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LYAPUNOV DENSITY CRITERIA FOR TIME-VARYING AND PERIODICALLY TIME-VARYING NONLINEAR SYSTEMS WITH CONVERSE RESULTS*

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Abstract. This paper presents criteria for the convergence of trajectories of time-varying nonlinear systems in terms of Lyapunov densities. The results are provided without assuming local stability and forward completeness of trajectories. As well as general time-varying nonlinear systems, periodically time-varying systems are also considered in this paper, where a weaker criterion is proposed for periodically time-varying systems. Also the existence of Lyapunov densities is proved for general and periodic time-varying nonlinear systems under the asymptotic stability of the equilibrium.

Key words. nonlinear time-varying systems, periodically time-varying systems, stability, almost attraction, Lyapunov density

AMS subject classifications. 93C10, 93D05

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1. Introduction.

Stability analysis is one of the most fundamental and important issues in analysis of nonlinear systems. The most fruitful and widely used approach to stability analysis of nonlinear systems is Lyapunov methods. Plenty of useful results have been provided with various applications to control problems. (See, e.g., [6].) On the other hand, a different methodology of Lyapunov densities has received a considerable amount of attention in the last two decades. For time-invariant nonlinear systems,

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \tag{1.1}$$

the method of Lyapunov densities [17] guarantees the almost attraction of an equilibrium $x = 0$, namely the convergence of trajectories of time-invariant nonlinear system (1.1) starting from almost all initial states to the origin. The convergence of trajectories is deduced via a measure on the state space of system (1.1) defined with a Lyapunov density, where the measure is monotonically increasing along the trajectories of nonlinear system (1.1).

An advantage of Lyapunov densities is that a convex formulation of nonlinear state feedback synthesis is available via Lyapunov densities via sum-of-squares programs [13]. Various results based on Lyapunov densities have been presented for input-to-output stability [1], positive invariance of trajectories [7], converse results [10, 7, 2], the convergence to invariant sets [18, 11, 3, 5], finite-time stability [5, 4], and so forth. Also a stability criterion of stochastic differential equations is provided in [20] and analysis of coupled systems is shown in [15]. Results on Lyapunov densities for discrete-time nonlinear systems can be found in [19], where continuous-time systems

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are considered via approximation. Nonlinear systems with a class of vector fields that can have nondifferentiable points are considered in [9]. Abstract extensions with Perron–Frobenius transfer operators [16, 14] and monotone measures for abstract dynamical systems [3] have been studied based on Lyapunov densities. In [21], a combined application of Lyapunov functions and Lyapunov densities is proposed for stability analysis of a rotation motion.

In the original work of Rantzer [17], the almost attraction of nonlinear time-invariant system (1.1) is considered, where a result is provided for instability of nonlinear time-varying systems. In [12], assuming the local stability of the equilibrium, a condition for the almost attraction of nonlinear time-varying systems is proposed in terms of Lyapunov densities that depend on both time and state. Systems with external inputs are considered in [1], where the local stability is also assumed. The existence of Lyapunov densities considered in [12] is investigated in [2].

In this paper, we focus on nonlinear time-varying systems,

$$\dot{x} = f(t, x), \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \tag{1.2}$$

and provide criteria on Lyapunov densities under which almost all of the trajectories of (1.2) converge to equilibrium $x = 0$, without assuming the local stability. We also do not assume the forward completeness of trajectories of (1.2) but the proposed criterion guarantees the existence of almost all trajectories for all t greater than the initial time. The proposed condition is similar to that of [12] but the integrability conditions of Lyapunov densities are different. More specifically, our results involve the integrability of Lyapunov densities on the product space of the state space and the time axis. Then we prove that, under the existence of a Lyapunov density satisfying the proposed criterion, the set of initial data for which the corresponding trajectory does not converge to the equilibrium has zero Lebesgue measure in $\mathbb{R}^{n+1}$.

Moreover, we consider nonlinear systems (1.2) which are periodic in time, namely f satisfies, for some constant $T > 0$,

$$f(t + T, x) = f(t, x) \qquad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^n.$$

We show that a weaker criterion in terms of Lyapunov densities guarantees the almost attraction to the origin, where the integrability condition is posed on integrals in $\mathbb{R}^n$. In these results on time-varying and periodically time-varying nonlinear systems, the proposed criteria on Lyapunov densities also guarantee the positive invariance of a subset of the product space of the state space and the time axis. The criteria apply to global almost attraction if the subset is chosen as the whole product space. Notice that the conference version of this paper [8] only handles the global attraction of nonlinear time-varying system (1.2), where periodically time-varying systems are not considered.

We also prove the existence of Lyapunov densities that satisfy each of the criteria proposed in this paper under the asymptotic stability of the equilibrium of the system. Unlike previous converse results [10, 7, 2], exponential stability or exponential dichotomy at the equilibrium is not required. The existence of a Lyapunov density is proved for both general and periodically time-varying systems.

The rest of the paper is organized as follows. In section 2, we provide a criterion for general time-varying system (1.2) in terms of Lyapunov densities, while section 3 shows a criterion for periodically time-varying systems. Section 4 is devoted for the existence proof of Lyapunov densities, where we begin with the local existence of Lyapunov densities and then extend the results to the existence proof for the region of attraction and general positively invariant sets. We conclude the paper in section 5.

Notation. Let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}_0$, and $\mathbb{N}$ be the set of all real numbers, integers, nonnegative integers, and natural numbers, respectively, Denote by $\mathbb{R}^n$ the set of n-dimensional real vectors and by 0_n the zero-vector of $\mathbb{R}^n$. For set S, S^{c}, S°, S^{cl}, and ∂S stand for the complement, the interior, the closure, and the boundary of S, respectively. Let $\|\cdot\|$ and $\|\cdot\|_s$ denote the Euclidean norm of $\mathbb{R}^n$ and the spectral norm of matrices, respectively. For a vector x, x_i stands for the ith element. For an n-dimensional real-vector-valued function $h(x) = (h_1(x), h_2(x), \dots, h_n(x))$ of the variable of n-dimensional real vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, define $[\nabla \cdot h](x) = \sum_{i=1}^n \frac{\partial h_i(x)}{\partial x_i}$. If f is an $(m+n)$-dimensional real-vector-valued function of variable $x = (y, z) = (y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_n) \in \mathbb{R}^{m+n}$, we define $[\nabla_y \cdot f](x) = \sum_{i=1}^m \frac{\partial f_i(x)}{\partial y_i}$ and $[\nabla_z \cdot f](x) = \sum_{i=1}^n \frac{\partial f_{m+i}(x)}{\partial z_i}$. We mean by $C^k(D, \mathbb{R}^m)$ the set of k-times continuously differentiable functions from D to $\mathbb{R}^m$, where $C(D, \mathbb{R}^m)$ is the set of continuous functions. Notation of differentials as f_x is used to represent $\frac{\partial f}{\partial x}$. For a square matrix M, let $|M|$ denote the determinant of M. We say that proposition $P(x)$ with $x \in \mathbb{R}^n$ is true for almost all $x \in D \subset \mathbb{R}^n$ if the Lebesgue measure of the subset of $x \in D$ for which $P(x)$ is false is zero. Let $B^n(r; x)$ be the open ball of $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius r.

2. Lyapunov density criterion for general time-varying systems.

Let $n \in \mathbb{N}$ and consider the following time-varying nonlinear system:

$$\dot{x} = f(t, x), \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \tag{2.1}$$

where $f \in C^1(\mathbb{R} \times (\mathbb{R}^n \setminus \{0_n\}), \mathbb{R}^n)$. We assume that $f(t, 0_n) = 0_n$ and f is locally Lipschitz continuous in x at $(t, 0_n)$ for all $t \in \mathbb{R}$, i.e., for each $t \in \mathbb{R}$, there exists $a > 0$ and $L > 0$ such that $\|f(\tau, x_1) - f(\tau, x_2)\| < L\|x_1 - x_2\|$ holds for all $\tau \in (t-a, t+a)$ and for all $x_1, x_2 \in B^n(a; 0)$. Denote by $\varphi(t; t_0, x_0)$ the solution to system (2.1) satisfying initial condition $x(t_0) = x_0$. For each (t_0, x_0), the maximal interval of t in which $\varphi(t; t_0, x_0)$ exists is represented as $(T_{-\infty}(t_0, x_0), T_{+\infty}(t_0, x_0))$, where $T_{-\infty}(t_0, x_0) \in [-\infty, t_0)$ and $T_{+\infty}(t_0, x_0) \in (t_0, \infty]$. Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \{t \in \mathbb{R} : t \geq \underline{t}_0\}$ for a given $\underline{t}_0 \in \mathbb{R}$ and consider family $\{S(t)\}_{t \in \mathbb{T}}$, $S(t) \subset \mathbb{R}^n$. A family $\{S(t)\}_{t \in \mathbb{T}}$ is said to be positively invariant of system (2.1) if $\varphi(t; t_0, x_0) \in S(t)$ for all $t \in [t_0, T_{+\infty}(t_0; x_0))$ for all $t_0 \geq \underline{t}_0$ and $x_0 \in S(t_0)$.

Assumption 2.1. Family $\{S(t)\}_{t \in \mathbb{T}}$ fulfills the following conditions: (i) There exists a continuous function $b : [\underline{t}_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(t) = \{x \in \mathbb{R}^n : b(t, x) > 0\} \quad \forall t \in [\underline{t}_0, \infty) \tag{2.2}$$

with $\sup_{(t,x) \in \mathbb{R}^{n+1}} b(t, x) < \infty$. (ii) For each $t \in \mathbb{T}$, $S(t)$ is a connected open set of $\mathbb{R}^n$ and contains 0_n.

Define

$$\widehat{S} = \{(t, x) \in \mathbb{R}^{n+1} : t \in \mathbb{T},\ x \in S(t)\}. \tag{2.3}$$

If $\{S(t)\}_{t \in \mathbb{T}}$ is a positively invariant family, $\widehat{S}$ is the set of positively invariant initial data (t, x). For convenience, define also $\partial_+ \widehat{S} = \{(t, x) \in \mathbb{R}^{n+1} : t > \underline{t}_0,\ x \in \partial S(t)\}$. Let $\widehat{E} = \mathbb{R} \times \{0_n\}$ and $\widehat{M}_r = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t \geq -1/r, \|x\| \geq r\}$, where $r > 0$. Note that $\bigcup_{r>0} \widehat{M}_r = \widehat{E}^{\mathrm{c}}$. Below we state a criterion of Lyapunov densities for general time-varying systems.

