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ALGEBRO-GEOMETRIC ASPECTS OF THE CHRISTOFFEL-DARBOUX KERNELS FOR CLASSICAL ORTHOGONAL POLYNOMIALS

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Abstract. In this paper we study algebro-geometric aspects of the Christoffel-Darboux kernels for classical orthogonal polynomials with rational coefficients. We find a novel connection between a projective curve defined by the Christoffel-Darboux kernel and a system of Diophantine equations, which was originally designed by Hausdorff (1909) for applications to Waring's problem, and which is closely related to quadrature formulas in numerical analysis and Gaussian designs in algebraic combinatorics. We prove some nonsolvability results of such Hausdorff-type equations.

1. Introduction

For real numbers $a < b$, let $\{\Phi_l\}$ be a sequence of orthogonal polynomials with respect to a given positive weight function w over the interval (a, b). The *Christoffel-Darboux kernel*, more precisely the *reproducing kernel* for polynomials of degree at most l, is the bivariate polynomial defined by

$$K_l(x, y) := \sum_{k=0}^{l} h_l h_k^{-1} \Phi_k(x)\Phi_k(y),$$

where $h_k = \int_a^b \Phi_k(x)^2 \; w(x)dx$; for example see [29]. Christoffel-Darboux kernels have considerable interest in functional analysis and play a crucial role in the theory of orthogonal polynomials. For various analytic properties and interesting questions concerning Christoffel-Darboux kernels, we refer the readers to Szegő's book *Orthogonal Polynomials* [32].

We are here particularly concerned with algebro-geometric aspects of Christoffel-Darboux kernels for *classical orthogonal polynomials*. By Bochner's Theorem, such polynomials are completely classified by Jacobi polynomials, (generalized) Laguerre polynomials and Hermite polynomials; see Subsection 2.1. Here and throughout this paper, we mainly deal with classical orthogonal polynomials with rational coefficients, which we call *rational classical polynomials*.

The following is one of the main results in this paper.

Theorem 1.1. *For $r \geq 2$, let K_r be the Christoffel-Darboux kernel for a series of rational classical polynomials. Let C_r be the projective curve defined by $Z^{2r} K_r(X/Z, Y/Z) = 0$. Then the following hold:*

(i) *The curve C_r is absolutely irreducible;*

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(ii) *The genus of* C_r *is* $(r-1)^2$.

In this paper, we also consider a certain system of Diophantine equations which seemingly look irrelevant to the above result:

Problem 1.2 (Hausdorff-type equations)**.** Does there exist a solution $x_1, \dots, x_m$, $y_1, \dots, y_m \in \mathbb{Q}$ such that

$$\sum_{i=1}^{m} x_i y_i^j = \int_a^b t^j w(t)\, dt, \quad j = 0, 1, \dots, n, \tag{1}$$

where the integration on the right corresponds to a series of rational classical polynomials?

We remark that the equations (1) have a rational solution if and only if there exists a quadrature formula of degree n with rational nodes y_i and weights x_i. Here a *quadrature formula of degree* n *with nodes* y_i *and weights* x_i is an integration formula of type

$$\sum_{i=1}^{m} x_i f(y_i) = \int_a^b f(t) w(t)\, dt \tag{2}$$

in which f ranges over all polynomials of degree at most n [12, 30]. By a Stroud-type bound, we obtain $n \le 2m-1$; see Proposition 2.7 of the present paper. The 'tight' situation, namely $n = 2m-1$, corresponds to the Gaussian quadrature whose nodes are the zeros of the orthogonal polynomial of degree m with respect to the weight function w. This is a special case of classical results by Riesz [23] and Shohat [27] concerning *quasi-orthogonal polynomials*. These facts will be summarized in Proposition 2.4 later.

Problem 1.2 goes back to Hausdorff [16], who examined the pair $(m, n) = (r+1, r)$ for Gaussian integration with respect to $w(t) = e^{-t^2}/\sqrt{\pi}$ and proved an existence theorem of solutions. His motivation was a simplification of Hilbert's solutions of Waring's problem in number theory. Interest was revived by Nesterenko [19, p.4700] who modified Hausdorff's arguments, and Problem 1.2 was extensively studied for Gaussian integration in [24]. For a brief summary concerning Hausdorff's work, we refer the readers to Section 2.3 of [24]. It is remarked that a solution of the equations (1) not only gives a constructive solution of Waring's problem, but also leads to constructions of isometric embeddings of classical finite-dimensional Banach spaces and other related objects in algebraic combinatorics; for example, see [22, 25].

Schur [26] proved that Hermite polynomials are irreducible over $\mathbb{Q}$. This implies these polynomials have no rational zeros and so the equations (1) have no rational solutions for $(m, n) = (r+1, 2r+1)$. Thus in [24], the present authors examined the case $(m, n) = (r+1, 2r)$ and proved a nonexistence theorem of solutions. A quadrature formula of degree $2r$ with $r+1$ nodes for Gaussian integration is also called an *almost tight Gaussian design*, which is of special interest in algebraic combinatorics [1].

Another interesting case is the definite integration on the interval $[-1, 1]$, corresponding to Legendre polynomials. Unlike Hermite polynomials, the question of the irreducibility of Legendre polynomials is still open (cf. [9]). However, by a classical result of Holt [17], Legendre polynomials cannot be factored into rational linear forms and so the equations (1) have no rational solutions for $(m, n) = (r+1, 2r+1)$. We are again concerned with the 'almost tight' situation.

In this paper, as a unification of these results, we investigate the almost tight situation for all rational classical polynomials. For this purpose, we derive a key lemma that creates a connection between the equations (1) and the projective curve C_r, and then prove the following theorems.

Theorem 1.3. *Let $r \geq 3$ be an integer. For rational classical polynomials, there exist at most finitely many rational solutions for $(m, n) = (r + 1, 2r)$.*

A stronger result holds for Hermite, Legendre and Laguerre polynomials:

Theorem 1.4. *For $(m, n) = (r + 1, 2r)$ with $r \geq 2$, nonexistence of rational solutions hold in the following three cases:*

(i) $w(t) = e^{-t^2}/\sqrt{\pi}$ *on* $(-\infty, \infty)$*;*
(ii) $w(t) \equiv 1/2$ *on* $[-1, 1]$*;*
(iii) $w(t) = e^{-t}$ *on* $(0, \infty)$.

In [24], the present authors examined the almost-tight case for Gaussian integration. By establishing an explicit formula for the discriminant of a quasi-Hermite polynomial, they showed that the equations (1) have no rational solutions for $r \equiv 2, 3, 4, 5, 6 \pmod 8$ (Corollary 4.7 of [24]). Theorem 1.4 (i) improves this and gives a complete answer to the almost-tight case for Gaussian integration.

This paper is organized as follows. Section 2 gives preliminaries, where we review some basic terminology and facts for further arguments that follow later. We give a key lemma (Lemma 2.5) and reduce Problem 1.2 to finding rational points on the projective curve C_r. Sections 3 and 4 are the main body of this paper. In Section 3 we first prove Theorem 1.1 by utilizing basic properties on classical polynomials and Bézout's Theorem in algebraic geometry. We then show Theorem 1.3 using Faltings's Theorem. In Section 4 we prove Theorem 1.4 using the theory of Newton polygons. Section 5 is the conclusion, where we make further remarks; for example, we give an answer to the problem, due to Bannai and Ito [5], of classifying rational classical polynomials for which the projective curves C_r actually have rational points. We also discuss a version of Problem 1.2 for the Gegenbauer (ultraspherical) weight function $w(t) = (1-t^2)^{(d-3)/2}$ on $[-1, 1]$ in connection with the spherical design theory, together with another proof of a famous theorem by Delsarte et al. [10, Theorem 7.7].

A novel feature of this paper is that it takes algebro-geometric approaches to examine the existence of rational quadrature formulas and Gaussian designs. Such fundamental problems in analysis and combinatorics are not fully recognized by researchers in other related areas, to whom this paper will provide new research topics. As will be clear in the present paper, the Riesz-Shohat theorem (Proposition 2.4) deserves to be known more, which is however not fully understood in combinatorics and elementary number theory. Thus, to bring Riesz-Shohat theorem into those areas is an important aim of this paper.

2. Preliminary and definitions

2.1. **Classical orthogonal polynomials.** Let μ be a positive Borel measure on an interval (a, b) with finite moments. For convenience, we assume that $\int_a^b d\mu = 1$. Let $\{\Phi_l\}$ be a sequence of *orthogonal polynomials* with respect to μ; namely, Φ_l is a polynomial of degree l, orthogonal to all polynomials of degree at most $l - 1$. A *quasi-orthogonal polynomial of degree l, of order k*, is a polynomial of the form

$$\Phi_{l,k}(x) = \Phi_l(x) + b_1\Phi_{l-1}(x) + \cdots + b_k\Phi_{l-k}(x)$$

in which $b_1, \ldots, b_k \in \mathbb{R}$ and in particular $b_k \neq 0$ [33]. For convenience, we set $\Phi_{l,0}(x) = \Phi_l(x)$. The polynomial $\Phi_{l,k}(x)$ is orthogonal to all polynomials of degree at most $l - k - 1$. Some authors use the terminology 'order' in a bit different meaning; for example, see [28].

