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Euler and Laplace integral representations of GKZ hypergeometric functions I

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Abstract: We introduce an interpolation between Euler integral and Laplace integral: Euler-Laplace integral. We claim that, when parameters δ of the integrand are non-resonant, the $\mathcal{D}$-module corresponding to Euler-Laplace integral is naturally isomorphic to GKZ hypergeometric system $M_A(\delta)$ where A is a generalization of Cayley configuration. As a topological counterpart of this isomorphism, we establish an isomorphism between certain rapid decay homology group and holomorphic solutions of $M_A(\delta)$.

Key words: GKZ hypergeometric systems; integral representations; twisted Gauß-Manin connections.

1. Introduction. GKZ hypergeometric system $M_A(\delta)$ is a system of linear partial differential equations introduced by I. M. Gel'fand, M. I. Graev, M. M. Kapranov, and A. V. Zelevinskiĭ in [GGZ87] and [GZK89]. This system is determined by two inputs: a $d \times n$ $(d < n)$ integer matrix $A = (\mathbf{a}(1)|\cdots|\mathbf{a}(n)) = (a_{ij})$ and a parameter vector $\delta \in \mathbf{C}^{d\times 1}$. GKZ system $M_A(\delta)$ is defined by

$$(1)\qquad \begin{cases} E_i \cdot f(z) = 0 & (i = 1, \ldots, n)\\ \Box_u \cdot f(z) = 0 & (u = {}^t(u_1, \ldots, u_n) \in L_A),\end{cases}$$

where $L_A = \mathrm{Ker}(A\times : \mathbf{Z}^{n\times 1} \to \mathbf{Z}^{d\times 1})$ and E_i and $\Box_u$ are differential operators defined by $E_i = \sum_{j=1}^n a_{ij} z_j \frac{\partial}{\partial z_j} + \delta_i$ and $\Box_u = \prod_{u_j>0}(\frac{\partial}{\partial z_j})^{u_j} - \prod_{u_j<0}(\frac{\partial}{\partial z_j})^{-u_j}$. Throughout this paper, we assume an additional condition $\mathbf{Z}A \overset{def}{=} \mathbf{Z}\mathbf{a}(1) + \cdots + \mathbf{Z}\mathbf{a}(n) = \mathbf{Z}^{d\times 1}$. Writing $\mathcal{D}_{\mathbf{A}^n}$ for the Weyl algebra on $\mathbf{A}^n$ and $H_A(\delta)$ for the left ideal of $\mathcal{D}_{\mathbf{A}^n}$ generated by the differential operators (1), we also call the left $\mathcal{D}_{\mathbf{A}^n}$-module $M_A(\delta) = \mathcal{D}_{\mathbf{A}^n}/H_A(\delta)$ GKZ system.

There have been discussed Euler and Laplace integral representations of GKZ systems by several authors ([GKZ90], [ET15]). In this paper, we introduce an integral representation which generalizes both Euler and Laplace integral representations: Euler-Laplace integral representation. In the language of $\mathcal{D}$-modules, integral representations can be defined as direct images of integrable connections commonly referred to as Gauß-Manin connections. Therefore, we need to establish an isomorphism between GKZ system and a certain Gauß-Manin connection. This is formulated in §2. In §3, we describe the space of rapid decay cycles generalizing the argument of [ET15]. The detailed argument is quite technical and we only outline the construction of the toric compactification. The readers can find proofs in [MHa, §2, §3].

