General Solutions of Painleve Equations(I)～(V) and Nonlinear 2-Systems without Poincare's Condition at an Irregular singular point

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DOCTORAL DISSERTATION

GENERAL SOLUTIONS OF PAINLEVÉ EQUATIONS (I) ∿ (V) AND NONLINEAR 2-SYSTEMS WITHOUT POINCARÉ'S CONDITION AT AN IRREGULAR SINGULAR POINT

パンルベ方程式（I）～（V）及びポアンカレ条件をみたさない２連立非線形微分方程式系の不確定特異点における一般解

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INTRODUCTION

This paper studies general solutions at an irregular singular point of Painlevé differential equations and nonlinear 2-systems relating to them.

As is well known, Painlevé equations were discovered by P. Painlevé and B. Gambier in order to define new transcendental functions. Since then, the new transcendents, called Painlevé transcendents, have been a subject of many investigations. However, the research so far done seems to be far from completion. Furthermore it has been known that Painlevé transcendents now frequently appear in physics such as in quantum field theory and gravitational field theory ([9],[19]).

The author is interested in studying the fixed singular points of Painlevé equations, in particular, in constructing analytic expressions of general solutions near the fixed singular points, for the movable singular points are known to be merely poles. There are two types of fixed singular points: one is regular type and the other irregular type. The general solutions of Painlevé equations at their fixed singular points of regular type were obtained by several authors ([15],[10],[18]). Concerning their singular points of irregular type, K. Takano ([17]) first obtained an analytic expression of a general solution for Painvevé (V) by using the idea of M. Iwano ([5]). The analogous result was also obtained by the present author for Painlevé (IV). Recently, S. Shimomura ([16]) proved that the formal general solution due to K. Takano converges without the additional assumption introduced by him.

In this paper we shall first establish a general theory for constructing a general solution of a certain 2-system of nonlinear differential equations with a fixed singular point of irregular type, and then apply this theory to all Painlevé equations at each irregular singular point. The general theory is described in Chapter I and its application in Chapter II.

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CONTENTS

CHAPTER I.

A GENERAL SOLUTION OF 3-SYSTEM (E).

§ 1. Statement of Main Results in Chapter I.

As is shown in Chapter II, every Hamiltonian system associated with each Painlevé equation can be reduced to a system of the form

$$(1.1)\quad x^{\sigma+1}dy_j/dx = \{(-1)^{j-1}\lambda(x) + \alpha_j x^\sigma\}y_j + f_j(x,y_1,y_2),\ j=1,2$$

at each fixed singular point of irregular type by a biholomorphic transformation. Here σ is a positive integer, x, y_1, y_2 are complex variables, $\lambda(x)$ is a polynomial of x of which the degree is less than σ, $\lambda(0)\neq 0$, and $f_j(x,y_1,y_2)$'s are bounded holomorphic functions of (x,y_1,y_2) for

$$(1.2)\quad 0 < |x| < r_0,\ |\arg x - \sigma^{-1}\arg\lambda(0) - (1/2 + m)\pi/\sigma| < \pi - \varepsilon_0$$

$$(1.3)\quad |y| = \max(|y_1|,|y_2|) < \rho_0$$

with

$$(1.4)\quad f_j(x,y_1,y_2) = O(|y|^2),\ j=1,2$$

m being an integer with $0\leq m\leq 2\sigma-1$, and the coefficients of their Taylor expansions in y_1 and y_2 at $y_1=y_2=0$ admit asymptotic expansions in powers of x as $x\to 0$ in (1.2), the symbol O in (1.4) denotes Landau's symbol. We note that (1.1) is a system having a singularity of irregular type at $x=0$ which does not satisfy Poincaré's condition. The purpose of this chapter is to obtain a bounded general solution of (1.1) in a subdomain of (1.2) satisfying the condition

$$(1.5)\quad y_1y_2 = O(x^\sigma),$$

for it is strongly expected that the solution of (1.1) with the property $y_j \to 0$ $(j=1,2)$ as $x \to 0$ satisfies this condition. We notice that, for any preassigned value $a_0 \neq 0$, system (1.1) can be reduced to a system of the same form with $\lambda(0)=a_0$ by a scale transformation of x. Remark that the constants α_j's are invariant under this transformation. Therefore, we can suppose that $\lambda(0)=\sigma$ and $\arg\lambda(0)=0$. We denote the sector (1.2) by $S_m(\varepsilon_0, r_0)$.

By using Iwano's symbol $\mathbb{1}(\cdot)$, system (1.1) can be written in a vector form as

$$(1.1) \qquad x^{\sigma+1}dy/dx = (\lambda(x)\mathbb{1}(1,-1) + x^{\sigma}\mathbb{1}(\alpha))y + f(x,y),$$

where y, α and f are 2 dimensional column vectors, and, for a vector α, $\mathbb{1}(\alpha)$ denotes a diagonal matrix of which the j-th diagonal component is the j-th component of α. The symbol $\mathbb{1}(\cdot)$ is frequently used in this paper. We use the usual notation, for example, for a multiple index $k=(k_1,k_2,\cdots)$ and a column vector $y={}^t(y_1,y_2,\cdots)$, we put

$$(1.6) \qquad \begin{aligned} |k| &= k_1 + k_2 + \cdots, \quad y^k = y_1^{k_1} y_2^{k_2} \cdots, \\ |y| &= \max(|y_1|, |y_2|, \cdots), \end{aligned}$$

and $k \geq k'$ means $k_j \geq k_j'$ for every j.

By a similar method as in [17], we see that (1.1) can be reduced to

$$(1.7) \qquad \begin{aligned} x^{\sigma+1}dz/dx = &\{\lambda(x)\mathbb{1}(1,-1) + x^{\sigma}\mathbb{1}(\alpha) + z_1z_2\mathbb{1}(\alpha'(x))\}z \\ &+ (z_1z_2)^2\mathbb{1}(z)g(x,z) \end{aligned}$$

by a holomorphic transformation of the form

$$(1.8) \qquad y = z + \sum_{|k|\geq 2} p_k(x)z^k.$$

Here, $p_k(x)$'s and $\alpha'(x)$ are vector functions holomorphic and bounded in $S_m(\varepsilon_0,r)$ admitting asymptotic expansions in powers of x as $x \to 0$ in $S_m(\varepsilon_0,r)$, the series of the right side of (1.8) is absolutely and uniformly convergent for

$$x \in S_m(\varepsilon_0,r), \quad |z| < \rho \tag{1.9}$$

and g has the same properties as f in (1.9), provided that r, $\rho > 0$ are sufficiently small. The existence of such a holomorphic transformation will be proved at the end of this chapter.

In order to construct a general solution of (1.7) satisfying the condition analogous to (1.5), we introduce a new variable z_3 given by

$$z_3 = x^{-\sigma} z_1 z_2. \tag{1.10}$$

Then, by putting

$$\begin{aligned} w &= {}^t(z_1,z_2,z_3),\ h(t,w) = {}^t(g_1(x,z),g_2(x,z),g_1(x,z)+g_2(x,z)) \\ \beta &= {}^t(\alpha_1,\alpha_2,\alpha_1+\alpha_2-\sigma),\ \beta'(x) = {}^t(\alpha_1'(x),\alpha_2'(x),\alpha_1'(x)+\alpha_2'(x)), \end{aligned} \tag{1.11}$$

we obtain the 3-system

$$\begin{aligned} x^{\sigma+1} dw/dx = \{&\lambda(x)\mathbb{1}(1,-1,0) + x^{\sigma}\mathbb{1}(\beta) + x^{\sigma} w_3 \mathbb{1}(\beta'(x))\}w \\ &+ x^{2\sigma} w_3{}^2 \mathbb{1}(w) h(x,w). \end{aligned} \tag{1.12}$$

Thus, for any solution ${}^t(z_1(x),z_2(x))$ of (1.7), the vector function ${}^t(z_1(x),z_2(x),z_3(x))$ with (1.10) becomes a solution of (1.12). Conversely, it can be verified that if a solution $w = {}^t(z_1(x),z_2(x),z_3(x))$ of the system (1.12), when z_1, z_2, z_3

considered as dependent variables, satisfies the relation (1.10) at some point $x=x_0$, then the $z = {}^t(z_1(x),z_2(x))$ represents a solution of the system (1.7). Therefore, in order to have a general solution of (1.7), it is sufficient to obtain a 2-parameter family of solutions of (1.12) satisfying (1.10) at some point $x=x_0$. Here, we remark that by making a simple transformation, we can suppose $\beta'(x)$ in (1.12) is equal to a constant vector $\beta' = \lim_{x\to 0, x\in S} \beta(x)$. Thus we have arrived at our starting point of this chapter: we would like to construct a formal general solution of (1.12) and prove its convergence.

In order to avoid unnecessary complication, we consider a 3-system of more general form than (1.12). The system which we are going to study is of the form

$$\text{(E)} \qquad x^{\sigma+1}dy/dx = \{\lambda(x)\mathbb{1}(1,-1,0) + x^{\sigma}\mathbb{1}(\alpha) + x^{\sigma}y_3\mathbb{1}(\alpha')\}y + x^{\sigma+1}\mathbb{1}(y)f(x,y).$$

Here, we assume:

(i) σ is a positive interger,

(ii) $\lambda(x)$ is a polynomial of the form

$$\text{(1.13)} \qquad \lambda(x) = \sigma + \sum_{i=1}^{\sigma-1} a_i x^i,$$

(iii) $y = {}^t(y_j)$, $f = {}^t(f_j)$ are 3-vectors and $\alpha = {}^t(\alpha_j)$, $\alpha' = {}^t(\alpha'_j)$ are constant 3-vectors,

(iv) $f(x,y)$ is a vector function holomorphic in a domain $D_m(\varepsilon_0,r_0,\rho_0)$ in the (x,y)-space defined by

$$\text{(1.14)} \qquad x \in S_m(\varepsilon_0,r_0),\ |y| < \rho_0$$

with

(1.15) $$f(x,y) = O(|y|^2),$$

where $S_m(\varepsilon_0, r_0)$ is a sector in the x-plane defined by

(1.2) $$0 < |x| < r_0, \quad |\arg x - (1/2 + m)\pi/\sigma| < \pi - \varepsilon_0,$$

(v) the coefficients $f_k(x)$'s of the Taylor expansion of f,

(1.16) $$f(x,y) = \sum_{|k|\geq 2} f_k(x) y^k,$$

have asymptotic expansions in powers of x as $x \to 0$ in $S_m(\varepsilon_0, r_0)$.

The first theorem we obtain is

<u>Theorem 1</u> (formal transformation). Assume that $k_1(\alpha_1+\alpha_2)+k_3\alpha_3$ is not a negative integer for all nonnegative integers k_1, k_3 such that $2k_1+k_3 \geq 2$. Then there exists a formal transformation

(T) $$y = \mathbb{1}(Y)\left(\sum_{k\geq 0} p_k(x) Y^k\right)$$

with $p_0(x) = {}^t(1,1,1)$ and $p_k(x) = 0$ ($|k|=1$), which reduces system (E) to the system

(E') $$x^{\sigma+1} dY/dx = \{\lambda(x)\mathbb{1}(1,-1,0) + x^\sigma \mathbb{1}(\alpha) + x^\sigma Y_3 \mathbb{1}(\alpha')\} Y.$$

Here, every $p_k(x)$ is a holomorphic vector function in $S_m = S_m(\varepsilon_0, r_0)$ having asymptotic expansions in powers of x as $x \to 0$ in S_m with

(1.17) $$p_k(x) = O(x^{(1-\delta(k))\sigma+1}), \quad |k| \geq 2,$$

where $\delta(k) = \delta_{k_1 k_2}$, $\delta_{k_1 k_2}$ being Kronecker's delta.

We can obtain the general solution of (E') by quadratures. In particular, in the case $\alpha_3 = \alpha_3' = 0$, we have

$$(1.18) \qquad \begin{aligned} Y_j(x) &= C_j e^{(-1)^j \Lambda(x)} x^{\alpha_j + \alpha_j' C_3}, \quad j=1,2, \\ Y_3(x) &= C_3, \end{aligned}$$

and, in the case $\alpha_3 \neq 0$, we have

$$(1.19) \qquad \begin{aligned} Y_j(x) &= C_j e^{(-1)^j \Lambda(x)} x^{\alpha_j} Q_j(Y_3(x)), \quad j=1,2, \\ Y_3(x) &= C_3 x^{\alpha_3} / \{1 - (\alpha_3'/\alpha_3) C_3 x^{\alpha_3}\}, \end{aligned}$$

where

$$(1.20) \qquad \Lambda(x) = -\int_{\infty}^{x} x^{-\sigma-1} \lambda(x) dx = 1 + \sum_{i=1}^{\sigma-1} a_i x^i / (\sigma - i),$$

$$(1.21) \qquad Q_j(\eta) = \begin{cases} \{1 + (\alpha_3'/\alpha_3)\eta\}^{\alpha_j'/\alpha_3'} & (\alpha_3' \neq 0) \\ e^{(\alpha_j'/\alpha_3)\eta} & (\alpha_3' = 0), \end{cases}$$

$j=1,2$, and C_1, C_2, C_3 are integral constants. Let $Y(x)$ be the general solution of (E'), then, by substituing $Y = Y(x)$ in the right side of (T), we obtain a formal general solution of (E). The following theorem is the main theorem of this paper.

Theorem 2. Assume that (a) $\alpha_3 = \alpha_3' = 0$, $\mathrm{Re}(\alpha_1 + \alpha_2) > 0$, or (b) $\alpha_3 \neq 0$, $\mathrm{Re}\,\alpha_3 \geq 0$, $\mathrm{Re}(\alpha_1 + \alpha_2) > 0$ holds. Then, for every ε with $\varepsilon > \varepsilon_0$, there exist sufficiently small r and $\rho > 0$ such that the power series $\sum_{k \geq 0} p_k(x) Y^k$ in (T) converges absolutely and uniformly in $D_m = D_m(\varepsilon, r, \rho)$ and represents there a holomorphic bounded vector function.

Remark 1. Each assumption (a) or (b) implies the assumption in Theorem 1.

Remark 2. Combining Theorems 1 and 2 and observing the behavior of the general solution of (E'), we obtain a bounded general solution of (E) under assumption (a) or (b).

