Reflection Principles, Generic Large Cardinals, and the Continuum Problem

Fuchino, Sakaé

Rodrigues, André Ottenbreit Maschio

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Sakaé Fuchino[1] and André Ottenbreit Maschio Rodrigues[2]

[1] Kobe University, Kobe, Japan
fuchino@diamond.kobe-u.ac.jp,
WWW home page: https://fuchino.ddo.jp/
[2] Kobe University, Kobe, Japan
andreomr@gmail.com

Abstract. Strong reflection principles with the reflection cardinal $\leq \aleph_1$ or $< 2^{\aleph_0}$ imply that the size of the continuum is either $\aleph_1$ or $\aleph_2$ or very large. Thus, the stipulation, that a strong reflection principle should hold, seems to support the trichotomy on the possible size of the continuum. In this article, we examine the situation with the reflection principles and related notions of generic large cardinals.

Keywords: Continuum Problem, Laver-generically large cardinals, forcing axioms, reflection principles

1 Gödel's Program and large cardinals

The Continuum Problem has been considered to be one of the central problems in set theory. Georg Cantor tried till the end of his mathematical carrier to prove his "theorem" which claims, formulated in present terminology, the continuum, the cardinality $2^{\aleph_0}$ of the set of all real numbers, is the first uncountable cardinal $\aleph_1$. This statement is now called the Continuum Hypothesis (CH). By Gödel [27], [28] [29], and Cohen [4], [5], [6], it is proven hat CH is independent from the axiom system ZFC of Zermelo-Fraenkel set theory with the Axiom of Choice.[5]

Although the majority of the non-set theorists apparently believes that the results by Gödel and Cohen were the final solutions of the Continuum Problem, Gödel maintained in [30] that the conclusive solution to the problem is yet to be obtained in that a "right" extension of ZFC will be found which will decide

[3] An extended version of the manuscript of the paper with some more details and proofs is downloadable as:
https://fuchino.ddo.jp/papers/refl_principles_gen_large_cardinals_continuum_problem-x.pdf

[4] The authors would like to thank Hiroshi Sakai and the anonymous referee for many valuable comments.

[5] Due to the Incompleteness Theorems, if we would like to formulate this statement precisely, we have to put it under the assumption that ZFC is consistent (which we not only assume but do believe).

the size of the continuum. Today the research program of searching for possible legitimate extensions of ZFC to settle the Continuum Problem is called Gödel's Program. Now that, besides CH, a multitude of mathematically significant statements is known to be independent from ZFC, the program should aim to decide not only the size of the continuum but also many of these independent mathematical statements. For modern views on Gödel's Program, the reader may consult e.g. Bagaria [2], Steel [39].

Gödel suggested in [30] that the large cardinal axioms are good candidates of axioms to be added to the axiom system ZFC. Unfortunately large cardinals do not decide the size of the continuum which Gödel also admits in the postscript to [30] added in 1966. Nevertheless, it is known today that some notable structural aspects of the continuum like the Projective Determinacy are decided under the existence of certain large large cardinals.

In this paper, we discuss about a new notion of generic large cardinals introduced in Fuchino, Ottenbreit and Sakai [20] and called there Laver-generic large cardinals (see Section 6 below). Reasonable instances of (the existential statement of a) Laver-generic large cardinal decide the size of the continuum to be either $\aleph_1$ or $\aleph_2$ or fairly large. We show that these three possible scenarios of Laver-generic large cardinal are in accordance with respective strong reflection properties with reflection cardinal $< \aleph_2$ or $< 2^{\aleph_0}$.

In connection with the view-point of set-theoretical multiverse (see Fuchino [16]), our trichotomy theorems, or some further developments of them, have certain possibility to become the final answer to the Continuum Problem. As is well-known, Hugh Woodin is creating a theory which should support CH from the point of view of what should hold in a canonical model of the set theory. It should be emphasized that our trichotomy is not directly in contradiction with the possible outcome of his research program. In any case, it should be mathematical results in the future which should decide the matter definitively (if ever?).

2 Reflection Principles

The following type of mathematical reflection properties are considered in many different mathematical contexts.

(2.1) If a structure $\mathfrak{A}$ in the class $\mathcal{C}$ has the property $\mathcal{P}$, then there is a structure $\mathfrak{B}$ in relation $\mathcal{Q}$ to $\mathfrak{A}$ such that $\mathfrak{B}$ has the cardinality $< \kappa$ and $\mathfrak{B}$ also has the property $\mathcal{P}$.

We shall call "$< \kappa$" in (2.1) above the *reflection cardinal* of the reflection property. If κ is a successor cardinal μ^+ we shall also say that the reflection cardinal is $\leq \mu$.

An example of an instance of (2.1) is, when $\mathcal{C}$ = "compact Hausdorff topological spaces", $\mathcal{P}$ = "non-metrizable", $\mathcal{Q}$ = "subspace" and $\kappa = \aleph_2$, that is, with the reflection cardinal $\leq \aleph_1$. In this case, we obtain the statement:

(2.2) For any compact Hausdorff topological space, if X is non-metrizable, then there is a subspace Y of X of cardinality $< \aleph_2$ such that Y is also non-metrizable.

This assertion is known to be a theorem in ZFC (see Dow [11]).

If we extend the class $\mathcal{C}$ in (2.2) to $\mathcal{C} =$ "locally compact Hausdorff space", the statement thus obtained

(2.3) For any locally compact Hausdorff topological space, if X is non-metrizable, then there is a subspace Y of X of cardinality $< \aleph_2$ such that Y is also non-metrizable

is no more a theorem in ZFC: we can construct a counterexample to (2.3), using a non-reflecting stationary subset S of $E^\kappa_\omega = \{\alpha < \kappa : cf(\alpha) = \omega\}$ for some regular $\kappa > \omega_1$ (Fuchino, Juhász, Soukup, Szentmiklóssy and Usuba [18]). Note that $\square_\lambda$ for any uncountable λ implies that there is such S for $\kappa = \lambda^+$. In particular, (2.3) implies the total failure of the square principles and thus we need very large large cardinals to obtain the consistency of this reflection principle. Actually, a known consistency proof of this principle requires the existence of a strongly compact cardinal[6].

(2.3) is equivalent to the stationarity reflection principle called Fodor-type Reflection Principle (FRP) introduced in [18].[7] This principle can be formulated as follows (see [24]).

For a regular uncountable cardinal λ and $E \subseteq E^\lambda_\omega = \{\gamma \in \lambda : cf(\gamma) = \omega\}$, a mapping $g : E \to [\lambda]^{\aleph_0}$ is said to be a *ladder system on* E if, for all $\alpha \in E$, $g(\alpha)$ is a cofinal subset of α and $otp(g(\alpha)) = \omega$.

(FRP): *For any regular $\lambda > \aleph_1$, stationary $E \subseteq E^\lambda_\omega$, and a ladder system $g : E \to \lambda^{\aleph_0}$ on E, there is an $\alpha^* \in E^\lambda_{\omega_1}$ such that*

$$\{x \in [\alpha^*]^{\aleph_0} : \sup(x) \in E,\ g(\sup(x)) \subseteq x\}$$

is stationary in $[\alpha^*]^{\aleph_0}$.

Besides (2.3), there are many mathematical reflection principles in the literature which have been previously studied rather separately but which are now all shown to be equivalent to FRP and hence also equivalent to each other (see [14], [15], [22], [24]). The equivalence of (2.3) to FRP is established in [24] via a further characterization of FRP by non existence of a ladder system with a strong property of disjointness from which a counterexample to (2.3) (and other reflection properties proved to be equivalent to FRP) can be constructed. Here we want to mention only one other reflection statement also equivalent to FRP:

For a graph $G = \langle G, \mathcal{E}\rangle$, where $\mathcal{E} \subseteq G^2$ is the adjacency relation of the graph, is said to be *of countable coloring number* if there is a well-ordering $\sqsubset$ on G such that, for each $g \in G$, $\{h \in G : h\ \mathcal{E}\ g$ and $h \sqsubset g\}$ is finite.

[6] The existence of a strongly compact cardinal is enough to force Rado's Conjecture discussed below and Rado's Conjecture implies the reflection statement (2.3).

[7] Here, we are not only talking about equiconsistency but really about equivalence over ZFC.

The following assertion is also equivalent to FRP ([18], Fuchino, Sakai Soukup and Usuba [24]):

(2.4) For any graph G, if G is not of countable coloring number, then there is a subgraph H of cardinality $< \aleph_2$ such that H is neither of countable coloring number.

In particular, it follows that the assertions (2.3) and (2.4) are equivalent to each other over ZFC.

(Strong) Downward Löwenheim Skolem Theorems of extended logics can be seen also as instances of the scheme (2.1). The following is a theorem in ZFC:

$\mathsf{SDLS}(\mathcal{L}(Q), {<}\,\aleph_2)$: *For any uncountable first-order structure $\mathfrak{A}$ in a countable signature, there is an elementary submodel $\mathfrak{B}$ of $\mathfrak{A}$ with respect to the logic $\mathcal{L}(Q)$ of cardinality*[8] $< \aleph_2$ *where the quantifier Q in a formula "$Qx\,\varphi$" is to be interpreted as "there are uncountably many x such that φ".*

Adopting the notation of Fuchino, Ottenbreit and Sakai [19], let $\mathcal{L}^{\aleph_0}_{stat}$ be the logic with monadic (weak) second order variable where the second order variables are to be interpreted as they are running over countable subsets of the structure in consideration. The logic has the built-in predicate ε where atomic formulas of the form $x \varepsilon X$ is allowed for first and second order variables x and X respectively. The logic also has the unique second order quantifier *stat* which is interpreted by

(2.5) for a structure $\mathfrak{A} = \langle A, ...\rangle$, $\mathfrak{A} \models stat\, X\, \varphi[X, ...]$ holds if and only if $\{U \in [A]^{\aleph_0} : \mathfrak{A} \models \varphi[U, ...]\}$ is stationary in $[A]^{\aleph_0}$.

Note that $\mathcal{L}^{\aleph_0}_{stat}$ extends $\mathcal{L}(Q)$ above, since $Qx\,\varphi$ can be expressed by $stat X \exists x\, (x \not\varepsilon X \wedge \varphi)$.

In $\mathcal{L}^{\aleph_0}_{stat}$ we have two natural generalizations of the notion of elementary substructure. For (first order) structures $\mathfrak{A} = \langle A, ...\rangle$ and $\mathfrak{B} = \langle B, ...\rangle$ with $\mathfrak{B} \subseteq \mathfrak{A}$, let

(2.6) $\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$ if and only if, for all $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, ..., X_0, ...)$ in the signature of $\mathfrak{A}$, $b_0, ... \in B$, and $U_0, ... \in [B]^{\aleph_0}$, we have $\mathfrak{B} \models \varphi[b_0, ..., U_0, ...] \Leftrightarrow \mathfrak{A} \models \varphi[b_0, ..., U_0, ...]$.

