Construction of Fuzzy Sets for Topological and Algebraic Structures

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位相および代数構造に適したファジイ集合の構成

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<u>CONTENTS</u>

0. Introduction.

0.1. Background

This paper examines construction of the theory of various effective fuzzy sets.

The notion of fuzzy sets was introduced to represent inexact or vague concepts. Similarly, the theory of probability was constructed to analyze uncertain phenomena. In this sense, fuzzy theory is related to the theory of probability. However, there is a great difference between them. Probability theory can be thought of as being concerned with quantity and fuzzy theory with quality. On one hand, probability theory is a branch of analysis. On the other hand, fuzzy set theory can be interpreted as a branch of axiomatic set theory. Scott and Solovay have used a somewhat similar construction "Boolean-valued sets", to obtain a proof of Cohen's famous result, that the Continuum Hypothesis is independent of the rest of formalized Zermelo-Fraenkel Set Theory (see [38] etc).

Now, before giving the aim and standpoint for this investigation, the definition of fuzzy sets will be introduced.

Let X be a non-empty set (a usual set of objects) whose generic elements are denoted by x. Membership of A, a classical subset of X, is often viewed as a characteristic function from X to $\{0,1\}$ such that

$$A(x) = \begin{cases} 1 & \text{if and only if} \quad x \in A \\ 0 & \text{if and only if} \quad x \notin A. \end{cases}$$

$\{0,1\}$ is called a truth value set. If the truth value set is allowed to be the real unit interval $[0,1]$, A is called a fuzzy set (Zadeh [44]). In this paper, $\{0,1\}$ is denoted by 2 and $[0,1]$

by I. This fuzzy set is called an I-fuzzy set.

Let L be a partially ordered set with order relation $\leq$. More general fuzzy sets can be define using the set L.

Definition (Goguen [10]). An L-fuzzy subset of X is a function from X to L. X is called the carrier (or universe), and L is called the truth value set.

In order to be able to extend the concept of set operators ('union' and 'intersection'), a lattice should be adopted for the truth value set. Goguen adopted a cl_{∞}-monoid, but many mathematicians adopt a completely distributive lattice with (order reversing) involution as the truth value set, with 'union' and 'intersection' on the same level.

Zadeh and many engineers have proposed various set-like operations for application to engineering problems such as pattern recognition, optimization, decision making, fuzzy algorithms and systems theory. However, most of their works used I-fuzzy sets. The unit interval I has frequently been used in various areas of mathematics, but it is not the best for the truth value set of fuzzy theory. I is a subset of R (the family of real numbers). R represents the intuitive concept of "continuity". The methods of construction used by Cantor and Dedekind are well known. After all, R is constructed by classifying, by some equivalence relation, objects with uncountable cardinality. It is not necessary for engineers to use concepts such as truth value sets. On the contrary, very complex truth value sets are difficult for engineers and non-mathematicians to use. For practicality, fuzzy sets should be constructed that are both suitable for various applications and mathematically (algebraically and topologically) easy to deal with. The problems is if it is possible to construct such a fuzzy theory and how it can be done.

0.2. Summary

The notation used in this thesis, is described in sections 3 and 4 of chapter 0. Section 3 outlines each chapter from the point of view of operators. Section 4 introduces the basic concept of fuzzy topological spaces.

Chapter 1 characterizes the class of operators which are pointwise, commutative, associative, compatible and averaging. This class of operators coincides with the class of mode-type operators. Consequently, the existential problem of averaging operators with good property can be solved.

In chapter 2, the concepts of II-fuzzy sets, II-fuzzy topological spaces and II-fuzzy linear topologies on vector spaces are introduced. In addition, some of the basic properties of II-fuzzy topological vector spaces are investigated using the notion of i-neighborhoods. Consequently, some problems in fuzzy topological space can be solved.

Section 1 of the last chapter, describes the discovery of subsets (denoted by $IQ_{\langle m\rangle}$) lying between I and II with the following properties:

(1) $II \supset IQ_{\langle m\rangle} \supset I$ as sublattice;

(2) there are suitable easy equivalence relations R_1 and R_2 such that II/R_1 is lattice isomorphic to $IQ_{\langle m\rangle}$ and $IQ_{\langle m\rangle}/R_2$ is lattice isomorphic to I;

(3) ${IQ_{\langle m\rangle}}^X$ form an Abelian group under a simple operation.

Section 2 of the last chapter introduces the concept of 'particle' in fuzzy sets to replace fuzzy points.

0.3. Operators of fuzzy sets

Fuzzy sets were made to extend the concept of sets, so it is important to consider set operators. The classical union and intersection of ordinary subsets of X can be extended by the following formulae, for A, B $\in L^X$

$$(A \vee B)(x) = \max[A(x), B(x)] \quad \text{for all} \quad x \in X$$

$$(A \wedge B)(x) = \min[A(x), B(x)] \quad \text{for all} \quad x \in X.$$

Also, the inclusion can be extended by

$$A \leq B \text{ if and only if } A = A \wedge B.$$

N.B. The symbols $\vee$, $\wedge$ and $\leq$ will be used here instead of $\cup$, $\cap$ and $\subset$, to distinguish between fuzzy set operators and ordinary set operators.

However, when L = I, many other operators can be defined on the basis of operators on I, for union and intersection.

Firstly, there are the following probabilistic-like operators:

$$(A \dot{+} B)(x) = A(x)+B(x)-A(x)\cdot B(x) \quad \text{for every} \quad x \in X$$

$$(A \bullet B)(x) = A(x)\cdot B(x) \quad \text{for every} \quad x \in X.$$

Secondly, there are the following operators which R.Giles [9] called bold union and intersection:

$$(A \oplus B)(x) = \min[1, A(x)+B(x)] \quad \text{for every} \quad x \in X$$

$$(A \odot B)(x) = \max[0, A(x)+B(x)-1] \quad \text{for every} \quad x \in X.$$

Other operators defined by

$$(A \dot{\vee} B)(x) = \begin{cases} A(x) & B(x)=0 \\ B(x) & A(x)=0 \\ 1 & \text{otherwise} \end{cases}$$

$$(A \dot{\wedge} B)(x) = \begin{cases} A(x) & B(x)=1 \\ B(x) & A(x)=1 \\ 0 & \text{otherwise.} \end{cases}$$

These operators satisfy the next inequality

$$A \underset{\bullet}{\wedge} B \leqq A \odot B \leqq A \cdot B \leqq A \wedge B \leqq A \vee B \leqq A \dot{+} B \leqq A \oplus B \leqq A \overset{\bullet}{\vee} B$$

To generalize the operators, some operators with parameter p were proposed. Yager proposed the following operators: for $p > 0$,

$$(A \overset{Y}{\vee} B)(x) = \min[1, \sqrt[p]{A(x)^p + B(X)^p}]$$

$$(A \underset{Y}{\wedge} B)(x) = 1 - \min[1, \sqrt[p]{(1-A(x))^p + (1-B(X))^p}].$$

Weber proposed the following: $p \leqq -1$,

$$(A \overset{W}{\vee} B)(x) = \min[1, A(x)+B(x)+p \cdot A(x) \cdot B(x)]$$

$$(A \underset{W}{\wedge} B)(x) = \max[0, (1+p)(A(x)+B(x)-1)-p \cdot A(x) \cdot B(x)].$$

Yager and Weber used these operators as measures of fuzziness. The aforementioned intersection operators satisfy the conditions of triangular norms. A triangular norm T is a 2-place function from $I \times I$ to I such that

(1) $T(0,0) = 0$; $T(a,1) = a$;

(2) $T(a,b) \leqq T(c,d)$ whenever $a \leqq c$, $b \leqq d$;

(3) $T(a,b) = T(b,a)$;

(4) $T(T(a,b),c) = T(a,T(b,c))$.

A corresponding concept can be used for the aforementioned union operators. Thus, various algebraic investigations into union and intersection operators are discussed. (See [24], [3], [32], [35], [36] and [37].) In Chapter 1, averaging operators with good algebraic properties are considered.

The complement of an ordinary subset of X can be expanded as follows:

$$A'(x) = 1 - A(x)$$

There are, of course, other definitions of complement. For example, $A'(x) = \frac{(1 - A(x))}{(1 + P \cdot A(x))}$ $\quad -1 < P < \infty$.

Which operators and complements should by selected and used? The answer to this question is different for different cases. It is important to examine properties of operators and to define new operators with good properties.

The infinite union and intersection of X was usually defined by for $A_j \in I^X$ $(j \in J)$,

$$(\bigvee_{j \in J} A_j)(x) = \sup_{j \in J} A_j(x)$$

$$(\bigwedge_{j \in J} A_j)(x) = \inf_{j \in J} A_j(x).$$

However, this definition causes some problems (see, example 1 in the next section) when the concept of topology is introduced. R. Lowen and many others researched I-fuzzy sets. In Chapter 2, the use of the two arrows set II instead of the unit interval I as the truth value set is proposed to avoid these problems, from the point of view of general topology and topological vector spaces. The two arrows set II is a completely distributive lattice with involution.

Chapter 3 deals with symmetric difference operators for fuzzy sets. In the framework of fuzzy set theory there may be different ways to define a symmetric difference.

Firstly, the fuzzy set $A \ominus B$ of element that belong more to A than to B or conversely is defined as

$$(A \ominus B)(x) = |A(x) - B(x)| \quad \text{for all } x \in X.$$

but this $\ominus$ is not associative.

Secondly, the fuzzy set $A \triangle B$ of the elements that approximately belong to A and not to B or conversely to B and not to A is defined as

$$(A \triangle B)(x) = (A'(x) \wedge B(x)) \vee (A(x) \wedge B'(x)) \text{ for all } x \in X.$$

It can be shown that this $\underline{\wedge}$ is associative. However, this operator is not satisfactory.

Throughout Chapter 2 and Chapter 3, it is asserted that good operators require good truth value sets.

0.4. Basic concepts and notation of fuzzy topological spaces

Let X be a non-empty set and $(L; \bigvee, \bigwedge, ')$ be a completely distributive lattice with involution , i.e. $x'' = x$, and if $x < y$ then $x' > y'$ $(x, y \in L)$. A constant fuzzy set is denoted by h_X, for $h \in L$. The supremum of L is denoted by 1 and the infimum of L is denoted by 0.

Definition 0.4.1. (Chang[2] and Goguen[12]) An L-fuzzy topology is a family $\mathcal{O}\iota$ of L-fuzzy sets of X which satisfies the following conditions:

(1) $0_X, 1_X \in \mathcal{O}\iota$,

(2) If $A, B \in \mathcal{O}\iota$, then $A \wedge B \in \mathcal{O}\iota$,

(3) If $A_j \in \mathcal{O}\iota$ for each $j \in J$, then $\bigvee_{j \in J} A_j \in \mathcal{O}\iota$.

An L-fuzzy topological space is denoted by $(X, \mathcal{O}\iota, L)$. Every member of $\mathcal{O}\iota$ is called an open fuzzy set. A fuzzy set is closed if and only if its involution is open. An ordinary topological space is a 2-fuzzy topological space, it will be denoted by $(X, \mathcal{O}\iota, 2)$.

Note.1. The above fuzzy topology $\mathcal{O}\iota$ is a subset of L^X, i.e. an element of ${}_2L^X$. Properly speaking, a fuzzy topology should be an element of ${}_LL^X$.

Note.2. Lowen's definition of fuzzy topology adopted condition (1') below instead of condition (1) (see [31], [25], [26], [27], [28], [29] and [30]),

(1') every constant fuzzy set $h_X \in \mathcal{O}\iota$.

Most of the concepts of ordinary topology can be generalized for fuzzy topology.

Let $(X, \mathfrak{T}, L)$ be an L-fuzzy topological space. The interior and closure of a fuzzy set are defined in same ways, and denoted by 'Int' and 'Cl' respectably, i.e. for $A \in L^X$,

$$\text{Int } A = \bigvee \{ B \in L^X \mid B \in \mathfrak{T} \text{ and } B \leq A \}$$
$$\text{Cl } A = \bigwedge \{ B \in L^X \mid B' \in \mathfrak{T} \text{ and } B \geq A \}.$$

A subfamily $\mathcal{B}$ of $\mathfrak{T}$ is a base for $\mathfrak{T}$ if and only if each member of $\mathfrak{T}$ can be expressed as the union of some member of $\mathcal{B}$. A subfamily $\mathcal{S}$ of $\mathcal{B}$ is a subbase for $\mathfrak{T}$ if and only if the family of finite intersections of member of $\mathcal{S}$ forms a base for $\mathfrak{T}$. A family $\mathcal{C}$ of fuzzy sets is a cover of a fuzzy set B if and only if $B \leq \bigvee \{A \mid A \in \mathcal{C}\}$. It is an open cover if and only if each member of $\mathcal{C}$ is an open fuzzy set. A subcover of $\mathcal{C}$ is a subfamily of $\mathcal{C}$ which is also a cover.

