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Chekrouna, Abdennasser

Kuniya, Toshikazu

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# Global threshold dynamics of an infection age-structured SIR epidemic model with diffusion under the Dirichlet boundary condition

Abdennasser Chekroun [a], Toshikazu Kuniya [b,*]

[a] Laboratoire d'Analyse Nonlinéaire et Mathématiques Appliquées, University of Tlemcen, Tlemcen 13000, Algeria
[b] Graduate School of System Informatics, Kobe University, 1-1 Rokkodai-cho, Nada-ku, Kobe 657-8501, Japan

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## Abstract

In this paper, we are concerned with the global asymptotic behavior of an infection age-structured SIR epidemic model with diffusion in a general $n$-dimensional bounded spatial domain under the homogeneous Dirichlet boundary condition. By using the method of characteristics, we reformulate the model into a system of a reaction-diffusion equation and a Volterra integral equation. We define the basic reproduction number $\mathcal{R}_0$ by the spectral radius of a compact positive linear operator and show that if $\mathcal{R}_0 < 1$, then the disease-free steady state is globally attractive, whereas if $\mathcal{R}_0 > 1$, then a positive endemic steady state exists and the system is uniformly persistent. By numerical simulation for the 2-dimensional case, we show that $\mathcal{R}_0$ depends on the shape of the spatial domain. This result is in contrast with the case of the homogeneous Neumann boundary condition, in which $\mathcal{R}_0$ is independent of the spatial domain.


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*Keywords:* SIR epidemic model; Infection age; Diffusion; Dirichlet boundary condition; Basic reproduction number

* Corresponding author.
*E-mail addresses:* chekroun@math.univ-lyon1.fr (A. Chekroun), tkuniya@port.kobe-u.ac.jp (T. Kuniya).

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## 1. Introduction

The SIR epidemic model is one of the most basic mathematical models for infectious disease dynamics, in which the total population is divided into three subpopulations called susceptible, infective and removed. In the first study [28] of an SIR epidemic model in 1927, Kermack and McKendrick considered the effect of the time elapsed since the infection, which is nowadays called the infection age [23,24]. Typical epidemic models with infection age are nonlinear systems of partial differential equations, and mathematical analysis for them is usually more difficult than that for ordinary differential equations models without infection age. More precisely, it is a classical fact that the global asymptotic behavior of an SIR epidemic model without infection age is completely determined by a threshold value called the basic reproduction number $\mathcal{R}_0$ [12], that is, the trivial disease-free steady state is globally asymptotically stable if $\mathcal{R}_0 \le 1$, whereas a positive endemic steady state is so if $\mathcal{R}_0 > 1$ [24, Section 5.5.2]. However, whether $\mathcal{R}_0$ for infection age-structured models also plays the role of such a threshold value had been an open problem for decades. In 2010, Magal et al. [31] studied an infection age-structured SIR epidemic model and proved that the disease-free steady state is globally asymptotically stable if $\mathcal{R}_0 < 1$, whereas the unique positive endemic steady state is so if $\mathcal{R}_0 > 1$ by using an integrated semigroup approach and constructing a suitable Lyapunov functional. After their work, the global asymptotic behavior of various kinds of infection age-structured epidemic models has been broadly studied by many authors (see, e.g., [3,21,26,32,33,44,46,48,52,53]).

Epidemic models with spatial diffusion have also been studied by many authors (see, e.g., [1,6,16,19,20,30,35,37,47,50]). In fact, such models have been applied to consider real infectious diseases such as rabies [27], Dengue [45], Lyme disease [8] and so on (see [39,40] for the detailed reviews on the related studies). However, relatively few authors have focused on epidemic models with both of the infection age and spatial diffusion (see [17,49,51] for models without birth and death processes and [13–15,54] for models in spatially unbounded domains). In this paper, we are concerned with the mathematical analysis of an infection age-structured SIR epidemic model with spatial diffusion in a spatially bounded domain, which can be regarded as a generalization of the model studied in [11] for 1-dimensional spatial domain to $n$-dimensional spatial domain. To our knowledge, most of the previous studies on spatially diffusive epidemic models in spatially bounded domains assumed the homogeneous Neumann (zero-flux) boundary condition. In contrast, in this paper, we assume the homogeneous Dirichlet boundary condition. Due to this assumption, the trivial disease-free steady state for our model is nonconstant, and the basic reproduction number $\mathcal{R}_0$ for our model can not be explicitly given by a positive constant even if all parameters are spatially homogeneous. This is in contrast to the case of homogeneous Neumann boundary condition, in which the disease-free steady state is constant and $\mathcal{R}_0$ is explicitly given by a positive constant if all parameters are spatially homogeneous (see, e.g., [10]). Thus, $\mathcal{R}_0$ under the homogeneous Neumann boundary condition with spatially homogeneous parameters is not affected by the shape of the spatial domain. This seems unrealistic because the infection is thought to be more likely to spread in a region with a suitable shape for individuals to contact each other. In this paper, we obtain $\mathcal{R}_0$ that depends on the spatial domain, and show that it plays the role of a threshold value for the global asymptotic behavior of our model: if $\mathcal{R}_0 < 1$, then the disease-free steady state is globally attractive, whereas if $\mathcal{R}_0 > 1$, then the system is uniformly persistent and a positive endemic steady state exists.

The organization of this paper is as follows. In Section 2, we first formulate our model as a system of parabolic differential equations, and reformulate it into a system of a reaction-diffusion equation and a Volterra integral equation by using the method of characteristics. In Section 3, we prove the existence and uniqueness of the positive bounded solution of the system. In Section 4, we define the basic reproduction number $\mathcal{R}_0$ by the spectral radius of a compact positive linear operator called the next generation operator. In Section 5, we prove the global attractivity of the disease-free steady state for $\mathcal{R}_0 < 1$. In Section 6, we prove the uniform persistence of the disease for $\mathcal{R}_0 > 1$. In Section 7, we prove the existence of a positive endemic steady state for $\mathcal{R}_0 > 1$. In Section 8, by numerical simulation, we confirm that our theoretical results are valid and $\mathcal{R}_0$ is affected by the shape of the spatial domain for the 2-dimensional case.

## 2. The model

Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}^* = \{1, 2, \ldots\}$ be a bounded, open and connected set (domain) with smooth boundary $\partial\Omega$. Let $S(t,x)$ and $R(t,x)$ denote the densities of susceptible and removed individuals in position $x \in \overline{\Omega}$ at time $t \geq 0$, respectively. Let $I(t,a,x)$ denote the density of infective individuals of infection age $a \geq 0$ in position $x \in \overline{\Omega}$ at time $t \geq 0$. We first construct the following SIR epidemic model, for $t > 0$, $a > 0$ and $x \in \Omega$,

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = d_1 \Delta S(t,x) + b - S(t,x) \displaystyle\int_0^{+\infty} \beta(a) I(t,a,x) \mathrm{d}a - \mu S(t,x), \\ \dfrac{\partial I(t,a,x)}{\partial t} + \dfrac{\partial I(t,a,x)}{\partial a} = d_2 \Delta I(t,a,x) - \left[\mu + \gamma(a)\right] I(t,a,x), \\ I(t,0,x) = S(t,x) \displaystyle\int_0^{+\infty} \beta(a) I(t,a,x) \mathrm{d}a, \\ \dfrac{\partial R(t,x)}{\partial t} = d_3 \Delta R(t,x) + \displaystyle\int_0^{+\infty} \gamma(a) I(t,a,x) \mathrm{d}a - \mu R(t,x), \end{cases} \tag{2.1}$$

with initial condition, for $a \geq 0$ and $x \in \overline{\Omega}$,

$$S(0,x) = \phi_1(x), \quad I(0,a,x) = \phi_2(a,x), \quad R(0,x) = \phi_3(x),$$

and the homogeneous Dirichlet boundary condition, for $t > 0$, $a \geq 0$ and $x \in \partial\Omega$,

$$S(t,x) = I(t,a,x) = R(t,x) = 0.$$

Here $d_1$, $d_2$ and $d_3$ denote the diffusion coefficients for susceptible, infective and removed individuals, respectively. $b$ denotes the birth rate, $\beta(a)$ denotes the infection age-dependent transmission rate, $\mu$ denotes the mortality rate and $\gamma(a)$ denotes the infection age-dependent removal rate. Throughout this paper, we make the following assumptions.

**(A1)** $b > 0$, $\mu > 0$ and $d_i > 0$, $i = 1, 2, 3$.
**(A2)** $\beta(\cdot) \in L^\infty_+(\mathbb{R}_+) \cap L^1_+(\mathbb{R}_+)$ and there exist $0 < a_1 < a_2 < +\infty$ such that $\beta(a) > 0$ for all $a \in (a_1, a_2)$.
**(A3)** $\gamma(\cdot) \in L^\infty_+(\mathbb{R}_+)$.

Let $\beta^\infty$ and $\gamma^\infty$ denote the essential supremum of $\beta(\cdot)$ and $\gamma(\cdot)$, respectively. That is,

$$\beta^\infty := \operatorname*{ess.sup}_{a\in\mathbb{R}_+} \beta(a) < +\infty \quad \text{and} \quad \gamma^\infty := \operatorname*{ess.sup}_{a\in\mathbb{R}_+} \gamma(a) < +\infty.$$

Let $\mathbb{X} := C(\overline{\Omega}, \mathbb{R})$, equipped with norm

$$\|\varphi\|_{\mathbb{X}} := \sup_{x\in\Omega} |\varphi(x)|, \qquad \varphi \in \mathbb{X}.$$

Let $w_i(t,x)$, $i = 1,2,3$ be the solution of the following initial-boundary value problem, for $i = 1,2,3$,

$$\begin{cases} \dfrac{\partial w_i(t,x)}{\partial t} = d_i \Delta w_i(t,x), & t > 0, \quad x \in \Omega, \\ w_i(0,x) = w_{i,0}(x), & x \in \overline{\Omega}, \\ w_i(t,x) = 0, & t > 0, \quad x \in \partial\Omega. \end{cases} \tag{2.2}$$

By [25, Theorem 7.1], if $w_{i,0} \in \mathbb{X}$, $i = 1,2,3$, then there exist fundamental solutions $\Gamma_i(t,x,y)$ to (2.2) such that, for $i = 1,2,3$,

$$w_i(t,x) = \int_\Omega \Gamma_i(t,x,y) w_{i,0}(y) \mathrm{d}y, \qquad t > 0, \quad x \in \overline{\Omega},$$

solves (2.2) in the sense that $\lim_{t\to+0} w_i(t,x) = w_{i,0}(x)$, $x \in \overline{\Omega}$. From the arguments in [25, Sections 8 and 10], we see that $\Gamma_i(t,x,y)$ is unique and satisfies the following properties, for $i = 1,2,3$,

**(G1)** $\Gamma_i(t,x,y) \ge 0$ for all $t > 0$ and $x, y \in \overline{\Omega}$. Moreover, $\Gamma_i(t,x,y) > 0$ for all $t > 0$ and $x, y \in \Omega$.
**(G2)** $\int_\Omega \Gamma_i(t,x,y)\mathrm{d}y \le 1$ for all $t > 0$ and $x \in \overline{\Omega}$.
**(G3)** $\int_\Omega \Gamma_i(t,x,z)\Gamma_i(s,z,y)\mathrm{d}z = \Gamma_i(t+s,x,y)$ for all $t > 0$ and $x, y \in \overline{\Omega}$.
**(G4)** $\Gamma_i(\cdot,x,\cdot) = 0$ for all $x \in \partial\Omega$.
**(G5)** $\Gamma_i(\cdot,x,y) = \Gamma_i(\cdot,y,x)$ for all $x, y \in \overline{\Omega}$.

For instance, if $\Omega$ is an $n$-dimensional rectangular domain such as

$$\Omega = \left\{ x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : 0 < x_i < \ell_i < +\infty, \ i = 1,2,\dots,n \right\},$$

then $\Gamma_i(t,x,y)$ is given as follows [25, Section 16], for $i = 1,2,3$, $t > 0$ and $x, y \in \overline{\Omega}$,

$$\Gamma_i(t,x,y) = \frac{2^n}{\prod_{j=1}^n \ell_j} \sum_{k_1,k_2,\dots,k_n=1}^{+\infty} \prod_{j,m=1}^{n} \sin\frac{k_j \pi x_j}{\ell_j} \sin\frac{k_m \pi y_m}{\ell_m} \mathrm{e}^{-d_i\left[\sum_{j=1}^n \left(\frac{k_j}{\ell_j}\right)^2\right]\pi^2 t}. \tag{2.3}$$

We now reformulate model (2.1) into a system of a reaction-diffusion equation and a Volterra integral equation by using the method of characteristics. Let $a - t = c \in \mathbb{R}$, $\hat{I}(t, x) := I(t, t + c, x)$ and $\hat{\gamma}(t) := \gamma(t + c)$. The second equation in (2.1) can then be rewritten as follows, for $t > t_0 := \max(0, -c)$ and $x \in \Omega$,

$$\frac{\partial \hat{I}(t,x)}{\partial t} = d_2 \Delta \hat{I}(t,x) - \left[\mu + \hat{\gamma}(t)\right] \hat{I}(t,x).$$

Using the fundamental solution $\Gamma_2$, we obtain, for $t > t_0$ and $x \in \Omega$,

$$\hat{I}(t,x) = \mathrm{e}^{-\int_{t_0}^{t} [\mu + \hat{\gamma}(\sigma)] \mathrm{d}\sigma} \int_{\Omega} \Gamma_2(t - t_0, x, y) \hat{I}(t_0, y) \mathrm{d}y.$$

Hence, for each of the cases of $t - a > 0$ and $a - t \geq 0$, we have the following expression,

$$I(t,a,x) = \begin{cases} \mathrm{e}^{-\int_0^a [\mu + \gamma(\sigma)] \mathrm{d}\sigma} \int_{\Omega} \Gamma_2(a, x, y) I(t - a, 0, y) \mathrm{d}y, & t - a > 0, \\ \mathrm{e}^{-\int_0^t [\mu + \gamma(\sigma + a - t)] \mathrm{d}\sigma} \int_{\Omega} \Gamma_2(t, x, y) \phi_2(a - t, y) \mathrm{d}y, & a - t \geq 0. \end{cases} \tag{2.4}$$