2. Gauß-Manin connections and GKZ $\mathcal{D}$-modules. We follow the notation of [HTT08]. Let $h_{l,z^{(l)}}(x) = \sum_{j=1}^{N_l} z_j^{(l)} x^{\mathbf{a}^{(l)}(j)}$ $(l = 0, 1, \ldots, k)$ be Laurent polynomials on the algebraic torus $(\mathbf{G}_m)_x^n$. Here, the subindex x means that we use the symbol x as a coordinate of the torus. The coefficients $z = (z_j^{(l)})_{j,l}$ are regarded as independent variables of the affine space $\mathbf{A}_z^N$ with $N = N_0 + \cdots + N_k$. We put $X_0 = \mathbf{A}_z^N \times (\mathbf{G}_m)_x^n \setminus \{(z, x) \in \mathbf{A}^N \times (\mathbf{G}_m)^n \mid h_{1,z^{(1)}}(x) \cdots h_{k,z^{(k)}}(x) = 0\}$ and write $\pi : X_0 \to \mathbf{A}_z^N$ for the natural projection. The Euler-Laplace integral representation is defined as a complex of $\mathcal{D}$-modules $\int_\pi \mathcal{O}_{X_0} e^{h_{0,z^{(0)}}(x)} h_{1,z^{(1)}}(x)^{-\gamma_1} \cdots h_{k,z^{(k)}}(x)^{-\gamma_k} x^c$ where $\gamma_l \in \mathbf{C}$ and $c \in \mathbf{C}^{n\times 1}$ are parameters.

The following theorem proves the equivalence of Laplace integral representation and Euler-Laplace integral representation.

Theorem 2.1 (Cayley trick for Euler-Laplace integrals)**.** *We put $X_k = \mathbf{A}_z^N \times (\mathbf{G}_m)_y^k \times (\mathbf{G}_m)_x^n$. Let $\varpi : X_k \to \mathbf{A}_z^N$ be the natural projection and*

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$\gamma_l \in \mathbf{C}\setminus\mathbf{Z}$, $\delta \in \mathbf{C}^{n\times 1}$ be parameters. Then, one has a natural isomorphism

$$(2)\quad \int_\pi \mathcal{O}_{X_0} e^{h_{0,z^{(0)}}(x)} h_{1,z^{(1)}}(x)^{-\gamma_1}\cdots h_{k,z^{(k)}}(x)^{-\gamma_k} x^c \simeq \int_\varpi \mathcal{O}_{X_k} y^\gamma x^c e^{h_z(y,x)},$$

where $h_z(y,x) = h_{0,z^{(0)}}(x) + \sum_{l=1}^{k} y_l h_{l,z^{(l)}}(x)$.

Corollary 2.2. *Under the assumption of Theorem 2.1, one has a natural isomorphism*

$$(3)\quad \int_{\pi!} \mathcal{O}_{X_0} e^{h_{0,z^{(0)}}(x)} h_{1,z^{(1)}}(x)^{-\gamma_1}\cdots h_{k,z^{(k)}}(x)^{-\gamma_k} x^c \simeq \int_{\varpi!} \mathcal{O}_{X_k} y^\gamma x^c e^{h_z(y,x)}.$$

We put $\Phi = \Phi(z,x) = e^{h_{0,z^{(0)}}(x)} h_{1,z^{(1)}}(x)^{-\gamma_1}\cdots h_{k,z^{(k)}}(x)^{-\gamma_k} x^c$, $\Phi_k = y^\gamma x^c e^{h_z(y,x)}$ to simplify the notation. Let us formulate the main theorem of this section. We define an $n\times N_l$ matrix A_l by $A_l = (\mathbf{a}^{(l)}(1) \mid \cdots \mid \mathbf{a}^{(l)}(N_l))$. Then, we define the Cayley configuration A as an $(n+k)\times N$ matrix by

$$(4)\quad A = \left(\begin{array}{c|c|c|c|c} 0\cdots 0 & 1\cdots 1 & 0\cdots 0 & \cdots & 0\cdots 0\\ \hline 0\cdots 0 & 0\cdots 0 & 1\cdots 1 & \cdots & 0\cdots 0\\ \hline \vdots & \vdots & \vdots & \ddots & \vdots\\ \hline 0\cdots 0 & 0\cdots 0 & 0\cdots 0 & \cdots & 1\cdots 1\\ \hline A_0 & A_1 & A_2 & \cdots & A_k \end{array}\right).$$

Combining Theorem 2.1 with the result of [SW09], we can obtain the following main result of this section.