Now we consider the 2-system (1.7). By applying the above theorems to the system, we have

Theorem 3. Assume that (a') $\alpha_1+\alpha_2-\sigma=0$, $\alpha_1'+\alpha_2'=0$ or (b') $\alpha_1+\alpha_2-\sigma\neq 0$, $\mathrm{Re}(\alpha_1+\alpha_2-\sigma)\geq 0$ holds. Then, for every ε with $\varepsilon > \varepsilon_0$, there exists a general solution $z = \Psi(x,Y(x))$ of (1.7) with the properties that:

(i) $\Psi(x,Y)$ is a holomorphic vector function of (x,Y) in $D_m(\varepsilon,r,\rho)$, and has there a uniformly convergent expansion in powers of Y of the form

$$\Psi(x,Y) = \mathfrak{I}(Y_1,Y_2)\Big(\sum_{k\geq 0} q_k(x)Y^k\Big), \tag{1.22}$$

r and ρ being small positive numbers. Here, $q_0(x) = {}^t(1,1)$, and every $q_k(x)$ is a holomorphic vector function in $S_m=S_m(\varepsilon,r)$ having asymptotic expansions in powers of x as $x\to 0$ in S_m with

$$q_k(x) = O(x^{(1-\delta(k))\sigma+1}), \quad |k| \geq 1. \tag{1.23}$$

(ii) $Y(x) = {}^t(Y_1(x),Y_2(x),Y_3(x))$ is the vector function given by (1.18) or (1.19) according as (a') or (b') holds with

$$C_3 = C_1C_2. \tag{1.24}$$

In order to obtain this theorem, we have only to prove that (1.24) is equivalent to (1.10) at some point $x=x_0$, which is verified as follows. For every solution $w=w(x)$ of (1.12) and every solution $Y=Y(x)$ of its simplified system, it holds that

$$d/dx \log\{z_1(x)z_2(x)/(x^\sigma z_3(x))\} = d/dx \log\{Y_1(x)Y_2(x)/(x^\sigma Y_3(x))\} = 0.$$

Hence we have

$$z_1(x)z_2(x)/(x^\sigma z_3(x)) \equiv K\cdot Y_1(x)Y_2(x)/(x^\sigma Y_3(x)) \equiv K\cdot C_1C_2/C_3$$

with some constant K. By taking the limit along the curve $\Gamma(x_0)$ defined in § 4 or § 7 and by using (1.23) we obtain $K = 1$.

We shall prove Theorems 1 and 2 only for $m=0$ and denote S_0 and D_0 simply by S and D, respectively.

In § 2, we prove Theorem 1. In order to show Theorem 2, we derive the so-called truncated systems of differential equations and prove a fundamental existence lemma (hereafter called Fundamental Lemma) for such systems. In § 3, we first state Fundamental Lemma and next show how to prove Theorem 2 by using this lemma. In §§ 4, 5 and 6, we prove Fundamental Lemma in detail in the case where assumption (a) is satisfied, because the case corresponds to Painlevé equations. The proof of Fundamental Lemma under assumption (b) is simply shown in § 7.

In order to show Fundamental Lemma by our method, we have to determine a path $\Gamma(x_0)$ of integration and the so-called stable domain $\mathcal{D}$ associated with (E'). We determine $\Gamma(x_0)$ and $\mathcal{D}$ by the pull back of a curve and a stable domain in the case $\sigma=1$, and then we have to consider the inverse mapping $x = \Xi(\omega)$ of the mapping

$$\omega = 1/\Lambda(x) = x^{\sigma}/\{1 + \sum_{i=1}^{\sigma-1} a_i x^i/(\sigma-i)\},$$

where the branch of Ξ is determined by the relation that arg $x \to \pi/(2\sigma)$ as $\omega \to 0$, $\arg\omega = \pi/2$. It is convenient to represent $x = \Xi(\omega)$ as

$$x = \omega^{1/\sigma}\xi(\omega^{1/\sigma}) \tag{1.25}$$

where $\xi(\mu)$ is holomophic at $\mu=0$ with $\xi(0)=1$.

§ 2. Proof of Theorem 1.

In this section, we shall prove Theorem 1. We denote by $A(\varepsilon_0, r_0)$ the set of all functions or vector functions of x which are holomorphic and bounded in a sector $S(\varepsilon_0, r_0)$, and having asymptotic expansions in powers of x as x tends to the origin in $S(\varepsilon_0, r_0)$.

<u>2.1.</u> We use the following lemmas which are special versions of a more general theorem due to M. Hukuhara ([1]). We notice that $S(\varepsilon_0, r_0)$ contains a singular direction of the equation in Lemma 1.

<u>Lemma 1.</u> Consider a linear differential equation of the form

$$x^{\sigma+1}dp/dx = (\sum_{i=0}^{\sigma} a_i x^i)p + b(x)$$

where σ is a positive integer and a_i's are complex constants. If a_0 is a non-zero real number and $b \in A(\varepsilon_0, r_0)$, then the equation has a unique solution $p \in A(\varepsilon_0, r_0)$.

Lemma 2. Consider a linear differential equation of the form

$$x\, dp/dx = a_0 p + b(x),$$

where a_0 is a complex constant. If $b \in A(\varepsilon_0, r_0)$ and if there exists a formal power series solution of x , then there exists a unique solution $p \in A(\varepsilon_0, r_0)$ of which the asymptotic expansion coincides with the formal power series solution.

2.2. Recall that (1.16) is the Taylor expansion of f in powers of y . Then in order that formal transformation (T) reduces (E) to (E'), it is necessary and sufficient that the equation

$$(2.1)\qquad \begin{aligned} &\sum_{k\geq 0} [x^{\sigma+1} dp_k/dx + \{(k_1 - k_2)\lambda(x) + k\cdot\alpha x^{\sigma}\} p_k] Y^k \\ &= - \sum_{k\geq e_3} (k - e_3)\cdot\alpha' x^{\sigma} p_{k-e_3} Y^k \\ &\quad + x^{\sigma} Y_3 \Big(\sum_{|k'|\geq 2} p_{3k'} Y^{k'}\Big) \mathbb{1}(\alpha') \Big(\sum_{k''\geq 0} p_{k''} Y^{k''}\Big) \\ &\quad + x^{\sigma+1} \mathbb{1}\Big(\sum_{k'\geq 0} p_{k'} Y^{k'}\Big) \sum_{|m|\geq 2} f_m Y^m \Big(\sum_{k''\geq 0} p_{k''} Y^{k''}\Big)^m \end{aligned}$$

holds as formal power series of the Y. Here we denote by $k\cdot\alpha$ the inner product

$$(2.2)\qquad k\cdot\alpha = \sum_{j=1}^{3} k_j \alpha_j,$$

and by p_{3k} the third component of p_k, and put $e_3 = (0,0,1)$, $0 = (0,0,0)$. We see that constant terms on both sides of (2.1) are equal to zero. Next we can verify that if $p_k(x) = 0$ ($|k|=1$) the coefficients of Y^k ($|k|=1$) on both sides of (2.1) are equal to zero. By equating the coefficients of Y^k ($|k|\geq 2$) on both sides of (2.1), we have

$$(2.3)_k \quad x^{\sigma+1}dp_k/dx + \{(k_1 - k_2)\lambda(x) + k\cdot\alpha x^\sigma\}p_k = x^\sigma R_k + x^{\sigma+1}R_k' ,$$

where each component of R_k is a polynomial of components of $p_{k'}$'s with $0\leq|k'|\leq|k|-1$ of which each monomial has a component of some $p_{k'}$ with $|k'|\geq 2$ as its factor, and each component of R_k' is a polynomial of components of $p_{k'}$'s and $f_{k'}$'s with $0\leq|k'|\leq|k|-2$.

We shall show that $(2.3)_k$'s can be solvable successively. For every k with $|k|=2$, we see $R_k = 0$ and then the right hand side of $(2.3)_k$ is of order $O(x^{\sigma+1})$. If $k_1 \neq k_2$, then, by Lemma 1, we can determine uniquely $p_k \in A=A(\varepsilon_0,r_0)$ with $p_k = O(x^{\sigma+1})$. If $k_1 = k_2$, then $(2.3)_k$ becomes

$$x\,dp_k/dx + \{(k_1 + k_2)\alpha_1 + k_3\alpha_3\}p_k = b(x)$$

where $b \in A$, $b = O(x)$. From the assumption of Theorem 1, it follows that the equation has a unique formal solution of the form $\sum_{i\geq 1} p_{k,i}x^i$ with $p_{k,i} \in C^3$. Therefore, by Lemma 2, we can obtain a unique solution $p_k \in A$ with $p_k = O(x)$.

Suppose that for every k' with $|k'|\leq N$ ($N\geq 2$), we have determined $p_{k'} \in A$ satisfying (1.17). Then, for every k with $|k|=N+1$, the right side of $(2.3)_k$ is of order $O(x^{\sigma+1})$. Therefore, by the same argument as for $|k|=2$, we see that we can uniquely determine the solution $p_k \in A$ of $(2.3)_k$ with (1.17). Thus, we have proved Theorem 1.

§ 3. Fundamental Lemma for truncated differential systems.

3.1. Systems of truncated differential equations.

For every integer $N \geq 2$, set

$$(3.1) \qquad P_N(x,Y) = \sum_{|k| \leq N} p_k(x) Y^k.$$

Then, in order that

$$y = \mathbb{1}(Y)(P_N(x,Y) + \varphi(x,Y))$$

transforms system (E) into system (E'), it is necessary and sufficient that φ satisfies the system of partial differential equations

$$(3.2)_N \quad x\, d\varphi/dx = -\, x\, dP_N/dx + Y_3\Big(\sum_{2 \leq |k| \leq N} p_{3k} Y^k + \varphi_3\Big)(P_N + \varphi) + x\mathbb{1}(P_N + \varphi) f(x, \mathbb{1}(Y)(P_N + \varphi)),$$

where the formal operator $x\, d/dx$ denotes the partial differential operator

$$x\, \partial/\partial x + x^{-\sigma} \sum_{j=1}^{2} \{(-1)^{j-1}\lambda(x) + \alpha_j x^{\sigma} + \alpha_j' x^{\sigma} Y_3)\partial/\partial Y_j\} + (\alpha_3 + \alpha_3' Y_3)\partial/\partial Y_3.$$

We denote by $f_N(x,Y,\varphi)$ the right side of $(3.2)_N$, namely,

$$(3.3)\qquad \begin{aligned} f_N(x,Y,\varphi) &= - x\, \partial P_N/\partial x \\ &- x^{-\sigma}\sum_{j=1}^{2}\{(-1)^{j-1}\lambda(x) + \alpha_j x^{\sigma} + \alpha_j' x^{\sigma} Y_3\}\partial P_N/\partial Y_j \\ &- (\alpha_3 + \alpha_3')\partial P_N/\partial Y_3 + Y_3\Big(\sum_{2\le|k|\le N} p_{3k}Y^k + \varphi_3\Big)(P_N + \varphi) \\ &+ x\mathbb{I}(P_N + \varphi)f(x,\mathbb{I}(Y)(P_N + \varphi)). \end{aligned}$$

It is easy to see that φ is a solution of $(3.2)_N$ if and only if $\varphi = \varphi(x,Y(x))$ is a solution of

$$(3.4)_N \qquad x\, d\varphi/dx = f_N(x,Y(x),\varphi)$$

where $Y(x)$ is a general solution of the reduced system (E'). Notice that on the left side of $(3.4)_N$, the operator d/dx is the usual ordinary differential operator.

From the fact that $(3.2)_N$ has a formal power series solution $\varphi = \sum_{|k|\ge N+1} p_k(x)Y^k$ with $p_k = O(x^{(1-\delta(k))\sigma+1})$ $(|k|\ge N+1)$ and from the definition of f_N, we obtain

<u>Proposition 1.</u> $f_N(x,Y,\varphi)$ is holomorphic and bounded for

$$(3.5)\qquad x \in S(\varepsilon_0,r_N^\circ),\quad |Y| < \rho_N^\circ,\quad |\varphi| < \Delta_N^\circ$$

and satisfies in this domain the equalities

$$(3.6)\qquad |f_N(x,Y,0)| \le C_N|x||Y|^{N+1}$$

$$(3.7)\qquad |f_N(x,Y,\varphi) - f_N(x,Y,\psi)| \le M(|x| + |Y|)|\varphi - \psi|,$$

for some constants C_N, $M > 0$, M being independent of N, provided that r_N°, ρ_N°, $\Delta_N^\circ > 0$ are sufficiently small.

From (3.6) and (3.7), it follows that

$$(3.8) \qquad |f_N(x,Y,\varphi)| \le C_N|x||Y|^{N+1} + M(|x| + |Y|)|\varphi|$$

for (x,Y,φ) in (3.5).

<u>3.2.</u> In this part, we first state the fundamental existence lemma for $(3.2)_N$.

<u>Fundamental Lemma.</u> Assume that (a) or (b) holds. Then, for every $N\geq 2$, there exists a solution $\varphi = \varphi_N(x,Y)$ of $(3.2)_N$ holomorphic and bounded in $D(\varepsilon, r_N, \rho_N)$ satisfying there

$$(3.9) \qquad \varphi_N(x,Y) = O(|x||Y|^{N+1}),$$

provided that r_N, $\rho_N > 0$ are sufficiently small. Further the solution of $(3.2)_N$ with (3.9) is unique.

Since $(3.2)_N$ is equivalent to $(3.4)_N$, we see that φ is a bounded holomorphic solution of $(3.2)_N$ in $D = D(\varepsilon, r_N, \rho_N)$ with (3.19) if and only if φ is a solution of the following integral equation

$$(3.10)_N \qquad \varphi(x_0,Y_0) = \int_{\Gamma(x_0)} x^{-1} f_N(x,Y(x),\varphi(x,Y(x)))dx$$

for an arbitrary point $(x_0,Y_0) \in D$. Here $Y(x)$ is the solution of (E') satisfying $Y(x_0) = Y_0$ and $\Gamma(x_0)$ is a suitable curve joining $x=0$ and $x=x_0$.

3.3. We shall show how to prove Theorem 2 by making use of Fundamental Lemma. Assume that Fundamental Lemma is valid. For every $N \geq 2$, put

$$\varphi^N(x,Y) = P_N(x,Y) + \varphi_N(x,Y),$$

where $\varphi_N(x,Y)$ is the unique solution of $(3.2)_N$ with (3.9). Then the transformation

$$y = \mathbb{I}(Y)\varphi^N(x,Y)$$

changes (E) to (E'). We can suppose that r_N and ρ_N are monotone decreasing in N. We shall show that φ^N is independent of N. For any N' and N with $N' \geq N \geq 2$,

$$(P_{N'} - P_N) + \varphi_{N'}$$

is a bounded solution of $(3.2)_N$ holomorphic and of order $O(|x||Y|^{N+1})$ in $D(\varepsilon, r_{N'}, \rho_{N'})$. Then by the uniqueness assertion in Fundamental Lemma, we have

$$(P_{N'} - P_N) + \varphi_{N'} = \varphi_N$$

in $D(\varepsilon, r_{N'}, \rho_{N'})$ which implies $\varphi^{N'} = \varphi^N$. Therefore, if we denote by φ this function independent of N and if we put $r = r_2$ and $\rho = \rho_2$, then φ is holomorphic and bounded in $D(\varepsilon, r, \rho)$ and satisfies in this domain

$$\varphi(x,Y) - \sum_{|k| \leq N} P_k(x)Y^k = O(|x||Y|^{N+1})$$

for every $N \geq 2$, which proves Theorem 2.

