2(\cdot)V^0(\cdot)]\varphi(\cdot), & \varphi\in\mathbb{X}_0,\quad x\in\Omega,\end{cases}$$

where $Dom(B^0):=\left\{\varphi\in\mathbb{X}_0\cap C^2(\Omega,\mathbb{R}):\ B^0\varphi\in\mathbb{X}_0\right\}$. Let B be the closure of B^0 in $\mathbb{X}_0$. It then generates the uniformly bounded C_0-semigroup $\{\mathscr{T}(t)\}_{t\ge0}$, where

$$[\mathscr{T}(t)\varphi](\cdot):=\int_\Omega \Gamma_3(t,\cdot,y)\varphi(y)\mathrm{d}y,\quad \varphi\in\mathbb{X}_0,\ (x,t)\in\overline{\Omega}\times(0,\infty),$$

and $Dom(B):=\left\{\varphi\in\mathbb{X}_0:\lim_{t\to+0}t^{-1}[\mathscr{T}(t)-I_d]\varphi \text{ exists}\right\}$. We easily see from (G3) that, for any $\Lambda>-\underline{r}_3$,

$$(\Lambda I_d-B)^{-1}\varphi(\cdot)=\int_0^{+\infty}\mathrm{e}^{-\Lambda s}\int_\Omega \Gamma_3(s,\cdot,y)\varphi(y)\mathrm{d}y\mathrm{d}s,\quad \varphi\in\mathbb{X}_0,\quad x\in\overline{\Omega}.$$

Thus, B is resolvent-positive ([32, Definition 3.1]). Further, the spectral bound $s(B)$ of B, satisfies $s(B)\le-\underline{r}_3<0$. Hence, $\mathscr{L}=-CB^{-1}$.

We are in position to prove the following result.

Proposition 3.1. *If $\Re_0<1$, then $E^0=(S^0,V^0,0)\in\mathbb{Y}_0^+\cap C^2(\Omega,\mathbb{R}^3)$ for system (1.8)–(1.9) is locally asymptotically stable (LAS). If $\Re_0>1$, then E^0 is unstable.*

Proof. Denote $\omega(\mathscr{T})$ as the exponential growth bound of $\mathscr{T}$, then $\omega(\mathscr{T})\le-\underline{r}_3<0$ by (G3). We can also easily see that operator C is positive, bounded and $C\mathscr{T}(t)$ is compact (see Lemma 3.1). Moreover, $B+C$ generates a positive C_0-semigroup $\{\mathscr{S}(t)\}_{t\ge0}$, which corresponds to the derivative of the continuous semiflow Φ at E^0. Then, the assertion holds by [32, Theorems 3.16 and 3.17], that is, if $\Re_0=\rho(\mathscr{L})=\rho(-CB^{-1})<1$, then $s(B+C)<0$ and E^0 is LAS, whereas if $\Re_0=\rho(-BC^{-1})>1$, then $s(B+C)>0$ and E^0 is unstable. □

We next prove that E^0 is globally asymptotically stable (GAS) in $\mathbb{Y}_0^+$.

Theorem 3.1. *If $\Re_0<1$, then $E^0=(S^0,V^0,0)\in\mathbb{Y}_0^+\cap C^2(\Omega,\mathbb{R}^3)$ is GAS in $\mathbb{Y}_0^+$, that is, E^0 is LAS and $\|\Phi(t,u_0)-(S^0,V^0,0)\|_{\mathbb{Y}_0}\to0$ $(t\to+\infty)$ for any $u_0\in\mathbb{Y}_0^+$.*

Proof. By Proposition 3.1, it remains to prove $\| \Phi(t, u_0) - (S^0, V^0, 0)\|_{\mathbb{Y}_0} \to 0$ $(t \to +\infty)$ for any $u_0 \in \mathbb{Y}_0^+$. Since $\Re_0 = \rho(\mathscr{L}) < 1$, choose $\varepsilon > 0$ small enough that $\rho(\mathscr{L}_\varepsilon) < 1$, where

$$\mathscr{L}_\varepsilon\varphi(\cdot) := \left[\beta_1(\cdot)\left(S^0(\cdot)+\varepsilon\right) + \beta_2(\cdot)\left(V^0(\cdot)+\varepsilon\right)\right] \int_0^{+\infty}\int_\Omega \Gamma_3(s,\cdot,y)\varphi(y)\mathrm{d}y\mathrm{d}s, \quad \varphi \in \mathbb{X}_0, \quad x \in \overline{\Omega}.$$

By (2.13), we see that there exists $t_0 > 0$ such that

$$S(\cdot,t) < S^0(\cdot)+\varepsilon, \quad V(\cdot,t) < V^0(\cdot)+\varepsilon, \ (x,t) \in \overline{\Omega}\times(t_0,\infty).$$

We then have that

$$\begin{cases} I_t \le d_3\Delta I + \left[\beta_1(\cdot)(S^0(\cdot)+\varepsilon)+\beta_2(\cdot)(V^0(\cdot)+\varepsilon)\right] I - (\mu(\cdot)+\delta(\cdot))I, & (x,t)\in\Omega\times(t_0,\infty) \\ I(\cdot,t) = 0, & (x,t)\in\partial\Omega\times(t_0,\infty). \end{cases} \tag{3.3}$$

Let $\{\mathscr{S}_\varepsilon(t)\}_{t\ge 0}$ be the positive C_0-semigroup generated by $B + C_\varepsilon$ with domain $Dom(B)$, where

$$C_\varepsilon\varphi(\cdot) := [\beta_1(\cdot)(S^0(\cdot)+\varepsilon)+\beta_2(\cdot)(V^0(\cdot)+\varepsilon)]\varphi(\cdot), \quad \varphi\in\mathbb{X}_0, \quad x\in\Omega.$$

We then see from (3.3) that $I(\cdot,t) \le [\mathscr{S}_\varepsilon(t-t_0)I(\cdot,t_0)](\cdot)$, $\forall (x,t)\in\overline{\Omega}\times(t_0,\infty)$. With the help of Proposition 3.1 and $\rho(\mathscr{L}_\varepsilon) = \rho(-C_\varepsilon B^{-1}) < 1$, we know that $s(B+C_\varepsilon) < 0$, and thus, $\omega(\mathscr{S}_\varepsilon) < 0$ (see [32, Theorem 3.15]). This implies that

$$0 \le I(\cdot,t) \le [\mathscr{S}_\varepsilon(t-t_0)I(\cdot,t_0)](\cdot) \to 0, \quad x\in\overline{\Omega},$$

as $t\to+\infty$ and thus, $\lim_{t\to+\infty}\|I(\cdot,t)\|_\infty = 0$.