(2.7) $\mathfrak{B} \prec^-_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$ if and only if, for all $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, ...)$ in the signature of $\mathfrak{A}$ without any free second order variables, and $b_0, ... \in B$, we have $\mathfrak{B} \models \varphi[b_0, ...] \Leftrightarrow \mathfrak{A} \models \varphi[b_0, ...]$.

By the remark after (2.5), the following principles are generalizations of $\mathsf{SDLS}(\mathcal{L}(Q), {<}\,\aleph_2)$:

$\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, {<}\,\aleph_2)$: *For any uncountable first-order structure $\mathfrak{A}$ in a countable signature, there is a submodel $\mathfrak{B}$ of $\mathfrak{A}$ of cardinality $< \aleph_2$ such that $\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.*

[8] The cardinality of a structure is defined to be the cardinality of the underlying set.

$\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$: *For any uncountable first-order structure* $\mathfrak{A}$ *in a countable signature, there is a submodel* $\mathfrak{B}$ *of* $\mathfrak{A}$ *of cardinality* $< \aleph_2$ *such that* $\mathfrak{B} \prec^-_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.

M. Magidor noticed that $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ implies (2.4) (see Magidor [37]). By the equivalence of (2.4) to FRP, we obtain

Theorem 1 $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *implies* FRP. □

Actually, it is also easy to see that the stationarity reflection principle RP (which is a strengthening of RP in Jech [31]) follows from $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$.

FRP follows from our RP ([18]) which is defined as follows:

RP : *For every regular* $\lambda \geq \aleph_2$*, stationary* $S \subseteq [\lambda]^{\aleph_0}$*, and* $X \in [\lambda]^{\aleph_1}$*, there is* $Y \in [\lambda]^{\aleph_1}$ *such that* $cf(Y) = \omega_1$*,* $X \subseteq Y$ *and* $S \cap [Y]^{\aleph_0}$ *is stationary in* $[Y]^{\aleph_0}$.

Jech's RP is just as our RP as defined above but without demanding the property "$cf(Y) = \omega_1$" for the reflection point Y.

Theorem 2 $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *implies* RP.

Sketch of the proof. Let λ, S, X be as in the definition of RP. Let $\mu > \lambda^{\aleph_0}$ be regular and $\mathfrak{A} = \langle \mathcal{H}(\mu), \lambda, S, X, \in \rangle$ where λ, S and X are thought to be interpretations of unary predicate symbols. Let $\mathfrak{B} = \langle B, ... \rangle$ be such that B is of cardinality $\aleph_1$ and $\mathfrak{B} \prec^-_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$. Then $Y = \lambda \cap B$ is as desired. For example, $cf(Y) = \omega_1$ follows from the fact that $\mathfrak{B} \models \psi$ by elementarity where ψ is the $\mathcal{L}^{\aleph_0}_{stat}$-sentence: $stat\, X\, \exists y (y \underset{\sqcup}{\in} \underset{\sqcup}{\lambda} \wedge \forall z\, ((z\ \varepsilon\ X \wedge z \underset{\sqcup}{\in} \underset{\sqcup}{\lambda}) \rightarrow z \underset{\sqcup}{\in} y))$ where $\underset{\sqcup}{\lambda}$ and $\underset{\sqcup}{\in}$ are constant and binary relation symbols corresponding to λ and $\in$ in the structure $\mathfrak{A}$. □ (Theorem 2)

By a theorem of Todorčević, RP in the sense of Jech implies $2^{\aleph_0} \leq \aleph_2$ (see Theorem 37.18 in [31]). Thus

Corollary 3 $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *implies* $2^{\aleph_0} \leq \aleph_2$. □

In contrast to Corollary 3, FRP does not put almost any restriction on the cardinality of the continuum since FRP is preserved by ccc forcing (see [18]).

A proof similar to that of Theorem 2 shows that $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ implies the Diagonal Reflection Principle down to an internally club reflection point of cardinality $<\aleph_2$ of S. Cox [8]. Conversely, we can also easily prove that the Diagonal Reflection Principle down to an internally club reflection point of cardinality $<\aleph_2$ implies $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$. The internally clubness of the reflection point is used to guarantee that the internal interpretation of the stationary logic coincides with the external correct interpretation of the logic in the small substructure to make it an elementary substructure (in the sense of $\prec^-_{\mathcal{L}^{\aleph_0}_{stat}}$) of the original structure. Thus we obtain (1) of the following theorem.

Theorem 4 (Theorem 1.1, (3) and (4) in [19])

(1) $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *is equivalent to the Diagonal Reflection Principle down to an internally club reflection point of cardinality* $<\aleph_2$.

(2) $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *is equivalent to* $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *plus* CH. □

S. Cox proved in [8] that the Diagonal Reflection Principle down to an internally club reflection point of cardinality $\aleph_1$ follows from $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$. Thus,

Corollary 5 (1) $\mathsf{MA}^{+\omega_1}(\sigma\text{-}closed)$ *implies* $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$.

(2) $\mathsf{MA}^{+\omega_1}(\sigma\text{-}closed) + \mathsf{CH}$ *implies* $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$. □

The reflection cardinal $<\aleph_2$ (or equivalently $\leq\aleph_1$) in the reflection principles above can be considered to be significant and even natural since, with this reflection cardinal, the reflection principles can be seen as statements claiming that the cardinality $\aleph_1$ is archetypical among uncountable cardinals, and hence that $\aleph_1$ already captures various phenomenon in uncountability in the sense that a certain type of properties of an uncountable structure can be reflected down to a substructure of the cardinality $\aleph_1$. From that point of view, it is interesting that one of the strongest reflection principles, namely the Strong Downward Löwenheim-Skolem Theorem for stationary logic with this reflection cardinal implies CH.

In a similar way, we can also argue that the reflection with the reflection cardinal $<2^{\aleph_0}$ or $\leq 2^{\aleph_0}$ should be regarded as significant and even natural since we can interpret the reflection with these reflection cardinals as a pronouncement of the richness of the continuum.

Let $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ and $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ be the principles obtained from $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ and $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ by replacing "$<\aleph_2$" with "$<2^{\aleph_0}$".

Theorem 6 (Proposition 2.1, Corollary 2.3, Corollary 2.4 in [20])

(1) $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ *implies* $2^{\aleph_0} = \aleph_2$. *In particular, if* $2^{\aleph_0} > \aleph_2$, *then* $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ *does not hold.*

(2) $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ *is inconsistent.* □

Note that $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ follows from $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed}) + \neg\mathsf{CH}$ which is e.g. a consequence of $\mathsf{PFA}^{+\omega_1}$.

Note that Lemma 9 implies that $\mathsf{GRP}^{\omega,\omega_1}(<2^{\aleph_0})$ is also inconsistent.

In contrast to the reflection down to $<2^{\aleph_0}$ whose strong version implies that the continuum is $\aleph_2$ (see Theorem 6, (2) above), the reflection down to $\leq 2^{\aleph_0}$ does not exert any such restriction on the size of the continuum as we will see this in the next section.

A slightly different type of reflection principle with reflection cardinal $<2^{\aleph_0}$ implies that the continuum is very large. We will see this in Section 5.

3 Game Reflection Principles and generically large cardinals

There is a further strengthening of $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ which is called (Strong) Game Reflection Principle[9] (GRP) introduced in B. König [34]. The following is a generalization of the principle:

For a regular uncountable cardinal μ, a set A, and $\mathcal{A} \subseteq {}^{\mu>}A$, $\mathcal{G}^{\,\mu>A}(\mathcal{A})$ is the following game of length μ for players I and II. A match in $\mathcal{G}^{\,\mu>A}(\mathcal{A})$ looks like:

$$\begin{array}{c|ccccccccc} \mathrm{I} & a_0 & & a_1 & & a_2 & \cdots & a_\xi & & \cdots \\ \hline \mathrm{II} & & b_0 & & b_1 & & b_2 \cdots & & b_\xi & \cdots \end{array} \qquad (\xi < \mu)$$

where a_ξ, $b_\xi \in A$ for $\xi < \mu$.

II wins this match if

(3.1) $\langle a_\xi, b_\xi \,:\, \xi < \eta\rangle \in \mathcal{A}$ and $\langle a_\xi, b_\xi \,:\, \xi < \eta\rangle^\frown\langle a_\eta\rangle \notin \mathcal{A}$ for some $\eta < \mu$; or $\langle a_\xi, b_\xi \,:\, \xi < \mu\rangle \in [\mathcal{A}]$

where $\langle a_\xi, b_\xi \,:\, \xi < \eta\rangle$ denotes the sequence $f \in {}^{2\cdot\eta}A$ such that $f(2\cdot\xi) = a_\xi$ and $f(2\cdot\xi+1) = b_\xi$ for all $\xi < \eta$ and $[\mathcal{A}] = \{f \in {}^{\mu}A \,:\, f \restriction \alpha \in \mathcal{A}\}$ for all $\alpha < \mu$.

For regular cardinals μ, κ with $\mu < \kappa$ $\mathcal{C} \subseteq [A]^{<\kappa}$ is said to be μ-club if $\mathcal{C}$ is cofinal in $[A]^{<\kappa}$ with respect to $\subseteq$ and closed with respect to the union of increasing $\subseteq$-chain of length ν for any regular $\mu \le \nu < \kappa$.

$\mathsf{GRP}^{<\mu}(<\kappa)$: *For any set A of regular cardinality $\ge \kappa$ and μ-club $\mathcal{C} \subseteq [A]^{<\kappa}$, if the player II has no winning strategy in $\mathcal{G}^{\,\mu>A}(\mathcal{A})$ for some $\mathcal{A} \subseteq {}^{\mu>}A$, there is $B \in \mathcal{C}$ such that the player II has no winning strategy in $\mathcal{G}^{\,\mu>B}(\mathcal{A} \cap {}^{\mu>}B)$.*

B. König's *Game Reflection Principle* (GRP) is $\mathsf{GRP}^{<\omega_1}(<\aleph_2)$.

Sometimes, the following variation of the games and the principles is useful: For a limit ordinal δ, a set A, and $\mathcal{A} \subseteq {}^{\delta\ge}A$, $\mathcal{G}^{\,\delta\ge A}(\mathcal{A})$ is the following game of length δ for players I and II. A match in $\mathcal{G}^{\,\delta\ge A}(\mathcal{A})$ looks like:

$$\begin{array}{c|ccccccccc} \mathrm{I} & a_0 & & a_1 & & a_2 & \cdots & a_\xi & & \cdots \\ \hline \mathrm{II} & & b_0 & & b_1 & & b_2 \cdots & & b_\xi & \cdots \end{array} \qquad (\xi < \delta)$$

where a_ξ, $b_\xi \in A$ for $\xi < \delta$.