§ 4. Sectorial domain $\mathcal{S}$ and path $\Gamma(x_0)$ of integration.

§§ 4, 5 and 6 will be devoted to the proof of Fundamental Lemma in the case where assumption (a) holds. For this purpose, we have to define a path $\Gamma(x_0)$ of integration from $x=0$ to $x=x_0$ and replace a domain D by a slightly modified domain $\mathcal{L}$ which is called a stable domain for (E'). In this section, we shall define a sectorial domain $\mathcal{S}$ and a curve $\Gamma(x_0)$.

We first take and fix a constant $\varepsilon' > 0$ so that

$$\varepsilon_0 < \varepsilon'/\sigma < \varepsilon \tag{4.1}$$

where $\varepsilon > 0$ is a constant given in Theorem 2.

4.1. Determination of constants $\varkappa$ and Ω. From the assumption $\mathrm{Re}(\alpha_1 + \alpha_2) > 0$, it follows $\mathrm{Re}\alpha_1 > 0$ or $\mathrm{Re}\alpha_2 > 0$. We put

$$\alpha_j^* = \mathrm{Re}(\alpha_j + \alpha_j' C_3), \quad j = 1, 2. \tag{4.2}$$

4.1.1. The case $\mathrm{Re}\alpha_2 > 0$. If we take $\rho° > 0$ sufficiently small, then we have

$$\mathrm{Re}\alpha_2 - \rho°|\alpha_2'| > 0 \tag{4.3}$$

$$\mathrm{Re}\alpha_2 - \rho°|\alpha_2'| > -\mathrm{Re}\alpha_1 + \rho°|\alpha_1'|. \tag{4.4}$$

Note that

$$\alpha_j^* \geq \mathrm{Re}\alpha_j - \rho°|\alpha_j'|, \quad j = 1, 2, \tag{4.5}$$

for $|C_3| < \rho°$.

First we determine a constant $\varkappa > 0$ so that

(4.6) $$\mathrm{Re}\alpha_2 - \rho^\circ|\alpha_2'| > \sigma/\varkappa > -\mathrm{Re}\alpha_1 + \rho^\circ|\alpha_1'|$$

holds. Put

(4.7) $$\begin{aligned} \nu_j &= (-1)^{j-1}\sigma + \varkappa(\mathrm{Re}\alpha_j - \rho^\circ|\alpha_j'|) \\ \nu_j^* &= (-1)^{j-1}\sigma + \varkappa\alpha_j^* \end{aligned} \qquad j = 1, 2,$$

then by (4.5) and (4.6),

(4.8) $$\nu_j^* \geq \nu_j > 0, \quad j = 1, 2,$$

for $|C_3| < \rho^\circ$.

We next define $0 < \Omega < \pi/2$ by

(4.9) $\tan\Omega$
$$= \{\sigma + (3\varkappa + 4)\max_{j=1,2}(|\alpha_j| + \rho^\circ|\alpha_j'|) + \max_{j=1,2}(\nu_j)\}/\min_{j=1,2}(\nu_j).$$

Note that

(4.10) $$\tan\Omega > 1.$$

<u>4.1.2.</u> The case $\mathrm{Re}\alpha_1 > 0$. We take $\rho^\circ > 0$ so small that

(4.3)' $$\mathrm{Re}\alpha_1 - \rho^\circ|\alpha_1'| > 0$$

(4.4)' $$\mathrm{Re}\alpha_1 - \rho^\circ|\alpha_1'| > -\mathrm{Re}\alpha_2 + \rho^\circ|\alpha_2'|$$

hold. We determine $\varkappa > 0$ so that

$$(4.6)' \qquad \mathrm{Re}\alpha_1 - \rho^\circ|\alpha_1'| > \sigma/\varkappa > -\mathrm{Re}\alpha_2 + \rho^\circ|\alpha_2'|.$$

If we put

$$(4.7)' \qquad \begin{aligned} \nu_j &= (-1)^j\sigma + \varkappa(\mathrm{Re}\alpha_j - \rho^\circ|\alpha_j'|), \\ \nu_j^* &= (-1)^j\sigma + \varkappa\alpha_j^*, \end{aligned} \qquad j = 1, 2,$$

then we have (4.8). The constant $0 < \Omega < \pi/2$ is determined by (4.9).

In the following discussion, small constants written as ρ' and ρ_N' will always be assumed to satisfy

$$(4.11) \qquad \rho', \rho_N' \leq \rho^\circ.$$

<u>4.2.</u> Sectorial domain $\mathscr{S}$. In order to determine a stable domain $\mathscr{D}$, we need to define a sectorial domain $\mathscr{S}$ which is a modification of a sector S. A domain $\mathscr{S} = \mathscr{S}(\varepsilon', r')$ in the x-plane is defined by

$$(4.12) \qquad x = \Xi(\omega),\ 0 < |\omega| < r'\chi(\arg\omega),\ |\arg\omega - \pi/2| < \pi - \varepsilon'$$

where

$$(4.13) \qquad \chi(\vartheta) = \begin{cases} \cos\Omega/\sin\varepsilon', & |\vartheta - \pi/2| < \pi/2 - \Omega \\ |\cos\vartheta|/\sin\varepsilon', & \pi/2 - \Omega \leq |\vartheta - \pi/2| \leq \pi - \varepsilon'. \end{cases}$$

Note that the opening angle of $\mathscr{S}$ at the origin is $2(\pi - \varepsilon')/\sigma$.

We divide $\mathcal{S}$ into three parts:

$$(4.14)\qquad \begin{aligned} \mathcal{S}_1 &= \mathcal{S} \cap \{x \in C \mid |\mathrm{Re}\,\omega| \tan\Omega \le \mathrm{Im}\,\omega, \quad \mathrm{Im}\,\omega > 0\} \\ \mathcal{S}_2 &= \mathcal{S} \cap \{x \in C \mid (\mathrm{Re}\,\omega) \tan\Omega > \mathrm{Im}\,\omega, \quad \mathrm{Re}\,\omega > 0\} \\ \mathcal{S}_3 &= \mathcal{S} \cap \{x \in C \mid (\mathrm{Re}\,\omega) \tan\Omega > \mathrm{Im}\,\omega, \quad \mathrm{Re}\,\omega < 0\}. \end{aligned}$$

<u>4.3.</u> Path $\Gamma(x_0)$ of integration. We shall define a curve $\Gamma(x_0)$ from $x=0$ to $x=x_0$ which generally consists of two parts $\Gamma'(x_0)$ and $\Gamma''(x_0)$. In case $x_0 \in \mathcal{S}_1$, $\Gamma(x_0)$ reduces to $\Gamma'(x_0)$.

<u>4.3.1.</u> The case $\mathrm{Re}\,\alpha_2 > 0$. If $x_0 \in \mathcal{S}_1$, the variable point $x = x(\tau)$ on $\Gamma(x_0) = \Gamma'(x_0)$ is defined by

$$(4.15)\qquad x(\tau) = \Xi(\omega(\tau)), \quad 1/\omega(\tau) = \tau + A - iBe^{\kappa\tau}, \quad \tau \ge 0,$$

where A and B are real numbers satisfying

$$(4.16)\qquad A - iB = \Lambda(x_0) \qquad (i = \sqrt{-1}).$$

If $x_0 \in \mathcal{S}_2$ (or $\mathcal{S}_3$), $\Gamma(x_0)$ consists of two parts $\Gamma'(x_0)$ and $\Gamma''(x_0)$. The variable point $x = x(\vartheta)$ on $\Gamma''(x_0)$ is defined by

$$(4.17)\qquad x(\vartheta) = \Xi(\omega(\vartheta)), \quad \omega(\vartheta) = (\cos\vartheta/\cos\vartheta_0)|\omega_0|e^{i\vartheta}$$

for $\vartheta_0 \le \vartheta \le \Omega$ (or $\pi - \Omega \le \vartheta \le \vartheta_0$) with $\omega_0 = 1/\Lambda(x_0)$, $\vartheta_0 = \arg\omega_0$. Then the point $x(\Omega)$ (or $x(\pi - \Omega)$) belongs to $\mathcal{S}_1$. $\Gamma'(x_0)$ is defined to be a curve defined as (4.15) joining $x=0$ and $x=x(\Omega)$ (or $x(\pi - \Omega)$).

4.3.2. The case $\mathrm{Re}\alpha_1 > 0$. If $x_0 \in \mathscr{S}_2$ (or $\mathscr{S}_3$), $\Gamma''(x_0)$ is defined by the same way as in the case $\mathrm{Re}\alpha_2 > 0$. Therefore, we have only to define $\Gamma(x_0) = \Gamma'(x_0)$ in the case $x_0 \in \mathscr{S}_1$. In this case the variable point $x = x(\tau)$ on $\Gamma'(x_0)$ is given by

$$(4.15)' \quad x(\tau) = \Xi(\omega(\tau)), \quad 1/\omega(\tau) = -\tau + A - iBe^{\kappa\tau}, \quad \tau \geq 0,$$

where A and B are real numbers satisfying (4.16).

4.4. We shall show some propositions concerning $\mathscr{S}$ and $\Gamma(x_0)$.

Proposition 2. For every $x_0 \in \mathscr{S}_1$, we have

$$(4.18) \qquad |A/B| \leq 1/\tan\Omega < 1$$

$$(4.19) \qquad B \geq (\tan\Omega\cdot\sin\varepsilon')/r' > \tan\Omega > 1$$

$$(4.20) \qquad \kappa B \geq \tan\Omega > 1$$

provided that $r' > 0$ is small.

Proof. $x_0 \in \mathscr{S}_1$ implies

$$\mathrm{Im}\omega_0 > 0, \quad |\arg\omega_0 - \pi/2| \leq \pi/2 - \Omega, \quad |\omega_0|^2 < (r'\cos\Omega/\sin\varepsilon')^2.$$

Since $\omega_0 = (A + iB)/(A^2 + B^2)$, we have

$$B > 0, \quad |\tan(\arg\omega_0)| = |A/B| \geq \tan\Omega$$

$$|\omega_0|^2 = 1/(A^2 + B^2) < (r'\cos\Omega/\sin\varepsilon')^2.$$

Hence

$$|A/B| \leq 1/\tan\Omega, \quad \{\sin\varepsilon'/(r'\cos\Omega)\}^2 < A^2 + B^2 \leq B^2/\sin^2\Omega,$$

which yield (4.18) and (4.19). Inequality (4.20) follows from (4.19), provided that $r' > 0$ is small.

Proposition 3. If $r' > 0$ is sufficiently small, then we have

(i) $x_0 \in \mathcal{S}_1$ implies $\Gamma(x_0) \subset \mathcal{S}_1$

(ii) $x_0 \in \mathcal{S}$ implies $\Gamma(x_0) \subset \mathcal{S}$.

Proof. Assertion (i) is verified by showing that $|\omega(\tau)|$ is monotone decreasing in τ and that the inequality

$$|\arg\omega(\tau) - \pi/2| \le \pi/2 - \Omega, \quad \tau \ge 0$$

holds in a similar way as in [5]. Assertion (ii) is easy to see if (i) is established.

The following proposition can be proved by an analogous method as in [17].

Proposition 4. There exists a positive constant L such that

$$\int_{\Gamma(x_0)} ds \le L|x_0| \tag{4.21}$$

for every $x_0 \in \mathcal{S}$, provided that $r' > 0$ is small. Here ds is the line element along the curve $\Gamma(x_0)$.

§ 5. Stable domain $\mathcal{B}$.

In this section, we shall define a domain $\mathcal{B}$ and show that it is a stable domain for system (E') in the case (a).

5.1. Definition of $\mathcal{B}$. The domain $\mathcal{B} = \mathcal{B}(\varepsilon', r', \rho')$ in the (x,Y)-space is defined by

$$(5.1)\qquad \begin{aligned} & x \in \mathscr{S}(\varepsilon',r') \\ & |Y_j| < \rho' c_j(\omega(x),Y_3)e_j(x,Y_3), \quad j = 1,\ 2, \\ & |Y_3| < \rho' \end{aligned}$$

where

$$c_j(\omega,\eta) = \begin{cases} |\xi(\omega^{1/\sigma})|^{\mathrm{Re}(\alpha_j+\alpha_j'\eta)}, & |\vartheta - \pi/2| \le \pi/2 - \Omega \\ \{(|\cos\vartheta|/\cos\Omega)^{1/\sigma}|\xi(\omega^{1/\sigma})|\}^{\mathrm{Re}(\alpha_j+\alpha_j'\eta)}, & \\ \qquad \pi/2 - \Omega \le |\vartheta - \pi/2| \le \pi - \varepsilon' & \end{cases}$$

with $\vartheta = \arg\omega$ and

$$e_j(x,\eta) = \exp\{-(\arg x)\mathrm{Im}(\alpha_j + \alpha_j'\eta)\}, \quad j = 1,\ 2.$$

Recall inequality (4.1). Then we can verify that, for any r, $\rho > 0$, there exist r', $\rho' > 0$ such that $\mathscr{D}(\varepsilon',r',\rho') \subset D(\varepsilon_0,r,\rho)$ and that, for any r', $\rho' > 0$, there exist r, $\rho > 0$ such that $D(\varepsilon,r,\rho) \subset \mathscr{D}(\varepsilon',r',\rho')$. This fact is a kind of equivalence of D and $\mathscr{D}$. Moreover, we note that there exists a positive constant R such that

$$(5.3)\qquad \sup_{x\in\mathscr{S}(\varepsilon',r')} |x| \le Rr'^{1/\sigma}, \quad \sup_{(x,Y)\in\mathscr{D}(\varepsilon',r',\rho')} |Y| \le R\rho'$$

for all sufficiently small r' and $\rho' > 0$.

5.2. Stability. We shall prove a kind of stability property of $\mathscr{D}$

with respect to the solution of (E') and the curve $\Gamma(x_0)$. Under assumption (a), the general solution of (E') is given by (1.18). Then by putting

$$u_j(x) = C_j e^{(-1)^j \Lambda(x)} \Lambda(x)^{-\alpha_j^*/\sigma}, \quad j = 1, 2, \tag{5.4}$$

we have

$$|Y_j(x)| = |u_j(x)||\xi(\omega^{1/\sigma})|^{\alpha_j^*} \exp\{- (\arg x)\mathrm{Im}(\alpha_j+\alpha_j' C_3)\}. \quad (j = 1, 2) \tag{5.5}$$

The following proposition is the most essential one in proving the stability of the solution $Y(x)$ in the domain $\mathcal{D}$.

<u>Proposition 5.</u> If $|C_3| < \rho'$, then

$$d \log|u_j|/d\tau < - (3/4\sigma)\nu_j^*, \quad j = 1, 2 \tag{5.6}$$

on the curve $\Gamma'(x_0)$ with $x_0 \in \mathcal{S}_1$, provided that r', $\rho' > 0$ are small.

