Links and Tangles

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LINKS AND TANGLES

（絡み輪とタングル）

BY

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Introduction and Preliminaries

Recently, we made progress in researching on non-singular flows on a closed 3-manifolds, especially, on a 3-sphere. A leading method is observing how closed orbits are linked.

We know that closed orbits on a 2-, 4-, or higher dimensional manifold are essentially unlinked by the argument of general position. However 3-dimension is very attractive and is hard to research.

We know that any pair of an oriented closed 3-manifold and a link in it induces non-singular flows on the 3-manifold whose closed orbits form the given link. On the other hand, we do not still have the complete solution of Seifert conjecture: A non-singular flows on a 3-sphere always has at least one closed orbit.

Last summer M. Wada [64] showed a neccesary and sufficient condition of links which consist of closed orbits of non-singular Morse-Smale flows on a 3-sphere. The class of such links consists of links obtained from Hopf-links by finite operations — a split sum, connected sum and/or cabling — (of course, further limited).

We shall consider a decomposition of a link into elemental links, which are called "inseparable", "prime" and "simple" corresponding to the above operations.

R. Kirby and W. B. R. Lickorish [27] introduced the method of tangles to prove the primeness of certain knots. R. Myers [35], [36] and T. Soma [57] proved the simplicity of certain links by the same method.

In Section 1, we shall generalize the notion of tangles and obtain a more applicable technics to prove the inseparability, primeness and simplicity of links. For example, we shall easily show that some links have the Brunnian property in Section 2.

We know in [41] that any link is concordant to a prime link with the same Alexander invariant. In Section 3, we shall show that any n-component link ($n \geq 2$) is concordant to a prime link with the same Alexander invariant preserving the knot types of components and the symmetric union of this concordance can be assumed to be ambient isotopic to the direct product of the given link in the 3-sphere and an interval.

F. Hosokawa [21] defined ∇-polynomials of links and characterized them as reciprocal polynomials of even degree. In Section 4, we shall characterize ∇-polynomials of ribbon (slice) links in the weak sense as reciprocal polynomials of even degree.

In Section 5, we shall prove *"For each integer n, there are distinct prime knots with the same n-fold cyclic branched covering space."*

In Section 6, we shall give an inequality between unknotting numbers and another invariants of knots: $0 \leq m(k) \leq sd(k) \leq u(k)$. K. Murasugi's formula in [33]: $0 \leq \frac{1}{2}|\sigma(k)| \leq g^*(k) \leq u(k)$ and the above will be shown to be rough but best possible from examples. In 6.15, we shall negatively answer to [17, Problem 18], i.e. *Is $g^*(k) = g(k')$ for some k' concordant to k?*

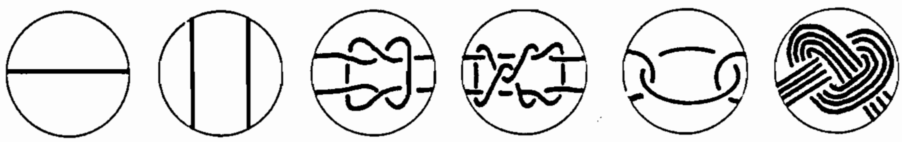
In Appendices, we shall show some observation and questions.

Throughout the paper, we work in the smooth or p.l. category.

1. Tangles

1.1. In this paper, the word *tangle* is borrowed from [8] and is used to mean a pair (B, t) where B is a 3-ball and t is a set of arcs and zero or more loops properly embedded in B, provided that the number of components of t is finite. Two tangles (B_1, t_1) and (B_2, t_2) are said to be *equivalent* if there is a homeomorphism of pairs from (B_1, t_1) to (B_2, t_2).

1.2. We show examples of tangles in Fig. 1.1; (a) is named *the unknotted tangle*, (b) is *the trivial tangle*, (c) is *the clasp* [27], (d) is *the K-T grabber* [5], and (e) is *the chain*.



(a) (b) (c) (d) (e) (f)

Fig. 1.1

1.3. A tangle (B, t) is said to be *prime* (resp. *simple*) if it has the following properties P_1, P_2 and P_3 (resp. P_1, P_2 and P_4):

P_1 (*Inseparable*): The arcs and loops of t cannot be separated by a disc properly embedded in B.

P_2 (*Locally Trivial*): Any 2-sphere in B, which meets t transversely in two points, bounds in B the unknotted tangle.

P_3 (*Indivisible*): Any disc properly embedded in B, which meets t transversely in a single point, divides (B, t) into the unknotted tangle and another.

P_4 (*Atoroidal*): The followings are satisfied;

a: Any annulus properly embedded in B, which does not meet t, is compressible, isotopic to a component of $FrN(t)$, or parallel to $\partial B - \partial t$, in $B - t$,

b: Any torus in B, which does not meet t, is compressible or isotopic to a component of $FrN(t)$ in $B - t$.

1.4. Remark. These definitions are slightly different from those of [5], [27], [29], [35], [36], [41], and [57].

1.5. Proposition. *P_4 implies P_3. If t consists of only two arcs, then P_1 and P_2 imply P_3, and then P_2 and P_4b imply P_4a.*

Proof. Suppose that a tangle (B, t) does not satisfy P_3; there is a disc properly embedded in B, which meets t transversely in a single point and divides (B, t) into two tangles each of which is not the unknotted tangle. Let t' is a component of t which meets this disc. Then $FrN(t')$ does not satisfy P_4a. So P_4 implies P_3.

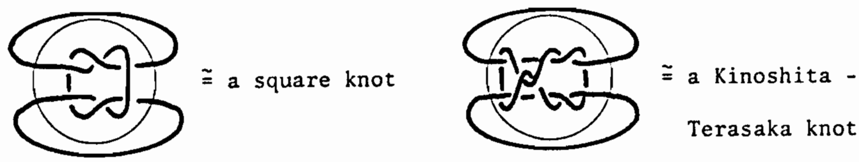
If t consists of two arcs, then any properly embedded disc, meeting t in a single point, divides (B, t) into the unknotted tangle and another. Because the given tangle has only two arcs, one of the tangles divided has only one arc. From locally triviality, it is the unknotted tangle.

Any incompressible annulus R properly embedded in B, which does not meet t, divides ∂B into one annulus R^* and two discs δ_1 and δ_2. If $R^* \cap t \neq \emptyset$, then $(\delta_1 \cup \delta_2) \cap t$ consists of two points. $\delta_1 \cup \delta_2 \cup R$ is a 2-sphere, which meets t transversely in two points. From locally triviality, it bounds the unknotted tangle. Hence R is isotopic to a component of $FrN(t)$. If $R^* \cap t = \emptyset$, then $R \cup R^*$ is a torus in B, which does not meet t. Since t has no loops, this torus is compressible from P_4b. Hence R is isotopic to R^*. This completes the proof.

From the above, a simple tangle is a prime tangle.

1.6. Proposition. *In Fig.* 1.1, (a), (c) *and* (d) *are simple,* (b) *does not satisfy* P_3, (e) *and* (f) *do not satisfy* P_4.

Proof. (a), (b): The proof is obvious. (c), (d): It is easily checked that they are locally trivial from the unknottedness of each arc. If they are separable, then they are equivalent to the trivial tangle. Then they induces two-bridge knots with ears as in Fig. 1.2. But this is false.



(c) with ears (d) with ears

Fig. 1.2

P_4 was checked by Soma [57]. (e): The loop of this tangle is linking each arc, so this tangle is inseparable. It is easily checked that we have P_2 but P_4. (f): We show the primeness of a pair of each two arcs and the ball. If it does not satisfy P_1 or P_2, then there is a ball meeting one arc in a trefoil. But it induces a trivial knot with ears as in Fig. 1.3. Hence it is prime. Each two arcs are inseparable in the ball. So this tangle (f) is indivisible. It is easily checked that P_4 is not satisfied. The proof is complete.

1.7. Remark. In the above argument on (f), "a trefoil" is not essential. Hence parallel arcs on a non-trivial knot in a ball is prime.

≅ a trivial knot

Fig. 1.3

1.8. A link is said to be *prime* (resp. *simple*) if it has the following properties Q_1 and Q_2 (resp. Q_1, Q_2 and Q_3):

Q_1 (*Inseparable*): There is no 2-sphere in S^3 that separates the components of L.

Q_2 (*Locally trivial*): Any 2-sphere in S^3, which meets L transversely in two points, bounds in S^3 the unknotted tangle.

Q_3 (*Atoroidal*): Any torus in S^3, which does not meet L, is compressible or isotopic to a component of $FrN(L)$ in $S^3 - L$.

1.9. Remark. This definition is slightly different from those in [29], [35], [36], [41] and [57]. A trivial knot is prime, and a trivial 2-component link is not prime. A connected sum of two Hopf-links is not simple. A non-trivial torus link is simple.

1.10. Theorem. *Let L be a link in S^3. Suppose that S^2 is a 2-sphere in S^3 meeting L transversely and separating S^3 into two 3-balls A and B. (1) If $(A, A \cap L)$ and $(B, B \cap L)$ are both inseparable tangles, then L is an inseparable link. (2) If $(A, A \cap L)$ and $(B, B \cap L)$ are both prime tangles, then L is a prime link. (3) If $(A, A \cap L)$ and $(B, B \cap L)$ are both simple tangles, then L is a simple link.*

Proof. (1) The proof is contained in the following.

(2) Suppose that F is a 2-sphere in S^3 meeting L transversely in two points; it is required to show that F bounds a ball meeting L in an unknotted arc. The definition of a prime tangle implies that this is true if F is entirely contained in A or in B, so it is in order to proceed by induction on the number of components of $F \cap S^2$.