THEOREM 2.2. *Let $\{S(t)\}_{t\in\mathbb{T}}$ be a family of subsets $S(t) \subset \mathbb{R}^n$ satisfying Assumption* 2.1 *and define $\widehat{S}$ as in* (2.3)*. Suppose that there exists a function $\rho \in C^1(\widehat{S}\setminus\widehat{E}, \mathbb{R})$ that satisfies the following conditions:* (i) *$\rho(t,x) > 0$ for all $(t,x) \in \widehat{S}^\circ \setminus \widehat{E}$.* (ii) *For every $(t_b, x_b) \in \partial_+\widehat{S}$, it holds that $\rho(t,x) \to 0$ as $(t,x) \to (t_b,x_b)$ with $(t,x) \in \widehat{S}$.* (iii) *It holds that*

$$[\rho_t + \nabla_x \cdot (f\rho)](t,x) > 0$$

for almost all $(t,x) \in \widehat{S} \setminus \widehat{E}$. Then, for all $(t_0,x_0) \in \widehat{S}$, $\varphi(t;t_0,x_0)$ belongs to $S(t)$ for all $t \in [t_0, T_{+\infty}(t_0,x_0))$. Next, in addition to (i)–(iii)*, suppose that* (iv) *the following integral is finite for all $r > 0$:*

$$I(\widehat{S}, r) = \int_{\widehat{S}\cap\widehat{M}_r} \frac{1 + \|f(t,x)\|}{1 + \|x\|} \rho(t,x) dx\, dt. \tag{2.4}$$

Then, for almost all $(t_0,x_0) \in \widehat{S}$, solution $\varphi(t;t_0,x_0)$ is defined and belongs to $S(t)$ for all $t \in [t_0,\infty)$ and $\lim_{t\to\infty} \varphi(t;t_0,x_0) = 0_n$.

Proof. (I) First, we prove the positive invariance of $\{S(t)\}_{t\in\mathbb{T}}$. Let $(t_0,x_0) \in \widehat{S}\setminus\widehat{E}$. From Assumption 2.1, there exists a $t_a > t_0 \in \mathbb{T}$ such that $\varphi(t;t_0,x_0) \in S(t)$ for all $t \in [t_0,t_a]$. Let $x_a = \varphi(t_a;t_0,x_0)$ and assume that there exists a $t_b \in (t_a, T_{+\infty}(t_0,x_0))$ for which $\varphi(t;t_0,x_0) \in S(t)$ for all $t \in [t_a,t_b)$ and $x_b := \varphi(t_b;t_0,x_0) \in \partial S(t_b)$. Define

$$\tilde{\rho}(t) = \rho(t, \varphi(t;t_a,x_a)) \left| \frac{\partial\varphi(t;t_a,x_a)}{\partial x_a} \right|.$$

Then conditions (i) and (ii) imply that $\tilde{\rho}(t_a) = \rho(t_a,x_a) > 0$ and $\lim_{t\to t_b-0} \tilde{\rho}(t_b) = 0$, respectively. From condition (iii),

$$\frac{d\tilde{\rho}(t)}{dt} = [\rho_t + \nabla_x \cdot (f\rho)](t,\varphi(t;t_a,x_a)) \left| \frac{\partial\varphi(t;t_a,x_a)}{\partial x_a} \right| \geq 0 \quad \forall t \in [t_a,t_b),$$

i.e., $\tilde{\rho}(t)$ is monotonically increasing for $t \in [t_a,t_b)$. Hence $\lim_{t\to t_b-0} \tilde{\rho}(t)$ can never be zero, which is a contradiction. Therefore $\varphi(t;t_0,x_0) \in S(t)$ for all $t \in [t_0, T_{+\infty}(t_0,x_0))$.

(II) To prove the convergence of trajectories, we utilize an augmented system defined as follows. Let

$$a(t,x) = \frac{1 + \|x\|}{1 + \|f(t,x)\|},$$

which is strictly positive on $\mathbb{R}^{n+1}$, and consider the following system:

$$\frac{ds}{d\tau} = a(s,y), \qquad \frac{dy}{d\tau} = a(s,y)f(s,y), \qquad \tau \in \mathbb{R}. \tag{2.5}$$

Let $(s(0),y(0)) = (t_0,x_0)$ and let $t = s(\tau)$. This is bijective since $\frac{dt}{d\tau} = \frac{ds(\tau)}{d\tau} = a(s(\tau),y(\tau)) > 0$. Then, setting $x(t) = y(\tau)$ with $t = s(\tau)$, we have $dx(t)/dt = f(t,x(t))$ and $x(t_0) = x_0$ and hence $x(t) = \varphi(t;t_0,x_0)$. Define

$$\xi = (s,y), \quad F(\xi) = (a(\xi),\, a(\xi)f(\xi)) = (a(s,y),\, a(s,y)f(s,y)), \quad \xi_0 = (t_0,x_0),$$

with which augmented system (2.5) is represented as

$$\frac{d\xi}{d\tau} = F(\xi), \qquad \xi(0) = \xi_0, \tag{2.6}$$

where $F \in C^1(\mathbb{R}^{n+1} \setminus \widehat{E}, \mathbb{R}^{n+1})$. Denote by $\Phi(\tau; \xi_0)$ the solution $\xi(\tau)$ of (2.6) for initial condition $\xi(0) = \xi_0$. We also write

$$\Phi(\tau; \xi_0) = (\Phi^s(\tau; \xi_0),\, \Phi^y(\tau; \xi_0)), \quad \Phi^s(\tau; \xi_0) \in \mathbb{R}, \quad \Phi^y(\tau; \xi_0) \in \mathbb{R}^n.$$

It holds for all $\xi = (s, y) \in \mathbb{R}^{n+1}$ that

$$\|F(\xi)\| = a(s, y)\sqrt{1 + \|f(s, y)\|^2} \le a(s, y)(1 + \|f(s, y)\|) = 1 + \|y\|,$$

which implies that solution $\Phi(\tau; \xi_0)$ is defined for all $\tau \in \mathbb{R}$ for all $\xi_0 \in \mathbb{R}^{n+1}$. In fact, it holds from the Gronwall–Bellman inequality [6] that

$$\begin{cases} \|\Phi(\tau; \xi) - \xi\| \le (1 + \|\xi\|)(e^{|\tau|} - 1), \\ \|\Phi^y(\tau; \xi) - y\| \le (1 + \|y\|)(e^{|\tau|} - 1) \end{cases} \tag{2.7}$$

for all $\xi = (s, y) \in \mathbb{R}^{n+1}$. We also note that $\Phi^s(\tau; \xi) \ge s$ for all $\tau \ge 0$.

If $\Phi^y(\tau; \xi_0) \to 0_n$ as $\tau \to \infty$ for $\xi_0 = (t_0, y_0)$, we have $\varphi(t; t_0, x_0) \to 0_n$ as $t \to T_{+\infty}(t_0, x_0)$. Then the Lipschitz continuity of $f(t, x)$ at $(t, 0_n)$, $t \in \mathbb{R}$, guarantees the uniqueness of the solution corresponding to the initial condition $x(t_0) = 0_n$, $t_0 \in \mathbb{T}$. Hence $T_{+\infty}(t_0, x_0) = \infty$ and $\lim_{t\to\infty} \varphi(t; t_0, x_0) = 0_n$.

Next, define a nonnegative function $R \in C^1(\widehat{S} \setminus \widehat{E}, \mathbb{R})$ as

$$R(\xi) = R(s, y) = \frac{\rho(s, y)}{a(s, y)} = \frac{1 + \|f(s, y)\|}{1 + \|y\|}\rho(s, y). \tag{2.8}$$

Then conditions (i)–(iv) are stated with $R(\xi)$ as follows, respectively: (i′) $R(\xi) > 0$ for all $\xi \in \widehat{S}^\circ \setminus \widehat{E}$. (ii′) For every $\xi_b \in \partial_+\widehat{S}$, it holds that $R(\xi) \to 0$ as $\xi \to \xi_b$ with $\xi \in \widehat{S}$. (iii′) $[\nabla_\xi \cdot (FR)](\xi) > 0$ for almost all $\xi \in \widehat{S} \setminus \widehat{E}$. (iv′) Integral $I(\widehat{S}, r) = \int_{\widehat{S}\cap\widehat{M}_r} R(\xi) d\xi$ is finite for all $r > 0$. We have seen the positive invariance of $\widehat{S}$ for system (2.6) as $\Phi(\tau; \xi) \in \widehat{S}$ for all $\xi \in \widehat{S}$ and $\tau \ge 0$.

(III) To proceed to the proof of the convergence, we refer to the following lemma, which is one of the key results in the original work of Rantzer [17].

LEMMA 2.3 (Rantzer [17, Theorem 2]). *Let $(X, \mathcal{A}, \mu)$ be a measure space and consider a measurable set $P \subset X$ and a mapping $\mathcal{T} : X \to X$. Let Z be the set of $x \in P$ for which $\mathcal{T}^k(x) \in P$ for infinitely many natural numbers k. Suppose that $\mu(P) < \infty$ and that $\mu(\mathcal{T}^{-1}U) \le \mu(U)$ holds for all measurable sets $U \subset X$. Then $\mu(\mathcal{T}^{-1}Z) = \mu(Z)$.*