Let $u(t, x) := I(t, 0, x)$ denote the density of newly infected individuals in position $x \in \overline{\Omega}$ at time $t \geq 0$. Substituting (2.4) into the third equation in (2.1), we have, for $t > 0$ and $x \in \Omega$,

$$\begin{aligned} u(t,x) = S(t,x) \int_0^t \beta(a) \mathrm{e}^{-\int_0^a [\mu + \gamma(\sigma)] \mathrm{d}\sigma} \int_{\Omega} \Gamma_2(a, x, y) u(t - a, y) \mathrm{d}y \mathrm{d}a \\ + S(t,x) F_{\phi_2}(t,x), \end{aligned} \tag{2.5}$$

where, for $t > 0$ and $x \in \Omega$,

$$F_{\phi_2}(t,x) := \int_t^{+\infty} \beta(a) \mathrm{e}^{-\int_0^t [\mu + \gamma(\sigma + a - t)] \mathrm{d}\sigma} \int_{\Omega} \Gamma_2(t, x, y) \phi_2(a - t, y) \mathrm{d}y \mathrm{d}a.$$

Combining the first equation in (2.1) and the above equation, we obtain the following system of a reaction-diffusion equation and a Volterra integral equation, for $t > 0$ and $x \in \Omega$,

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = d_1 \Delta S(t,x) + b - u(t,x) - \mu S(t,x), \\ u(t,x) = S(t,x) \displaystyle\int_0^t \beta(a) \mathrm{e}^{-\int_0^a [\mu + \gamma(\sigma)] \mathrm{d}\sigma} \int_{\Omega} \Gamma_2(a, x, y) u(t - a, y) \mathrm{d}y \mathrm{d}a \\ \qquad + S(t,x) F_{\phi_2}(t,x), \end{cases} \tag{2.6}$$

with initial condition, for $x \in \overline{\Omega}$,

$$S(0,x) = \phi_1(x), \quad u(0,x) = \phi_1(x) \int_0^{+\infty} \beta(a)\phi_2(a,x)\mathrm{d}a, \tag{2.7}$$

and the homogeneous Dirichlet boundary condition, for $t > 0$ and $x \in \partial\Omega$,

$$S(t,x) = u(t,x) = 0. \tag{2.8}$$

In the remainder of this paper, we focus on the mathematical analysis of system (2.6) with conditions (2.7) and (2.8).

## 3. Existence and uniqueness of the solution

Let $\mathbb{Y} := BC(\mathbb{R}_+, \mathbb{X})$ be the space of all bounded and continuous functions that take values in $\mathbb{X}$ and let $\|\psi\|_{\mathbb{Y}} := \sup_{a\geq 0} \|\psi(a)\|_{\mathbb{X}} = \sup_{(a,x)\in\mathbb{R}_+\times\Omega} |\psi(a,x)|$ for $\psi \in \mathbb{Y}$. Let

$$\mathbb{X}^0 := \{\varphi \in \mathbb{X} : \varphi(x) = 0 \text{ for all } x \in \partial\Omega\},$$
$$\mathbb{Y}^0 := \{\psi \in \mathbb{Y} : \psi(\cdot,x) = 0 \text{ for all } x \in \partial\Omega\}.$$

Let $\mathbb{X}^0_+$ and $\mathbb{Y}^0_+$ be the positive cones of $\mathbb{X}^0$ and $\mathbb{Y}^0$, respectively. Let us define the following linear operator on $\mathbb{Y}^0$, for $x \in \overline{\Omega}$,

$$(\Phi\psi)(t,x) := \int_0^t \beta(a)\mathrm{e}^{-\int_0^a[\mu+\gamma(\sigma)]\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y)\psi(t-a,y)\mathrm{d}y\mathrm{d}a, \quad \psi \in \mathbb{Y}^0.$$

By (A1)-(A3) and (G1)-(G2), we can easily check that $\Phi(\mathbb{Y}^0_+) \subset \mathbb{Y}^0_+$. We now prove the following proposition on the positivity of solution of (2.6).

**Proposition 3.1.** *Suppose that (A1)-(A3) hold and let $(S,u)$ be a solution of (2.6) corresponding to $(\phi_1,\phi_2) \in \mathbb{X}^0_+ \times \mathbb{Y}^0_+$ with the interval of existence $[0,T]$, where $T > 0$. Then, $S(t,x) > 0$ and $u(t,x) \geq 0$ for all $t \in (0,T]$ and $x \in \Omega$.*

**Proof.** Using operator $\Phi$, the first equation in (2.6) can be rewritten as follows, for $t \in (0,T]$ and $x \in \Omega$,

$$\frac{\partial S(t,x)}{\partial t} = d_1\Delta S(t,x) + b - \left[\mu + (\Phi u)(t,x) + F_{\phi_2}(t,x)\right] S(t,x). \tag{3.1}$$

By (A1), we have, for all $t \in (0,T]$ and $x \in \Omega$,

$$\frac{\partial S(t,x)}{\partial t} > d_1\Delta S(t,x) - \left[\mu + (\Phi u)(t,x) + F_{\phi_2}(t,x)\right] S(t,x).$$

Since $\mu + (\Phi u)(t,x) + F_{\phi_2}(t,x)$ is continuous and bounded with respect to $t$ and $x$, it follows from the strong maximum principle for parabolic equations [38] that $S(t,x) > 0$ for all $t \in (0,T]$ and $x \in \Omega$ or $S \equiv 0$. Since $S \equiv 0$ does not satisfy (3.1), we have that $S(t.x) > 0$ for all $t \in (0,T]$ and $x \in \Omega$.

To show that $u(t,x) \geq 0$ for all $t \in (0,T]$ and $x \in \Omega$, we suppose on the contrary that there exist $t_1 \in (0,T)$ and $x_1 \in \Omega$ such that $u(t,x) \geq 0$ for all $t \in (0,t_1)$ and $x \in \Omega$, and $u(t_1+\epsilon, x_1) < 0$ for $0 < \epsilon \ll 1$. By (A2) and (G1), we have for sufficiently small $\epsilon$ that

$$\begin{aligned} &u(t_1+\epsilon, x_1) \\ &= S(t_1+\epsilon, x_1) \int_0^{t_1+\epsilon} \beta(a) \mathrm{e}^{-\int_0^a \{\mu+\gamma(\sigma)\}\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y) u(t_1+\epsilon-a, y)\mathrm{d}y\mathrm{d}a \\ &\quad + S(t_1+\epsilon, x) F_{\phi_2}(t_1+\epsilon, x) \geq 0, \end{aligned}$$

which is a contradiction. Consequently, $u(t,x) \geq 0$ for all $t \in (0,T]$ and $x \in \Omega$. This completes the proof. □

Using the fundamental solution $\Gamma_1$, we can express $S$ in (2.6) as follows, for $t > 0$ and $x \in \overline{\Omega}$,

$$S(t,x) = \int_0^t \mathrm{e}^{-\mu a} \int_\Omega \Gamma_1(a,x,y)\,[b - u(t-a,y)]\,\mathrm{d}y\mathrm{d}a + F_{\phi_1}(t,x), \tag{3.2}$$

where, for $t > 0$ and $x \in \overline{\Omega}$,

$$F_{\phi_1}(t,x) := \mathrm{e}^{-\mu t} \int_\Omega \Gamma_1(t,x,y)\phi_1(y)\mathrm{d}y.$$

Substituting (3.2) into the second equation in (2.6), we obtain the following single equation of $u(t,x)$, for $t > 0$ and $x \in \overline{\Omega}$,

$$\begin{aligned} u(t,x) = &\left\{ \int_0^t \mathrm{e}^{-\mu a} \int_\Omega \Gamma_1(a,x,y)\,[b - u(t-a,y)]\,\mathrm{d}y\mathrm{d}a + F_{\phi_1}(t,x) \right\} \\ &\times \left\{ \int_0^t \beta(a) \mathrm{e}^{-\int_0^a [\mu+\gamma(\sigma)]\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y) u(t-a,y)\mathrm{d}y\mathrm{d}a + F_{\phi_2}(t,x) \right\}. \end{aligned} \tag{3.3}$$

If (3.3) has a solution $u$, then we can obtain $S$ from (3.2). We now prove the following proposition on the existence of the solution of system (2.6).

**Proposition 3.2.** *Suppose that (A1)-(A3) hold and let* $(\phi_1, \phi_2) \in \mathbb{X}_+^0 \times \mathbb{Y}_+^0$. *Then, system (2.6) has a positive solution* $(S,u)$ *defined on* $[0,+\infty) \times \Omega$.

**Proof.** Let $F^{\infty}_{\phi_i} := \sup_{(t,x)\in\mathbb{R}_+\times\Omega} F_{\phi_i}(t,x)$ for $i = 1, 2$. Since $(\phi_1, \phi_2) \in \mathbb{X}^0_+ \times \mathbb{Y}^0_+$, we see from (G2) that $F^{\infty}_{\phi_i} < +\infty$ for $i = 1, 2$. Let $M > F^{\infty}_{\phi_1} F^{\infty}_{\phi_2}$ be an arbitrary large positive constant. Let, for $t \geq 0$ and $x \in \overline{\Omega}$,

$$h_1(t,x) := \frac{b+M}{\mu}(1-\mathrm{e}^{-\mu t}) + F_{\phi_1}(t,x),\ \ h_2(t,x) := \frac{\beta^{\infty} M}{\mu}(1-\mathrm{e}^{-\mu t}) + F_{\phi_2}(t,x).$$

Since $h_i(0,x) = F_{\phi_i}(0,x)$ for $x \in \overline{\Omega}$ and $i = 1, 2$, from the continuity, we can choose sufficiently small $T > 0$ such that

$$\sup_{(t,x)\in(0,T)\times\Omega} [h_1(t,x)h_2(t,x)] < M \quad \text{and}$$
$$\tilde{h} := \sup_{(t,x)\in(0,T)\times\Omega} \frac{\left[\beta^{\infty} h_1(t,x) + h_2(t,x)\right]\left(1-\mathrm{e}^{-\mu t}\right)}{\mu} < 1.$$

For such $T > 0$, let $\mathbb{X}_T := C([0,T], \mathbb{X})$, equipped with norm

$$\|\varphi\|_{\mathbb{X}_T} := \sup_{t\in(0,T)} \|\varphi(t)\|_{\mathbb{X}} = \sup_{(t,x)\in(0,T)\times\Omega} |\varphi(t,x)|, \quad \varphi \in \mathbb{X}_T.$$

Let $\mathbb{X}^0_T := \{\varphi \in \mathbb{X}_T : \varphi(\cdot,x) = 0 \text{ for all } x \in \partial\Omega\}$ and let $\mathbb{X}^0_{T,M} := \{\varphi \in \mathbb{X}^0_T : \|\varphi\|_{\mathbb{X}_T} < M\}$. To show the existence of solution $u$ to (3.3), we define the following linear operator on $\mathbb{X}^0_{T,M}$, for $t \in (0,T]$ and $x \in \overline{\Omega}$,

$$\begin{aligned}(\mathcal{F}\varphi)(t,x) = &\left\{\int_0^t \mathrm{e}^{-\mu a}\int_\Omega \Gamma_1(a,x,y)\left[b-\varphi(t-a,y)\right]\mathrm{d}y\mathrm{d}a + F_{\phi_1}(t,x)\right\}\\ &\times\left\{\int_0^t \beta(a)\mathrm{e}^{-\int_0^a[\mu+\gamma(\sigma)]\mathrm{d}\sigma}\int_\Omega \Gamma_2(a,x,y)\varphi(t-a,y)\mathrm{d}y\mathrm{d}a + F_{\phi_2}(t,x)\right\},\end{aligned} \tag{3.4}$$

where $\varphi \in \mathbb{X}^0_{T,M}$.

We first show that $\mathcal{F}(\mathbb{X}^0_{T,M}) \subset \mathbb{X}^0_{T,M}$. Let $\varphi \in \mathbb{X}^0_{T,M}$. It is obvious from (G4) that $(\mathcal{F}\varphi)(\cdot,x) = 0$ for all $x \in \partial\Omega$. Moreover, we have

$$\begin{aligned}\|\mathcal{F}\varphi\|_{\mathbb{X}_T} &\leq \sup_{(t,x)\in(0,T)\times\Omega}\left\{\left[(b+M)\int_0^t \mathrm{e}^{-\mu a}\int_\Omega \Gamma_1(a,x,y)\mathrm{d}y\mathrm{d}a + F_{\phi_1}(t,x)\right]\right.\\ &\qquad\left.\times\left[\beta^{\infty} M\int_0^t \mathrm{e}^{-\mu a}\int_\Omega \Gamma_2(a,x,y)\mathrm{d}y\mathrm{d}a + F_{\phi_2}(t,x)\right]\right\}\\ &\leq \sup_{(t,x)\in(0,T)\times\Omega} [h_1(t,x)h_2(t,x)] < M,\end{aligned}$$

and thus, $\mathcal{F}\varphi \in \mathbb{X}^0_{T,M}$. This implies that $\mathcal{F}(\mathbb{X}^0_{T,M}) \subset \mathbb{X}^0_{T,M}$.