Let $f: X \longrightarrow Y$ be a function from X to Y. Then, f induces a function $F^{-1} : L^Y \longrightarrow L^X$ defined by

$$F^{-1}(B)(x) = B(f(x)) \quad (\text{i.e. } F^{-1}(B) = B \circ F).$$

Also, f induces another function $F : L^X \longrightarrow L^Y$ defined by

$$F(A)(y) = \begin{cases} \bigvee \{ A(x) \mid x \in f^{-1}(y) \} & \text{if } f^{-1}(y) \text{ is not empty} \\ 0 \text{ (minimal element of L)} & \text{otherwise.} \end{cases}$$

Then, $F^{-1}(F(A)) = A$. Thus, this F is called the fuzzy function from L^X to L^Y induced by the function f from X to Y, and F^{-1} is the inverse of F.

More generally, M.A.Erceg [5] defined fuzz relation and fuzz function without using the inducing function. (For distinction, we do not call it fuzzy function.)

Definition. 0.4.2. A fuzz relation between L^X and L^Y is a map

$$R : L^X \longrightarrow L^Y$$

such that

(1) $R(0_X) = 0_Y$ and

(2) $R(\bigvee_{j \in J} A_j) = \bigvee_{j \in J} R(A_j)$ for all $A_j \in L^X$

are satisfied.

The inverse fuzz relation R^{-1} is defined to be a map

$$R^{-1}: L^X \longrightarrow L^Y$$

where $R^{-1}(A) = \bigwedge\{B \in L^X | R(B') \leq A'\}$.

Definition. 0.4.3. A fuzz function from L^X to L^Y is a map

$$F: L^X \longrightarrow L^Y$$

such that

(1) $F(0_X) = 0_Y$,

(2) $F(\bigvee_{j \in J} A_j) = \bigvee_{j \in J} F(A_j)$ for all $A_j \in L^X$,

(3) $F^{-1}(A') = [F^{-1}(A)]'$

are satisfied.

In view of condition (3), we have $F^{-1}(A) = \bigvee\{B | F(B) \leq A\}$.

It is easy to prove that a fuzzy function F induced by the function f from X to Y is a fuzz function and its inverse fuzz function is the same as the inverse of F.

Definition. 0.4.4. Let $(X, \mathfrak{T}_1, L)$ and $(Y, \mathfrak{T}_2, L)$ be L-fuzzy topological spaces. A fuzzy function (or fuzz function) F from $(X, \mathfrak{T}_1, L)$ to $(Y, \mathfrak{T}_2, L)$ is fuzzy continuous if and only if $F^{-1}(U) \in \mathfrak{T}_1$ for every $U \in \mathfrak{T}_2$.

Definition. 0.4.5. A fuzzy topological space is compact if and only if each open cover of the space has a finite subcover.

In fuzzy topology, there have been many other definitions of compactness. The reason is that the above compact does not have good properties for I-fuzzy topology. (See example 1.)

In usual topological theory, there are some operations on topological spaces, i.e. methods of constructing new topological spaces from old ones. The most well-known and important six methods are "subspace", "sum of spaces", "Cartesian product", "inverse systems" and "function space". There are various extensions of above operations to fuzzy topology (see [42], [19], [5] etc). Wong[42] defined the following fuzzy product topology. If $(X_j, \mathcal{O}_j, L)$ are L-fuzzy topological spaces for $j \in J$, we define their product $\prod_{j \in J}(X_j, \mathcal{O}_j, L)$ to be the L-fuzzy topological space $(X, \mathcal{O}, L)$, where $X = \prod_{j \in J} X_j$ is the ordinary set product and $\mathcal{O}$ is the topology on X generated by the subbase

$$\mathcal{S} = \{ P_j^{-1}(A_j) \mid A_j \in \mathcal{O}_j,\ j \in J \},$$

where $P_j: L^X \longrightarrow L^X$ induced fuzzy function by the usual projection onto the i-th coordinate $p_j: X \longrightarrow X_j$. This $\mathcal{O}$ is the weakest topology such that each P_j is continuous.

Example 1 (in Goguen [12]). For $k, m \in \mathbb{N}$ (the family of natural numbers), let $P_{k,m}$ be a function from $\mathbb{N}$ to I such that

$$P_{k,m}(n) = \begin{cases} 0 & n > m \\ 1-1/k & n \leqq m. \end{cases}$$

Let $(X_j, \mathcal{O}_j, I)$ be an I-fuzzy topological space for $j \in N$, where $X_j = \mathbb{N}$ and $\mathcal{O}_j = \{ 0_X, 1_X \text{ and } P_{j,m}\ (m \in N)\}$.
Then, $(X_j, \mathcal{O}_j, I)$ is compact for every $j \in N$. However, $\prod_{j \in J}(X_j, \mathcal{O}_j, I)$ is not compact.

This example means that the Tychonoff theorem does not hold. To avoid this problem, Lowen changed the definition of topology and compact, Wang G.J. altered the definition of compact, Pu and Liu used another concept of cover and Hutton changed the definition of product. (See[27], [41], [34] and [19].) However, the fundamental cause of this problem is the truth value set I (which infinite union consist of supremum). The suggestion by Goguen [12] that a better lattice should be used is probably correct.

The definition of neighborhood is also a problem for I-fuzzy topological space theory.

Definition. 0.4.6 (in Warren [39]). A I-fuzzy set U in a I-fuzzy topological space $(X, \mathfrak{N}, I)$ is a neighborhood of a point $x \in X$ if and only if there exists $\theta \in \mathfrak{N}$ such that $U \geq \theta$ and $U(x) = \theta(x) > 0$.

$\mathcal{U}_x$ denotes the family of all neighborhoods of x which are determined by the fuzzy topology $\mathfrak{N}$ on X.

Theorem. 0.4.7 (in Warren [39]). Let $(X, \mathfrak{N}, I)$ be a I-fuzzy topological space. Then for each $x \in X$, $\mathcal{U}_x$ satisfies:

(1) $1_X \in \mathcal{U}_x$,

(2) if $U \in \mathcal{U}_x$, then $U(x) > 0$,

(3) if $U \in \mathcal{U}_x$, $U \leq V$ and $U(x) = V(x)$, then $V \in \mathcal{U}_x$,

(4) if $U_j \in \mathcal{U}_x$ $j \in J$, then $\bigvee \{U_j : j \in J\} \in \mathcal{U}_x$,

(5) if $U, V \in \mathcal{U}_x$, then $U \wedge V \in \mathcal{U}_x$,

(6) if $U \in \mathcal{U}_x$, then there exists $U \in \mathcal{U}_x$ such that $V \leq U$, $V(x) = U(x)$ and if $V(y) > 0$ then $V \in \mathcal{U}_y$.

Compare this theorem with theorem 2.3.2. proposed here.

Finally, the definitions (or equivalence definitions) of the separation axioms T_0 and T_1 in Hutton [20] should be examined.

Definition. 0.4.8. $(X, \mathfrak{A}, L)$ is T_0 if every $A \in L^X$ can be written in the form $A = \bigvee_{i \in I} \bigwedge_{j \in J} B_{ij}$, where B_{ij} is an open or closed set. Also, $(X, \mathfrak{A}, L)$ is T_1 if every $A \in L^X$ can be written in the form $A = \bigvee_{i \in I} B_i$, where B_i is a closed set.

There were two problems in the theory of I-fuzzy topological spaces. One problem is "how should fuzzy compactness be defined?", and the other problem is "how should fuzzy points be treated?". Both problems can be solved by using II-fuzzy sets.

1. Mode-Type Operators On Fuzzy Sets

This chapter introduce some new type operators on fuzzy sets whose truth value set is I (i.e., [0, 1] interval). There are many (aggregation) operators already in existence and some of these were selected and used for fuzzy decision-making, fuzzy logic and so on.

One of the axioms on aggregation operators in [4] or [6] is "continuity". Mathematically speaking, continuity is a very good property, but it is a restrictive condition. In section 3, we can define mode-type operators on fuzzy sets, which does not require continuity. We show that mode-type operators are precisely those operators which are pointwise, commutative, associative, compatible and averaging.

1.1. Definitions and Question

Let X be an ordinary set, I^X the family of all fuzzy subsets of X.

Definition 1.1.1. A binary operator $\oplus : I^X \times I^X \longrightarrow I^X$ is said to be

commutative if $A \oplus B = B \oplus A$ for all $A, B \in I^X$,

idempotent if $A \oplus A = A$ for all $A \in I^X$,

associative if $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ for all $A, B, C \in I^X$,

compatible if $A \oplus B \geq C \oplus D$ for all $A, B, C, D \in I^X$ such that $A \geq C$ and $B \geq D$,

averaging if $A \vee B \geq A \oplus B \geq A \wedge B$ for all $A, B \in I^X$.

It is obvious from the definition that every averaging operator is idempotent.

Definition 1.1.2. An operator $\oplus$ is said to be pointwise if there exists an operator $* : I \times I \longrightarrow I$ such that

$$(A \oplus B)(x) = A(x) * B(x) \quad \text{for all } A, B \in I^X.$$

It is clear that operators $\vee$ and $\wedge$ are pointwise. The + in [21] is not pointwise.

Definition 1.1.3. A binary operator $* : I \times I \to I$ is said to be

commutative if $a * b = b * a$ for all $a, b \in I$,

associative if $(a * b) * c = a * (b * c)$ for all $a, b, c \in I$,

compatible if $a * b \geq c * d$ for all $a, b, c, d \in I$ such that $a \geq c$ and $b \geq d$,

averaging if $\max(a,b) \geq a * b \geq \min(a,b)$ for all $a, b \in I$.

When an operator $\oplus$ is pointwise, it is obvious from the definition that $\oplus$ is commutative, associative, compatible and averaging if and only if the corresponding operator $*$ is commutative, associative, compatible and averaging respectively.

Axioms of aggregation operators similar to the definitions given above were introduced in [4] and [6].

Median-type operators on I^X are defined by

$$(A \oplus B)(x) = \text{median}\,[A(x), B(x), m] \quad \text{for some } m,\ 0 \leq m \leq 1.$$

The case when $0 < m < 1$ was studied in [4, p.222].

It is clear that operators $\vee$ and $\wedge$ are median-type operators. And every median-type operator is pointwise, commutative, associative, compatible and averaging.

Question. Does there exist pointwise, commutative, associative, compatible and averaging operator other than the median-type operators?

1.2. Mode-type operators

Let L be a linearly ordered set, and let f be a function from I to L satisfying the following two conditions:
for every closed interval [a, b] of I,

(1) there exists a $c \in [a, b]$ such that
$f(c) = m, \quad m = \sup\{f(x) \mid x \in [a, b]\}$,

and (2) for the subset $M = \{c \mid c \in [a, b], f(c) = m\}$ of [a, b],
$\sup\{d \mid d \in M\} \in M$.

Definition 1.2.1. Let $f: I \longrightarrow L$ be a function stated above. And let a, b be arbitrary elements of I. We define a binary operation $*_f$ on I by the following way:
if $a \leqq b$,

$a *_f b = \sup\{c \mid c \in [a, b], f(c) = m\}$, where $\sup\{f(x) \mid x \in [a, b]\}$,

if $b < a$,

$a *_f b = \sup\{c \mid c \in [b, a], f(c) = m\}$, where $\sup\{f(x) \mid x \in [b, a]\}$.

That is, $*_f$ is the supremum of the subset in I on which f takes its maximum.
The operator $*_f$ would then induce an operator $\oplus_f$ on I^X defined by

$$(A \oplus_f B)(x) = A(x) *_f B(x).$$

We shall call the operator $\oplus_f$ "mode-type operator" and the function f "mode function".

Note that every continuous function from I to I is a mode function.

Example 1. Let L = I. If $f: I \to I$ is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational or } 0 \text{ or } 1, \\ 1/n & \text{if } x = m/n \text{ s.t. } m \text{ and } n \text{ are irreducible.} \end{cases}$$

Then, $0 *_f 1 = \frac{1}{2}$, $\frac{1}{4} *_f \frac{2}{5} = \frac{1}{3}$. See figure 1.

We can define the corresponding $\oplus_f$ from this function f. This example indicates that mode-function does not need continuity.

We can define similar operators by using the supremum of the subset in I on which f takes its minimum, the infimum of the subset in I on which f takes its minimum or the infimum of the subset in I on which f takes its maximum. All these operators are essentially the same.

Proposition 1.2.1. Every mode-type operator $\oplus_f$ is pointwise, commutative, associative, compatible and averaging.

Proof. From the definition, the $\oplus_f$ is pointwise, commutative and averaging. Because $\oplus_f$ is pointwise, it is sufficient that $*_f$ is compatible and associative.