Consider a measure space $(\widehat{S}, \mathcal{B}(\widehat{S}), \mu)$ with

$$\mu(U) = \int_U R(\xi) d\xi, \qquad U \in \mathcal{B}(\widehat{S}), \tag{2.9}$$

where $\mathcal{B}(\widehat{S})$ stands for the Borel family of $\widehat{S} \subset \mathbb{R}^{n+1}$. From (iv′), $\mu(\widehat{S} \cap \widehat{M}_r)$ is finite. Let $\sigma > 0$ and define a mapping $\mathcal{T}_\sigma : \widehat{S} \to \widehat{S}$ as $\mathcal{T}_\sigma(\xi) = \Phi(\sigma; \xi)$, $\xi \in \widehat{S}$, where the positive invariance of $\widehat{S}$ justifies this definition. For $U \subset \widehat{S}$, $\mathcal{T}_\sigma^{-1}U = \{\xi \in \widehat{S} : \mathcal{T}_\sigma(\xi) \in U\}$ satisfies $\mathcal{T}_\sigma(\mathcal{T}_\sigma^{-1}(U)) \subset U$. Therefore, for $U \in \mathcal{B}(\widehat{S})$,

$$\begin{aligned} \mu(U) - \mu(\mathcal{T}_\sigma^{-1}(U)) &\ge \mu(\mathcal{T}_\sigma(\mathcal{T}_\sigma^{-1}(U))) - \mu(\mathcal{T}_\sigma^{-1}(U)) \\ &= \int_0^\sigma \int_{\Phi(\tau;\mathcal{T}_\sigma^{-1}(U))} [\nabla_\xi \cdot (FR)](\xi) d\xi d\tau \\ &= \int_0^\sigma \int_{\mathcal{T}_\sigma^{-1}(U)} [\nabla_\xi \cdot (FR)](\Phi(\tau; \eta)) \left|\frac{\partial \Phi(\tau; \eta)}{\partial \eta}\right| d\eta d\tau \ge 0, \end{aligned} \tag{2.10}$$

where the last inequality holds from $\Phi(\tau;\mathcal{T}_\sigma^{-1}(U))\subset\widehat{S}$ and condition (iii′). Let $Z_{r,\sigma}$ be the set of $\xi\in\widehat{S}\cap\widehat{M}_r$ such that $\mathcal{T}_\sigma^k\xi\in\widehat{S}\cap\widehat{M}_r$ for infinitely many natural numbers k. Then $\mu(\mathcal{T}_\sigma^{-1}Z_{r,\sigma})=\mu(Z_{r,\sigma})$ from Lemma 2.3. Therefore, from (2.10) with $U=Z_{r,\sigma}$, we see that $\mathcal{T}_\sigma^{-1}(Z_{r,\sigma})$ is a Lebesgue zero set of $\mathbb{R}^{n+1}$ for any $r>0$ and $\sigma>0$ since $[\nabla_\xi\cdot(FR)](\xi)>0$ for almost all $\xi\in\widehat{S}\setminus\widehat{E}$ from condition (iii). Define

$$
\begin{aligned}
\text{(2.11)}\qquad \widehat{Z} &= \left(\bigcup_{p,q\in\mathbb{N}}\mathcal{T}_{1/p}^{-1}(Z_{1/p,\,1/q})\right)\cup\widehat{E}\\
&=\Bigg\{\xi\in\widehat{S}:\ \exists p,q\in\mathbb{N}\quad \Phi(1/q;\xi)\in\widehat{S}\cap\widehat{M}_{1/p}\ \text{ and}\\
&\qquad\left[\forall k\in\mathbb{N}\quad \exists l\geq k,\quad \Phi((l+1)/q;\xi)\in\widehat{S}\cap\widehat{M}_{1/p}\right]\Bigg\}\cup\widehat{E},
\end{aligned}
$$

which is also a Lebesgue zero set. Taking into account that $\Phi^s(\tau;\xi)\geq s\geq -p$ for all $\tau\geq 0$ and $\xi\in\widehat{S}\cap\widehat{M}_{1/p}$, we see from (2.11) that, for all $\xi\in\widehat{S}\setminus\widehat{Z}$ and for all $p,q\in\mathbb{N}$,

$$
\text{(2.12)}\quad \|\Phi^y(1/q;\xi)\|<\frac{1}{p}\quad\text{or}\quad\left[\exists N_{p,q}\in\mathbb{N}\quad\forall k\geq N_{p,q}\quad\|\Phi^y(k/q;\xi)\|<\frac{1}{p}\right].
$$

Now we are ready to complete the proof of the convergence. Let $\xi=(s,y)\in\widehat{S}\setminus\widehat{Z}$ and $\varepsilon\in(0,1)$ be arbitrary. Then $y\neq 0$ and there exist $p,q\in\mathbb{N}$ such that

$$
\text{(2.13)}\qquad \frac{1}{p}<\min\left\{\frac{\|y\|}{2},\ \frac{\varepsilon}{4}\right\},\quad e^{1/q}-1<\min\left\{\frac{\|y\|}{2(1+\|y\|)},\ \frac{\varepsilon}{4}\right\}
$$

and $p>-s$. Then $\xi\in\widehat{M}_{1/p}$. First, from (2.7) and (2.13),

$$
\begin{aligned}
\|\Phi^y(1/q;\xi)\|&\geq\|y\|-\|\Phi^y(1/q;\xi)-y\|\\
&\geq\|y\|-(1+\|y\|)(e^{1/q}-1)>\frac{\|y\|}{2}>\frac{1}{p},
\end{aligned}
$$

which implies that the latter statement of (2.12) holds. Let $k\geq N_{p,q}$ and $\tau\in[k/q,(k+1)/q)$. Then, from (2.7), (2.13), and $\varepsilon\in(0,1)$,

$$
\begin{aligned}
\|\Phi^y(\tau;\xi)\|&\leq\|\Phi^y(k/q;\xi)\|+\|\Phi^y(\tau;\xi)-\Phi^y(k/q;\xi)\|\\
&\leq\|\Phi^y(k/q;\xi)\|+(1+\|\Phi^y(k/q;\xi)\|)(e^{\tau-k/q}-1)\\
&<\frac{1}{p}+\left(1+\frac{1}{p}\right)(e^{1/q}-1)<\frac{\varepsilon}{4}+\left(1+\frac{\varepsilon}{4}\right)\frac{\varepsilon}{4}<\varepsilon.
\end{aligned}
$$

Since $k\geq N_{p,q}$ and $\tau\in[k/q,(k+1)/q)$ are arbitrary, we see that, for almost all $\xi\in\widehat{S}\setminus\widehat{E}$ and for any $\varepsilon\in(0,1)$, $\|\Phi^y(\tau;\xi)\|<\varepsilon$ holds for all $\tau\geq\tau_\varepsilon:=N_{p,q}/q$. Hence $\lim_{\tau\to\infty}\Phi^y(\tau;\xi)=0_n$, which is interpreted as $\lim_{t\to\infty}\varphi(t;t_0,x_0)=0_n$ for almost all $(t_0,x_0)\in\widehat{S}\setminus\widehat{E}$. This completes the proof. ∎

COROLLARY 2.4. *Let $S(t)=\mathbb{R}^n$ for all $t\in\mathbb{T}$ in Theorem* 2.2. *Then the same results are drawn if $\rho(t,x)\geq 0$ for all $t\in\mathbb{T}$ and $x\neq 0$ and conditions* (iii) *and* (iv) *in Theorem* 2.2 *hold.*

Proof. The positive invariance of $\{S(t)\}_{t\in\mathbb{T}}$ is trivial. It is easy to see that the convergence proof of Theorem 2.2 is valid even with nonstrict positivity of $\rho(t,x)$. ∎

Example 2.5. Consider a system $\dot{x} = -(\sin t + 1)x^3$, whose trajectories converge to 0 as $t \to \infty$ for all initial data (t_0, x_0). For this system,

$$\rho(t,x) = \frac{1}{x^6} \exp\left(\frac{1}{x^2} - t + \cos t\right)$$

is a Lyapunov density satisfying Corollary 2.4 for any $\underline{t}_0 \in \mathbb{R}$ with $S(t) = \mathbb{R}$, $t \in \mathbb{T}$, where $\mathbb{T} = \mathbb{R}$. In fact, we can easily verify

$$[\rho_t + \nabla_x \cdot (f\rho)](t,x) = \frac{(1+\sin t)(1+3x^2)}{x^6} \exp\left(\frac{1}{x^2} - t + \cos t\right),$$
$$I(\widehat{S}, r) \le 2\exp\left(\frac{1}{r^2}+1\right)\left(\frac{1}{7r^7} + \frac{1}{2r^4}\right) e^{\frac{1}{r}} < \infty \quad \forall r > 0.$$

Next, let $S(t) = (-c, c)$, $c > 0$ for $t \in \mathbb{T} = \mathbb{R}$, which is positively invariant of the above system for all t. We have a Lyapunov density that guarantees the positive invariance of $S(t) = (-c, c)$ and the convergence of trajectories within this positively invariant family as

$$\rho(t,x) = \frac{c^2 - x^2}{x^6} \exp\left(\frac{1}{x^2} - t + \cos t\right), \quad x \in (-c, c),$$

for which

$$[\rho_t + \nabla_x \cdot (f\rho)](t,x)$$
$$= \frac{(1+\sin t)\{(c^2 - x^2)(1+x^2) + 2c^2x^2\}}{x^6} \exp\left(\frac{1}{x^2} - t + \cos t\right)$$

and (i)–(iv) of Theorem 2.2 are satisfied.

Example 2.6. Consider the following system with unknown function $h \in C^1(\mathbb{R}, \mathbb{R})$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_1^2 - x_2^2 \\ -2h(t)x_2 + 2x_1x_2 \end{bmatrix}.$$

This system has been investigated in [17] with $h(t)$ fixed as a constant. Here the vector field depends on time t. We can verify that Lyapunov density $\rho(t,x) = e^{-\alpha t}/(x_1^2+x_2^2)^2$ satisfies the assumptions of Corollary 2.4 provided that $h(t) \in [\underline{h}, \overline{h}]$ for all $t \in \mathbb{R}$, where $1/3 < \underline{h} < \overline{h} < 3$ and $\alpha > 0$ is chosen to be small enough so that $[\rho_t + \nabla_x \cdot (f\rho)](t,x)$ shown below is positive for almost all $(t,x) \in \mathbb{R} \times \mathbb{R}^2$:

$$[\rho_t + \nabla_x \cdot (f\rho)](t,x) = \frac{e^{-\alpha t}\{(6 - 2h(t) - \alpha)x_1^2 + (6h(t) - 2 - \alpha)x_2^2\}}{(x_1^2 + x_2^2)^2}.$$

Then Corollary 2.4 says that, for almost all initial data $(t_0, x_0) \in \mathbb{R}\times\mathbb{R}^2$, the trajectory $\varphi(t; t_0, x_0)$ exists and converges to the origin as $t \to \infty$, with arbitrary C^1 function $h(t)$ with $h(t) \in [\underline{h}, \overline{h}]$.

In [18], a condition of Lyapunov densities is proposed in Theorem 1 therein for the convergence of trajectories of time-invariant systems to a given invariant set. If we apply the conditions of Theorem 1 in [18] to the former instance in Example 2.5, $\rho(t,x)$ has to be integrable on $\widehat{A}_r = \mathbb{R} \times ((-\infty, -r] \cup [r, \infty))$ for all $r > 0$, while Corollary 2.4 requires the integrability on $\widehat{M}_r = [-1/r, \infty) \times ((-\infty, -r] \cup [r, \infty))$ for all $r > 0$, which is a proper subset of $\widehat{A}_r$. Theorem 2.2 also guarantees the positive invariance when $S(t) \neq \mathbb{R}^n$ for some $t \in \mathbb{T}$ as the latter instance in Example 2.5. The proof of the positive invariance and inequality (2.10) is a generalization of that of [8] for time-varying systems.