We next show that $\mathcal{F}$ is a strict contraction in $\mathbb{X}^0_{T,M}$. Let $u_1, u_2 \in \mathbb{X}^0_{T,M}$ and $\hat{u} := u_1 - u_2$. We then have, for $t \in (0, T]$ and $x \in \overline{\Omega}$,

$$(\mathcal{F}\hat{u})(t,x) = \left\{B(t,x) - U_1[u_1](t,x) + F_{\phi_1}(t,x)\right\} U_2[\hat{u}](t,x) - \left\{U_2[u_2](t,x) + F_{\phi_2}(t,x)\right\} U_1[\hat{u}](t,x), \tag{3.5}$$

where

$$B(t,x) := b\int_0^t \mathrm{e}^{-\mu a} \int_\Omega \Gamma_1(a,x,y)\mathrm{d}y\mathrm{d}a,$$

$$U_1[\varphi](t,x) := \int_0^t \mathrm{e}^{-\mu a} \int_\Omega \Gamma_1(a,x,y)\varphi(t-a,y)\mathrm{d}y\mathrm{d}a,$$

$$U_2[\varphi](t,x) := \int_0^t \beta(a)\mathrm{e}^{-\int_0^a [\mu+\gamma(\sigma)]\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y)\varphi(t-a,y)\mathrm{d}y\mathrm{d}a.$$

Thus, we have

$$\begin{aligned}\left\|\mathcal{F}\hat{u}\right\|_{\mathbb{X}_T} &\le \sup_{(t,x)\in(0,T)\times\Omega} \left\{ h_1(t,x)\beta^\infty \left\|\hat{u}\right\|_{\mathbb{X}_T} \int_0^t \mathrm{e}^{-\mu a} \int_\Omega \Gamma_2(a,x,y)\mathrm{d}y\mathrm{d}a \right. \\ &\qquad \left. + h_2(t,x)\left\|\hat{u}\right\|_{\mathbb{X}_T} \int_0^t \mathrm{e}^{-\mu a}\int_\Omega \Gamma_1(a,x,y)\mathrm{d}y\mathrm{d}a \right\} \\ &\le \tilde{h}\left\|\hat{u}\right\|_{\mathbb{X}_T}.\end{aligned}$$

Since $0 < \tilde{h} < 1$, $\mathcal{F}$ is a strict contraction in $\mathbb{X}^0_{T,M}$.

From the above discussion and the Banach fixed point theorem, we see that $\mathcal{F}$ has the unique fixed point in $\mathbb{X}^0_{T,M}$. This implies that (3.3) has the unique local solution. We can easily check the regularity of the solution, and thus, we see that (2.6) has the unique local solution $(S, u)$. The positivity of $(S, u)$ follows from Proposition 3.1. To extend the domain of existence, we now show that $(S, u)$ never blows up in finite time. By (3.2), we have, for $t > 0$ and $x \in \overline{\Omega}$,

$$\begin{aligned} S(t,x) &\le b\int_0^t \mathrm{e}^{-\mu a}\int_\Omega \Gamma_1(a,x,y)\mathrm{d}y\mathrm{d}a + F_{\phi_1}(t,x) \\ &\le \frac{b}{\mu}(1-\mathrm{e}^{-\mu t}) + \mathrm{e}^{-\mu t}\left\|\phi_1\right\|_{\mathbb{X}} < +\infty, \end{aligned} \tag{3.6}$$

and thus, $S$ never blows up. Suppose that there exist a $t^* > 0$ and an $x^* \in \Omega$ such that $\lim_{t\to t^*-0} u(t,x^*) = +\infty$. By the first equation in (2.6), $\lim_{t\to t^*-0} \partial_t S(t,x^*) = -\infty$. This implies that $S(t,x^*)$ becomes negative in the neighborhood of $t^*$, which contradicts to the positivity of $S$. Thus, $u$ also never blows up in finite time. This completes the proof. □

By a similar argument as in the proof of Proposition 3.2, we can show the uniform boundedness of $S$ and $u$.

**Proposition 3.3.** *Suppose that (A1)-(A3) hold. For any $(\phi_1,\phi_2)\in\mathbb{X}_+^0\times\mathbb{Y}_+^0$, there exist positive constants $M_1=M_1(\phi_1,\phi_2)>0$ and $M_2=M_2(\phi_1,\phi_2)>0$ such that, for all $t>0$ and $x\in\Omega$,*

$$0<S(t,x)\le M_1,\quad 0\le u(t,x)\le M_2. \tag{3.7}$$

**Proof.** From Proposition 3.1 and (3.6), we immediately obtain the first inequalities of (3.7) by setting $M_1:=b/\mu+\|\phi_1\|_{\mathbb{X}}$. To obtain the second inequalities of (3.7), on the contrary, we suppose that there exists an $\tilde{x}\in\Omega$ such that $\lim_{t\to+\infty}u(t,\tilde{x})=+\infty$. By the first equation of (2.6), we have that $\lim_{t\to+\infty}\partial_t S(t,\tilde{x})=-\infty$, which implies that $S(t,\tilde{x})$ becomes negative for sufficiently large $t>0$, and this is a contradiction. This completes the proof. □

On the uniqueness of the positive bounded solution, we prove the following theorem.

**Theorem 3.1.** *Suppose that (A1)-(A3) hold and let $(\phi_1,\phi_2)\in\mathbb{X}_+^0\times\mathbb{Y}_+^0$. Then, system (2.6) has the unique positive bounded solution $(S,u)$ defined on $[0,+\infty)\times\Omega$.*

**Proof.** The existence of a global classical solution follows from Proposition 3.2. Suppose, on the contrary, that there exist two solutions $(S_1,u_1)$ and $(S_2,u_2)$. Note that $u_1=\mathcal{F}u_1$ and $u_2=\mathcal{F}u_2$. By Proposition 3.3, there exists a positive constant $M^+>0$ such that $0\le u_1(t,x),u_2(t,x)\le M^+$ for all $t>0$ and $x\in\Omega$. By (3.5), we then have that, for $t>0$ and $x\in\Omega$,

$$\begin{aligned}
&|u_1(t,x)-u_2(t,x)|=\left|\mathcal{F}\hat{u}(t,x)\right|\\
&\le\left|B(t,x)+U_1[M^+](t,x)+F_{\phi_1}(t,x)\right|\left|U_2[\hat{u}](t,x)\right|\\
&\quad+\left|U_2[M^+](t,x)+F_{\phi_2}(t,x)\right|\left|U_1[\hat{u}](t,x)\right|\\
&\le\left(\frac{b+M^+}{\mu}+F_{\phi_1}^{\infty}\right)\beta^{\infty}\int_0^t\left\|\hat{u}(a)\right\|_{\mathbb{X}}\mathrm{d}a+\left(\frac{M^+\beta^{\infty}}{\mu}+F_{\phi_2}^{\infty}\right)\int_0^t\left\|\hat{u}(a)\right\|_{\mathbb{X}}\mathrm{d}a,
\end{aligned}$$

where $\hat{u}=u_1-u_2$. Thus, we have, for $t>0$,

$$\left\|\hat{u}(t)\right\|_{\mathbb{X}}\le C^+\int_0^t\left\|\hat{u}(a)\right\|_{\mathbb{X}}\mathrm{d}a,$$

where $C^+:=\left(b\beta^{\infty}+2M^+\beta^{\infty}\right)/\mu+F_{\phi_1}^{\infty}\beta^{\infty}+F_{\phi_2}^{\infty}$. Hence, by the Gronwall inequality, we obtain $\left\|\hat{u}(t)\right\|_{\mathbb{X}}=0$ for all $t>0$, which implies that $u_1=u_2$.

Let $\hat{S}:=S_1-S_2$. We then have that, for $t>0$ and $x\in\Omega$,

$$\frac{\partial\hat{S}(t,x)}{\partial t}=d_1\Delta\hat{S}(t,x)-\mu\hat{S}(t,x),$$

and $\hat{S}(t,x) = 0$ for $t > 0$ and $x \in \partial\Omega$ and $\hat{S}(0,x) = 0$ for $x \in \overline{\Omega}$. Hence, it follows from the maximum principle [38] that $\hat{S} = 0$, which implies that $S_1 = S_2$. The positivity and boundedness of the solution follow from Propositions 3.1 and 3.3, respectively. This completes the proof. □

By Theorem 3.1, we see that the problem (2.6) with conditions (2.7) and (2.8) is well-posed if $(\phi_1, \phi_2) \in \mathbb{X}_+^0 \times \mathbb{Y}_+^0$.

## 4. Basic reproduction number

The disease-free steady state for system (2.6) is given by $E_0 : (S, u) = (S_0, 0) \in \mathbb{X}_+^0 \times \mathbb{X}_+^0$, where $S_0$ is the solution of the following elliptic equation,

$$\begin{cases} 0 = d_1 \Delta S_0(x) + b - \mu S_0(x), & x \in \Omega, \\ S_0(x) = 0, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

We now prove the following proposition on the existence of the unique nonnegative solution $S_0 \in \mathbb{X}_+^0 \setminus \{0\}$ of (4.1), which is strictly positive on $\Omega$.

**Proposition 4.1.** *Suppose that (A1) holds. Then, (4.1) has the unique nonnegative solution $S_0 \in \mathbb{X}_+^0 \setminus \{0\}$ such that $S_0(x) > 0$ for all $x \in \Omega$.*

**Proof.** Let $\overline{S}_0(x) := b/\mu$ and $\underline{S}_0(x) := 0$ for all $x \in \overline{\Omega}$. We then have

$$\begin{cases} 0 \geq d_1 \Delta \overline{S}_0(x) + b - \mu \overline{S}_0(x), & x \in \Omega, \\ \overline{S}_0(x) \geq 0, & x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} 0 \leq d_1 \Delta \underline{S}_0(x) + b - \mu \underline{S}_0(x), & x \in \Omega, \\ \overline{S}_0(x) \leq 0, & x \in \partial\Omega. \end{cases}$$

Thus, $\overline{S}_0$ and $\underline{S}_0$ are upper and lower solutions to (4.1), respectively. By a classical argument (see, for instance, [34, Theorem 3]), we see that (4.1) has a nonnegative solution $S_0 \in \mathbb{X}_+^0 \setminus \{0\}$ such that $0 \leq S_0(x) \leq b/\mu$ for all $x \in \overline{\Omega}$. Moreover, we have

$$\begin{cases} 0 > d_1 \Delta S_0(x) - \mu S_0(x), & x \in \Omega, \\ S_0(x) = 0, & x \in \partial\Omega. \end{cases}$$

Thus, it follows from the strong maximum principle for elliptic equations [38] that $S_0(x) > 0$ for all $x \in \Omega$. The uniqueness also can be proved by using the maximum principle [38] as in the proof of Theorem 3.1. This completes the proof. □

By using the fundamental solution $\Gamma_1$, we obtain the following expression of $S_0(x)$, for $x \in \overline{\Omega}$,

$$S_0(x) = b \int_0^{+\infty} e^{-\mu a} \int_\Omega \Gamma_1(a, x, y) dy da. \tag{4.2}$$

In fact, since the fundamental solution $\Gamma_1(t, x, \cdot)$ satisfies (2.2), we have

$$\begin{aligned} d_1 \Delta S_0(x) &= b \int_0^{+\infty} e^{-\mu a} \int_\Omega d_1 \Delta \Gamma_1(a, x, y) dy da = b \int_0^{+\infty} e^{-\mu a} \int_\Omega \frac{\partial \Gamma_1(a, x, y)}{\partial a} dy da \\ &= b \left[ e^{-\mu a} \int_\Omega \Gamma_1(a, x, y) dy \right]_0^{+\infty} + \mu \left[ b \int_0^{+\infty} e^{-\mu a} \int_\Omega \Gamma_1(a, x, y) dy da \right] \\ &= -b + \mu S_0(x), \end{aligned}$$

and thus, $S_0$ satisfies (4.1). We now obtain the following lemma on the ultimate boundedness of $S$.

**Lemma 4.1.** *Suppose that (A1)-(A3) hold and let $(\phi_1, \phi_2) \in \mathbb{X}^0_+ \times \mathbb{Y}^0_+$. Then, we have, for $x \in \overline{\Omega}$,*

$$\limsup_{t \to +\infty} S(t, x) \le S_0(x). \tag{4.3}$$

**Proof.** Taking limit $t \to +\infty$ in (3.6), we immediately obtain (4.3) (note that $\lim_{t \to +\infty} F_{\phi_1}(t, x) = 0$ for all $x \in \overline{\Omega}$). This completes the proof. □

Linearizing the second equation in (2.6) around the disease-free steady state $E_0 : (S_0, 0)$, we have, for $t > 0$ and $x \in \overline{\Omega}$,

$$u(t, x) = S_0(x) \int_0^t \beta(a) e^{-\int_0^a [\mu + \gamma(\sigma)] d\sigma} \int_\Omega \Gamma_2(a, x, y) u(t - a, y) dy da.$$

Based on the classical theory of the basic reproduction number $\mathcal{R}_0$ in [12], we define the next generation operator $\mathcal{K} : \mathbb{X}^0 \to \mathbb{X}^0$ for system (2.6) by, for $x \in \overline{\Omega}$,

$$(\mathcal{K}\varphi)(x) := S_0(x) \int_0^{+\infty} \beta(a) e^{-\int_0^a [\mu + \gamma(\sigma)] d\sigma} \int_\Omega \Gamma_2(a, x, y) \varphi(y) dy da, \quad \varphi \in \mathbb{X}^0. \tag{4.4}$$

Following the argument in [12], the basic reproduction number $\mathcal{R}_0$ is defined by $\mathcal{R}_0 := r(\mathcal{K})$, where $r(\cdot)$ denotes the spectral radius of an operator. To show that $\mathcal{R}_0$ is an eigenvalue of operator $\mathcal{K}$, we now prove the following lemma.