To prove compatibility, let $a \leqq b \leqq c \leqq d$. It is clear that $f(b *_f c) \geqq f(x)$ for any $x \in [b, c]$, and $f(b *_f c) \leqq f(b *_f d)$ since $[b, c] \subset [b, d]$. Then, we have $b *_f d \in [b, b *_f c)$.

Hence $b *_f d \geqq b *_f c$.

Since $f(b *_f c) > f(x)$ for any $x \in (b *_f c, c]$ and $f(b *_f c) \leqq f(a *_f c)$, we have $a *_f c \in (b *_f c, c]$. Hence $a *_f c \leqq b *_f c$.

Therefore $a *_f c \leqq b *_f c \leqq b *_f d$.

Next, we shall show associativity. For any $x \in [a, b]$, $f(a *_f b)$

$\geq f(x)$. Hence, $f(a *_f b) = f((a *_f b) *_f x)$. By definition,

$$a *_f b = (a *_f b) *_f x \tag{1}$$

Let $a \leq b \leq c$. From (1), we have

$$(a *_f c) *_f b = a *_f c \tag{2}$$

Since $a *_f b \geq a$, we have $(a *_f b) *_f c \geq a *_f c$. By compatibility, $a *_f b \leq a *_f c$. So $(a *_f b) *_f c \leq (a *_f c) *_f c$. From (1), $(a *_f b) *_f c \leq a *_f c$. Hence

$$(a *_f b) *_f c = a *_f c \tag{3}$$

Similarly,

$$a *_f (b *_f c) = a *_f c \tag{4}$$

From (2), (3) and (4),

$$(a *_f b) *_f c = (a *_f c) *_f b = a *_f (b *_f c) = a *_f c.$$

This shows associativity.

Proposition 1.2.2. All median-type operators including $\vee$ and $\wedge$ are mode-type operators.

Proof. If f is a monotone increasing function from I to I, then the corresponding $\oplus_f$ is $\vee$. If f is a strictly monotone decreasing function from I to I, then the corresponding $\oplus_f$ is $\wedge$. For m in I, we define the function f from I to I by $f(x) = 1-(x-m)^2$. Then, $(A \oplus_f B)(x) = \text{median}\ [A(x), B(x), m]$.

Remark. Not every mode-type operator is median-type as seen by the following example.

Example 2. Let L = I, and suppose that $f(x) = \frac{1}{2}\sin 2\pi x + \frac{1}{2}$. Then,

$0 *_f \frac{1}{2} = \frac{1}{4}$ $\frac{1}{2} *_f 1 = 1$. See figure 2.

We can define the corresponding $\oplus_f$ from this function f. This mode-type operator is not median-type.

figure 1

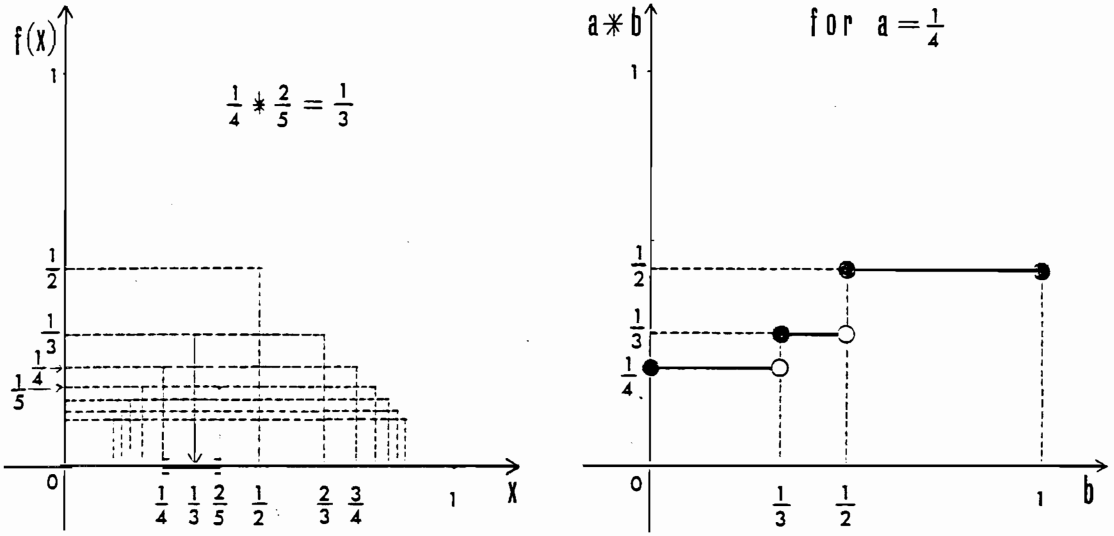
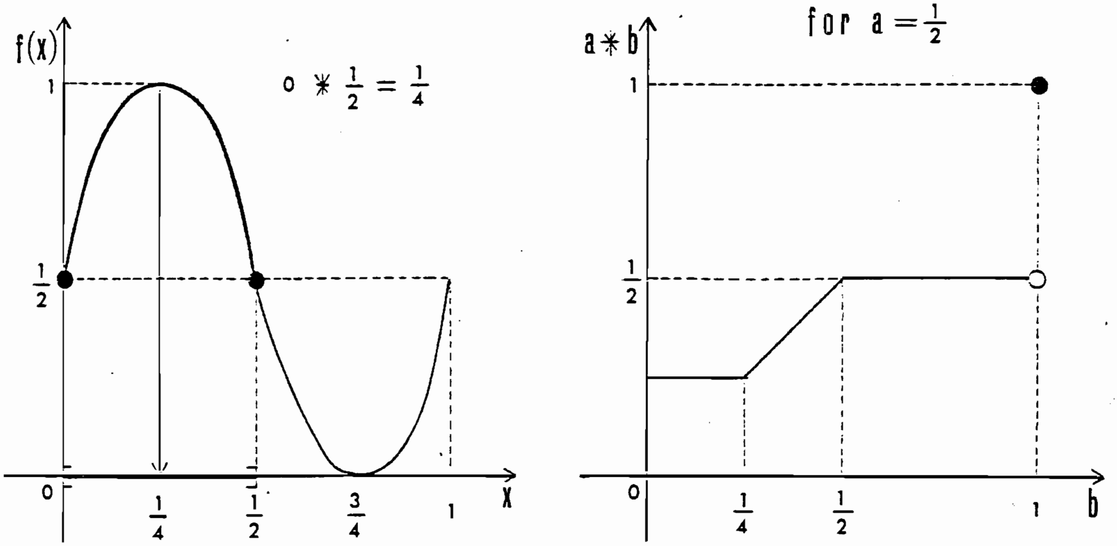


figure 2



1.3. Characterization of pointwise, commutative, associative, compatible and averaging operators

In this section, we shall show that the converse of Proposition 1.2.1 is also true. That is, all pointwise, commutative, associative, compatible and averaging operators are mode-type operators.

Let $\oplus$ be an pointwise, commutative, associative, compatible and averaging. operator on I. Since any operator $\oplus$ is pointwise, we see that there exists an operator * such that

$$(A \oplus B)(x) = A(x) * B(x).$$

Lemma 1.3.1. For any $x, y \in I$ such that $x * y = c$, it holds that $x * c = c$.

Proof. Suppose that x and y are two elements satisfying the condition that $x * y = c$. From the properties of the operator *, we can easily obtain $x * c = x * (x * y) = (x * x) * y = x * y = c$.

Lemma 1.3.2. If a, b, c belong to I and $a < b < c$, it holds that

$a * c = a * b$ or $b * c$.

Proof. We put $d = a * c$ and suppose that $d \leq b$. From the compatibility condition, we have $d = a * c \geq a * b$. On the other hand, by Lemma 1.3.1 and the compatibility condition, $d = a * d \leq a * b$. Therefore $d = a * b$. Similarly, in case that $d \geq b$, from Lemma 1.3.1 and the compatibility condition, we obtain $d = b * c$. This completes the proof.

Now, for every c of I, we set

$$i(c) = \inf\{x \mid \text{there exists } z \in I \text{ such that } x * z = c\}$$

and

$$s(c) = \sup\{x \mid \text{there exists } z \in I \text{ such that } x * z = c\}.$$

Lemma 1.3.3. Let x and c be two elements such that $i(c) < x \leqq c$, then $x * c = c$.

Proof. By the definition of $i(c)$, we can find y and z of I such that $i(c) \leqq y < x$ and $y * z = c$. Since $y * c = c$ by Lemma 1.3.1, we obtain that from the compatibility condition,

$$c = y * c \leqq x * c \leqq c * c = c.$$

This completes the proof.

We can prove the following Lemma analogously to Lemma 1.3.3.

Lemma 1.3.4. Let x and c be two elements such that $s(c) > x \geqq c$, then $x * c = c$.

Let $\{1,2,3\}$ be the three points set. We introduce the lexicographic order on the set $I \times \{1,2,3\}$, i.e.,

for every $i, j \in I$ such that $i < j$ iff $(i, x) < (j, x)$, and

for every $i \in I$, $(i, 1) < (i, 2) < (i, 3)$.

Then, we define the function f, which depends on the operator $*$, from I to $I \times \{1,2,3\}$ such that

$$f(c) = \begin{cases} (s(c)-i(c),\ 1) & \text{for } i(c) * c \neq c \text{ and } s(c) * c \neq c, \\ (s(c)-i(c),\ 3) & \text{for } i(c) * s(c) = c, \\ (s(c)-i(c),\ 2) & \text{otherwise.} \end{cases}$$

Now, we can prove the following,

Theorem 1.3.5. The family of all pointwise, commutative, associative, compatible and averaging operators coincides with that of all mode-type operators.

Proof. Since every mode-type operator satisfies the pointwise, commutative, associative, compatible and averaging conditions, by Proposition 1, it suffices to show that every pointwise, commutative, associative, compatible and averaging operator * is a mode-type operator corresponding to a function f from I to $I \times \{1,2,3\}$ defined above. To see this, we shall prove that $f(x) < f(a * b)$ for any closed subinterval [a, b] of I and for any element $x \in [a, b]$ with $x \neq a * b$, because the operator introduced by f is identical with the operator *.

Put $c = a * b$.

[I] Consider the first case that $a \neq c$ and $b \neq c$. Let x be any given element with $a \leqq x < c < b$.

Assume that $s(x) > c$. Then there is an element y such that $c < y < s(x)$. From Lemma 1.3.1, 1.3.4 and the compatibility condition, we get $c = a * c \leqq x * y = x$. But this contradicts the fact that $a \leqq x < c < b$. Hence we have

$$s(x) \leqq c. \quad (1)$$

If $i(x) < i(c)$, there exists an element z with

$$i(x) < z < i(c) \leqq a \leqq x \leqq s(x) \leqq c < b. \quad (2)$$

Combining (2) with Lemma 1.3.3 and the compatibility condition, we can easily verify that $z * a \leqq z * x = x$ and $z * b > x$. By Lemma 1.3.2 and (2) we see that $z * b = z * a$ or $z * b = a * b$. Since $z * a \leqq z * x = x$ and $z * b > x$, we have $z * b = a * b = c$. On the other hand, $z * b \neq c$ from the definition of i(c). Thus we have

$$i(x) \geq i(c). \tag{3}$$

Since $b > c$ and $a * b = c$, it is clear that $c < s(c)$. Combining this inequality with (1) and (3), we conclude that $s(x) - i(x) \leq c - i(c) < s(c) - i(c)$, which means $f(x) < f(c)$ as desired. When x is an element with $a < c < x \leq b$, we can analogously prove that $f(x) < f(c)$.

[II] Next we consider the second case that $a = c$.
Assume that there exists an element p such that $p < a$ and $p * b = a$. Then, by the result of the first case, we observe that, for every element $x \in [p, b]$ such that $x \neq a$, $f(x) < f(a)$. Hence $f(x) < f(a) = f(c)$ for every $x \in (a, b]$.

On the other hand, suppose that $p * b \neq a$ for all $p < a$. When $i(a) < a$, there exist two elements q and r such that $i(a) < q < a \leq r$ and $q * r = a$. Since $q * a = a$ by Lemma 1.3.1, using Lemma 1.3.2 and $a * b = c = a$, we obtain $q * b = a$. But this contradicts the assumption that $p * b \neq a$ for all $p < a$. Therefore we get

$$i(a) = a, \tag{4}$$

and hence

$$i(a) * a = a. \tag{5}$$

Let x be an element of $(a, b]$ and suppose that $i(x) < a$. Then we can take an element y such that $i(x) < y < a$. Thus we obtain $x = y * x \leq a * x \leq a * b = a$, which contradicts the fact that $a < x$. Hence we have

$$a \leq i(x). \tag{6}$$

By Lemma 1.3.1 and the fact that $a * b = c = a$, we can easily see that $a * x = a$. When $s(x) \leq b$, we have $s(x) \leq s(a)$ because $a * b = a$ and so $b \leq s(a)$.