It may be assumed that F meets S^2 transversely in simple closed curves. Let γ be such a curve innermost on F that bounds a disc δ in F; it may be assumed without loss of generality that F is in A. This γ may be chosen so that $\delta \cap L$ is empty or a single point. If $\delta \cap L$ is empty, from inseparability, δ cannot separate the components of $A \cap L$ and so there is an isotopy, fixed on L, which reduces the number of components of $F \cap S^2$. On the other hand, if $\delta \cap F$ is a single point, from indivisibility, δ divides $(A, A \cap L)$ into the unknotted tangle and another. Then there is an isotopy that reduces the number of components of $F \cap S^2$ and which keeps L set-wise fixed. Hence by induction, F bounds a ball meeting L in an unknotted arc.

Finally, if F is a 2-sphere in S^3 disjoint from L, then by similar argument as above, F may be isotoped into A keeping L fixed. Thus F bounds a ball disjoint from L. So L is a prime link.

(3) Suppose that R is a torus in S^3 disjoint from L; it is required to show that R is compressible or isotopic to a component of $FrN(L)$. The definition of a simple tangle implies that this is true if R is entirely contained in A or in B. So it may be assumed that R meets S^2 transversely in simple closed curves, and furthermore such a curve is essential on R from the similar argument as above.

Since essential simple closed curves on a torus are parallel, they divide the torus into annuli. If one of the annuli is compressible in $A-A\cap L$ or $B-B\cap L$, then R is compressible in S^3-L. If one of the annuli is parallel to $S^2-S^2\cap L$, then there is an isotopy reducing the number of the annuli. Hence it may be assumed that each one of the annuli is isotopic to a component of $FrN(L)\cap A$ or $FrN(L)\cap B$ from the definition of a simple tangle. Then it is easily checked that R is isotopic to a component of $FrN(L)$. Therefore L is a simple link. The proof is complete.

1.11. Remark. A prime link may not be always divided into two prime tangles (see Corollary 1.19). But a prime link can be divided into one trivial tangle and one prime tangle (see Lemma 3.1). It is the similar on a simple link.

1.12. Theorem. *Let (C, v) be a tangle and let D be a disc properly embedded in C that separates (C, v) into two tangles (A, t) and (B, u) and that the number of points of $\partial A-D\cap v$, $\partial B-D\cap v$, $D\cap v$ is greater than one respectively. And let (B, u) be an inseparable tangle.*

(1) *Suppose that, for any disc Δ properly embedded in A with $\Delta\cap\partial D=\emptyset$ and $\Delta\cap t=\emptyset$, Δ does not separate t in A. Then (C, v) is inseparable.*

(2) *Furthermore suppose that, for any disc Δ properly embedded in A such that $\Delta\cap\partial D=\emptyset$ and that $\Delta\cap t$ is a single point, Δ does not divide (A, t) into the unknotted tangle and another. If (B, u) is a prime tangle and (A, t) is a locally trivial tangle, then (C, v) is a prime tangle.*

(3) *Furthermore, if (A, t) is an atoroidal tangle and (B, u) is a simple tangle, then (C, v) is a simple tangle.*

Proof. (1) The proof is essentially contained in the following.

(2) Suppose that F is a 2-sphere in C meeting v transversely in two points. Then, in the similar way as Proof of Theorem 1.10, F may be isotoped into A or into B keeping v fixed. From locally triviality, F bounds a ball meeting v in an unknotted arc.

Suppose that Δ is a disc properly embedded in C, which is disjoint from v or meets v transversely in a single point, it is required to show that Δ does not separate v in C or does not divide (C, v) into the unknotted tangle and another.

First, we consider whether the number of components of $\Delta \cap D$ can be reduced or not. It may be assumed that Δ meets D transversely in simple closed curves and proper arcs. In the similar way as Proof of Theorem 1.10, simple closed curves are reducible. So it may be assumed that $\Delta \cap D$ contains only proper arcs. Let γ be such a curve innermost on Δ i.e. γ separates Δ into two discs, one of which, say δ, satisfies $\delta \cap D = \gamma$. This γ may be chosen so that $\delta \cap v$ is empty. If δ is in B, from inseparability of (B, u), δ cannot separate the components of u and so there is an isotopy, fixed on v, which reduces the components of $\Delta \cap D$.

It may be assumed that δ in A. If δ does not separate t in A, there is an isotopy, fixed on v, which reduces the components of $\Delta \cap D$. It may be assumed that δ separates t in A.

Consider a collection of such discs δ_i's properly embedded in A that $\delta_i \cap D$ has only one component for each i and $\delta_i \cap \delta_j = \emptyset$ for each $i \neq j$ and $\delta_i \cap t = \emptyset$ for each i. Since the number of components of t is finite, there are at most finite number of these discs essentially.

Let $\gamma_i = \delta_i \cap D$ and let γ_1 is innermost on D.

If δ_1 does not separate t in A, it is not essential.

δ_1 separates (A, t) into two tangles (A_1, t_1) and (A_2, t_2).

If they are equivalent to the unknotted tangle, then (C, v) itself is equivalent to (B, u), so is a prime tangle.

If only one of them, say (A_1, t_1), is equivalent to the unknotted tangle, then $(A_1 \cup B, t_1 \cup u)$ is equivalent to (B, u) and so prime.

If they are not equivalent to the unknotted tangle, we assume that γ_1 bounds an innermost disc, $A_1 \cap D$, on D. It is easily checked that $(A_1 \cup B, t_1 \cup u)$ is a prime tangle.

In the last two case, we obtain a new disc D' dividing (C, v) into a prime tangle and another, which satisfies the condition of this Theorem and the number of points of $D' \cap v$ is less than that of $D \cap v$.

Hence, by induction, (C, v) is shown to be a prime tangle.

(3) Supposed that R is an incompressible annulus in C with $R \cap t = \emptyset$. Then, by the similar argument as the above, it may be assumed that $R \cap D$ consists of only essential loops on R, or of only parallel arcs joining the distinct boundary of R. In the former case, the loops divide R into small annuli. If one of the small annuli is compressible, then R is also compressible. If one of the small annuli, R_i, is parallel to $\partial A - \partial t$ in $A - t$, or to $\partial B - \partial t$ in $B - u$, then there is an isotopy reducing the number of the small annuli. (If ∂R_i divides D into two discs and a disc with two holes, then $(\partial A - D) \cap L$ or $(\partial B - D) \cap L$ is empty. This contradicts the assumption. Otherwise, ∂R_i divides D into an annulus and a disc with a hole, or only one component of ∂R_i is on D. In the both cases, a required isotopy can easily be finded.)

From the definition of an atoroidal tangle, it may be assumed that each small annulus is isotopic to a component of $FrN(v) \cap A$ or $FrN(v) \cap B$ Then it is easily checked that R is isotopic to a component of $FrN(v)$.
In the later case, the arcs divides R into small discs δ_i's. If δ_i is parallel to D in $C - v$, then there is an isotopy reducing the number of small discs. So it may be assumed that each small disc is not parallel to D. From the definition of an inseparable tangle, all δ_i's in B are parallel. If the number of δ_i's in B is greater than one, then R has returns as shown in Fig. 1.4.

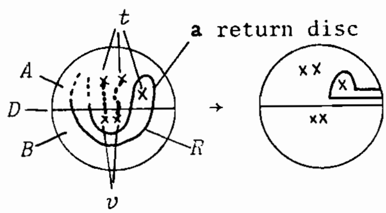


Fig. 1.4

There is an isotopy in $A - t$ carrying this return small disc away not to meet D (see Fig. 1.4). This contradicts the assumption. Hence the number of δ_i's in B is one, and the number of δ_i's in A is one.
From the assumption: for any disc Δ properly embedded in A with $\Delta \cap \partial D = \emptyset$, and $\Delta \cap t = \emptyset$, Δ does not separate t in A; v is in only one part of C separated by R. Hence R is parallel to $\partial C - \partial v$. It is obvious in the case that R is entirely contained in A or B.

Finally, if T is an incompressible torus in $C-v$, then by the similar argument as above, T is isotopic to a component of $FrN(v)$ in $C - v$.

Hence (C, v) is shown to be a simple tangle.

The proof is complete.

1.13. Corollary. *On the condition in the beginning part of Theorem* 1.12, *if both tangles* (A, t) *and* (B, u) *are inseparable* (*resp. prime, simple*), *then* (C, v) *is also inseparable* (*resp. prime, simple*).

1.14. Theorem. *Let* (C, v) *be a cycle sum of* n *tangles* (B_i, t_i)*'s* $(n \geq 3;\ i = 1, 2, \cdots, n)$ *as shown in Fig.* 1.5, *where the number of points* $t_i \cap \partial C$ *is greater than one and each* (B_i, t_i) *is joined to the adjacent tangles by a single arc.*

If (B_i, t_i)*'s are all inseparable (resp. prime, simple), then* (C, v) *is also inseparable (resp. prime, simple).*

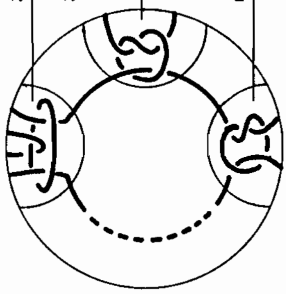


Fig. 1.5

The proof is very similar to the Proofs of Theorems 1.10 and 1.12, so we omit it.

1.15. Let G be a connected graph on a 2-sphere. A graph G is *prime* if it has the following properties:

O_1: Each vertex of G has (non-zero) even degree.

O_2: G has no loop.

O_3: No circle on a 2-sphere meets just two edges transversely each in a single point.

1.16. Theorem. *Let* L *be a link obtained by substituting inseparable (resp. prime, simple) tangles for vertices of a prime graph. Then* L *is an inseparable (resp. prime, simple) link.*

Proof. If the number $|G|$ of the vertices of a given graph G is two, then L is a prime link from Theorem 1.10. So it is in order to proceed by induction on the number $|G|$.

If a graph G has multiedges, then a neighbourhood of multiedges and adjacent vertices satisfies the condition of Corollary 1.13 from the definition of a prime graph. So it is sufficient to show the theorem for G' obtained by identifying the adjacent vertices from G.