Proof. We prove the proposition only in the case $\mathrm{Re}\alpha_2 > 0$. Let s be the arc length along the curve $\Gamma' = \Gamma'(x_0)$ measured from the origin to the variable point $x \in \Gamma'$. Then by the definition of Γ', we have

$$dx/d\tau = - x^{\sigma+1}\lambda(x)^{-1}(1 - i\kappa B e^{\kappa\tau})$$

so that

$$ds = - |x|^{\sigma+1}|\lambda(x)|^{-1}(1 + \kappa^2B^2e^{2\kappa\tau})^{1/2}d\tau.$$

Then, (5.6) is equivalent to

$$(5.7) \quad d \log|u_j|/ds > (3/4\sigma)\nu_j^*|\lambda(x)||x|^{-\sigma-1}(1 + \kappa^2B^2e^{2\kappa\tau})^{-1/2}. \quad (j = 1, 2)$$

From the definition of u_j, it follows

$$d \log u_j/dx = u_j^{-1}\, du_j/dx = x^{-\sigma-1}\lambda(x)\{(-1)^{j-1} + \alpha_j^*/(\sigma\Lambda(x))\},$$

which yields

$$\begin{aligned} d \log|u_j|/ds &= d\, \mathrm{Re}(\log u_j)/ds = \mathrm{Re}(d \log u_j/ds) \\ &= \mathrm{Re}\{(d \log u_j/dx)(dx/d\tau)(d\tau/ds)\} \\ &= |\lambda(x)||x|^{-\sigma-1}(1 + \kappa^2B^2e^{2\kappa\tau})^{-1/2} \\ &\quad \cdot\mathrm{Re}[\{(-1)^{j-1} + \alpha_j^*/(\sigma\Lambda(x))\}(1 - i\kappa Be^{\kappa\tau})]. \end{aligned}$$

Hence, (5.7) is equivalent to

$$(5.8) \qquad \mathrm{Re}[\{(-1)^{j-1} + \alpha_j^*/(\sigma\Lambda(x))\}(1 - i\kappa Be^{\kappa\tau})] > (3/4\sigma)\nu_j^*. \quad (j = 1, 2)$$

Therefore, by (4.7), we see that the proposition is valid if and only if

$$(5.9) \quad I_j(\tau) \equiv \nu_j^*B^2e^{2\kappa\tau} - \{(-1)^j\sigma + 3\kappa\alpha_j^*\}(\tau + A)^2 + 4\alpha_j^*(\tau + A) > 0, \qquad j = 1, 2.$$

Thus, in order to verify the proposition, it is sufficient to show

$$(5.10) \qquad I_j(0) = 0, \quad I_j'(0) > 0, \quad I_j''(\tau) > 0, \quad j = 1, 2.$$

Remark that

$$(5.11) \qquad \begin{aligned} &\tan\Omega > \{\sigma + (3\varkappa + 4)|\alpha_j^*|\}/\nu_j^*, \qquad j = 1, 2 \\ &\tan\Omega > 1, \end{aligned}$$

which are verified by (4.2), (4.8) and (4.9). We see

$$\begin{aligned} I_j'(\tau) &= 2\varkappa B^2\nu_j^* e^{2\varkappa\tau} - 2\{(-1)^j\sigma + 3\varkappa\alpha_j^*\}(\tau + A) + 4\alpha_j^* \\ I_j''(\tau) &= 4\varkappa^2 B^2\nu_j^* e^{2\varkappa\tau} - 2\{(-1)^j\sigma + 3\varkappa\alpha_j^*\}. \end{aligned}$$

First, we have

$$\begin{aligned} I_j(0) &= B^2[\nu_j^* - \{(-1)^j\sigma + 3\varkappa\alpha_j^*\}(B/A)^2 + 4\alpha_j^* A/B^2] \\ &> B^2[\nu_j^* - \{\sigma + 3\varkappa|\alpha_j^*|\}/\tan\Omega - 4|\alpha_j^*|/\tan\Omega] \quad \text{(by (4.18), (4.19))} \\ &> (B^2\nu_j^*/\tan\Omega)[\tan\Omega - \{\sigma + (3\varkappa + 4)|\alpha_j^*|\}/\nu_j^*] \\ &> 0 \qquad \text{(by (5.11))}. \end{aligned}$$

Next, we can verify

$$\begin{aligned} I_j'(0) &= 2B[\varkappa B\nu_j^* - \{(-1)^j\sigma + 3\varkappa\alpha_j^*\}A/B + 2\alpha_j^*/B] \\ &> 2B[\nu_j^* - \{\sigma + 3\varkappa|\alpha_j^*|\}/\tan\Omega - 2|\alpha_j^*|\tan\Omega] \\ &\qquad\qquad \text{(by (4.18), (4.19), (4,20))} \\ &= (2B\nu_j^*/\tan\Omega)[\tan\Omega - \{\sigma + (3\varkappa + 2)|\alpha_j^*|\}/\nu_j^*] \\ &> 4B|\alpha_j^*|/\tan\Omega \qquad \text{(by (5.11))} \end{aligned}$$

$\geq 0.$

Finally, we can obtain

$$
\begin{aligned}
I_j''(\tau) \geq I_j''(0) &= 2(\kappa B)^2[2\nu_j^* - \{(-1)^j\sigma + 3\kappa\alpha_j^*\}/(\kappa B)^2] \\
&> 2(\kappa B)^2\{2\nu_j^* - (\sigma + 3\kappa|\alpha_j^*|)/\tan\Omega\} \qquad \text{(by (4.20))} \\
&= \{2(\kappa B)^2\nu_j^*/\tan\Omega\}\{2\tan\Omega - (\sigma + 3\kappa|\alpha_j^*|)/\nu_j^*\} \\
&> 2(\kappa B)^2\{\sigma + (3\kappa + 8)|\alpha_j^*|\}/\tan\Omega \qquad \text{(by (5.11))} \\
&> 0.
\end{aligned}
$$

Thus, we have proved the proposition.

<u>Proposition 6.</u> (stability). Let $Y(x)$ be the general solution of (E') under assumption (a). Then, $(x_0,Y(x_0)) \in \mathcal{D}(\varepsilon',r',\rho')$ implies $(x,Y(x)) \in \mathcal{D}(\varepsilon',r',\rho')$ for every $x \in \Gamma(x_0)$, provided that r' and $\rho' > 0$ are sufficiently small.

Proof. Under assumption (a), the general solution of (E') is given by (1.18). Since

$$Y_3(x) = Y_3(x_0) = C_3,$$

$|Y(x_0)| < \rho'$ implies $|Y_3(x)| < \rho'$ for every $x \in \Gamma(x_0)$.

Suppose $x_0 \in \mathcal{S}_2$ (or $\mathcal{S}_3$) and $(x_0,Y(x_0)) \in \mathcal{D}(\varepsilon',r',\rho')$. Then we have

$$|Y_j(x_0)| < \rho' c_j(\omega_0,C_3)e_j(x_0,C_3), \qquad j = 1,\ 2$$

where $\omega_0 = 1/\Lambda(x_0)$. Remark that for every point $x \in \Gamma''(x_0)$,

$$\omega = (\cos\vartheta/\cos\vartheta_0)|\omega_0|e^{i\vartheta}$$

where $\omega_0 = 1/\Lambda(x_0)$, $\vartheta_0 = \arg\omega_0$, $\omega = 1/\Lambda(x)$, $\vartheta = \arg\omega$.

Hence we have

$$\mathrm{Re}\Lambda(x) = \mathrm{Re}\Lambda(x_0), \quad |x/x_0| = |\cos\vartheta/\cos\vartheta_0|^{1/\sigma}|\xi(\omega^{1/\sigma})/\xi(\omega_0{}^{1/\sigma})|.$$

Therefore, it follows

$$\begin{aligned}|Y_j(x)| &= |C_j|e^{(-1)^j\mathrm{Re}\Lambda(x)}|x|^{\mathrm{Re}(\alpha_j+\alpha_j'C_3)}\exp\{-(\arg x)\mathrm{Im}(\alpha_j+\alpha_j'C_3)\}\\ &= |Y_j(x_0)||x/x_0|^{\mathrm{Re}(\alpha_j+\alpha_j'C_3)}\exp\{-(\arg x - \arg x_0)\mathrm{Im}(\alpha_j+\alpha_j'C_3)\}\\ &= |Y_j(x_0)|\,\frac{c_j(\omega,C_3)\ e_j(x,C_3)}{c_j(\omega_0,C_3)e_j(x_0,C_3)}\\ &< \rho' c_j(\omega,C_3)e_j(x,C_3)\\ &\quad j = 1,\ 2\end{aligned}$$

for $x \in \Gamma''(x_0)$.

Next suppose $x_0 \in \mathcal{S}_1$ and $(x_0,Y(x_0)) \in \mathcal{D}(\varepsilon',r',\rho')$. Then, we have

$$|Y_j(x_0)| < \rho' c_j(\omega_0,C_3)e_j(x_0,C_3), \qquad j = 1,\ 2.$$

By (5.5), we see

$$\begin{aligned}|Y(x)| = {}&|Y(x_0)||u_j(x)/u_j(x_0)||\xi(\omega^{1/\sigma})/\xi(\omega_0{}^{1/\sigma})|^{\mathrm{Re}(\alpha_j+\alpha_j'C_3)}\\ &\cdot\exp\{-(\arg x - \arg x_0)\mathrm{Im}(\alpha_j+\alpha_j'C_3)\}.\end{aligned}$$

By virtue of Proposition 5, $|u_j(x(\tau))|$ is monotone decreasing in τ. Therefore, we have

$$\begin{aligned}|Y(x_0)| &\le \{|Y_j(x_0)|/(c_j(\omega_0,C_3)e_j(x_0,C_3))\}c_j(\omega,C_3)e_j(x,C_3)\\ &< \rho' c_j(\omega,C_3)e_j(x,C_3), \qquad j = 1,\ 2\end{aligned}$$

for every $x \in \Gamma'(x_0)$. Thus we have proved Proposition 6.

§ 6. Proof of Fundamental Lemma under assumption (a).

First we shall show the following proposition.

Proposition 7. There exists a positive constant J_N such that for every $x_0 \in \mathcal{S}(\varepsilon', r')$ and for the general solution (1.18) of (E') with $|C_3| < \rho'$, the inequality

$$\int_{\Gamma(x_0)} |Y(x)|^{N+1}\, dx \le J_N |x_0| |Y(x_0)|^{N+1} \tag{6.1}$$

holds, provided that r' and $\rho' > 0$ are small.

Proof. Since $Y_3(x) = C_3$, we have by Proposition 4

$$\int_{\Gamma(x_0)} |Y_3(x)|^{N+1} ds \le L|x_0| |Y_3(x_0)|^{N+1}.$$

Recall that

$$|Y_j(x)| = |u_j(x)| |\xi(\omega^{1/\sigma})|^{\mathrm{Re}(\alpha_j+\alpha_j' C_3)} \exp\{-(\arg x)\mathrm{Im}(\alpha_j+\alpha_j' C_3)\},$$
$$j = 1,\ 2$$

and notice that $|\xi(\omega^{1/\sigma})|^{\alpha_j^*} \exp\{-(\arg x)\mathrm{Im}(\alpha_j+\alpha_j' C_3)\}$ is bounded from below and above by constants L_1, $L_2 > 0$, respectively. Hence on the curve $\Gamma'=\Gamma'(x_0)$, we have

$$\int_{\Gamma'} |Y_j(x)|^{N+1} ds \le L_2^{N+1} \int_{\Gamma'} |u_j(x)|^{N+1} ds$$
$$\le L_2^{N+1} |u_j(x_0)|^{N+1} \int_{\Gamma'} ds \le L_2^{N+1} |u(x_0)|^{N+1} \cdot L|x_0|$$
$$\le (L_2/L_1)^{N+1} L|x_0| |Y_j(x_0)|^{N+1}.$$

On the other hand, on the curve $\Gamma''=\Gamma''(x_0)$, we can verify

$$|Y_j(x)| = |Y_j(x_0)| c_j(\omega, C_3) e_j(x, C_3)/(c_j(\omega_0, C_3) e_j(x_0, C_3))$$
$$\le |Y_j(x_0)| L_3$$

for some constant $L_3 > 0$. Then we obtain

$$\int_{\Gamma''} |Y_j(x)|^{N+1} ds < L_3^{N+1} L |x_0| |Y_j(x_0)|^{N+1}, \quad j = 1, 2.$$

Thus, we have proved the proposition.

Now, we are ready to prove Fundamental Lemma under assumption (a) by using a fixed point theorem due to M. Hukuhara ([2]).

Recall that $\varepsilon' > 0$ be a constant such that (4.1) holds. We denote by $\mathfrak{F}$ the family of all vector functions $\varphi(x,Y)$ of which the components are holomorphic and satisfy

$$(6.2) \qquad |\varphi(x,Y)| \leq K_N |x| |Y|^{N+1}$$

in $\mathcal{D} = \mathcal{D}(\varepsilon', r_N', \rho_N')$, where K_N is a positive constant which will be determined later and r_N', ρ_N' are sufficiently small positive constants. We note that r_N' and ρ_N' must be chosen so small that (4.11) and all the propositions in §§ 4 and 5 hold.

It is easy to see that $\mathfrak{F}$ is not empty and is convex. Moreover, it is closed and normal with respect to the topology of uniform convergence on every compact subset of $\mathcal{D}$.

We next define a mapping $\mathcal{J}$. For a vector function $\varphi \in \mathfrak{F}$, we define a vector function Φ by

$$(6.3) \quad \Phi(x_0,Y_0) = \int_{\Gamma(x_0)} x^{-1} f_N(x,Y(x),\varphi(x,Y(x)))dx, \quad (x_0,Y_0) \in \mathcal{D},$$

where $Y(x)$ is the solution of (E') with $Y(x_0) = Y_0$. In order that $(x,Y(x),\varphi(x,Y(x)))$ stays in the domain of definition of f_N, it is sufficient that

$$(6.4) \qquad K_N R^{N+2} r_N'^{1/\sigma} \rho_N'^{N+1} \leq \Delta_N^\circ, \quad R r_N'^{1/\sigma} \leq r_N^\circ, \quad R\rho_N' \leq \rho_N^\circ$$

where Δ_N°, r_N°, ρ_N° are the constants in Proposition 1.

We next estimate $\Phi(x_0,Y_0)$. By (6.3), (3.8), (5.3), (6.1), we have

$$|\Phi(x_0,Y_0)| \le \int_{\Gamma(x_0)} |x|^{-1}|f_N(x,Y(x),\varphi(x,Y(x)))||dx|$$
$$\le \int_{\Gamma(x_0)} \{C_N|Y(x)|^{N+1} + K_N M(|x| + |Y(x)|)|Y(x)|^{N+1}\}|dx|$$
$$\le \{C_N + K_N MR(r_N'^{1/\sigma} + \rho_N')\}\int_{\Gamma(x_0)} |Y(x)|^{N+1}|dx|$$
$$\le \{C_N + K_N MR(r_N'^{1/\sigma} + \rho_N')\}J_N|x_0||Y_0|^{N+1}.$$

Hence, if r_N', $\rho_N' > 0$ are small enough to satisfy

$$MRJ_N(r_N'^{1/\sigma} + \rho_N') < 1/2 \tag{6.5}$$

and if K_N is determined by

$$K_N = 2J_N C_N \tag{6.6}$$

then Φ satisfies the order condition (6.2).