II wins this match if

(3.2) $\langle a_\xi, b_\xi \,:\, \xi < \eta\rangle \in \mathcal{A}$ and $\langle a_\xi, b_\xi \,:\, \xi < \eta\rangle^\frown\langle a_\eta\rangle \notin \mathcal{A}$ for some $\eta < \delta$; or $\langle a_\xi, b_\xi \,:\, \xi < \eta\rangle \in \mathcal{A}$ for all $\eta \le \delta$.

[9] In [34], B. König originally called the principle introduced here the Strong Game Reflection Principle and the local version of the principle the Game Reflection Principle.

where $\langle a_\xi, b_\xi : \xi < \eta\rangle$ is defined as above.

For a limit ordinal δ, and uncountable regular cardinals μ, κ with $\delta \leq \mu < \kappa$,

$\mathsf{GRP}^{\delta,\mu}(<\kappa)$: *For any set A of regular cardinality $\geq \kappa$ and μ-club $\mathcal{C} \subseteq [A]^{<\kappa}$, if the player II has no winning strategy in $\mathcal{G}^{\,{}^{\delta\geq}A}(\mathcal{A})$ for some $\mathcal{A} \subseteq {}^{\delta\geq}A$, there is $B \in \mathcal{C}$ such that the player II has no winning strategy in $\mathcal{G}^{\,{}^{\delta\geq}B}(\mathcal{A} \cap {}^{\delta\geq}B)$.*

The next Lemma follows immediately from the definitions:

Lemma 7 *Suppose that δ and δ' are limit ordinals and μ, μ', κ, κ' are regular cardinals such that $\delta \leq \delta' < \mu \leq \mu' < \kappa$. Then we have*

$$\text{(3.3)} \qquad \mathsf{GRP}^{<\mu'}(<\kappa) \;\Rightarrow\; \mathsf{GRP}^{<\mu}(<\kappa) \;\Rightarrow\; \mathsf{GRP}^{\delta',\mu}(<\kappa) \;\Rightarrow\; \mathsf{GRP}^{\delta,\mu}(<\kappa) \quad \Box$$

GRP is indeed a strengthening of $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$. The following Theorem 8, Lemma 9 and Corollary 10 are slight generalizations of results in B. König [34].

Theorem 8 (Theorem 4.7 in [19]) *Suppose that κ is a regular uncountable cardinal such that*

(3.4) $\quad \mu^{\aleph_0} < \kappa$ *for all* $\mu < \kappa$, *and*

(3.5) $\quad \mathsf{GRP}^{\omega,\omega_1}(<\kappa)$ *holds.*

Then $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ *holds.*[10] $\Box$

Lemma 9 (Lemma 4.2 in [19]) *For a regular cardinal κ, $\mathsf{GRP}^{\omega,\omega_1}(<\kappa)$ implies $2^{\aleph_0} < \kappa$.* $\Box$

Remember that GRP is the principle $\mathsf{GRP}^{<\omega_1}(<\aleph_2)$. For a regular cardinal $\kappa > \aleph_1$ we shall write $\mathsf{GRP}(<\kappa)$ for $\mathsf{GRP}^{<\omega_1}(<\kappa)$. Thus GRP is $\mathsf{GRP}(<\aleph_2)$.

Corollary 10 (1) GRP *implies* $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$.
(2) $\mathsf{GRP}(<(2^{\aleph_0})^+)$ *implies* $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, \leq 2^{\aleph_0})$.

Proof. (1): By Lemma 9, GRP implies CH. Thus, under GRP, (3.4) holds for $\kappa = \aleph_2$. By Lemma 7, GRP implies $\mathsf{GRP}^{\omega,\omega_1}(<\aleph_2)$. By Theorem 8, it follows that $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$.

(2): Note that, for $\mu < (2^{\aleph_0})^+$, $\mu^{\aleph_0} \leq 2^{\aleph_0} < (2^{\aleph_0})^+$ holds. By Lemma 7, $\mathsf{GRP}(<(2^{\aleph_0})^+)$ implies $\mathsf{GRP}^{\omega,\omega_1}(<(2^{\aleph_0})^+)$. Thus, by Theorem 8, it follows that $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <(2^{\aleph_0})^+)$, or $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, \leq 2^{\aleph_0})$ in the other notation, holds.

$\Box$ (Corollary 10)

GRP also implies another prominent reflection principle which is called Rado's Conjecture.

[10] Actually we can prove a slight strengthening of $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ (see [19]).

We call a partial ordering $T = \langle T, \leq_T \rangle$ a *tree* if the initial segment below any element is a well-ordering. A tree $T = \langle T, \leq_T \rangle$ is said to be *special* if it can be partitioned into countably many antichains (i.e. pairwise incomparable sets). Note that every special tree has height $\leq \omega_1$.

For a regular cardinal $\kappa > \aleph_1$, we define Rado's Conjecture with reflection cardinal $< \kappa$ as

$\mathsf{RC}(<\kappa)$: *For any tree T, if T is not special then there is $B \in [T]^{<\kappa}$ such that B (as the tree $\langle B, \leq_T \cap B^2 \rangle$) is not special.*

The original *Rado's Conjecture* (RC) is $\mathsf{RC}(<\aleph_2)$.

Theorem 11 (B. König [34], see also Theorem 4.3 in [19]) *For a regular cardinal $\kappa > \aleph_1$, $\mathsf{GRP}^{<\omega_1}(<\kappa)$ implies $\mathsf{RC}(<\kappa)$.* □

FRP is also a consequence of GRP. This is simply because FRP follows from RC (see [25]).

Game Reflection Principles are characterizations of certain instances of the existence of generically supercompact cardinals.

Let $\mathcal{P}$ be a class of posets. A cardinal κ is said to be a *generically supercompact cardinal by $\mathcal{P}$*, if, for any regular λ, there is a poset $\mathbb{P} \in \mathcal{P}$ such that, for any $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$, there are classes M, $j \subseteq \mathsf{V}[\mathbb{G}]$ such that M is an inner model of $\mathsf{V}[\mathbb{G}]$, $j : \mathsf{V} \xrightarrow{\preceq} M$, $crit(j) = \kappa$, $j(\kappa) > \lambda$ and $j{}''\lambda \in M$.

Theorem 12 ([19]) *For a regular uncountable κ, the following are equivalent:*

(a) $2^{<\kappa} = \kappa$ *and* $\mathsf{GRP}^{<\kappa}(<\kappa^+)$ *holds.*

(b) κ^+ *is generically supercompact by $<\kappa$-closed posets.* □

Corollary 13 (B. König [34]) *The following are equivalent:*

(a) GRP *holds.*

(b) $\aleph_2$ *is generically supercompact by σ-closed posets.*

Proof. Assume that GRP holds (remember that GRP denotes $\mathsf{GRP}^{<\omega_1}(<\aleph_2)$). Then, by Corollary 10, (1), $2^{<\aleph_1} = 2^{\aleph_0} = \aleph_1$. Thus, by Theorem 12, "(a) $\Rightarrow$ (b)" for $\kappa = \aleph_1$, it follows that $\aleph_2 = (\aleph_1)^+$ is generically supercompact by σ-closed forcing. The implication "(b) $\Rightarrow$ (a)" follows from "(b) $\Rightarrow$ (a)" of Theorem 12 for $\kappa = \aleph_1$. □ (Corollary 13)

4 Simultaneous reflection down to $< 2^{\aleph_0}$ and $\leq 2^{\aleph_0}$

As we discussed in Section 2, the reflection down to $< 2^{\aleph_0}$ as well as the reflection down to $\leq 2^{\aleph_0}$ can be regarded as significant being principles which claim certain richness of the continuum.

One of the strong form of reflection principles with reflection cardinal $< 2^{\aleph_0}$ implies that the continuum is equal to $\aleph_2$ (Theorem 6, (2)) while there is a limitation on the possible types of reflection (Theorem 6, (3)).

In contrast, as we see below, the reflection down to $\leq 2^{\aleph_0}$ can be established in one of its strongest forms without almost any restriction on the size of the continuum: (a) of Theorem 12 can be easily realized starting from a supercompact cardinal.

The following is well-known.

Lemma 14 (Lemma 4.10 in [19]) *If κ is a supercompact and $\mu < \kappa$ is an uncountable regular cardinal then for $\mathbb{P} = \mathrm{Col}(\mu, \kappa)$ and $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$, we have $\mathsf{V}[\mathbb{G}] \models$ "$\kappa = \mu^+$ and κ* is generically supercompact by $<\mu$-closed posets"*.* $\square$

Suppose now that κ_1 is a supercompact cardinal and $2^{\aleph_0}$ is a regular cardinal. Let $\mathbb{Q} = \mathrm{Col}(2^{\aleph_0}, \kappa_1)$ and let $\mathbb{H}$ be a $(\mathsf{V}, \mathbb{Q})$-generic filter. By $<2^{\aleph_0}$-closedness of $\mathbb{Q}$, we have $(2^{\aleph_0})^{\mathsf{V}} = (2^{\aleph_0})^{\mathsf{V}[\mathbb{H}]}$ and $\mathsf{V}[\mathbb{H}] \models \kappa_1 = (2^{\aleph_0})^+$. By Lemma 14, $\mathsf{V}[\mathbb{H}] \models$ "$(2^{\aleph_0})^+$ is a generically supercompact cardinal by $<2^{\aleph_0}$-closed posets". By Theorem 12, it follows that $\mathsf{V}[\mathbb{H}] \models$ "$\mathsf{GRP}^{<2^{\aleph_0}}(<(2^{\aleph_0})^+)$".

By Corollary 10, (2), Lemma 7 and Theorem 11, we have, in particular,

$$\mathsf{V}[\mathbb{H}] \models \text{“}\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, \leq 2^{\aleph_0}) \ \wedge\ \mathsf{RC}(\leq 2^{\aleph_0})\text{”}. \tag{4.1}$$

Note that the continuum can be forced to be practically anything of uncountable cofinality below κ_1 prior to the generic extension by $\mathbb{Q}$.

The following Proposition 15 should also belong to the folklore (for similar statements, see Theorem 4.1 in König and Yoshinobu [35] or Theorem 4.3 in Larson [36]).

Recall that, for a regular cardinal μ, a poset $\mathbb{P}$ is *$<\mu$-directed closed* if any downward directed subset of $\mathbb{P}$ of cardinality $<\mu$ has a lower bound (in $\mathbb{P}$).