Assume that $s(x) > b$. For any element z such that $a < x \leqq b < z < s(x)$, by using Lemma 1.3.2, we can verify that

$$a * z = a * x \quad \text{or} \quad a * z = x * z.$$

If $a * z = x * z$, then $a * z = x * z = x$ by Lemma 1.3.4. From Lemma 1.3.1, $a * x = x$. But $a * x \leqq a * b = a$. This contradicts $a < x$. Hence $a * z = a * x = a$. Therefore,

$$s(a) \geqq s(x) \qquad (7)$$

We shall now show that $f(x) < f(c)$. By (4), (6) and (7), we can observe that

$$s(a) - i(a) \geqq s(a) - a \geqq s(x) - i(x).$$

In particular, when $s(a) - i(a) = s(x) - i(x)$, it holds that $i(a) = a = i(x)$, and therefore we deduce that

$$a * a \leqq a * x = i(x) * x \leqq a * b = a. \qquad (8)$$

From (5) and (8), we obtain $i(a) * a = a$ and $i(x) * x = a \neq x$. When $s(a) * a = a$, $i(a) * s(a) = a$ by Lemma 1.3.2. Then, $i(x) * s(x) = i(a) * s(a) = a \neq x$. Hence,

$$f(a) = (s(a)-i(a), 3)$$

$$f(x) = (s(x)-i(x), 1) \quad \text{or} \quad (s(x)-i(x), 2).$$

When $s(a) * a \neq a$, put $p = s(a) * a$. From Lemma 1.3.1, $a * p = p$. If $a < p < s(a)$, then $a * p = a$ by Lemma 1.3.4. This contradicts that $a \neq p$. Hence $p \geqq s(a)$. So $a \leqq x \leqq b < s(x) = s(a) \leqq p$, but $p = s(a) * a \leqq s(a) * x = s(x) * x$. Hence, $s(x) * x \neq x$. Therefore,

$$f(a) = (s(a)-i(a), 2)$$

$$f(x) = (s(x)-i(x), 1).$$

We obtain $f(x) < f(c)$.

[III] In case that $b = c$, we can similarly show that $f(x) < f(c)$ as in the second case. This completes the proof.

From Proposition 1.2.1 and Theorem 1.3.5, we conclude that mode-type operators are precisely those operators which are pointwise, commutative, associative, compatible and averaging. However, there are operators which do not satisfy some of these properties as illustrated by the following example.

Example 3. Let $*_1$, $*_2$, $*_3$, $*_4$ and $*_5$ be operators on I defined by

$$a *_1 b = a$$

$$a *_2 b = \frac{1}{2}(a + b)$$

$$a *_3 b = \begin{cases} a & (\text{if } \frac{1}{2} \leqq a \leqq b \text{ or } b < \frac{1}{2} \leqq a \text{ or } a \leqq b < \frac{1}{2}) \\ b & (\text{otherwise}). \end{cases}$$

And

$$a *_4 b = \begin{cases} 1 & (\text{if } a = 1 \text{ or } b = 1) \\ a & (\text{if } a = b) \\ a & (\text{if } 3a-[3a] < 3b-[3b]) \\ b & (\text{if } 3a-[3a] > 3b-[3b]) \\ \frac{1}{3}(1+3a-[3a]) & (\text{otherwise}). \end{cases}$$

For the operators $*_1$, $*_2$, $*_3$ and $*_4$, we can define the corresponding operators $\oplus_1$, $\oplus_2$, $\oplus_3$ and $\oplus_4$ on I^X by $(A \oplus_i B)(x) = A(x) *_i B(x)$ $(i=1,2,3,4)$. Then $\oplus_1$ is not commutative, $\oplus_2$ is not associative, $\oplus_3$ and $\oplus_4$ are not compatible. Triangular norms in section 0.3 are not averaging.

2. II-FUZZY TOPOLOGICAL SPACES AND II-FUZZY LINEAR SPACES

In this chapter we induce the concept of II-fuzzy sets , II-fuzzy topological spaces and II-fuzzy linear topologies on vector spaces. And, using the notion of i-neighborhoods, some of the basic properties of II-fuzzy topological vector spaces are investigated.

The concept of fuzzy sets, whose truth value set is the unit interval I = [0,1], was introduced by Zadeh [44], and that of L-fuzzy sets was introduced by Goguen [10]. Later, Chang [2] introduced fuzzy topologies on a set of fuzzy sets. Since then, much work has been done on fuzzy topological spaces. Then, the notion of fuzzy topological vector space on I-fuzzy set was given by Katsaras [21], [22] and [23]. In this paper, we consider special fuzzy sets whose truth value set is not I but II (two arrows set). And we investigate the fuzzy topological vector space on II-fuzzy set. As the II-fuzzy sets have the representation sets, it is easy to imagine various topological structures. Then, we explain II-fuzzy topological vector spaces systematically.

2.1. Definition of II-fuzzy topological spaces

We consider a set of two I's, where I=[0,1], placed side by side, and denote the interval on the right by I^+ and the other interval on the left by I^-. Points in I^+ are denoted by x^+, and points in I^- are denoted by x^-. The two arrows set II is the union of I^+ and I^-, that is,

$$II = I^+ \cup I^- = \{x^+ | x \in I\} \cup \{x^- | x \in I\} = \{x^+, x^- | x \in I\}.$$

We introduce the following order relation on II. For all $a \in I$, we define $a^- < a^+$. If $a < b$ $(a, b \in I)$, then $a^+ < b^-$. It is easy to see that the set II is a linearly ordered set by this relation with the least element 0^- and the greatest element 1^+. It is well known that any complete linearly ordered set is a completely distributive lattice (see Birkhoff [1] etc). Thus, the set II is a completely distributive lattice.

We define the operation $'$ from II to II, by

$$(a^+)' = (1-a)^- \text{ and } (a^-)' = (1-a)^+ \quad \text{where } a^+ \in I^+ \subset II,\ a^- \in I^- \subset II.$$

Since the operation has the involution property, the set II is a completely distributive lattice with involution. Let X be a non-empty set. The lattice operations $\bigvee$, $\bigwedge$ and $'$ in II induce the corresponding lattice operations in II^X (the set of all function from X into II) which we also denote by $\bigvee$, $\bigwedge$ and $'$ respectively.

That is,

$$(\bigvee_{j \in J} A_j)(x) = \bigvee_{j \in J} A_j(x),$$

$$(\bigwedge_{j \in J} A_j)(x) = \bigwedge_{j \in J} A_j(x)$$

and

$$(A')(x) = (A(x))'.$$

In the sequel, we shall consider the II-fuzzy topological space $(X, \mathfrak{T}, II)$.

2.2. Representation Theorems

Let X be a set (i.e. a set of points). We denote the set $\{A \subset X \times I \mid (x,i) \in A \text{ implies } (x,j) \in A \text{ for each } j < i\}$ by $M(X)$.

It is clear that $M(X) \subset P(X \times I)$ (the power set of $X \times I$) and $M(X)$ is a complete lattice with $\bigcup$ (set union) and $\bigcap$ (set intersection). We define the involution $'$ from $P(X \times I)$ to $P(X \times I)$ by

$$A' = \{(x,i) \in X \times I \mid (x,1-i) \notin A\} \quad \text{for } A \in P(X \times I).$$

The lattice $(M(X); \bigcup, \bigcap, ')$ is a completely distributive lattice with involution.

Theorem 2.2.1. The function $k : II^X \longrightarrow P(X \times I)$ defined by

$$k(B) = \{(x,i) \in X \times I \mid B(x) \geq i+\} \quad (B \in II^X)$$

has the following properties:

(0) k is an isomorphism from $(II^X; \bigvee, \bigwedge, ')$ onto $(M(X); \bigcup, \bigcap, ')$,

(1) $k(\bigvee_{j \in J} B_j) = \bigcup_{j \in J} k(B_j) \quad (B_j \in II^X)$,

(2) $k(\bigwedge_{j \in J} B_j) = \bigcap_{j \in J} k(B_j) \quad (B_j \in II^X)$,

(3) $k(B') = [k(B)]' \quad (B \in II^X)$.

Proof. (0) Let $A, B \in II^X$ such that $A \neq B$, then there exists $x \in X$ such that $A(x) \neq B(x)$. When $A(x) > B(x)$, there exists an $i \in I$ such that $A(x) \geq i^+ > B(x)$. Then $(x,i) \in k(A)$ but $(x,i) \notin k(B)$. Hence, k is one-to-one. Suppose $C \in M(X)$, and let $D \in I^X$ such that $D(x) = \sup_{(x,i) \in C} i$.

We define $E \in II^X$ by

$$E(x) = \begin{cases} [D(x)]^+ & (x,D(x)) \in C \\ [D(x)]^- & (x,D(x)) \notin C. \end{cases}$$

Then $k(E) = C$, therefore k is onto.

(1) Let $A, B \in II^X$ be such that $A(x) \leq B(x)$ for every $x \in X$. For each $(x,i) \in k(A)$, we have $i^+ < k(A) \leq k(B)$, hence $(x,i) \in k(B)$. This shows that $k(A) \subset k(B)$. Therefore k is order preserving. Let $B_j \in II^X$, then $\bigvee_{j \in J} B_j \geq B_j$. Since k is order preserving, we have $k(\bigvee_{j \in J} B_j) \supset k(B_j)$. Hence $k(\bigvee_{j \in J} B_j) \supset \bigcup_{j \in J} k(B_j)$. Let $(x,i) \in k(\bigvee_{j \in J} B_j)$, then $[\bigvee_{j \in J} B_j](x) > i^+$. Then, there exists $j_0 \in J$ such that $B_{j_0}(x) > i^+$. Thus $(x,i) \in k(B_{j_0})$. Therefore we have $k(\bigvee_{j \in J} B_j) \subset \bigcup_{j \in J} k(B_j)$.

We can prove (2) similarly.

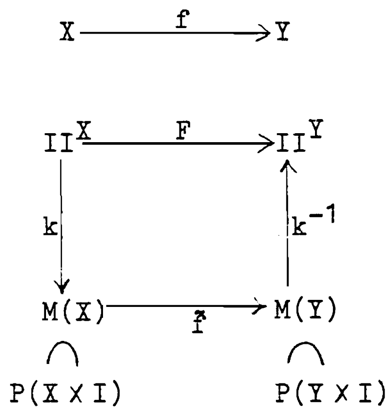
(3)
$$
\begin{aligned}
k(B') &= \{(x,i) \in X \times I \mid B'(x) \geq i+\} \\
&= \{(x,i) \in X \times I \mid B(x) \leq (1-i)^- < (1-i)^+\} \\
&= \{(x,i) \in X \times I \mid (x, 1-i) \notin k(B)\} \\
&= [k(B)]'.
\end{aligned}
$$

We shall call the set $k(B)$ the representation of the $I\bar{I}$-fuzzy subset B of X.

Since I is a distributive lattice, it is isomorphic to a ring of sets (by Birkhoff-Stone representation theorem). However this isomorphism is not sup-inf-preserving. Therefore we see that II^X has a better property than I^X.

Let f be a function from X to Y. We will define the function $\tilde{f}$ from $X \times I$ to $Y \times I$ by $\tilde{f}(x,i) = (f(x),i)$.

Proposition 2.2.2. Let F be a fuzzy function from II^X to II^Y generated by the function f. Then $F = k^{-1} \circ \tilde{f} \circ k$.



Proposition 2.2.3. Let $(X, \mathcal{T}, II)$ be a II-fuzzy topological space, and $K(\mathcal{T})$ the set $\{A \subset X \times I \mid A = k(B),\ B \in \mathcal{T}\}$. Then $(X \times I, K(\mathcal{T}), 2)$ is a topological space.

Proof. Since 0^-_X, $1^+_X \in II^X$, we have $\emptyset = k(0^-_X) \in K(\mathcal{T})$ and $X \times I = k(1^-_X) \in K(\mathcal{T})$. Let $U_j \in K(\mathcal{T})$. Then, for each j, there exists a $V_j \in \mathcal{T}$ s.t. $k(V_j) = U_j$. We see that $U_1 \cap U_2 = k(V_1) \cap k(V_2) = k(V_1 \wedge V_2) \in K(\mathcal{T})$ and $\bigcup_{j \in J} U_j = \bigcup_{j \in J} k(V_j) = k(\bigvee_{j \in J} V_j) \in K(\mathcal{T})$.

Proposition 2.2.4. Let $(X, \mathcal{T}, II)$ be a II-fuzzy topological space, and $K'(\mathcal{T})$ the set $\{A \subset X \times I \mid A = k(B')^c,\ B \in \mathcal{T}\}$, where $k(B')^c$ is the complement of $k(B')$. Then $(X \times I, K'(\mathcal{T}), 2)$ is a topological space.

We can prove this in the same way.