We can also verify that Φ is holomorphic in $\mathcal{D}$. Therefore, if various constants are suitably chosen as above, $\mathcal{J}$ becomes a mapping from $\mathcal{F}$ into itself. Moreover, it can be seen by Lipschitz inequality (3.7) that $\mathcal{J}$ is a continuous mapping. Then, the fixed point theorem referred above shows that $\mathcal{J}$ admits a fixed point, which proves the existence of φ satisfying the properties stated in Fundamental Lemma where $D(\varepsilon,r_N,\rho_N)$ must be replaced by $\mathcal{D}(\varepsilon',r_N',\rho_N')$. Since $\varepsilon > \sigma\varepsilon'$, we can choose r_N and $\rho_N > 0$ so small that $D(\varepsilon,r_N,\rho_N) \subset \mathcal{D}(\varepsilon',r_N',\rho_N')$. Thus we have proved the existence assertion in Fundamental Lemma.

Now, it remains only to show the uniqueness. Let $\varphi^{(1)}$ and $\varphi^{(2)}$ be two solutions satisfying the properties in Fundamental

Lemma and put

$$\psi = \varphi^{(1)} - \varphi^{(2)}.$$

Then ψ is holomorphic in a domain $\mathcal{D} = \mathcal{D}(\varepsilon'', r'', \rho'')$ where $\varepsilon''/\sigma \geq \varepsilon$, $r'' \leq r'_N$, $\rho'' \leq \rho'_N$,and of order $O(|x||Y|^{N+1})$. Let H be a constant defined by

$$H = \inf\{ H'\geq 0 \mid |\psi(x,Y)| \leq H'|x||Y|^{N+1}, \quad (x,Y) \in \mathcal{D} \}.$$

Then, we have for every point $(x_0,Y_0) \in \mathcal{D}$

$$\begin{aligned}
&|\psi(x_0,Y_0)| \\
&\leq \int_{\Gamma(x_0)} |x|^{-1} |f_N(x,Y(x),\varphi^{(1)}(x,Y(x))) - \\
&\qquad\qquad\qquad f_N(x,Y(x),\varphi^{(2)}(x,Y(x)))||dx| \\
&\leq \int_{\Gamma(x_0)} |x|^{-1} M(|x| + |Y(x)|)|\varphi^{(1)}(x,Y(x)) - \varphi^{(2)}(x,Y(x))||dx| \\
&\qquad\qquad\qquad \text{(by (3.7))} \\
&\leq MR(r''^{1/\sigma} + \rho'')H \int_{\Gamma(x_0)} |Y(x)|^{N+1}|dx| \\
&\leq MRJ_N(r''^{1/\sigma} + \rho'')\, H|x_0||Y_0|^{N+1} \\
&\leq (H/2)|x_0||Y_0|^{N+1}.
\end{aligned}$$

Therefore, by the definition of H, we obtain $H = 0$, which implies $\psi = 0$. Thus we have completed the proof of Fundamental Lemma under assumption (a).

§ 7. Proof of Fundamental Lemma under assumption (b).

From the assumption $\mathrm{Re}(\alpha_1 + \alpha_2) > 0$, it follows that $\mathrm{Re}\alpha_1 > 0$ or $\mathrm{Re}\alpha_2 > 0$. We shall only show the case $\mathrm{Re}\alpha_2 > 0$.

7.1. Path $\Gamma(x_0)$ of integration. Take and fix a constant $\varepsilon' > 0$ so that (4.1) holds. Put

(7.1) $$\alpha_j^* = \mathrm{Re}\alpha_j, \quad j = 1, 2.$$

Take $\varkappa > 0$ so that

(7.2) $$\alpha_2^* > \sigma/\varkappa > -\alpha_1^*.$$

If we put

(7.3) $$\nu_j = \nu_j^* = (-1)^{j-1}\sigma + \varkappa\alpha_j^*, \quad j = 1, 2,$$

then we have

(7.4) $$\nu_j = \nu_j^* > 0, \quad j = 1, 2.$$

We determine $0 < \Omega < \pi/2$ by

(7.5) $$\tan\Omega = \{\sigma + (3\varkappa + 4)\max_{j=1,2}(|\alpha_j|) + \max_{j=1,2}(\nu_j)\}/\min_{j=1,2}(\nu_j).$$

Note that

(7.6) $$\tan\Omega > 1.$$

For these constants $\varkappa$ and Ω, we define $\mathcal{S}(\varepsilon', r')$ by (4.12) and (4.13), $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ by (4.14), and the path of integration in the similar way as in § 4. Then we can verify that Propositions 2, 3 and 4 are valid for these new domains and path of integration.

7.2. Stable domain. We define a domain $\mathcal{D} = \mathcal{D}(\varepsilon', r', \rho')$ by

$$(7.7) \qquad \begin{aligned} & x \in \mathcal{S}(\varepsilon', r') \\ & |Y_j| < \rho' c_j(\omega(x)) e_j(x) |Q_j(Y_3)|, \quad j = 1, 2, 3. \end{aligned}$$

Here

$$(7.8) \qquad c_j(x) = \begin{cases} |\xi(\omega^{1/\sigma})|^{\mathrm{Re}\alpha_j}, & |\vartheta - \pi/2| \le \pi/2 - \Omega \\ \{(|\cos\vartheta|/\cos\Omega)^{1/\sigma} |\xi(\omega^{1/\sigma})|\}^{\mathrm{Re}\alpha_j} & \\ \pi/2 - \Omega \le |\vartheta - \pi/2| \le \pi - \varepsilon' & \end{cases}$$

with $\vartheta = \arg\omega$,

$$e_j(x) = \exp\{- (\arg x)\mathrm{Im}\alpha_j\}$$

and $Q_j(\eta)$'s $(j=1,2)$ are the functions given by (1.21) and

$$(7.9) \qquad Q_3(\eta) = 1 + (\alpha_3'/\alpha_3)\eta.$$

Recalling (4.1), we see that, for any $r, \rho > 0$, there exist $r', \rho' > 0$ such that $\mathcal{D}(\varepsilon', r', \rho') \subset D(\varepsilon_0, r, \rho)$ and that, for any $r', \rho' > 0$, there exist $r, \rho > 0$ such that $D(\varepsilon, r, \rho) \subset \mathcal{D}(\varepsilon', r', \rho')$. We can also verify that there exists a constant $R > 0$ such that (5.3) is valid.

For the solution of (E') given by (1.19), if we put

$$(7.10) \qquad u_j(x) = C_j e^{(-1)^j \Lambda(x)} \Lambda(x)^{-\alpha_j^*/\sigma} \qquad j = 1, 2$$

then we have

$$(7.11) \quad |Y_j(x)| = |u_j(x)| |\xi(\omega^{1/\sigma})|^{\alpha_j^*} \exp\{- (\arg x)\mathrm{Im}\alpha_j\} |Q_j(Y_3(x))|$$

$$j = 1, 2$$

with $\omega = 1/\Lambda(x)$. For these u_j's, we can verify that Proposition 5 is valid and we can see that Proposition 6 holds for the domain $\mathcal{D}$.

7.3. Proof of Fundamental Lemma under assumption (b). In order to show Proposition 7 in our case (b), it is necessary to verify the following proposition.

Proposition 8. There exists a positive constant L_4 such that, for every solution $Y(x)$ of (E'), $(x_0, Y(x_0)) \in \mathcal{B}(\varepsilon', r', \rho')$ implies

$$|Y_3(x)| \le L_4 |Y_3(x_0)|$$

for every $x \in \Gamma(x_0)$. Here, r', $\rho' > 0$ are supposed to be sufficiently small.

Proof. The proposition follows from

$$|Y_3(x)/Y_3(x_0)| = |x/x_0|^{\mathrm{Re}\alpha_3} \exp\{-(\arg x - \arg x_0)\mathrm{Im}\alpha_3\}$$

$$\cdot |1 - (\alpha_3'/\alpha_3)C_3 x_0^{\alpha_3}| / |1 - (\alpha_3'/\alpha_3)C_3 x_0^{\alpha_3}|,$$

$$|x/x_0| = |\omega/\omega_0|^{1/\sigma} |\xi(\omega^{1/\sigma})/\xi(\omega_0^{1/\sigma})|,$$

$$\mathrm{Re}\alpha_3 \ge 0,$$

and that $\omega(\tau)$ is monotone decreasing in τ on $\Gamma'(x_0)$, and

$$|\omega/\omega_0| = |\cos\vartheta/\cos\vartheta_0|, \quad \vartheta = \arg\omega$$

on $\Gamma''(x_0)$.

By making use of this proposition, we can show Proposition 7 in a similar way as in § 6. Then, by virtue of the propositions established in the case (b), we can prove Fundamental Lemma by the same method as in the case (a).

Appendix.

The existence of the holomorphic transformation (1.8) will be proved by constructing the following three transformations.

The first transformation of the form

$$y = y' + \sum_{i=1}^{2} P^{(i)}(x, y'_i) \tag{8.1}$$

is determined so that it changes system (1.1) to a system of the form

$$x^{\sigma+1} dy'/dx = (\lambda(x)\mathbb{1}(1,-1) + x^{\sigma}\mathbb{1}(\alpha) + \mathbb{1}(f'(x,y')))y'. \tag{8.2}$$

The second and third transformations of the form

$$y' = \mathbb{1}(y'')({}^{t}(1,1) + \sum_{i=1}^{2} P'^{(i)}(x, y''_i)) \tag{8.3}$$

$$y'' = \mathbb{1}(z)({}^{t}(1,1) + z_1 z_2 \sum_{i=1}^{2} P''^{(i)}(x, z_i)) \tag{8.4}$$

are determined so that the former reduces system (8.2) to a system of the form

$$x^{\sigma+1}\, dy''/dx = \{\lambda(x)\mathbb{1}(1,-1) + x^{\sigma}\mathbb{1}(\alpha) + y''_1 y''_2 \mathbb{1}(f''(x,y''))\}y'' \tag{8.5}$$

and the latter reduces (8.5) to a system of the form (1.7). Here f', f'' and g are vector functions with the same analyticity as f.

We shall prove the existence of (8.1) only, since the other transformations are obtained by a similar method. In order to determine (8.1), we consider the following partial differential equations

$$(8.6)\qquad x^{\sigma+1}\frac{\partial P^{(i)}(x,y_i')}{\partial x}+\frac{\partial P^{(i)}(x,y_i')}{\partial y_i'}((-1)^{i-1}\lambda(x)+\alpha_i x^{\sigma})y_i'$$

$$=(\lambda(x)\mathbb{1}(1,-1)+x^{\sigma}\mathbb{1}(\alpha))P^{(i)}+f(x,{}^t(\delta_{i1}y_1',\delta_{i2}y_2')+P^{(i)}),\ i=1,\ 2,$$

δ_{ij} being Kronecker's delta.

We first obtain a formal solution of (8.6) of the form

$$(8.7)\qquad P^{(i)}(x,y_i')=\sum_{k\geq 2}p_k^{(i)}(x)y_i'^k.$$

By substituting (8.7) in (8.6) and by equating the coefficients of $y_i'^k$ ($k\geq 2$), we have

$(8.8)_k$

$$x^{\sigma+1}\frac{dp_k^{(i)}}{dx}+\{\lambda(x)\mathbb{1}((-1)^{i-1}k-1,(-1)^{i-1}k+1)+x^{\sigma}\mathbb{1}(k\alpha_i-\alpha_1,k\alpha_i-\alpha_2)\}p_k^{(i)}$$
$$=c_k^{(i)},$$

where each component of $c_k^{(i)}$ is a polynomial of components of $p_h^{(i)}$'s with $2\leq h\leq k-1$. It is easy to see that we can determine successively the unique solutions $p_k^{(i)}$'s of $(8.8)_k$'s with $p_k^{(i)}\in A(\varepsilon,r)$ by using Lemma 1.

We next prove the convergence of (8.7). Let $Y_i(x)$ be the general solution of

$$x^{\sigma+1}\,dY_i/dx=((-1)^{i-1}\lambda(x)+\alpha_i x^{\sigma})Y_i,$$

then it follows that the formal series $P^{(i)}=P^{(i)}(x,Y_i(x))$ formally satisfies

$$x^{\sigma+1}\frac{dP^{(i)}}{dx}=(\lambda(x)\mathbb{1}(1,-1)+x^{\sigma}\mathbb{1}(\alpha))P^{(i)}$$

$$+ f(x, {}^t(\delta_{i1}Y_1(x), \delta_{i2}Y_2(x)) + P^{(i)}).$$

Therefore, by virtue of a theorem due to Iwano [3], we see that $P^{(i)}(x, y_i')$ converges absolutely and uniformly for $x \in S(\varepsilon, r)$, $|y_i'| < \rho$, provided that $r, \rho > 0$ are sufficiently small.

Finally we show that, for the $P^{(i)}$'s determined above, transformation (8.1) reduces system (1.1) to a system of the form (8.2). By differentiating the both sides of (8.1) and by using the fact that $P^{(i)}$'s are the solutions of (8.6), we have

$$(1 + (\frac{\partial P^{(1)}}{\partial y_1'}, \frac{\partial P^{(2)}}{\partial y_2'})) \cdot x^{\sigma+1} dy'/dx$$
$$= (1 + (\frac{\partial P^{(1)}}{\partial y_1'}, \frac{\partial P^{(2)}}{\partial y_2'})) \cdot (\lambda(x) \mathbb{I}(1,-1) + x^{\sigma} \mathbb{I}(\alpha)) y' + F(x, y'),$$

where

$$F(x, y') = f(x, y' + \sum_{i=1}^{2} P^{(i)}) - \sum_{i=1}^{2} f(x, {}^t(\delta_{i1}y_1', \delta_{i2}y_2') + P^{(i)}).$$

We see that $F(x, y')$ vanishes on $y_1'y_2' = 0$ and hence $F(x, y') = O(y_1'y_2')$. Therefore, since $(\partial P^{(i)}/\partial y_i')(x, 0) = 0$, the system transformed by (8.1) is of the form (8.2) with $f_1' = O(y_2')$, $f_2' = O(y_1')$. Thus we have proved the existence of the first transformation.