Proposition 15 *Suppose that $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$ (or $\mathsf{PFA}^{+\omega_1}$, or $\mathsf{MM}^{+\omega_1}$, resp.) holds. If $\mathbb{P}$ is $<\aleph_2$-directed closed, then we have*

$$\Vdash_{\mathbb{P}} \text{“}\mathsf{MA}^{+\omega_1}(\sigma\textit{-closed}) \ \ (\textit{or } \mathsf{PFA}^{+\omega_1}, \textit{ or } \mathsf{MM}^{+\omega_1}, \textit{ resp.})\text{”}. \tag{4.2}$$

Proof. We prove the case of $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$. Other cases can be proved by the same argument.

Suppose that $\mathbb{P}$ is a $<\aleph_2$-directed closed poset and let $\underset{\sim}{\mathbb{Q}}$, $\langle \underset{\sim}{D}_\alpha : \alpha < \omega_1\rangle$, $\langle \underset{\sim}{S}_\beta : \beta < \omega_1\rangle$ be $\mathbb{P}$-names such that

$$\begin{aligned}\Vdash_{\mathbb{P}} \text{“}&\underset{\sim}{\mathbb{Q}} \text{ is a } \sigma\text{-closed poset,}\\ &\underset{\sim}{D}_\alpha\ (\alpha < \omega_1) \text{ is a dense subset of } \underset{\sim}{\mathbb{Q}} \text{ for all } \alpha < \omega_1, \text{ and}\\ &\underset{\sim}{S}_\beta\ (\beta < \omega_1) \text{ is a } \mathbb{Q}\text{-name of a stationary subset of } \omega_1\\ &\text{for all } \beta < \omega_1\text{”}\end{aligned} \tag{4.3}$$

Let $\mathbb{P}^* = \mathbb{P} * \underset{\sim}{\mathbb{Q}}$. For $\alpha < \omega_1$, let

$$D^*_\alpha = \{\langle \mathbb{p}, \underset{\sim}{\mathbb{q}}\rangle \in \mathbb{P}^* : \mathbb{p} \Vdash_{\mathbb{P}} \text{“}\underset{\sim}{\mathbb{q}} \in \underset{\sim}{D}_\alpha\text{”}\}. \tag{4.4}$$

For $\beta < \omega_1$, let

$$(4.5) \qquad \underset{\sim}{S}^*_\beta = \{\langle\langle \mathbb{p}, \underset{\sim}{\mathbb{q}}\rangle, \check{\alpha}\rangle \,:\, \langle \mathbb{p}, \underset{\sim}{\mathbb{q}}\rangle \in \mathbb{P}^*,\ \mathbb{p} \Vdash_{\mathbb{P}} \text{“}\underset{\sim}{\mathbb{q}} \Vdash_{\underset{\sim}{\mathbb{Q}}} \text{“}\check{\alpha}\ \varepsilon\ \underset{\sim}{S}_\beta\text{” ”}\}.$$

By the definition of $\mathbb{P}^*$, $\langle D^*_\alpha \,:\, \alpha < \omega_1\rangle$, and $\langle \underset{\sim}{S}^*_\beta \,:\, \beta < \omega_1\rangle$, the following is easy to show:

Claim. $\mathbb{P}^*$ is a σ-closed poset, D^*_α is a dense subset of $\mathbb{P}^*$ for all $\alpha < \omega_1$, and $\underset{\sim}{S}^*_\beta$ is a $\mathbb{P}^*$-name with $\Vdash_{\mathbb{P}^*}$ “$\underset{\sim}{S}^*_\beta$ is a stationary subset of ω_1” for all $\beta < \omega_1$. □

Let $\mathcal{D}^* = \{D^*_\alpha \,:\, \alpha < \omega_1\}$. By $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$, there is a $\mathcal{D}^*$-generic filter $\mathbb{G}^*$ on $\mathbb{P}^*$ such that $\underset{\sim}{S}^*_\beta[\mathbb{G}^*]$ is a stationary subset of ω_1 for all $\beta < \omega_1$.

Let θ be a sufficiently large regular cardinal and let $M \prec \mathcal{H}(\theta)$ be of cardinality $\aleph_1$ such that $\omega_1 \subseteq M$ and M contains everything relevant (in particular, $\mathbb{G}^*$, D^*_α, $\underset{\sim}{S}^*_\beta \in M$ for $\alpha, \beta < \omega_1$).

Let $\mathbb{G}_0 = \mathbb{G}^* \cap M$ and let $\mathbb{G}$ be the filter on $\mathbb{P}^*$ generated by $\mathbb{G}_0$. By the choice of M, we have $\underset{\sim}{S}^*_\beta[\mathbb{G}^*] = \underset{\sim}{S}^*_\beta[\mathbb{G}_0] = \underset{\sim}{S}^*_\beta[\mathbb{G}]$.

Let $G = \{\mathbb{p} \in \mathbb{P} \,:\, \langle \mathbb{p}, \underset{\sim}{\mathbb{q}}\rangle \in \mathbb{G}$ for some $\underset{\sim}{\mathbb{q}}\}$. Since $|\,G\,| \leq |\,M\,| < \aleph_2$ and G is downward directed, there is a lower bound $\mathbb{p}_0 \in \mathbb{P}$ of G.

Let

$$(4.6) \qquad \underset{\sim}{\mathbb{H}} = \{\langle \underset{\sim}{\mathbb{q}}, \mathbb{1}_{\mathbb{P}}\rangle \,:\, \langle \mathbb{p}, \underset{\sim}{\mathbb{q}}\rangle \in \mathbb{G} \text{ for some } \mathbb{p} \in \mathbb{P}\}.$$

Then $\underset{\sim}{\mathbb{H}}$ is a $\mathbb{P}$-name and we have

$$(4.7) \qquad \mathbb{p}_0 \Vdash_{\mathbb{P}} \text{“}\underset{\sim}{\mathbb{H}} \text{ is a } \{\underset{\sim}{D}_\alpha \,:\, \alpha < \omega_1\}^\bullet\text{-generic filter on } \underset{\sim}{\mathbb{Q}} \text{ such that } \underset{\sim}{S}_\beta[\underset{\sim}{\mathbb{H}}] \text{ is a stationary subset of } \omega_1 \text{ for all } \beta < \omega_1\text{”}.$$

Since the argument above can be also performed in $\mathbb{P} \restriction \mathbb{r}$ instead of in $\mathbb{P}$ for any $\mathbb{r} \in \mathbb{P}$. It follows that

$$(4.8) \qquad \Vdash_{\mathbb{P}} \text{“there is a } \{\underset{\sim}{D}_\alpha \,:\, \alpha < \omega_1\}^\bullet\text{-generic filter } H \text{ on } \underset{\sim}{\mathbb{Q}} \text{ such that } \underset{\sim}{S}_\beta[H] \text{ is a stationary subset of } \omega_1 \text{ for all } \beta < \omega_1\text{”}.$$

□ (Proposition 15)

Theorem 16 *Suppose that κ and κ_1 with $\kappa < \kappa_1$ are two supercompact cardinals. Then there is a generic extension $\mathsf{V}[\mathbb{G} * \mathbb{H}]$ such that*

$$\mathsf{V}[\mathbb{G} * \mathbb{H}] \models \mathsf{MM}^{+\omega_1} + \mathsf{GRP}^{<2^{\aleph_0}}(\leq 2^{\aleph_0}).$$

Note that, by Corollary 5, (1), we have

$$\mathsf{V}[\mathbb{G} * \mathbb{H}] \models \mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0}) + \mathsf{GRP}^{<2^{\aleph_0}}(\leq 2^{\aleph_0}).$$

Proof of Theorem 16. Let $\mathsf{V}[\mathbb{G}]$ be a standard model of MM obtained by a reverse countable iteration of length κ along with a fixed Laver-function $\kappa \to V_\kappa$. It is easy to see that $\mathsf{V}[\mathbb{G}]$ also satisfies $\mathsf{MM}^{+\omega_1}$. Note that we have $\mathsf{V}[\mathbb{G}] \models \kappa = \aleph_2 = 2^{\aleph_0}$. In $\mathsf{V}[\mathbb{G}]$, κ_1 is still supercompact. Thus, working in $\mathsf{V}[\mathbb{G}]$, let $\mathbb{Q} = \mathrm{Col}(2^{\aleph_0}, \kappa_1)$. Let $\mathbb{H}$ be a $(\mathsf{V}[\mathbb{G}], \mathbb{Q})$-generic filter. By Proposition 15, we have $\mathsf{V}[\mathbb{G} * \mathbb{H}] \models \mathsf{MM}^{+\omega_1}$. By Lemma 14 and Theorem 12, we have $\mathsf{V}[\mathbb{G} * \mathbb{H}] = (\mathsf{V}[\mathbb{G}])[\mathbb{H}] \models \mathsf{GRP}^{<2^{\aleph_0}}(\leq 2^{\aleph_0})$. □ (Theorem 16)

5 Reflection principles under large continuum

The continuum can be "very large" as a cardinal number. For example, this is the case in the model $\mathsf{V}[\mathbb{G}]$ obtained by starting from a supercompact κ and then adding κ many Cohen reals. In this model, we have $2^{\aleph_0} = \kappa$ and there is a countably saturated normal fine filter over $\mathcal{P}_\kappa(\lambda)$ for all regular $\lambda \geq \kappa$. The last property of $\mathsf{V}[\mathbb{G}]$ implies that κ there is still fairly large (e.g. κ-weakly Mahlo and more, see e.g. Proposition 16.8 in Kanamori [32]).

If the ground model satisfies FRP then $\mathsf{V}[\mathbb{G}]$ also satisfies FRP since FRP is preserved by ccc extensions (see [18]). On the other hand, as we already have seen, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, {<}\,\aleph_2)$ or even $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, {<}\,2^{\aleph_0})$ is incompatible with large continuum. In particular, these reflection principles do not hold in our model $\mathsf{V}[\mathbb{G}]$.

A weakening of $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, {<}\,2^{\aleph_0})$ is compatible with large continuum. Let us begin with the diagonal reflection principle which characterizes the version of the strong downward Löwenheim-Skolem theorem with reflection points of cardinality $<$ large continuum. The following is a weakening of Cox's Diagonal Reflection Principle down to an internally club reflection point.

For regular cardinals κ, λ with $\kappa \leq \lambda$, let

$(*)^{int+}_{<\kappa,\lambda}$: *For any countable expansion $\tilde{\mathfrak{A}}$ of $\langle \mathcal{H}(\lambda), \in \rangle$ and sequence $\langle S_a : a \in \mathcal{H}(\lambda) \rangle$ such that S_a is a stationary subset of $[\mathcal{H}(\lambda)]^{\aleph_0}$ for all $a \in \mathcal{H}(\lambda)$, there are stationarily many $M \in [\mathcal{H}(\lambda)]^{<\kappa}$ such that*

(1) *$\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}$; and*

(2) *$S_a \cap M$ is stationary in $[M]^{\aleph_0}$ for all $a \in M$.*

Note that (1) implies that $c \subseteq M$ holds for all $c \in [M]^{\aleph_0} \cap M$.