Hence, for arbitrary $\eta > 0$, there exists $t_1 > t_0$ such that $0 \le I(x,t) \le \eta$, for $(x,t)\in\overline{\Omega}\times(t_1,\infty)$. We then have that

$$\begin{cases} S_t \ge d_1\Delta S + \Lambda(\cdot) - \eta\beta_1(\cdot)S - (\mu(\cdot)+\alpha(\cdot))S, & (x,t)\in\Omega\times(t_1,\infty), \\ S(\cdot,t) = 0, & (x,t)\in\partial\Omega\times(t_1,\infty). \end{cases}$$

Solving above equation yields that,

$$\begin{aligned} S(\cdot,t) &\ge \int_\Omega \Gamma_1(t-t_1,\cdot,y)S(y,t_1)\mathrm{d}y + \int_{t_1}^t\int_\Omega \Gamma_1(t-s,\cdot,y)\left[\Lambda(y) - \eta\beta_1(y)S(y,s)\right]\mathrm{d}y\mathrm{d}s \\ &\ge \int_0^{t-t_1}\int_\Omega \Gamma_1(u,\cdot,y)\Lambda(y)\mathrm{d}y\mathrm{d}u - \eta\int_0^{t-t_1}\int_\Omega \Gamma_1(u,\cdot,y)\beta_1(y)[S^0(y)+\varepsilon]\mathrm{d}y\mathrm{d}u \\ &\to S^0(\cdot) - \eta\int_0^{+\infty}\int_\Omega \Gamma_1(u,\cdot,y)\beta_1(y)[S^0(y)+\varepsilon]\mathrm{d}y\mathrm{d}u, \quad (t\to+\infty). \end{aligned}$$

for $(x,t)\in\overline{\Omega}\times(t_1,\infty)$. Since η is arbitrary, we have that $\liminf_{t\to+\infty} S(\cdot,t) \ge S^0(\cdot)$ for all $x\in\overline{\Omega}$. Together with (2.13), we obtain that $\lim_{t\to+\infty}\|S(\cdot,t)-S^0(\cdot)\|_\infty = 0$. Similarly, we can show that $\lim_{t\to+\infty}\|V(\cdot,t)-V^0(\cdot)\|_\infty = 0$. This proves Theorem 3.1. □

Theorem 3.1 indicates that the disease in our model will be extinct if $\Re_0 < 1$.

4. Persistence of disease and endemic equilibrium

This section is devoting to prove that the BRN $\Re_0$ is a threshold index in determining disease persistence and the existence of endemic equilibrium. Firstly, we give the following lemma.

Lemma 4.1. *For any $u_0 \in \mathbb{Y}_0^+$, there exist functions $\varepsilon_1(\cdot), \varepsilon_2(\cdot) \in C(\overline{\Omega},\mathbb{R}_+)\cap C^2(\Omega,\mathbb{R}_+)$ such that $\varepsilon_1(\cdot) = \varepsilon_2(\cdot) = 0$ for $x\in\partial\Omega$ and*

$$\liminf_{t\to+\infty} S(\cdot,t) > \varepsilon_1(\cdot) > 0 \quad \text{and} \quad \liminf_{t\to+\infty} V(\cdot,t) > \varepsilon_2(\cdot) > 0 \quad \text{for} \ \ x\in\Omega.$$

Proof. By Proposition 2.2, there exists $T_1 > 0$ such that $I(\cdot,t) < K_0$ for $(x,t)\in\overline{\Omega}\times(T_1,\infty)$. One has that

$$S_t > d_1\Delta S + \Lambda(\cdot) - \overline{\beta}_1 K_0 S - [\mu(\cdot)+\alpha(\cdot)]S, \quad (x,t)\in\overline{\Omega}\times(T_1,\infty).$$

Hence, we get the following estimate,

$$S(\cdot,t) > \int_0^t \mathrm{e}^{-\overline{\beta}_1 K_0 s} \int_\Omega \Gamma_1(s,\cdot,y)\Lambda(y)\mathrm{d}y\mathrm{d}s, \quad (x,t) \in \overline{\Omega} \times (T_1,\infty),$$

thus,

$$\liminf_{t\to+\infty} S(\cdot,t) > \int_0^{+\infty} \mathrm{e}^{-\overline{\beta}_1 K_0 s} \int_\Omega \Gamma_1(s,\cdot,y)\Lambda(y)\mathrm{d}y\mathrm{d}s =: \varepsilon_1(\cdot) > 0, \quad x \in \Omega.$$

Further, there exists $T_2 > T_1$ such that

$$V_t > d_2\Delta V + \alpha(\cdot)\varepsilon_1(\cdot) - \overline{\beta}_2 K_0 V - [\mu(\cdot)+\gamma(\cdot)]V, \quad (x,t) \in \overline{\Omega} \times (T_2,\infty).$$

Similarly,

$$\liminf_{t\to+\infty} V(\cdot,t) > \int_0^{+\infty} \mathrm{e}^{-\overline{\beta}_2 K_0 s} \int_\Omega \Gamma_2(s,\cdot,y)\alpha(y)\varepsilon_1(y)\mathrm{d}y\mathrm{d}s =: \varepsilon_2(\cdot) > 0, \quad x \in \Omega.$$

As a result, $\varepsilon_1(x)$ and $\varepsilon_2(x)$ are twice continuously differentiable on Ω and vanish on $\partial\Omega$ as the fundamental solutions $\Gamma_1(\cdot,x,\cdot)$ and $\Gamma_2(\cdot,x,\cdot)$ are so in x. This proves Lemma 4.1. □

Let

$$\mathbb{Y}_0^{++} := \left\{(\varphi_1,\varphi_2,\varphi_3) \in \mathbb{Y}_0^+ : \varphi_2 \neq 0\right\}.$$

We next pay attention to the uniform weak $\|\cdot\|_\infty$-persistence of $I(\cdot,t)$.