Theorem 2.3. *Suppose that the parameter $\delta = {}^t(\gamma_1,\ldots,\gamma_k,c)$ is non-resonant and $\gamma_l \notin \mathbf{Z}$ for $l = 1,\ldots,k$. Then, one has a sequence of canonical isomorphisms of $\mathcal{D}_{\mathbf{A}_z^N}$-modules*

$$(5)\quad M_A(\delta) \simeq \int_\varpi \mathcal{O}_{X_k}\Phi_k \simeq \int_\pi \mathcal{O}_{X_0}\Phi.$$

Moreover, the regularization conditions

$$(6)\quad \int_\varpi \mathcal{O}_{X_k}\Phi_k \simeq \int_{\varpi!} \mathcal{O}_{X_k}\Phi_k$$

and

$$(7)\quad \int_\pi \mathcal{O}_{X_0}\Phi \simeq \int_{\pi!} \mathcal{O}_{X_0}\Phi$$

hold.

In the proof of the last two isomorphisms of Theorem 2.3, we use the proper direct image description of Fourier transform of $\mathcal{D}$-modules ([Dai00, Proposition 2.2.3.2]).

3. Description of the rapid decay homology groups of Euler-Laplace integrals. We formulate the isomorphism between the space of holomorphic solutions of GKZ system and a certain rapid decay homology group. To begin with, let us remark that there is a concrete version of the isomorphism (5).

Theorem 3.1. *There is an isomorphism*

$$(8)\quad \int_\pi^0 \mathcal{O}_{X_0}\Phi \to \int_\varpi^0 \mathcal{O}_{X_k}\Phi_k,$$

of $\mathcal{D}_{\mathbf{A}_z^N}$-modules which sends $[\frac{dx}{x}]$ to $[\frac{dy}{y}\wedge\frac{dx}{x}]$.

Here, we realize $\int_\pi^0 \mathcal{O}_{X_0}\Phi$ and $\int_\varpi^0 \mathcal{O}_{X_k}\Phi_k$ in terms of relative de Rham cohomology groups and we set $\frac{dx}{x} = \frac{dx_1}{x_1}\wedge\cdots\wedge\frac{dx_n}{x_n}$ and $\frac{dy}{y} = \frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_k}{y_k}$.

Corollary 3.2. *If the parameter δ is non-resonant and $\gamma_l \notin \mathbf{Z}$ for any $l = 1,\ldots,k$, $M_A(\delta) \ni [1] \mapsto [\frac{dx}{x}] \in \int_\pi^0 \mathcal{O}_{X_0}\Phi$ defines an isomorphism of $\mathcal{D}_{\mathbf{A}_z^N}$-modules.*

The proof of Theorem 3.1 is based on tedious calculations of relative de Rham complexes.

Now we discuss the solutions of Laplace-Gauss-Manin connection $\int_\pi \mathcal{O}_{X_0}\Phi$. We repeat the relevant material from [ET15] and [Hie09] without proofs. Let U be a smooth complex affine variety, let $f : U \to \mathbf{A}^1$ be a non-constant morphism, and let $M = (E,\nabla)$ be a regular integrable connection on U. We consider an embedding of U into a smooth projective variety X with a meromorphic prolongation $\bar{f} : X \to \mathbf{P}^1$ of f. We assume that $D = X\setminus U$ is a normal crossing divisor. Any projective variety X with these conditions is called a good compactification. We decompose D as $D = f^{-1}(\infty)\cup D_{irr}$. Then, we write $\widetilde{X^D} = \widetilde{X}$ for the real oriented blow-up of X along D and write $\pi_X : \widetilde{X} \to X$ for the associated morphism ([Sab13, §8.2]). Let $\widetilde{\mathbf{P}^1}$ denote the real oriented blow-up of $\mathbf{P}^1$ at infinity and $\pi_\infty : \widetilde{\mathbf{P}^1} \to \mathbf{P}^1$ denote the associated morphism. Note that the closure of the ray $[0,\infty)e^{\sqrt{-1}\theta}$ in $\widetilde{\mathbf{P}^1}$ and $\widetilde{\mathbf{P}^1}\setminus\mathbf{C}$ has a unique intersection point which we will denote by $e^{\sqrt{-1}\theta}\infty$. Now, a morphism $\tilde{f} : \widetilde{X} \to X$ is naturally induced so that it fits into a commutative diagram

$$(9)\quad \begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{\mathbf{P}^1}\\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_\infty}\\ \bar{X} & \xrightarrow{\bar{f}} & \mathbf{P}^1. \end{array}$$