In the notation above, "int" (internal) refers to the condition (2) in which not $S_a \cap [M]^{\aleph_0}$ but $S_a \cap M$ is declared to be stationary in $[M]^{\aleph_0}$; "+" refers to the condition that $M \in [\mathcal{H}(\lambda)]^{<\kappa}$ with (1) and (2) not only exists but there are stationarily many such M.

That $(*)^{int+}_{<\kappa,\lambda}$ is compatible with $\kappa = 2^{\aleph_0}$ and it is arbitrarily large is seen in the following Theorem 17 together with Lemma 18 below:

Theorem 17 (Theorem 2.10 in [20])*Suppose that κ is a generically supercompact cardinal by proper posets. Then $(*)^{int+}_{<\kappa,\lambda}$ holds for all regular $\lambda \geq \kappa$.* □

Similarly to Lemma 14, starting from a supercompact cardinal, it is easy to force that the continuum is generically supercompact cardinal by ccc-posets. Let us call a poset $\mathbb{P}$ appropriate for κ, if we have $j''\mathbb{P} \lessdot j(\mathbb{P})$ for all supercompact embedding j for κ.

Lemma 18 *If κ is a supercompact and $\mu < \kappa$ is an uncountable regular cardinal then for any ${<}\,\mu$-cc poset $\mathbb{P}$ appropriate for κ, adding $\geq \kappa$ many reals, we have* $\mathsf{V}[\mathbb{G}] \models$ "$\kappa \leq 2^{\aleph_0}$ *and* κ is generically supercompact by ${<}\,\mu$-cc posets". □

"$(*)^{int+}_{<\kappa,\lambda}$ holds for all regular $\lambda \geq \kappa$" characterizes the strong downward Löwenheim-Skolem theorem for internal interpretation of stationary logic defined in the following.

For a structure $\mathfrak{A} = \langle A, ...\rangle$ of a countable signature, an $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, ..., X_0, ...)$[11] and $a_0, ... \in A$, $U_0, ... \in [A]^{\aleph_0} \cap A$, we define the internal interpretation of $\varphi(a_0, ..., U_0, ...)$ in $\mathfrak{A}$ (notation: $\mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ...)$ for "$\varphi(a_0, ..., U_0, ...)$ holds internally in $\mathfrak{A}$") by induction on the construction of φ as follows:

If φ is "$x_i \,\varepsilon\, X_j$" then

(5.1) $\quad \mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ...) \;\Leftrightarrow\; a_i \in U_j$

for a structure $\mathfrak{A} = \langle A, ...\rangle$, $a_0, ... \in A$ and $U_0, ... \in [A]^{\aleph_0} \cap A$.

For first-order connectives and quantifiers in $\mathcal{L}^{\aleph_0}_{stat}$, the semantics "$\models^{int}$" is defined exactly as for the first order "$\models$".

For an $\mathcal{L}^{\aleph_0}_{stat}$ formula φ with $\varphi = \varphi(x_0, ..., X_0, ..., X)$, assuming that the notion of $\mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ..., U)$ has been defined for all $a_0, ... \in A$, $U_0, ..., U \in [A]^{\aleph_0} \cap A$, we stipulate

(5.2) $\quad \mathfrak{A} \models^{int} \mathit{stat}\, X\, \varphi(a_0, ..., U_0, ..., X) \;\Leftrightarrow$
$\qquad \{U \in [A]^{\aleph_0} \cap A : \mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ..., U)\}$ is stationary in $[A]^{\aleph_0}$

for a structure $\mathfrak{A} = \langle A, ...\rangle$ of a relevant signature, $a_0, ... \in A$ and $U_0, ... \in [A]^{\aleph_0} \cap A$.

For structures $\mathfrak{A}$, $\mathfrak{B}$ of the same signature with $\mathfrak{B} = \langle B, ...\rangle$ and $\mathfrak{B} \subseteq \mathfrak{A}$, we define

(5.3) $\quad \mathfrak{B} \prec^{int}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A} \;\Leftrightarrow$

$\qquad \mathfrak{B} \models^{int} \varphi(b_0, ..., U_0, ...)$ *if and only if* $\mathfrak{A} \models^{int} \varphi(b_0, ..., U_0, ...)$
$\qquad$ *for all* $\mathcal{L}^{\aleph_0}_{stat}$*-formulas* φ *in the signature of the structures with*
$\qquad \varphi = \varphi(x_0, ..., X_0, ...)$, $b_0, ... \in B$ *and* $U_0, ... \in [B]^{\aleph_0} \cap B$.

Finally, for a regular $\kappa > \aleph_1$, the internal strong downward Löwenheim-Skolem Theorem $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ is defined by

$\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$: *For any structure* $\mathfrak{A} = \langle A, ...\rangle$ *of countable signature with* $|A| \geq \kappa$, *there are stationarily many* $M \in [A]^{<\kappa}$ *such that* $\mathfrak{A} \restriction M \prec^{int}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.

Similarly to the $+$ in "$(*)^{int+}_{<\kappa,\lambda}$", '$+$' in "$\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$" refers to the existence of "stationarily many" reflection points M. This additional condition can be dropped if $\kappa = \aleph_2$. This is because the quantifier $Qx\,\varphi$ defined by $\mathit{stat}\, X \exists x\, (x \not\varepsilon\, X \wedge \varphi$, $\mathfrak{A} \models^{int} Qx\,\varphi(x, ...))$ still implies that "there are uncountably many $a \in A$ with $\varphi(a, ...)$". Note that, if $\mathfrak{A} \models^{int} \neg \mathit{stat}\, X\, (x \equiv x)$, for a structure $\mathfrak{A} = \langle A, ...\rangle$, we can easily find even club many $X \in [A]^{<\kappa}$ for any regular $\aleph_1 \leq \kappa \leq |A|$ such that $\mathfrak{A} \restriction X \prec^{int}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.

[11] As before, when we write $\varphi = \varphi(x_0, ..., X_0, ...)$, we always assume that the list $x_0, ...$ contains all the free first order variables of φ and $X_0, ...$ all the free weak second order variables of φ.

Proposition 19 (Proposition 3.1 in [20]) *For a regular cardinal $\kappa > \aleph_1$, the following are equivalent:*

(a) $(*)^{int+}_{<\kappa,\lambda}$ *holds for all regular $\lambda \geq \kappa$.*

(b) $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ *holds.* $\square$

Although $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ is compatible with large continuum, as a weakening of $\mathsf{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$, this principle does not imply the largeness of the continuum. The strong Löwenheim-Skolem theorem for the following variation of stationary logic does.

For sets s and t we denote with $\mathcal{P}_s(t)$ the set $[t]^{<|s|} = \{a \in \mathcal{P}(t) : |a| < |s|\}$. We say $S \subseteq \mathcal{P}_s(t)$ is stationary if it is stationary in the sense of Jech [31].

The logic $\mathcal{L}^{\mathsf{PKL}}_{stat}$ has a built-in unary predicate symbol $\underbracket{K}(\cdot)$.[12] For a structure $\mathfrak{A} = \langle A, \underbracket{K}^{\mathfrak{A}}, ...\rangle$, the weak second-order variables X, Y,... run over elements of $\mathcal{P}_{\underbracket{K}^{\mathfrak{A}}}(A)$.

We shall call a structure $\mathfrak{A}$ with $\underbracket{K}$ in its signature as a unary predicate symbol such that $|\underbracket{K}^{\mathfrak{A}}|$ is a regular uncountable cardinal, a PKL-*structure*.

$\mathcal{L}^{\mathsf{PKL}}_{stat}$ has the unique second-order quantifier "*stat*" and the internal interpretation $\models^{int}$ of formulas in this logic is defined similarly to $\mathcal{L}^{\aleph_0}_{stat}$ with the crucial step in the inductive definition being

$$(5.4)\qquad \mathfrak{A} \models^{int} stat\, X\, \varphi(a_0, ..., U_0, ..., X) \;\Leftrightarrow\; \{U \in \mathcal{P}_{\underbracket{K}^{\mathfrak{A}}}(A) \cap A : \mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ..., U)\} \text{ is stationary in } \mathcal{P}_{\underbracket{K}^{\mathfrak{A}}}(A)$$

for an $\mathcal{L}^{\mathsf{PKL}}_{stat}$-formula $\varphi = \varphi(x_0, ..., X_0, ..., X)$ (for which the relation $\models^{int}$ has been defined), a PKL-structure $\mathfrak{A} = \langle A, \underbracket{K}^{\mathfrak{A}}, ...\rangle$ of a relevant signature, $a_0, ... \in A$ and $U_0, ... \in \mathcal{P}_{\underbracket{K}^{\mathfrak{A}}}(A) \cap A$.

For PKL-structures $\mathfrak{A}$, $\mathfrak{B}$ of the same signature with $\mathfrak{B} = \langle B, \underbracket{K}^{\mathfrak{B}}, ...\rangle$ and $\mathfrak{B} \subseteq \mathfrak{A}$, we define:

$$(5.5)\qquad \mathfrak{B} \prec^{int}_{\mathcal{L}^{\mathsf{PKL}}_{stat}} \mathfrak{A} \;\Leftrightarrow$$

$$\mathfrak{B} \models^{int} \varphi(b_0, ..., U_0, ...) \textit{ if and only if } \mathfrak{A} \models^{int} \varphi(b_0, ..., U_0, ...)$$
for all $\mathcal{L}^{\mathsf{PKL}}_{stat}$-formulas φ in the signature of the structures with $\varphi = \varphi(x_0, ..., X_0, ...)$, $b_0, ... \in B$ and $U_0, ... \in \mathcal{P}_{\underbracket{K}^{\mathfrak{B}}}(B) \cap B$.

Finally, we define the internal strong downward Löwenheim-Skolem theorem for this logic as follows:

Suppose that κ is a regular cardinal $> \aleph_1$.

$\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\mathsf{PKL}}_{stat}, <\kappa)$: *For any PKL-structure $\mathfrak{A} = \langle A, \underbracket{K}^{\mathfrak{A}}, ...\rangle$ of countable signature with $|A| \geq \kappa$ and $|\underbracket{K}^{\mathfrak{A}}| = \kappa$, there are stationarily many $M \in [A]^{<\kappa}$ such that $\mathfrak{A} \restriction M$ is a PKL-structure and $\mathfrak{A} \restriction M \prec^{int}_{\mathcal{L}^{\mathsf{PKL}}_{stat}} \mathfrak{A}$.*

[12] PKL stands here for "pi-kappa-lambda" in the sense of "$\mathcal{P}_\kappa(\lambda)$".