We shall call the space $(X \times I, K(\mathcal{T}), 2)$ the open representation space of $(X, \mathcal{T}, II)$, and the space $(X \times I, K'(\mathcal{T}), 2)$ the closed representation space of $(X, \mathcal{T}, II)$. We see that the closed representation space is the inversion of the open representation space. In fact, the function from $(X \times I, K(\mathfrak{O}), 2)$ to $(X \times I, K'(\mathfrak{O}), 2)$ defined by $f((x,i)) = (x,1-i)$, is a homeomorphism.

Proposition 2.2.5. Let F be a fuzzy function from $(X, \mathfrak{O}_1, II)$ to $(Y, \mathfrak{O}_2, II)$ induced by the function f from X to Y. Then the following conditions are equivalent:

(1) F is a fuzzy continuous function,

(2) $\hat{f}$ is a continuous function from $(X \times I, K(\mathfrak{O}_1), 2)$ to $(Y \times I, K(\mathfrak{O}_2), 2)$,

(3) $\hat{f}$ is a continuous function from $(X \times I, K(\mathfrak{O}_1), 2)$ to $(Y \times I, K(\mathfrak{O}_2), 2)$.

2.3. Neighborhood system of a II-fuzzy topological space

Definition 2.3.1. Let $(X, \mathfrak{O}, II)$ be a II-fuzzy topological space and $i \in I$. A II-fuzzy set U in X is an i-neighborhood of x in X if and only if there exists an open II-fuzzy set $O \in \mathfrak{O}$ such that $U(y) \geqq O(y)$ for every $y \in X$ and $O(x) \geqq i^+$.

The concept of i-neighborhoods of x in $(X, \mathfrak{O}, II)$ is the inverse image by k of the concept of neighborhoods of (x,i) in $(X \times I, K(\mathfrak{O}), 2)$.

Hence, this notion of i-neighborhoods is easier for us to understand than the notion of neighborhoods in Warren [39] [40], Lowen [30] or Ghanim [8].

Next results are clear too.

Theorem. 2.3.2. Let $(X, \mathfrak{N}, II)$ be a II-fuzzy topological space. If, for each $x \in X$ and $i \in I$, $\mathcal{U}_i(x)$ denotes the family of all i-neighborhoods of x, then the family $\{\mathcal{U}_i(x) \mid x \in X, i \in I\}$ has the following properties:

(N1) $U(x) \geqq i^+$ for each $U \in \mathcal{U}_i(x)$;

(N2) if $U_1, U_2 \in \mathcal{U}_i(x)$, then $U_1 \wedge U_2 \in \mathcal{U}_i(x)$;

(N3) if $U \in \mathcal{U}_i(x)$ and $U \leqq V$, then $V \in \mathcal{U}_i(x)$;

(N4) for each $U \in \mathcal{U}_i(x)$ there exists $W \in \mathcal{U}_i(x)$ such that $U \in \mathcal{U}_j(y)$ for each y and j for which $W(y) \geqq j^+$.

Conversely, for any non empty set X, if the family $\{\mathcal{U}_i(x) \mid x \in X, i \in I\}$ is a family satisfying (N1)-(N4), then there exists a unique II-fuzzy topology $\mathfrak{N}$ on X such that, for each $x \in X$ and $i \in I$, $\mathcal{U}_i(x)$ coincides with the family of all i-neighborhoods of x.

Proposition 2.3.3. Let F be a fuzzy function from $(X, \mathfrak{N}_1, II)$ to $(Y, \mathfrak{N}_2, II)$ induced by $f: X \longrightarrow Y$. Then, F is fuzzy continuous if and only if $f^{-1}(N)$ is an i-neighborhood of x for every i-neighborhood N of f(x) and each $x \in X$.

From the general topological point of view, I-fuzzy topological spaces do not have good properties. II-fuzzy topological spaces are better. As we can think of the family of I-fuzzy sets as the quotient sets of the family of II-fuzzy sets, we investigate II-fuzzy topological spaces instead of I-fuzzy topological spaces. And if we must consider L-fuzzy topology, we should construct a fuzzy set whose truth value set is a special lattice with desirable properties like II.

2.4. Slice topology and Product theorem

It may seem that II-fuzzy topological spaces are very similar to general topological spaces, for a II-fuzzy topological space has a representation space. However, they are quite different because the notion of involution in II-fuzzy sets, which would correspond to the notion of the complement in usual sets, is essentially different from it. It is clear that the notion of involution is the key to fuzzy set theory. There exists another difference between II-fuzzy topological spaces and general topological spaces, because of the definition of product spaces. In fact, this definition gives rise to difficult problems. Another product was defined in Hutton [19]. Butin II-fuzzy topological spaces, we can easily check Tychonoff theorem under Wong's definitions of product and compact.

Let $(X\times I, K(\mathfrak{N}), 2)$ be the open representation space of $(X, \mathfrak{N}, II)$. For each $i\in I$, we denote the set $\{(x,i)\mid x\in X\}\subset X\times I$ by X^i and $\{A\subset X^i\mid A=k(B)\cap X^i, B\in\mathfrak{N}\}$ by $\mathfrak{N}^i$. It is clear that $(X^i, \mathfrak{N}^i, 2)$ is a topological space. The space $(X^i, \mathfrak{N}^i, 2)$ will be called an i-slice space.

Proposition 2.4.1. The following three conditions are equivalent:

(1) $(X, \mathfrak{N}, II)$ is fuzzy compact,

(2) $(X\times I, K(\mathfrak{N}), 2)$ is compact,

(3) $(X^1, \mathfrak{N}^1, 2)$ is compact.

Proof. (1)==>(2). Let $\mathcal{G}=\{G_j\mid j\in J\}$ be an open cover of $X\times I$. Then $X = k^{-1}(X\times I) = k^{-1}(\bigcup_{j\in J}\{G_j\mid j\in J\}) = \bigvee_{j\in J}\{k^{-1}(G_j)\mid j\in J\}$. Hence

$\{k^{-1}(G_j) \mid j \in J\}$ is a fuzzy open cover of X. Since $(X, \mathcal{O}, II)$ is fuzzy compact, there exist finite $k^{-1}(G_{j_0}), \cdots, k^{-1}(G_{j_n})$ s.t. $X = k^{-1}(G_{j_0}) \vee \cdots \vee k^{-1}(G_{j_n})$. Therefore we have

$$X \times I = k(X) = k(k^{-1}(G_{j_0}) \vee \cdots \vee k^{-1}(G_{j_n}))$$

$$= k(k^{-1}(G_{j_0})) \vee \cdots \vee k(k^{-1}(G_{j_n}))$$

$$= G_{j_0} \cup \cdots \cup G_{j_n}.$$

This shows that $\mathcal{G}$ has finite subcover $\{G_j, \cdots, G_j\}$.

(2)==>(1). Similar to (1)==>(2) but in the converse way.

(2)<==>(3). Trivial.

Corollary 2.4.2. Let $(X_j, \mathcal{O}_j, II)$ be a fuzzy compact space for each $j \in J$. Then the product space $\prod_{j \in J}(X_j, \mathcal{O}_j, II)$ is fuzzy compact.

Proof. $(X_j, \mathcal{O}_j, II)$ is fuzzy compact for each $j \in J$

<==> $(X_j^1, \mathcal{O}_j^1, 2)$ is compact for each $j \in J$

<==> $\prod_{j \in J}(X_j^1, \mathcal{O}_j^1, 2)$ is compact

<==> $\prod_{j \in J}(X_j, \mathcal{O}_j, II)$ is fuzzy compact

This was already proved by J.A.Goguen[10], but our proof using proposition 1 shown above is much easier.

Let F be a fuzzy function from II^X to II^Y induced by $f: X \longrightarrow Y$. We can define natural function $f^i: X^i \longrightarrow Y^i$ by

$$f^i(x,i) = (f(x),i).$$

Then, next proposition is clear.

Proposition 2.4.3. If F be a fuzzy continuous function from $(X, \mathcal{O}_1, II)$ to $(Y, \mathcal{O}_2, II)$, then f^i is a continuous function $(X^i, \mathcal{O}_1{}^i, 2)$ to $(X^i, \mathcal{O}_2{}^i, 2)$ for every $i \in I$.

But the converse of this proposition is not true.

Example. Let R be the set of all real numbers. For each $r, s \in R$ s.t. $r<s$, we define r^s and ${}^r s$ functions from R to II by for $x \in R$

$$r^s(x) = \begin{cases} 0^- & x < r \\ \left(\frac{x-r}{s-r}\right)^- & r \leqq x \leqq s \\ 1^+ & x > s \end{cases}$$

$${}^r s(x) = \begin{cases} 0^- & x > s \\ \left(\frac{s-x}{s-r}\right)^- & r \leqq x \leqq s \\ 1^+ & x < r . \end{cases}$$

And we define the II-fuzzy topologies $\mathcal{O}_1$ and $\mathcal{O}_2$ generated by the subbase

$$\mathcal{S}_1 = \{r^{r+1}, {}^{r-1}r \mid r \in R\} \text{ and}$$
$$\mathcal{S}_2 = \{r^{r+2}, {}^{r-2}r \mid r \in R\} \text{ respectively.}$$

Suppose $f: R \longrightarrow R$ be identity function. Then, the fuzzy function F from $(R, \mathcal{O}_2, II)$ to $(R, \mathcal{O}_1, II)$ is not continuous. But, for each $i \in I$, the function f^i from $(R, \mathcal{O}_2{}^i, 2)$ to $(R, \mathcal{O}_1{}^i, 2)$ is continuous.

We can define the following cardinal functions, the network weight, the Lindelöf number and the cellularity, for every L-fuzzy space $(X, \mathcal{O}, L)$.

$nw((X,\mathcal{O},L)) = \min\{|\mathcal{G}| \mid$ For any U fuzzy open in X, there exists $\mathcal{L} \subset \mathcal{G}$ s.t. $\bigvee \mathcal{L} = U.\}$

$\ell((X,\mathcal{O},L)) = \min\{\tau \mid$ For any $\mathcal{G}$ fuzzy open cover of X there exist $\mathcal{F}$ subcover of $\mathcal{G}$ s.t. $|\mathcal{F}| \leq \tau\}$

$c((X,\mathcal{O},L)) = \min\{\tau \mid$ For every $\mathcal{G}$ disjoint family of non-empty fuzzy open sets of X s.t. $|\mathcal{G}| \leq \tau\}$.

These functions nw, ℓ and c are generalized from those in the general topology.

Theorem 2.4.4. Let $(X,\mathcal{O},II)$ be a L-fuzzy topological space and $(X^i,\mathcal{O}^i,2)$ its i-slice space. Then,

$$\ell((X,\mathcal{O},II)) = \ell((X^1,\mathcal{O}^1,2)) \quad \text{and}$$
$$c((X,\mathcal{O},II)) = c((X^0,\mathcal{O}^0,2)).$$

Also, if $nw((X,\mathcal{O},II)) > \mathcal{L}$, then there exists an $i_0 \in I$ s.t.

$$nw((X,\mathcal{O},II)) = nw((X^{i_0},\mathcal{O}^{i_0},2)).$$

Proof. Let $\mathcal{G}$ be a fuzzy open cover of X, $\mathcal{L}$ a subcover of $\mathcal{G}$. Then $K(\mathcal{G})(=\{k(U)|U\in\mathcal{G}\})$ is an open cover of $X \times I$ and $K(\mathcal{L})$ $(=\{k(U)|U\in\mathcal{L}\})$ is a subcover of $K(\mathcal{G})$. $\mathcal{G}^1(=\{k(U)\wedge X^1|U\in\mathcal{G}\})$ is an open cover of X^1, and $\mathcal{L}^1(=\{k(U)\wedge X^1|U\in\mathcal{L}\})$ is a subcover of $\mathcal{G}^1$. Conversely, if $\mathcal{U}$ is an open cover of X^1, there exists $\mathcal{G}$ a fuzzy open cover of X s.t. $\mathcal{U} = \mathcal{G}^1$. Therefore $\ell((X,\mathcal{O},II))$ = $\ell((X^1,\mathcal{O}^1,2))$. In a similar way we can prove that $c((X,\mathcal{O},II))$ $= c((X^1,\mathcal{O}^1,2))$.

From the definition of nw, we can easily obtain that $nw((X,\mathcal{O},II)) \geq nw((X^i,\mathcal{O}^i,2))$ for every $i\in I$. For $U^i \subset X^i$, let $g(U^i) = \{(x,j)|(x,i)\in U^i \text{ and } j<i\} \in M(X) \subset P(X\times I)$, and $\mathcal{G}$ be the family $\{k^{-1}(g(U^i))|U^i\in\mathcal{O}^i$ for any $i\in I\}$. Then for every

fuzzy open set U there is a subfamily $\mathcal{L}$ of $\mathcal{G}$ s.t. $\bigvee \mathcal{L} = U$. If, for every $i \in I$, $nw((X, \mathcal{O}, II)) > nw((X^i, \mathcal{O}^i, 2))$, we must have $nw((X, \mathcal{O}, II)) > |\mathcal{G}|$. It is a contradiction.