**Lemma 4.2.** *Suppose that (A1)-(A3) hold and let $\mathcal{K}$ be defined by (4.4). $\mathcal{K}$ is strongly positive, bounded and compact.*

**Proof.** The strong positivity of $\mathcal{K}$ (that is, $\mathcal{K}(\mathbb{X}_+^0 \setminus \{0\}) \subset \mathbb{X}_+^0 \setminus \{0\}$) follows from (A2), (G1) and Proposition 4.1. As shown in the proof of Proposition 4.1, we have $S_0(x) \le b/\mu$ for all $x \in \overline{\Omega}$, and thus,

$$\|\mathcal{K}\varphi\|_{\mathbb{X}} \le \sup_{x\in\Omega}\left[\frac{b}{\mu}\int_0^{+\infty}\beta^{\infty}\mathrm{e}^{-\mu a}\int_{\Omega}\Gamma_2(a,x,y)\mathrm{d}y\mathrm{d}a\,\|\varphi\|_{\mathbb{X}}\right] \le \frac{b\beta^{\infty}}{\mu^2}\|\varphi\|_{\mathbb{X}},$$

which implies that $\mathcal{K}$ is bounded. By the semigroup property (G3), we can define the semigroup $T_2(a) : \mathbb{X}^0 \to \mathbb{X}^0$, $a > 0$ generated by $d_2\Delta$ subject to the homogeneous Dirichlet boundary condition as follows, for $a > 0$ and $x \in \overline{\Omega}$,

$$[T_2(a)\varphi](x) := \int_{\Omega}\Gamma_2(a,x,y)\varphi(y)\mathrm{d}y, \quad \varphi \in \mathbb{X}^0.$$

By [42, Chapter 7], we see that $T_2(a)$ is a compact operator for $a > 0$. Hence, $\mathcal{K}$ is also compact since it is a composition of a bounded operator and a compact operator (see, e.g., [41, Theorem 4.18]). This completes the proof. □

By Lemma 4.2, we can apply the Krein-Rutman theorem to establish the following proposition, which states that $\mathcal{R}_0 = r(\mathcal{K})$ is an eigenvalue associated with a positive eigenvector.

**Proposition 4.2.** *Suppose that (A1)-(A3) hold and let $\mathcal{K}$ be defined by (4.4). The basic reproduction number $\mathcal{R}_0 = r(\mathcal{K})$ is a simple positive eigenvalue of operator $\mathcal{K}$, associated with a positive eigenvector $v_0 \in \mathbb{X}_+^0 \setminus \{0\}$. Moreover, there is no other eigenvalue associated with a positive eigenvector.*

**Proof.** The assertion directly follows from Lemma 4.2 and the Krein-Rutman theorem (see, e.g., [2, Theorem 3.2]). □

By (4.4), we see that $\mathcal{K}$ does not have any positive constant eigenvector. Hence, from Proposition 4.2, we can expect that $\mathcal{R}_0$ is affected by the spatial domain $\Omega$ (see Section 8). In what follows, we show that $\mathcal{R}_0$ is the threshold value for the global asymptotic behavior of system (2.6) with conditions (2.7) and (2.8).

## 5. Global attractivity of the disease-free steady state

For arbitrary large positive constant $\Lambda > 0$, let us define the following set,

$$\begin{aligned} C_\Lambda := \{(\phi_1,\phi_2) \in \mathbb{X}_+^0 \times \mathbb{Y}_+^0 :\ & \phi_1(x) \le S_0(x) \text{ for all } x \in \overline{\Omega}, \\ & \phi_2(a,x) \le \Lambda \mathrm{e}^{-\int_0^a[\mu+\gamma(\sigma)]\mathrm{d}\sigma}\int_{\Omega}\Gamma_2(a,x,y)v_0(y)\mathrm{d}y \ \text{ for all } a \ge 0 \text{ and } x \in \overline{\Omega}\}, \end{aligned} \tag{5.1}$$

where $v_0 \in \mathbb{X}_+^0 \setminus \{0\}$ is the positive eigenvector of operator $\mathcal{K}$, associated with the eigenvalue $\mathcal{R}_0 = r(\mathcal{K})$ (see Proposition 4.2). We now prove the following lemma.

**Lemma 5.1.** *Suppose that (A1)-(A3) hold and let $C_\Lambda$ be defined by (5.1). If $(\phi_1, \phi_2) \in C_\Lambda$ and $\mathcal{R}_0 < 1$, then $0 \le S(t,x) \le S_0(x)$ and $0 \le u(t,x) \le \Lambda v_0(x)$ for all $t > 0$ and $x \in \overline{\Omega}$.*

**Proof.** By the first equation in (2.6), we have, for $t > 0$,

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} \le d_1 \Delta S(t,x) + b - \mu S(t,x), & x \in \Omega, \\ S(0,x) = \phi_1(x), & x \in \overline{\Omega}, \\ S(t,x) = 0, & x \in \partial\Omega. \end{cases}$$

Hence, from (4.1), we see that $S_0$ is an upper solution to $S$. Thus, we have that $0 \le S(t,x) \le S_0(x)$ for all $t > 0$ and $x \in \overline{\Omega}$.

By (2.7), we have, for $x \in \Omega$,

$$\begin{aligned} u(0,x) &= \phi_1(x) \int_0^{+\infty} \beta(a) \phi_2(a,x) \mathrm{d}a \\ &\le S_0(x) \int_0^{+\infty} \beta(a) \Lambda \mathrm{e}^{-\int_0^a [\mu + \gamma(\sigma)] \mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y) v_0(y) \mathrm{d}y \mathrm{d}a \\ &= \Lambda (\mathcal{K} v_0)(x) = \mathcal{R}_0 \Lambda v_0(x) < \Lambda v_0(x). \end{aligned}$$

Suppose on the contrary that there exist $t^* > 0$ and $x^* \in \Omega$ such that $u(t,x) < \Lambda v_0(x)$ for all $t \in (0, t^*)$ and $x \in \Omega$ and $u(t^* + \epsilon, x^*) > \Lambda v_0(x^*)$ for sufficiently small $0 < \epsilon \ll 1$. We then have from the second equation in (2.6) that, for $x \in \Omega$ and sufficiently small $\epsilon > 0$,

$$\begin{aligned} &u(t^* + \epsilon, x) \le S_0(x) \int_0^{t^*+\epsilon} \beta(a) \mathrm{e}^{-\int_0^a [\mu + \gamma(\sigma)] \mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y) u(t^* + \epsilon - a, y) \mathrm{d}y \mathrm{d}a \\ &+ S_0(x) \int_{t^*+\epsilon}^{+\infty} \beta(a) \mathrm{e}^{-\int_0^{t^*+\epsilon} [\mu + \gamma(\sigma + a - t^* - \epsilon)] \mathrm{d}\sigma} \int_\Omega \Gamma_2(t^* + \epsilon, x, y) \phi_2(a - t^* - \epsilon, y) \mathrm{d}y \mathrm{d}a \\ &\le S_0(x) \int_0^{t^*+\epsilon} \beta(a) \mathrm{e}^{-\int_0^a [\mu + \gamma(\sigma)] \mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y) \Lambda v_0(y) \mathrm{d}y \mathrm{d}a + S_0(x) \\ &\times \int_{t^*+\epsilon}^{+\infty} \beta(a) \mathrm{e}^{-\int_0^a [\mu + \gamma(\sigma)] \mathrm{d}\sigma} \int_\Omega \Gamma_2(t^* + \epsilon, x, y) \int_\Omega \Gamma_2(a - t^* - \epsilon, y, z) \Lambda v_0(z) \mathrm{d}z \mathrm{d}y \mathrm{d}a \end{aligned}$$

$$
\begin{aligned}
&= S_0(x) \int_0^{+\infty} \beta(a) \mathrm{e}^{-\int_0^a [\mu+\gamma(\sigma)]\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y) \Lambda v_0(y) \mathrm{d}y \mathrm{d}a \\
&= \Lambda(\mathcal{K} v_0)(x) = \Lambda \mathcal{R}_0 v_0(x) < \Lambda v_0(x),
\end{aligned}
$$

which is a contradiction. This completes the proof. □

Using Lemma 5.1, we now prove the global attractivity of the disease-free steady state in $C_\Lambda$ for $\mathcal{R}_0 < 1$.

**Theorem 5.1.** *Suppose that (A1)-(A3) hold and let $C_\Lambda$ be defined by (5.1). If $(\phi_1, \phi_2) \in C_\Lambda$ and $\mathcal{R}_0 < 1$, then the disease-free steady state $E_0 : (S, u) = (S_0, 0)$ is globally attractive, that is,*

$$
\lim_{t \to +\infty} \|S(t,\cdot) - S_0(\cdot)\|_{\mathbb{X}} = 0 \text{ and } \lim_{t \to +\infty} \|u(t,\cdot)\|_{\mathbb{X}} = 0.
$$

**Proof.** Let $u^\infty(x) := \limsup_{t \to +\infty} u(t,x)$, $x \in \overline{\Omega}$. By Lemma 5.1, we have that $u^\infty(x) \le \Lambda v_0(x)$ for all $x \in \overline{\Omega}$. Hence, we have, for $x \in \overline{\Omega}$,

$$
\begin{aligned}
u^\infty(x) &= \limsup_{t \to +\infty} \Bigg\{ S(t,x) \\
&\quad \times \left[ \int_0^t \beta(a) \mathrm{e}^{-\int_0^a [\mu+\gamma(\sigma)]\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y) u(t-a,y) \mathrm{d}y \mathrm{d}a + F_{\phi_2}(t,x) \right] \Bigg\} \\
&\le S_0(x) \int_0^{+\infty} \beta(a) \mathrm{e}^{-\int_0^a [\mu+\gamma(\sigma)]\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y) \left[ \limsup_{t \to +\infty} u(t,y) \right] \mathrm{d}y \mathrm{d}a \\
&\le \Lambda S_0(x) \int_0^{+\infty} \beta(a) \mathrm{e}^{-\int_0^a [\mu+\gamma(\sigma)]\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y) v_0(y) \mathrm{d}y \mathrm{d}a \\
&= \Lambda(\mathcal{K} v_0)(x) = \mathcal{R}_0 \Lambda v_0(x).
\end{aligned}
$$

By iteration, we obtain $u^\infty(x) \le \mathcal{R}_0^n \Lambda v_0(x)$ for all $n \in \mathbb{N}$ and $x \in \overline{\Omega}$. Since $\mathcal{R}_0 < 1$, this implies that $u^\infty(x) = 0$ for all $x \in \overline{\Omega}$, and thus, $\lim_{t \to +\infty} \|u(t,\cdot)\|_{\mathbb{X}} = 0$ for all $x \in \overline{\Omega}$.

From the above result, we see that for any $\epsilon > 0$, there exists a $T > 0$ such that $\|u(t,\cdot)\|_{\mathbb{X}} \le \epsilon$ for all $t \ge T$. As in (3.2), we then have, for $t \ge T$ and $x \in \overline{\Omega}$,

$$
\begin{aligned}
S(t,x) = &\int_T^t \mathrm{e}^{-\mu(a-T)} \int_\Omega \Gamma_1(a-T,x,y) \left[ b - u(t-a+T,y) \right] \mathrm{d}y \mathrm{d}a \\
&+ \mathrm{e}^{-\mu(t-T)} \int_\Omega \Gamma_1(t-T,x,y) S(T,y) \mathrm{d}y
\end{aligned}
$$

$$\geq (b-\epsilon)\int_T^t \mathrm{e}^{-\mu(a-T)}\int_\Omega \Gamma_1(a-T,x,y)\mathrm{d}y\mathrm{d}a$$

$$=(b-\epsilon)\int_0^{t-T} \mathrm{e}^{-\mu a}\int_\Omega \Gamma_1(a,x,y)\mathrm{d}y\mathrm{d}a.$$

Hence, we have, for all $x \in \overline{\Omega}$,

$$S_0^\epsilon(x) \leq \liminf_{t\to+\infty} S(t,x) \leq \limsup_{t\to+\infty} S(t,x) \leq S_0(x),$$

where, for $x \in \overline{\Omega}$,

$$S_0^\epsilon(x) := (b-\epsilon)\int_0^{+\infty} \mathrm{e}^{-\mu a}\int_\Omega \Gamma_1(a,x,y)\mathrm{d}y\mathrm{d}a.$$

Since $\epsilon > 0$ is arbitrary and $S_0^\epsilon \to S_0$ as $\epsilon \to +0$, we have that $\lim_{t\to+\infty} \|S(t,\cdot) - S_0(\cdot)\|_{\mathbb{X}} = 0$. This completes the proof. □

By Theorem 5.1, we see that $\mathcal{R}_0 < 1$ can be a quantitative target for the eradication of diseases.

## 6. Uniform persistence of the disease

Based on (2.4), we define the following function, for $t \geq 0$, $a \geq 0$ and $x \in \overline{\Omega}$,

$$i(t,a,x) = \begin{cases} \mathrm{e}^{-\int_0^a [\mu+\gamma(\sigma)]\mathrm{d}\sigma} \displaystyle\int_\Omega \Gamma_2(a,x,y)u(t-a,y)\mathrm{d}y, & t-a>0, \\ \mathrm{e}^{-\int_0^t [\mu+\gamma(\sigma+a-t)]\mathrm{d}\sigma} \displaystyle\int_\Omega \Gamma_2(t,x,y)\phi_2(a-t,y)\mathrm{d}y, & a-t \geq 0. \end{cases} \tag{6.1}$$

As in [10, Lemma 5.1], we can prove the following lemma on the existence of a continuous semiflow.

**Lemma 6.1.** *Suppose that (A1)-(A3) hold and let $(\phi_1,\phi_2) \in \mathbb{X}_+^0 \times \mathbb{Y}_+^0$. Then, there exists a continuous semiflow defined by $\Sigma(t,\phi_1,\phi_2) := (S(t,\cdot), i(t,\cdot,\cdot)) \in \mathbb{X}_+^0 \times \mathbb{Y}_+^0$ for all $t \geq 0$.*

**Proof.** We omit the proof because it is almost the same as that of [10, Lemma 5.1]. □

We next prove the following lemma on the compactness of the semiflow.