The following facts are well known (cf. Theorems 3.2.2, 3.3.1, 3.3.2 of [32]):

Proposition 2.1 (Christoffel-Darboux formula)**.**

$$K_l(x,y) = \frac{k_l}{k_{l+1}} \cdot \frac{\Phi_{l+1}(x)\Phi_l(y) - \Phi_l(x)\Phi_{l+1}(y)}{x-y} \tag{3}$$

where k_i is the leading coefficient of Φ_i.

Proposition 2.2. *The roots of $\Phi_l(x)$ are real and distinct and are located in the open interval (a,b).*

Proposition 2.3. *Let $x_1 < x_2 < \cdots < x_l$ be the roots of $\Phi_l(x)$. Let $x_0 = a$ and $x_{l+1} = b$. Then each open interval (x_k, x_{k+1}), $k = 0,1,2,\ldots,l$, contains exactly one root of $\Phi_{l+1}(x)$.*

Bochner [6] completely classified all polynomials which are solutions of a second-order Sturm-Liouville type differential equation. Among such polynomial solutions, the only orthogonal polynomials with respect to a positive definite linear functional are *Jacobi polynomials* $P_l^{(\alpha,\beta)}(x)$, *(generalized) Laguerre polynomials* $L_l^{(\alpha)}(x)$ and *Hermite polynomials* $H_l(x)$; for example see [18]. These polynomials are often called *classical orthogonal polynomials*; without assuming positive definiteness, some authors also consider Bessel polynomials as a class of classical polynomials. For further arguments below, note that $P_l^{(\alpha,\beta)}(x), L_l^{(\alpha)}(x), H_l(x) \in \mathbb{Q}[x]$ for all $\alpha, \beta \in \mathbb{Q}$.

In Appendix A we summarize basic properties on classical orthogonal polynomials which are used in the next sections.

2.2. **Riesz-Shohat theorem.** We use the same notation a, b, Φ_l, K_l as in the previous subsection.

The following was first obtained by Riesz [23, p.23] for $k=2$ and generalized by Shohat [27, Theorem I] for $k \geq 3$, which plays a key role throughout this paper.

Proposition 2.4 (Riesz-Shohat Theorem)**.** *Let $c_1, \ldots, c_{r+1}$ be distinct real numbers and $\omega_{r+1}(x) = \prod_{i=1}^{r+1}(x - c_i)$. Let k be an integer with $1 \leq k \leq r+2$. The following are equivalent.*

(i) *There exist real numbers $\gamma_1, \ldots, \gamma_{r+1}$ such that*

$$\sum_{i=1}^{r+1} \gamma_i f(c_i) = \int_a^b f(x) d\mu(x), \tag{4}$$

is a quadrature formula of degree $2r+2-k$, that is, (4) *holds for all polynomials $f(x)$ of degree at most $2r+2-k$.*

(ii) *For all polynomials $g(x)$ of degree at most $r+1-k$, $\int_a^b \omega_{r+1}(x) g(x) d\mu(x) = 0$.*

(iii) *The polynomial $\omega_{r+1}(x)$ is a quasi-orthogonal polynomial of degree $r+1$ and order $k-1$, that is, there exist real numbers $b_1, \ldots, b_{k-1}$ such that $\omega_{r+1}(x) = \Phi_{r+1}(x) + b_1\Phi_r(x) + \cdots + b_{k-1}\Phi_{r+2-k}(x)$.*

Furthermore, if the above equivalent conditions hold, then $\gamma_1, \ldots, \gamma_{r+1}$ in (i) *satisfy*

$$\gamma_i = \int_a^b \frac{\omega_{r+1}(x)}{(x-c_i)\omega'_{r+1}(c_i)} d\mu(x).$$

Quadrature formulas, appearing in the context of numerical integrations, are often *positive*, namely quadrature formulas with positive weights γ_i. Positive quadrature for Gaussian integration $\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-t^2}\, dt$ are also called *Gaussian designs* in algebraic combinatorics [1]. Some of the standard facts about quadrature formulas, used in this paper without detailed explanation, can be found in Dunkl-Xu [12] and Stroud [30].

Let r be a positive integer. We consider a 'normalized' Christoffel-Darboux kernel given by

$$f_r(x,y) = \frac{\Phi_{r+1}(x)\Phi_r(y) - \Phi_r(x)\Phi_{r+1}(y)}{x-y}. \tag{5}$$

By Proposition 2.1, f_r is a bivariate polynomial in x and y.

The following lemma is simple but creates a connection between the equations (1) and a projective curve associated with f_r.

Lemma 2.5. *Let $x_1, \dots, x_{r+1}, y_1, \dots, y_{r+1}$ be real numbers satisfying* (1) *for* $(m,n) = (r+1, 2r)$. *Assume that $y_1, \dots, y_{r+1}$ are distinct. Then we have*

$$f_r(y_i, y_j) = 0 \quad \textit{for every distinct } i, j.$$

Proof. Let $\omega_{r+1}(x) = \prod_{i=1}^{r+1}(x - y_i)$. By Proposition 2.4, there exists a real number c such that $\omega_{r+1}(x) = \Phi_{r+1}(x) + c\Phi_r(x)$. Hence we have

$$\Phi_{r+1}(y_i) + c\Phi_r(y_i) = 0$$

for every i. Eliminating c, we see that $f_r(y_i, y_j) = 0$ for every distinct i, j. □

2.3. **Hausdorff-type equations.** The following fact was originally designed by Hausdorff [16] as a simplification of Hilbert's solution of Waring problem in number theory. For a brief summary concerning Hausdorff's work, we refer the readers to [19, p.4700] and Section 2.3 of [24].

Lemma 2.6 (Hausdorff's Lemma)**.** *Let r be a positive integer. Then there exist rationals $x_1, \dots, x_{r+1}, y_1, \dots, y_{r+1}$ such that*

$$\sum_{i=1}^{r+1} x_i y_i^j = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^j e^{-t^2}\, dt, \qquad j = 0, 1, \dots, r. \tag{6}$$

This is a special case of the equations (1). In this paper, as a full generalization of (6), we consider the equations (1) for all rational classical polynomials.

The following result given in [24, Proposition 4.3] (cf. [12, Theorem 3.7.1] and [30]) is a variation of *Stroud bound* for positive quadrature formulas (cf. [24, Proposition 5.3]); this is also called a *Fisher-type bound* in algebraic combinatorics [1].

Proposition 2.7 (Stroud-type bound)**.** *Assume there exists a rational solution of the equations (1). Then it holds that $n \le 2m - 1$.*

The 'tight' case $n = 2m - 1$ corresponds to the Gaussian quadrature and the nodes y_i are the zeros of the orthogonal polynomial of degree m with respect to the weight function w (cf. [12, 32]). This is a special case of Riesz-Shohat theorem (Proposition 2.4).

A remarkable case is the Gaussian integration, corresponding to the Hermite polynomials. Then by a classical result of Schur [26] (see also [31]), the polynomials $H_{2k}(x)$ and $H_{2k+1}(x)/x$ are irreducible over $\mathbb{Q}$. Therefore the equations (1) have no rational solutions.

Thus, the present authors [24] considered the case $n = 2m - 2$ and proved a nonexistence theorem of rational solutions.

Theorem 2.8 ([24])**.** *If $r \equiv 2, 3, 4, 5, 6 \pmod 8$, then there do not exist rationals $x_1, \dots, x_{r+1}, y_1, \dots, y_{r+1}$ such that*

$$\sum_{i=1}^{r+1} x_i y_i^j = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^j e^{-t^2}\, dt, \quad j = 0, 1, \dots, 2r. \tag{7}$$

In [24], Theorem 2.8 was proved by utilizing an elegant explicit formula for the discriminant of a quasi-Hermite polynomial of order one. In Section 4, we improve this and prove a complete nonexistence theorem by using Newton polygons, and similarly for Legendre and Laguerre polynomials.

3. Proof of Theorems 1.1 and 1.3

Hereafter and throughout this section, let $\{\Phi_l\}$ be a series of rational classical polynomials and fix a positive integer r. Let $F_r(X, Y, Z)$ be the homogenization of the (normalized) Christoffel-Darboux kernel $f_r(x, y)$, that is,

$$F_r(X, Y, Z) = Z^{2r} f_r\left(\frac{X}{Z}, \frac{Y}{Z}\right). \tag{8}$$

We denote by C_r the projective curve defined by $F_r(X, Y, Z) = 0$. We say that a singular point P on a curve C is an *ordinary multiple point* of multiplicity m if C has m distinct simple tangent lines at P (cf. [14, Chapter 3, § 1]).