If a graph G has no multiedges, then G divides a 2-sphere into many regions, the number of whose surrounding edges is three or more. Let Ω be a region that such number is minimum. Then a neighbourhood of Ω satisfies the condition of Theorem 1.14 from the definition of a prime graph. So it is sufficient to show the theorem for G^* obtained by identifying the vertices surrounding Ω from G.

These operations reduce a given graph to a required prime graph with two vertices. The proof is complete.

1.17. Generating Prime Tangles. Let (B, t) be a prime tangle. And let t^* be the union of t and a parallel arc (or parallel loop) to a component of t in B. Then (B, t^*) is also a prime tangle, except the case (B, t) is the unknotted tangle. (We omit a proof.)

1.18. Theorem. *A tangle (B, t) is prime if and only if the double branched covering space of B branched over (all components of) t is both irreducible and boundary-irreducible.*

The following proof is essentially due to Lickorish [29]. He proved the theorem in the case t consists of two arcs.

Proof. Suppose that (B, t) is a prime tangle and let $p: M \to B$ be the double branched covering spaces of B branched over t. Now, Z_2 acts on M with generator g as the group of covering translations; g is an involution with proper 1-submanifold as its fixed point-set. If M is reducible, there exists a 2-sphere S in M, not bounding a ball, which may be chosen so that either $gS = S$ or $gS \cap S = \emptyset$ by the Z_2 sphere theorem [24]. If $gS \cap S = \emptyset$, then pS is a sphere embedded in $B - t$; then pS bounds a ball in $B - t$ which lifts to a ball in M with S as its boundary, and this cannot exist. If $gS = S$, then $g|S$ is an involution with two fixed points, so pS is a 2-sphere in B meeting t in two points. By the locally triviality of (B, t), it follows that pS bounds the unknotted tangle. The lift of this tangle is a ball with S as its boundary. This is a contradiction and therefore M is irreducible. It remains to check that M is boundary-irreducible. Suppose that M is boundary-reducible, there is a disc D properly embedded in M, with ∂D essential on ∂M, such that either $gD = D$ or $gD \cap D = \emptyset$ (this uses the Z_2 loop theorem [24]). If $gD = D$, then pD is a proper disc in B meeting t in a single point; then pD divides (B, t) into the unknotted tangle and another by the indivisibility of (B, t). Then $\partial(pD)$ bounds a disc in ∂B meeting t in a single point, and this disc lifts to a disc in ∂M with ∂D as its boundary. This contradicts that ∂D is essential on ∂M. If $gD \cap D = \emptyset$, then pD is an embedded disc disjoint from t. As pD cannot separate t in B by the inseparability of (B, t), $\partial(pD)$ bounds a disc in $\partial A - \partial t$, and this lifts a disc in M with ∂D as its boundary. This is again a contradiction, so M is boundary-irreducible.

Conversely, suppose that $p\colon M \to B$ is the double branched covering space of a 3-ball B branched over a tangle t, and that M is irreducible and boundary-irreducible. If (B, t) does not satisfy P_1, there is a disc D separating t in B. Then a lift of D is also a proper disc in M, and a lift of ∂D is essential on ∂M because ∂D separates ∂B into two discs meeting t in two or more even points. This contradicts to the boundary-irreducibility of M. If (B, t) does not satisfy P_2, there is a sphere S meeting t in two points such that S does not bound the unknotted tangle. Then the lift of S is a sphere and does not bound a ball by the solution to the Z_2 Smith conjecture. This contradicts to the irreducibility of M. If (B, t) does not satisfy P_3 but P_2, there is a disc D dividing (B, t) into two tangles, the number of whose arcs are more than two. Then the lift of D is also a proper disc in M, and a lift of ∂D is essential on ∂M because ∂D separates ∂B into two discs meeting t in three or more odd points. This contradicts to the boundary-irreducibility of M. So (B, t) is a prime tangle. The proof is complete.

1.19. Corollary. *A 2-bridge link cannot have a decomposition into prime tangles.*

Proof. If a 2-bridge link has a decomposition into prime tangles, then there is a decomposing sphere S. A sphere S meets t in four or more points, and its lift in the double branched covering space M of the link is a surface F with genus ≥ 1. A surface F separates M into two irreducible and boundary-irreducible manifolds by Theorem 1.18. Hence F is incompressible in M. So the fundamental group $\pi_1(M)$ has $\pi_1(F)$ as a subgroup. But M is a lens space and $\pi_1(M)$ is a cyclic group. This is a contradiction. The proof is complete.

1.20. A link L in $S^1 \times B^2$ is *locally trivial* if any 2-sphere in $S^1 \times B^2$, which meets L transversely in two points, bounds in $S^1 \times B^2$ a ball meeting L in an unknotted spanning arc. The *wrapping number* of L is the minimum number of intersections of L with a meridian disc of $S^1 \times B^2$. If L is a link in $S^1 \times B^2$ and K is a knot in S^3, the L *cable of* K is a link in S^3 formed by mapping $S^1 \times B^2$ into $N(K)$, and considering the image of L under this map (cf. [30]).

1.21. Theorem. (Cf. [30,Theorem 4.2]) *Let L be a link in $S^1 \times B^2$ with wrapping number* > 1, *and let K be any non-trivial knot in S^3. The L cable of K is prime if and only if L is locally trivial in $S^1 \times B^2$.*

The proof is parallel to the argument in [30], so we omit it.

1.22. Remark. On the above, if K is not prime, the L cable of K has a decomposition into prime tangles from 1.7 and 1.12.

1.23. Remark. The author does not know whether the followings are true or not. "A torus link does not have a decomposition into prime tangles." "A link, which have a decomposition into prime tangles, does not have a two-generator group-presentation."

2. Prime Links and Brunnian Property

2.1. If a link is non-trivial, yet every proper sublink is trivial, we say that it has the *Brunnian property* or that it is a *Brunnian link.*

2.2. A Brunnian link is easily shown to be a prime link. Conversely, if a link, whose proper sublinks are all trivial links, is inseparable, it is a Brunnian link. In the following examples, it is easily checked that every proper sublink is trivial. So, by showing it inseparable (prime), we know them to be Brunnian links.

2.3. The *Borromean rings*, as in Fig. 2.1, is a well-known Brunnian link. It has a decomposition into prime tangles; the chain (see 1.6), as in Fig. 2.1. From Theorem 1.10, it is prime, and so is a Brunnian link.

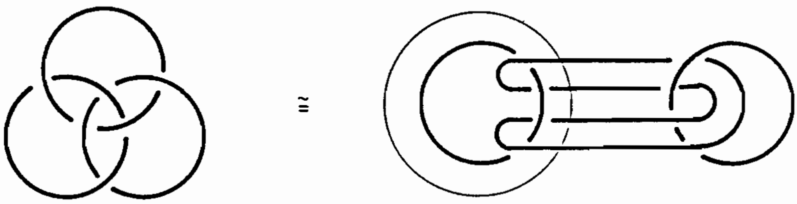


Fig. 2.1

2.4. In the above, each component is homotopically linked from the remaining components. The followings are not so.

2.5. Consider the links ${}_1L_n$ and ${}_2L_n$ as in Fig. 2.2 where the subscripts n means the number of components. They have a decomposition into prime tangles; the clasp or the K - T grabber (see 1.6), as in Fig. 2.2 respectively. From Theorems 1.10 and 1.12, they are prime links, and so are Brunnian links.

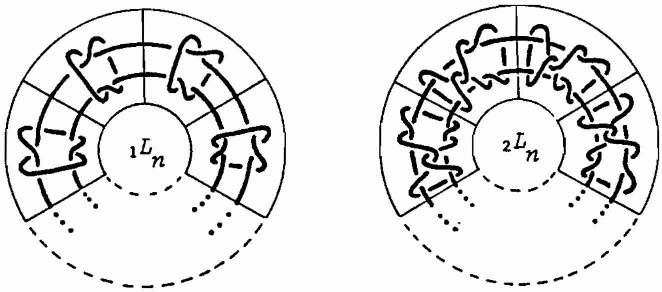


Fig. 2.2

2.6. Consider the link $_3L_n$ as in Fig. 2.3, where the subscript n means the number of components. $_3L_2$ is equivalent to $_1L_2$. Here we show that $_3L_n$ $(n \geq 4)$ has a decomposition into prime tangles.

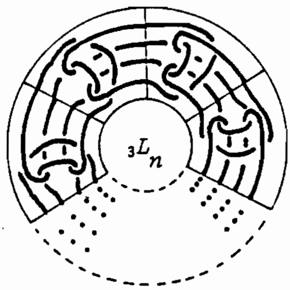
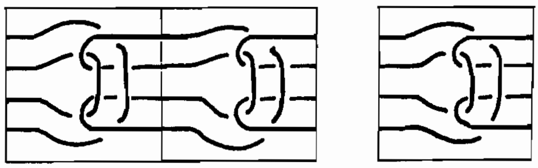
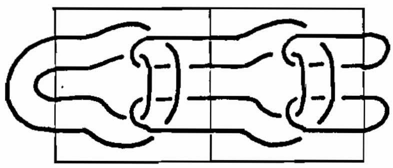


Fig. 2.3



(i) (ii)

Fig. 2.4



(i) with ears

Fig. 2.5

Consider the tangle (i) in Fig. 2.4. Suppose that an arc is separable from the remainder, then the remainder itself is completely separable, and so the tangle itself is completely separable. Hence it induces a link separated from the loop with ears as in Fig. 2.5. But this is false.

Since the tangle, which is obtained by excluding the loop from (i), is completely separable and the tangle (i) itself is inseparable, the tangle (i) is locally trivial from the unknottedness of each arc.

Suppose that a disc, which meets an arc transversely in a single point, divides the tangle (i) into two tangles which are not the unknotted tangle. The tangle divided without the loop is completely separable from the above argument, so the arcs in this tangle divided are separable if they do not meet the disc. This contradicts the first claim.