Lemma 4.2. *If $\Re_0 > 1$, then there exists an $\varepsilon > 0$ such that*

$$\limsup_{t\to+\infty} \|I(\cdot,t)\|_\infty > \varepsilon, \tag{4.1}$$

provided that $u_0 \in \mathbb{Y}_0^{++}$.

Proof. Suppose to contrary that there exist $T_3 > 0$ such that $\|I(\cdot,t)\|_\infty \le \varepsilon$ with $0 < \varepsilon \ll 1$ for $(x,t) \in \overline{\Omega}\times(T_3,\infty)$. Hence we have

$$S_t \ge d_1\Delta S + \Lambda(\cdot) - \overline{\beta}_1\varepsilon S - [\mu(\cdot)+\alpha(\cdot)]S, \quad (x,t) \in \Omega \times (T_3,\infty).$$

Similar to Lemma 4.1,

$$S(\cdot,t) \ge \mathrm{e}^{-\overline{\beta}_1\varepsilon t}\int_\Omega \Gamma_1(t,\cdot,y)\phi(y)\mathrm{d}y + \int_0^t \mathrm{e}^{-\overline{\beta}_1\varepsilon s}\int_\Omega \Gamma_1(s,\cdot,y)\Lambda(y)\mathrm{d}y\mathrm{d}s, \quad (x,t) \in \Omega \times (T_3,\infty),$$

which deduces that

$$\liminf_{t\to+\infty} S(\cdot,t) \ge \int_0^{+\infty} \mathrm{e}^{-\overline{\beta}_1\varepsilon s}\int_\Omega \Gamma_1(s,\cdot,y)\Lambda(y)\mathrm{d}y\mathrm{d}s =: S_\varepsilon^0(\cdot), \quad x \in \Omega.$$

Note that $S_\varepsilon^0 \to S^0$ as $\varepsilon \to +0$ uniformly in $x \in \Omega$. We can conclude that there exists $T_4 > T_3$ such that $S(\cdot,t) \ge (1-\varepsilon)S_\varepsilon^0(\cdot)$ for $(x,t) \in \Omega \times (T_4,\infty)$. We then have that

$$V_t \ge d_2\Delta V + \alpha(\cdot)(1-\varepsilon)S_\varepsilon^0(\cdot) - \overline{\beta}_2\varepsilon V - [\mu(\cdot)+\gamma(\cdot)]V, \quad (x,t) \in \Omega \times (T_4,\infty).$$

Hence,

$$\limsup_{t\to+\infty} V(\cdot,t) \ge \int_0^{+\infty} \mathrm{e}^{-\overline{\beta}_2\varepsilon s}\int_\Omega \Gamma_2(s,\cdot,y)\alpha(y)(1-\varepsilon)S_\varepsilon^0(y)\mathrm{d}y\mathrm{d}s =: V_\varepsilon^0(\cdot)$$

for all $x \in \Omega$. Note that $V_\varepsilon^0 \to V^0$ as $\varepsilon \to +0$ uniformly with respect to $x \in \Omega$. We then see that there is a $T_5 > T_4$ such that $V(\cdot,t) \ge (1-\varepsilon)V_\varepsilon^0(\cdot)$ for $(x,t) \in \Omega \times (T_5,\infty)$. It follows that

$$I_t \ge d_3\Delta I + (1-\varepsilon)[\beta_1(\cdot)S_\varepsilon^0(\cdot) + \beta_2(\cdot)V_\varepsilon^0(\cdot)]I - [\mu(\cdot)+\delta(\cdot)]I, \quad (x,t) \in \Omega \times (T_5,\infty).$$

Without loss of generality, after using time shifting, the above inequality holds for all $t \in (0,\infty)$, that is,

$$I_t \ge d_3\Delta I + (1-\varepsilon)[\beta_1(\cdot)S_\varepsilon^0(\cdot) + \beta_2(\cdot)V_\varepsilon^0(\cdot)]I - [\mu(\cdot)+\delta(\cdot)]I, \quad (x,t) \in \Omega \times (0,\infty).$$

Hence, we have

$$I(\cdot,t)\geq(1-\varepsilon)\int_0^t\int_\Omega\Gamma_3(s,\cdot,y)[\beta_1(y)S_\varepsilon^0(y)+\beta_2(y)V_\varepsilon^0(y)]I(y,t-s)\mathrm{d}y\mathrm{d}s,\quad(x,t)\in\Omega\times(0,\infty).\tag{4.2}$$

For $\Lambda>0$, we see from Proposition 2.2 that the Laplace transform $L_\Lambda[I](x):=\int_0^{+\infty}\mathrm{e}^{-\Lambda t}I(x,t)\mathrm{d}t<+\infty$ exists for all $x\in\Omega$. It follows from (4.2) that

$$L_\Lambda[I](x)\geq(1-\varepsilon)\int_0^{+\infty}\mathrm{e}^{-\Lambda s}\int_\Omega\Gamma_3(s,x,y)[\beta_1(y)S_\varepsilon^0(y)+\beta_2(y)V_\varepsilon^0(y)]L_\Lambda[I](y)\mathrm{d}y\mathrm{d}s,\quad x\in\Omega.\tag{4.3}$$

We now define

$$\overline{\mathscr{L}}_{\varepsilon,\Lambda}\varphi(\cdot):=(1-\varepsilon)\left[\beta_1(\cdot)S_\varepsilon^0(\cdot)+\beta_2(\cdot)V_\varepsilon^0(\cdot)\right]\int_0^{+\infty}\mathrm{e}^{-\Lambda s}\int_\Omega\Gamma_3(s,\cdot,y)\varphi(y)\mathrm{d}y\mathrm{d}s.$$