Lemma 3.1. *Let $r \geq 2$. Then the points $(0 : 1 : 0)$ and $(1 : 0 : 0)$ on C_r are ordinary multiple points of multiplicity r.*

Proof. It is sufficient to prove the lemma for the point $(0 : 1 : 0)$ by symmetry. By (3), we have

$$\begin{aligned} F_r(x, 1, z) &= \sum_{k=0}^{r} c_k z^{2r} \Phi_k\left(\frac{x}{z}\right) \Phi_k\left(\frac{1}{z}\right) \\ &= c z^r \Phi_r\left(\frac{x}{z}\right) + (\text{terms of degree greater than } r), \end{aligned}$$

where $c_0, c_1, \ldots, c_r$ and c are non-zero real numbers. Since $\Phi_r(x)$ has no multiple roots by Proposition 2.2, the point $(0 : 1 : 0)$ is an ordinary multiple point of multiplicity r. □

Lemma 3.2. *Let $r \geq 2$. Then the curve C_r has only two singular points $(0 : 1 : 0)$ and $(1 : 0 : 0)$.*

Proof. We first consider the case where $Z = 0$. By definition, we have $F_r(X, Y, 0) = cX^r Y^r$, where c is some constant. Hence the points on C_r with $Z = 0$ are $(0 : 1 : 0)$ and $(1 : 0 : 0)$. These two points are singular points by Lemma 3.1.

Next we consider the case where $Z \neq 0$. It is sufficient to prove that the system of equations $f_r(x, y) = (\partial f_r/\partial x)(x, y) = (\partial f_r/\partial y)(x, y) = 0$ has no solutions.

By the definition of f_r,

$$(x - y) f_r(x, y) = \Phi_{r+1}(x)\Phi_r(y) - \Phi_{r+1}(y)\Phi_r(x). \tag{9}$$

Taking the derivatives of the both sides with respect to x, we have

$$f_r(x, y) + (x - y)\frac{\partial f_r}{\partial x}(x, y) = \Phi'_{r+1}(x)\Phi_r(y) - \Phi_{r+1}(y)\Phi'_r(x). \tag{10}$$

Similarly taking the derivatives of the both sides with respect to y, we have

$$-f_r(x, y) + (x - y)\frac{\partial f_r}{\partial y}(x, y) = \Phi_{r+1}(x)\Phi'_r(y) - \Phi'_{r+1}(y)\Phi_r(x). \tag{11}$$

Let $(x, y) = (\xi, \eta)$ be a solution of $f_r(x, y) = (\partial f_r/\partial x)(x, y) = (\partial f_r/\partial y)(x, y) = 0$. By (9), (10) and (11), we have

$$\Phi_{r+1}(\xi)\Phi_r(\eta) = \Phi_{r+1}(\eta)\Phi_r(\xi), \tag{12}$$

$$\Phi'_{r+1}(\xi)\Phi_r(\eta) = \Phi_{r+1}(\eta)\Phi'_r(\xi), \tag{13}$$

$$\Phi_{r+1}(\xi)\Phi'_r(\eta) = \Phi'_{r+1}(\eta)\Phi_r(\xi). \tag{14}$$

If $\Phi_r(\xi) = 0$, then we have $\Phi_{r+1}(\xi)\Phi_r(\eta) = 0$ by (12). By Proposition 2.2, we have $\Phi_r(\eta) = 0$. Hence we have $\Phi_{r+1}(\eta)\Phi_r'(\xi) = 0$ by (13), which contradicts Propositions 2.2 and 2.3. Therefore $\Phi_r(\xi) \neq 0$. By similar arguments, we have $\Phi_r(\eta) \neq 0$, $\Phi_{r+1}(\xi) \neq 0$ and $\Phi_{r+1}(\eta) \neq 0$.

Let $\gamma = \Phi_{r+1}(\xi)/\Phi_r(\xi) = \Phi_{r+1}(\eta)/\Phi_r(\eta) \neq 0$. By (13) and (14), we have

$$\Phi_{r+1}'(\xi) = \gamma\Phi_r'(\xi), \quad \Phi_{r+1}'(\eta) = \gamma\Phi_r'(\eta). \tag{15}$$

Claim. It holds that $\xi = \eta$.

Proof of Claim. By the derivative formulas and the three-term relations in Appendix A, we have

$$\begin{aligned} \rho(x)\Phi_{r+1}'(x) &= (a_r x + b_r)\Phi_{r+1}(x) + c_r\Phi_r(x), \\ \rho(x)\Phi_r'(x) &= (d_r x + e_r)\Phi_r(x) + f_r\Phi_{r+1}(x), \end{aligned} \tag{16}$$

where $a_r, b_r, c_r, d_r, e_r, f_r$ are constants and

$$\rho(x) = \begin{cases} 1 - x^2 & \text{if } \Phi_n = P_n^{(\alpha,\beta)}, \\ x & \text{if } \Phi_n = L_n^{(\alpha)}, \\ 1 & \text{if } \Phi_n = H_n. \end{cases}$$

By substituting $x = \xi$ in (16) and then combining them and (15), we have

$$((a_r\xi + b_r)\gamma + c_r)\,\Phi_r(\xi) = \left((d_r\xi + e_r)\gamma + f_r\gamma^2\right)\Phi_r(\xi).$$

Since $\Phi_r(\xi) \neq 0$, we have

$$(d_r - a_r)\gamma\xi = -f_r\gamma^2 + (b_r - e_r)\gamma + c_r.$$

By computing the coefficients with the formulas in Appendix A,

$$d_r - a_r = \begin{cases} 2r + \alpha + \beta + 2 & \text{if } \Phi_n = P_n^{(\alpha,\beta)}, \\ 1 & \text{if } \Phi_n = L_n^{(\alpha)}, \\ 2 & \text{if } \Phi_n = H_n. \end{cases}$$

Since $\alpha > -1$ and $\beta > -1$, we have $2r + \alpha + \beta + 2 \neq 0$. Since $\gamma \neq 0$,

$$\xi = \frac{-f_r\gamma^2 + (b_r - e_r)\gamma + c_r}{(d_r - a_r)\gamma}. \tag{17}$$

By the same argument, it is seen that the right-hand side of (17) equals to η. Therefore we obtain the claim.

Now, to consider the case where $\xi = \eta$, we take the derivatives of both sides of (10) with respect to x. Then we have

$$2\frac{\partial f_r}{\partial x}(x,y) + (x - y)\frac{\partial^2 f_r}{\partial x^2}(x,y) = \Phi_{r+1}''(x)\Phi_r(y) - \Phi_{r+1}(y)\Phi_r''(x).$$

Substituting $x = y = \xi$, we have $\Phi_{r+1}''(\xi)\Phi_r(\xi) = \Phi_{r+1}(\xi)\Phi_r''(\xi)$. Since $\gamma = \Phi_{r+1}(\xi)/\Phi_r(\xi)$, we have

$$\Phi_{r+1}''(\xi) = \gamma\Phi_r''(\xi). \tag{18}$$

By using the second-order linear differential equations (40), (44) and (48), Φ_n satisfies the Sturm-Liouville equation:

$$\rho(x)\Phi_n''(x) + q(x)\Phi_n'(x) - \lambda_n\Phi_n(x) = 0, \tag{19}$$

where $q(x)$ is a polynomial and λ_n is the eigenvalue corresponding to Φ_n.

Then it follows by (19) that

$$\rho(\xi)\Phi_r''(\xi) + q(\xi)\Phi_r'(\xi) - \lambda_r\Phi_r(\xi) = 0,$$
$$\rho(\xi)\Phi_{r+1}''(\xi) + q(\xi)\Phi_{r+1}'(\xi) - \lambda_{r+1}\Phi_{r+1}(\xi) = 0.$$

Since $(d^k\Phi_{r+1}/dx^k)(\xi) = \gamma(d^k\Phi_r/dx^k)(\xi)$ for $k = 0, 1, 2$, we obtain

$$(\lambda_{r+1} - \lambda_r)\gamma\Phi_r(\xi) = 0.$$

We have $\lambda_r \neq \lambda_{r+1}$ since the eigenvalues λ_r and λ_{r+1} correspond to linearly independent eigenfunctions $\Phi_r(x)$ and $\Phi_{r+1}(x)$ respectively. Hence we have $\gamma\Phi_r(\xi) = 0$, which is a contradiction. This completes the proof of this lemma. □

We are now ready to complete the proof of Theorem 1.1:

Proof of Theorem 1.1 (i). The statement is easily verified for $r = 1$. In fact, by Proposition 2.1, we have $f_1(x, y) = c\Phi_1(x)\Phi_1(y) + d$, where c and d are nonzero constants. Since $\Phi_1(x)$ is an affine linear polynomial, $f_1(x, y)$ is absolutely irreducible.

We consider the case where $r \geq 2$. Assume that C_r is not absolutely irreducible. Then there exist homogeneous polynomials G and H of positive degree with coefficients in $\overline{\mathbb{Q}}$ such that $F_r = GH$. If G and H have a common factor, then C_r has infinitely many singular points. This contradicts Lemma 3.2. Hence G and H have no common factors.