We set $\widetilde{D^{r.d.}} = \tilde{f}^{-1}(\{e^{\sqrt{-1}\theta}\infty \mid \theta\in(\frac{\pi}{2},\frac{3\pi}{2})\})\setminus \pi_X^{-1}(D_{irr}) \subset \widetilde{X}$. We put $\mathcal{L} = (\mathrm{Ker}(\nabla^{an} :$

$\mathcal{O}_{X^{an}}(E^{an}) \to \Omega^1_{X^{an}}(E^{an})))^\vee$, where $^\vee$ stands for the dual local system. We consider the natural inclusion $U^{an} \overset{j}{\hookrightarrow} U^{an} \cup \widetilde{D^{r.d.}}$. Then, the rapid decay homology group of M. Hien $\mathrm{H}^{r.d.}_*(U^{an}, (Me^f)^\vee)$ is defined in this setting by

$$\mathrm{H}^{r.d.}_*(U^{an}, (Me^f)^\vee) = \mathrm{H}_*(U^{an} \cup \widetilde{D^{r.d.}}, \widetilde{D^{r.d.}}; j_*\mathcal{L}) \tag{10}$$

([Hie09], see also [ET15] and [MHb]). Note that $U^{an} \cup \widetilde{D^{r.d.}}$ is a topological manifold with boundary and that $j_*\mathcal{L}$ is a local system on $U^{an} \cup \widetilde{D^{r.d.}}$. We set $\mathrm{H}^{*+\dim U}_{dR}(U, Me^f) = \mathrm{H}^*(\mathrm{DR}_{U/\mathrm{pt}}(Me^f))$. The main result of [Hie09] states that the period pairing $\mathrm{H}^{r.d.}_*(U^{an}, (Me^f)^\vee) \times \mathrm{H}^*_{dR}(U, Me^f) \to \mathbf{C}$ is perfect.

Remark 3.3. We put $\widetilde{D^{r.d.}_0} = \tilde{f}^{-1}(\{e^{\sqrt{-1}\theta}\infty \mid \theta \in (\frac{\pi}{2}, \frac{3\pi}{2})\})$ and write $\bar{j}$ for the natural inclusion $U^{an} \hookrightarrow U^{an} \cup \widetilde{D^{r.d.}_0}$. It can easily be seen that the inclusion $(U^{an} \cup \widetilde{D^{r.d.}}, \widetilde{D^{r.d.}}) \hookrightarrow (U^{an} \cup \widetilde{D^{r.d.}_0}, \widetilde{D^{r.d.}_0})$ is a homotopy equivalence ([MHb, Lemma 2.3]). Therefore, the rapid decay homology group can be computed by the formula $\mathrm{H}^{r.d.}_*(U^{an}, (Me^f)^\vee) = \mathrm{H}_*(U^{an} \cup \widetilde{D^{r.d.}_0}, \widetilde{D^{r.d.}_0}; \bar{j}_*\mathcal{L})$. Note that this realization is compatible with the period pairing.

We construct a family of good compactifications X associated to the Laplace-Gauss-Manin connection $\int^0_\pi \mathcal{O}_{X_0}\Phi$. First, we write Δ_0 for the convex hull of the set $\{\mathbf{a}^{(0)}(1), \dots, \mathbf{a}^{(0)}(N_0)\}$ and the origin and write Δ_l for the convex hull of the set $\{\mathbf{a}^{(l)}(1), \dots, \mathbf{a}^{(l)}(N_l)\}$ $(l = 1, \dots, k)$. For any covector $\xi \in (\mathbf{R}^n)^*$, we set $\Delta^\xi_l = \{v \in \Delta_l \mid \langle \xi, v\rangle = \min_{w \in \Delta_l} \langle \xi, w\rangle\}$ and $h^\xi_{l,z^{(l)}}(x) = \sum_{\mathbf{a}^{(l)}(j) \in \Delta^\xi_l} z_j x^{\mathbf{a}^{(l)}(j)}$. Now, we consider the dual fan Σ of the Minkowski sum $\Delta_0 + \Delta_1 + \cdots + \Delta_k$. By taking a refinement if necessary, we may assume that Σ is a smooth fan. Then, the associated toric variety $X = X(\Sigma)$ is sufficiently full for any Δ_l in the sense of [Hov77]. We write $\{D_j\}_{j \in J}$ for the set of torus invariant divisors of X.