**Lemma 6.2.** *Suppose that (A1)-(A3) hold and let $(\phi_1,\phi_2) \in \mathbb{X}_+^0 \times \mathbb{Y}_+^0$. Then, $\Sigma(t,\cdot,\cdot)$ is a compact operator on $\mathbb{X}_+^0 \times \mathbb{Y}_+^0$ for each $t > 0$.*

**Proof.** Let $\mathcal{B} \subset \mathbb{X}_+^0 \times \mathbb{Y}_+^0$ be an arbitrary bounded set. By Proposition 3.3, there exist positive constants $\tilde{M}_1 > 0$ and $\tilde{M}_2 > 0$ defined by

$$\tilde{M}_i := \sup_{(\phi_1,\phi_2)\in\mathcal{B}} M_i(\phi_1,\phi_2), \quad i = 1, 2.$$

Let $\{\phi^n := (\phi_1^n, \phi_2^n)\}_{n=1}^{+\infty}$ be an arbitrary sequence in $\mathcal{B}$. We then see that there exists a positive constant $\mathcal{M} > 0$ such that $\max\left(\|\phi_1^n\|_{\mathbb{X}}, \|\phi_2^n\|_{\mathbb{Y}}\right) < \mathcal{M}$ for all $n \in \{1, 2, \ldots\}$. Let $(S_n, u_n)$ be the solution of (2.6) for $(\phi_1, \phi_2) = (\phi_1^n, \phi_2^n)$ and let, for $t \geq 0$, $a \geq 0$ and $x \in \overline{\Omega}$,

$$i_n(t,a,x) = \begin{cases} e^{-\int_0^a [\mu+\gamma(\sigma)]d\sigma} \int\limits_\Omega \Gamma_2(a,x,y) u_n(t-a,y)dy, & t-a > 0, \\ e^{-\int_0^t [\mu+\gamma(\sigma+a-t)]d\sigma} \int\limits_\Omega \Gamma_2(t,x,y)\phi_2^n(a-t,y)dy, & a-t \geq 0. \end{cases} \tag{6.2}$$

It then suffices to show that, for each fixed $t > 0$, there exists a convergent subsequence of $\{(S_n(t,\cdot), i_n(t,\cdot,\cdot))\}_{n=1}^{+\infty}$. By the above definition and Proposition 3.3, we have that $0 \leq S_n(t,x) \leq \tilde{M}_1$ and $0 \leq u_n(t,x) \leq \tilde{M}_2$ for all $t > 0$ and $x \in \overline{\Omega}$. This implies that sequences $\{S_n(t,\cdot)\}_{n=1}^{+\infty}$ and $\{u_n(t,\cdot)\}_{n=1}^{+\infty}$ are uniformly bounded. On the other hand, by the continuity of $\Gamma_1(\cdot, x, \cdot)$ on $\overline{\Omega}$, for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\int\limits_\Omega \left|\Gamma_1(t,x,y) - \Gamma_1(t,x',y)\right| dy < \epsilon,$$

provided $|x - x'| < \delta$ and $x, x' \in \overline{\Omega}$. Hence, by (3.2), we obtain

$$\begin{aligned} \left|S_n(t,x) - S_n(t,x')\right| \leq & \int\limits_0^t e^{-\mu a} \int\limits_\Omega \left|\Gamma_1(a,x,y) - \Gamma_1(a,x',y)\right| dy da \left(b + \tilde{M}_2\right) \\ & + e^{-\mu t} \int\limits_\Omega \left|\Gamma_1(t,x,y) - \Gamma_1(t,x',y)\right| dy \mathcal{M}, \\ \leq & \epsilon \left(\frac{b + \tilde{M}_2}{\mu} + \mathcal{M}\right), \end{aligned}$$

provided $|x - x'| < \delta$ and $x, x' \in \overline{\Omega}$. This implies that $\{S_n(t,\cdot)\}_{n=1}^{+\infty}$ is equi-continuous. In a similar manner, by using (3.3), we can show that $\{u_n(t,\cdot)\}_{n=1}^{+\infty}$ is also equi-continuous. Hence, by the Ascoli-Arzela theorem, $\{S_n(t,\cdot)\}_{n=1}^{+\infty}$ and $\{u_n(t,\cdot)\}_{n=1}^{+\infty}$ have convergent subsequences. By (6.2), we see that $\{i_n(t,\cdot,\cdot)\}_{n=1}^{+\infty}$ also has a convergent subsequence. This completes the proof. □

Let us define the following set.

$$D := \left\{ (\phi_1, \phi_2) \in \mathbb{X}^0_+ \times \mathbb{Y}^0_+ : \phi_1(x) \int_0^{+\infty} \beta(a)\phi_2(a,x)\mathrm{d}a > 0 \text{ for some } x \in \Omega \right\}. \tag{6.3}$$

We now prove the following proposition on the uniform weak $\|\cdot\|_{\mathbb{X}}$-persistence of the disease in system (2.6).

**Proposition 6.1.** *Suppose that (A1)-(A3) hold and let D be defined by (6.3). If* $\mathcal{R}_0 > 1$, *then there exists a positive constant* $\epsilon_1 > 0$ *such that*

$$\limsup_{t\to+\infty} \|u(t,\cdot)\|_{\mathbb{X}} > \epsilon_1, \tag{6.4}$$

*provided* $(\phi_1, \phi_2) \in D$.

**Proof.** For $\epsilon_1 > 0$, $h > 0$ and $x \in \overline{\Omega}$, let

$$S_0^{\epsilon_1,h}(x) := (b - \epsilon_1) \int_0^h \mathrm{e}^{-\mu a} \int_\Omega \Gamma_1(a,x,y)\mathrm{d}y\mathrm{d}a.$$

Since $S_0^{\epsilon_1,h}(\cdot) \to S_0(\cdot)$ as $\epsilon_1 \to 0$ and $h \to +\infty$, it follows from the Krein-Rutman theorem [2, Theorem 3.2] and the continuity that there exist sufficiently small $\epsilon_1 > 0$, $\lambda > 0$ and large $h > 0$ such that $r(\tilde{\mathcal{K}}) > 1$, where, for $x \in \overline{\Omega}$,

$$(\tilde{\mathcal{K}}\varphi)(x) := S_0^{\epsilon_1,h}(x) \int_0^{+\infty} \beta(a)\mathrm{e}^{-\int_0^a[\mu+\gamma(\sigma)+\lambda]\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y)\varphi(y)\mathrm{d}y\mathrm{d}a, \quad \varphi \in \mathbb{X}^0.$$

In what follows, we fix such $\epsilon_1$, $\lambda$ and $h$. Let $\mathcal{R}_0^* := r(\tilde{\mathcal{K}}^*)$, where, for $x \in \overline{\Omega}$,

$$(\tilde{\mathcal{K}}^*\varphi)(x) := \int_0^{+\infty} \beta(a)\mathrm{e}^{-\int_0^a[\mu+\gamma(\sigma)+\lambda]\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y)S_0^{\epsilon_1,h}(y)\varphi(y)\mathrm{d}y\mathrm{d}a, \quad \varphi \in \mathbb{X}^0.$$

It is easy to see that $\mathcal{R}_0^* = r(\tilde{\mathcal{K}}^*) = r(\tilde{\mathcal{K}}) > 1$. By the Krein-Rutman theorem, there exists a positive eigenvector $v_0^* \in \mathbb{X}^0_+ \setminus \{0\}$ such that $\mathcal{R}_0^* v_0^*(\cdot) = \tilde{K}^* v_0^*(\cdot)$. We see from (G1) that $v_0^*(x) > 0$ for all $x \in \Omega$.

Suppose on the contrary that (6.4) does not hold. Then, there exists a sufficiently large $T_1 > 0$ such that $u(t,x) \le \epsilon_1$ for all $t \ge T_1$ and $x \in \overline{\Omega}$. As in the proof of Theorem 5.1, we have that, for $t \ge T_1$ and $x \in \overline{\Omega}$,

$$S(t,x) \ge (b - \epsilon_1) \int_0^{t-T_1} \mathrm{e}^{-\mu a} \int_\Omega \Gamma_1(a,x,y)\mathrm{d}y\mathrm{d}a.$$

Let $T_2 := T_1 + h$. We then have $S(t,x) \geq S_0^{\epsilon_1,h}(x)$ for all $t \geq T_2$ and $x \in \overline{\Omega}$. By Lemma 6.1, without loss of generality, we can assume that $S(t,x) \geq S_0^{\epsilon_1,h}(x)$ for all $t \geq 0$ and $x \in \overline{\Omega}$ by regarding $S(T_2,\cdot)$ and $i(T_2,\cdot,\cdot)$ as new initial conditions. We then have that, for $t > 0$ and $x \in \overline{\Omega}$,

$$u(t,x) \geq S_0^{\epsilon_1,h}(x) \int_0^t \beta(a) \mathrm{e}^{-\int_0^a [\mu+\gamma(\sigma)]\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y) u(t-a,y) \mathrm{d}y \mathrm{d}a.$$

Multiplying by $\mathrm{e}^{-\lambda t}$ and integrating from $t = 0$ and $t = +\infty$, we obtain, for $x \in \overline{\Omega}$,

$$\begin{aligned}
&\mathcal{L}[u(\cdot,x)](\lambda) \\
&\geq \int_0^{+\infty} \mathrm{e}^{-\lambda t} S_0^{\epsilon_1,h}(x) \int_0^t \beta(a) \mathrm{e}^{-\int_0^a [\mu+\gamma(\sigma)]\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y) u(t-a,y) \mathrm{d}y \mathrm{d}a \mathrm{d}t \\
&= \int_0^{+\infty} S_0^{\epsilon_1,h}(x) \beta(a) \mathrm{e}^{-\int_0^a [\mu+\gamma(\sigma)+\lambda]\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y) \mathcal{L}[u(\cdot,y)](\lambda) \mathrm{d}y \mathrm{d}a,
\end{aligned}$$

where $\mathcal{L}[u(\cdot,x)](\lambda) := \int_0^{+\infty} \mathrm{e}^{-\lambda t} u(t,x) \mathrm{d}t$ denotes the Laplace transform of $u(\cdot,x)$ with respect to $\lambda$ for all $x \in \overline{\Omega}$. Note that $\mathcal{L}[u(\cdot,x)](\lambda) < +\infty$ for all $x \in \overline{\Omega}$ by virtue of Proposition 3.3. Multiplying by $v_0^*(x)$, integrating over $\Omega$ and using (G5), we obtain

$$\begin{aligned}
&\int_\Omega v_0^*(x) \mathcal{L}[u(\cdot,x)](\lambda) \mathrm{d}x \\
&\geq \int_\Omega v_0^*(x) \int_0^{+\infty} S_0^{\epsilon_1,h}(x) \beta(a) \mathrm{e}^{-\int_0^a [\mu+\gamma(\sigma)+\lambda]\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y) \mathcal{L}[u(\cdot,y)](\lambda) \mathrm{d}y \mathrm{d}a \mathrm{d}x \\
&= \int_\Omega \left[ \int_0^{+\infty} \beta(a) \mathrm{e}^{-\int_0^a [\mu+\gamma(\sigma)+\lambda]\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y) S_0^{\epsilon_1,h}(x) v_0^*(x) \mathrm{d}x \mathrm{d}a \right] \mathcal{L}[u(\cdot,y)](\lambda) \mathrm{d}y \\
&= \int_\Omega \left[ \int_0^{+\infty} \beta(a) \mathrm{e}^{-\int_0^a [\mu+\gamma(\sigma)+\lambda]\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,y,x) S_0^{\epsilon_1,h}(x) v_0^*(x) \mathrm{d}x \mathrm{d}a \right] \mathcal{L}[u(\cdot,y)](\lambda) \mathrm{d}y \\
&= \int_\Omega (\tilde{\mathcal{K}}^* v_0^*)(y) \mathcal{L}[u(\cdot,y)](\lambda) \mathrm{d}y = \mathcal{R}_0^* \int_\Omega v_0^*(x) \mathcal{L}[u(\cdot,x)](\lambda) \mathrm{d}x.
\end{aligned}$$

Since $\mathcal{R}_0^* > 1$ and $\int_\Omega v_0^*(x) \mathcal{L}[u(\cdot,x)](\lambda) \mathrm{d}x > 0$, this leads to a contradiction. This completes the proof. □

From Proposition 6.1, we can prove the uniform strong $\|\cdot\|_{\mathbb{X}}$-persistence of the disease as in [10, Proposition 5.3] (see also [18, Proof of Theorem 1]).

**Theorem 6.1.** *Suppose that (A1)-(A3) hold and let $D$ be defined by (6.3). If $\mathcal{R}_0 > 1$, then there exists a positive constant $\epsilon_2 > 0$ such that*

$$\liminf_{t\to+\infty} \|u(t,\cdot)\|_{\mathbb{X}} > \epsilon_2,$$

*provided $(\phi_1, \phi_2) \in D$.*

**Proof.** We omit the proof because it is almost the same as that of [10, Proposition 5.3]. □

From Theorem 6.1, we see that $i(t,a,x)$ is also uniformly persistent. In fact, let $\mathcal{X}(t) := (S(t,\cdot), i(t,\cdot,\cdot))$, $t \in \mathbb{R}$ be a total trajectory of semiflow $\Sigma$ such that $\Sigma(t, \mathcal{X}(r)) = \mathcal{X}(t+r)$ for all $t \geq 0$ and $r \in \mathbb{R}$. We then have, for all $t \in \mathbb{R}$, $a \geq 0$ and $x \in \overline{\Omega}$,

$$i(t,a,x) = \mathrm{e}^{-\int_0^a [\mu+\gamma(\sigma)]\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y)u(t-a,y)\mathrm{d}y, \tag{6.5}$$

where $u(t,x) = i(t,0,x)$, $t \in \mathbb{R}$, $x \in \overline{\Omega}$. For fixed $a > 0$ and $x \in \Omega$, let us define $\varrho_1, \varrho_2 : \mathbb{X}^0_+ \times \mathbb{Y}^0_+ \to \mathbb{R}_+$ by, for $t \in \mathbb{R}$,

$$\begin{aligned}
&\varrho_1(\mathcal{X}(t)) := \|u(t,\cdot)\|_{\mathbb{X}}, \\
&\varrho_2(\mathcal{X}(t)) := \mathrm{e}^{-\int_0^a [\mu+\gamma(\sigma)]\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y)u(t-a,y)\mathrm{d}y.
\end{aligned}$$

By Theorem 6.1, the uniform $\varrho_1$-persistence for $(\phi_1,\phi_2) \in D$ holds. To apply [43, Corollary 4.22], we assume that $\mathcal{X}(t)$ is a total trajectory with pre-compact range and $\inf_{t\in\mathbb{R}} \varrho_1(\mathcal{X}(t)) = \inf_{t\in\mathbb{R}} \|u(t,\cdot)\|_{\mathbb{X}} > 0$. It is then obvious that $\varrho_2(\mathcal{X}(0)) > 0$ by the positivity of $\Gamma_2$ (see (G1)). Hence, we can apply [43, Corollary 4.22] to conclude that there exists a positive $\epsilon_3 = \epsilon_3(a,x) > 0$ such that $\liminf_{t\to+\infty} \varrho_2(\mathcal{X}(t)) > \epsilon_3$. By (6.5), this implies that, for all $a > 0$ and $x \in \Omega$,

$$\liminf_{t\to+\infty} i(t,a,x) > \epsilon_3(a,x). \tag{6.6}$$

Theorem 6.1 implies that the basic reproduction number $\mathcal{R}_0$ plays the role of a threshold value not only for the eradication but also for the persistence of the disease.