Let D and E be the curves defined by $G = 0$ and $H = 0$ respectively. We denote by $I(P, D \cap E)$ the intersection number of D and E at P. By Bézout's theorem [14, Chapter 5, § 3],

$$\sum_P I(P, D \cap E) = (\deg G)(\deg H).$$

Let $P_1 = (0 : 1 : 0)$ and $P_2 = (1 : 0 : 0)$. If $I(P, D \cap E) > 0$, then P is a singular point on C_r. Therefore, by Lemma 3.2, we have

$$I(P_1, D \cap E) + I(P_2, D \cap E) = (\deg G)(\deg H).$$

Let $i = 1, 2$. By Lemma 3.1, P_i is an ordinary multiple point. By [14, Chapter 3, § 3, (5)], we have

$$I(P_i, D \cap E) = m_{P_i}(D)m_{P_i}(E),$$

where $m_{P_i}(D)$ and $m_{P_i}(E)$ are the multiplicities of D and E at P_i respectively. Therefore we have

$$m_{P_1}(D)m_{P_1}(E) + m_{P_2}(D)m_{P_2}(E) = (\deg G)(\deg H). \tag{20}$$

Let $k = \deg G (\geq 1)$. Then we have $\deg H = 2r - k$. We may assume $k \leq r$ without loss of generality. Note that

$$m_{P_i}(D) + m_{P_i}(E) = m_{P_i}(C_r) = r, \quad 0 \leq m_{P_i}(D) \leq k.$$

When $1 \leq k \leq r/2$,

$$m_{P_i}(D)m_{P_i}(E) = m_{P_i}(D)\left(r - m_{P_i}(D)\right) \leq k(r - k).$$

Therefore,

$$m_{P_1}(D)m_{P_1}(E) + m_{P_2}(D)m_{P_2}(E) \leq 2k(r - k) < k(2r - k) = (\deg G)(\deg H).$$

This contradicts (20).

When $r/2 < k \leq r$,

$$m_{P_i}(D)m_{P_i}(E) = m_{P_i}(D)\left(r - m_{P_i}(D)\right) \leq \frac{1}{4}r^2.$$

However,

$$(\deg G)(\deg H) = k(2r - k) > \frac{3}{4}r^2.$$

Therefore,

$$m_{P_1}(D)m_{P_1}(E) + m_{P_2}(D)m_{P_2}(E) \le \frac{1}{2}r^2 < \frac{3}{4}r^2 < (\deg G)(\deg H).$$

This contradicts (20). □

Proof of Theorem 1.1 (ii). Let g be the genus of C_r. When $r = 1$, since C_r is a conic, we have $g = 0$.

Let $r \ge 2$. By Lemmas 3.1 and 3.2, and by Theorem 1.1 (i), C_r is an absolutely irreducible curve with only ordinary multiple points. By [14, § 8.3, Proposition 5],

$$g = \frac{(2r-1)(2r-2)}{2} - \sum_{P \in C_r} \frac{m_P(C_r)(m_P(C_r)-1)}{2},$$

where $m_P(C_r)$ is the multiplicity of C_r at P. By Lemma 3.1,

$$m_P(C_r) = \begin{cases} r & \text{if } P = (0:1:0), (1:0:0), \\ 1 & \text{if } P \in C_r \setminus \{(0:1:0), (1:0:0)\}. \end{cases}$$

Therefore we have

$$g = (2r-1)(r-1) - r(r-1) = (r-1)^2.$$ □

Finally, we prove Theorem 1.3 using Lemma 2.5.

Proof of Theorem 1.3. When $r \ge 3$, the genus of C_r is at least 4 by Theorem 1.1 (ii). Since α and β are rational numbers, C_r is defined over $\mathbb{Q}$. Therefore, by Faltings's Theorem [13], C_r has only finitely many rational points. Hence the theorem follows from Theorem 1.1 (ii) and Lemma 2.5. □

Remark 3.3. Bannai and Ito [5] posed the problem of classifying rational classical polynomials for which the projective curves C_r actually have rational points. Theorem 1.4, together with Lemma 2.5, implies that the curve C_r has no rational points for the Hermite and Legendre polynomials. Whereas, it has rational points for Chebyshev polynomials of the second kind, or more generally for Gegenbauer (ultraspherical) polynomials, as will be shown in Section 5 later. The problem depends on the choice of classical polynomials.

4. Proof of Theorem 1.4

In this section we prove Theorem 1.4. We first recall the definition and some properties of Newton polygons.

Let $v_p\colon \mathbb{Q}^\times \to \mathbb{Z}$ be the p-adic valuation which satisfies $v_p(p) = 1$. We use the convention that $v_p(0) = \infty$. For a positive integer l, we denote by $l!!$ the double factorial of l, that is, $l = l(l-2)\cdots 3 \cdot 1$ if l is odd, $l = l(l-2)\cdots 4 \cdot 2$ if l is even. We use the convention that $(-1)!! = 0!! = 1$.

Let

$$f(x) = \sum_{k=0}^{l} a_k x^{l-k}$$

be a polynomial with integer coefficients with $a_0 a_l \neq 0$. Let p be a prime number. We consider a set of points

$$S = \{(k, v_p(a_k)) \bigm| a_k \neq 0\}$$

in $\mathbb{R}^2$. The lower edges of the convex hull of S is called the *Newton polygon* of $f(x)$ with respect to p. The Newton polygon is a broken line from $(0, v_p(a_0))$ to $(l, v_p(a_l))$.

The following fact is due to Dumas [11].

Proposition 4.1 (Dumas's Theorem)**.** *Let $g(x)$ and $h(x)$ be polynomials with integer coefficients with $g(0)h(0) \neq 0$. Let p be a prime number. Then the Newton polygon of $g(x)h(x)$ with respect to p is obtained by joining the edges of the Newton polygons of $g(x)$ and $h(x)$ in order of non-decreasing slope.*

Proof. See [11] or [21, Theorem 2.2.1]. □

As a consequence of Proposition 4.1, we obtain a lemma used in the proof of Theorem 1.4.

Lemma 4.2 (cf. [4])**.** *Let $f(x)$ be a polynomial with integer coefficients. If the zeros of $f(x)$ are all rational, then all edges of the Newton polygon of $f(x)$ have integral slope.*

To prove Theorem 1.4 (ii), we need some more preparations. Let $P_l(x) := P_l^{(0,0)}(x)$, the *Legendre polynomial* of degree l. Then

$$P_l(x) = \sum_{k=0}^{\lfloor l/2 \rfloor} (-1)^k \frac{(2l-2k)!}{2^l k!(l-k)!(l-2k)!} x^{l-2k}. \tag{21}$$

Note that $(2l-2k)!/(k!(l-k)!(l-2k)!)$ is an integer since it is a multinomial coefficient. Let

$$R_l(x) = 2^{e_l} P_l(x), \tag{22}$$

where e_l is the smallest integer such that $R_l(x)$ has integer coefficients.

Proposition 4.3 ([17])**.** *Let a_k be the coefficient of x^k in $R_l(x)$, that is,*

$$R_l(x) = \sum_{k=0}^{l} a_k x^k.$$

Let p be an odd prime number. Then the following hold:

(i) *The coefficients $a_0, a_1, \ldots, a_l$ have no common factor.*
(ii) *Assume that $\lfloor l/p \rfloor$ is odd and less than p. If l is even, then $v_p(a_0) = 1$ and $v_p(a_k) \geq 1$ for $k = 1, 2, \ldots, p$. If l is odd, then $v_p(a_1) = 1$ and $v_p(a_k) \geq 1$ for $k = 2, 3, \ldots, p-1$.*

The following fact was proved by Chebyshev (cf. [15, Theorem 418]).

Proposition 4.4 (Bertrand's postulate)**.** *For any integer $l \geq 1$, there exists a prime number p such that $l < p \leq 2l$.*

Lemma 4.5. *For any integer $l \geq 2$, there exists a prime number p such that $(l+1)/2 < p \leq l$.*

Proof. If l is even, there exists a prime number p such that $l/2 < p \leq l$. Since $l/2$ and p are integers, $p \geq l/2 + 1 > (l+1)/2$. Similarly, if l is odd, we have a prime number p such that $(l+1)/2 < p \leq l+1$. Since $l \geq 3$, we have $p > 2$. Since $l+1$ is even, we have $p \leq l$. □

We are now in a position to complete the proof of Theorem 1.4.

Proof of Theorem 1.4 (i). Assume there exist rationals $x_1, \ldots, x_{r+1}, y_1, \ldots, y_{r+1}$ which satisfy (7). We may assume that $y_1, \ldots, y_{r+1}$ are distinct since Problem 1.2 has no solutions for $(m, n) = (r, 2r)$ by Proposition 2.7. By Proposition 2.4, there exists $c \in \mathbb{R}$ such that the zeros of the quasi-Hermite polynomial $H_{r+1;c}(x)$ are $y_1, \ldots, y_{r+1}$. Since $y_1, \ldots, y_{r+1}$ are rational, c is also rational. Write $c = s/t$ in lowest terms. Let

$$f(x) = tH_{r+1;c}(x) = tH_{r+1}(x) + sH_r(x).$$

Then $f(x)$ is a polynomial with integer coefficients whose zeros are all rational.