In [12], a similar criterion is proposed for time-varying systems with an integrability condition of $\rho(t,x)$ in x for each t. However, the integrability condition (iv) in (t,x) of Theorem 2.2 is crucial, as is seen in the following example.

Example 2.7. Consider a system $\dot{x} = -e^{-t}x$ [8]. Let $\rho(t,x) = 1/x^2$ and $S(t) = \mathbb{R}$. Then $\rho(t,x) > 0$ and $[\rho_t + \nabla_x \cdot (f\rho)](t,x) = e^{-t}/x^2 > 0$ for all $x \neq 0$. However, the integrability condition (iv) does not hold, while $\rho(t,x)$ is integrable on $\{x \in \mathbb{R} : |x| \geq r\}$ for each $r > 0$ and t. In fact, $\varphi(t;t_0,x_0) = \exp(e^{-t_0} - e^{-t})x_0$ for this system and hence the origin is stable but trajectories do not converge to 0 if $x_0 \neq 0$.

3. Lyapunov density criterion for periodically time-varying systems. In this section, we consider periodically time-varying systems, namely systems (2.1) whose vector field satisfies

$$f(t+T,x) = f(t,x) \qquad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n \tag{3.1}$$

for some $T > 0$. Then it holds that

$$\varphi(\tau+T; t+T, x) = \varphi(\tau; t, x) \qquad \forall \tau \geq t \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

Accordingly we consider positive invariant families $\{S(t)\}_{t\in\mathbb{R}}$ periodic in t.

Assumption 3.1. Family $\{S(t)\}_{t\in\mathbb{R}}$ satisfies (2.2) in Assumption 2.1 for $\mathbb{T} = \mathbb{R}$ and it holds that $S(t+T) = S(t)$ for all $t \in \mathbb{R}$.

Let $\widehat{E}_r = \mathbb{R} \times B^n(r; 0_n)$. Below we provide a criterion of positive invariance and convergence in terms of Lyapunov densities for periodically time-varying system (2.1). Define the following sets:

$$\widehat{S} = \{(t,x) : t \in \mathbb{R}, x \in S(t)\}, \qquad \widehat{S}_T = \{(t,x) : t \in [0,T), x \in S(t)\}. \tag{3.2}$$

THEOREM 3.2. *Suppose that system* (2.1) *is periodically time-varying with period* T *and family* $\{S(t)\}_{t\in\mathbb{R}}$ *satisfies Assumption* 3.1. *Let function* $\rho \in C^1(\widehat{S} \setminus \widehat{E}, \mathbb{R})$ *be periodic as*

$$\rho(t+T,x) = \rho(t,x) \qquad \forall (t,x) \in \widehat{S} \setminus \widehat{E}$$

and satisfy the following conditions: (i) $\rho(t,x) > 0$ *for all* $(t,x) \in \widehat{S}_T \setminus \widehat{E}$. (ii) *For every* $(t_b,x_b) \in \partial_+\widehat{S}$, *it holds that* $\rho(t,x) \to 0$ *as* $(t,x) \to (t_b,x_b)$ *with* $(t,x) \in \widehat{S}$. (iii) $[\rho_t + \nabla_x \cdot (f\rho)](t,x) > 0$ *for almost all* $(t,x) \in \widehat{S}_T \setminus \widehat{E}$. *Then, for all* $(t_0,x_0) \in \widehat{S}$, $\varphi(t;t_0,x_0) \in S(t)$ *for all* $t \in [t_0, T_{+\infty}(t_0,x_0))$. *Moreover, if* (iv) *for all* $t \in [0,T)$ *and* $r > 0$, $\rho(t,x)(1+\|f(t,x)\|)/(1+\|x\|)$ *is integrable on* $\{x \in S(t) : \|x\| \geq r\}$, *then, for almost all* $(t_0,x_0) \in \widehat{S}$, *solution* $\varphi(t;t_0,x_0)$ *is defined for* $t \in [t_0,\infty)$, *satisfies* $\varphi(t;t_0,x_0) \in S(t)$ *for all* $t \in [t_0,\infty)$, *and* $\lim_{t\to\infty} \varphi(t;t_0,x_0) = 0$.

Proof. (I) Conditions (i)–(iii) of Theorem 3.2 imply conditions (i)–(iii) of Theorem 2.2, respectively, for arbitrary $\underline{t}_0 \in \mathbb{R}$. Hence the positive invariance of $\{S(t)\}_{t\in\mathbb{R}}$ is obvious.

(II) Consider augmented system (2.6). Apparently $F(s+T,y) = F(s,y)$ holds for all $(s,y) \in \mathbb{R} \times \mathbb{R}^n$ and solutions $\Phi(\tau;\xi_0)$ of system (2.6) satisfy

$$\Phi(\tau; \xi_0 + (T, 0_n)) = \Phi(\tau; \xi_0) + (T, 0_n) \qquad \forall \tau \in \mathbb{R} \quad \forall \xi_0 \in \mathbb{R}^{n+1}. \tag{3.3}$$

Set $R \in C^1(\widehat{S} \setminus \widehat{E})$ as in (2.8). Then $R(\xi + (T, 0_n)) = R(\xi)$ for all $\xi \in \widehat{S} \setminus \widehat{E}$. Conditions (i′)–(iii′) in the proof of Theorem 2.2 are fulfilled for any $\underline{t}_0 \in \mathbb{R}$ from conditions (i)–(iii) of this theorem, respectively.

To prove the convergence, first, let $\widehat{X}_T = [0, T) \times \mathbb{R}^n$ and $\sigma \neq 0$ and define a mapping $\mathcal{T}_\sigma : \widehat{X}_T \to \widehat{X}_T$ as

$$\mathcal{T}_\sigma(\xi) = \Pi_T \Phi(\sigma; \xi),$$

where $\Pi_T : \mathbb{R}^{n+1} \to \widehat{X}_T$ is given by

$$\Pi_T(s, y) = (s - kT, y), \quad s \in [kT,\ (k+1)T), \quad k \in \mathbb{Z}.$$

Let $U \subset \widehat{X}_T$ and define

$$U_k = \{(s, y) \in \Phi(\sigma; U) : s \in [kT, (k+1)T)\}, \quad k \in \mathbb{Z}.$$

Then $\Phi(\sigma; U)$ is decomposed as

$$\Phi(\sigma; U) = \bigcup_{k=-\infty}^{\infty} U_k, \qquad U_k \cap U_l = \emptyset, \qquad k \neq l,$$

where we notice that $U_k = \emptyset$ for all $k < 0$ if $\sigma > 0$ and that $U_k = \emptyset$ for all $k > 0$ if $\sigma < 0$. We have

$$\mathcal{T}_\sigma(U) = \Pi_T \Phi(\sigma; U) = \bigcup_{k=-\infty}^{\infty} \Pi_T U_k. \tag{3.4}$$

This is seen as

$$\begin{aligned} (s, y) \in \Pi_T \Phi(\sigma; U) &\iff s \in [0, T), \quad \exists k \in \mathbb{Z} \quad (s + kT, y) \in \Phi(\sigma; U) \\ &\iff s \in [0, T), \quad \exists k \in \mathbb{Z} \quad (s + kT, y) \in U_k \\ &\iff \exists k \in \mathbb{Z} \quad (s, y) \in \Pi_T U_k. \end{aligned}$$

Moreover, it holds that

$$\Pi_T U_{k_1} \cap \Pi_T U_{k_2} = \emptyset, \qquad k_1 \neq k_2. \tag{3.5}$$

To see this, first notice that U_k can be represented as

$$U_k = \{\Phi(\sigma; \zeta) : \Phi(\sigma; \zeta) - k(T, 0_n) \in \widehat{X}_T, \quad \zeta \in U\}.$$

Let $\eta \in U_k$. Then there exists a $\zeta \in U$ such that $\eta = \Phi(\sigma; \zeta)$ with $\eta - k(T, 0_n) \in \widehat{X}_T$. Hence $\Pi_T \eta = \Phi(\sigma; \zeta_k) - k(T, 0_n) = \Phi(\sigma; \zeta_k - k(T, 0_n))$. Now assume that there exists a ξ that belongs to $\Pi_T U_{k_1} \cap \Pi_T U_{k_2}$. Then there exist $\zeta_i = (u_i, z_i) \in U \subset \widehat{X}_T$, $i = 1, 2$, such that

$$\xi = \Phi(\sigma; \zeta_i - k_i(T, 0_n)), \quad i = 1, 2,$$

and hence $\zeta_1 - \zeta_2 = (k_1 - k_2)(T, 0_n)$, i.e.,

$$u_1 - u_2 = T(k_1 - k_2), \quad z_1 - z_2 = 0.$$

This implies that $u_1 = u_2$ since $u_1, u_2 \in [0, T)$ from $\zeta_1, \zeta_2 \in \widehat{X}_T$ and k_1 and k_2 are integers. Hence $k_1 = k_2$, which contradicts the assumption $k_1 \neq k_2$ and thus (3.5) is proved.