2.5. Relation between general topology and II-fuzzy topology

In relation to II-fuzzy topological spaces, we can consider various concepts that we consider in relation to I-fuzzy topological spaces. In most cases, they are easier to consider than those in I-fuzzy topological space. In this section, we consider the concept which corresponds to the functions ι and ω in R.Lowen [25].

Let $(X, \mathcal{O}, II)$ be a II-fuzzy topological space, then we can consider $\mathcal{O}^i$ $(i \in I)$ introduced in section 2.4 as topologies on X. We define $\iota(\mathcal{O})$ as the topology generated by $\bigcup_{i \in I} \mathcal{O}^i$.

Let $(X, \mathcal{O}, 2)$ be a general topological space, then for $U \in \mathcal{O}$, we define $U[i] \in II^X$ $(i \in I)$ by

$$U[i](x) = \begin{cases} i+ & (x \in U) \\ 0^- & (x \notin U). \end{cases}$$

We define $\omega(\mathcal{O})$ as the II-fuzzy topology generated by $\{U[i] \in II^X \mid U \in \mathcal{O},\ i \in I\}$.

We can easily check $\iota(\omega(\mathcal{O})) = \mathcal{O}$ for every topological space $(X, \mathcal{O}, 2)$. For a II-fuzzy topological space $(X, \mathcal{O}, II)$, we say that $(X, \mathcal{O}, II)$ is topologically generated when $\omega(\iota(\mathcal{O})) = \mathcal{O}$. For an order preserving function g from I to I, let G be the function from II to II, defined by

$$G(x^+) = (g(x))^+ \text{ and } G(x^-) = (g(x))^-.$$

We say that G is a sign and order preserving function.

Theorem 2.5.1. Let $(X, \mathfrak{O}, II)$ be a II-fuzzy topological space s.t. $i^+_X \in \mathfrak{O}$ for every $i \in I$. The space $(X, \mathfrak{O}, II)$ is topologically generated if and only if, for each sign and order preserving function $G: II \longrightarrow II$ and $U \in \mathfrak{O}$, $G \circ U \in \mathfrak{O}$.

Proof. Let $(X, \mathfrak{O}, II)$ be topologically generated. Since $U \in \mathfrak{O}$ can be written as $U = \bigcup_{i \in I} (U^i[i])$, we have $G \circ U = \bigcup_{i \in I} (U^i[g(i)])$. From $U^i \in \mathfrak{O}^i \subset \iota(\mathfrak{O})$ and $g(i) \in I$, we see that $U^i[g(i)] \in \omega(\iota(\mathfrak{O}))$. Thus we have $G \circ U \in \omega(\iota(\mathfrak{O})) = \mathfrak{O}$.

Conversely, let $G \circ U \in \mathfrak{O}$ for each sign and order preserving G and $U \in \mathfrak{O}$. For every $V \in \mathfrak{O}^i$, there exists a $U \in \mathfrak{O}$ s.t. $U^i = V$. Let G be the sign and order preserving map s.t. $G(i^+) = j^+$. From $G \circ U \in \mathfrak{O}$, we see that $(F \circ U)^j \in \mathfrak{O}^j$. Hence $(F \circ U)^j = U^i \in \mathfrak{O}^j$, and therefore we have $V \in \mathfrak{O}^j$. This indicates that $\mathfrak{O}^i = \mathfrak{O}^j$ for every $i, j \in I$. Consequently the space $(X, \mathfrak{O}, II)$ is topologically generated.

Since every continuous function from I_r to I_r in R.Lowen[25] is an order preserving function from I to I, this theorem correspond to theorem 2.2.2. in R.Lowen[25].

2.6. II-fuzzy linear topological spaces

In this section, we use II-fuzzy topologies which contain all the constant fuzzy sets. (Lowen adopt only this type as fuzzy topology set in [25].) One of the reasons for doing this is to make sure that the topology of such a space is translation invariant. This assumption is not necessary but sufficient. So, we don't adopt Lowen's definition of fuzzy topology in section 1.

Let E be a vector space over K, where K is the field of either the real numbers R or the complex numbers C.

Let $f : E \times E \to E \quad (x,y) \mapsto x+y,$

$g : K \times E \to E \quad (k,x) \mapsto kx,$

be the vector addition and the scalar multiplication in E.

If α is a II-fuzzy set in K and A, B are II-fuzzy sets in E, then we denote $F(A \times B)$ (F is the fuzzy function induced by f) by A+B and $G(\alpha \times A)$ (G is the fuzzy function induced by g) by αA. Then we have,

(1) $(A + B) + C = A + (B + C)$;

(2) $(\{t\}A)(x) = A(x/t) \quad (t \neq 0)$;

(3) $(\{0\}A)(x) = \begin{cases} 0^- & (x \neq 0) \\ \bigvee_{y \in E} A(y) & (x = 0) \end{cases}$

From now on, we denote $\{t\}A$ by tA, $A+\{x\}$ by A+x. And for brevity, we use f (instead of F) for function induced by f.

Let E and F be vector spaces over K. And let f be a linear map from E to F, we have,

(4) $f(sA + tB) = sf(A) + tf(B)$,

specially, $t(A + B) = tA + tB$;

(5) $f^{-1}(tB) = tf^{-1}(B) \quad (t \neq 0)$;

(6) $\underbrace{A + \cdots + A}_{n \text{ terms}} \leq B \quad A \leq \frac{1}{n} B.$

A II-fuzzy set A in E over K is called balanced if $tA \leq A$ for each scalar t with $|t| \leq 1$. A is balanced if and only if $A(tx) \geq A(x)$ for each t with $|t| \leq 1$. Hence, $A(0) \geq A(x)$.
A II-fuzzy set A in E over K is call r-absorbing if $\bigvee_{k>0} kA \geq r^+$.

Definition 2.6.1. A II-fuzzy linear topology on a vector space E over K is a II-fuzzy topology (containing all the constant fuzzy sets) such that the two mappings

$+ : E \times E \longrightarrow E \quad (x,y) \longmapsto x+y$

$\cdot : K \times E \longrightarrow E \quad (t,x) \longmapsto tx$

are fuzzy continuous when K is equipped with minimum II-fuzzy topology which includes the usual topology and all the constant fuzzy sets, and $K \times E$, $E \times E$ have the corresponding product fuzzy topologies. We denote the the family of all i-neighborhoods of $t \in K$ by $\mathcal{V}_i(t)$.
A linear space with a II-fuzzy linear topology is called a II-fuzzy linear space or a II-fuzzy topological vector space.

Let E be a II-fuzzy topological vector space. Then the maps

$f : E \longrightarrow E \quad f(x) = t_0 x \quad (t_0 \in K,\ t_0 \neq 0)$

$g : E \longrightarrow E \quad g(x) = x + x_0 \quad (x_0 \in E)$

are topological homeomorphism.
Furthermore the sum of open fuzzy set and any fuzzy set is again an open fuzzy set. (See Katsaras [22].)

Example 2.6.2. Let $(R, \mathcal{O}, 2)$ be a usual topological spaces. We define the II-fuzzy topological spaces $(R, \mathcal{O}_\#, II)$, where $\mathcal{O}_\#$

is the II-fuzzy topology on X generated by the subbase

$$\mathcal{S} = \{1^+_A \mid A \in \mathfrak{A}\} \cup \{i^-_A \mid i \in I \text{ and } A \in \mathfrak{A} \text{ s.t. } A \neq 0\},$$

where $1^+_A(x) = \begin{cases} 1^+ & x \in A \\ 0^- & x \notin A \end{cases}$ and $i^-_A(x) = \begin{cases} i^- & x \in A \\ 0^- & x \notin A. \end{cases}$

We denote $(R, \mathfrak{A}_\#, II)$ by E, then the two mappings,

$f : E \times E \longrightarrow E \quad (x,y) \longmapsto x+y$ and $\cdot : R \times E \longrightarrow E \quad (r,x) \longmapsto rx$

are fuzzy continuous.

But, the fuzzy topology $\mathfrak{A}_\#$ on E is not translation invariant. Therefore, we use II-fuzzy topologies which contain all the constant fuzzy sets.

Proposition 2.6.3. Let A be an i-neighborhood of $z_0 = x_0 + y_0$, i.e. $A \in \mathcal{U}_i(z_0)$ and $z_0 = x_0 + y_0$, in a II-fuzzy topological vector space E. Then, there exist $A_1 \in \mathcal{U}_i(x_0)$ and $A_2 \in \mathcal{U}_i(y_0)$ such that $A_1 + A_2 \leqq A$. In case $x_0 = y_0 = 0$, there exists $B \in \mathcal{U}_i(0)$ such that $B + B \leqq A$.

Proof. Since the map $+ : E \times E \longrightarrow E$, $(x,y) \longmapsto x+y$ is continuous, the proof is obvious.

Proposition 2.6.4. In a II-fuzzy topological vector space E, if $A \in \mathcal{U}_i(0)$, then there exists $B \in \mathcal{U}_i(0)$ such that B is balanced and $B \leqq A$.

Proof. Without loss of generality, we may assume that A is open. The map $g : K \times E \longrightarrow E$, $g(t,x) = tx$ is continuous. Since $g^{-1}(A)(0,0) = A(0) \geqq i^+$, there exist $\alpha \in \mathcal{V}_i(0)$ and $D \in \mathcal{U}_i(0)$ such that $\alpha \times D \leqq g^{-1}(A)$. From the definition of fuzzy topology on K, there exists $\delta > 0$ such that $\{t \in K \mid |t| \leqq \delta\} \subset \{t \in K \mid \alpha(t) \geqq i^+\}$. Now, let $C = A \wedge D \wedge i^+ \in \mathcal{U}_i(0)$. If $|t| \leqq \delta$ then

$A(tx) = g^{-1}(A)(t,x) \geqq (\alpha \times D)(t,x) \geqq (\alpha \times C)(t,x) = \alpha(t) \wedge C(x) = C(x)$.

Then we take $B = \delta C$.

Proposition 2.6.5. Let A be an i-neighborhood of 0 in a II-fuzzy topological vector space E. Then, for every $x_0 \in E$, there exists $t_0 \in K$ such that $t_0 A(x_0) \geqq i^+$.

Proof. Let $f : K \times E \longrightarrow E$, $f(t,x) = tx$. Since f is continuous and $f(0,x_0) = 0$, $f^{-1}(A)$ is an i-neighborhood of $(0,x_0)$. $f^{-1}(A)(0,x_0) \geqq i^+$. Hence there exist $\alpha \in \mathcal{V}_i^{\sim}(0)$ and $B \in \mathcal{U}_i(x_0)$ such that $\alpha \times B \leqq f^{-1}(A)$.

From the definition of fuzzy topology on K, there exists $\delta > 0$ such that $\beta \leqq \alpha$, where,

$$\beta(t) = \begin{cases} i^+ & (|t| < \delta) \\ 0^- & (|t| \geqq \delta). \end{cases}$$

Clearly, $\beta \times B \leqq f^{-1}(A)$.

If $|t| < \delta$, $A(tx_0) = f^{-1}(A)(tx_0) \geqq \beta(t) \wedge A(x_0) \geqq i^+$. Therefore, $t_0 A(x_0) \geqq i^+$ for t_0 with $0 < |1/t_0| < \delta$.

C. Omoto proved the following

Theorem 2.6.6. Let E be a II-fuzzy topological vector space over K, and $\mathcal{U}_i(0)$ be the family of all i-neighborhoods of 0 for $i \in [0,1]$. Then $\mathcal{U}_i(0)$ has the following properties:

(1) every constant fuzzy set j_X with $j \geqq i^+$ belongs to $\mathcal{U}_i(0)$. And $A(0) \geqq i^+$ for each $A \in \mathcal{U}_i(0)$;

(2) if $A_1, A_2 \in \mathcal{U}_i(0)$, then $A_1 \wedge A_2 \in \mathcal{U}_i(0)$;

(3) for each $A \in \mathcal{U}_i(0)$, there exists $B \in \mathcal{U}_i(0)$ which is balanced with $B \leqq A$;

(4) for each $A \in \mathcal{U}_i(0)$, there exists $B \in \mathcal{U}_i(0)$ such that B + B

$\leqq A$;

(5) let $A \in II^E$, if there exists $B \in \mathcal{U}_i(0)$ with $B \leq A$, then $A \in \mathcal{U}_i(0)$;

(6) let $A \in \mathcal{U}_i(0)$ and $x_0 \in E$, then there exists a positive number δ such that $A(tx_0) \geqq i^+$ for all $|t| < \delta$.

Conversely, if $\{\mathcal{U}_i(0)\}_{i\in[0,1]}$ is a family of families of II-fuzzy sets in a vector space E over K satisfying (1)-(6), then there exists a unique II-fuzzy linear topology $\mathfrak{A}$ on E.