Concerning transformations (8.3) and (8.4), we only state that we can obtain them by solving the following equations

$$x^{\sigma+1} \frac{\partial P'^{(i)}(x, y_i'')}{\partial x} + \frac{\partial P'^{(i)}(x, y_i'')}{\partial y_i''}((-1)^{i-1}\lambda(x) + \alpha_i x^{\sigma}) y_i''$$
$$= \mathbb{I}(f'(x, {}^t(\delta_{i1}y_1'', \delta_{i2}y_2'') + P'^{(i)}))({}^t(1,1) + P'^{(i)}), \qquad i = 1, 2,$$

and

$$x^{\sigma+1} \frac{\partial P''^{(i)}(x, z_i)}{\partial x} + \frac{\partial P''^{(i)}(x, z_i)}{\partial z_i}((-1)^{i-1}\lambda(x) + \alpha_i x^{\sigma}) z_i$$

$$= \mathbb{1}(f''(x, {}^t(\delta_{i1}z_1, \delta_{i2}z_2) - \alpha'(x))) \cdot {}^t(1,1) - (\alpha_1 + \alpha_2)x^\sigma \cdot p''^{(i)}, \quad i=1,\ 2,$$

where

$$\alpha'(x) = f''(x,0) = (\alpha_1 + \alpha_2)x^\sigma \cdot {}^t(1,1).$$

CHAPTER II.

GENERAL SOLUTIONS OF PAINLEVÉ EQUATIONS (I) ∿ (V)

§ 1. Contents of Chapter II.

The purpose of this chapter is to show that we can construct general solutions of Painlevé differential equations at fixed singular points of irregular type. For this purpose we first recall an immediate consequence of a general theory established in Chapter I.

We consider a 2-system of the form

$$(1.1)\qquad w^2\, dv/dw = (\mathbb{1}(1,-1) + w\mathbb{1}(\alpha))v + \sum_{|k|\geq 2} h_k(w)v^k$$

having an irregular singular point of Poincaré rank 1 at $w=0$. Here v and $h_k(w)={}^t(a_k(w),b_k(w))$ $(|k|\geq 2)$ are 2-vectors, α is a constant 2-vector ${}^t(\alpha_1,\alpha_2)$, $h_k(w)\in\mathcal{A}(\varepsilon,r)$ $(|k|\geq 2)$ and $\sum h_k(w)v^k$ converges absolutely and uniformly for

$$(1.2)\qquad w\in S(\varepsilon,r),\ |v|\equiv \max(|v_1|,|v_2|) < \rho,$$

and represents there a bounded holomorphic vector function. $\mathcal{A}(\varepsilon,r)$ denotes the set of all 2-vector functions of which the components are the elements of the $A(\varepsilon,r)$ defined in § 2 in Chapter I.

For system (1.1), we define the constants α_1' and α_2' by

$$(1.3)\qquad \begin{aligned} \alpha_1' &= \lim_{w\to 0,\, w\in S(\varepsilon,r)} (a_{21} - a_{20}a_{11} + a_{11}b_{11} + 2a_{02}b_{20}/3) \\ \alpha_2' &= \lim_{w\to 0,\, w\in S(\varepsilon,r)} (b_{12} + b_{02}b_{11} - a_{11}b_{11} - 2a_{02}b_{20}/3). \end{aligned}$$

Then, from the transformation (1.8) and Theorem 3 in Chapter I, we obtain the following theorem: <u>Assume that</u>

$$(A)\qquad \alpha_1 + \alpha_2 = 1,\quad \alpha_1' + \alpha_2' = 0$$

is satisfied, then for every ε' with $\varepsilon' > \varepsilon$, there exists a general solution $v = \psi(w,V(w))$ of (1.2) having the properties that (i) $\psi(w,V)$ is a bounded holomorphic function of w and $V = {}^t(V_1,V_2,V_3)$ in a domain in the (w,V)-space defined by

$$(1.4) \qquad w \in S(\varepsilon',r'), \quad |V| < \rho'$$

and admits there a uniformly convergent expansion in powers of V of the form

$$\psi(w,V) = \mathbb{D}(V_1,V_2)\Big(\sum_{|k|\geq 0} p_k(w)V^k\Big) + \mathbb{D}(V_2{}^2,V_1{}^2)\Big(\sum_{|k|\geq 0} p_k'(w)V^k\Big),$$

r' and ρ' being small positive constants. Here $p_0(w) = {}^t(1,1)$ and $p_k(w)$, $p_k'(w) \in \mathcal{A}(\varepsilon',r')$ $(|k|\geq 0)$.

(ii) $V(w) = {}^t(V_1(w),V_2(w),V_3(w))$ is the vector function defined by

$$(1.5) \qquad \begin{aligned} V_j(w) &= C_j \exp((-1)^j/w) w^{\alpha_j+\alpha_j' C_1 C_2}, \quad j = 1, 2, \\ V_3(w) &= C_1 C_2, \end{aligned}$$

C_1 and C_2 being arbitrary constants.

As was noticed by K. Okamoto [14], every Painlevé equation can be expressed as a Hamiltonian system

$$(1.6) \qquad dy_1/dt = \partial H/\partial y_2, \quad dy_2/dt = -\partial H/\partial y_1,$$

where H is a polynomial of y_1 and y_2 of which the coefficients are rational functions of t. If we eliminate the variable y_2 in (1.6), then we have the corresponding Painlevé equation. Therefore, in order to obtain general solutions of Painlevé equations near the fixed singular points of irregular type, it is sufficient to show that the Hamiltonian system (1.6) associated with each Painlevé equation can be reduced to a system (1.1) satisfying assumption (A). We note that Painlevé equation (VI) has no irregular singular point and that each of other Painlevé equations has only one irregular singular point which is the point at infinity.

The transformation for each Painlevé equation is decomposed into the product of four transformations. The first transformation maps the point at infinity to the origin. The second transformation is a singular one which normalizes the system (1.6) into a system of the form

$$(1.7) \qquad w^2dz/dw = a(w) + (\mathbb{1}(1,-1) + A(w))z + \sum_{|k|\geq 2} f_k(w)z^k.$$

We make the third transformation in order to eliminate the a(w) and the fourth one in order to change the system thus obtained to a system of the form (1.1). Among these four transformations, the singular transformation is most essential and it will be found out by observing a formal transformation which changes system (1.6) into system (1.1). In 3.2 we shall explain in detail how to obtain this singular one for the system (1.6) associated with Painlevé equation (I).

In § 2, we give two lemmas which are useful for constructing the second and third transformations. The subsequent sections are devoted to reducing the Hamiltonian system (1.6) associated with each Painlevé equation to a system of the form (1.1) satisfying (A). In each section we denote the system (1.6) by (E_0) and the reduced system of the form (1.1) by (E_4). It is shown in 5.2, 5.3 or 7.2 that the singular point of Painlevé equation (III) or (V) reduces to that of regular type for some special values of parameters. Since the reduction devised by several authors ([16],[10],[18]) can be applied in these exceptional cases, we do not make further investigations of the cases.

§ 2. Lemmas.

In this section we state two known lemmas which will be used to obtain transformations (T_3) and (T_4) in each of the subsequent sections.

<u>Lemma 1.</u> (Malmquist [11],[12],[13]). Let

$$w^2 dz/dw = a(w) + (\mathbb{1}(1,-1) + A(w))z + f(w,z) \tag{2.1}$$

be a 2-system of nonlinear differential equations, where we assume:

(i) $z={}^t(z_j)$, $a={}^t(a_j)$ and $f={}^t(f_j)$ are 2-vectors.

(ii) $a(w) \in \mathcal{A}(\varepsilon,r)$ and $a(w) = O(w)$, O denoting Landau's symbol.

(iii) $A = (a_{ij})$ is a 2×2 matrix with components $a_{ij}(w) \in \mathcal{A}(\varepsilon,r)$ and $a_{ij}(w) = O(w)$ $(1 \leq i,j \leq 2)$.

(iv) $f(w,z)$ is a bounded holomorphic vector function in a domain

$$w \in S(\varepsilon,r),\ |z| < \rho, \tag{2.2}$$

and has there a Taylor expansion of the form

$$f(w,z) = \sum_{|k|\geq 2} f_k(w) z^k,$$

with $f_k(w) \in \mathcal{A}(\varepsilon,r)$ $(|k|\geq 2)$. Then there exists a unique holomorphic solution $z = p(w)$ of (2.1) in $S(\varepsilon,r')$ which is asymptotically developable into the formal power series solution of w, as w tends to 0 through $S(\varepsilon,r')$ for sufficiently small positive r'.

Lemma 2. (Hukuhara [1]). Consider a 2-system of nonlinear differential equations

$$(2.2) \qquad w^2 du/dw = (\mathbb{1}(1,-1) + B(w))u + g(w,u).$$

Let $B(w) = (b_{ij}(w))$ be a 2×2 matrix with $b_{ij}(w) \in \mathcal{A}(\varepsilon,r)$ and

$$b_{11}(w) = \alpha_1 w + O(w^2),\ b_{22}(w) = \alpha_2 w + O(w^2),$$

$$b_{12}(w),\ b_{21}(w) = O(w),$$

$\alpha={}^t(\alpha_j)$ a constant 2-vector and let g have the same properties as f has in Lemma 1. Then there exists a linear transformation

$$(2.3) \qquad u = P(w)v$$

which changes (2.2) to a system of the form

$$(2.4) \qquad w^2 dv/dw = (\mathbb{1}(1,-1) + w\mathbb{1}(\alpha))v + h(w,v).$$

Here $P(w)$ is a bounded holomorphic matrix with components in $\mathcal{A}(\varepsilon,r')$ such that $\lim\limits_{w\to 0,\, w\in S(\varepsilon,r')} P(w) = 1$ and h is a vector function having the same properties as g in a domain

$$w \in S(\varepsilon,r'),\quad |v| < \rho,$$

provided that r' and ρ are sufficiently small positive.

We remark that these lemmas hold in the sector $S(\varepsilon,r')$ containing a singular direction of the equations (2.1) and (2.2).

In Lemma 2 we see that h is given by

$$(2.5) \qquad h(w,v) = P^{-1}(w)g(w,P(w)v),$$

so that we have

$$(2.6) \qquad \lim_{w\to 0,\, w\in S(\varepsilon,r')} h(w,v) = \lim_{w\to 0,\, w\in S(\varepsilon,r')} g(w,v).$$

§ 3. Painlevé equation (I).

<u>3.1.</u> The first equation of Painlevé is given by

$$\text{(P.I)} \qquad \frac{d^2\lambda}{dt^2} = 6\lambda^2 + t,$$

which is equivalent to the 2-system

$$(E_0) \qquad \frac{dy}{dt} = \begin{pmatrix} 0 \\ t \end{pmatrix} + \begin{pmatrix} 0,1 \\ 0,0 \end{pmatrix} y + \begin{pmatrix} 0 \\ 6y_1{}^2 \end{pmatrix},$$

where $y_1 = \lambda$, y_1 being the first component of the 2-vector y.

By the change of variables

$$(T_1) \qquad t = x^{-1},$$

(E_0) is transformed into

$$(E_1) \qquad x^3 \frac{dy}{dx} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0,x \\ 0,0 \end{pmatrix} y - \begin{pmatrix} 0 \\ 6xy_1{}^2 \end{pmatrix}.$$

We define a complex constant $\varkappa$ and a transformation (T_2) from the variables (x,y) to (w,z) by

$$(3.1) \qquad \varkappa = (-24)^{1/4}$$

$$(T_2) \qquad \begin{aligned} x &= (4\varkappa w/5)^{4/5} \\ y &= \mathbb{1}(x^{-1/2}, \varkappa x^{-3/4})\left(\begin{pmatrix} \varkappa^2/12 \\ 0 \end{pmatrix} + \begin{pmatrix} 1,1 \\ -1,1 \end{pmatrix} z\right). \end{aligned}$$

Then (T_2) transforms (E_1) into

$$(E_2) \qquad w^2 \frac{dz}{dw} = \begin{pmatrix} \varkappa^2 w/60 \\ \varkappa^2 w/60 \end{pmatrix} + \left(\mathbb{1}(1,-1) + \begin{pmatrix} w/2, & -w/10 \\ -w/10, & w/2 \end{pmatrix}\right) z + f(w,z),$$

where

$$(3.2) \qquad f(w,z) = 3(z_1 + z_2)^2/\varkappa^2 \cdot {}^t(1,-1).$$

It follows from Lemma 1 that there exists a bounded holomorphic solution $z = p(w)$ of (E_2) admitting the asymptotic expansion as

$$(3.3) \qquad p(w) \sim {}^t(-\varkappa^2 w/60 + \cdots,\ \varkappa^2 w/60 + \cdots)$$

in $S=S(\varepsilon,r')$, $r' > 0$ being small. Then the transformation

$$(T_3) \qquad z = u + p(w)$$

transforms (E_2) into a system of the form

$$(E_3) \qquad w^2 du/dw = (\mathbb{1}(1,-1) + B(w))u + g(w,u),$$

where $B(w)$ and $g(w,u)$ are given by

$$(3.4) \qquad B(w) = \begin{pmatrix} w/2 + 6(p_1+p_2)/\varkappa^2, & -w/10 + 6(p_1+p_2)/\varkappa^2 \\ -w/10 - 6(p_1+p_2)/\varkappa^2, & w/2 - 6(p_1+p_2)/\varkappa^2 \end{pmatrix}$$

$$(3.5) \qquad g(w,u) = f(w,u).$$

Note that $p_1(w) + p_2(w) = O(w^2)$ as is shown by (3.3).

By using Lemma 2 we can choose a holomorphic matrix $P(w) \in \mathcal{A}(\varepsilon,r)$ with $\lim_{w\to 0, w\in S} P(w) = 1$ so that the transformation

$$(T_4) \qquad u = P(w)v$$

changes (E_3) to a system of the form

$$(E_4) \qquad w^2 dv/dw = (\mathbb{1}(1,-1) + w\mathbb{1}(1/2,1/2))v + h(w,v).$$

Here we see

$$(3.6) \qquad \lim_{w\to 0, w\in S} h(w,v) = 3(v_1 + v_2)^2/\varkappa^2 \cdot {}^t(1,-1),$$

by (3.5) and (2.6). Therefore by (1.3) and (3.6), we obtain $\alpha' = (5/2,-5/2)$.

Thus we have proved that (E_0) can be reduced to a system (E_4) satisfying assumption (A) where $\alpha = (1/2,1/2)$ and $\alpha' = (5/2,-5/2)$.

3.2. In this subsection, we explain how to obtain the singular transformation (T_2). For this purpose, we shall show the procedure of constructing a formal transformation which formally changes (E_1) to a formal system of the form (1.1).

First we notice that (E_1) has a formal solution of the form

(3.7) $y = \mathbb{1}(x^{-1/2},x^{1/2})$

$\cdot$(2-vector whose components are formal power series of $x^{5/2}$).