Consider measure space $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}), \mu)$, where μ is an arbitrary measure that is shift invariant as $\mu(U) = \mu(U + (T, 0_n))$ for $U \in \mathcal{B}(\mathbb{R}^{n+1})$. Suppose that $U \subset \widehat{X}_T$ is measurable and $\mu(U) < \infty$. Then

$$\begin{aligned}(3.6)\quad \mu(\Phi(\sigma; U)) &= \mu\left(\bigcup_{k=-\infty}^{\infty} U_k\right) = \sum_{k=-\infty}^{\infty} \mu(U_k) = \sum_{k=-\infty}^{\infty} \mu(U_k - k(T, 0_n)) \\ &= \sum_{k=-\infty}^{\infty} \mu(\Pi_T U_k) = \mu\left(\bigcup_{k=-\infty}^{\infty} \Pi_T U_k\right) = \mu\left(\Pi_T \Phi(\sigma; U)\right) = \mu(\mathcal{T}_\sigma(U)).\end{aligned}$$

(III) Now consider measure space $(\widehat{S}_T, \mathcal{B}(\widehat{S}_T), \mu_T)$ with

$$\mu_T(U) = \int_U R(\xi) d\xi, \quad U \in \mathcal{B}(\widehat{S}_T).$$

From condition (iv) and Fubini's theorem, $R(\xi)$ is integrable on $\widehat{S}_T \setminus \widehat{E}_r = \{(s, y) \in \widehat{S}_T : \|y\| \geq r\}$ for all $r > 0$ and hence $\mu_T(\widehat{S}_T \setminus \widehat{E}_r) < \infty$. Let $\sigma > 0$ and consider $\mathcal{T}_\sigma = \Pi_T \Phi(\sigma; \cdot)$ as a mapping from $\widehat{S}_T$ to $\widehat{S}_T$, based on the positive invariance of $\widehat{S}$ and the definition of Π_T. Let $\widehat{Z}_{r,\sigma} \subset \widehat{S}_T$ be the set of $\xi \in \widehat{S}_T \setminus \widehat{E}_r \subset \widehat{S}_T$ such that $\mathcal{T}_\sigma^k(\xi) \in \widehat{S}_T \setminus \widehat{E}_r$ for infinitely many natural numbers k. From Lemma 2.3, it holds that $\mu_T(\mathcal{T}_\sigma^{-1} \widehat{Z}_{r,\sigma}) = \mu_T(\widehat{Z}_{r,\sigma})$, where $\mathcal{T}_\sigma^{-1}$ is defined as $\mathcal{T}_\sigma^{-1} U = \{\xi \in \widehat{S}_T : \mathcal{T}_\sigma \xi \in U\}$ for $U \subset \widehat{S}_T$. Similarly to (2.10), we derive

$$(3.7)\quad 0 = \mu_T(\widehat{Z}_{r,\sigma}) - \mu_T(\mathcal{T}_\sigma^{-1}(\widehat{Z}_{r,\sigma})) \geq \mu_T(\mathcal{T}_\sigma(\mathcal{T}_\sigma^{-1}(\widehat{Z}_{r,\sigma}))) - \mu_T(\mathcal{T}_\sigma^{-1}(\widehat{Z}_{r,\sigma})).$$

Here we go back to measure space $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}), \mu)$ with

$$\mu(U) = \int_{U \cap \widehat{S}} R(\xi) d\xi, \quad U \in \mathcal{B}(\mathbb{R}^{n+1}),$$

which is shift invariant as $\mu(U) = \mu(U + (T, 0_n))$ and it holds that $\mu(U) = \mu_T(U)$ if $U \in \widehat{S}_T$. From (3.6), (3.7), and the fact that $\mathcal{T}_\sigma^{-1}(\widehat{Z}_{r,\sigma})$ and $\mathcal{T}_\sigma(\mathcal{T}_\sigma^{-1}(\widehat{Z}_{r,\sigma}))$ are included in $\widehat{S}_T$,

$$\begin{aligned}0 &\geq \mu_T(\mathcal{T}_\sigma(\mathcal{T}_\sigma^{-1}(\widehat{Z}_{r,\sigma}))) - \mu_T(\mathcal{T}_\sigma^{-1}(\widehat{Z}_{r,\sigma})) \\ &\geq \mu(\mathcal{T}_\sigma(\mathcal{T}_\sigma^{-1}(\widehat{Z}_{r,\sigma}))) - \mu(\mathcal{T}_\sigma^{-1}(\widehat{Z}_{r,\sigma})) \\ &= \mu(\Phi(\sigma; (\mathcal{T}_\sigma^{-1}(\widehat{Z}_{r,\sigma})))) - \mu(\mathcal{T}_\sigma^{-1}(\widehat{Z}_{r,\sigma})) \\ &= \int_0^\sigma \int_{\mathcal{T}_\sigma^{-1}(\widehat{Z}_{r,\sigma})} [\nabla_\xi \cdot (FR)](\Phi(\tau; \eta)) \left| \frac{\partial \Phi(\tau; \eta)}{\partial \eta} \right| d\eta d\tau \geq 0,\end{aligned}$$

where we also exploited that $\Phi(\sigma; \mathcal{T}_\sigma^{-1}(\widehat{Z}_{\sigma,r}))$ and $\mathcal{T}_\sigma^{-1}(\widehat{Z}_{\sigma,r})$ are included in $\widehat{S}$. Therefore condition (iii) implies that $\mathcal{T}_\sigma^{-1}(\widehat{Z}_{r,\sigma})$ is a Lebesgue zero set of $\mathbb{R}^{n+1}$. Observe that

$$\mathcal{T}_\sigma^{-1}(\widehat{Z}_{r,\sigma}) = \Big\{ \xi \in \widehat{S}_T : \mathcal{T}_\sigma(\xi) \in \widehat{S}_T \setminus \widehat{E}_r \quad \text{and} \\ \left[\forall k \in \mathbb{N} \quad \exists l \geq k, \quad \mathcal{T}_\sigma^{l+1}(\xi) \in \widehat{S}_T \setminus \widehat{E}_r \right] \Big\},$$

and $\mathcal{T}_\sigma(\xi) \in \widehat{S}_T \setminus \widehat{E}_r$ if and only if $\|\Phi^y(\sigma;\xi)\| \geq r$ from the definitions of $\widehat{S}_T \setminus \widehat{E}_r$ and $\mathcal{T}_\sigma$. Hence

$$\begin{aligned}(3.8)\qquad \widehat{Z} &:= \left(\bigcup_{p,q\in\mathbb{N}} \mathcal{T}_{1/p}^{-1}(\widehat{Z}_{1/p,\,1/q})\right) \cup \widehat{E} \\ &= \Big\{\xi \in \widehat{S}_T :\ \ \exists p,q \in \mathbb{N} \quad \|\Phi^y(1/q;\xi)\| \geq 1/p \quad \text{and} \\ &\qquad \left[\forall k \in \mathbb{N} \quad \exists l \geq k, \quad \|\Phi^y((l+1)/q;\xi)\| \geq 1/p\right]\Big\} \cup \widehat{E}\end{aligned}$$

holds and $\widehat{Z}$ is a Lebesgue zero set. From (3.8), we see (2.12) in the proof of Theorem 2.2 for all $\xi \in \widehat{S}_T \setminus \widehat{Z}$ and for all $p, q \in \mathbb{N}$. Therefore we can conclude the convergence $\Phi(\tau;\xi) \to 0_n$ as $\tau \to \infty$ similarly to the proof of Theorem 2.2 for $\xi = (s,y) \in \widehat{S}_T$ with $y \neq 0$. Then the convergence of $\Phi(\tau;\xi)$ for almost all $\xi \in \widehat{S} \setminus \widehat{E}$ is readily seen from the periodicity of Φ and the proof is completed. ∎

Example 3.3. Consider the system in Example 2.5, $\dot{x} = -(1+\sin t)x^3$, again but here we exploit that this system is periodic with period $T = 2\pi$. Let $S(t) = \mathbb{R}$ and $\underline{t}_0$ be arbitrary. We have a Lyapunov density $\rho(t,x) = 1/x^4$, for which $[\rho_t + \nabla_x \cdot (f\rho)](t,x) = (1+\sin t)/x^2$ and $\rho(t,x)$ is integrable on $|x| \geq r$ for any $r > 0$ and $t \in [0, 2\pi)$. Hence conditions (i)–(iv) of Theorem 3.2 are satisfied. Thus, with the periodicity of the system, we may have a much simpler Lyapunov density satisfying the conditions of Theorem 3.2 than such ρ as in Example 2.5 that satisfies Theorem 2.2.

4. Converse results.

In this section, we prove the existence of Lyapunov densities for general time-varying systems and periodically time-varying systems under the asymptotic stability of trajectories that we will state in subsection 4.1. For general time-varying systems, we show first a local existence of Lyapunov density in subsection 4.2, with guaranteeing the positive invariance of a certain region with the convergence of trajectories. Then the region is extended in subsection 4.3 to the region of attraction in subsection 4.4. Last, subsection 4.5 provides a converse result for any positively invariant family satisfying Assumption 2.1. Converse results for periodically time-varying systems are then obtained in subsection 4.6 similarly to those for general time-varying nonlinear systems.

4.1. Assumption on the convergence.

In this subsection, we state the standing assumption for this section with which we consider the existence of Lyapunov densities.

DEFINITION 4.1. *Let $a \in (0,\infty]$.* (i) *A continuous function $\alpha : [0,a) \to [0,\infty)$ is said to be class $\mathcal{K}$ if α is strictly monotonically increasing and $\alpha(0) = 0$. If in addition $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$, then α is said to be class $\mathcal{K}_\infty$.* (ii) *A continuous function $\beta : [0,a) \times [0,\infty) \to [0,\infty)$ is said to be class $\mathcal{KL}$ if the following two conditions are satisfied: For each fixed s, mapping $\beta(r,s)$ is class $\mathcal{K}$ with respect to r, and, for each fixed r, mapping $\beta(r,s)$ is monotonically decreasing and $\beta(r,s) \to 0$ as $s \to \infty$.*

Let $\underline{t}_0 \in \mathbb{R}$ and define $\widehat{X}(t,c) = [t,\infty) \times B^n(c;0_n)$ for $t \in \mathbb{R}$ and $c > 0$. Assume the following for system (2.1).

Assumption 4.2. (i) $f \in C^2(\mathbb{R}^{n+1}, \mathbb{R}^n)$. (ii) There exist class $\mathcal{K}_\infty$ functions α_f, α_d such that $\|f\|$ and $\|f_x\|_s$ are bounded as

$$\|f(t,x)\| \le \alpha_f(\|x\|), \quad \|f_x(t,x)\|_s \le \alpha_d(\|x\|) \qquad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n. \tag{4.1}$$

(iii) There exists a class $\mathcal{KL}$ function β defined on $[0, \overline{c}_0] \times [0, \infty)$ for some $\overline{c}_0 > 0$ such that trajectories $\varphi(t; t_0, x_0)$ are defined for all $t \ge t_0$ and satisfies

$$\|\varphi(t; t_0, x_0)\| \le \beta(\|x_0\|, t - t_0) \quad \forall t \ge t_0 \quad \forall (t_0, x_0) \in \widehat{X}(\underline{t}_0, \overline{c}_0)^{\mathrm{cl}} \tag{4.2}$$

if $(t_0, x_0) \in \widehat{X}(\underline{t}_0, \overline{c}_0)$.

Below let c be a number such that $\beta(\overline{c}_0, 0) < c$. In Theorem 4.16 of Khalil [6], a local version of these conditions is assumed to provide a converse theorem that guarantees the local existence of a Lyapunov function for time-varying nonlinear systems. More specifically, condition (ii) is assumed only for x with $\|x\| < c$. Below we refer to this theorem.