Proof. It is clear that $\mathcal{U}_i(0)$ has the property (1)-(6).

Conversely let $\{\mathcal{U}_i(0)\}_{i\in[0,1]}$ be a family satisfying (1)-(6). For $A \in \mathcal{U}_i(0)$, from (3) and (4), there exist sequence (A_n) such that $A_n \in \mathcal{U}_i(0)$, A_n is balanced, $A_1 = A$ and $A_{n+1}+A_{n+1} \leqq A_n$. Since A_n is balanced, $A_n(0) \geqq A_{n+1}(0) \geqq i^+$, $A(0) \geqq A_1(0)$. Let B be $\sum_{n=1}^{\infty} A_n$, where $\sum_{n=1}^{\infty} A_n = \bigvee_{n=1}^{\infty} \sum_{k=1}^{n} A_k$. Then, $B \in \mathcal{U}_i(0)$.

We need to show that, if $B(x_0) \geqq j^+$ then $B-x_0 \in \mathcal{U}_j(0)$. Since $j^+ \neq \bigvee\{q \mid q<j^+\}$, there exist A_n and $n_0 \in N$ such that

$(\sum_{n=1}^{n_0} A_n)(x_0) \geqq j^+$, $(\sum_{n=1}^{n_0} A_n)(0) \geqq i^+$. Now,

$$B \geqq A_1 + A_2 + \cdot \cdot \cdot + A_n + A_{n+1}.$$

Hence,

$$\begin{aligned} B(x) &\geqq (A_1 + A_2 + \cdot \cdot \cdot + A_n + A_{n+1})(x) \\ &= \bigvee_{x=y+z} [(A_1 + A_2 + \cdot \cdot \cdot + A_n)(y) \wedge A_{n+1}(z)] \\ &\geqq (A_1 + A_2 + \cdot \cdot \cdot + A_n)(x_0) \wedge A_{n+1}(x-x_0) \\ &\geqq (j^+ \wedge A_{n+1})(x-x_0). \end{aligned}$$

So, $j^+ \wedge A_{n+1} \leqq B-x_0$. From $j^+ \wedge A_{n+1} \in \mathcal{U}_j(0)$ and (5), $B-x_0 \in \mathcal{U}_j(0)$. For each $x \in E$, let $\mathcal{U}_i(x) = \{ x+A \mid A \in \mathcal{U}_i(0)\}$. It is easy to see that the family $\{\mathcal{U}_i(x) \mid i \in I,\ x \in E\}$ has the properties (N1)-(N4) of theorem 2.3.2. Therefore, there exists a unique fuzzy topology $\mathfrak{A}$ on E. It is easy to see that $\mathfrak{A}$ is translation invariant.

It remains to prove that the mappings + and $\cdot$ are continuous.

Continuity of sum:

Let the map $f : E \times E \longrightarrow E$, $f(x,y) = x+y$. If $D \in \mathcal{U}_i(x_0+y_0)$, then there exists $A \in \mathcal{U}_i(0)$ such that $D = x_0 + y_0 + A$. From (4), there exists $B \in \mathcal{U}_i(0)$ such that $B + B \leqq A$. So, $(x_0+B) \times (y_0+B) \leqq f^{-1}(x_0+y_0+A)$. Hence f is continuous.

Continuity of scalar multiplication:

We must show that, if $A \in \mathcal{U}_i(0)$ then $tA \in \mathcal{U}_i(0)$ $(t \neq 0)$. Let A be balanced. When $|t| \geqq 1$, the result is clear. Let $|t| < 1$. For n such that $2^{-n} < |t|$, there exists $B \in \mathcal{U}_i(0)$ such that

$$\underbrace{B + B + \cdots + B}_{2^n} \leqq A.$$

Hence $B \leqq 2^{-n}A$. So $tA = 2^n t 2^{-n} A \geqq 2^n tB \geqq B$.

Therefore $tA \in \mathcal{U}_i(0)$.

Let the map $g : K \times E \longrightarrow E$, $g(tx) = tx$. And let $(t_0, x_0) \in K \times E$, $\underline{A} = t_0x_0 + A$, $A \in \mathcal{U}_i(0)$. From (3) and (4), there exists a balanced set $B \in \mathcal{U}_i(0)$ such that $B + B + B + B \leqq A$. Since B is balanced, $B \leqq B + B$. Hence $B + B + B \leqq A$. From (6), there exists $\delta \in (0,1]$ such that $B(tx_0) \geqq i^+$ for $|t| \leqq \delta$.

Let
$$C = \begin{cases} (1/t_0)B \wedge B & (t_0 \neq 0) \\ B & (t_0 = 0) \end{cases}$$

$$\alpha(t) = \begin{cases} i^+ & (|t| < \delta) \\ i^- & (|t| \geqq \delta). \end{cases}$$

Then C is balanced, $C \in \mathcal{U}_i(0)$ and $\alpha \in \mathcal{V}_i(0)$.

If $D = (t_0+\alpha) \times (x_0+C)$, then $D \in \mathcal{U}_i(t_0, x_0)$. We need to show that

$\alpha(t-t_0) \wedge C(x-x_0) \leq \underline{A}(tx) = A(tx-t_0x_0)$

When $|t-t_0| \geq \delta$, it is obvious from $\alpha(t-t_0) = 0$. Let $|t-t_0| < \delta$.

We have $tx-t_0x_0 = t_0(x-x_0) + (t-t_0)x_0 + (t-t_0)(x-x_0)$.

Then, $B(t_0(x-x_0)) = \begin{cases} (B/t_0)(x-x_0) \geq C(x-x_0) & (t_0 \neq 0) \\ B(0) \geq C(x-x_0) & (t_0 = 0). \end{cases}$

Hence, $B(t_0(x-x_0)) \geq C(x-x_0) \geq D(t,x)$.

Similarly, from $|t-t_0| \leq \delta$,

$B((t-t_0)x_0) \geq i^+ \geq (t-t_0) \geq D(t,x)$.

Since B is balanced,

$$\begin{aligned} B((t-t_0)(x-x_0)) &\geq B(x-x_0) \\ &\geq C(x-x_0) \\ &\geq D(t,x). \end{aligned}$$

Hence, $$\begin{aligned} A(tx-t_0x_0) &= A(t_0(x-x_0) + (t-t_0)x_0 + (t-t_0)(x-x_0)) \\ &\geq B(t_0(x-x_0)) \wedge B((t-t_0)x_0) \wedge B((t-t_0)(x-x_0)) \\ &\geq D(t,x). \end{aligned}$$

Therefore g is continuous.

Definition 2.6.7. A sequence of fuzzy sets $\mathcal{U} = (U_n)$ is called a fuzzy string (abbreviate string), if

(0) $U_n(0) \in I^+$ and $U_n(0) = U_{n+1}(0)$ for all $n \in N$,

(1) every $U_n \in \mathcal{U}$ is balanced,

(2) every U_n is weak absorbing, that means U_n is $U_n(0)$-absorbing,

(3) (U_n) is summative, that means $U_{n+1} + U_{n+1} \leq U_n$ for all $n \in N$.

U_n is called the n-th knot.

If $\mathcal{U} = (U_n)$ and $\mathcal{V} = (V_n)$ are strings in E and $t \in K$, we define

$t\mathcal{U} = (tU_n)$,

$\mathcal{U} + \mathcal{V} = (U_n + V_n)$,

$\mathcal{U} \cap \mathcal{V} = (U_n \cap V_n)$.

$\mathcal{U} + \mathcal{V}$ is called the sum and $\mathcal{U} \cap \mathcal{V}$ the intersection of the strings $\mathcal{U}$ and $\mathcal{V}$. Of course $\mathcal{U} + \mathcal{V}$, $\mathcal{U} \cap \mathcal{V}$ and $t\mathcal{U}$ $(t \neq 0)$ are again strings in E.

Corollary 2.6.8. Let $\mathcal{F}$ be a set of strings in a vector space E, then finite intersections of the knots of the strings in generate a unique fuzzy linear topology.

3. $IQ_{<M>}$-FUZZY SET AND PARTICLE.

3.1. $IQ_{<m>}$-fuzzy set

When we apply the fuzzy theory to various fields, we often adopt the unit interval I (that is, [0,1]) as the truth value set of the fuzzy sets (see [44], [33] and [3]). Why do we adopt the set I as the truth value set? The reason may simply be that I is a totally ordered continuum in which operations like algebraic operations such as addition and multiplication are built-in. It is natural to define "all" by 1 and "nothing" by 0, and anything in between by a number between 0 and 1. However, for constructing a mathematical system where we can formulate our problems in a better way, we may have to consider more suitable truth value sets. For example, as is shown in [12], [19], [27] and [7], it is not easy to introduce the concept of compactness using I-fuzzy sets in general topology. On the other hand, the notion of L-fuzzy sets defined in [10], where L is a general completely distributive lattice with involution, is too abstract and general for some applications. Our aim is to construct fuzzy sets that are at the same time suitable for various applications and mathematically easy to deal with.

Since the unit interval I plays an important role in fuzzy sets, it is desirable that a truth value set L has the following two properties:

(1) L includes I as a sublattice;

(2) there is a suitable equivalence relation R such that the quotient lattice L/R is lattice isomorphic to I.

The two arrow set II introduced in chapter 2 is an example of the set satisfying the above conditions.

Let X be an ordinary set, and $P(X)$ the ordinary power set. For $A, B \in P(X)$, we denote the symmetric difference $(A \cap B^c) \cup (A^c \cap B)$ by $A \Delta B$. Then, $P(X)$ is an Abelian group relative to Δ. That is ;

(0) if $A, B \in P(X)$, then $A \Delta B \in P(X)$,

(1) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$,

(2) $A \Delta \emptyset = \emptyset \Delta A = A$ ($\emptyset$ is the unity element),

(3) $A \Delta A = \emptyset$ (A is the inverse element of itself),

(4) $A \Delta B = B \Delta A$.

A natural question arises. Can we extend this operation to the family of fuzzy sets ? In the family of I-fuzzy sets, if we define $A \Delta B \equiv (A \wedge B') \vee (A' \wedge B)$, then I^X does not form a group relative to Δ. Even if we define $(A \ominus B)(x) \equiv |A(x) - B(x)|$, I^X is not a group under $\ominus$. It is difficult to find a simple operation which is an extension of the symmetric difference in the family of I-fuzzy sets, by which the family becomes an Abelian group. The above mentioned difficulties also occur in II-fuzzy sets. Hence, it is natural to look for truth value set L lying between I and II such that L^X forms an Abelian group under a simple operation. Indeed we show that there exists a family of truth value sets L with :

(1) $II \supset L \supset I$ as sublattice;

(2) there are suitable easy equivalence relations R_1 and R_2 such that II/R_1 is lattice isomorphic to L and L/R_2 is lattice isomorphic to I;

(3) L^X form an Abelian group under a simple operation.

For a fixed natural number m, let $Q_{<m>}$ be the set $\{k/m^n \in I \mid n \in N, k \in N$ such that $0 < k < m^n\}$. We denote $\{x^+ \in I^+ \mid x \in Q_{<m>}\}$ by $Q^+_{<m>}$ and $\{x^- \in I^- \mid x \in Q_{<m>}\}$ by $Q^-_{<m>}$. Then, the set $I^- \cup Q^+_{<m>}$ by $IQ_{<m>}$. The following results are immediate.

Proposition. 3.1.1. $IQ_{<m>}$ is a sublattice of II.

Proposition. 3.1.2. I is a sublattice of $IQ_{<m>}$.

Proof. Regard I^- as I.

Proposition. 3.1.3. Let R_1 be the equivalence relation in II such that $x^- \sim x^+$ for every $x \in I - Q_{<m>}$. Then, II/R_1 is isomorphic to $IQ_{<m>}$.

Proposition. 3.1.4. Let R_2 be the equivalence relation in $IQ_{<m>}$ such that $x^- \sim x^+$ for every $x \in Q_{<m>}$. Then, $IQ_{<m>}/R_2$ is isomorphic to I.

We define the operation $'$ from $IQ_{<m>}$ to $IQ_{<m>}$ by :

$$(x^+)' = (1-x)^- \in Q^-_{<m>} \quad \text{for} \quad x^+ \in Q^+_{<m>}.$$
$$(x^-)' = (1-x)^+ \in Q^+_{<m>} \quad \text{for} \quad x^- \in Q^-_{<m>}$$
$$(x^-)' = (1-x)^- \in I^- - Q^-_{<m>} \quad \text{for} \quad x^- \in I^- - Q^-_{<m>}.$$

Since the operation $'$ has the involution property, the set $IQ_{<m>}$ is a completely distributive lattice with involution.