In the following, we consider the Newton polygon of $f(x)$ with respect to 2. Let a_k be the coefficient of x^{r+1-k} in $f(x)$, that is, $f(x) = \sum_{k=0}^{r+1} a_k x^{r+1-k}$. We denote by P_k the point $(k, v_2(a_k))$ in $\mathbb{R}^2$. By (45), we have

$$f(x) = t \sum_{k=0}^{\lfloor (r+1)/2 \rfloor} (-1)^k \frac{(r+1)!}{k!(r+1-2k)!}(2x)^{r+1-2k} + s \sum_{k=0}^{\lfloor r/2 \rfloor} (-1)^k \frac{r!}{k!(r-2k)!}(2x)^{r-2k}.$$

Since

$$\frac{n!}{k!(n-2k)!} = \frac{n!}{(2k)!(n-2k)!}\frac{(2k)!}{k!} = \binom{n}{2k} 2^k (2k-1)!!,$$

we have

$$\begin{aligned} v_2(a_{2k}) &= v_2\left(\binom{r+1}{2k}\right) - k + r + 1 + v_2(t), \\ v_2(a_{2k+1}) &= v_2\left(\binom{r}{2k}\right) - k + r + v_2(s). \end{aligned} \tag{23}$$

We first consider the case where r is even. By (23), we have

$$v_2(a_0) = r + 1 + v_2(t), \quad v_2(a_1) = r + v_2(s),$$

$$v_2(a_{2k}) \geq -k + r + 1 + v_2(t),\ v_2(a_{2k+1}) \geq -k + r + v_2(s) \ \text{ for } 1 \leq k \leq \frac{r-2}{2},$$

$$v_2(a_r) = \frac{r}{2} + 1 + v_2(t), \quad v_2(a_{r+1}) = \frac{r}{2} + v_2(s).$$

Hence, if $1 + v_2(t) \leq v_2(s)$, then the Newton polygon of $f(x)$ is the broken line $P_0 P_r P_{r+1}$. The edge $P_0 P_r$ has slope $-1/2$, which contradicts Lemma 4.2. Similarly, if $1 + v_2(t) > v_2(s)$, then the Newton polygon of $f(x)$ is the broken line $P_0 P_1 P_{r+1}$. The edge $P_1 P_{r+1}$ has slope $-1/2$, which is a contradiction.

Next, we consider the case where r is odd. Note that we have $r \geq 3$. By (23), we have

$$v_2(a_0) = r + 1 + v_2(t), \quad v_2(a_1) = r + v_2(s),$$

$$v_2(a_{2k}) \geq -k + r + 1 + v_2(t),\ v_2(a_{2k+1}) \geq -k + r + v_2(s) \text{ for } 1 \leq k \leq \frac{r-3}{2},$$

$$v_2(a_{r-1}) = \frac{r+3}{2} + v_2(t), \quad v_2(a_r) = \frac{r+1}{2} + v_2(s), \quad v_2(a_{r+1}) = \frac{r+1}{2} + v_2(t).$$

Hence, if $1 + v_2(t) \leq v_2(s)$, then the Newton polygon of $f(x)$ is the segment $P_0 P_{r+1}$, whose slope is equal to $-1/2$. This contradicts Lemma 4.2. Similarly, if $1 + v_2(t) > v_2(s)$, then the Newton polygon of $f(x)$ is the broken line $P_0 P_1 P_r P_{r+1}$. The edge $P_1 P_r$ has slope $-1/2$, which is a contradiction. $\square$

Proof of Theorem 1.4 (ii). Assume that there exist rationals $x_1, \ldots, x_{r+1}, y_1, \ldots, y_{r+1}$ which satisfy (1) with $w(t) = \frac{1}{2}\chi_{(-1,1)}(t)$. By Proposition 2.4, there exists $c \in \mathbb{Q}$ such that the zeros of $P_{r+1;c}(x)$ are $y_1, \ldots, y_{r+1}$.

We first assume that $r \geq 3$. Since $R_n(x) = 2^{e_n} P_n(x)$, there exists a rational number u and relatively prime integers s and t such that

$$P_{r+1;c}(x) = u f(x), \quad f(x) = t R_{r+1}(x) + s R_r(x).$$

Then $f(x)$ has integer coefficients and the zeros of $f(x)$ are all rational. By Lemma 4.5, there exists a prime number p such that $(r+1)/2 < p \leq r$. Since $r \geq 3$, we have $p > 2$. We consider the Newton polygon of $f(x)$ with respect to p.

Let

$$R_{r+1}(x) = \sum_{k=0}^{\lfloor (r+1)/2 \rfloor} a_{r+1-2k} x^{r+1-2k}, \quad R_r(x) = \sum_{k=0}^{\lfloor r/2 \rfloor} a_{r-2k} x^{r-2k}.$$

By Proposition 4.3 (ii), $v_p(a_0) = v_p(a_1) = 1$ and $v_p(a_k) \geq 1$ for $2 \leq k \leq p-1$. Let b_k be the coefficient of x^{r+1-k} in $f(x)$, that is,

$$f(x) = \sum_{k=0}^{r+1} b_k x^{r+1-k}.$$

Then we have $b_k = ta_{r+1-k}$ if k is even; $b_k = sa_{r+1-k}$ if k is odd. We denote by P_k the point $(k, v_p(b_k))$ in $\mathbb{R}^2$.

Since s and t are relatively prime, we have $v_p(s) = 0$ or $v_p(t) = 0$. Hence it suffices to consider four cases: (1) r is even and $v_p(s) = 0$; (2) r is even and $v_p(s) \geq 1$; (3) r is odd and $v_p(t) = 0$; (4) r is odd and $v_p(t) \geq 1$.

If either (1) or (3) holds, then we have $v_p(b_{r+1}) = 1$ and $v_p(b_{r+1-k}) \geq 1$ for $1 \leq k \leq p-1$. By Proposition 4.3 (i), there exists an index i such that $v_p(b_i) = 0$. Let i be the largest index such that $v_p(b_i) = 0$. Then the Newton polygon of $f(x)$ has the edge P_iP_{r+1}. Since $i \leq r+1-p$, the slope of P_iP_{r+1} satisfies the inequality

$$0 < \frac{v_p(b_{r+1}) - v_p(b_i)}{r+1-i} \leq \frac{1}{p}.$$

This contradicts Lemma 4.2.

If either (2) or (4) holds, then we have $v_p(b_{r+1}) \geq 2$, $v_p(b_r) = 1$, and $v_p(b_{r+1-k}) \geq 1$ for $2 \leq k \leq p-1$. By Proposition 4.3 (i), there exists an index i such that $v_p(b_i) = 0$. Let i be the largest index such that $v_p(b_i) = 0$. Then we have $i \leq r+1-p$. Hence the slope of P_iP_r satisfies the inequality

$$0 < \frac{v_p(b_r) - v_p(b_i)}{r-i} \leq \frac{1}{p-1}.$$

Since the slope of the segment P_rP_{r+1} is greater than or equal to 1, the Newton polygon of $f(x)$ has the edges P_iP_r and P_rP_{r+1}. The slope of P_iP_r is not an integer, which contradicts Lemma 4.2.

Next, we consider the case where $r = 2$. Let $Q_n(x) = P_n(2x+1)$. Write $c = s/t$ in lowest terms. Let $f(x) = tP_{3;c}(2x+1)$. Then all zeros of $f(x)$ are rational. Since

$$Q_2(x) = 6x^2 + 6x + 1, \quad Q_3(x) = 20x^3 + 30x^2 + 12x + 1,$$

we have

$$f(x) = tQ_3(x) + sQ_2(x) = 20tx^3 + (30t + 6s)x^2 + (12t + 6s)x + (t + s).$$

Hence $f(x)$ has integer coefficients. We consider the Newton polygon of $f(x)$ with respect to the prime 2. Let b_k be the coefficient of x^{3-k} in $f(x)$ and let P_k be the point $(k, v_2(b_k))$ in $\mathbb{R}^2$.

Since s and t are relatively prime, we have $v_2(s) = 0$ or $v_2(t) = 0$. It follows that

$$v_2(20t) \geq 3,\ v_2(30t+6s) = v_2(12t+6s) = 1,\ v_2(t+s) = 0, \quad \text{if } v_2(t) \geq 1, v_2(s) = 0,$$

$$v_2(20t) = 2,\ v_2(30t+6s) \geq 2,\ v_2(12t+6s) = 1,\ v_2(t+s) \geq 1, \quad \text{if } v_2(t) = v_2(s) = 0,$$

$$v_2(20t) = 2,\ v_2(30t+6s) = 1,\ v_2(12t+6s) \geq 2,\ v_2(t+s) = 0, \quad \text{if } v_2(t) = 0, v_2(s) \geq 1.$$

Hence the Newton polygon of $f(x)$ has the edge P_1P_3, P_0P_2, and P_1P_3 with slope $-1/2$, respectively.