Definition 3.4. We say that a point $z = (z^{(0)}, z^{(1)}, \dots, z^{(k)}) \in \mathbf{C}^N$ is nonsingular if the following two conditions are both satisfied:

(a) For any $1 \le l_1 < \cdots < l_s \le k$, the Laurent polynomials $h_{l_1,z^{(l_1)}}(x), \dots, h_{l_s,z^{(l_s)}}(x)$ are nonsingular in the sense of [Hov77], i.e., for any covector $\xi \in (\mathbf{R}^n)^*$, the s-form $d_x h^\xi_{l_1,z^{(l_1)}}(x) \wedge \cdots \wedge d_x h^\xi_{l_s,z^{(l_s)}}(x)$ never vanishes on the set $\{x \in (\mathbf{C}^\times)^n \mid h^\xi_{l_1,z^{(l_1)}}(x) = \cdots = h^\xi_{l_s,z^{(l_s)}}(x) = 0\}$.

(b) For any covector $\xi \in (\mathbf{R}^n)^*$ such that $0 \notin \Delta^\xi_0$ and for any $1 \le l_1 < \cdots < l_s \le k$ (s can be 0), the $(s+1)$-form $dh^\xi_{0,z^{(0)}}(x) \wedge dh^\xi_{l_1,z^{(l_1)}}(x) \wedge \cdots \wedge dh^\xi_{l_s,z^{(l_s)}}(x)$ never vanishes on the set $\{x \in (\mathbf{C}^\times)^n \mid h^\xi_{l_1,z^{(l_1)}}(x) = \cdots = h^\xi_{l_s,z^{(l_s)}}(x) = 0\}$.

Remark 3.5. If $k = 0$, the nonsingularity condition is equivalent to the non-degenerate condition of [Ado94, p.274]. In general, nonsingularity condition is stronger than non-degenerate condition. Nevertheless, it is still a Zariski open dense condition. Note that this is proved in the case when $h_{0,z^{(0)}} \equiv 1$ in [Hov77] (see also [Oka97, Chapter V, COROLLARY (3.2.1)]).

In the following, we fix a nonsingular z and a small positive real number ε. Let $\Delta(z;\varepsilon)$ denote the disk with center at z and with radius ε. By abuse of notation, we write D_j for the product $\Delta(z;\varepsilon) \times D_j$. For any $l = 0, \dots, k$, we set $Z_l = \overline{\{(z', x) \in \Delta(z;\varepsilon) \times (\mathbf{C}^\times)^n_x \mid h_{l,z'^{(l)}}(x) = 0\}} \subset \Delta(z;\varepsilon) \times X$.