## 7. Existence of the endemic steady state

Let $(S^*, I^*, R^*)$ denote the positive endemic steady state for the original model (2.1). We then have, for $a > 0$ and $x \in \Omega$,

$$\begin{cases} 0 = d_1 \Delta S^*(x) + b - S^*(x) \int_0^{+\infty} \beta(a) I^*(a,x) \mathrm{d}a - \mu S^*(x), \\ \dfrac{\partial I^*(a,x)}{\partial a} = d_2 \Delta I^*(a,x) - \left[\mu + \gamma(a)\right] I^*(a,x), \\ I^*(0,x) = S^*(x) \int_0^{+\infty} \beta(a) I^*(a,x) \mathrm{d}a, \\ \dfrac{\partial R^*(x)}{\partial t} = d_3 \Delta R^*(x) + \int_0^{+\infty} \gamma(a) I^*(a,x) \mathrm{d}a - \mu R^*(x), \end{cases} \tag{7.1}$$

and $S^*(x) = I^*(a,x) = R^*(x) = 0$ for all $a \geq 0$ and $x \in \partial\Omega$. Let $u^*(x) := I^*(0,x)$, $x \in \overline{\Omega}$. We then have, by using the fundamental solution $\Gamma_2$, for $x \in \Omega$,

$$\begin{cases} 0 = d_1 \Delta S^*(x) + b - u^*(x) - \mu S^*(x), \\ u^*(x) = S^*(x) \int_0^{+\infty} \beta(a) \mathrm{e}^{-\int_0^a [\mu + \gamma(\sigma)] \mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y) u^*(y) \mathrm{d}y \mathrm{d}a, \end{cases} \tag{7.2}$$

and $S^*(x) = u^*(x) = 0$ for all $x \in \partial\Omega$. We can regard the positive solution $E^* : (S^*, u^*)$ of (7.2) as the endemic steady state for system (2.6).

If a system is dissipative and uniformly persistent (permanent), then we can apply the asymptotic Schauder fixed point theorem to prove that there exists a positive (endemic) steady state [7,22]. However, we can not directly follow such a way as Proposition 3.3 states that the upper bound $M_2 = M_2(\phi_1, \phi_2)$ for $u$ depends on $(\phi_1, \phi_2)$, and thus, the dissipativity of the system is unknown. We now make the following additional assumption.

**(A4)** $d_1 = d_2 = d_3$.

We next prove the following theorem on the existence of the endemic steady state $E^*$ for system (2.6) (see also [7, Proof of Theorem 6.2]).

**Theorem 7.1.** *Suppose that (A1)-(A4) hold. If $\mathcal{R}_0 > 1$, then system (2.6) has at least one endemic steady state $E^* : (S, u) = (S^*, u^*)$.*

**Proof.** Under (A4), we can easily check that the total population $N(t,x) = S(t,x) + \int_0^{+\infty} I(t,a,x)\mathrm{d}a + R(t,x)$, $t \geq 0$, $x \in \overline{\Omega}$ for the original model (2.1) satisfies the differential equation $N_t = d\Delta N + b - \mu N$, and thus, $N \to S_0$ as $t \to +\infty$. This implies that $\limsup_{t\to+\infty} u(t,x) = \limsup_{t\to+\infty} I(t,0,x) \leq \beta^\infty [S_0(x)]^2 \leq \beta^\infty (b/\mu)^2$ for all $x \in \overline{\Omega}$. Moreover, since $S_t \geq d_1 \Delta S + b - (\beta^\infty S_0 + \mu)S$, there exists an $\epsilon_4(x) > 0$, $x \in \Omega$ such that $\liminf_{t\to+\infty} S(t,x) > \epsilon_4(x)$ for all $x \in \Omega$. Hence, together with Lemma 4.1, we can say that the system is dissipative. We then see that there exist sufficiently smooth functions $e_1(x)$ and $e_2(a,x)$ such that, for all $a > 0$ and $x \in \Omega$,

$$0 < \underline{\alpha} e_1(x) < \liminf_{t\to+\infty} S(t,x) \le \limsup_{t\to+\infty} S(t,x) < \overline{\alpha} e_1(x),$$

$$0 < \underline{\alpha} e_2(a,x) < \liminf_{t\to+\infty} i(t,a,x) \le \limsup_{t\to+\infty} i(t,a,x) < \overline{\alpha} e_2(a,x),$$

provided $(\phi_1, \phi_2) \in D$ and $\mathcal{R}_0 > 1$, where $\underline{\alpha}, \overline{\alpha} \in (0, +\infty)$. Let us define a set $D_0 \subset D$ by

$$D_0 := \Big\{ (\phi_1, \phi_2) \in \mathbb{X}_+^0 \times \mathbb{Y}_+^0 : \frac{\underline{\alpha}}{2} e_1(x) < \phi_1(x) < 2\overline{\alpha} e_1(x) \text{ for all } x \in \Omega$$
$$\frac{\underline{\alpha}}{2} e_2(a,x) < \phi_2(a,x) < 2\overline{\alpha} e_2(a,x) \text{ for all } a > 0 \text{ and } x \in \Omega \Big\}.$$

By Lemmas 6.1-6.2, the semiflow $\Sigma(t,\cdot,\cdot)$ is continuous and compact for all $t > 0$. Moreover, we see from the above argument that, for each fixed $t > 0$, there exists an $r_0 \ge 0$ such that $\Sigma(rt, \overline{D_0}) \subset D_0$ for any $r \ge r_0$. Hence, it follows from the asymptotic Schauder fixed point theorem [7, Theorem 6.1] that the operator $\mathcal{T}_t$ defined by $\mathcal{T}_t(\phi_1, \phi_2) := \Sigma(t, \phi_1, \phi_2)$ has a fixed point in $D_0$. Since $t > 0$ is arbitrary, we can construct a sequence of periodic trajectories in $D_0$ with period $t_n > 0$ such that $t_n \to 0$, $n \to +\infty$. Hence, by [5, Lemma 3.7 in Chapter V], the semiflow $\Sigma$ has a steady state in $D_0$. This implies the existence of the endemic steady state $E^*$. This completes the proof. □

Next, to investigate the existence of the endemic steady state $E^*$ without the additional assumption (A4), we employ an operator-based method. Let us define the following linear operator on $\mathbb{X}_+^0$, for $x \in \overline{\Omega}$,

$$(\Psi\varphi)(x) := \int_0^{+\infty} \beta(a) \mathrm{e}^{-\int_0^a [\mu+\gamma(\sigma)]\mathrm{d}\sigma} \int_\Omega \Gamma_2(a,x,y)\varphi(y)\mathrm{d}y\mathrm{d}a, \quad \varphi \in \mathbb{X}_+^0.$$

(7.2) can then be rewritten as, for $x \in \Omega$,

$$\begin{cases} 0 = d_1 \Delta S^*(x) + b - \left[\mu + (\Psi u^*)(x)\right] S^*(x), \\ u^*(x) = S^*(x)(\Psi u^*)(x). \end{cases} \tag{7.3}$$

By using the Feynman-Kac formula for boundary value problems [36, Chapter 9], we obtain, for $x \in \Omega$,

$$S^*(x) = E^x \left[ b \int_0^{\tau_\Omega} \mathrm{e}^{-\int_0^s [\mu+(\Psi u^*)(X_u)]\mathrm{d}u} \mathrm{d}s \right], \tag{7.4}$$

where $\{X_t\}$ denotes the Itô diffusion corresponding to $d_1\Delta$, $E^x$ denotes the expectation with respect to the probability law of $X_t$ starting at $x$ and $\tau_\Omega$ denotes the first exit time defined by $\tau_\Omega := \inf\{t > 0 : X_t \notin \Omega\}$. From (7.3) and (7.4), we obtain, for $x \in \Omega$,

$$u^*(x) = E^x \left[ b \int_0^{\tau_\Omega} \mathrm{e}^{-\int_0^s [\mu+(\Psi u^*)(X_u)]\mathrm{d}u} \mathrm{d}s \right] (\Psi u^*)(x). \tag{7.5}$$

Let us define the following nonlinear operator on $L^1_+(\Omega)$, for $x \in \overline{\Omega}$,

$$\Upsilon(\varphi)(x) := E^x \left[ b \int_0^{\tau_\Omega} \mathrm{e}^{-\int_0^s [\mu + (\Psi\varphi)(X_u)]\mathrm{d}u} \mathrm{d}s \right] (\Psi\varphi)(x), \quad \varphi \in L^1_+(\Omega).$$

If $\Upsilon$ has a positive fixed point $\varphi^*$ in $L^1_+(\Omega) \setminus \{0\}$, then $\varphi^*$ satisfies (7.5). Moreover, such $\varphi^*$ is twice continuously differentiable and equal to zero on $\partial\Omega$ since $\Psi\varphi^*$ is so. By (7.4) and (7.5), we can then obtain the endemic steady state $(S^*, u^*)$ for system (2.6). For this reason, we focus on the fixed point problem of $\Upsilon$.

Let $\{\Omega_\epsilon\}_{\epsilon>0}$ be a family of proper sub-domains of $\Omega$ such that $d(\Omega_\epsilon, \partial\Omega) > 0$ for all $\epsilon > 0$, $\Omega_{\epsilon_1} \subset \Omega_{\epsilon_2}$ for $\epsilon_1 > \epsilon_2$ and $|\Omega \setminus \Omega_\epsilon| \to 0$ as $\epsilon \to 0$. Note that, for $a > 0$, $\epsilon > 0$ and $x \in \Omega$,

$$0 < \Gamma^-_{2,\epsilon}(a, x) := \inf_{y\in\Omega_\epsilon} \Gamma_2(a, x, y) \le \Gamma^+_{2,\epsilon}(a, x) := \sup_{y\in\Omega_\epsilon} \Gamma_2(a, x, y) < +\infty.$$

For $\epsilon > 0$, let us define the following operator on $L^1_+(\Omega_\epsilon)$ to $L^1_+(\Omega)$:

$$(\Psi_\epsilon\varphi)(x) := \int_0^{+\infty} \beta(a)\mathrm{e}^{-\int_0^a [\mu+\gamma(\sigma)]\mathrm{d}\sigma} \int_{\Omega_\epsilon} \Gamma_2(a, x, y)\varphi(y)\mathrm{d}y\mathrm{d}a, \quad \varphi \in L^1_+(\Omega_\epsilon),$$

and the following operators on $L^1_+(\Omega_\epsilon)$ to itself: for $\varphi \in L^1_+(\Omega_\epsilon)$,

$$\Upsilon_\epsilon(\varphi)(x) := E^x \left[ b \int_0^{\tau_\Omega} \mathrm{e}^{-\int_0^s [\mu+(\Psi_\epsilon\varphi)(X_u)]\mathrm{d}u} \mathrm{d}s \right] (\Psi_\epsilon\varphi)(x) \Bigg|_{\Omega_\epsilon},$$

$$(\mathcal{K}_\epsilon\varphi)(x) := E^x \left[ b \int_0^{\tau_\Omega} \mathrm{e}^{-\mu s} \mathrm{d}s \right] (\Psi_\epsilon\varphi)(x) \Bigg|_{\Omega_\epsilon} = S_0(x)(\Psi_\epsilon\varphi)(x)\,|_{\Omega_\epsilon},$$

where $|_{\Omega_\epsilon}$ denotes the restriction of the domain to $\Omega_\epsilon$. In what follows, for simplicity, we omit the symbol $|_{\Omega_\epsilon}$. If $\Upsilon_\epsilon$ has a positive fixed point $\varphi^*_\epsilon$ in $L^1_+(\Omega_\epsilon) \setminus \{0\}$ for arbitrary small $\epsilon > 0$, then we can conclude by the continuity that $\Upsilon$ has a positive fixed point, and thus, the endemic steady state exists. We now prove the following lemma on $\Upsilon_\epsilon$ for $\epsilon > 0$.