Since $\Re_0=\rho(\mathscr{L})>1$, choosing ε and Λ small enough that $\rho(\overline{\mathscr{L}}_{\varepsilon,\Lambda})>1$. Similar to Lemma 3.1, corresponding to $\rho(\overline{\mathscr{L}}_{\varepsilon,\Lambda})$, there is a strongly positive eigenvector $v\in\mathbb{X}_0$, that is, $\overline{\mathscr{L}}_{\varepsilon,\Lambda}v=\rho(\overline{\mathscr{L}}_{\varepsilon,\Lambda})v$. We multiply both sides of (4.3) with v and then integrate over Ω,

$$\begin{aligned}\int_\Omega v(\cdot)L_\Lambda[I](\cdot)\mathrm{d}x&\geq\int_\Omega v(\cdot)(1-\varepsilon)\int_0^{+\infty}\mathrm{e}^{-\Lambda s}\int_\Omega\Gamma_3(s,\cdot,y)[\beta_1(y)S_\varepsilon^0(y)+\beta_2(y)V_\varepsilon^0(y)]L_\Lambda[I](y)\mathrm{d}y\mathrm{d}s\mathrm{d}x\\&=\int_\Omega(1-\varepsilon)[\beta_1(\cdot)S_\varepsilon^0(\cdot)+\beta_2(\cdot)V_\varepsilon^0(\cdot)]\int_0^{+\infty}\mathrm{e}^{-\Lambda s}\int_\Omega\Gamma_3(s,\cdot,y)v(y)\mathrm{d}y\mathrm{d}sL_\Lambda[I](\cdot)\mathrm{d}x\\&=\int_\Omega\overline{\mathscr{L}}_{\varepsilon,\Lambda}v(\cdot)L_\Lambda[I](\cdot)\mathrm{d}x=\rho(\overline{\mathscr{L}}_{\varepsilon,\Lambda})\int_\Omega v(\cdot)L_\Lambda[I](\cdot)\mathrm{d}x,\end{aligned}$$

here we applied the symmetry property $\Gamma_3(\cdot,x,y)=\Gamma_3(\cdot,y,x)$, $x,y\in\Omega$ (see, e.g. [17, Theorem 8.5]). Since $u_0\in\mathbb{Y}_0^{++}$, we obtain $\int_\Omega v(\cdot)L_\Lambda[I](\cdot)\mathrm{d}x>0$. Consequently,

$$\int_\Omega v(\cdot)L_\Lambda[I](\cdot)\mathrm{d}x\geq\rho(\overline{\mathscr{L}}_{\varepsilon,\Lambda})\int_\Omega v(\cdot)L_\Lambda[I](\cdot)\mathrm{d}x>\int_\Omega v(\cdot)L_\Lambda[I](\cdot)\mathrm{d}x,$$

a contradiction. This proves Lemma 4.2. □

By Lemma 4.2 and [28, Theorem 4.13], the uniform strong $\|\cdot\|_\infty$-persistence of $I(\cdot,t)$ is shown as the following proposition.

Proposition 4.1. *If $\Re_0>1$, then there exists an $\varepsilon_0>0$ such that*

$$\liminf_{t\to+\infty}\|I(\cdot,t)\|_\infty>\varepsilon_0,$$

provided that $u_0\in\mathbb{Y}_0^{++}$.

Proof. Let $f(\psi):=\|\psi_2\|_\infty$, $\psi=(\psi_1,\psi_2,\psi_3)\in\mathbb{Y}_0$. It is obvious that f is uniformly continuous and $f\circ\Phi:\mathbb{R}_+\times\mathbb{Y}_0^+\to\mathbb{Y}_0^+$ is continuous. Thanks to Proposition 2.3, $f(u_0)>0$, $d(\Phi(t,u_0),\mathscr{A})\to0$ as $t\to+\infty$ for any $u_0\in\mathbb{Y}_0^+$, where $d(\psi,\mathscr{A}):=\inf_{\tilde{\psi}\in\mathscr{A}}d(\psi,\tilde{\psi})$ for any $\psi\in\mathbb{Y}_0$. Moreover, since $\mathscr{A}$ is compact, the closed subset $\mathscr{A}\cap\{\psi\in\mathbb{Y}_0^+:a_1\leq f(\psi)\leq a_2\}$ for any $0<a_1<a_2<+\infty$ is also compact. Further, from (2.6), we know that $f(\Phi(t,u_0))>0$, $\forall t>0$, provided that $u_0\in\mathscr{A}$ and $f(u_0)>0$. Thus, all of the conditions in [28, Theorem 4.13] are satisfied, and hence, the uniform strong $\|\cdot\|_\infty$-persistence is implied by the uniform weak $\|\cdot\|_\infty$-persistence. This proves Proposition 4.1. □

Set $\Re_0>1$. Following [5, Definition 2.2], we can say from Proposition 4.1 that the semiflow Φ is uniformly persistent. Moreover, by [5, Theorem 2.3]. Furthermore, semiflow Φ possesses a global attractor $\mathscr{A}_0\subset\mathscr{A}$ relative to strongly bounded sets. Let $e(\cdot)\in C^2(\overline{\Omega})$ with

$$e(\cdot)>0,\ \text{for } x\in\Omega \text{ and } \partial e(\cdot)/\partial\mathbf{n}<-r<0 \text{ for } x\in\partial\Omega,$$

Thanks to [5, Lemma 3.6], we can easily show from Propositions 2.2 and 4.1 that if $\Re_0>1$, then there exist $0<k_1<k_2<+\infty$ such that for any $\psi=(\psi_1,\psi_2,\psi_3)\in\mathscr{A}_0$,

$$k_1e(\cdot)\leq\psi_2(\cdot)\leq k_2e(\cdot)\text{ for }\forall x\in\overline{\Omega}.$$

Following the line of [5, Theorem 6.2], we are now in position to confirm the system admits an endemic equilibrium.