The following diagonal reflection characterizes $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\mathsf{PKL}}_{stat}, <\kappa)$. For regular cardinals κ, λ with $\kappa \leq \lambda$, let

$(*)^{int+\mathsf{PKL}}_{<\kappa,\lambda}$: *For any countable expansion $\mathfrak{A}$ of the structure $\langle \mathcal{H}(\lambda), \kappa, \in\rangle$ and any family $\langle S_a : a \in \mathcal{H}(\lambda)\rangle$ such that S_a is a stationary subset of $\mathcal{P}_\kappa(\mathcal{H}(\lambda))$ for all $a \in \mathcal{H}(\lambda)$, there are stationarily many $M \in \mathcal{P}_\kappa(\mathcal{H}(\lambda))$ such that $|\,\kappa \cap M\,|$ is regular, $\mathfrak{A} \restriction M \prec \mathfrak{A}$ and $S_a \cap \mathcal{P}_{\kappa\cap M}(M) \cap M$ is stationary in $\mathcal{P}_{\kappa\cap M}(M)$.*

Proposition 20 (Proposition 4.1 in [20]) *For a regular cardinal $\kappa > \aleph_1$, the following are equivalent:*

(a) $(*)^{int+\mathsf{PKL}}_{<\kappa,\lambda}$ *holds for all regular $\lambda \geq \kappa$.*

(b) $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\mathsf{PKL}}_{stat}, <\kappa)$ *holds.* □

For a regular cardinal κ and a cardinal $\lambda \geq \kappa$, $\mathcal{S} \subseteq \mathcal{P}_\kappa(\lambda)$ is said to be 2-*stationary* if, for any stationary $\mathcal{T} \subseteq \mathcal{P}_\kappa(\lambda)$, there is an $a \in \mathcal{S}$ such that $|\,\kappa \cap a\,|$ is a regular uncountable cardinal and $\mathcal{T} \cap \mathcal{P}_{\kappa\cap a}(a)$ is stationary in $\mathcal{P}_{\kappa\cap a}(a)$. A regular cardinal κ has the 2-*stationarity property* if $\mathcal{P}_\kappa(\lambda)$ is 2-stationary (as a subset of itself) for all $\lambda \geq \kappa$.

Since the property (a) in Proposition 20 is an extension of the 2-stationarity of κ, we obtain:

Lemma 21 *For a regular uncountable κ, $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\mathsf{PKL}}_{stat}, <\kappa)$ implies that κ is 2-stationary.* □

This implies that a regular uncountable κ with $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\mathsf{PKL}}_{stat}, <\kappa)$ must be a fairly large cardinal:

Lemma 22 (Lemma 4.3 in [20]) *Suppose that κ is a regular uncountable cardinal. If κ is 2-stationary then κ is a weakly Mahlo cardinal.* □

Actually the proof of Lemma 22 (in [20]) shows that κ is weakly hyper Mahlo, weakly hyper hyper Mahlo, etc.

Corollary 23 $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\mathsf{PKL}}_{stat}, <2^{\aleph_0})$ *implies that $2^{\aleph_0}$ is weakly Mahlo, weakly hyper Mahlo, weakly hyper hyper Mahlo, etc.* □

Using the characterization Proposition 20 of $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\mathsf{PKL}}_{stat}, <\kappa)$, a proof similar to that of Theorem 12 shows the following:

Theorem 24 *Suppose that κ is a generically supercompact cardinal by $<\mu$-cc posets for some $\mu < \kappa$. Then $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\mathsf{PKL}}_{stat}, <\kappa)$ holds.* □

Since ccc posets are proper, we obtain the following by Theorem 17, Proposition 19 and Theorem 24:

Corollary 25 *Suppose that κ is a generically supercompact cardinal by ccc posets. Then $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ and $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\mathsf{PKL}}_{stat}, <\kappa)$ hold.* □

By Lemma 18, it follows that

Corollary 26 *If* ZFC + *"there is a supercompact cardinal" is consistent then so is* $\mathsf{ZFC} + \mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ *and* $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\mathsf{PKL}}_{stat}, <2^{\aleph_0})$. *Note that the continuum is fairly large in the latter axiom system by Corollary 23.* □

6 Laver-generic large cardinals

The reflection properties we presented so far in connection with the size of the continuum can be summarized in three possible scenarios:

(A) GRP .... *This implies* CH *(Lemma 9).*
(B) $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ .... *This implies* $2^{\aleph_0} = \aleph_2$ *(Theorem 6,* (2)*).*
(C) $\mathsf{SDLS}^{int}_+(\mathcal{L}^{\mathsf{PKL}}_{stat}, <2^{\aleph_0})$ *This implies that* $2^{\aleph_0}$ *is fairly large (Corollary 23).*

These three possible scenarios can be treated in a uniform way from the point of view of the Laver-generic large cardinals defined below.

We shall call a class $\mathcal{P}$ of posets *iterable* if

(6.1) $\mathcal{P}$ is closed with respect to forcing equivalence. That is, if $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P}'$ is forcing equivalent to $\mathbb{P}$ then $\mathbb{P}' \in \mathcal{P}$;
(6.2) For any $\mathbb{P} \in \mathcal{P}$ and $\mathbb{p} \in \mathbb{P}$, $\mathbb{P} \restriction \mathbb{p} \in \mathcal{P}$;
(6.3) If $\mathbb{P} \in \mathcal{P}$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}}\ \varepsilon\ \mathcal{P}$" then $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathcal{P}$.

Note that most of natural classes of posets like σ-closed posets, ccc posets, proper posets, stationary preserving posets etc. are iterable.

For a cardinal κ and an iterable class $\mathcal{P}$ of posets, we call κ a *Laver-generically supercompact for* $\mathcal{P}$ if, for any $\lambda \geq \kappa$ and any $\mathbb{P} \in \mathcal{P}$, there is a poset $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{P} \lessdot \mathbb{Q}$ such that, for any $(\mathsf{V}, \mathbb{Q})$-generic filter $\mathbb{H}$, there are M, $j \subseteq \mathsf{V}[\mathbb{H}]$ such that

(6.4) M is an inner model of $\mathsf{V}[\mathbb{H}]$,
(6.5) $j : \mathsf{V} \overset{\prec}{\to} M$,
(6.6) $crit(j) = \kappa$, $j(\kappa) > \lambda$,
(6.7) $\mathbb{P}$, $\mathbb{H} \in M$ and
(6.8) $j''\lambda \in M$.[13]

κ is *Laver-generically superhuge* (*Laver-generically super almost-huge* resp.) *for* $\mathcal{P}$ if κ satisfies the definition of Laver-generic supercompactness for $\mathcal{P}$ with (6.8) replaced by

(6.8)′ $j''j(\kappa) \in M$ ($j''\mu \in M$ for all $\mu < j(\kappa)$ resp.).

κ is *tightly Laver-generically supercompact* (*tightly Laver-generically superhuge, tightly Laver-generically super almost-huge*, resp.) if the definition of Laver-generically supercompact (Laver-generically superhuge, Laver-generically super almost-huge, resp.) holds with (6.6) replaced by

(6.6)′ $crit(j) = \kappa$, $j(\kappa) = |\,\mathbb{Q}\,| > \lambda$.

[13] This definition of Laver-generic supercompactness for $\mathcal{P}$ is different from the one given in [20]. However, it is easy to show that the present definition is equivalent to the one in [20] for an iterable $\mathcal{P}$. Note that, strictly speaking, this equivalence is used at the end of the proof of Theorem 34 below.

All consistency proofs of the existence of Laver-generic large cardinals we know actually show the existence of tightly Laver-generic large cardinals (see the proof of Theorem 5.2 in [20]).

The consistency of the existence of a Laver-generic large cardinal can be proved from the assumption of the existence of the corresponding genuine large cardinals except the case of the Laver-generic large cardinals by proper posets. This case will be further discussed in [21].

Theorem 27 ([20]) (1) *Suppose that* ZFC + *"there exists a supercompact cardinal" is consistent. Then* ZFC + *"there exists a Laver-generically supercompact cardinal for* σ*-closed posets" is consistent as well.*

(2) *Suppose that* ZFC + *"there exists a superhuge cardinal" is consistent. Then* ZFC + *"there exists a Laver-generically super almost-huge cardinal for proper posets" is consistent as well.*

(3) *Suppose that* ZFC + *"there exists a supercompact cardinal" is consistent. Then* ZFC + *"there exists a Laver-generically supercompact cardinal* κ *for c.c.c. posets" with* $\kappa = 2^{\aleph_0}$ *is consistent as well.*

Sketch of the proof. Let us consider (2). The other assertions are similarly and easier to prove.

Starting from a model of ZFC with a superhuge cardinal κ, we can obtain models of respective assertions by iterating in countable support with proper posets κ times along a Laver function for super almost-hugeness which exists by a result in Corazza [7].

In the resulting model, we obtain Laver-generically super almost-hugeness in terms of proper poset $\mathbb{Q}$ in each respective inner model $M[\mathbb{G}]$ of $\mathsf{V}[\mathbb{G}]$. The closedness of M in V in terms of super almost-hugeness implies that $\mathbb{Q}$ is also proper in $\mathsf{V}[\mathbb{G}]$ (this is the place where we need the super almost-hugeness: for (1) and (3) we do not need this much closedness of M).

This shows that κ is Laver-generically super almost-huge of proper posets.