**Lemma 7.1.** *Suppose that (A1)-(A3) hold. For $\epsilon > 0$, the following statements hold.*

**(i)** $\Upsilon'_\epsilon[0] = \mathcal{K}_\epsilon$, *where $\Upsilon'_\epsilon[0]$ denotes the strong Fréchet derivative of $\Upsilon_\epsilon$ at $0 \in L^1_+(\Omega_\epsilon)$.*
**(ii)** $\Upsilon'_\epsilon[\infty] = 0$, *where $\Upsilon'_\epsilon[\infty]$ denotes the derivative at infinity of $\Upsilon_\epsilon$ with respect to the cone $L^1_+(\Omega_\epsilon)$. Moreover, $\Upsilon'_\epsilon[\infty] = 0$ is the strong asymptotic derivative of $\Upsilon_\epsilon$.*
**(iii)** $\Upsilon_\epsilon$ *is compact.*

**Proof.** (i) Let $\varphi \in L^1_+(\Omega_\epsilon) \setminus \{0\}$. By the Taylor expansion, we have, for all $x \in \Omega_\epsilon$,

$$\Upsilon_\epsilon(\varphi)(x) = (\mathcal{K}_\epsilon\varphi)(x) + E^x\left[b\int_0^{\tau_\Omega} \mathrm{e}^{-\mu s}\sum_{n=1}^{+\infty}\frac{\left[-\int_0^s(\Psi_\epsilon\varphi)(X_u)\mathrm{d}u\right]^n}{n!}\mathrm{d}s\right](\Psi_\epsilon\varphi)(x).$$

Note that $\Upsilon_\epsilon(0) = 0$. We then have that, for $x \in \Omega_\epsilon$,

$$\begin{aligned}&|\Upsilon_\epsilon(\varphi)(x) - \Upsilon_\epsilon(0)(x) - (\mathcal{K}_\epsilon\varphi)(x)|\\ &\le E^x\left[b\int_0^{\tau_\Omega} \mathrm{e}^{-\mu s}\left(\mathrm{e}^{\int_0^s(\Psi_\epsilon\varphi)(X_u)\mathrm{d}u} - 1\right)\mathrm{d}s\right](\Psi_\epsilon\varphi)(x).\end{aligned}$$

Since $\Psi_\epsilon$ is a bounded linear operator, we see that

$$\lim_{\varphi\to 0}\frac{\|\Upsilon_\epsilon(\varphi) - \Upsilon_\epsilon(0) - \mathcal{K}_\epsilon\varphi\|_{L^1(\Omega_\epsilon)}}{\|\varphi\|_{L^1(\Omega_\epsilon)}} = 0.$$

This implies that $\Upsilon'_\epsilon[0] = \mathcal{K}_\epsilon$.

(ii) Let $\varphi \in L^1_+(\Omega_\epsilon) \setminus \{0\}$. For $h > 0$, note that $\Psi_\epsilon(h\varphi)(x) = h(\Psi_\epsilon\varphi)(x) > 0$ for all $x \in \Omega_\epsilon$. We then have that

$$\frac{\Upsilon_\epsilon(h\varphi)}{h} = E^x\left[b\int_0^{\tau_\Omega} \mathrm{e}^{-\int_0^s[\mu + h(\Psi_\epsilon\varphi)(X_u)]\mathrm{d}u}\mathrm{d}s\right](\Psi_\epsilon\varphi)(x) \to 0 \text{ as } h \to +\infty.$$

This implies that $\Upsilon'_\epsilon[\infty] = 0$. Moreover, we have that, for $x \in \Omega_\epsilon$,

$$\left|\Upsilon_\epsilon(\varphi)(x) - (\Upsilon'_\epsilon[\infty]\varphi)(x)\right| \le E^x\left[b\int_0^{\tau_\Omega} \mathrm{e}^{-\int_0^s\left[\mu + \hat{\Gamma}^-_{2,\epsilon}(X_u)\|\varphi\|_{L^1(\Omega_\epsilon)}\right]\mathrm{d}u}\mathrm{d}s\right](\Psi_\epsilon\varphi)(x),$$

where $\hat{\Gamma}^-_{2,\epsilon}(x) := \int_0^{+\infty}\beta(a)\mathrm{e}^{-\int_0^a[\mu+\gamma(\sigma)]\mathrm{d}\sigma}\Gamma^-_{2,\epsilon}(a,x)\mathrm{d}a > 0$, $x \in \Omega_\epsilon$. Since $\Psi_\epsilon$ is a bounded linear operator, we then have

$$\lim_{r\to+\infty}\sup_{\|\varphi\|_{L^1(\Omega_\epsilon)}\ge r}\frac{\left\|\Upsilon_\epsilon(\varphi) - \Upsilon'_\epsilon[\infty]\varphi\right\|_{L^1(\Omega_\epsilon)}}{\|\varphi\|_{L^1(\Omega_\epsilon)}} = 0.$$

This implies that $\Upsilon'_\epsilon[\infty] = 0$ is the strong asymptotic derivative of $\Upsilon_\epsilon$.

(iii) Let $\mathcal{C} := \left\{\varphi \in L^1_+(\Omega_\epsilon) : \|\varphi\|_{L^1(\Omega_\epsilon)} \le 1\right\}$, let $\{\varphi_n\}_{n=1}^{+\infty}$ be an arbitrary sequence in $\mathcal{C}$ and let $v_n := \Upsilon_\epsilon(\varphi_n)$. We then have that, for all $x \in \Omega_\epsilon$,

$$
\begin{aligned}
v_n(x) \leq & E^x\left[b\int_0^{\tau_\Omega} \mathrm{e}^{-\mu s}\mathrm{d}s\right]\int_0^{+\infty}\beta(a)\mathrm{e}^{-\int_0^a[\mu+\gamma(\sigma)]\mathrm{d}\sigma}\Gamma_{2,\epsilon}^{+}(a,x)\mathrm{d}a\,\|\varphi_n\|_{L^1(\Omega_\epsilon)} \\
\leq & \sup_{x\in\Omega_\epsilon}\left[S_0(x)\hat{\Gamma}_{2,\epsilon}^{+}(x)\right] < +\infty,
\end{aligned}
$$

where $\hat{\Gamma}_{2,\epsilon}^{+}(x) := \int_0^{+\infty}\beta(a)\mathrm{e}^{-\int_0^a[\mu+\gamma(\sigma)]\mathrm{d}\sigma}\Gamma_{2,\epsilon}^{+}(a,x)\mathrm{d}a$. This implies that $\{v_n\}$ is uniformly bounded. Moreover, for any $x, h \in \Omega_\epsilon$ such that $x+h \in \Omega_\epsilon$, we have

$$
\begin{aligned}
|v_n(x+h) - v_n(x)| = & \left|S_\epsilon^*(x+h)(\Psi_\epsilon\varphi_n)(x+h) - S_\epsilon^*(x)(\Psi_\epsilon\varphi_n)(x)\right| \\
\leq & \left|S_\epsilon^*(x+h) - S_\epsilon^*(x)\right|(\Psi_\epsilon\varphi_n)(x+h) \\
& + S_\epsilon^*(x)\left|(\Psi_\epsilon\varphi_n)(x+h) - (\Psi_\epsilon\varphi_n)(x)\right|,
\end{aligned}
\tag{7.6}
$$

where $S_\epsilon^*(x) := E^x\left[b\int_0^{\tau_\Omega}\mathrm{e}^{-\int_0^s[\mu+\Psi_\epsilon(\varphi_n)(X_u)]\mathrm{d}u}\mathrm{d}s\right]$. Note that $S_\epsilon^*(x)$ satisfies the following boundary value problem:

$$
0 = d_1\Delta S_\epsilon^*(x) + b - [\mu + (\Psi\varphi_n)(x)]\,S_\epsilon^*(x), \quad x \in \Omega, \qquad S_\epsilon^*(x) = 0, \quad x \in \partial\Omega.
$$

Hence, $0 < S_\epsilon^*(x) \leq S_0(x)$ for all $x \in \Omega$. Moreover, regarding $b - (\Psi\varphi_n)(x)S_\epsilon^*(x)$ as the non-homogeneous term, we obtain

$$
S_\epsilon^*(x) = \int_0^{+\infty}\mathrm{e}^{-\mu a}\int_\Omega \Gamma_1(a,x,y)\left[b - (\Psi\varphi_n)(y)S_\epsilon^*(y)\right]\mathrm{d}y\mathrm{d}a, \quad x \in \Omega.
$$

Recalling that $(\Psi\varphi_n)(x) \leq \hat{\Gamma}_{2,\epsilon}^{+}(x)$, we obtain

$$
\begin{aligned}
\left|S_\epsilon^*(x+h) - S_\epsilon^*(x)\right| \leq & \int_0^{+\infty}\mathrm{e}^{-\mu a}\int_\Omega |\Gamma_1(a,x+h,y) - \Gamma_1(a,x,y)|\,\mathrm{d}y\mathrm{d}a \\
& \times \sup_{x\in\Omega_\epsilon}\left[b + \hat{\Gamma}_{2,\epsilon}^{+}(x)S_0(x)\right].
\end{aligned}
\tag{7.7}
$$

Moreover, we obtain

$$
\begin{aligned}
& |(\Psi_\epsilon\varphi_n)(x+h) - (\Psi_\epsilon\varphi_n)(x)| \\
\leq & \int_0^{+\infty}\beta(a)\mathrm{e}^{-\int_0^a[\mu+\gamma(\sigma)]\mathrm{d}\sigma}\int_{\Omega_\epsilon}|\Gamma_2(a,x+h,y) - \Gamma_2(a,x,y)|\,\varphi_n(y)\mathrm{d}y\mathrm{d}a \\
\leq & \int_0^{+\infty}\beta(a)\mathrm{e}^{-\int_0^a[\mu+\gamma(\sigma)]\mathrm{d}\sigma}\sup_{y\in\Omega_\epsilon}|\Gamma_2(a,x+h,y) - \Gamma_2(a,x,y)|\,\mathrm{d}a.
\end{aligned}
\tag{7.8}
$$

From (7.6)-(7.8) and the boundedness of $\Psi_\epsilon \varphi_n$ and $S^*_\epsilon$, we obtain

$$\lim_{\|h\| \to 0} \|\tau_h v_n - v_n\|_{L^1(\Omega_\epsilon)} = 0 \text{ uniformly in } \{v_n\},$$

where $\tau_h v_n(x) := v_n(x+h)$, $x, h \in \Omega_\epsilon$ denotes the shift of the function. Hence, from the Fréchet-Kolmogorov theorem [4, Theorem 4.26], we can conclude that $\Upsilon_\epsilon$ is compact. This completes the proof. □

Using Lemma 7.1, we prove the following theorem on the existence of the endemic steady state $E^* : (S, u) = (S^*, u^*)$ for system (2.6).

**Theorem 7.2.** *Suppose that (A1)-(A3) hold. If $\mathcal{R}_0 > 1$, then system (2.6) has at least one endemic steady state $E^* : (S, u) = (S^*, u^*)$.*

**Proof.** Since $\mathcal{R}_0 = r(\mathcal{K}) > 1$, it follows from the continuity that there exists a sufficiently small $\epsilon > 0$ such that $r(\mathcal{K}_\epsilon) > 1$. For such $\epsilon$, it is obvious that $\Upsilon_\epsilon$ is positive (that is, $\Upsilon_\epsilon(L^1_+(\Omega_\epsilon)) \subset L^1_+(\Omega_\epsilon)$) and $\Upsilon_\epsilon(0) = 0$. By Lemma 7.1, we see that $\Upsilon_\epsilon$ has the strong Fréchet derivative $\Upsilon'_\epsilon[0] = \mathcal{K}_\epsilon$ and the strong asymptotic derivative $\Upsilon'_\epsilon[\infty] = 0$ with respect to cone $L^1_+(\Omega_\epsilon)$. Since $\Upsilon'_\epsilon[\infty] = 0$, the spectrum of $\Upsilon'_\epsilon[\infty]$ is 0, and thus, lies in the circle centered at 0 with radius less than 1. As in Proposition 4.2, we easily see that $\Upsilon'_\epsilon[0] = \mathcal{K}_\epsilon$ has a positive eigenvector $v_{0,\epsilon} \in L^1_+(\Omega_\epsilon) \setminus \{0\}$ such that $\Upsilon'_\epsilon[0] v_{0,\epsilon} = \mathcal{K}_\epsilon v_{0,\epsilon} = r(\mathcal{K}_\epsilon) v_{0,\epsilon}$. Since $r(\mathcal{K}_\epsilon) > 1$, it follows from the Krein-Rutman theorem [2, Section 3] that $\Upsilon'_\epsilon[0]$ does not have any eigenvectors in $L^1_+(\Omega_\epsilon)$ corresponding to the eigenvalue 1. Moreover, by Lemma 7.1, $\Upsilon_\epsilon$ is compact. We then see from the Krasnoselskii fixed point theorem [29, Theorem 4.11] that $\Upsilon_\epsilon$ has at least one non-zero fixed point $\varphi^*_\epsilon \in L^1_+(\Omega_\epsilon) \setminus \{0\}$. By the above argument, we can conclude that system (2.6) has at least one endemic steady state $E^* : (S, u) = (S^*, u^*)$. This completes the proof. □

## 8. Numerical simulation

In this section, we provide numerical examples that support our theoretical results. In what follows, we fix the following parameters.

$$b = 1, \quad \mu = 1, \quad \gamma = 1, \quad d_1 = 0.05, \quad d_2 = 0.05, \quad \beta(a) = \begin{cases} \beta, & a \le 1, \\ 0, & a > 1, \end{cases} \tag{8.1}$$

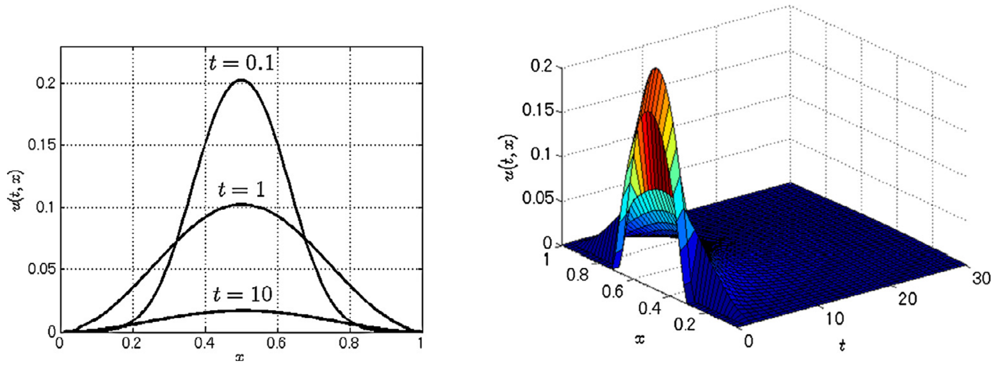
where $\beta > 0$ is a positive constant.

### *8.1. Spatially 1-dimensional case*

Let us consider the 1-dimensional spatial domain $\Omega = (0, \ell)$. In this case, from (2.3), the fundamental solutions $\Gamma_i(t, x, y)$ are given by, for $i = 1, 2, 3$, $t > 0$ and $x, y \in [0, \ell]$,

$$\Gamma_i(t, x, y) = \frac{2}{\ell} \sum_{k=1}^{+\infty} \sin \frac{k\pi x}{\ell} \sin \frac{k\pi y}{\ell} \mathrm{e}^{-d_i \frac{k^2}{\ell^2} \pi^2 t}.$$

We choose the initial conditions as follows.