Theorem 4.1. *If* $\Re_0 > 1$, *then* (1.8)–(1.9) *admits an endemic equilibrium* $E^* = (S^*, V^*, I^*) \in \mathbb{Y}_0^+ \cap C^2(\Omega, \mathbb{R}^3)$, *where* $I^* \neq 0$.

Proof. Let $\tilde{k}_1 < k_1$, $\tilde{k}_2 > k_2$ and

$$U := \{\psi = (\psi_1, \psi_2, \psi_3) \in \mathbb{Y}_0^+ : \ \tilde{k}_1 e(\cdot) < \psi_2(\cdot) < \tilde{k}_2 e(\cdot), \quad \forall x \in \Omega\}$$

For any $t > 0$, define $\mathscr{F}_t : \mathbb{Y}_0^+ \to \mathbb{Y}_0^+$ by $\mathscr{F}_t \psi = \Phi(t, \psi)$, $\psi \in \mathbb{Y}_0^+$. Since $\mathscr{A}_0$ is a global attractor relative to strongly bounded sets and U is strongly bounded, there exists a $m_0 > 0$ such that $\mathscr{F}_t^m(\overline{U}) \subset \mathscr{A}_0 \subset U$ for $m \geq m_0$. Since U is open and convex and $\mathscr{F}_t$ is continuous and compact (see the proof of Proposition 2.2), by Schauder's fixed point theorem [5, Theorem 6.1], we conclude that $\mathscr{F}_t$ has a fixed point in U. By the arbitrariness of $t > 0$ and [4, Chapter V, Lemma 3.7], semiflow Φ admits a fixed point in U. This proves Theorem 4.1. □

5. Numerical simulation

This section is spent on verifying our theoretical results by performing numerical simulation. In what follows, we concentrate on 2-dimensional circular disc $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ for convenience and fix the following parameters.

$$\begin{cases} d_1 = d_2 = 1, \quad d_3 = 0.1, \quad \gamma(x) = \gamma = 100, \quad \mu(x) = \mu = 1, \quad \delta(x) = \delta = 100, \\ \Lambda(x) = \sqrt{1 - (x_1^2 + x_2^2)}, \quad \alpha(x) = 100\sqrt{1 - (x_1^2 + x_2^2)}, \\ \phi_1(x) = 0.999 \cos \dfrac{\pi \left(x_1^2 + x_2^2\right)}{2}, \quad \phi_2(x) = 0, \quad \phi_3(x) = 0.001 \cos \dfrac{\pi \left(x_1^2 + x_2^2\right)}{2}, \end{cases} \quad x = (x_1, x_2) \in \overline{\Omega}. \tag{5.1}$$

Note that they are technical parameters satisfying the assumptions required in the previous sections and there are less biological bases for the choice of them. We employ the disease transmission coefficients in the following forms.

$$\beta_1(x) = k\left(x_1^2 + x_2^2\right) \times 10^4, \quad \beta_2(x) = 0.1\,\beta_1(x), \quad x = (x_1, x_2) \in \overline{\Omega}, \tag{5.2}$$

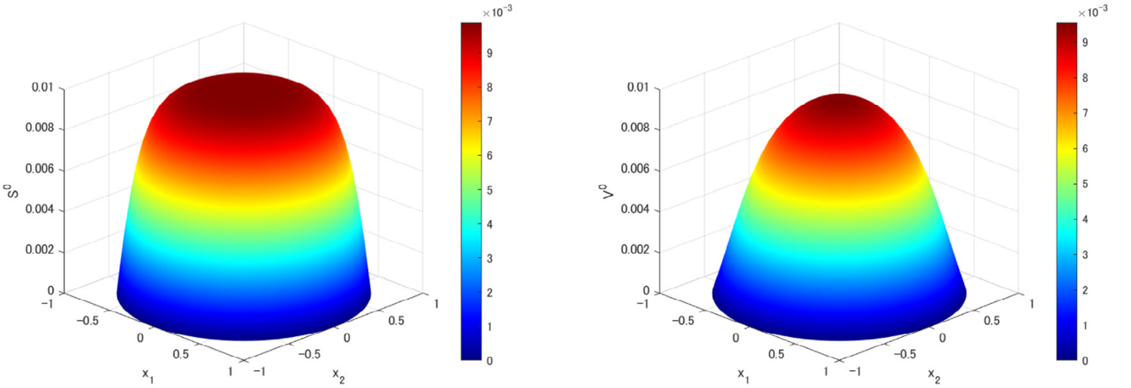
where $k > 0$ is a positive manipulated parameter. The numerical values of S^0 and V^0 are obtained as shown in Fig. 1. Following the arguments in [17, Section 16], we obtain, for $x = (x_1, x_2) \in \overline{\Omega}$ and $y = (y_1, y_2) \in \overline{\Omega}$,

$$\begin{aligned} \int_0^{+\infty} \Gamma_3(t, x, y)\mathrm{d}t = \int_0^{+\infty} \Gamma_3(t, x_1, x_2, y_1, y_2)\mathrm{d}t &= \frac{1}{\pi}\sum_{k=1}^{+\infty} \frac{J_0\left(\alpha_{0k} d_3^{-\frac{1}{2}}\sqrt{x_1^2 + x_2^2}\right) J_0\left(\alpha_{0k} d_3^{-\frac{1}{2}}\sqrt{y_1^2 + y_2^2}\right)}{\left(\mu + \delta + \alpha_{0k}^2\right)[J_1(\alpha_{0k} d_3^{-\frac{1}{2}})]^2} \\ &+ \frac{2}{\pi}\sum_{n=1}^{+\infty}\sum_{k=1}^{+\infty} \frac{J_n\left(\alpha_{nk} d_3^{-\frac{1}{2}}\sqrt{x_1^2 + x_2^2}\right) J_n\left(\alpha_{nk} d_3^{-\frac{1}{2}}\sqrt{y_1^2 + y_2^2}\right)}{\left(\mu + \delta + \alpha_{nk}^2\right)[J_{n+1}(\alpha_{nk} d_3^{-\frac{1}{2}})]^2} \cos n(\theta - \theta'), \end{aligned}$$

where J_n, $n = 0, 1, \ldots$ are the first kind Bessel functions, α_{nk}, $k = 1, 2, \ldots$ are the positive roots of equation $J_n(\alpha d_3^{-\frac{1}{2}}) = 0$, θ and θ' are the arguments of x and y, respectively. Using these numerical values, the BRN $\Re_0$ is identified by applying the Fredholm discretization method [6, Section 3] to the next generation operator $\mathscr{L}$ defined by (3.2).