Therefore, the Newton polygon of $f(x)$ has an edge whose slope is not an integer. This contradicts Lemma 4.2. □

Proof of Theorem 1.4 (iii). Assume that there exist rationals $x_1, \ldots, x_{r+1}, y_1, \ldots, y_{r+1}$ which satisfy (1) with $w(t) = e^{-t}\chi_{(0,\infty)}(t)$. By Proposition 2.4, there exists $c \in \mathbb{Q}$ such that the zeros of $L^{(0)}_{r+1;c}(x)$ are $y_1, \ldots, y_{r+1}$. Write $c = s/t$ in lowest terms. Let

$$f(x) = t(r+1)!L^{(0)}_{r+1;c}(x) = t(r+1)!L^{(0)}_{r+1}(x) + s(r+1)!L^{(0)}_{r}(x).$$

Then $f(x)$ is a polynomial with integer coefficients. Let a_k be the coefficient of x^{r+1-k} in $f(x)$, that is,

$$f(x) = \sum_{k=0}^{r+1} a_k x^{r+1-k}.$$

By (41), for $1 \leq k \leq r+1$,

$$a_k = (-1)^{r+1-k} \frac{(r+1)!}{(r+1-k)!} \left(t\binom{r+1}{k} + s\binom{r}{k-1} \right). \tag{24}$$

We also have

$$\begin{aligned} a_0 &= (-1)^{r+1} t, & a_1 &= (-1)^r (r+1)(t(r+1)+s), \\ a_r &= -(r+1)!(r(t+s)+t), & a_{r+1} &= (r+1)!(t+s). \end{aligned} \tag{25}$$

Since $r \geq 2$, there exists a prime number p such that $(r+1)/2 < p \leq r$ by Lemma 4.5. We consider the Newton polygon of $f(x)$ with respect to p. We denote by P_k the point $(k, v_p(a_k))$ in $\mathbb{R}^2$.

Since $(r+1)/2 < p \leq r$, we have $v_p((r+1)!) = 1$. By (24), $v_p(a_k) \geq 1$ for $r+2-p \leq k \leq r+1$. By (25), if $t+s \not\equiv 0 \pmod{p}$, then $v_p(a_{r+1}) = 1$. If $t+s \equiv 0 \pmod{p}$, then $v_p(s) = v_p(t) = 0$ since s and t are relatively prime. Then, by (25), we have $v_p(a_{r+1}) \geq 2$ and $v_p(a_r) = 1$. Let $j = r$ if $t+s \equiv 0 \pmod{p}$ and $j = r+1$ otherwise.

If $v_p(t) = 0$, then $v_p(a_0) = 0$ by (25). If $v_p(t) \geq 1$, then $v_p(s) = 0$ and $v_p(a_1) = 0$ by (25). Let i be the largest index such that $v_p(a_i) = 0$. Then $i \leq r+1-p$ since $v_p(a_k) \geq 1$ for $r+2-p \leq k \leq r+1$.

The Newton polygon of $f(x)$ has the edge P_iP_j. Since $i \leq r+1-p$ and $j \geq r$, the slope of P_iP_j satisfies the inequality

$$0 < \frac{v_p(a_j) - v_p(a_i)}{j-i} \leq \frac{1}{p-1}.$$

If $r \geq 3$, then $p \geq 3$ and the above inequality contradicts Lemma 4.2.

Assume $r = 2$. Then we have $p = 2$. If $t+s \not\equiv 0 \pmod{2}$, then $j = r+1 = 3$. Hence the slope of the edge P_iP_j is less than or equal to $1/2$. If $t+s \equiv 0 \pmod{2}$, then $j = r = 2$ and $s \equiv t \equiv 1 \pmod{2}$. It then follows that $a_1 = 3(3t+s) \equiv 0 \pmod{2}$, that is, $v_2(a_1) \geq 1$. Hence we have $i = 0$. Therefore the Newton polygon of $f(x)$ has the edge P_0P_2 with slope $1/2$. This contradicts Lemma 4.2. ☐

Remark 4.6. (i) By the argument in the proof, we can obtain the following results. If r is even, then $H_{r+1;c}(x)$ has at most one rational zero. On the other hand, if r is odd, then $H_{r+1;c}(x)$ has at most two rational zeros. Moreover, if $c = s/t$ and $1 + v_2(t) > v_2(s)$, then $H_{r+1;c}(x)$ has two zeros in $\mathbb{Q}_2$ by the theory of Newton polygons; see [20, Chapter II, Propositions 6.3 and 6.4]. (ii) The polynomials $Q_n(x)$ in the above proof were used to prove the irreducibility of Legendre polynomials in certain cases by Wahab [34].

5. Concluding remarks and future works

We have shown algebro-geometric aspects of the Christoffel-Darboux kernels for rational classical polynomials. We have also found a novel connection between the projective curve C_r and the equations (1), and thereby proved nonexistence theorems for rational solutions.

We find, as a consequence of Theorem 1.1, that there exist at most finitely many rational points on the projective curve C_r. Bannai and Ito [5] posed the problem of classifying classical polynomials for which the curves C_r have rational points. We here provide an answer to the quadratic case. First, by Theorem 1.4 and Lemma 2.5, the curve C_2 has no rational points for the Hermite and Legendre polynomials.

However, the same conclusion does not hold for the Chebyshev polynomials of the second kind. In this case, the Christoffel-Darboux kernel $f_2(x,y)$ is given by

$$f_2(x,y) = 32x^2y^2 - 8x^2 + 8xy - 8y^2 + 4. \tag{26}$$

Clearly we have $f_2(1/2,-1/2) = f_2(-1/2,1/2) = 0$.

Theorem 5.1. *The equation $f_2(x,y) = 0$ has no rational solutions other than $(x,y) = (1/2,-1/2),(-1/2,1/2)$.*

Proof. Let (x,y) be a rational solution of $f_2(x,y) = 0$. We transform $f_2(x,y) = 0$ into a Weierstrass equation by imitating the transformation for Edwards curves (cf. [35, § 2.6.3]).

Multiplying the both sides of $f_2(x,y) = 0$ by $(8y^2-2)/4$,

$$(8y^2-2)^2x^2 + 2(8y^2-2)yx + (8y^2-2)(-2y^2+1) = 0.$$

Let $w = (8y^2-2)x + y$. Then we have $w^2 = 16y^4 - 11y^2 + 2$, which has a solution $(y,w) = (1/2,1/2)$. Following [35, § 2.5.3], let

$$y' = y - 1/2, \quad X' = \frac{w + 1/2 - 3y'}{(y')^2}, \quad Y' = \frac{w + 1/2 - 3y' + 4(y')^2}{(y')^3}.$$

Then we have a Weierstrass equation

$$(Y')^2 - 6X'Y' + 32Y' = (X')^3 + 4(X')^2 - 16X' - 64.$$

This equation defines an elliptic curve. By using Tate's algorithm (or the Laska-Kraus-Connell algorithm), we obtain a global minimal model

$$Y^2 + XY + Y = X^3 - 11X + 12, \tag{27}$$

where $X' = 4X - 4$ and $Y' = 8Y + 16X - 24$. Here X and Y are written in terms of x and y as follows:

$$X = \frac{8xy^2 + 4y^2 - 6y - 2x + 3}{(2y-1)^2}, \; Y = -\frac{32xy^3 - 8y^3 - 24xy^2 - 8xy + 12y + 6x - 6}{(2y-1)^3}.$$

The minimal model (27) coincides with the curve labeled 14a6 in Cremona's Elliptic Curve Data [8]. We denote by E the elliptic curve defined by (27). According to [8], $E(\mathbb{Q}) \cong \mathbb{Z}/6\mathbb{Z}$ and $(0,3)$ is a generator of $E(\mathbb{Q})$. Hence we have

$$E(\mathbb{Q}) = \left\{(0,3),(2,-2),\left(\frac{7}{4},-\frac{11}{8}\right),(2,-1),(0,-4),\mathcal{O}\right\},$$

where $\mathcal{O}$ is the point at infinity. The points $(0,3)$, $(2,-2)$, $(2,-1)$, and $(0,-4)$ do not correspond to points on $f_2(x,y) = 0$. The points $(7/4,-11/8)$ and $\mathcal{O}$ correspond to $(1/2,-1/2)$ and $(-1/2,1/2)$ respectively. □

What about the Chebyshev polynomials of the first kind? In this case, the Christoffel-Darboux kernel $f_2(x,y)$ is given by $f_2(x,y) = 8x^2y^2 - 4x^2 + 2xy - 4y^2 + 3$. Then by computing the 3-adic valuation of f_2, we conclude that f_2 has no rational points, unlike the degree-two Chebyshev polynomial of the second kind. We here omit to discuss higher degree cases and leave them as future works. Nevertheless, the question fairly depends on the choice of classical polynomials.