Hence the tangle (i) is a prime tangle.

The tangle (ii) in Fig. 2.4 satisfies the condition of (A, t) in Theorem 1.12. So, the sum of (i) and (ii) is a prime tangle from Theorem 1.12.

Hence ${}_3L_n$ $(n \geq 4)$ has a decomposition into prime tangles.

3. Primeness and Alexander Invariants of Concordant Links

3.1. Lemma. (Cf. [5], [27], [29]) *Let* L *be a link in* S^3. *Then there exists, embedded in* S^3, *a 2-sphere meeting* L *transversely in four points and separating* S^3 *into two 3-balls* A *and* B *such that*

(*i*) $(A, A \cap L)$ *is equivalent to the trivial tangle,*

(*ii*) $(B, B \cap L)$ *is a prime tangle.*

Proof. We know the existence and uniqueness of prime decomposition of inseparable links by Y. Hashizume [18]. Hence there is a locally trivial decomposition of a link without uniqueness. First, we show the above for locally trivial links. Let denote by $n(L)$ the number of components of a link L.

In the case $n(L) = 1$, a locally trivial knot is a prime knot. If it is a trivial knot, we take a 2-sphere as in Fig. 3.1. If it is not a trivial knot, there is two arcs whose meridians are not homotopic in the exterior of the knot. We take a 2-sphere that bounds the trivial tangle whose arcs are just the above arcs. From the locally triviality of the knot, the exterior (B', t') of the trivial tangle is a locally trivial tangle. We illustrate it as in Fig. 3.2. It may be deformed into Fig. 3.3. Since they satisfy the condition in Theorem 1.12, (B, t) in Fig. 3.3 is prime.

Fig. 3.1

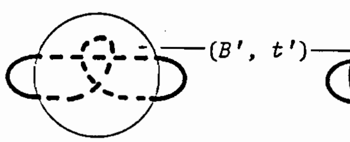


Fig. 3.2

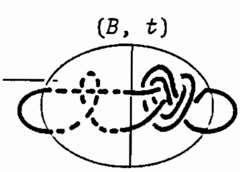


Fig. 3.3

In the case $n(L) = 2$, a locally trivial 2-component link which is not prime is only a trivial 2-component link. If L is a trivial 2-component link, we take a 2-sphere as in Fig. 3.4. If L is not so, it is a prime link. If L has a knotted component, we take a 3-ball A in S^3 such that $(A, A \cap L)$ is equivalent to the trivial tangle and that A contains a subarc of each component of L. Let B be the exterior of A. We will show that $(B, B \cap L)$ is a prime tangle. Locally triviality is immediate as L is locally trivial. $B \cap L$ contains a knotted arc from the assumption. This fact and locally triviality imply inseparability. From Proposition 1.5, $(B, B \cap L)$ is a prime tangle. If L has no knotted components, we take a 3-ball as similar to the above. $(B, B \cap L)$ is easily checked to be a locally trivial tangle. We illustrate it as in Fig. 3.5. It may be deformed into Fig. 3.6. Since they satisfy the condition in Theorem 1.12, (B^*, t^*) in Fig. 3.6 is a prime tangle.

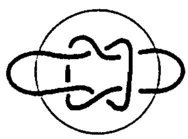


Fig. 3.4

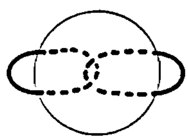


Fig. 3.5

(B^*, t^*)

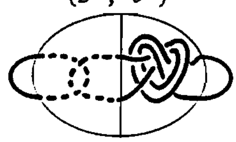


Fig. 3.6

In the case $n(L) \geq 3$, a locally trivial link is a prime link. We take a 3-ball as similar to the above. We will show that $(B, B \cap L)$ is a prime tangle. Locally triviality is immediate as L is locally trivial. Suppose that a proper disc in B separates $B \cap L$ in B. From locally triviality, it separates arcs. But the part containing loops includes one arcs and loops, so this part is not locally trivial. This is a contradiction. Hence $(B, B \cap L)$ is inseparable and so is prime.

For each locally trivial factor of a given link, we can construct a prime tangle as above, and a sum of these prime tangles with the trivial tangle reproduces the given link in S^3. From Theorem 1.12, this sum is a prime tangle, too. The proof is complete.

3.2. Lemma. ([41, Lemma 2]) *Let* $L' = L \cup k$ *be a link, where* k *is a trivial knot. Suppose that* k *has linking number* 0 *with each component of* L *and that* k *bounds a ribbon disc in* $S^3 - L$. *We denote by* L^* *a link which is obtained from* L *by a surgery along* k *in* S^3. *Then both* L *and* L^* *have the same Alexander invariants.*

3.3. Lemma. *The substitution the K-T grabber for the trivial tangle on a regular projection of a link does not change the concordance class and the Alexander invariant.*

Proof. From a surgical description of the K-T grabber as in Fig. 3.7, which is deformed to Fig. 3.8, we see this surgery curve bounds a ribbon disc in the complement of the tangle in the 3-ball. From Lemma 3.2, this substitution does not change the Alexander invariant. Ribbon move at β in Fig. 3.7 shows that the K-T grabber is concordant to the trivial tangle.

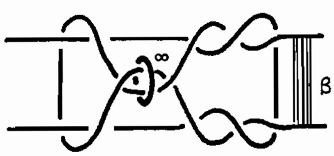


Fig. 3.7

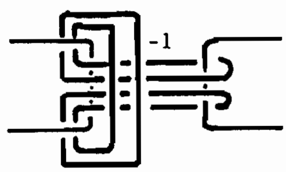


Fig. 3.8

3.4. Lemma. *The symmetric union of the above concordance with equator the K - T grabber is ambient isotopic to (the trivial tangle) × I.*

Proof. The K-T grabber and the band β in Fig. 3.7 can be regarded as in Fig. 3.9. The above concordance is a ribbon concordance obtained by a fission along the band β. We obtain β' from β by interchanging over- and under- crossings of the band as in Fig. 3.10. The symmetric union of concordance obtained by a fission along β and that along β' are ambient isotopic (cf. [58, Proposition 2.15]). Fig. 3.10 is deformed to Fig. 3.11. It is easily checked that the symmetric union is ambient isotopic to the direct product. This completes the proof.

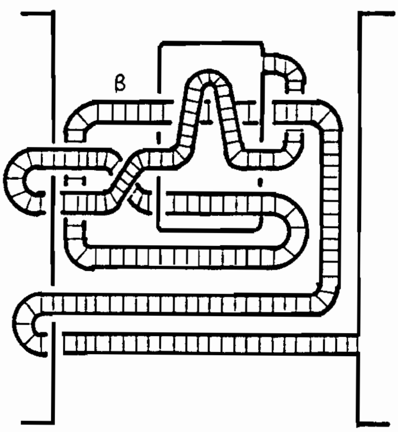


Fig. 3.9

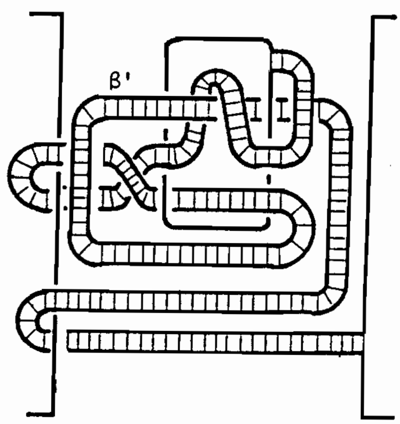


Fig. 3.10

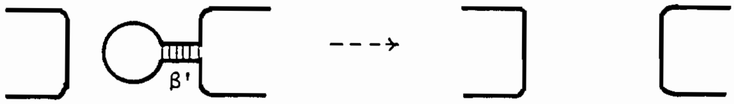


Fig. 3.11

3.5. Theorem. *Any n-component link L is concordant to a prime link L^* with the same Alexander invariant and the symmetric union of this concordance can be assumed to be ambient isotopic to* $(S^3, L) \times$ *(an interval). Furthermore, if n is greater than one, this concordance can be assumed to preserve the knot types of components of a link L.*

Proof. By Lemma 3.1, we have a tangle decomposition of a link in S^3. From Lemma 3.3, we substitute the K - T grabber for $(A, A \cap L)$, and obtain a new link, which is concordant to the given link and has the same Alexander invariant. From Lemma 3.4, the symmetric union of this concordance is ambient isotopic to the direct product. From Theorem 1.10, this link is a prime link. Since each arc of the K - T grabber is unknotted, the concordance preserves the knot types of components if $n \geq 2$. The proof is ended.

3.6. In the case $n = 1$; *"Any knot is concordant to a prime knot with the same Alexander polynomial"* was proved by S. A. Bleiler [5].

3.7. Corollary. *Any set of surfaces embedded in a 4-space has the cross-section which is a prime link up to ambient isotopy.*

3.8. Corollary. (With [69, Theorem 2.8]) *Any ribbon 2-knot has a prime equatorial knot.*

Proof. T. Yanagawa [69] proved *"Any ribbon n-knot* $(n \geq 2)$ *has an equatorial knot."* Since the above concordance preserves ribbon property, we obtain a required one by applying the argument in this section.

3.9. Corollary. *For any knot K, there is a prime knot K′ with the same Alexander invariant such that K # K′ is doubly null-cobordant.*

Proof. It is well-known that $K \# K^*$ is doubly null-cobordant where K^* means the reflection of K. Apply the argument in this section to K^*, and we obtain a required one.

3.10. Note. There is another construction to show Theorem 3.5 without preserving the knot types of components as follows.

It is known that any link is represented as a closed braid. Its wrapping number, denoted by n, may be assumed to be greater than one. Consider the $(n, 0)$-cable link of a Kinoshita-Terasaka knot. We obtain a new link from the above closed braid and the above $(n, 0)$-cable link by natural complete fusion as illustrated in Fig. 3.12. On the analogy of Lemmas 3.3 and 3.4, we know this new link to be a required link by Theorem 1.21.