LEMMA 4.3. *There exists a continuously differentiable function* $v : \widehat{X}(\underline{t}_0, \overline{c}_0) \to \mathbb{R}$ *that satisfies* $\alpha_1(\|x\|) \le v(t,x) \le \alpha_2(\|x\|)$, $[v_t + v_x f](t,x) \le -\alpha_3(\|x\|)$, *and* $\|v_x(t,x)\| \le \alpha_4(\|x\|)$ *for all* $(t,x) \in \widehat{X}(\underline{t}_0, \overline{c}_0)$ *under Assumption* 4.2, *where* α_i, $i = 1, 2, 3, 4$ *are class* $\mathcal{K}$ *functions defined on* $[0, \overline{c}_0]$.

Proof. The proof can be seen in [6]. ∎

We utilize this theorem in order to guarantee the existence of a certain positively invariant family for time-varying systems. Then we prove in the next subsection the local existence of a Lyapunov density. This result is further extended to show Lyapunov densities for general positively invariant sets.

4.2. Local existence of a Lyapunov density. Consider Lyapunov function v in Lemma 4.3. Define a positively invariant family $\{S(t)\}_{t \ge \underline{t}_0}$ as

$$S(t) = \{x \in \mathbb{R}^n : v(t,x) < \alpha_1(\overline{c}_0)\}, \qquad t \in [\underline{t}_0, \infty), \tag{4.3}$$

via Lyapunov function $v(t,x)$ in Lemma 4.3. Let $\widehat{S} = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t \in [\underline{t}_0, \infty),\ v(t,x) < \alpha_1(\overline{c}_0)\}$. It is easy to see that

$$B^n(\underline{c}_0; 0) \subset S(t) \subset B^n(\overline{c}_0; 0), \quad t \in [\underline{t}_0, \infty), \tag{4.4}$$

or equivalently $\widehat{X}(\underline{t}_0, \underline{c}_0) \subset \widehat{S} \subset \widehat{X}(\underline{t}_0, \overline{c}_0)$, where $\underline{c}_0 = \alpha_2^{-1}(\alpha_1(\overline{c}_0))$.

LEMMA 4.4. *Under Assumption* 4.2, *there exists a* $\rho \in C^1(\widehat{S} \setminus \widehat{E}, \mathbb{R})$ *that satisfies the conditions* (i)–(iv) *of Theorem* 2.2 *with family* $\{S(t)\}_{t \ge \underline{t}_0}$ *defined in* (4.3).

Proof. Let $t_1 > \underline{t}_0$ and let $\nu : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing C^∞ function satisfying $\nu(t) = 0$, $t \le \underline{t}_0$, and $\nu(t) = 1$, $t \ge t_1$. Define

$$\lambda(t,x) = \begin{cases} (\alpha_1(\overline{c}_0) - v(t,x))^2 \nu(t) e^{-t}, & (t,x) \in \widehat{S}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

It is easy to check that $\lambda \in C^1(\mathbb{R}^{n+1}, \mathbb{R})$, $\lambda(t,x) > 0$ for $(t,x) \in \widehat{S}^\circ$ and

$$0 \le \lambda(t,x) \le \overline{\alpha} e^{-t} \qquad \forall (t,x) \in \mathbb{R}^{n+1}, \quad \overline{\alpha} := \alpha_1(\overline{c}_0)^2. \tag{4.6}$$

Define $\rho(t,x)$ for $(t,x) \in \mathbb{R}^{n+1} \setminus \widehat{E}$ as

$$\rho(t,x) = \int_{T_{-\infty}(t,x)}^{t} \lambda(\tau, \varphi(\tau;t,x)) \left| \frac{\partial \varphi(\tau;t,x)}{\partial x} \right| d\tau. \tag{4.7}$$

Since $\widehat{S}$ is positively invariant and $\lambda(t,x) = 0$ for $(t,x) \notin \widehat{S}^{\circ}$, we have $\rho(t,x) = 0$ if $(t,x) \notin \widehat{S}^{\circ}$. Let $(t,x) \in \widehat{S}^{\circ} \setminus \widehat{E}$. It is easy to see that there exist a neighborhood $\widehat{N}$ of (t,x) included in $\widehat{S}^{\circ}$ and a positive number T_d such that

$$(t',x') \in \widehat{N} \implies \varphi(t' - T_d; t', x') \in (\widehat{S}^{\mathrm{c}})^{\circ},$$

by which we have

$$\rho(t',x') = \int_{t-T_d}^{t} \lambda(\tau, \varphi(\tau;t',x')) \left| \frac{\partial \varphi(\tau;t',x')}{\partial x'} \right| d\tau \qquad \forall (t',x') \in \widehat{N}.$$

This implies that ρ is continuously differentiable in $\widehat{N}$ and hence also is in $\widehat{S} \setminus \widehat{E}$. We can obtain

$$[\rho_t + \nabla_x \cdot (f\rho)](t,x) = \lambda(t,x) \qquad \forall (t,x) \in \mathbb{R}^{n+1} \setminus \widehat{E}.$$

By what we have seen so far, conditions (i)–(iii) of Theorem 2.2 are confirmed. In order to prove integrability condition (iv), we derive an upper bound of $\rho(t,x)$. Let

$$\zeta(u) = \max\{s \ge 0 : \beta(\overline{c}_0, s) \ge u\}, \quad u \in (0,c).$$

Since $\beta(\overline{c}_0, s)$ is continuous and monotonically decreasing in $s \ge 0$ and $\lim_{s\to\infty} \beta(\overline{c}_0, s) = 0$, $\zeta(u)$ is well-defined and monotonically decreasing. Next, let $(t,x) \in \widehat{S} \setminus \widehat{E}_r$ and define

$$t_e = \inf\{\tau < t : (\tau, \varphi(\tau;t,x)) \in \widehat{S}\}, \qquad x_e = \varphi(t_e;t,x). \tag{4.8}$$

Apparently $t_e \ge \underline{t}_0$, $x_e \in \widehat{S}^{\mathrm{cl}}$, and $\|x_e\| \le \overline{c}_0$ from (4.4). Assumption 4.2 and the continuity of β on $[0,\overline{c}_0] \times [0,\infty)$ imply that $\|x\| \le \beta(\|x_e\|, t - t_e)$. Moreover, $(t,x) \in \widehat{S} \setminus \widehat{E}_r$ means $\|x\| \ge r$ and hence $r \le \beta(\overline{c}_0, t - t_e)$. Therefore

$$t - t_e \le \zeta(r) \tag{4.9}$$

holds from the definition of ζ.

Now consider $\rho(t,x)$ in (4.7). Since $\widehat{S}$ is positively invariant, the interval of integration $(T_{-\infty}(t,x), t]$ in (4.7) can be replaced with $[t_e, t]$ and the length of this interval satisfies $t - t_e \le \zeta(r) < \infty$. Moreover, from (ii) of Assumption 4.2, it holds for $\tau \le t$ that

$$\left| \frac{\partial \varphi(\tau;t,x)}{\partial x} \right| \le e^{L_0(t-\tau)}, \qquad L_0 := \alpha_d(\overline{c}_0). \tag{4.10}$$

From (4.6) and (4.10),

$$\begin{aligned} \rho(t,x) &= \int_{t_e}^{t} \lambda(\tau, \varphi(\tau;t,x)) \left| \frac{\partial \varphi(\tau;t,x)}{\partial x} \right| d\tau \\ &\le \int_{t-\zeta(r)}^{t} \overline{\alpha} e^{-\tau} e^{L_0(t-\tau)} d\tau = p_0 e^{-t} \qquad \forall (t,x) \in \widehat{S} \setminus \widehat{E}_r, \end{aligned} \tag{4.11}$$

where $p_0 := \overline{\alpha}(e^{(1+L_0)\zeta(r)} - 1)/(1 + L_0)$. We also have

$$\frac{1 + \|f(t,x)\|}{1 + \|x\|} \leq H_0, \qquad H_0 := 1 + \alpha_f(\overline{c}_0),$$

for $(t, x) \in \widehat{S} \setminus \widehat{E}$. Using these inequalities, we obtain

$$I(\widehat{S}, r) = \int_{\widehat{S}\setminus\widehat{E}_r} \frac{1 + \|f(t,x)\|}{1 + \|x\|}\rho(t,x)dxdt$$
$$\leq \int_{\underline{t}_0}^{\infty} \int_{r\leq\|x\|\leq\overline{c}_0} H_0 p_0 e^{-t} dxdt \leq V(\overline{c}_0)H_0 p_0 e^{-\underline{t}_0} =: q_0,$$

where $V(\overline{c}_0)$ stands for the Lebesgue measure of balls in $\mathbb{R}^n$ with radius $\overline{c}_0$. Thus all the conditions of Theorem 2.2 are satisfied with family $\{S(t)\}_{t\geq\underline{t}_0}$ of (4.3). ∎

4.3. Lyapunov density defined on a sequence of positively invariant families. In this subsection, we extend the positively invariant family considered in the previous subsection along the trajectories to the negative direction of time, which will lead to the result of the next section that proves the existence of a Lyapunov density on the region of attraction. Define a sequence of sets $\widehat{S}_k$ via the augmented system of section 2 as

$$\widehat{S}_k = \{\xi = (t,x) : \Phi(k;\xi) \in \widehat{S}\} = \Phi(-k;\widehat{S}), \qquad k \in \mathbb{N}_0. \tag{4.12}$$

Clearly each $\widehat{S}_k$ is positively invariant and $\widehat{S} = \widehat{S}_0 \subset \widehat{S}_1 \subset \cdots \subset \widehat{S}_k \subset \cdots$. Since $\widehat{S} \subset \widehat{X}(\underline{t}_0, \overline{c}_0)$, it holds for $k \in \mathbb{N}_0$ that

$$\|\Phi(-k;\xi) - \xi\| \leq (1 + \|\xi\|)(e^k - 1) \leq (1 + \overline{c}_0)(e^k - 1) =: h_k \quad \forall \xi \in \widehat{S} \tag{4.13}$$

from (2.7). Hence

$$\widehat{S}_k \subset \widehat{X}(\underline{t}_k, \overline{c}_k), \qquad k \in \mathbb{N}_0, \tag{4.14}$$

holds for $\underline{t}_k := \underline{t}_0 - (1 + \overline{c}_0)(e^k - 1)$ and $\overline{c}_k := \overline{c}_0 + (1 + \overline{c}_0)(e^k - 1)$.

We extend Lemma 4.4 to sequence $\widehat{S}_k$, $k \in \mathbb{N}_0$.