If we put

$$(T_2') \qquad x^{5/2} = \zeta, \quad y = \mathbb{1}(x^{-1/2},x^{1/2})z,$$

then (E_1) is changed to

$$(E_2') \qquad \zeta^2 \frac{dz}{d\zeta} = \frac{1}{5}\cdot\left(\begin{pmatrix}0\\-2\end{pmatrix} + \begin{pmatrix}\zeta, & -2\zeta\\ 0, & -\zeta\end{pmatrix}z - \begin{pmatrix}0\\ 12{z_1}^2\end{pmatrix}\right).$$

It is easy to verify that (E_2') has a formal power series solution of ζ of the form

$$(3.8) \qquad z = q(\zeta) \equiv {}^t((-1/6)^{1/2},(-1/6)^{1/2}/2) + O(\zeta).$$

Here $O(\zeta)$ denotes a formal power series of ζ of which the lowest power is equal or greater than 1. In the following, we also use the obvious notation $O(\zeta^j)$ or $O(|v|^j)$. Then the formal change of variables

$$(T_3') \qquad z = s + q(\zeta)$$

transforms (E_2') into

$$(E_3') \qquad \zeta^2 \frac{ds}{d\zeta} = -\frac{1}{5}\cdot\left(\begin{pmatrix}-\zeta, & 2\zeta\\ 24q_1(\zeta), & \zeta\end{pmatrix}s + \begin{pmatrix}0\\ 12{s_1}^2\end{pmatrix}\right)$$

which has no constant term with respect to s.

Now we consider the shearing transformation

$$(T_4')\qquad s = \mathbb{1}(1,\zeta^{-1/2})u,$$

which changes (E_3') to

$$(E_4')\qquad \zeta^{3/2}\frac{du}{d\zeta} = -\frac{1}{5}\cdot\left(\begin{pmatrix}-\zeta^{1/2}, & 2\\ 24q_1(\zeta), & 3\zeta^{1/2}/2\end{pmatrix}u + \begin{pmatrix}0\\ 12u_1{}^2\end{pmatrix}\right).$$

Putting

$$(T_5')\qquad \zeta^{1/2} = \xi,$$

we have

$$(E_5')\quad \xi^2\frac{du}{d\xi} = -\frac{4}{5}\left(\begin{pmatrix}0 & ,1\\ (-24)^{1/2} & ,0\end{pmatrix} + \begin{pmatrix}-1/2, & 0\\ 0 & ,3/4\end{pmatrix}\xi + O(\xi^2)\right)u - \frac{24}{5}\begin{pmatrix}0\\ u_1{}^2\end{pmatrix}.$$

Next we diagonalize the leading matrix of system (E_5'). We see that the transformation of the form

$$(T_6')\qquad u = \left(\begin{pmatrix}1, & 1\\ -\varkappa, & \varkappa\end{pmatrix} + O(\xi)\right)v,$$

$\varkappa$ being a constant defined by (3.1), changes (E_5') to a system of the form

$$(E_6')\qquad \xi^2\frac{dv}{d\xi} = (\mathbb{1}(4\varkappa/5,-4\varkappa/5) + \xi\mathbb{1}(1/2,1/2))v + O(|v|^2).$$

Finally the scale transformation

$$(T_7')\qquad \xi = 4\varkappa w/5$$

changes (E_6') to a system of the desired form

$$w^2 dv/dw = (\mathbb{1}(1,-1) + w\mathbb{1}(1/2,1/2))v + O(|v|^2).$$

Now by composing the above transformations (T_j') $(j=2,\cdots,7)$ we see that there exists a formal transformation of the form

$$(3.9)\quad \begin{aligned} x &= (4\varkappa w/5)^{4/5} \\ y &= \mathbb{I}(x^{-1/2},\varkappa x^{-3/4})\left\{\begin{pmatrix}\varkappa^2/12\\0\end{pmatrix} + O(w) + \left(\begin{pmatrix}1, & 1\\-1, & 1\end{pmatrix} + O(w)\right)v\right\}, \end{aligned}$$

which transforms formally system (E_1) into a formal system of the form (1.1). We remark that we obtain the singular transformation (T_2) by omitting the two terms written by $O(w)$ in (3.9).

§ 4. Painlevé equation (II).

The second equation of Painlevé is given by

$$(\mathrm{P.II})\qquad \frac{d^2\lambda}{dt^2} = 2\lambda^3 + t\lambda + \alpha,$$

which is equivalent to the 2-system

$$(E_0)\qquad \frac{dy}{dt} = \begin{pmatrix}-\ t/2\\ \alpha + 1/2\end{pmatrix} + \begin{pmatrix}0,1\\0,0\end{pmatrix}y + \begin{pmatrix}-y_1^2\\ 2y_1y_2\end{pmatrix}.$$

where $y_1 = \lambda$ (cf. [14]).

The change of variables

$$(T_1)\qquad t = x^{-1}$$

reduces (E_0) to

$$(E_1)\qquad x^3\frac{dy}{dx} = \begin{pmatrix}1/2\\ -(\alpha + 1/2)x\end{pmatrix} + \begin{pmatrix}0, & -x\\0, & 0\end{pmatrix}y + \begin{pmatrix}xy_1^2\\ -2xy_1y_2\end{pmatrix}.$$

We define a transformation (T_2) from (x,y) to (w,z) by

$$x = (2w/3)^{2/3}$$

(T_2)

$$y = \mathbb{1}(x^{-1/2}, x^{-1})\left(\begin{pmatrix}0\\1/2\end{pmatrix} + \begin{pmatrix}1,1\\-1,1\end{pmatrix}z\right).$$

Then (T_2) transforms (E_1) into

$$(E_2) \qquad w^2\frac{dz}{dw} = \begin{pmatrix}\alpha w/3\\-\alpha w/3\end{pmatrix} + \left(\mathbb{1}(1,-1) + w\cdot\begin{pmatrix}1/2,-1/6\\-1/6,\ 1/2\end{pmatrix}\right)z + f(w,z),$$

where

$$(4.1) \qquad f(w,z) = \tfrac{1}{2}\cdot{}^t(-z_1{}^2 + 2z_1z_2 + 3z_2{}^2,\ 3z_1{}^2 + 2z_1z_2 - z_2{}^2).$$

It follows from Lemma 1 that there exists a bounded holomorphic solution $z = p(w)$ of (E_2) having the asymptotic expansion as

$$(4.2) \qquad p(w) \sim {}^t(-\alpha w/3 + \cdots,\ -\alpha w/3 + \cdots)$$

in $S=S(\varepsilon,r')$. Then the transformation

$$(T_3) \qquad z = u + p(w)$$

transforms (E_2) into

$$(E_3) \qquad w^2du/dw = (\mathbb{1}(1,-1) + B(w))u + g(w,u),$$

where $B(w)$ and $g(w,u)$ are given by

$$(4.3) \qquad B(w) = \begin{pmatrix}w/2 - p_1 + p_2, & -w/6 + p_1 + 3p_2\\-w/6 + 3p_1 + p_2, & w/2 + p_1 - p_2\end{pmatrix}$$

$$(4.4) \qquad g(w,u) = f(w,u).$$

Note that $p_1(w) - p_2(w) = O(w^2)$, which is a consequence of (4.2).

By Lemma 2 the linear term in (E_3) is simplified by a transformation of the form

$$(T_4) \qquad u = P(w)v,$$

namely, (T_4) transforms (E_3) into

$$(E_4) \qquad w^2 dv/dw = (\mathbb{1}(1,-1) + w\mathbb{1}(1/2,1/2))v + h(w,v).$$

Here it follows from (4.4) and (2.6) that $h(w,v)$ satisfies

$$(4.5) \qquad \lim_{w\to 0, w\in S} h(w,v) = \lim_{w\to 0, w\in S} f(w,v).$$

By using (4.5), we can verify that (E_4) satisfies assumption (A) where $\alpha = (1/2,1/2)$ and $\alpha' = (3,-3)$.

§ 5. Painlevé equation (III).

The third equation of Painlevé is given by

$$(\mathrm{P.III}) \qquad \frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\cdot\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\cdot\frac{d\lambda}{dt} + \frac{1}{t}\cdot(\alpha\lambda^2 + \beta) + \gamma\lambda^3 + \frac{\delta}{\lambda},$$

which is equivalent to the 2-system

$$(E_0) \qquad \frac{dy}{dt} = \begin{pmatrix} 2\eta_0 \\ -\eta_\infty(\vartheta_\infty+\vartheta_0) \end{pmatrix} + \begin{pmatrix} -(2\vartheta_0+1)t^{-1}, & 0 \\ 0, & (2\vartheta_0+1)t^{-1} \end{pmatrix} y$$

$$+ \begin{pmatrix} -2\eta_\infty y_1{}^2 + 4t^{-1}y_1{}^2 y_2 \\ 4\eta_\infty y_1 y_2 - 4t^{-1}y_1 y_2{}^2 \end{pmatrix},$$

where the following relations hold ([14]) among the constants α, β, γ, δ and ϑ_0, ϑ_∞, η_0, η_∞:

$$(5.1) \qquad \alpha = -4\eta_\infty\vartheta_\infty, \ \beta = 4\eta_0(\vartheta_0 + 1), \ \gamma = 4\eta_\infty{}^2, \ \delta = -4\eta_0{}^2.$$

Put

$$(T_1) \qquad t = x^{-1}.$$

Then (E_0) is written as

$$(E_1) \qquad x^2 \frac{dy}{dx} = \begin{pmatrix} -2\eta_0 \\ \eta_\infty(\vartheta_\infty+\vartheta_0) \end{pmatrix} + \begin{pmatrix} (2\vartheta_0+1)x, & 0 \\ 0 \quad , & -(2\vartheta_0+1)x \end{pmatrix} y$$
$$+ \begin{pmatrix} 2\eta_\infty y_1{}^2 - 4xy_1{}^2 y_2 \\ -4\eta_\infty y_1 y_2 + 4xy_1 y_2{}^2 \end{pmatrix}.$$

<u>5.1.</u> The case where $\eta_0 \neq 0$ and $\eta_\infty \neq 0$. In this case, we define a complex constant $\varkappa$ and a transformation (T_2) from (x,y) to (w,z) by

$$(5.2) \qquad \varkappa = (-\eta_0\eta_\infty)^{1/2}$$

$$(T_2) \qquad \begin{aligned} x &= -4\varkappa w \\ y &= \mathbb{1}(1,\eta_\infty x^{-1})\left(\begin{pmatrix} \varkappa/\eta_\infty \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \ , & \eta_0 \\ 0 \ , & 2\varkappa \end{pmatrix} z\right). \end{aligned}$$

Then (T_2) transforms (E_1) into

$$(E_2) \qquad w^2 \frac{dz}{dw} = a(w) + (\mathbb{1}(1,-1) + A(w))z + f(w,z),$$

where

$$a(w) = {}^t((3\vartheta_0 + \vartheta_\infty + 2)\varkappa w/2\eta_\infty,\ (\vartheta_\infty - \vartheta_0)w/2\varkappa),$$

$$A(w) = w \begin{pmatrix} 2\vartheta_0 + 1, & (4\vartheta_0 + 1)\eta_0 \\ 0 \quad , & -2\vartheta_0 \end{pmatrix},$$

$$(5.3) \qquad f(w,z) = {}^t(\eta_\infty z_1{}^2/2 + 2\varkappa z_1 z_2 + 9\eta_0\varkappa z_2{}^2/2 + 2\eta_\infty z_1{}^2 z_2 - 6\varkappa^2 z_1 z_2{}^2 - 4\varkappa^2\eta_0 z_2{}^3,\ -\eta_\infty z_1 z_2/\varkappa - \varkappa z_2{}^2 - 2\eta_\infty z_1 z_2{}^2 + 2\varkappa^2 z_2{}^3).$$

It follows from Lemma 1 that there exists a bounded holomorphic solution $z = p(w)$ of (E_2) admitting the asymptotic expansion as

$$(5.4) \qquad p(w) \sim {}^t(-(3\vartheta_0+\vartheta_\infty+2)\varkappa w/2\eta_\infty + \cdots,\ (\vartheta_\infty-\vartheta_0)w/2\varkappa + \cdots)$$

in $S=S(\varepsilon,r')$. The transformation

$$(T_2) \qquad z = u + p(w)$$

changes (E_2) to

$$(E_3) \qquad w^2 du/dw = (\mathbb{1}(1,-1) + B(w))u + g(w,u),$$

where $B(w)$ is given by

$$B(w) = \begin{pmatrix} (\vartheta_\infty - \vartheta_0)w/2 + O(w^2), & O(w) \\ O(w) & , (1 - (\vartheta_\infty - \vartheta_0)/2)w + O(w^2) \end{pmatrix}$$

and $g(w,u)$ satisfies

$$(5.5) \qquad \lim_{w\to 0, w\in S} g(w,u) = \lim_{w\to 0, w\in S} f(w,u).$$

By the same method as in §§ 3 and 4, the linear term in system (E_3) is simplified by a transformation of the form,

$$(T_4) \qquad u = P(w)v,$$

namely, (T_4) transforms (E_3) into

$$(E_4) \qquad w^2 dv/dw = (\mathbb{1}(1,-1) + C(w))v + h(w,v).$$

Here it holds that

$$C(w) = w\cdot\mathbb{1}((\vartheta_\infty - \vartheta_0)/2,\ 1 - (\vartheta_\infty - \vartheta_0)/2),$$

and

$$\lim_{w\to 0, w\in S} h(w,v) = \lim_{w\to 0, w\in S} f(w,v).$$

We can verify that system (E_4) satisfies assumption (A) where

$\alpha = ((\vartheta_\infty - \vartheta_0)/2,\ (1-(\vartheta_\infty - \vartheta_0))/2)$ and $\alpha' = (-\eta_\infty, \eta_\infty)$.

5.2. The case where $\eta_0 \neq 0$ and $\eta_\infty = 0$. In this case, we define a transformation (T_a) from (x,y) to (x,z) by

$$(T_a) \qquad y = \mathbb{1}(x^{-1}, x)z.$$

Then (T_a) transforms system (E_1) into a system of regular type

$$(E_a) \qquad x\, dz/dx = {}^t(-2\eta_0, 0) + \mathbb{1}(2(\vartheta_0+1), -2(\vartheta_0+1))z + {}^t(-4z_1{}^2 z_2, 4z_1 z_2{}^2).$$

5.3. The remaining cases. In these cases, we define a transformation (T_b) from (x,y) to (x,z) by

$$(T_b) \qquad y = \mathbb{1}(x, x^{-1})z.$$

Then (T_b) transforms system (E_1) into a system of regular type,

$$(E_b) \qquad x\, dz/dx = {}^t(\eta_\infty(\vartheta_\infty + \vartheta_0),\ 0) + \mathbb{1}(2\vartheta_0, -2\vartheta_0)z + 2\cdot{}^t(\eta_\infty z_1{}^2 - 2z_1{}^2 z_2,\ -2\eta_\infty z_1 z_2 + 2z_1 z_2{}^2).$$

§ 6. Painlevé equation (IV).