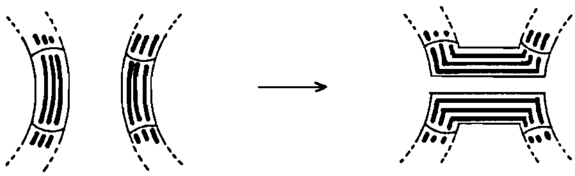


Fig. 3.12

3.11. Note. Soma [57] proved that "Any link is concordant to a simple link with the same Alexander invariant". Here we show another construction. A link has a link diagram L (cf. 6.1). By regarding crossings as vertices, we consider L as a graph on a 2-sphere. Up to ambient isotopy, it can be assumed that L is a prime graph. By substituting vertices of L for the K-T grabbers suitably, we have a required link. From Theorem 1.16, the link is a simple link. From Lemma 3.3, the link has the same Alexander invariant.

4. ∇-Polynomials of Ribbon Links in the Weak Sense

4.1. A link is a *ribbon link in the weak sense* iff it bounds an immersed oriented surface of genus 0 whose singuralities are all ribbon-types, or equivalently, iff it introduces a ribbon knot by only fusions.

The *∇-polynomial* $\nabla(t)$ of a link is $\Delta(t_1, \cdots, t_n) / (t-1)^{n-2}$ where $\Delta(t_1, \cdots, t_n)$ is the Alexander polynomial of a given link and n is the number of components.

4.2. Theorem. *For any reciprocal polynomial of even degree $f(t)$, there exists an n-component prime ribbon link in the weak sense whose ∇-polynomial is equal to $f(t)$ for an arbitrally integer $n \geq 2$.*

Proof. Adding the following condition (*) to Hosokawa's construction in [21], we obtain a ribbon link in the weak sense.

(*) $a_2 \cup a_4 \cup \cdots \cup a_{2h}$ is a trivial link.

Because we can obtain a ribbon knot from the link by fusions along B_{2h+i}'s $(1 \leq i \leq n-1)$.

We can construct a prime link from the above link as in Section 3. Since ribbon concordance preserves ribbon property in the weak sense, this link is a required one. This completes the proof.

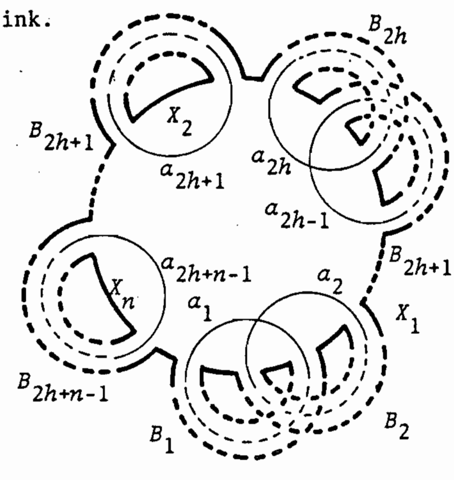


Fig. 4.1

4.3. Corollary. *∇-polynomials of ribbon links in the weak sense are characterized as reciprocal polynomials of even degree. Those of slice links in the weak sense are also characterized as the same.*

Proof. F. Hosokawa characterized ∇-polynomials of links as reciprocal polynomials of even degree in [21]. A ribbon link in the weak sense is one of slice links in the weak sense, and is one of links. From Theorem 4.2, those ∇-polynomials are characterized as the above.

5. Prime Knots with the Same Cyclic Branched Covering Spaces

5.1. Preliminaries. Let $L = J \cup K$ be a 2-component link such that J and K are trivial knots with $lk(J, K) = 1$. Let denote by $\Sigma_n(K)$ (resp. $\Sigma_n(J)$) the n-fold cyclic branched covering space of K (resp. J), and let denote by J_n (resp. K_n) the lift of J (resp. K) in $\Sigma_n(K)$ (resp. $\Sigma_n(J)$). Since K and J are trivial, $\Sigma_n(K)$ and $\Sigma_n(J)$ are homeomorphic to S^3, J_n and K_n are also knots in S^3.

The n-fold cyclic branched covering spaces of J_n and K_n are homeomorphic to the $Z_n \oplus Z_n$ - covering space of L, and so J_n and K_n have the homeomorphic n-fold cyclic branched covering spaces.

If $L = J \cup K$ is not interchangeable, i.e. there is no homeomorphism of S^3 mapping J to K, then J_n and K_n may have distinct knot types.

5.2. Examples. ([39], [49]) Consider the link 9^2_{34}, 9^2_{35}, or 9^2_{42} in the table of [47, Appendix C]. Let denote by $L = J \cup K$ one of them. J_n and K_n are known to have distinct knot types from the fact: their polynomials are distinct (see, for example, Sakuma [51]).

5.3. Primeness of J_n and K_n in the case 9^2_{34}. 9^2_{34} is illustrated in Fig. 5.1. Let denote by $\tilde{J}_i$ (resp. $\tilde{K}_i$) an i-tuple of a fundamental region of the infinite cyclic covering space of K (resp. J). We consider them as tangles. $\tilde{J}_1$, $\tilde{J}_3$, $\tilde{K}_1$, and $\tilde{K}_3$ is illustrated in Fig. 5.2.

First we show that $\tilde{J}_3$ is a prime tangle. $\tilde{J}_3$ is considered as a sum of the tangles (i) and (ii) in Fig. 5.3. We denote the tangle (i) by $(B, t_1 \cup t_2 \cup t_3)$ as in Fig. 5.3. The arc t_1 has knotted prime factors 3_1 and 6_3 in B. The arcs t_2 and t_3 are unknotted in B.

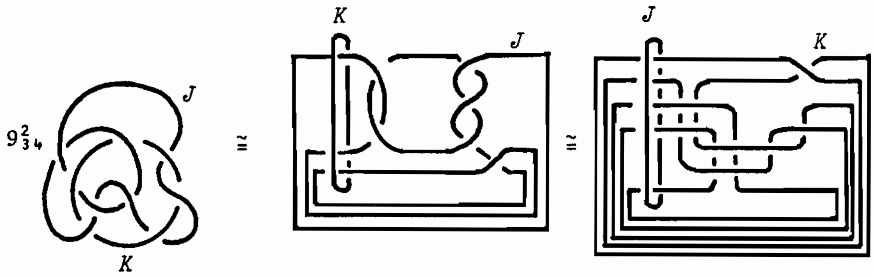


Fig. 5.1

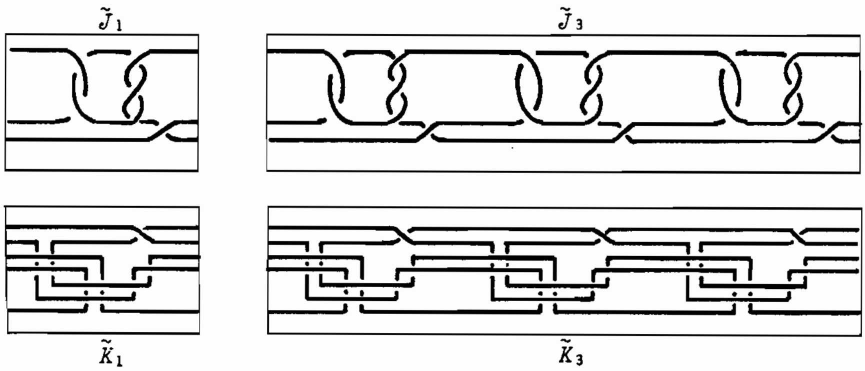
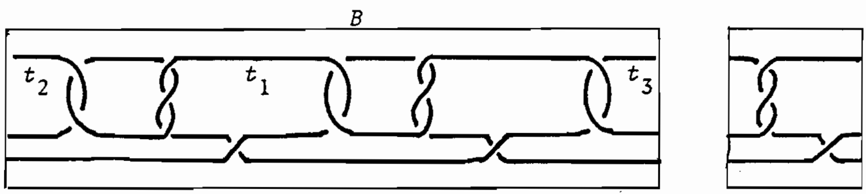


Fig. 5.2



(i) (ii)

Fig. 5.3

5.4. Lemma. *Let* $(B, t_1 \cup t_2)$ *be a tangle such that at least one of* t_1 *and* t_2 *is non-trivial in* B. *If* $(B, t_1 \cup t_2)$ *with ears is a prime knot and has no knotted factor of* t_1 *nor* t_2, *then this tangle is prime. Here* (B, t) *with ears means a unoin of* (B, t) *and the trivial tangle.*

Proof. If $(B,\ t_1 \cup t_2)$ does not satisfy locally triviality or inseparability, then $(B,\ t_1 \cup t_2)$ with ears must have at least one knotted factor of t_1 or t_2 from non-triviality of t_1 or t_2. This contradicts the assumption. Hence $(B,\ t_1 \cup t_2)$ is a prime tangle.

5.5. Lemma. *Let* $(B,\ t_1 \cup t_2)$ *be a tangle such that at least one of* t_1 *and* t_2 *is non-trivial in* B. *If* $(B,\ t_1 \cup t_2)$ *with ears does not have all knotted factors of* t_1 *and* t_2, *then this tangle is inseparable.*

The proof is obvious, so we omit it.

5.6. Lemma. *Let* $(B,\ t_1 \cup t_2 \cup t_3 \cup \cdots \cup t_n)$ *be a tangle such that* t_1 *is a non-trivial arc and* t_i*'s* $(2 \le i \le n)$ *are unknotted arcs.*
If $(B,\ t_1 \cup t_2)$ *is a prime tangle and if* $(B,\ t_1 \cup t_i)$ $(3 \le i \le n)$ *are inseparable tangles and at least one of knotted factors of* t_1 *is locally trivial in* $(B,\ t_1 \cup t_i)$ *respectively for each* i, *then this tangle is a prime tangle.*

Proof. Locally triviality and inseparability is immediate.