The *Gegenbauer polynomial* is a natural generalization of the Chebyshev polynomial of the second kind, defined as

$$C_m^{((d-2)/2)}(t) = \frac{\Gamma((d-1)/2)}{\Gamma(d-2)} \frac{\Gamma(m+d-2)}{\Gamma((2m+d-1)/2)} P_m^{((d-3)/2,(d-3)/2)}(t).$$

In the reminder of this section, we use the notation C_m for $C_m^{((d-2)/2)}$ and mainly deal with the case where d is an integer ≥ 3. Polynomials C_m satisfy the orthogonality condition (cf. [32, Section 4.7]) that

$$\int_{-1}^{1} C_m(t)C_n(t)(1-t^2)^{(d-3)/2}\,dt = \frac{\pi 2^{-d+4}\Gamma(d+n-2)}{n!(d+2n-2)(\Gamma(\frac{d-2}{2}))^2}\delta_{mn}. \tag{28}$$

A normalized version of this polynomial, defined by

$$Q_n(t) = \frac{d+2n-2}{d-2}C_n(t), \tag{29}$$

often appears in the context of the spherical design theory (cf. [10, Remark 2.2]). A finite subset X of the $(d-1)$-dimensional unit sphere $\mathbb{S}^{d-1}$ is called a *spherical n-design* if

$$\frac{1}{|X|}\sum_{x\in X} f(x) = \frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbb{S}^{d-1}} f(\omega)\,d\rho(\omega)$$

holds for every polynomial f of degree at most n, where ρ is the surface measure and $|\mathbb{S}^{d-1}|$ is the surface area of $\mathbb{S}^{d-1}$. It is well known (cf. [10, Theorem 2.4]) that

$$Q_n(1) = \binom{d+n-1}{d-1} - \binom{d+n-3}{d-1}. \tag{30}$$

A conclusion emerging from (28), (30) and the Christoffel-Darboux formula is the following. (This can also be proved using the three-term relation as in Definition 2.1 of [10].)

Proposition 5.2. *Let $R_r(t) := \sum_{k=0}^{r} Q_k(t)$ and $Q_{r+1;c}(t) = Q_{r+1}(t) + cQ_r(t)$. There exist some positive real number a and real number c, such that*

$$R_r(t) = \frac{aQ_{r+1;c}(t)}{t-1}. \tag{31}$$

Proof. It follows from direct calculations with (28), (30) that

$$R_r(t) = \sum_{k=0}^{r} Q_k(t) = \frac{d+2r-2}{(d-2)C_r(1)}K_r(t,1).$$

The result then follows from the Christoffel-Darboux formula (Proposition 2.1). □

Under the assumption that there exists a 'tight' spherical $2r$-design on $\mathbb{S}^{d-1}$, i.e. $2r$-design with equality in a certain lower bound called *Fisher-type bound* (cf. [10, Theorem 5.11]), Bannai and Damerell [3, Theorem 2] showed the rationality of all zeros of polynomial R_r. Combining this with 'interlacing property' of zeros of C_{r+1} and C_r, they finally proved a famous conjecture, Conjecture 7.8 of [10], that for any integers $d \geq 3$ and $r \geq 3$, there exists no tight $2r$-design on $\mathbb{S}^{d-1}$. Thus the following problem, which is also a special case of Problem 2 of [2, p.250], is of profound theoretical interest from the combinatorial viewpoint.

Problem 5.3. Let $d \geq 3$ be an integer. Then, do there exist rationals $x_1, \ldots, x_{r+1}$ and $y_1, \ldots, y_r$ such that

$$\sum_{i=1}^{r+1} x_i y_i^j = \frac{1}{\int_{-1}^{1}(1-t^2)^{(d-3)/2}\,dt}\int_{-1}^{1} t^j(1-t^2)^{(d-3)/2}\,dt, \quad j = 0,1,\ldots,2r \tag{32}$$

where $y_{r+1} = 1$?

By the Riesz-Shohat theorem (Proposition 2.4), (32) implies

$$C_{r+1;c}(t) = a\prod_{i=1}^{r+1}(t-y_i)$$

for some $a, c \in \mathbb{R}$. When $y_{r+1} = 1$, we have $C_{r+1;c}(1) = 0$ and hence by (29) and (30),

$$c = -\frac{r+d-2}{r+1}.$$

Delsarte et al. [10, Theorem 5.11] proved that $X \subset \mathbb{S}^{d-1}$ is a tight spherical $2r$-design if and only if the set $A(X)$ of Euclidean inner products between all distinct points in X, called the *distance set* of X, coincides with the set of all zeros of polynomial R_r. A remarkable theorem by Bannai and Damerell [3, Theorem 1] states the non-existence of tight $2r$-designs on $\mathbb{S}^{d-1}$ for $r \geq 3$ and $d \geq 3$, and hence in this case, Problem 5.3 has no rational solutions. In the case where $r = 2$, there actually exist a (27-points) tight 4-design on $\mathbb{S}^5$ with $A(X) = \{-1/2, 1/4\}$ and a (275-points) tight 4-design on $\mathbb{S}^{21}$ with $A(X) = \{-1/4, 1/6\}$ (cf. [10, Example 8.3]), and therefore equations (32) have a rational solution with

$$\text{(i)}\quad d = 6,\ x_1 = \frac{10}{27},\ x_2 = \frac{16}{27},\ x_3 = \frac{1}{27},\ y_1 = -\frac{1}{2},\ y_2 = \frac{1}{4},\ y_3 = 1, \quad \text{or}$$

$$\text{(ii)}\quad d = 22,\ x_1 = \frac{112}{275},\ x_2 = \frac{162}{275},\ x_3 = \frac{1}{275},\ y_1 = -\frac{1}{4},\ y_2 = \frac{1}{6},\ y_3 = 1.$$

(Pairs $(y_1, 1)$, $(y_2, 1)$, $(1, y_1)$, $(1, y_2)$ are rational points on $f_2(x, y) = 0$.) Equations (32) have a rational solution for $r = 2$ if and only if $d + 3$ is a square, since the quasi-Gegenbauer polynomial $C_{3,-d/3}(t)$ can be factored as

$$C_{3,-d/3}(t) = \frac{(d-2)d}{6}(t-1)((d+2)t^2 + 2t - 1)$$

and the discriminant of $(d+2)t^2 + 2t - 1$ is equal to $4(d+3)$.

What about ignoring the restriction $y_{r+1} = 1$? For example when $r = 2$, it follows (from (34) below) that equations (32) have a rational solution whenever $(d+2)/3$ is a square in $\mathbb{Q}$. Note that if equations (32) have a rational solution, then there exists some index $1 \leq i \leq 3$ for which

$$d(d+2)^2 y_i^4 - 3(d^2 + 4d + 1)y_i^2 + 3(d+2) \text{ is a square in } \mathbb{Q}. \tag{33}$$

Indeed, it follows from direct calculation with (34) that

$$C_{3;c}(t) = \frac{d-2}{6}(d(d+2)t^3 + 3cdt^2 - 3dt - 3c).$$

Since $C_{3;c}(y_i) = 0$, we have $c = dy_i((d+2)y_i^2 - 3)/(3(1 - dy_i^2))$. Therefore

$$C_{3;c}(t) = \frac{d(d-2)}{6(1-dy_i^2)}(t - y_i)((d+2)(1 - dy_i^2)t^2 + 2y_i(1-d)t + (d+2)y_i^2 - 3).$$

By considering the discriminant of the quadratic factor, we obtain (33). When $d \equiv 2 \pmod 3$ or $d \equiv 3 \pmod 4$, this is however impossible from standard arguments in elementary number theory.

The situation becomes quite different when $r \geq 3$. For example when $r = 3$, it follows from direct calculation that

$$C_{4;-(d+1)/4}(t) = \frac{1}{24}(d-2)d(t-1)((d+2)(d+4)t^3 + 3(d+2)t^2 - 3(d+2)t - 3)$$

and

$$R_3(t) = \frac{1}{6}d((d+2)(d+4)t^3 + 3(d+2)t^2 - 3(d+2)t - 3).$$

Then the following hold:

Theorem 5.4. *Let d be a* rational number *with $d \neq -2$ and let*

$$\tilde{R}(t) = (d+2)(d+4)t^3 + 3(d+2)t^2 - 3(d+2)t - 3.$$

If all the zeros of $\tilde{R}(t)$ are rational, then $d = -5$ or $d = -1$.

Note that the case where $d = -2$ is excluded because $\tilde{R}(t) = -3$ is a constant. In the following proof, Magma [7] was used for computation.