The fourth equation of Painlevé is given by

$$(\text{P.IV}) \qquad \frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\cdot\left(\frac{d\lambda}{dt}\right)^2 + \frac{3}{2}\lambda^3 + 4t\lambda^2 + 2(t^2 - \alpha)\lambda + \frac{\beta}{\lambda},$$

which is equivalent to the 2-system

$$(E_0) \qquad \frac{dy}{dt} = -\begin{pmatrix} 2\vartheta_0 \\ \vartheta_\infty \end{pmatrix} + \begin{pmatrix} -2t, & 0 \\ 0, & 2t \end{pmatrix} y + \begin{pmatrix} -y_1{}^2 + 4y_1 y_2 \\ 2y_1 y_2 - 2y_2{}^2 \end{pmatrix},$$

where the relations

(6.1) $$\alpha = -\vartheta_0 + 2\vartheta_\infty + 1, \quad \beta = -2\vartheta_0^2$$

hold ([14]).

Put

(T$_1$) $$t = x^{-1}.$$

Then (E$_0$) is written by

(E$_1$) $$x^3 \frac{dy}{dx} = x \cdot \begin{pmatrix} 2\vartheta_0 \\ \vartheta_\infty \end{pmatrix} + \mathbb{1}(2,-2)y + x \cdot \begin{pmatrix} y_1^2 - 4y_1y_2 \\ -2y_1y_2 + 2y_2^2 \end{pmatrix}.$$

We define a transformation (T$_2$) from (x,y) to (w,z) by

(T$_2$) $$x = w^{1/2}, \quad y = \mathbb{1}(x^{-1}, x^{-1})z.$$

Then (T$_2$) transforms (E$_1$) into

(E$_2$) $$w^2 \frac{dz}{dw} = {}^t(\vartheta_0 w, \vartheta_\infty w/2) + (\mathbb{1}(1,-1) + w \cdot \mathbb{1}(1/2,1/2))z + f(w,z),$$

where $f(w,z)$ is given by

(6.2) $$f(w,z) = {}^t(z_1^2/2 - 2z_1z_2, \; -z_1z_2 + z_2^2).$$

It follows from Lemma 1 that there exists a bounded holomorphic solution $z = p(w)$ of (E$_2$) admitting the asymptotic expansion as

(6.3) $$p(w) \sim {}^t(-\vartheta_0 w + \cdots, \; \vartheta_\infty w/2 + \cdots)$$

in $S = S(\varepsilon, r')$. Then the change of variables

(T$_3$) $$z = u + p(w)$$

transforms (E$_2$) into

(E$_3$) $$w^2 du/dw = (\mathbb{1}(1,-1) + B(w))u + g(w,u),$$

where $B(w)$ is given by

$$(6.4) \qquad B(w) = \begin{pmatrix} w/2 + p_1 - 2p_2, & -2p_1 \\ -2p_2 & , w/2 - p_1 + 2p_2 \end{pmatrix},$$

and $g(w,u)$ satisfies

$$(6.5) \qquad g(w,u) = f(w,u).$$

Note that $p_1(w) - 2p_2(w) = -(\vartheta_0 + \vartheta_\infty)w + O(w^2)$.

By the same method as in §§ 3, 4 and 5, the linear term in system (E_3) is simplified by a transformation of the form,

$$(T_3) \qquad u = P(w)v,$$

namely, (T_3) transforms (E_3) into

$$(E_4) \qquad w^2 dv/dw = (\mathbb{1}(1,-1) + C(w))v + h(w,v).$$

Here $C(w)$ is given by

$$C(w) = w \cdot \mathbb{1}(1/2 - \vartheta_0 - \vartheta_\infty, 1/2 + \vartheta_0 + \vartheta_\infty),$$

and, by using (2.6) and (6.5), we have

$$\lim_{w \to 0, w \in S} h(w,v) = \lim_{w \to 0, w \in S} f(w,v).$$

Then we can verify that system (E_4) satisfies assumption (A) where $\alpha = (1/2 - \vartheta_0 - \vartheta_\infty,\ 1/2 + \vartheta_0 + \vartheta_\infty)$ and $\alpha' = (3,-3)$.

§ 7. Painlevé equation (V).

The fifth equation of Painlevé is given by

(P.V) $$\frac{d^2\lambda}{dt^2} = \left(\frac{1}{2\lambda} + \frac{1}{\lambda - 1}\right)\cdot\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\cdot\frac{d\lambda}{dt} + \frac{1}{t^2}\cdot(\lambda - 1)^2\left(\alpha\lambda + \frac{\beta}{\lambda}\right) + \gamma\,\frac{\lambda}{t} + \delta\,\frac{\lambda(\lambda + 1)}{\lambda - 1},$$

which is equivalent to the 2-system

(E_0) $$\frac{dy}{dt} =$$

$$- t^{-1}\cdot\begin{pmatrix} \vartheta_0 \\ \{(\vartheta_0+\vartheta_1)^2-\vartheta_\infty^2\}/4 \end{pmatrix} + \begin{pmatrix} \eta_1+(2\vartheta_0+\vartheta_1)t^{-1}, & 0 \\ 0 & ,-\{\eta_1+(2\vartheta_0+\vartheta_1)t^{-1}\} \end{pmatrix} y$$
$$+ t^{-1}\cdot\begin{pmatrix} -(\vartheta_0+\vartheta_1)y_1^2 + 2y_1y_2 - 4y_1^2y_2 + 2y_1^3y_2 \\ 2(\vartheta_0+\vartheta_1)y_1y_2 - y_2^2 + 4y_1y_2^2 - 3y_1^2y_2^2 \end{pmatrix}.$$

where the relations

(7.1) $$\alpha = \vartheta_\infty^2/2,\ \beta = -\vartheta_0^2/2,\ \gamma = -\eta_1(\vartheta_1 + 1)\ ,\ \delta = -\eta_1^2/2$$

hold ([14]).

Put

(T_1) $$t = x^{-1}.$$

Then (E_0) is written by

(E_1) $$x^2\frac{dy}{dx} =$$

$$x\cdot\begin{pmatrix} \vartheta_0 \\ \{(\vartheta_0+\vartheta_1)^2-\vartheta_\infty^2\}/4 \end{pmatrix} + \begin{pmatrix} -\eta_1-(2\vartheta_0+\vartheta_1)x\ , & 0 \\ 0 & ,\ \eta_1+(2\vartheta_0+\vartheta_1)x \end{pmatrix} y$$
$$- x\cdot\begin{pmatrix} -(\vartheta_0+\vartheta_1)y_1^2 + 2y_1y_2 - 4y_1^2y_2 + 2y_1^3y_2 \\ 2(\vartheta_0+\vartheta_1)y_1y_2 - y_2^2 + 4y_1y_2^2 - 3y_1^2y_2^2 \end{pmatrix}$$

7.1. The case where $\eta_1 \neq 0$. In this case, we define a transformation (T_2) from (x,y) to (w,z) by

(T_2) $$x = -\eta_1 w, \quad y = 1\!\!1(1,x^{-1})z.$$

Then (T_2) transforms (E_1) into

(E_2) $$w^2 \frac{dz}{dw} = a(w) + (1\!\!1(1,-1) + A(w))z + f(w,z),$$

where

$$a(w) = {}^t(\vartheta_0, -\eta_1\{(\vartheta_0+\vartheta_1)^2 - \vartheta_\infty{}^2\}w^2/4),$$

$$A(w) = w\cdot\begin{pmatrix} -2\vartheta_0 - \vartheta_1, & 0 \\ 0 & , 2\vartheta_0 + \vartheta_1 + 1 \end{pmatrix},$$

(7.2) $$f(w,z) = (\vartheta_0+\vartheta_1)w\cdot{}^t(z_1{}^2, -2z_1z_2)$$

$$+ \frac{1}{\eta_1}\cdot{}^t(2(z_1z_2 - 2z_1{}^2z_2 + z_1{}^3z_2), - z_1{}^2 + 4z_1z_2{}^2 - 3z_1{}^2z_2{}^2).$$

By the same method as in the previous sections, we can verify that there exists a bounded holomorphic solution $z = p(w)$ of (E_2) having the asymptotic expansion as

(7.3) $$p(w) \sim {}^t(-\vartheta_0 w + \cdots, -\eta_1\{(\vartheta_\infty+\vartheta_1)^2 - \vartheta_\infty\}^2 w^2 + \cdots)$$

in $S=S(\varepsilon,r')$. Then the transformation

(T_3) $$z = u + p(w)$$

changes (E_2) to

(E_3) $$w^2du/dw = (1\!\!1(1,-1) + B(w))u + g(w,u),$$

where $B(w)$ is given by

(7.4) $$B(w) = \begin{pmatrix} -(2\vartheta_0 + \vartheta_1)w + O(w^2), & O(w) \\ O(w^3) & , (1 + 2\vartheta_0 + \vartheta_1)w + O(w^2) \end{pmatrix}$$

and $g(w,u)$ satisfies

$$\lim_{w\to 0, w\in S} g(w,u) = \lim_{w\to 0, w\in S} f(w,u). \tag{7.5}$$

In a similar way as in the previous sections, the linear term in system (E_3) is simplified by a transformation of the form,

$$u = P(w)v, \tag{T_4}$$

namely, (T_4) transforms (E_3) into

$$w^2 dv/dw = (\mathbb{I}(1,-1) + C(w))v + h(w,v). \tag{E_4}$$

Here $C(w)$ is given by

$$C(w) = w\cdot\mathbb{I}(2\vartheta_0 + \vartheta_1, 1 + 2\vartheta_0 + \vartheta_1),$$

and from (2.6) and (7.5), it follows that

$$\lim_{w\to 0, w\in S} h(w,v) = \lim_{w\to 0, w\in S} f(w,v).$$

Then we can verify that system (E_4) satisfies assumption (A) where $\alpha = (-2\vartheta_0 + \vartheta_1, 1 + 2\vartheta_0 + \vartheta_1)$ and $\alpha' = (-4/\eta_1, 4/\eta_1)$.

7.2. The case where $\eta_1 = 0$. In this case, system (E_1) is rewritten as

$$x\frac{dy}{dx} = \begin{pmatrix} \vartheta_0 \\ \{(\vartheta_0+\vartheta_1)^2 - \vartheta_\infty^2\}/4 \end{pmatrix} + \begin{pmatrix} -2\vartheta_0 - \vartheta_1, & 0 \\ 0 & , 2\vartheta_0+\vartheta_1 \end{pmatrix} y - \begin{pmatrix} -(\vartheta_0+\vartheta_1)y_1^2 + 2y_1y_2 - 4y_1^2y_2 + 2y_1^3y_2 \\ 2(\vartheta_0+\vartheta_1)y_1y_2 - y_2^2 + 4y_1y_2^2 - 3y_1^2y_2^2 \end{pmatrix}, \tag{E_1}$$

which has a regular singular point at $x=0$ $(t=\infty)$.

REFERENCES

[1] M. Hukuhara, Sur les points singuliers des équation différentielles linéaires, III, Mém. Fac. Sci., Kyushu Univ., 2(1941),. pp. 125-137.

[2] M. Hukuhara, T. Kimura and T. Matsuda, Equations différentielles ordinaires du premier ordre dans le champ complexe, Publications 7(1961), Japan Math. Soc., Tokyo.

[3] M. Iwano, Intégration analytique d'un système d'équations différentielles non linéaires dans le voisinage d'un point singulier, I, Ann. Mat. Pura Appl. Serie 4, 44(1957), pp. 261-292.

[4] M. Iwano, Intégration analytique d'un système d'équations différentielles non linéaires dans le voisinage d'un point singulier, II, ibid., 47(1959), pp. 91-150.

[5] M. Iwano, On a general solution of a nonlinear 2-system of the form $x^2dw/dx = \Lambda w + xh(x,w)$ with a constant diagonal matrix Λ of signature (1,1), Tôhoku Math. J.(2), 32(1980), pp. 453-486.

[6] M. Iwano, On general solution of a first-order non-linear differential equation of the form $x(dy/dx) = y(\lambda + f(x,y))$ with negative rational λ, Ann. Mat. Pura Appl. Serie 4, 126(1980), pp. 19-80.

[7] M. Iwano, On a general solution of a nonlinear n-system of the form $x^2dy/dx = (\mathbb{1}_n(\mu) + x\mathbb{1}_n(\alpha))y + xf(x,y)$ with a constant diagonal matrix $\mathbb{1}_n(\mu)$ of signatyre (m,n-m), Ann. Mat. Pura Appl. Serie 4, 130(1982), pp. 331-384.

[8] M. Iwano, On an n-parameter family of solutions of a nonlinear n-system with an irregular type singularity, to appear.

[9] M. Jimbo, T. Miwa, Y. Mori and M. Sato, Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent, Physica 1D(1980), pp. 80-158.

[10] H. Kimura, The construction of a general solution of a Hamiltonian system with regular type singularity and its application to Painlevé equations, Ann. Mat. Pura Appl. Serie 4, 134(1983), pp. 363-392.

[11] J. Malmquist, Sur l'étude analytique des solutions d'un système d'équations defférentielles dans le voisinage d'un point singulier d'indétermination, I, Acta Math. 73(1941), pp. 87-129.

[12] J. Malmquist, Sur l'étude analytique des solutions d'un système d'équations defférentielles dans le voisinage d'un point singulier d'indétermination, II, ibid., 74(1941), pp. 1-64.

[13] J. Malmquist, Sur l'étude analytique des solutions d'un système d'équations defférentielles dans le voisinage d'un point singulier d'indétermination, III, ibid., 74(1941), pp. 109-128.

[14] K. Okamoto, Polynomial Hamiltonians associated with Painlevé equations, I, Proc. Japan Acad. 56, Ser. A(1980), pp.264-268.

[15] S. Shimomura, Series expansions of Painlevé transcendents in the neighbourhood of a fixed singular point, Funkcial. Ekvac., 25(1982), pp. 185-197.

[16] S. Shimomura, Analytic integration of some nonlinear ordinary differetial equations and the fifth Painlevé equation in the neighbourhood of an irregular singular point, Funkcial. Ekvac., 26(1983), pp. 301-338.

[17] K. Takano, A 2-parameter family of solutions of Painlevé equation (V) near the point at infinity, Funkcial. Ekvac.,

26(1983), pp. 79-113.

[18] K. Takano, A normal form of a Hamiltonian system at a fixed singular point of the first kind, preprint.

[19] T. T. Wu, B. McCoy, C. A. Tracy and E. Barouch, Spin-spin correlation functions for the two-dimensional Ising model: Exact theory in the scaling region, Phys. Rev. B 13(1976), pp. 316-374.

[20] S. Yoshida, A general solution of a nonlinear 2-system without Poincaré's condition at an irregular singular point, Funkcial. Ekvac., 27(1984).

[21] S. Yoshida, 2-Parameter family of solutions for Painlevé equations (I) ∿ (V) at an irregular singular point, ibid., to appear.