$\square$ (Theorem 27)

In contrast to simple generic supercompactness, a Laver-generically supercompact cardinal for a natural class $\mathcal{P}$ of posets is determined uniquely if it exists:

Proposition 28 ([20]) (0) *If* κ *is generically measurable for some poset* $\mathbb{P}$*, then* κ *is regular.*

(1) *Suppose that* κ *is generically measurable by a* ω_1 *preserving* $\mathbb{P}$*. Then* $\kappa > \omega_1$*.*

(2) *Suppose that* κ *is Laver-generically supercompact for a class* $\mathcal{P}$ *of posets with* $\mathrm{Col}(\omega_1, \{\omega_2\}) \in \mathcal{P}$*. Then* $\kappa \leq \omega_2$*. If all elements of* $\mathcal{P}$ *are* ω_1*-preserving, then we have* $\kappa = \omega_2$*.*

(3) *Suppose that* $\mathcal{P}$ *is a class of posets containing a poset* $\mathbb{P}$ *such that any* $(\mathsf{V}, \mathbb{P})$*-generic filter* $\mathbb{G}$ *codes a new real. If* κ *is a Laver-generically supercompact for* $\mathcal{P}$*, then* $\kappa \leq 2^{\aleph_0}$*.*

(4) *Suppose that $\mathcal{P}$ is a class of posets such that elements of $\mathcal{P}$ do not add any reals. If κ is Laver-generically supercompact by $\mathcal{P}$, then we have $2^{\aleph_0} < \kappa$.* $\Box$

For a class $\mathcal{P}$ of posets and cardinals μ, κ,

$\mathsf{MA}^{+\mu}(\mathcal{P}, {<}\kappa)$: *For any $\mathbb{P} \in \mathcal{P}$, any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ and any family $\mathcal{S}$ of $\mathbb{P}$-names such that $|\mathcal{S}| \leq \mu$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{S}$* is a stationary subset of ω_1" *for all $\underset{\sim}{S} \in \mathcal{S}$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ over $\mathbb{P}$ such that $\underset{\sim}{S}[\mathbb{G}]$ is a stationary subset of ω_1 for all $\underset{\sim}{S} \in \mathcal{S}$.*

The following strengthening of the Laver-genericity is needed to obtain "++" versions of forcing axioms.

For a cardinal κ and an iterable class $\mathcal{P}$ of posets, we call κ a *strongly Laver-generically supercompact for $\mathcal{P}$* if, for any $\lambda \geq \kappa$ and any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name of a poset $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \ \varepsilon\ \mathcal{P}$" such that, for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic filter $\mathbb{H}$, there are M, $j \subseteq \mathsf{V}[\mathbb{H}]$ with

(6.9) M is an inner model of $\mathsf{V}[\mathbb{H}]$,

(6.10) $j : \mathsf{V} \overset{\preceq}{\to} M$,

(6.11) $crit(j) = \kappa$, $j(\kappa) > \lambda$,

(6.12) $\mathbb{P}$, $\mathbb{H} \in M$ and

(6.13) $({}^{\lambda}M)^{\mathsf{V}[\mathbb{H}]} \subseteq M$.

The notions of *strongly Laver-generically superhuge* and *strongly Laver-generically super almost-huge* are defined correspondingly. For example, κ is strongly Laver-generically super almost-huge if the definition of strongly Laver-generically supercompact cardinal holds with (6.13) replaced with

(6.14) $({}^{\mu}M)^{\mathsf{V}[\mathbb{H}]} \subseteq M$ holds for all $\mu < j(\kappa)$.

Note that, if $\mathcal{P}$ is the class of ccc posets, then the "strongly" version of the Laver-generically large cardinal is equivalent to the original version of the corresponding Laver-generic largeness. Note also that the construction in the proof of Theorem 27 actually provides models of strongly Laver-genericity.

Theorem 29 (Theorem 5.7 in [20]) *For a class $\mathcal{P}$ of proper posets, if $\kappa > \aleph_1$ is a strongly Laver-generically supercompact for $\mathcal{P}$, then $\mathsf{MA}^{+\mu}(\mathcal{P}, {<}\kappa)$ holds for all $\mu < \kappa$.* $\Box$

Lemma 30 *Suppose that κ is generically supercompact by a class $\mathcal{P}$ of posets such that all $\mathbb{P} \in \mathcal{P}$ has the ${<}\mu$-cc for some $\mu < \kappa$. Then for any regular $\lambda \geq \kappa$, $\mathcal{P}_\kappa(\lambda)$ has a μ-saturated normal fine filter F over $\mathcal{P}_\kappa(\lambda)$.*

Proof. For a regular $\lambda \geq \kappa$, let $\mathbb{P}$ be such that there are $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, and M, $j \subseteq \mathsf{V}[\mathbb{G}]$ such that $j : \mathsf{V} \overset{\preceq}{\to} M$, $crit(j) = \kappa$, $j(\kappa) > \lambda$ and $j''\lambda \in M$.

In V, let $F = \{A \subseteq \mathcal{P}_\kappa(\lambda) : \Vdash_{\mathbb{P}}$ "$j''\lambda \ \varepsilon\ j(\check{A})$"$\}$. By the ${<}\mu$-cc of $\mathbb{P}$, this F is as desired. $\Box$ (Lemma 30)

Combining Proposition 28, Theorem 29 and Lemma 30, we obtain:

Theorem 31 *Suppose that κ is strongly Laver-generically supercompact cardinal for an iterable class $\mathcal{P}$ of posets.*

(A$'$) *If all elements of $\mathcal{P}$ are ω_1-preserving and do not add any reals, and* $\mathrm{Col}(\omega_1, \{\omega_2\}) \in \mathcal{P}$*, then $\kappa = \aleph_2$ and* CH *holds.*

(B$'$) *If all elements of $\mathcal{P}$ are ω_1-preserving and $\mathcal{P}$ contains all proper posets then* $\mathsf{PFA}^{+\omega_1}$ *holds and $\kappa = 2^{\aleph_0} = \aleph_2$.*

(C$'$) *If all elements of $\mathcal{P}$ are $<\mu$-cc for some fixed $\mu < \kappa$ and $\mathbb{P}$ contains a poset which adds a new real then κ is fairly large (in the sense of Lemma 30) and $\kappa \leq 2^{\aleph_0}$.*

Proof. (A$'$): By Proposition 28, (2) and (4).

(B$'$): $\kappa = \aleph_2$ by Proposition 28, (2). $\mathsf{PFA}^{+\omega_1}$ holds by Theorem 29. PFA implies $2^{\aleph_0} = \aleph_2$.

(C$'$): $\kappa \leq 2^{\aleph_0}$ by Proposition 28, (3). κ is large by Lemma 30. □ (Theorem 31)

The three cases in Theorem 31 can be further modified to fit to the reflection principles discussed in earlier sections.

Theorem 32 (A$''$) *Suppose that κ is Laver-generically supercompact for σ-closed posets. Then $2^{\aleph_0} = \aleph_1$, $\kappa = \aleph_2$,* $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$ *and* GRP *holds. It follows that* RC *and* $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *hold.*

(B$''$) *Suppose that elements of $\mathcal{P}$ are ω_1-preserving and $\mathcal{P}$ contains all proper posets. If κ is strongly Laver-generically supercompact for $\mathcal{P}$, then $2^{\aleph_0} = \kappa = \aleph_2$,* $\mathsf{PFA}^{+\omega_1}$ *and hence also* $\mathsf{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ *holds.*

(C$''$) *Suppose that κ is Laver-generically supercompact for ccc posets. Then $\kappa \leq 2^{\aleph_0}$ and $\mathcal{P}_\kappa(\lambda)$ for any regular $\lambda \geq \kappa$ carries an $\aleph_1$-saturated normal ideal. In particular, κ is κ-weakly Mahlo.* $\mathsf{MA}^{+\mu}(ccc, <\kappa)$ *for all $\mu < \kappa$,* $\mathsf{SDLS}^{int}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ *and* $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\mathsf{PKL}}_{stat}, <\kappa)$ *also hold.*

Proof.(A$''$): $2^{\aleph_0} = \aleph_1$ and $\kappa = \aleph_2$ follows from Theorem 31, (A). $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$ holds by Theorem 29. GRP holds by Corollary 13. RC and $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ follow from GRP by Theorem 11 and Corollary 10, (1).

(B$''$): This is just as (B) in Theorem 31. $\mathsf{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ holds by Corollary 5, (1).

(C$''$): The first half of the assertion follows from Theorem 31, (C). $\mathsf{MA}^{+\mu}(ccc, <\kappa)$ for all $\mu < \kappa$ holds by Theorem 29. $\mathsf{SDLS}^{int}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ and $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\mathsf{PKL}}_{stat}, <\kappa)$ hold by Corollary 25. □ (Theorem 32)

At the moment we do not know whether the assumption in (C) in Theorem 32 implies $\kappa = 2^{\aleph_0}$. The following partial answer is obtained in [20]:

Theorem 33 (Theorem 5.3 in [20]) *If κ is tightly Laver-generically superhuge for ccc posets, then $\kappa = 2^{\aleph_0}$.* □

The following Theorem 34 is a Laver-generic version of Theorem 16. The proof of the theorem is a typical application of the master condition argument (see e.g. Cummings [9]).

Theorem 34 *Suppose that κ is strongly Laver-generically almost super-huge for an iterable $\mathcal{P}$ which provably contains all posets of the form $\mathrm{Col}(\mu,\lambda)$ for all regular μ, λ with $\kappa \leq \mu < \lambda$. For a regular λ_0, let $\mathbb{P}_0 = \mathrm{Col}(\kappa,\lambda_0)$ and let $\mathbb{G}_0$ be a $(\mathsf{V},\mathbb{P}_0)$-generic filter.*

Then we have

(6.15) $\mathsf{V}[\mathbb{G}_0] \models$ "*κ is strongly Laver-generically super almost-huge for $\mathcal{P}$*".

Proof. Suppose that $\mathsf{V}[\mathbb{G}_0] \models$ "$\mathbb{P} \in \mathcal{P}$" and let $\underset{\sim}{\mathbb{P}}$ be a $\mathbb{P}_0$-name of $\mathbb{P}$ such that $\Vdash_{\mathbb{P}_0}$ "$\underset{\sim}{\mathbb{P}}\ \varepsilon\ \mathcal{P}$". Let $\lambda \geq \kappa$ be regular. Without loss of generality, we may assume that $\lambda \geq |\,\mathbb{P}\,| \geq \lambda_0$. $\mathbb{P}_0 * \underset{\sim}{\mathbb{P}} \in \mathcal{P}$ by iterability of $\mathcal{P}$. Since κ is strongly Laver-generically super almost-huge for $\mathcal{P}$, there are $\mathbb{P}_0 * \underset{\sim}{\mathbb{P}}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}_0 * \underset{\sim}{\mathbb{P}}}$ "$\underset{\sim}{\mathbb{Q}}\ \varepsilon\ \mathcal{P}$" and $(\mathsf{V},\mathbb{P}_0 * \underset{\sim}{\mathbb{P}} * \underset{\sim}{\mathbb{Q}})$-generic filter $\mathbb{H}$ such that $\mathbb{G}_0 \subseteq \mathbb{H}$ and such that there are M, $j \subseteq \mathsf{V}[\mathbb{H}]$ with (6.9), (6.10), (6.11), (6.14) and

(6.16) $\mathbb{P}_0 * \underset{\sim}{\mathbb{P}},\ \mathbb{H} \in M$.

We have $j''\mathbb{G}_0 \in M$. Let $\mathbb{P}_1 = j(\mathbb{P}_0)$ by (6.14).