Lemma 4.5. *Under Assumption* 4.2, *there exists a* $\rho_k \in \mathbb{R}^{n+1} \setminus \widehat{E}$ *that satisfies the conditions* (i)–(iv) *of Theorem* 2.2 *with* $\widehat{S}_k$ *defined in* (4.12) *for each* $k \in \mathbb{N}_0$.

Proof. Using λ in (4.5), define λ_k, $k \in \mathbb{N}_0$ as

$$\lambda_k(t,x) = \lambda(\Phi(k;(t,x))), \tag{4.15}$$

for which $\lambda_k \in C^1(\mathbb{R}^{n+1}, \mathbb{R})$, $\lambda_k(t,x) > 0$ in $\widehat{S}_k^{\circ}$, and $\lambda_k(t,x) = 0$ in $\widehat{S}_k^{c}$. From (4.6),

$$\lambda_k(t,x) = \lambda(\Phi(k;(t,x))) \leq \overline{\alpha}e^{-\Phi^s(k;\xi)} \leq \overline{\alpha}e^{-t}$$

for $\xi = (t,x) \in \mathbb{R}^{n+1}$, where the latter inequality holds since $\Phi^s(k;\xi) \geq t$. Now define

$$\rho_k(t,x) = \int_{T_{-\infty}(t,x)}^{t} \lambda_k(\tau, \varphi(\tau;t,x)) \left|\frac{\partial\varphi(\tau;t,x)}{\partial x}\right| d\tau, \quad k \in \mathbb{N}_0. \tag{4.16}$$

As in the proof of Lemma 4.4, we can show $\rho_k \in C^1(\mathbb{R}^{n+1} \setminus \widehat{E}, \mathbb{R})$, $\rho_k(t,x) > 0$ for $(t,x) \in \widehat{S}_k^\circ \setminus \widehat{E}$, $\rho_k(t,x) = 0$ for $(t,x) \in \widehat{S}_k^c \setminus \widehat{E}$, and the identity

$$[(\rho_k)_t + \nabla_x \cdot (f\rho_k)](t,x) = \lambda_k(t,x)$$

for all $(t,x) \in \mathbb{R}^{n+1} \setminus \widehat{E}$. Thus we have seen conditions (i)–(iii).

Let us derive an upper bound of ρ_k on $\widehat{S}_k \setminus \widehat{E}_r$ for $k \in \mathbb{N}_0$ and $r > 0$. First, let $(t,x) \in (\widehat{S}_k \setminus \widehat{E}_r) \setminus \widehat{S}$. From the definition of $\widehat{S}_k$, we have $\Phi(-k;(t,x)) \notin \widehat{S}_k$. Therefore τ that satisfies $\tau \le t$ and $\varphi(\tau;t,x) \in \widehat{S}_k \setminus \widehat{E}_r$ belongs to interval $[\Phi^s(-k;(t,x)), t]$, whose length is less than h_k from inequality (4.13). Next, suppose that $(t,x) \in \widehat{S} \setminus \widehat{E}_r$ and define t_e and x_e as in (4.8). Then the interval of τ for which $\varphi(\tau;t,x) \in \widehat{S}_k \setminus \widehat{E}_r$ is included in the union of $[\Phi^s(-k;(t_e,x_e)), t_e]$ and $[t_e, t]$. The length of each interval is bounded by h_k and $\zeta(r)$, respectively, where the latter is obtained as in the previous subsection.

From (4.14), it holds for $k \in \mathbb{N}_0$ that $\|f_x(\tau, \varphi(\tau;t,x))\|_s \le \alpha_d(\bar{c}_k) =: L_k$ as far as $\varphi(\tau;t,x) \in \widehat{S}_k$. Therefore $|\varphi_x(\tau;t,x)| \le e^{L_k(t-\tau)}$ if $\tau \le t$ and $\varphi(\tau;t,x) \in \widehat{S}_k$. From these observations, we obtain

$$\begin{aligned} \rho_k(t,x) &\le \int_{T_{-\infty}(t,x)}^{t} \lambda_k(\tau, \varphi(\tau;t,x)) \left| \frac{\partial \varphi(\tau;t,x)}{\partial x} \right| d\tau \\ &\le \int_{t-(h_k+\zeta(r))}^{t} \overline{\alpha} e^{-\tau} e^{L_k(t-\tau)} d\tau = p_k e^{-t}, \quad k \in \mathbb{N}_0, \end{aligned} \tag{4.17}$$

where $p_k = \overline{\alpha}(e^{(1+L_k)(h_k+\zeta(r))} - 1)/(1+L_k)$.

Recall (ii) of Assumption 4.2, by which $1 + \|f(t,x)\| \le 1 + \alpha_f(\bar{c}_k) =: H_k$ for $(t,x) \in \widehat{S}_k \setminus \widehat{E}_r$. Using (4.17), we derive an upper bound of the integral of $\rho_k(t,x)(1 + \|f(t,x)\|)/(1+\|x\|)$ on $\widehat{S}_k \setminus \widehat{E}_r$ as

$$\begin{aligned} I(\widehat{S}_k, r) &= \int_{\widehat{S}_k \setminus \widehat{E}_r} \frac{1+\|f(t,x)\|}{1+\|x\|} \rho_k(t,x) dx dt \le \int_{\widehat{S}_k \setminus \widehat{E}_r} H_k p_k e^{-t} dx dt \\ &\le V(\bar{c}_k) P_k H_k e^{\underline{t}_k} =: q_k, \quad k \in \mathbb{N}_0, \end{aligned}$$

which completes the proof. ∎

4.4. Lyapunov density on the region of attraction. Denote by $\widehat{R}_A$ the set of initial data (t,x) for which $\varphi(\tau;t,x)$ is defined for all $\tau \ge t$ and $\varphi(\tau;t,x) \to 0$ as $\tau \to \infty$, which set we call the region of attraction of system (2.1).

LEMMA 4.6. *Under Assumption* 4.2*, there exists a* $\rho \in \mathbb{R}^{n+1} \setminus \widehat{E}$ *that satisfies the conditions* (i)–(iv) *of Theorem* 2.2 *for* $\widehat{S} = \widehat{R}_A$.

Proof. We have seen in (4.4) that $\widehat{X}(\underline{t}_0, \underline{c}_0) \subset \widehat{S}$ holds for some $\underline{c}_0 > 0$. Therefore, if $(t,x) \in \widehat{R}_A$, there exists a time $t_1 \ge \underline{t}_0$ for which $\varphi(t_1;t,x) \in \widehat{S}$. This means that $(t,x) \in \widehat{R}_A$ belongs to $\widehat{S}_k$ for some integer k. Thus $\{\widehat{S}_k\}$, which is monotonically increasing, satisfies $\widehat{R}_A = \bigcup_{k=1}^\infty \widehat{S}_k$. Moreover, the sequence of compact sets

$$\widehat{J}_k = \left\{ (t,x) \in \mathbb{R} \times \mathbb{R}^n : \quad \underline{t}_0 - k \le t \le \underline{t}_0 + k, \quad \frac{1}{k} \le \|x\| \le k \right\}, \quad k \in \mathbb{N},$$

is monotonically increasing and satisfies $\bigcup_{k=1}^\infty \widehat{J}_k = \mathbb{R}^{n+1} \setminus \widehat{E}$. Therefore

$$\widehat{R}_A \setminus \widehat{E} = \bigcup_{k=1}^{\infty} (\widehat{S}_k \cap \widehat{J}_k).$$

Next, define for $k \in \mathbb{N}$ the following numbers:

$$d_k^1 = \max_{(t,x)\in\widehat{J}_k} \rho_k(t,x), \quad d_k^2 = \max_{(t,x)\in\widehat{J}_k} \left|\frac{\partial \rho_k(t,x)}{\partial t}\right|, \quad d_k^3 = \max_{(t,x)\in\widehat{J}_k} \left\|\frac{\partial \rho_k(t,x)}{\partial x}\right\|,$$

where the functions that appear in the right-hand sides are continuous on compact sets $\widehat{J}_k$. Let positive numbers C_k be

$$C_k = 2^{-k}[\max\{d_k^1,\, d_k^2,\, d_k^3,\, d_k^2 + d_k^3\alpha_f(k) + d_k^1\alpha_d(k),\, q_k\}]^{-1}, \quad k \in \mathbb{N}, \tag{4.18}$$

and define

$$\rho(t,x) = \sum_{k=1}^{\infty} C_k \rho_k(t,x). \tag{4.19}$$

Below we verify the convergence of this series. Let $\widehat{K} \subset \mathbb{R}^{n+1} \setminus \widehat{E}$ be an arbitrary compact set. Then, there exists an integer m for which $\widehat{K} \subset \widehat{J}_k$ holds for all $k \geq m+1$. Hence, if $p > m$,

$$\begin{aligned}\sum_{k=1}^{p} C_k\rho_k(t,x) &= \sum_{k=1}^{m} C_k\rho_k(t,x) + \sum_{k=m+1}^{p} C_k\rho_k(t,x)\\ &\leq \sum_{k=1}^{m} C_k \max_{(t,x)\in\widehat{K}} \rho_k(t,x) + 2^{-m}\end{aligned}$$

for $(t,x) \in \widehat{K}$. Similarly,

$$\sum_{k=1}^{p} C_k \left|\frac{\partial\rho_k(t,x)}{\partial t}\right| \leq \sum_{k=1}^{m} C_k \max_{(t,x)\in\widehat{K}} \left|\frac{\partial\rho_k(t,x)}{\partial t}\right| + 2^{-m},$$

$$\sum_{k=1}^{p} C_k \left\|\frac{\partial\rho_k(t,x)}{\partial x}\right\| \leq \sum_{k=1}^{m} C_k \max_{(t,x)\in\widehat{K}} \left\|\frac{\partial\rho_k(t,x)}{\partial x}\right\| + 2^{-m}.$$

The right-hand sides of these inequalities are constants that depend only on set $\widehat{K}$. Therefore series $\rho(t,x)$ converges uniformly on $\widehat{K}$. Moreover, $\frac{\partial\rho_k(t,x)}{\partial t}$ and $\frac{\partial\rho_k(t,x)}{\partial x}$ are also series that uniformly converge on $\widehat{K}$ and hence ρ is continuously differentiable on $\widehat{K}$. Consider the following series:

$$\sum_{k=1}^{\infty} C_k \left(\frac{\partial\rho_k(t,x)}{\partial t} + [\nabla_x \cdot (f\rho_k)](t,x)\right) = \sum_{k=1}^{\infty} C_k\lambda_k(t,x).$$