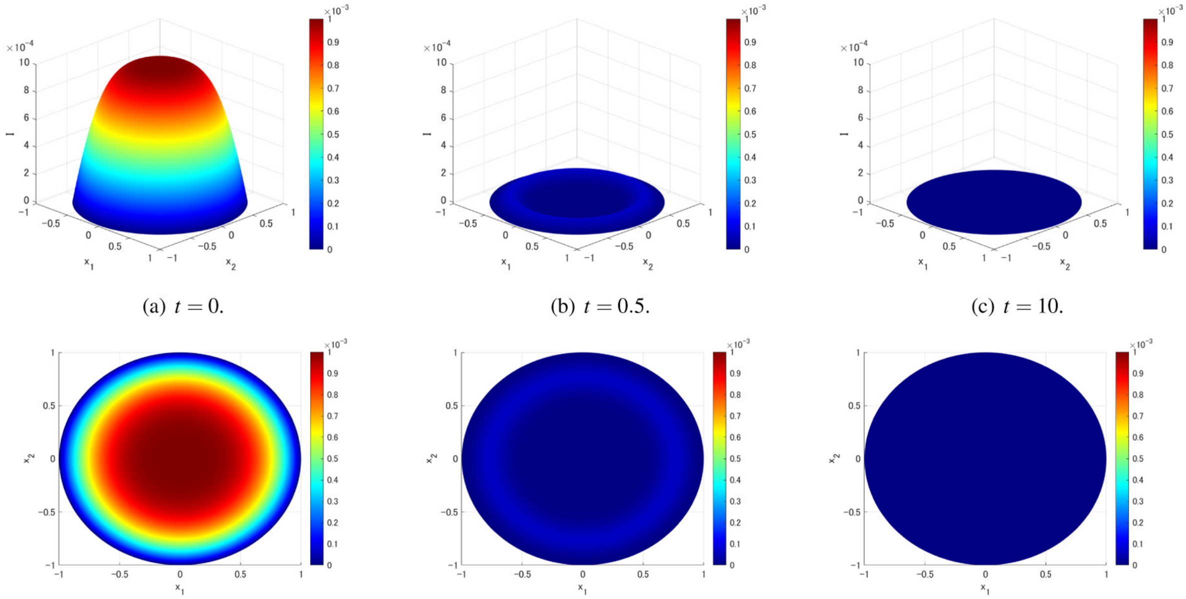
For $k = 2.2$, we obtain $\Re_0 \approx 0.922 < 1$. In this case, we can expect from Theorem 3.1 that E^0 is GAS. Fig. 2 demonstrates that the numerical solution of $I(\cdot, t)$ in this case converges to zero as time evolves.

For $k = 2.5$, we obtain $\Re_0 \approx 1.048 > 1$. In this case, we can expect from Proposition 4.1 and Theorem 4.1 that the $I(\cdot, t)$ is uniformly strongly $\|\cdot\|_\infty$-persistent and $I(\cdot, t)$ approaches an endemic equilibrium E^*. Fig. 3 shows that $I(\cdot, t)$ in this case converges in a non-homogeneous way to a spatially distribution as time evolves.



(a) Susceptible population $S^0 = S^0(x) = S^0(x_1, x_2)$. (b) Vaccinated population $V^0 = V^0(x) = V^0(x_1, x_2)$.

Fig. 1. Susceptible population S^0 and vaccinated population V^0 in the E^0 for parameters (5.1).



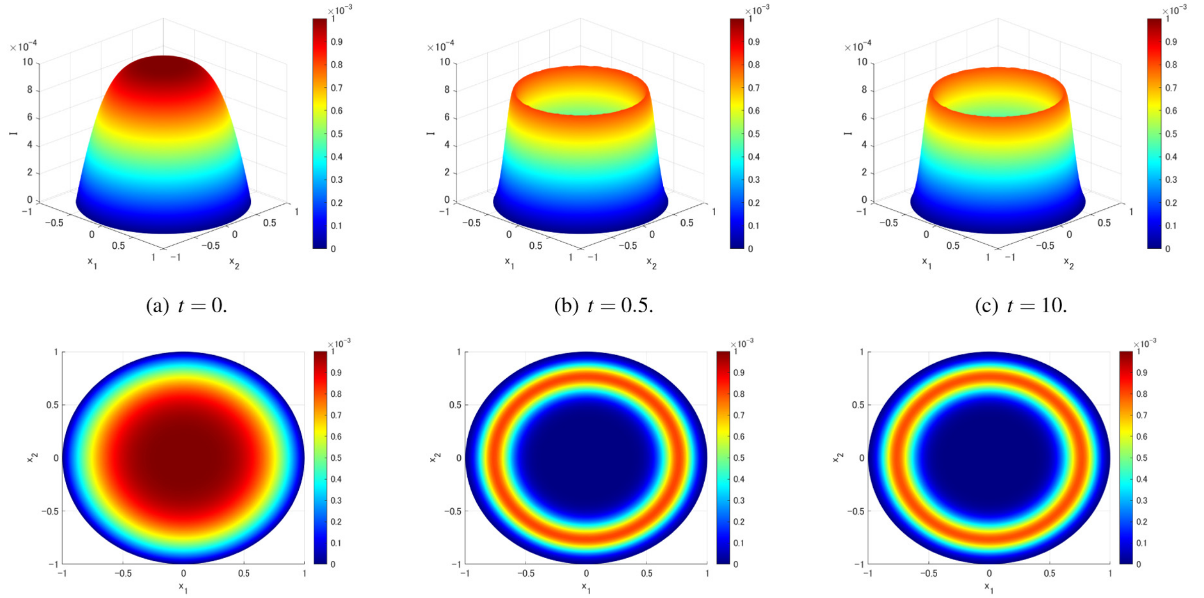
(a) $t = 0$. (b) $t = 0.5$. (c) $t = 10$.