By elementarity, we have $M \models$ "$\mathbb{P}_1 = \mathrm{Col}(j(\kappa), j(\lambda_0))$". Note that we also have $\mathsf{V}[\mathbb{H}] \models$ "$\mathbb{P}_1 = \mathrm{Col}(j(\kappa), j(\lambda_0))$" by (6.14). Since $M \models$ "$j''\mathbb{G}_0$ has the fip" by elementarity and $M \models$ "$|\,j''\mathbb{G}_0\,| \leq |\,\mathbb{P}\,| \leq \lambda < j(\kappa)$", there is $\mathbb{q}^* \in \mathbb{P}_1$ in M such that $\mathbb{q}^* \leq_{\mathbb{P}_1} j(\mathbb{p})$ for all $\mathbb{p} \in \mathbb{G}_0$. Let $\mathbb{G}_1$ be $(\mathsf{V}[\mathbb{H}],\mathbb{P}_1)$-generic filter with $\mathbb{q}^* \in \mathbb{G}_1$. In $\mathsf{V}[\mathbb{H} * \mathbb{G}_1]$, let

(6.17) $\tilde{j} : \mathsf{V}[\mathbb{G}_0] \overset{\preceq}{\rightarrow} M[\mathbb{G}_1];\ \underset{\sim}{a}[\mathbb{G}_0] \mapsto j(\underset{\sim}{a})[\mathbb{G}_1]$.

Since $\mathbb{P} \in \mathcal{P}^{\mathsf{V}[\mathbb{G}_0]}$ and λ were arbitrary, the elementary embedding $\tilde{j}$ above witnesses the Laver-generic super almost-hugeness of κ for $\mathcal{P}$ in $\mathsf{V}[\mathbb{G}_0]$. □ (Theorem 34)

Corollary 35 *Suppose that $\mathcal{P}$ is an iterable class of posets which provably contains all posets of the form $\mathrm{Col}(\mu,\lambda)$ for all regular uncountable μ, λ with $\kappa \leq \mu < \lambda$. If the theory* ZFC + *"there is a strongly Laver-generically super almost-huge κ for $\mathcal{P}$" + "there is a supercompact $\kappa_1 > \kappa$" is consistent, then so is the theory* ZFC + *"there is a strongly Laver-generically super almost-huge κ for $\mathcal{P}$" + "κ^+ is generically supercompact by $<\kappa$-closed posets". In particular,* $\mathsf{GRP}^{<\kappa}(<\kappa^+)$ *follows from this theory.*

Proof. Suppose that κ is strongly Laver-generically supercompact for $\mathcal{P}$ and κ_1 is a supercompact cardinal. Let $\mathbb{P}_0 = \mathrm{Col}(\kappa,\kappa_1)$ and let $\mathbb{G}_0$ be a $(\mathsf{V},\mathbb{P}_0)$-generic filter. By Theorem 34, $\mathsf{V}[\mathbb{G}_0] \models$ "κ is strongly Laver-generically supercompact for $\mathcal{P}$". $\mathsf{V}[\mathbb{G}_0] \models \kappa_1 = \kappa^+$ and $\mathsf{V}[\mathbb{G}_0] \models$ "κ^+ is generically supercompact by $<\kappa$-closed posets" by Lemma 14. By Theorem 12, $\mathsf{GRP}^{<\kappa}(<\kappa^+)$ follows.□ (Corollary 35)

Theorem 34 and Corollary 35 have many variants with similar proofs. For example:

Theorem 36 *Suppose that κ is strongly Laver-generically supercompact for an iterable $\mathcal{P}$ which provably contains all σ-closed posets. For a regular λ_0, let $\mathbb{P}_0 = \mathrm{Col}(\kappa,\lambda_0)$ and let $\mathbb{G}_0$ be a $(\mathsf{V},\mathbb{P}_0)$-generic filter.*

Then we have

(6.18) $\quad \mathsf{V}[\mathbb{G}_0] \models$ "*κ is strongly Laver-generically supercompact for $\mathcal{P}$*". □

Corollary 37 *Suppose that $\mathcal{P}$ is an iterable class of posets which provably contains all σ-closed posets. If the theory ZFC + "there is a strongly Laver-generically supercompact κ for $\mathcal{P}$" + "there is a supercompact $\kappa_1 > \kappa$" is consistent, then so is the theory ZFC + "there is a strongly Laver-generically supercompact κ for $\mathcal{P}$" + "κ^+ is generically supercompact by $<\kappa$-closed posets". In particular, $\mathsf{GRP}^{<\kappa}(<\kappa^+)$ follows from this theory.* □

Note that, by Proposition 28, (2), we have $\kappa \leq \aleph_2$ in Theorem 36 and Corollary 37 above.

In [21], we show that the combination of the principles $\mathsf{SDLS}^{int}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ and $\mathsf{GRP}^{<2^{\aleph_0}}(\leq 2^{\aleph_0})$ is also consistent under large continuum assuming the consistency of two supercompact cardinals.

7 Some open problems and Takeuti's account on Gödel's contribution to the continuum problem

Let us mention some open problems. Some of them will be addressed in [21].

The following problem is already mentioned in the previous section:

Problem 1. If κ is Laver-generically supercompact for ccc posets, does this imply $\kappa = 2^{\aleph_0}$?

Forcing axioms have some characterizations which may be interpreted as suggestions of the correctness of the axioms. See e.g. Bagaria [1], Fuchino [13].

Problem 2. Is there any nice characterizations of "$+\mu$" versions of forcing axioms?

Any meaningful answer to this problem would enhance the relevance of the trichotomy in Theorem 32.

The trichotomy (A), (B), (C) of reflection principles mentioned at the beginning of Section 6 has an alternative trichotomy (A), (B*), (C*), where

(B*) $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ *and* RC *hold.*

(C*) $\mathsf{SDLS}^{int}_+(\mathcal{L}^{\mathsf{PKL}}_{stat}, <2^{\aleph_0})$ *and* $\mathsf{RC}(<2^{\aleph_0})$ *hold.*

Note that RC in (B*) is equivalent to $\mathsf{RC}(<2^{\aleph_0})$ since $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ implies $2^{\aleph_0} = \aleph_2$.

(B*) and (C*) are not compatible with (B$''$) and (C$''$) respectively, since $\mathsf{MA}(\kappa)$ — i.e. MA (for ccc posets) for $\leq\kappa$ many dense sets — implies the negation of $\mathsf{RC}(\leq\kappa^+)$ (see section 5 in Fuchino [17]).

(B*) and (C*) can be realized by starting from a supercompact cardinal and then forcing with a Mitchell type mixed support iteration (for (B*), this is mentioned in [43], for (C*), see [21]). Actually, (C*) can be realized much easier by simply adding supercompact many Cohen reals, but we do need mixed support iteration to obtain a model of

(C†) $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\mathsf{PKL}}_{stat}, <2^{\aleph_0})$, $\mathsf{RC}(<2^{\aleph_0})$ *and* $\mathsf{GRP}^{<2^{\aleph_0}}(\leq 2^{\aleph_0})$ *hold.*

Models obtained in this way seem to be much more artificial than the models for (B$''$) and (C$''$) as in the proof of Theorem 27. Even so, we have the feeling that we do not understand yet much about the models constructed by the mixed support iteration and its variations. Thus

Problem 3. What is possible with variations of mixed support iteration?

The result of König in [34] cited here as Corollary 13 can be also regarded as a characterization of $\aleph_2$ being generically supercompact by σ-closed posets. Thus we may further ask:

Problem 4. Is there any characterizations of Laver-generically large cardinals in terms of some strong reflection principles?

The following essential problem might be much harder than the other problems:

Problem 5. Are Laver-generically large cardinals equiconsistent with corresponding genuine large cardinals?

The consistency of the reflection of non-metrizability of a first countable topological spaces down to $<\aleph_2$ is an open problem known as Hamburger's problem. The consistency of the reflection of the property of partial orderings that they are not represented as countable union of chains down to $<\aleph_2$ is also an open problem known as Galvin's conjecture. In case of Hamburger's problem, it is known that the reflection of non-metrizability of a first countable topological spaces down to $<2^{\aleph_0}$ is realized in the model obtained by adding supercompact many Cohen reals (Dow, Tall and Weiss [12]). The reflection cardinal of uncountable chromatic number of graphs is known to be $\geq \beth_\omega$ (Erdős and Hajnal, see [23] for a detailed proof in ZFC). There are many open problems in connection with reflection of these and some other mathematical properties. Some of them seem to be extremely difficult. Let us mention here merely one problem which may have some connection to Problem 3:

Problem 6. Is the reflection of non-metrizability of first countable topological spaces down to $<2^{\aleph_0}$ consistent with $\mathsf{RC}(<2^{\aleph_0})$?

The first author of this article belongs to the generation of Japanese logicians who were strongly inspired by the writings of late Professor Gaishi Takeuti who published many expository articles and books in Japanese from 1960's to the end of 1990's. Although set theory was not his main field, Professor Takeuti wrote many expositions and told his views on the subject. Gödel's program was one of the issues he discussed repeatedly there.

Gödel, who usually refused to publish papers which he thought was not yet perfect, tried once to publish a quite unfinished note in 1970 with the title "Some considerations leading to the probable conclusion that the true power of the continuum is $\aleph_2$." It is said that, being seriously ill, he did so under the fear that he would soon die. After this crisis, Gödel withdrew the note finding out some inaccuracy in it but he continued the study on the problem. Oskar Morgenstern noticed in his diary on 20. September, 1975 that Gödel told him in a telephone call that he was finally convinced that (the newest version of) his axiom implies that the continuum is “different from $\aleph_1$” and that he will write it up ([38]). Dawson [10] contains some accounts about this development.

Takeuti wrote about the details of what he understood from the 1970 note on pp.99–124 in his book [40] published in 1972. According to Takeuti [42], he was then invited by Gödel in 1975 or 1976 shortly before Gödel's retirement from IAS and discussed with him about the results on the Continuum Problem. Takeuti [41] in 1978 must be closely related to this discussion.

Twenty years later, in May 1998, the first author of the present article obtained a letter from Professor Takeuti with a copy of his handwritten manuscript in Japanese, which contained a further development of the material in [41] among other things. Unfortunately, the first author could not give any reasonable comments to the manuscript at that time. The part of the manuscript on “Gödel's Continuum Hypothesis” was then published in the new edition of [42] as an appendix in September 1998.

Modern treatment of Gödel's axioms is to be found in Brendle, Larson and Todorčević [3]. [41] is cited in [3] but neither [40] nor [42] is mentioned there.

Though the technical details of the present article are rather orthogonal to the Gödel-Takeuti line of the support of $2^{\aleph_0} = \aleph_2$, the first author considers the results presented in this article as his belated reply to the letter in 1998 and would like to dedicate this article to the memory of Professor Gaishi Takeuti.

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