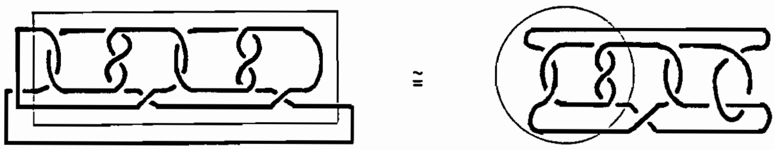
If a disc D meeting t in a single point divides $(B,\ t = t_1 \cup \cdots \cup t_n)$ into two tangles which are not the unknotted tangles, then D must meet only with t_1 from inseparability of $t_1 \cup t_i$ $(2 \le i \le n)$ in B. We denote by B_1 and B_2 two tangles divided, such that B_2 contains t_2. By the assumption, there is t_j $(3 \le j \le n)$ in B_1. Since $(B,\ t_1 \cup t_2)$ is prime, $(B_1,\ t_1 \cap B_1)$ is the unknotted tangle. Then all knotted factors of t_1 are not locally trivial in $(B,\ t_1 \cup t_j)$. This contradicts the assumption. Hence $(B,\ t_1 \cup \cdots \cup t_n)$ is a prime tangle.

5.7. Proposition. *The tangle in Fig.* 5.4 *is a prime tangle.*

Fig. 5.4

Proof. A union of this tangle and the trivial tangle is a trivial knot which has no knotted factor of the arcs. From 5.4, this tangle is a prime tangle.

5.8. Continued from 5.3. $(B, t_1 \cup t_2)$ with ears as in Fig. 5.5 is a prime knot from 1.10 and 5.7, and has no knotted factor of t_1. From 5.4, this tangle is a prime tangle. $(B, t_1 \cup t_3)$ with ears as in Fig. 5.6 is a knot which does not have a knotted factor 6_3 of t_1. From 5.5, this tangle is an inseparable tangle. From 5.6, the tangle (i) is a prime tangle.



11_{443} (21, 2+)(3, 2-) in [8] with $\Delta(t)=t^6-3t^5+5t^4-5t^3+5t^2-3t+1$

Fig. 5.5

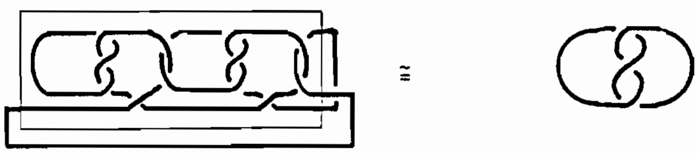


Fig. 5.6

$\tilde{J}_3$: a sum of the tangles (i) and (ii) in Fig. 5.3 satisfies the condition in Theorem 1.12, and so is a prime tangle.

$\tilde{J}_4$: a sum of $\tilde{J}_1$ and $\tilde{J}_3$ satisfies the condition in Theorem 1.12, and so is a prime tangle. In the same way, we can show that $\tilde{J}_n$ $(n \geq 3)$ is a prime tangle.

Since the knot J_n may be considered as a union of $\tilde{J}_i$ and $\tilde{J}_j$ for $i+j=n$, J_n $(n \geq 6)$ are prime knots from Theorem 1.10.

J_2, J_3, J_4, and J_5 have a decomposition into prime tangles as in Fig. 5.7. Hence they are prime knots from Theorem 1.10.

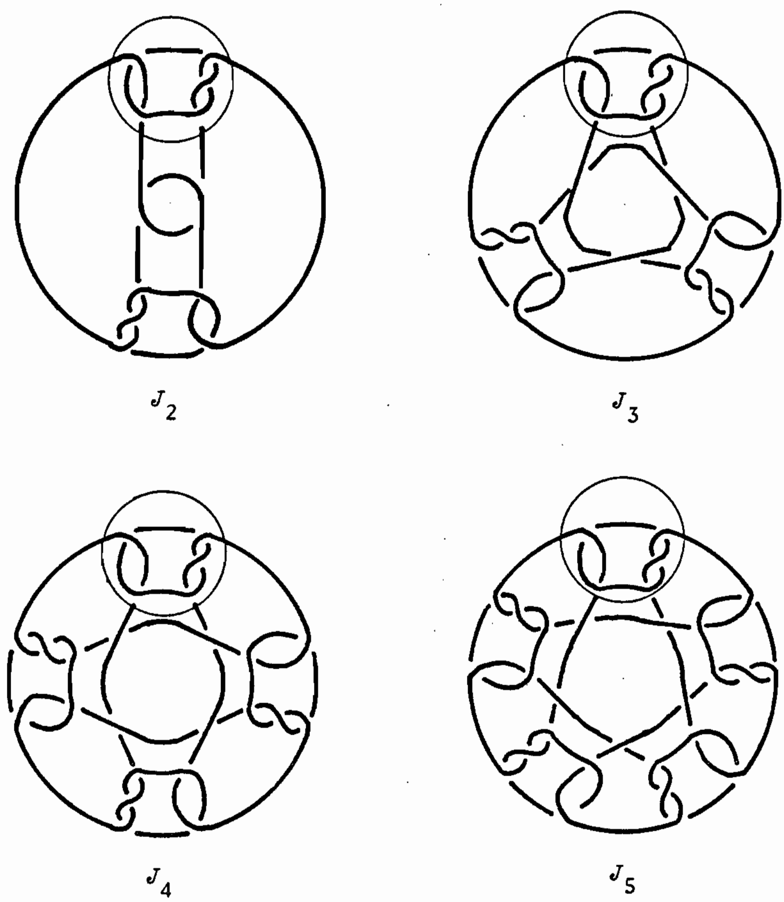


Fig. 5.7

Next we show that $\tilde{K}_3$ is a prime tangle. We denote $\tilde{K}_3$ by $(B, t = t_1 \cup t_2 \cup t_3 \cup t_4 \cup t_5)$ as in Fig. 5.8. t_1 is 4_1 and t_i $(i = 2,3,4,5)$ are unknotted arcs in B. $(B, t_1 \cup t_i)$ $(i = 2, 3, 4, 5)$ with ears as in Fig. 5.9 have no knotted factor of t_1 and are prime knots. From Lemma 5.4, these tangles are prime tangles. From Lemma 5.6, (B, t) is a prime tangle.

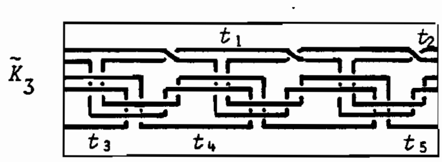


Fig. 5.8

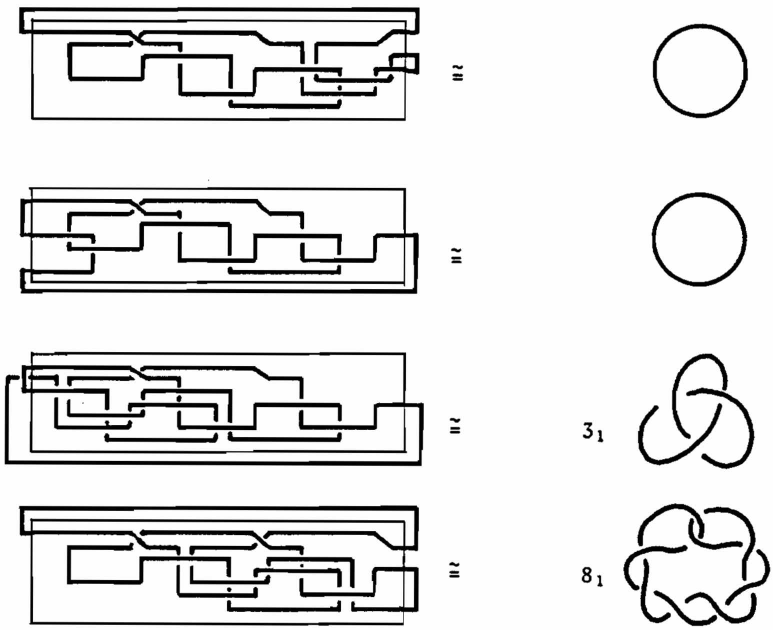


Fig. 5.9

$\tilde{K}_4$: a sum of $\tilde{K}_1$ and $\tilde{K}_3$ satisfies the condition in Theorem 1.12, and so is a prime tangle. In the same way, we can show that $\tilde{K}_n$ $(n \geq 3)$ is a prime tangle.

Since the knot K_n may be considered as a union of $\tilde{K}_i$ and $\tilde{K}_j$ for $i + j = n$, K_n $(n \geq 6)$ are prime knots from Theorem 1.10.

K_2 is a doubled knot of a trefoil knot with 6-twists as illustrated in Fig. 5.10, and so is a prime knot from Theorem 1.14.

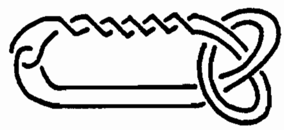


Fig. 5.10

The author does not know whether K_3, K_4, and K_5 have a decomposition into prime tangles or not. So the proof of their primeness is complicated, and we omit it here.

5.9. In the case 9^2_{35}: 9^2_{35} is illustrated in Fig. 5.11.

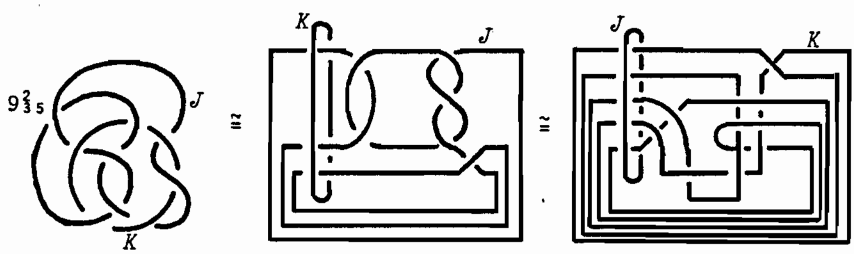


Fig. 5.11

In a similar way as 9^2_{34}, $\tilde{J}_n$ and $\tilde{K}_n$ $(n \geq 3)$ are prime tangles. Hence J_n and K_n $(n \geq 6)$ have a decomposition into prime tangles. Also J_2, J_3, J_4, and J_5 have a decomposition into prime tangles. K_2 is a doubled knot of a trefoil knot with -6-twists.