(d) $t = 0$. (e) $t = 0.5$. (f) $t = 10$.

Fig. 2. Time variation of the infective population $I(x, t)$ with parameters (5.1)–(5.2) and $k = 2.2$ ($\Re_0 \approx 0.922 < 1$).

6. Discussion and conclusion

In the current work, we have studied the global dynamics of a diffusive SVIR epidemic model with space dependent parameters and homogeneous Dirichlet boundary condition. The BRN, $\Re_0$, has been identified as the spectral radius of the next generation operator $\mathscr{L}$, and also characterized by a simple eigenvalue of $\mathscr{L}$. We have shown that threshold dynamics is achieved in terms of BRN $\Re_0$. That is, if $\Re_0 < 1$, then the disease-free equilibrium E^0 is globally asymptotically stable (see Proposition 3.1 and Theorem 3.1) and the disease in our model will go extinct, whereas if $\Re_0 > 1$, then the system is uniformly strong $\|\cdot\|_\infty$-persistent (see Lemma 4.2 and Proposition 4.1) and the disease will persist. Moreover, by directly applying the Schauder's fixed point theorem [5, Theorem 6.1], we have confirmed that if $\Re_0 > 1$, then the semiflow Φ admits a nontrivial fixed point, that is, (1.8) admits an endemic equilibrium. To test our theoretical results, we have presented numerical simulations in the case of a 2-dimensional circular disc $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$.



(a) $t = 0$. (b) $t = 0.5$. (c) $t = 10$.

(d) $t = 0$. (e) $t = 0.5$. (f) $t = 10$.

Fig. 3. Time variation of the infective population $I(x, t)$ with parameters (5.1)–(5.2) and $k = 2.5$ ($\Re_0 \approx 1.048 > 1$).

We should point out that the parameters in our simulation were chosen for illustrative purpose only, and may not necessarily be epidemiologically realistic. Although the comparison between the numerical simulation and the real experimental data would be interesting and significant, it would exceed the range of this work because our purpose here was mathematical modeling and analysis. We shall leave the fitting of our model to a realistic data as an important future work.

Compared to the results in [44,45], our results have enriched the dynamical results of a diffusive epidemic model on the homogeneous Dirichlet boundary condition. In fact, our results have revealed that the disease control strategies should focus on decreasing the BRN $\Re_0$. There have remained a challenging and interesting problem to revisit (1.8) with high dimensional domain for specific formula of $\Re_0$. In a homogeneous case, there have also remained a technical problem to perform Lyapunov functional to study the stability of steady states. Analysis on the cases of other boundary conditions such as the Robin boundary condition and mixed boundary condition would be also an important future work.

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References

[1] N.D. Alikakos, An application of the invariance principle to reaction–diffusion equations, J. Differential Equations 33 (1979) 201–225.
[2] L.J.S. Allen, B.M. Bolker, Y. Lou, A.L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic reaction–diffusion model, Discrete Contin. Dyn. Syst. 21 (2008) 1–20.
[3] Z. Bai, S.-L. Wu, Traveling waves in a delayed SIR epidemic model with nonlinear incidence, Appl. Math. Comput. 263 (2015) 221–232.
[4] N.P. Bhatia, G.P. Szegö, Stability Theory of Dynamical Systems, Springer, Berlin, 1970.