Now we take a small positive real number ε and consider the canonical projection $p : \Delta(z;\varepsilon) \times X \to \Delta(z;\varepsilon)$. Following [ET15], we consider a sequence of blow-ups along codimension 2 divisors $Z_0 \cap D_j$ for D_j contained in the pole divisor of $h_{0,z'^{(0)}}(x)$. If the pole order of $h_{0,z'^{(0)}}(x)$ along D_j is $m_j \in \mathbf{Z}_{>0}$, one needs at most m_j blow-ups along $Z_0 \cap D_j$. Repeating this process finitely many times, we obtain a non-singular complex manifold $\bar{X}$. We write $\bar{p} : \bar{X} \to \Delta(z;\varepsilon)$ for the composition of the natural morphism $\bar{X} \to \Delta(z;\varepsilon) \times X$ with the canonical projection $\Delta(z;\varepsilon) \times X \to \Delta(z;\varepsilon)$. We also write $\bar{Z}_l$ and $\bar{D}_j$ for the proper transforms of Z_l and D_j. We equip $\bar{X}$ with the Whitney stratification coming from the normal crossing divisor $\bar{D} = \{\bar{Z}_l\}^k_{l=1} \cup \{\bar{D}_j\}_{j \in J} \cup \{\text{exceptional divisors of blow-ups}\}$. We have a diagram $\Delta(z;\varepsilon) \overset{\bar{p}}{\leftarrow} \bar{X} \overset{\bar{h}_{0,z'^{(0)}}}{\to} \mathbf{P}^1$. Let us consider a real oriented blow-up $\widetilde{X} = \widetilde{\bar{X}^{\bar{D}}}$ of $\bar{X}$ along $\bar{D}$. We naturally have the following commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{h}_{0,z'^{(0)}}} & \widetilde{\mathbf{P}^1} \\ {\scriptstyle \varpi_X}\downarrow & & \downarrow{\scriptstyle \pi_\infty} \\ \bar{X} & \xrightarrow{\bar{h}_{0,z'^{(0)}}} & \mathbf{P}^1. \end{array} \tag{11}$$

We can show that $\bar{p}^{-1}(z')$ is a good compactification of $\pi^{-1}(z')$ for any $z' \in \Delta(z;\varepsilon)$. We define $\widetilde{D^{r.d.}} \subset \widetilde{X}$ by the formula $\widetilde{D^{r.d.}} = \tilde{h}^{-1}_{0,z'^{(0)}}((\frac{\pi}{2}, \frac{3\pi}{2})\infty)$, put $\tilde{p} =$

$p \circ \varpi_X$ and put $\widetilde{D^{r.d.}_z} = \widetilde{D^{r.d.}} \cap \tilde{p}^{-1}(z)$. For any $z' \in \mathbf{A}^N$, let $\Phi_{z'}$ denote the multivalued function on $\pi^{-1}(z')$ defined by $\pi^{-1}(z') \ni x \mapsto \Phi(z', x)$. Writing $j_z : \pi^{-1}(z)^{an} \hookrightarrow \pi^{-1}(z)^{an} \cup \widetilde{D^{r.d.}_z}$ for the natural inclusion, we set

$$\mathrm{H}^{r.d.}_{*,z} = \mathrm{H}_*(\pi^{-1}(z)^{an} \cup \widetilde{D^{r.d.}_z}, \widetilde{D^{r.d.}_z}; j_{z*}(\underline{\mathbf{C}}\Phi_z)). \tag{12}$$

Theorem 3.6. *Suppose the parameter vector δ is non-resonant and $\gamma_l \notin \mathbf{Z}$ for any $l = 1, \ldots, k$. Suppose that $z \in \mathbf{C}^N$ is nonsingular. Then the morphism*

$$\mathrm{H}^{r.d.}_{n,z} \xrightarrow{\int} \mathrm{Hom}_{\mathcal{D}_{\mathbf{C}^N_z}}(M_A(\delta), \mathcal{O}_{\mathbf{C}^N})_z \tag{13}$$

given by

$$[\Gamma] \mapsto \int_\Gamma \Phi \frac{dx}{x} \tag{14}$$

is an isomorphism of **C***-vector spaces.*

Remark 3.7. Let Ω denote the Zariski open dense subset of $\mathbf{A}^N$ consisting of nonsingular points. It is straightforward to construct a local system $\mathcal{H}^{r.d.}_n = \bigcup_{z \in \Omega^{an}} \mathrm{H}^{r.d.}_{n,z} \to \Omega^{an}$ and an isomorphism $\mathcal{H}^{r.d.}_n \xrightarrow{\int} \mathrm{Hom}_{\mathcal{D}_{\mathbf{C}^N_z}}(M_A(\delta), \mathcal{O}_{\mathbf{C}^N})_{\Omega^{an}}$ whose stalks are identical with (13). See [HR08].

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