5.10. In the case 9^2_{42}. 9^2_{42} is illustrated in Fig. 5.12.

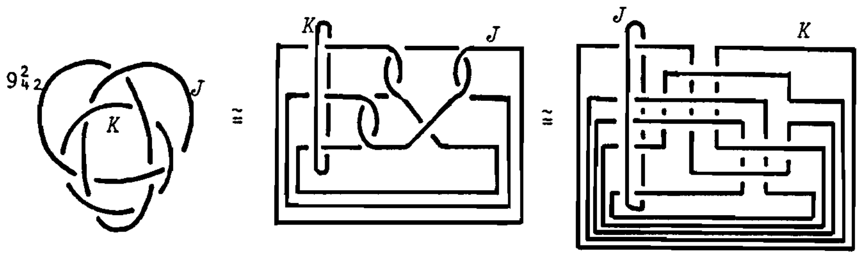


Fig. 5.12

In a similar way as 9^2_{34}, $\tilde{J}_n$ and $\tilde{K}_{n+2}$ $(n \geq 1)$ are prime tangles. Hence J_n and K_{n+4} $(n \geq 2)$ have a decomposition into prime tangles.

5.11. Theorem. *For each integer n, there exists distinct prime knots whose n-fold cyclic branched covering spaces are homeomorphic.*

The argument in this section proves the above except $n = 3, 4, 5$.

5.12. Remark. Sakuma showed the same result as Theorem 5.11 in [51].

5.13. Notes. (1) Theorem 5.11 is an answer to a question of Viro [63].
(2) In [1] it was announced that Gordon-Litherland had given a similar construction.
(3) Livingston [31] announced that the r-fold branched cyclic covering space of the (p, r)-cable of the (q, r)-torus knot and the (q, r)-cable of the (p, r)-torus knot are homeomorphic. This also proves Theorem 5.11.

(4) These examples were exhibited in the Master Theses of Sakuma [49] and the author [39], without proving the primeness of them.

6. Unknotting Numbers

6.1. Let k be an oriented tame knot in a 3-sphere S^3, and let K be a diagram i.e. the image under a regular projection of S^3 into S^2. On all diagrams representing k, the minimum number of exchanges of over- and under- crossings required to deform k into a trivial knot is called the *unknotting number* of k, denoted by $u(k)$, and the minimum number of crossings is called the *crossing number* of k, denoted by $cr(k)$ [45].

6.2. The minimum genus of Seifert surface of k is called the 3-*genus* of k, denoted by $g(k)$ ([15], [54]). Fox defined the 4-*genus* of k, denoted by $g^*(k)$ ($h^*(k)$ in [11]).

6.3. **Proposition.** ([11]) *For any knot* k, *we have*

(1. 1) $$0 \le g^*(k) \le g(k).$$

6.4. The *signature* of k, denoted by $\sigma(k)$, is known to be an invariant of the knot type [62]. Murasugi showed the following in [33].

6.5. **Proposition.** ([33]) *For any knot* k, *we have*

(1. 2) $$0 \le \frac{1}{2}|\sigma(k)| \le g^*(k) \le u(k).$$

6.6. A *surgical description* of k is as follows [46].

Let k be a knot in S^3, there are disjoint solid tori $T_1, \cdots, T_n$ in $S^3 - k$ and a homeomorphism $h: S^3 - ({}^oT_1 \cup \cdots \cup {}^oT_n) \to S^3 - ({}^oT_1 \cup \cdots \cup {}^oT_n)$ so that

(i) $h(k)$ is unknotted in S^3,

(ii) the T_i's are unknotted and pairwise unlinked,

(iii) $lk(T_i, k) = lk(T_i, h(k)) = 0$ for all i,

(iv) $h(\partial T_i) = \partial T_i$ and $lk(\mu_i', T_i) = 1$, where μ_i is a curve on T_i meridinal to T_i and $\mu_i' = h(\mu_i)$.

The minimum number of these solid tori is called the *surgical description number* of k, denoted by $sd(k)$.

6.7. The *minor index* of k is the minimum size among all square Alexander matrices of k, denoted by $m(k)$, provided that $m(k) = 0$ iff an Alexander matix of k is equivalent to $|1|$ as presentation matrices.

6.8. Theorem. *For any knot* k, *we have*

$$0 \le m(k) \le sd(k) \le u(k). \tag{1. 3}$$

Proof. A surgical view of Alexander matrices shows the existence of an Alexander matrix of k with size $sd(k)$. We can see that an exchange of an over- and under- crossing is realized by surgery arround the crossing. The proof is complete.

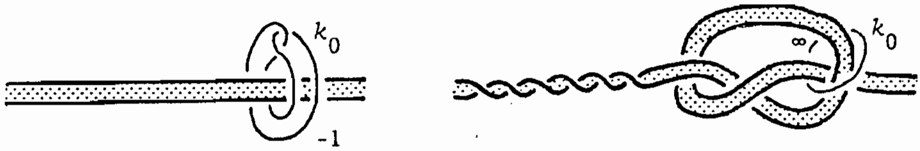
6.9. Proposition. ([54]) *For any knot* k, *we have*

$$0 \le (\text{the degree of } \Delta_k(t)) \le 2g(k), \tag{1. 4}$$

where $\Delta_k(t)$ *is the Alexander polynomial of* k.

6.10. Lemma. *In Fig.* 6.1, *the parallel strings are subarcs of a knot and the shaded area between them is a part of a Seifert surface of this knot respectively. Deformation* (a) *to* (b) *preserves* $\sigma(k)$ *and* $m(k)$.

The proof is obvious, hence we omit it (cf. [42]).

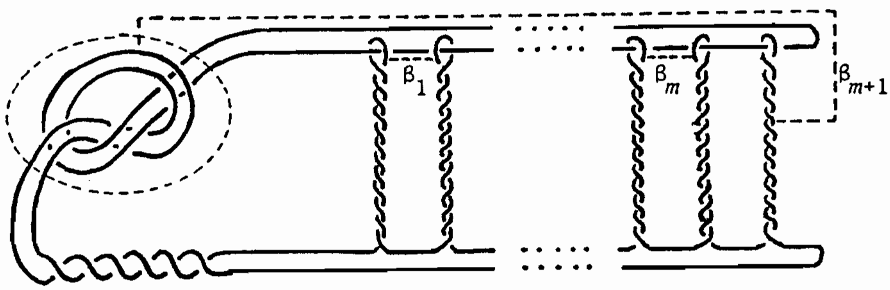


(a) (b)

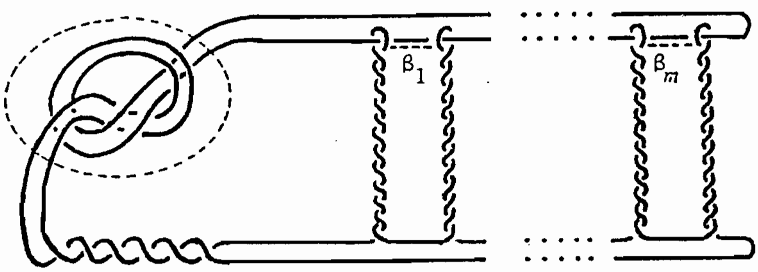
Fig. 6.1

6.11. Example. *For any positive integer* n, *there is a prime knot* k *such that* $g^*(k) = 0$ *and* $u(k) = n$.

By Lemma 6.10, the knot k as in Fig. 6.2 has the same minor index as a connected sum of n copies of the -6-twisted knot i.e. $m(k) = n$. With observation on the diagram, we have $m(k) = n$ from (1. 3). On the other hand, ribbon moves at $\beta_1, \cdots, \beta_{[\frac{n+1}{2}]}$ show that k is slice.



when $n = 2m+1$ for some integer m



when $n = 2m$ for some integer m

Fig. 6.2

In a similar way we can see the fact: *For any pair of positive integers* $m \leq n$, *there is a prime knot* k *such that* $g^*(k) = m$ *and* $u(k) = n$.

6.12. Example. *For any integer* n, *there is a prime knot* k *such that* $g^*(k) = u(k) = sd(k) = n$.

By Lemma 6.10, the knot k as in Fig. 6.3 has the same minor index and the signature as a connected sum of n copies of the right-hand trefoil knot i.e. $m(k) = \frac{1}{2}|\sigma(k)| = n$. With observation on the diagram, we have $u(k) = g^*(k) = sd(k) = n$ from (1. 2 and 3).

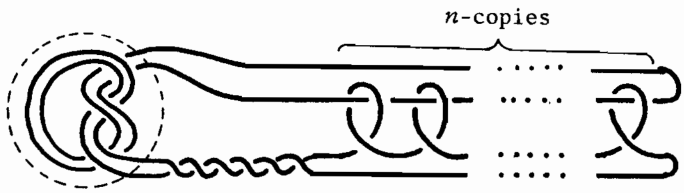


Fig. 6.3

In Examples 6.11 and 12, the interior and the exterior of a dotted line in the figures are prime tangles respectively. From Theorem 1.10, the above knots are prime.

6.13. Example. *For any positive integer* n, *there is a prime knot* k *such that* $m(k) = 1$ *and* $u(k) = n$.

The $(2, 2n+1)$-torus knot k has $\frac{1}{2}|\sigma(k)| = u(k) = n$ [33]. A group of the torus knot has a one-relator group-presentation. Fox's free differential calculus [10] shows $m(k) = 1$.