Remark Let f be the canonical injection from I to $IQ_{<m>}$, and g be the canonical projection from $IQ_{<m>}$ to $IQ_{<m>}/R_2$. Then, $f(a') \neq [f(a)]'$ for $a \in I$. But $g(b') = [g(b)]'$ for $b \in IQ_{<m>}$ and $g(f(a')) = g([f(a)]') = [g(f(a))]'$ for $a \in I$. Let h be the canonical injection from $IQ_{<m>}$ to II, and k be the canonical projection from II to II/R_1. Then, $h(a') \neq [h(a)]'$ for $a \in IQ_{<m>}$. But $k(b') = [k(b)]'$ for $b \in II$ and $k(h(a')) = k([h(a)]') = [k(h(a))]'$ for $a \in IQ_{<m>}$. Hence, the involution in $IQ_{<m>}$ stated above is natural.

Let Z_m^N be the set of infinite sequences $a = (a_n)$, where $a_n \in \{0,1,\ldots m-1\}$, $n \in N$. The order in Z_m^N is defined lexicographically. Thus for any two elements $a, b \in Z_m^N$, $a > b$ if there exists n such that $a_1 = b_1$, $a_2 = b_2$, ..., $a_n = b_n$ but $a_{n+1} > b_{n+1}$.

Theorem. 3.1.5. Z_m^N is lattice isomorphic to $IQ_{<m>}$.

Proof. Define the function from Z_m^N to $IQ_{<m>}$ by,

$$f((a_n)) = \begin{cases} (a_0/m + a_1/m^2 + \ldots + a_n/m^n)^+ \in Q^+_{<m>} \\ \qquad \text{for } a=(a_n)\ Z_m^N \text{ with finitely many nonzero } a_n\text{s}, \\ (a_0/m + a_1/m^2 + \ldots + a_n/m^n + \ldots)^- \in I^- \\ \qquad \text{otherwise.} \end{cases}$$

Then, f is a isomorphism from Z_m^N to $IQ_{<m>}$.

Therefore, $IQ_{<m>}$ inherit algebraic properties from Z_m^N. Z_m forms an Abelian group under usual addition $\dotplus$ modulo m. This induce an operation $\ddagger$ in Z_m^N defined by $(a_n) \ddagger (b_n) = (a_n \dotplus b_n)$. It is easily seen that $(Z_m^N, \ddagger)$ forms an Abelian group. Hence by the isomorphism given in Theorem 1, $IQ_{<m>}$ forms an Abelian group under the induced operation : $(A \oplus B)(x) = A(x) \ddagger B(x)$.
Hence we have,

Theorem 3.1.6. There exists an operation on the family of $IQ_{<2>}$-fuzzy sets, which is an extension of symmetric difference, by which the family forms an Abelian group.

3.2. Particle

The concept of fuzzy points was defined in [43]. However, it causes some problems (see [13]). We introduce another new approach to the concept of points. In our real world (in physics), "point" is an imaginary concept to indicate a position. It does not exist in material world. On the other hand, we found out (or thought out) "atoms" and "elementary particles" as constituent elements of matter. We introduce the concept of "particles" in fuzzy spaces in the same manner. This is not the extension of the concept of the usual "point" but the extension of the concept of the usual "one-point-set" as a minimal set. The concept of "one-point-set" in usual space has at least two properties. One is "any set can be represented as the union of one-point-sets". The other is "every one-point-set cannot be represented as the union of other one-point-sets". We construct "particles" with these properties.

Let S be a non-empty subfamily of L^X, R a subfamily of S. We say that R (subset of S) is a particle family of S (in L^X) if the following two conditions are satisfied:

(P1) for any $s \in S$, there exists a subfamily R_0 of R s.t. $s = \bigvee R_0$ (that is R is a base of S),

(P2) if $r \in R$, then $r \neq \bigvee S_0$ for any subfamily S_0 of $S-\{r\}$.

The condition (P1) means "any set can be represented as the union of one-point-sets". The condition (P2) means "every one-point-set cannot be represented as the union of other one-point-sets". We call an element of R a particle of S. When $R_0=\emptyset$, we have $\bigvee R_0 = 0_X$, and from the condition (P2) we see that $0_X \notin R$.

We can define the co-particle as the dual concept of the particle. We say that R is a co-particle family of S (in L^X) if the following two conditions are satisfied:

(CP1) for any $s \in S$, there exists a subfamily R_0 of R s.t. $s = \bigwedge R_0$

(CP2) if $r \in R$, then $r \neq \bigwedge S_0$ for any subfamily S_0 of $S-\{r\}$.

We call an element of R a co-particle of S. If R is particle of S, then $\{r' \mid r \in R\}$ is a co-particle of $\{s' \mid s \in S\}$.

Proposition 3.2.1. If there exists a particle family of S, it is unique.

Proof. Let R and Q are particle families of S. Suppose there exists $A \in Q$ s.t. $A \notin R$. Then from the property (P1) in R, the A is written in the form $A = \bigvee R_0$ where R_0 is subfamily of R. But this contradicts the property (P2) of Q. So every element of Q is in R. We can prove that every element of R is in Q in the same way.

Proposition 3.2.2. If there exists a co-particle family of S, it is unique.

The proof is similar to that of the proposition 3.2.1.

In the sequel, we write the particle family of S by S^p, the co-particle family of S by S_p.

Example 1. Let L = 2. The particle family $(2^X)^p$ is the family of the all one-point-sets of X (i.e. $\{\{x\} \mid x \in X\}$), and the co-particle family $(2^X)_p$ is the family $\{X-\{x\} \mid x \in X\}$.

Example 2. Let L=II. Let p_y be a II-fuzzy set defined by

$$p_y(x) = \begin{cases} p^+ & x=y \\ 0^- & x \neq y. \end{cases}$$

The particle family $(II^X)^p$ is $\{p_y \mid 0 \leqq p \leqq 1,\ y \in X\}$, and the co-particle family $(II^X)_p$ is $\{(p_y)' \mid 0 \leqq p \leqq 1,\ y \in X\}$. Similarly, the particle family $(IQ_{\langle m \rangle}{}^X)^p$ is $\{p_y \mid p \in Q_{\langle m \rangle}, y \in X\}$, and the co-particle family $(IQ_{\langle m \rangle}{}^X)_p$ is $\{(p_y)' \mid p \in Q_{\langle m \rangle},\ y \in X\}$.

It is not always true that the particle family and the co-particle family of L^X exist.

Example 3. When L = I, the particle family and the co-particle family of L^X do not exist.

Example 4. When X = L = I, let C_y be a I-fuzzy set defined by

$$C_y(x) = \begin{cases} y & x < y \\ 1 & x \geqq y, \end{cases}$$

and D be the set $\{C_y \mid y \in I\}$. Then D^p is $D-\{1_x\}$.

Next, we describe how to construct a particle. Let S be a non-empty subfamily of L^X. We define $S^{\#} = \{A \in S \mid A > \bigwedge\{B \in S \mid B < A\}\}$ and $S_{\#} = \{A \in S \mid A < \bigvee\{B \in S \mid B > A\}\}$.

Proposition 3.2.3. If there exists the particle family of S, then $S^p = S^{\#}$.

Proof. Suppose $A \in S^p$, then can not be written as a union of subfamily of $S-\{A\}$, by the condition (P2). Hence $A > \bigvee\{B \in S \mid B < A\}$, and so $A \in S^{\#}$. Conversely, suppose that there exists an $A \in S^{\#}$ s.t. $A \notin S^{P}$. By the condition (P1), there exist a subset $\{B_j\}$ of S^{P} s.t. $A = \bigvee_{j \in J} B_j$. Then $B_j \wedge A < A$ and $\bigvee_{j \in J} (B_j \wedge A) = (\bigvee_{j \in J} B_j) \wedge A = A$. That is, $\bigvee \{(B_j \wedge A) \in S \mid (B_j \wedge A) < A\} = A$. It is a contradiction.

Proposition 3.2.4. If there exists the co-particle family of S, then $S_p = S_{\#}$.

The proof is similar to that of the proposition 4.1.

Corollary 3.2.5. If there exists the particle family of L^X, then $(L^X)^P = (L^X)^{\#}$, and if there exists the co-particle family of L^X, then $(L^X)_p = (L^X)_{\#}$.

For an L-fuzzy topological space $(X, \mathfrak{O}, L)$, let K_0 be the set $\{A \in L_X \mid A = \bigvee_{i \in I} \bigwedge_{j \in J} B_{ij}$ where $B_{ij} \in \mathfrak{O}$ or $(B_{ij})' \in \mathfrak{O}\}$, K^*_0 be the set $\{A \in L_X \mid A = \bigwedge_{j \in J} B_j$ where $B_j \in \mathfrak{O}$ or $(B_j)' \in \mathfrak{O}\}$, K_1 be the set $\{A \in L_X \mid A = \bigvee_{i \in I} \bigwedge_{j \in J} B_{ij}$ where $B_{ij} \in \mathfrak{O}\}$, and K^*_1 be the set $\{A \in L_X \mid A = \bigwedge_{j \in J} B_j$ where $B_j \in \mathfrak{O}\}$.

Proposition 3.2.6. $(X, \mathfrak{O}, L)$ is T_0 if and only if $L^X = K_0$. $(X, \mathfrak{O}, L)$ is T_1 if and only if $L^X = K_1$. (The definitions of T_0 and T_1 are given in section 0.4.)

Proof. It is obvious.

Proposition 3.2.7. $(K_0)^{\#} = (K^*_0)^{\#}$ and $(K_1)^{\#} = (K^*_1)^{\#}$.

Proof. Suppose that there exists an $A \in (K_0)^{\#}$ s.t. $A \notin (K^*_0)^{\#}$. From $A \in K_0$, there exist $B_{ij} \in \mathfrak{O} \cup \mathfrak{O}'$ s.t. $A = \bigvee_{i \in I} \bigwedge_{j \in J} B_{ij}$. Since

$A > \{B \in K_0 | B < A\}$, there exists a $J_0 \subset J$ s.t. $A = \bigwedge_{j \in J_0} B_j$. Hence $A \in K^*_0$. From $A \notin (K^*_0)^\#$, we have $A \leq \bigvee\{B\ K^*_0 | B < A\}$. By $K^*_0 \subset K_0$, $A \leq \bigvee\{B\ K_0 | B < A\}$, which leads a contradiction. $(K_0)^\# \subset (K^*_0)^\#$. Conversely, suppose that there exists an $A \in (K^*_0)^\#$ s.t. $A \notin (K_0)^\#$. From $A \notin (K_0)^\#$, $A = \bigvee\{B \in K_0 | B < A\}$. We can write $B = \bigvee_{i \in I} B_i$ where $B_i \in K^*_0$, and hence $A \leq \bigvee\{B \in K^*_0 | B < A\}$. It is a contradiction. Hence, $(K_0)^\# \supset (K^*_0)^\#$. Therefore $(K_0)^\# = (K^*_0)^\#$. Similarly, we can prove $(K_1)^\# = (K^*_1)^\#$.

Proposition 3.2.8. If $(X, \mathcal{O}\!\iota, L)$ is T_0, then $(L^X)^\# = (K^*_0)^\#$. If $(X, \mathcal{O}\!\iota, L)$ is T_1, then $(L^X)^\# = (K^*_1)^\#$.

proof. It follows from Proposition 3.2.6 and Proposition 3.2.7.

Theorem 3.2.9. Suppose $(X, \mathcal{O}\!\iota, L)$ is an L-fuzzy topological space, and there exists the particle family of L^X. Then $(X, \mathcal{O}\!\iota, L)$ is T_0 if and only if $(L^X)^P = (K^*_0)^\#$. Also $(X, \mathcal{O}\!\iota, L)$ is T_1 if and only if $(L^X)^P = (K^*_1)^\#$.

Proof. By Corollary 3.2.4 and Proposition 3.2.8.

This theorem indicate particles of spaces which are T_0 space can be constructed by open and closed sets, particles of spaces which are T_1 space can be constructed by open and sets.

Let L^X be the family of L-fuzzy sets of X, L^Y be the family of L-fuzzy sets of Y. Suppose there exists the particle families of L^X and L^Y. $(L^X)^P$ and $(L^Y)^P$ are ordered sets.

Proposition 3.2.10. If f is a order preserving function from $(L^X)^p$ to $(L^Y)^p$, then we can define a fuzz relation F from L^X to L^Y by:

$$F(A)=\bigvee_{j\in J} f(A_j)$$

for any $A\in L^X$, where $A = \bigvee_{j\in J} A_j$, $A_j\in (L^X)^p$.

Proof. If $\bigvee_{i\in I} A_i = \bigvee_{j\in J} B_j$, A_i, $B_j\in (L^X)^p$, there exists a j_i such that $A_i \leqq B_{j_i}$, since A_i is a particle for each $i\in I$, Hence $f(A_i) \leqq F(\bigvee_{j\in J} B_j)$. And so $F(\bigvee_{i\in I} A_i) \leqq F(\bigvee_{j\in J} B_j)$. Therefore $F(\bigvee_{i\in I} A_i) = F(\bigvee_{j\in J} B_j)$.

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