Proof. Let $d = Y/Z$ and $t = Z/X$. Then we have

$$X^3\tilde{R}(t) = -3X^3 - 3X^2Y - 6X^2Z + 3XYZ + 6XZ^2 + Y^2Z + 6YZ^2 + 8Z^3.$$

We denote by $F(X, Y, Z)$ the right-hand side. Let C be the projective curve defined by $F(X, Y, Z) = 0$. Applying the coordinate transformation

$$X = \frac{x-3}{3}, \quad Y = \frac{2x+y+5}{3}, \quad Z = \frac{-2x-y+1}{6}$$

to $F(X, Y, Z) = 0$, we obtain a Weierstrass equation

$$y^2 + xy + y = x^3 - x^2 - 14x + 29.$$

We denote by E the elliptic curve defined by the above equation. Then E coincides with the elliptic curve labeled 54b3 in Cremona's Elliptic Curve Data [8]. It is known that $E(\mathbb{Q})$ is isomorphic to $\mathbb{Z}/9\mathbb{Z}$ and generated by the point $(3, 1)$. Hence we have

$$E(\mathbb{Q}) = \{(3, 1), (-3, 7), (1, -5), (9, -29), (9, 19), (1, 3), (-3, -5), (3, -5), \mathcal{O}\}.$$

Therefore, since $d = 2(2x + y + 5)/(-2x - y + 1)$, if all the zeros of $\tilde{R}$ are rational, then d must belong to the set $\{-5, -4, -7/3, -1, 1\}$. By actually factoring $\tilde{R}$ for these values, we obtain the theorem. $\square$

Delsarte et al. [10, Theorem 7.7] proved the non-existence of tight 6-designs on $\mathbb{S}^{d-1}$ for $d \geq 3$, by using 'interlacing property' of zeros of Gegenbauer polynomials C_3 and C_4. Theorem 5.4 provides not only another proof but also a strengthening of this classical result. After the pioneering work by Delsarte et al., Bannai and Damerell [3] proved the non-existence of tight $2r$-designs on $\mathbb{S}^{d-1}$ for $d \geq 3$ and $r \geq 3$. Their proof is similar to the proof of Theorem 7.7 of [10], using 'interlacing property' of C_r and C_{r+1}. To present another proof of the Bannai-Damerell theorem is thus of natural interest from the spherical design theory. A possible approach for this could be the use of a certain expression of discriminants of quasi-Gegenbauer polynomials developed in [24, Corollary 3.3]. For example, it is shown that if d is divisible by 3, the discriminant of $C_{4;c}$ is not a square in $\mathbb{Q}$ for any rational c. Then, as in the proof of Corollary 4.7 of [24] (Theorem 2.8 of the present paper), we can prove, without assuming $y_4 = 1$, that Problem 5.3 has no rational solutions. The general cases will be left as challenging future works.

We may consider Problem 5.3 for $(m, n) = (r + 1, 2r + 1)$. It is known (see [4, Theorem 2]) that polynomial $C_{r+1}^{(d/2)}$ has irrational zeros for any $d \geq 0$ and $r \geq 3$, except for the case $(d, r) = (24, 4)$ in which all zeros of $C_5^{(12)}$ are rational (cf. [10, Example 8.5]). The Gegenbauer polynomials of degree 2 and 3 with respect to $(1 - t^2)^{(d-3)/2}dt$ are given by

$$\begin{aligned} C_2(t) &= -\frac{d-2}{2} + \frac{(d-2)d}{2}t^2, \\ C_3(t) &= -\frac{(d-2)d}{2}t + \frac{(d-2)d(d+2)}{6}t^3 \end{aligned} \tag{34}$$

and therefore all zeros of C_2 (resp., C_3) are rational if and only if d (resp., $(d + 2)/3$) is a square in $\mathbb{Q}$. In summary, as with the Hermite polynomial case, we can completely solve Problem 5.3 for the tight case, without assuming $y_{r+1} = 1$.

Although we have mainly discussed the situation $(m, n) = (r + 1, 2r)$, Problem 1.2 still has considerable interest for other pairs of m and n. For example, as already seen in Subsection 2.3, Hausdorff [16] proved that the equations (1) have a rational solution for $(m,n)=(r+1,r)$ (for Gaussian integration). We are naturally concerned with the pair $(m,n)=(r+1,r+1)$. Then by slightly modifying Hausdorff's proof, we can prove the following theorem.

Theorem 5.5. *Assume that the integration $\int_a^b w(t)\,dt$ corresponds to a series of rational classical polynomials. Then for $r \geq 0$, there exist rationals $x_1,\dots,x_{r+1}$, $y_1,\dots,y_{r+1}$ such that*

$$\sum_{i=1}^{r+1} x_i y_i^j = \int_a^b t^j w(t)\,dt, \quad j=0,1,\dots,r+1. \tag{35}$$

The authors do not know what would happen for $m+1 \leq n \leq 2m-3$, which is again left for future work. We emphasize that a constructive solution of Problem 1.2 creates a constructive solution of Waring's problem.

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Appendix A. The classical orthogonal polynomials and some basic properties

We here describe some basic properties on Jacobi polynomials, Laguerre polynomials, Hermite polynomials, which are used in the proof of our results.

A.1. Jacobi polynomials.

The following informations can be found in [32, Chapter IV].

Explicit form

$$P_l^{(\alpha,\beta)}(x) = \sum_{k=0}^{l} \binom{l+\alpha}{l-k}\binom{l+\beta}{k}\Big(\frac{x-1}{2}\Big)^k\Big(\frac{x+1}{2}\Big)^{l-k}. \tag{36}$$

Three-term relation

$$\begin{aligned} &2l(l+\alpha+\beta)(2l+\alpha+\beta-2)P_l^{(\alpha,\beta)}(x) \\ &= (2l+\alpha+\beta-1)\left((2l+\alpha+\beta)(2l+\alpha+\beta-2)x+\alpha^2-\beta^2\right)P_{l-1}^{(\alpha,\beta)}(x) \\ &\qquad - 2(l+\alpha-1)(l+\beta-1)(2l+\alpha+\beta)P_{l-2}^{(\alpha,\beta)}(x). \end{aligned} \tag{37}$$

Derivative formulas

$$\begin{aligned} &(2l+\alpha+\beta)(1-x^2)\frac{d}{dx}P_l^{(\alpha,\beta)}(x) \\ &= -l\left((2l+\alpha+\beta)x+\beta-\alpha\right)P_l^{(\alpha,\beta)}(x) + 2(n+\alpha)(l+\beta)P_{l-1}^{(\alpha,\beta)}(x). \end{aligned} \tag{38}$$

$$\begin{aligned} &(2l+\alpha+\beta+2)(1-x^2)\frac{d}{dx}P_l^{(\alpha,\beta)}(x) \\ &= (l+\alpha+\beta+1)\left((2l+\alpha+\beta+2)x+\alpha-\beta\right)P_l^{(\alpha,\beta)}(x) \\ &\qquad - 2(l+1)(l+\alpha+\beta+1)P_{l+1}^{(\alpha,\beta)}(x). \end{aligned} \tag{39}$$

Sturm-Liouville equation

$$(1-x^2)\frac{d^2}{dx^2}P_l^{(\alpha,\beta)}(x)+(\beta-\alpha-(\alpha+\beta+2)x)\frac{d}{dx}P_l^{(\alpha,\beta)}(x) + l(l+\alpha+\beta+1)P_l^{(\alpha,\beta)}(x)=0. \tag{40}$$

A.2. **Laguerre polynomials.** The following informations can be found in [32, § 5.1].

Explicit form

$$L_l^{(\alpha)}(x)=\sum_{k=0}^{l}\binom{l+\alpha}{l-k}\frac{(-x)^k}{k!}. \tag{41}$$

Three-term relation

$$lL_l^{(\alpha)}(x)=(-x+2l+\alpha-1)L_{l-1}^{(\alpha)}(x)-(l+\alpha-1)L_{l-2}^{(\alpha)}(x). \tag{42}$$

Derivative formula

$$\frac{d}{dx}L_l^{(\alpha)}(x)=x^{-1}\left(lL_l^{(\alpha)}(x)-(l+\alpha)L_{l-1}^{(\alpha)}(x)\right). \tag{43}$$

Sturm-Liouville equation

$$x\frac{d^2}{dx^2}L_l^{(\alpha)}(x)+(\alpha+1-x)\frac{d}{dx}L_l^{(\alpha)}(x)+lL_l^{(\alpha)}(x)=0. \tag{44}$$

A.3. **Hermite polynomials.** The following informations can be found in [32, § 5.5].

Explicit form

$$H_l(x)=\sum_{k=0}^{\lfloor l/2\rfloor}(-1)^k\frac{l!}{k!(l-2k)!}(2x)^{l-2k}. \tag{45}$$

Three-term relation

$$H_l(x)-2xH_{l-1}(x)+2(l-1)H_{l-2}(x)=0. \tag{46}$$

Derivative formula

$$H_l'(x)=2lH_{l-1}(x). \tag{47}$$

Sturm-Liouville equation

$$H_l''(x)-2xH_l'(x)+2lH_l(x)=0. \tag{48}$$

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