6.14. Example. *For any positive integer* n*, there is a prime knot* k *such that* $u(k) = 1$ *and* $g(k) = n$.

The knot k as in Fig. 6.4 has the Alexander polynomial with degree $2n$. With observation on the diagram, we have $g(k) = n$ from (1. 4).

We remark this knot is fibered.

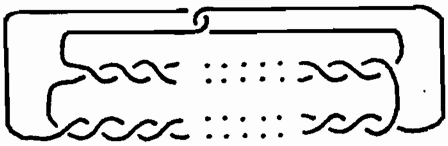


Fig. 6.4

In Examples 6.13 and 14, the above knots are 2-bridge, so are prime.

From above arguments, we know (1. 2 and 3) are rough but best possible.

6.15. Remark. If $m(k) = 0$, then $\sigma(k) = 0$. So we cannot decide the unknotting number of such a knot by (1. 2 and 3).

6.15. *For any positive integer* n, *there is a prime knot* k *such that* $n = g^*(k) < g(k')$ *for any* k' *concordant to* k.

Kondo [28] and Sakai [48] showed "*For any knot polynomial* $\Delta(t)$, *there is a knot* k *with* $u(k) = 1$ *and* $\Delta_k(t) = \Delta(t)$".

Hence, for any positive integer i, there is a knot k_i with $u(k_i) = 1$ and $\Delta_{k_i}(t) = t^{2i} - t^{2i-1} + t^{2i-2} - t^{2i-3} \cdots +1$. Since all roots of this polynomial are mutualy distinct roots of unity, we see $\sigma(k_i) \equiv 2i \pmod 4$ from Milnor's arguments in [32]. Hence $|\sigma(k_i)| = 2$ if i is odd, from $u(k_i) = 1$ and (1. 2). Let m be an odd integer greater than $n + 1$.

There is a connected sum of k_m and $(n-1)$ copies of k_1 (or its mirror image), denoted by k^*, with $|\sigma(k^*)| = 2n$, from the additivity of the signature. Since we easily see $u(k^*) \leq n$, we have $g^*(k^*) = n$ from (1. 2). On the other hand, we know $\Delta_{k^*}(t) = \Delta_{k_m}(t)\cdot\{\Delta_{k_1}(t)\}^{n-1}$. Since the Alexander polynomials of concordant knots are unique up to $f(t)f(t^{-1})$-factors [13] where $f(t)$ is a Laurent polynomial with $f(1) = \pm 1$, and $\Delta_{k_m}(t)$ has no $f(t)f(t^{-1})$-factors, the degree of $\Delta_{k^{*\prime}}(t)$ for any $k^{*\prime}$ concordant to k^* is not smaller than $2(m-1)$. From (1. 4), the genus of $k^{*\prime}$ is not smaller than $m-1$, so is greater than n. Futhermore we can regard k^* as a prime knot from Theorem 3.5.

Thus we have obtained a required knot.

Appendix A. Fibered Ribbon Knots

There is a conjecture that a fibered ribbon 1-knot always introduces a fibered ribbon 2-knot. This is partially answered by Yoshikawa [71], but is still open. We know the fact: Certain ribbon 1-knots introduce many distinct ribbon 2-knots, in [44]. We show here that some fibered ribbon 1-knots introduce distinct fibered ribbon 2-knots.

Theorem. *There is a fibered ribbon 1-knot k which introduces distinct fibered ribbon 2-knots associated with k.*

Proof. Let k be a connected sum of 8_{20} and its mirror image (see Fig. A.1). The knot k has distinct ribbon presentations as in Fig. A. 2. Let K_1 and K_2 be ribbon 2-knots associated with each ribbon presentation. Then K_1 is the spun knot of 8_{20} and K_2 is the spun knot of the connected sum of a trefoil knot and its mirror image. Here we know that 8_{20} is a fibered 1-knot and that the spun knot of a 1-knot is a fibered 2-knot. So the condition are satisfied. It is sufficient to show that K_1 and K_2 are distinct. The minor index of them are 1 and 2. The proof is complete.

Remark. It is the same in the case $k = 10_{140} \# r(10_{140})$, etc.

Question. Can the fibering structure of k be extended to those of K_1 and K_2 in the above case? In other words, K_i in the upper space is a (2, 4)-ball pair. Is it a fibered (2, 4)-ball pair?

Question. For a prime knot k, does a similar situation occur?

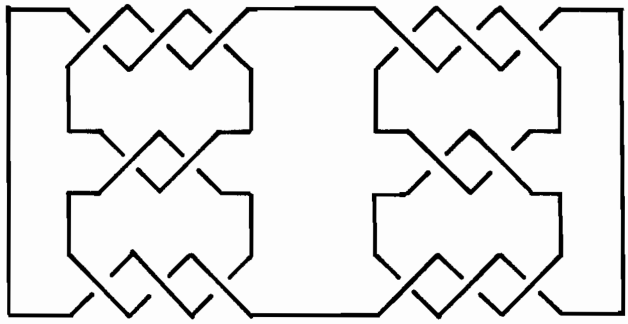


Fig. A.1

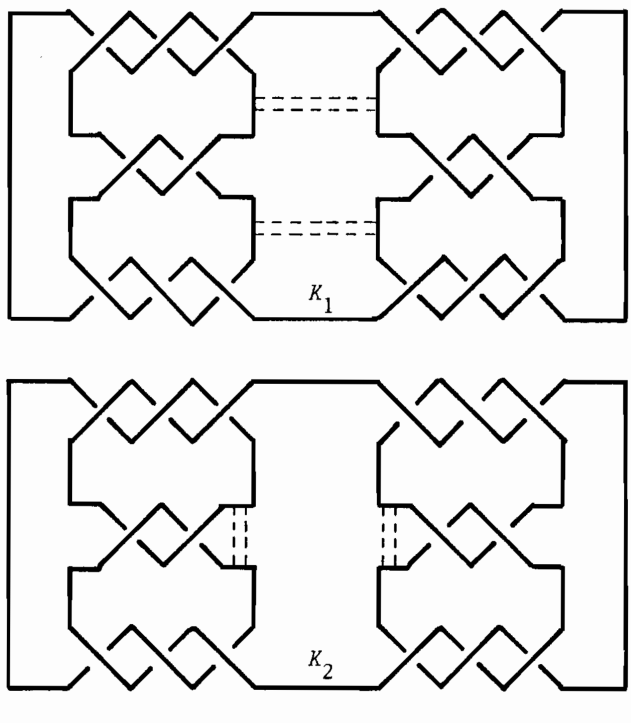


Fig. A.2

Appendix B.

Unknotting Numbers and Knot Diagrams

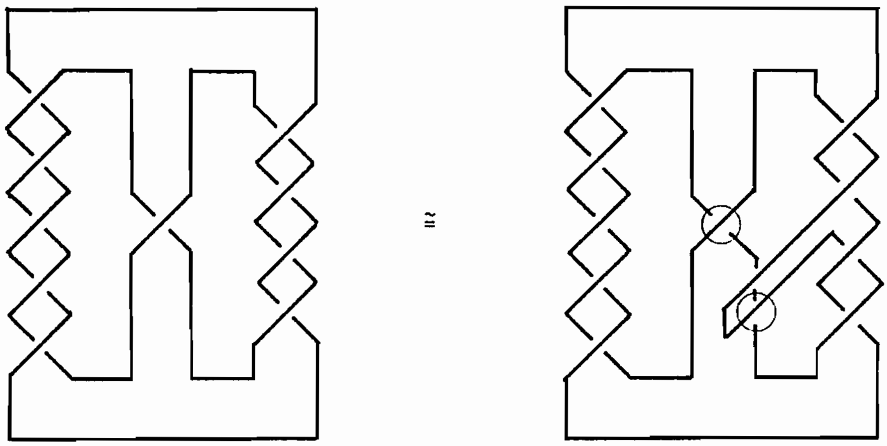
The unknotting number $u(k)$ of a knot k is defined to be the minimum number of exchanges of over- and under- crossings required to deform k into a trivial knot over all knot diagrams representing k.

If we limit "all knot diagrams" to "all knot diagrams with the minimum crossings", we cannot obtain the unknotting number. In fact, we consider the knot in Fig. B.1 (it is named 514 by Conway [8]). The knot diagram of 514 with the minimum crossings is Fig. B.1 (i) and is unique. (Its uniqueness can be shown by Conway's method [8].) In Fig. B.1 (i), at least three exchanges of crossings are required to deform the knot into a trivial knot. But at most two exchanges of crossings in Fig. B.1 (ii) deform it into a trivial knot.

Here we raise the following problem:

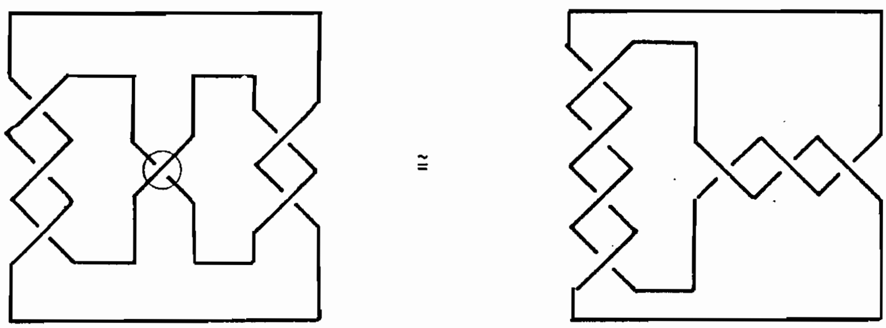
Problem. Let K be a knot diagram of a knot k with the minimum crossings. Can we deform K into K' with $u(k') < u(k)$ by one exchange of a crossing?

In the above, it is important that K is a diagram with the minimum crossings. We consider the knot in Fig. B.2 (named 312). In Fig. B.2 (ii), we cannot deform it into a trivial knot by *one* exchange of a crossing. But its unknotting number is one as shown in Fig. B.2 (i).



(i) (ii)

Fig. B.1



(i) (ii)

Fig. B.2

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