Since, for $k \geq m+1$, the left-hand side is bounded as

$$\begin{aligned}\left|\frac{\partial\rho_k(t,x)}{\partial t} + [\nabla_x \cdot (f\rho_k)](t,x)\right| &\leq \left|\frac{\partial\rho_k(t,x)}{\partial t}\right| + \left\|\frac{\partial\rho_k(t,x)}{\partial x}\right\| \|f(t,x)\| + \rho_k(t,x)\|f_x(t,x)\|_s\\ &\leq d_k^2 + d_k^3\alpha_f(k) + d_k^1\alpha_d(k)\end{aligned}$$

on $\widehat{K}$, we see that $\lambda(t,x) = \sum_{k=1}^{\infty} C_k\lambda_k(t,x)$ uniformly converges on $\widehat{K}$ and hence λ is continuous. The above convergence results are valid for any compact sets $\widehat{K} \subset \mathbb{R}^{n+1} \setminus \widehat{E}$ and therefore

$$\frac{\partial\rho(t,x)}{\partial t} + [\nabla_x \cdot (f\rho)](t,x) = \lambda(t,x) \quad \forall (t,x) \in \mathbb{R}^{n+1} \setminus \widehat{E}.$$

Let $(t,x) \in \widehat{R}_A \backslash \widehat{E}$. Then there exists an integer m for which $(t,x) \in \widehat{S}_k$, $k \geq m+1$. Since $\rho_k(t,x)$ and $\lambda_k(t,x)$ are strictly positive on $\widehat{S}_k \setminus \widehat{E}$, it holds that $\rho(t,x) > 0$ and $\lambda(t,x) > 0$. Obviously $\rho(t,x) = \lambda(t,x) = 0$ for $(t,x) \notin \widehat{R}_A$. Thus conditions (i) and (iii) are verified, while it is easy to see (ii). Condition (iv), the integrability on $R_A \setminus \widehat{E}_r$, is obtained from (4.18) as

$$\int_{\widehat{R}_A \backslash \widehat{E}_r} \frac{1 + \|f(t,x)\|}{1 + \|x\|}\, \rho(t,x)dxdt = \sum_{k=1}^{\infty} C_k \int_{\widehat{S}_k \backslash \widehat{E}_r} \frac{1 + \|f(t,x)\|}{1 + \|x\|}\, \rho_k(t,x)dxdt$$
$$\leq \sum_{k=1}^{\infty} C_k q_k < \infty,$$

which completes the proof. ∎

4.5. Lyapunov density for general positively invariant family. Lemma 4.6 is easily generalized to guarantee the existence of Lyapunov densities for a general positively invariant family satisfying Assumption 2.1. Let $\{U(t)\}_{t \geq \underline{t}_0}$ be a positively invariant family that satisfies Assumption 2.1, given as $U(t) = \{x \in \mathbb{R}^n : u(t,x) > 0\}$ for all $t \in [\underline{t}_0, \infty)$, where u is a continuous function on $[\underline{t}_0, \infty) \times \mathbb{R}^n$ for some $\underline{t}_0 \in \mathbb{R}$ with $\overline{u} := \sup_{(t,x) \in \mathbb{R}^{n+1}} u(t,x) < \infty$. Let $\widehat{U} = \{(t,x) \in \mathbb{R}^{n+1} : t \geq \mathbb{R},\ x \in U(t)\} \subset \widehat{R}_A$. The results of subsections 4.2, 4.3, and 4.4 are generalized as follows.

THEOREM 4.7. *Consider time-varying system* (2.1). *Suppose that Assumption* 4.2 *is satisfied and a positively invariant family* $\{U(t)\}_{t \in \mathbb{R}}$ *is given as above. Then there exists a* $\rho \in \mathbb{R}^{n+1} \setminus \widehat{E}$ *that satisfies the conditions* (i)–(iv) *of Theorem* 2.2 *for* $\widehat{S} = \widehat{U}$.

Proof. The proof follows from Lemmas 4.4, 4.5, and 4.6 with modifying λ_k as

$$\lambda_k(t,x) = \begin{cases} \lambda(\Phi(k;(t,x)))u(t,x)^2/\overline{u}^2, & (t,x) \in \widehat{U}, \\ 0 & \text{otherwise} \end{cases} \tag{4.20}$$

in (4.15). ∎

4.6. Converse result for periodically time-varying systems. Here we consider family $\{S(t)\}_{t \in \mathbb{R}}$, which is periodic and defined for all $t \in \mathbb{R}$, and let $\widehat{S}$ be as in (3.2). In the proof of Theorem 4.16 of [6], which we refer to as Lemma 4.3 in this paper, a Lyapunov function is defined as

$$v(t,x) = \int_t^{\infty} g(\|\varphi(\tau;t,x)\|)d\tau,$$

where $g \in C^1([0,\infty), \mathbb{R})$ is a function constructed via Massera's lemma. Since $\varphi(\tau; t+T, x) = \varphi(\tau - T; t, x)$ holds for periodically time-varying systems, it holds that $v(t+T, x) = v(t,x)$. Hence v is extended to a function defined for all $t \in \mathbb{R}$ periodic in t with period T. Then define $\lambda(t,x)$ as

$$\lambda(t,x) = \begin{cases} (\alpha_1(\overline{c}_0) - v(t,x))^2, & (t,x) \in \widehat{S}, \\ 0, & (t,x) \in \widehat{S}^{\mathrm{c}}, \end{cases} \tag{4.21}$$

which satisfies $0 \leq \lambda(t,x) \leq \overline{\alpha} = \alpha_1(\overline{c}_0)^2$. Next, if $(t,x) \in \widehat{S} \setminus \widehat{E}$, there exists a $\tau < t$ such that $\varphi(\tau; t, x) \notin S(\tau)^{\mathrm{cl}}$. To see this, assume that $(t,x) \in \widehat{S} \setminus \widehat{E}$ and

$\varphi(\tau;t,x)\in S(\tau)^{\mathrm{cl}}$ for all $\tau<t$. Then $\|\varphi(\tau;t,x)\|\le\bar{c}_0$ and hence

$$\|x\|\le\beta(\|\varphi(\tau;t,x)\|,t-\tau)\le\beta(\bar{c}_0,t-\tau)\to 0\quad(\tau\to-\infty)$$

holds from the definition of class $\mathcal{KL}$ functions. Thus $\|x\|=0$, which contradicts $(t,x)\in\widehat{S}\setminus\widehat{E}$. Therefore we can define $t_e\in(T_{-\infty}(t,x),t)$ and x_e as in (4.8).

Define λ_k as in (4.15) with λ in (4.21). Then $\lambda_k\in C^1(\mathbb{R}^{n+1},\mathbb{R})$ and $\lambda_k(t,x)\le\overline{\alpha}$. Setting ρ_k by (4.16), we can easily derive conditions (i) and (ii) of Theorem 3.2 and the periodicity of ρ_k. Instead of (4.17), we have

$$\begin{aligned}\rho_k(t,x)&\le\int_{T_{-\infty}(t,x)}^{t}\lambda_k(\tau,\varphi(\tau;t,x))\left|\frac{\partial\varphi(\tau;t,x)}{\partial x}\right|d\tau\\&\le\int_{t-(h_k+\zeta(r))}^{t}\overline{\alpha}e^{L_k(t-\tau)}d\tau=\tilde{p}_k\end{aligned}$$

with $\tilde{p}_k=\overline{\alpha}/L_k(e^{L_k(h_k+\zeta(r))}-1)$. Note that an upper bound of the length of the interval of the above integration is obtained as $h_k+\zeta(r)$ similarly to that of (4.17). Thus ρ_k is defined on $\mathbb{R}^{n+1}\setminus\widehat{E}$. We can also prove that $\rho_k\in C^1(\mathbb{R}^{n+1}\setminus\widehat{E},\mathbb{R})$ and condition (iii) holds. Since $\rho_k(t,x)$ is bounded on $\{x\in\mathbb{R}^n:\|x\|\ge r\}$ for each t and $r>0$, integrability condition (iv) is obvious. Let

$$\tilde{q}_k:=\sup_{t\in[0,T)}\int_{x\in S(t),\|x\|\ge r}\lambda(\tau,\varphi(\tau;t,x))\left|\frac{\partial\varphi(\tau;t,x)}{\partial x}\right|d\tau\ (<\infty).$$

Lyapunov densities for the region of attraction is derived via (4.19) with $\rho_k(t,x)$ obtained above, where coefficients C_k are defined as in (4.18) with replacing q_k with $\tilde{q}_k$. We can also construct Lyapunov densities for a positive invariant family $\{U(t)\}_{t\in\mathbb{R}}$ that is periodic in t with period T, setting λ_k as in (4.20). The proof is similar to those of previous subsections. We summarize the results in the following theorem.

THEOREM 4.8. *Consider time-varying system* (2.1) *that is periodic as in* (3.1) *with period T. Suppose that Assumption* 4.2 *is satisfied and a positively invariant family $\{U(t)\}_{t\in\mathbb{R}}$ is given as $U(t)=\{x\in\mathbb{R}^n:u(t,x)>0\}$ for all $t\in[\underline{t}_0,\infty)$, where u is a continuous function on $[\underline{t}_0,\infty)\times\mathbb{R}^n$ for some $\underline{t}_0\in\mathbb{R}$ with $\overline{u}:=\sup_{(t,x)\in\mathbb{R}^{n+1}}u(t,x)<\infty$ and $u(t)=u(t+T)$ for all $t\in\mathbb{R}$. Then there exists a $\rho\in\mathbb{R}^{n+1}\setminus\widehat{E}$ that satisfies the conditions* (i)–(iv) *of Theorem* 3.2 *for $\widehat{S}=\widehat{U}$.*

5. Conclusion. In this paper, first we showed a convergence criterion in Theorem 2.2 on trajectories of general time-varying nonlinear systems in terms of Lyapunov densities. We do not need the assumptions of local stability and forward completeness of trajectories. Then we showed a criterion in Theorem 3.2 for periodically time-varying systems, where the integrability condition (iv) is weakened by exploiting the periodicity of the vector field. We also proved the existence of Lyapunov densities for general and periodically time-varying systems in Theorems 4.7 and 4.8, respectively, under the asymptotic stability.

Future work can include applications of Lyapunov densities for time-varying and periodically systems to state feedback synthesis. The inequalities proposed in this paper can be applied to state feedback of control system $\dot{x}=f(t,x)+g(t,x)u$ with input $u=u(t,x)$ similarly to the time-invariant case [13] through the linearization technique. An important open issue is the proof of the existence of Lyapunov densities without assuming even asymptotic stability.

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