[5] R.S. Cantrell, C. Cosner, Permanence in ecological systems with spatial heterogeneity, Proc. Roy. Soc. Edinb. A 123 (1993) 533–559.
[6] F. Chatelin, The spectral approximation of linear operators with applications to the computation of eigenelements of differential and integral operators, SIAM Rev. 23 (1981) 495–522.
[7] A. Chekroun, T. Kuniya, Global threshold dynamics of an infection age-structured SIR epidemic model with diffusion under the Dirichlet boundary condition, J. Differential Equations 269 (2020) 117–148.
[8] K. Deng, Y. Wu, Dynamics of a susceptible-infected-susceptible epidemic reaction–diffusion model, Proc. Roy. Soc. Edinburgh Sect. A 146 (2016) 929–946.
[9] O. Diekmann, J. Heesterbeek, J. Metz, On the definition and the computation of the basic reproduction ratio $\Re_0$ in models for infectious diseases in heterogeneous populations, J. Math. Biol. 28 (1990) 365–382.
[10] X. Duan, S. Yuan, X. Li, Global stability of an SVIR model with age of vaccination, Appl. Math. Comput. 226 (2014) 528–540.
[11] Z. Guo, F.-B. Wang, X. Zou, Threshold dynamics of an infective disease model with a fixed latent period and non-local infections, J. Math. Biol. 65 (2012) 1387–1410.
[12] J.K. Hale, Asymptotic Behavior of Dissipative Systems, in: Mathematical Surveys and Monographs, vol. 25, American Mathematical Society, Providence, RI, 1988.
[13] S. Henshaw, C.C. McCluskey, Global stability of a vaccination model with immigration, Electron. J. Differential Equations 2015 (92) (2015).
[14] Y. Hosono, B. Ilyas, Travelling waves for a simple diffusive epidemic model, Math. Model Methods Appl. Sci. 5 (1995) 935–966.
[15] W. Huang, M. Han, K. Liu, Dynamics of an SIS reaction–diffusion epidemic model for disease transmission, Math. Biosci. Eng. 7 (2010) 51–66.
[16] H. Inaba, Age-Structured Population Dynamics in Demography and Epidemiology, Springer, Singapore, 2017.
[17] S. Itô, Diffusion Equations, in: Translations of Mathematical Monographs, vol. 114, American Mathematical Society, Providence, RI, 1992.
[18] W.O. Kermack, A.G. McKendrick, A contribution to the mathematical theory of epidemics, Proc. R. Soc. Lond. Ser. A 115 (1927) 700–721.
[19] T. Kuniya, Global stability of a multi-group SVIR epidemic model, Nonlinear Anal. RWA 14 (2013) 1135–1143.
[20] T. Kuniya, On the relationship between the basic reproduction number and the shape of the spatial domain, in: K. Hattaf, H. Dutta (Eds.), Mathematical Modelling and Analysis of Infectious Diseases, Springer, Cham, 2020, pp. 37–59.
[21] T. Kuniya, J. Wang, Lyapunov Functions and global stability for a spatially diffusive SIR epidemic model, Appl. Anal. 96 (2017) 1935–1960.
[22] X. Liu, Y. Takeuchi, S. Iwami, SVIR Epidemic models with vaccination strategies, J. Theoret. Biol. 253 (2008) 1–11.
[23] Y. Lou, X.-Q. Zhao, A reaction–diffusion malaria model with incubation period in the vector population, J. Math. Biol. 62 (2011) 543–568.
[24] A. Pazy, Semigroups of Linear Operators and Application To Partial Differential Equations, in: Applied Mathematical Sciences, vol. 44, Springer, Berlin, 1983.
[25] R. Peng, S. Liu, Global stability of the steady states of an SIS epidemic reaction–diffusion model, Nonlinear Anal. 71 (2009) 239–247.
[26] N.Y. Rots, A.J. Wijmenga-Monsuur, W. Luytjes, et al., Hepatitis b vaccination strategies tailored to different endemicity levels: some considerations, Vaccine 28 (2010) 893–900.
[27] H.L. Smith, Monotone Dynamical Systems: An Introduction To the Theory of Competitive and Cooperative Systems, in: Mathematical Surveys and Monographs, vol. 41, American Mathematical Society, Providence, RI, 1995.
[28] H.L. Smith, H.R. Thieme, Dynamical Systems and Population Persistence, in: Graduate Studies in Mathematics, vol. 118, American Mathematical Society, Providence, RI, 2011.
[29] P. Song, Y. Lou, Y. Xiao, A spatial SEIRS reaction–diffusion model in heterogeneous environment, J. Differential Equations 267 (2019) 5084–5114.
[30] The RTS, S clinical trials partnership, RTS, s/AS01 malaria vaccine during 18 months after vaccination: a phase 3 randomized, controlled trial in children and young infants at 11 african sites, PLoS Med. 11 (2014) e1001685.
[31] The RTS, S clinical trials partnership, efficacy and safety of RTS, s/AS01 malaria vaccine with or without a booster dose in infants and children in africa: final results of a phase 3, individually randomised, controlled trial, Lancet 386 (2015) 31–45.
[32] H.R. Thieme, Spectral bound and reproduction number for infinite-dimensional population structure and time heterogeneity, SIAM J. Appl. Math. 70 (2009) 188–211.
[33] Y. Tong, C. Lei, An SIS epidemic reaction–diffusion model with spontaneous infection in a spatially heterogeneous environment, Nonlinear Anal. RWA 41 (2018) 443–460.
[34] N.K. Vaidya, X. Li, F.-B. Wang, Impact of spatially heterogeneous temperature on the dynamics of dengue epidemics, Discrete Contin. Dyn. Syst. Ser. B 24 (2019) 321–349.
[35] J. Wang, M. Guo, S. Liu, SVIR Epidemic model with age structure in susceptibility, vaccination effects and relapse, IMA J. Appl. Math. 82 (2018) 945–970.
[36] J. Wang, Y. Takeuchi, S. Liu, A multi-group SVEIR epidemic model with distributed delay and vaccination, Intern. J. Biomath. 5 (2012) 1260001.
[37] J. Wang, J. Wang, Analysis of a reaction–diffusion cholera model with distinct dispersal rates in the human population, J. Dyn. Differential Equations (2020) http://dx.doi.org/10.1007/s10884-019-09820-8.
[38] J. Wang, F. Xie, T. Kuniya, Analysis of a reaction–diffusion cholera epidemic model in a spatially heterogeneous environment, Commun. Nonlinear Sci. Numer. Simul. 80 (2020) 104951.
[39] J. Wang, R. Zhang, T. Kuniya, The dynamics of an SVIR epidemiological model with infection age, IMA J. Appl. Math. 81 (2016) 321–343.

[40] M. Winkler, Y. Lou, Advantage and disadvantage of dispersal in two-species competition models, CSIAM Trans. Appl. Math. 1 (2020) 86–103.
[41] Y. Wu, X. Zou, Asymptotic profiles of steady states for a diffusive SIS epidemic model with mass action infection mechanism, J. Differential Equations 261 (2016) 4424–4447.
[42] Y. Wu, X. Zou, Dynamics and profiles of a diffusive host-pathogen system with distinct dispersal rates, J. Differential Equations 264 (2018) 4989–5024.
[43] J. Xu, Y. Geng, Y. Zhou, Global stability of a multi-group model with distributed delay and vaccination, Math. Methods Appl. Sci. 40 (2017) 1475–1486.
[44] Z. Xu, Y. Xu, Y. Huang, Stability and traveling waves of a vaccination model with nonlinear incidence, Comput. Math. Appl. 75 (2018) 561–581.
[45] C. Zhang, J. Gao, H. Sun, J. Wang, Dynamics of a reaction–diffusion SVIR model in a spatial heterogeneous environment, Physica A 533 (2019) 122049.
[46] R. Zhang, S. Liu, Traveling waves for SVIR epidemic model with nonlocal dispersal, Math. Biosci. Eng. 16 (2019) 1654–1682.
[47] J. Zhou, Y. Yang, C.H. Hsu, Traveling waves for a nonlocal dispersal vaccination model with general incidence, Discrete Contin. Dyn. Syst. Ser. B 25 (2020) 1469–1495.
[48] C.-C. Zhu, J. Zhu, X.-L. Liu, Influence of spatial heterogeneous environment on long-term dynamics of a reaction–diffusion SVIR epidemic model with relapse, Math. Biosci. Eng. 16 (2019) 5897–5922.