(a) (b)

Fig. 1. Time evolution of the density $u(t, x)$ of newly infected individuals for parameters (8.1), $\ell = 1$ and $\beta = 3.6$ ($\mathcal{R}_0 \approx 0.9516 < 1$).

$$\phi_1(x) = S_0(x), \quad x \in \overline{\Omega},$$

$$\phi_2(a, x) = \begin{cases} 5e^{-(\mu+\gamma)a}(x - 0.3\ell)(0.7\ell - x), & x \in \Omega_0, \\ 0, & \text{otherwise}, \end{cases}$$

where $\Omega_0 = (0.3\ell, 0.7\ell) \subset \Omega$. It is easy to check that (A1)-(A3) hold and $(\phi_1, \phi_2) \in C_\Lambda \cap D$ for sufficiently large $\Lambda > 0$ by the strict positivity of $\int_\Omega \Gamma_2(a, x, y)v_0(y)dy$ for $a > 0$ and $x, y \in \Omega$. For the numerical computation of the basic reproduction number $\mathcal{R}_0$, we use the Fredholm discretization method as in [9, Section 3.1.2].

For $\beta = 3.6$, we obtain $\mathcal{R}_0 \approx 0.9516 < 1$. Hence, by Theorem 5.1, we can expect that the disease-free steady state $E_0$ is globally attractive. In fact, Fig. 1 shows that the density $u(t, x)$ of newly infected individuals converges to zero as time evolves.

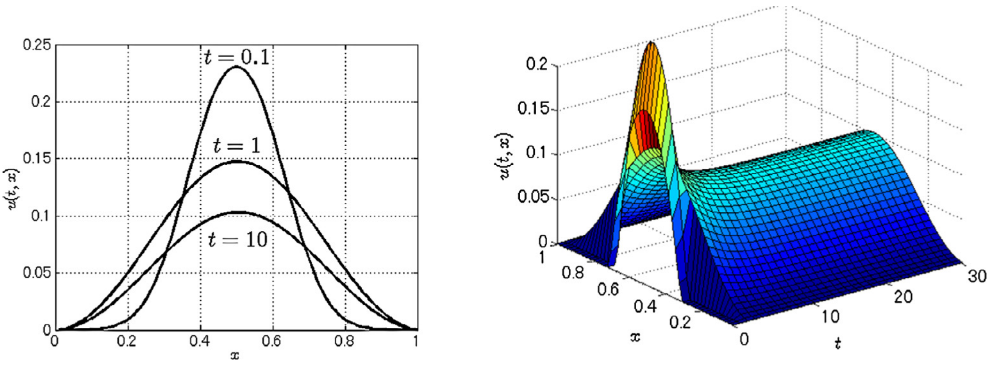
For $\beta = 4$, we obtain $\mathcal{R}_0 \approx 1.0574 > 1$. Hence, by Theorems 6.1 and 7.2, we can expect that the disease eventually persists and an endemic steady state $E^*$ exists. In fact, Fig. 2 shows that the density $u(t, x)$ of newly infected individuals converges to a positive distribution as time evolves. From this numerical result, we can conjecture that the endemic steady state $E^*$ is asymptotically stable.

### 8.2. *Spatially 2-dimensional case*

Let us consider the rectangular domain $\Omega = (0, \ell_1) \times (0, \ell_2) \subset \mathbb{R}^2$, $\ell_1, \ell_2 > 0$. From (2.3), we see that the fundamental solutions $\Gamma_i(t, x, y) = \Gamma_i(t, x_1, x_2, y_1, y_2)$, $i = 1, 2, 3$ are given by, for $i = 1, 2, 3$, $t > 0$, $x_1, y_1 \in [0, \ell_1]$ and $x_2, y_2 \in [0, \ell_2]$,

$$\Gamma_i(t, x_1, x_2, y_1, y_2) := \frac{4}{\ell_1\ell_2}\sum_{m,n=1}^{+\infty} \sin\frac{m\pi x_1}{\ell_1}\sin\frac{n\pi x_2}{\ell_2}\sin\frac{m\pi y_1}{\ell_1}\sin\frac{n\pi y_2}{\ell_2}e^{-d_i\left(\frac{m^2}{\ell_1^2}+\frac{n^2}{\ell_2^2}\right)\pi^2 t}.$$

We fix the following initial conditions.




(a) (b)

Fig. 2. Time evolution of the density $u(t, x)$ of newly infected individuals for parameters (8.1), $\ell = 1$ and $\beta = 4$ ($\mathcal{R}_0 \approx 1.0574 > 1$).

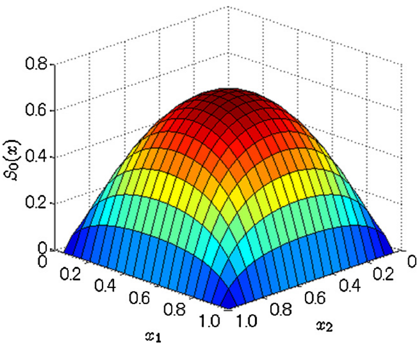


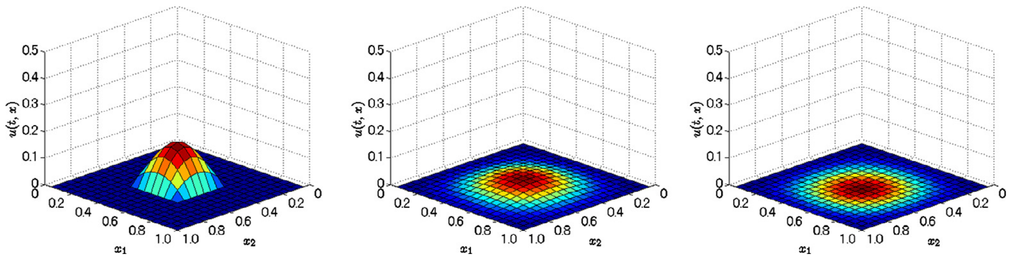

Fig. 3. Density $S_0(x)$ of susceptible individuals in the disease-free steady state $E_0$ for parameters (8.1) and $\ell_1 = \ell_2 = 1$.

$$\begin{cases} \phi_1(x) = S_0(x), \quad x \in \overline{\Omega}, \\ \phi_2(a, x) = \begin{cases} 100\mathrm{e}^{-(\mu+\gamma)a} \prod_{i=1}^{2} (x_i - 0.3\ell_i)(0.7\ell_i - x_i), & (x_1, x_2) \in \Omega_0, \\ 0, & \text{otherwise}, \end{cases} \end{cases}$$

where $\Omega_0 = (0.3\ell_1, 0.7\ell_1) \times (0.3\ell_2, 0.7\ell_2) \subset \Omega$. As in the previous subsection, it is easy to check that (A1)-(A3) hold and $(\phi_1, \phi_2) \in C_\Lambda \cap D$ for sufficiently large $\Lambda > 0$.

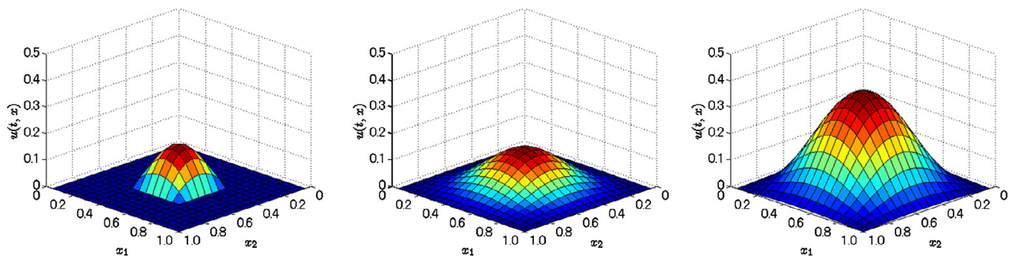
We first consider the case of square domain such that $\ell_1 = \ell_2 = 1$. In this case, $S_0(x) = S_0(x_1, x_2)$ can be calculated as shown in Fig. 3. For $\beta = 4$, we obtain $\mathcal{R}_0 \approx 0.8587 < 1$. Hence, by Theorem 5.1, we can expect that the disease-free steady state $E_0$ is globally attractive. In fact, Fig. 4 shows that the density $u(t, x)$ of newly infected individuals converges to zero as time evolves. On the other hand, for $\beta = 6$, we obtain $\mathcal{R}_0 \approx 1.1450 > 1$. Hence, by Theorems 6.1 and 7.2, we can expect that the disease eventually persists and an endemic steady state $E^*$ exists. In fact, Fig. 5 shows that the density $u(t, x)$ of newly infected individuals converges to a positive distribution as time evolves. This numerical result suggests that the endemic steady state $E^*$ is asymptotically stable.

We next consider the case of rectangular domain such that $\ell_1 = p$ and $\ell_2 = 1/p$ for $p > 0$. Other parameters are similar to those in the previous example for Fig. 5. Note that the area of



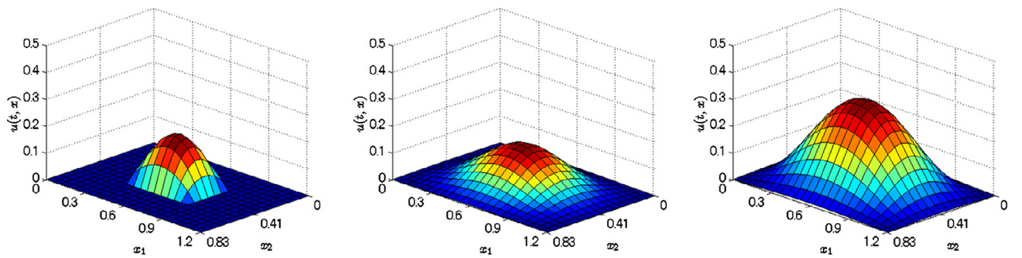

(a) $t = 0$. (b) $t = 1$. (c) $t = 10$.

Fig. 4. Time evolution of the density $u(t, x)$ of newly infected individuals for parameters (8.1), $\ell_1 = \ell_2 = 1$ and $\beta = 4$ ($\mathcal{R}_0 \approx 0.8587 < 1$).




(a) $t = 0$. (b) $t = 1$. (c) $t = 10$.

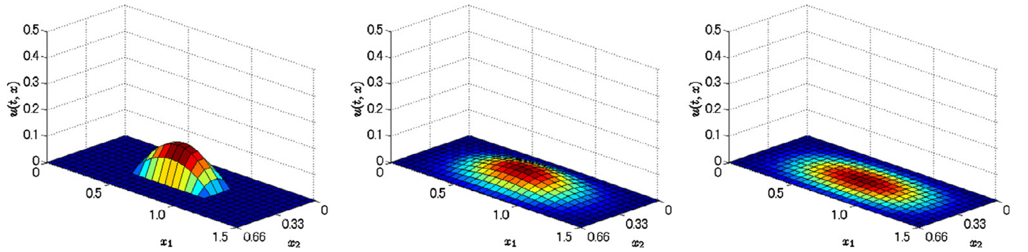
Fig. 5. Time evolution of the density $u(t, x)$ of newly infected individuals for parameters (8.1), $\ell_1 = \ell_2 = 1$ and $\beta = 6$ ($\mathcal{R}_0 \approx 1.1450 > 1$).




(a) $t = 0$. (b) $t = 1$. (c) $t = 10$.

Fig. 6. Time evolution of the density $u(t, x)$ of newly infected individuals for parameters (8.1), $\beta = 6$, $\ell_1 = p$, $\ell_2 = 1/p$ and $p = 1.2$ ($\mathcal{R}_0 \approx 1.0725 > 1$).

the domain is $|\Omega| = \ell_1 \ell_2 = 1$ and equal to that in the previous examples. For $p = 1.2$, we obtain $\mathcal{R}_0 \approx 1.0725 > 1$ and thus, the disease eventually persists (Fig. 6). On the other hand, for $p = 1.5$, we obtain $\mathcal{R}_0 \approx 0.8694 < 1$ and thus, the disease is eventually eradicated (Fig. 7). From these examples, we can conclude that the intensity of the disease spread is affected by the shape of the spatial domain. In particular, our examples suggest that the disease is most likely to spread in the square domain for $p = 1$ (Fig. 8). These results are in contrast to the case of homogeneous Neumann boundary condition with spatially homogeneous parameters in which the basic reproduction number $\mathcal{R}_0$ is explicitly given by a positive constant and independent of the shape of the spatial domain (see, e.g., [10]). Since the infection is thought to be more likely to spread in a re-




(a) $t = 0$. (b) $t = 1$. (c) $t = 10$.

Fig. 7. Time evolution of the density $u(t, x)$ of newly infected individuals for parameters (8.1), $\beta = 6$, $\ell_1 = p$, $\ell_2 = 1/p$ and $p = 1.5$ ($\mathcal{R}_0 \approx 0.8694 < 1$).

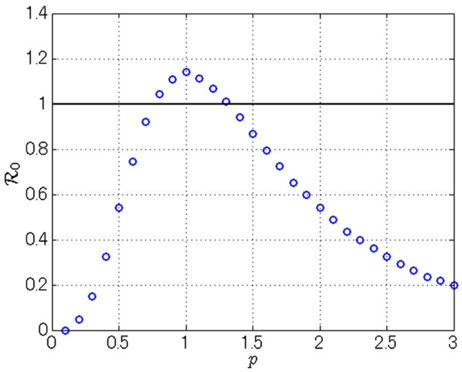



Fig. 8. $R_0$ for parameters (8.1) with $\beta = 6$, $\ell_1 = p$, $\ell_2 = 1/p$ and $p \in (0, 3]$.

gion with a suitable shape for individuals to contact each other, the Dirichlet boundary condition may be more realistic than the Neumann boundary condition in epidemiological modelling.

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