Background Field Equations in the String Theory and Higher Dimensional Cosmology

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Background Field Equations in the String Theory and Higher Dimensional Cosmology

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Doctoral Thesis

Background Field Equations in the String Theory and Higher Dimensional Cosmology

ストリング理論における背景場方程式と高次元宇宙論

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Abstract

The Weyl invariance of the two-dimensional σ-model, which describes the string propagation in a background, is a necessary condition for consistent quantization of the string theory and it restricts the background configurations. On the other hand, in the string theory, the vanishing one-point amplitude is the condition for a classical solution of the background fields by analogy with field theory. Thus it is natural to anticipate that the Weyl invariance condition is equivalent to the vanishing one-point amplitude including the string loop correction. But at the string loop level, this equivalence is not confirmed explicitly. Therefore, we calculate one-point amplitude and show that its vanishing provides the same background field equation as that obtained from the Weyl invariance condition to string one-loop order and $O(\alpha')$.

Next we consider the higher dimensional cosmology based on this string-loop corrected background field equation and find a cosmological evolution different from the ordinary Kaluza-Klein cosmology due to the string vacuum energy, which is a string loop correction to the equation.

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1. Introduction

String theories[1] are quantum theories of elementary one-dimensional objects, rather than points as in the conventional quantum theory. These theories originated in an attempt to describe hadron physics[2]. Strings consist of two distinct topologies, called open and closed. Open strings have free ends, whereas closed strings have the topology of a circle.

However, these theories contain massless vector and 2-tensor states, which arise from open and closed strings, respectively. In the zero slope limit (or in the low-energy limit) massless vector particles behave precisely as Yang-Mills gauge fields and massless symmetric 2-tensor state interacts appropriately to be identified as a graviton, so that the string theories are regarded as a unified theory including gravity[3].

There are two basic types of string theories: bosonic strings and superstrings. Bosonic string theories are consistently formulated in 26-dimensional space-time and superstring theories in 10-dimensional space-time. Superstring theories are classified into two types: type I and type II. Type I superstring theory (SST I) consists of open and closed strings and have N=1 space-time supersymmetry. In the low energy limit the SST I is reduced to D=10, N=1 supergravity coupled to super-Yang-Mills theory. Type II superstring theories (SST II) consist of closed strings only and have N=2 space-time supersymmetry. The type IIa theory has supercharges of opposite chirality and its low-energy limit is the non-chiral D=10, N=2 supergravity. The type IIb theory has supercharges of the same chirality and its low-energy limit is the chiral D=10, N=2 supergravity theory. The SST IIb was shown to be gravitational anomaly free[4], and it was thought that the SST I might have gauge and gravitational anomalies.

In 1984, however, Green and Schwarz[5] showed that the SST I is gauge and gravitational anomaly free and one-loop finite if the gauge group is SO(32) and that in the low-energy effective field theory those anomalies vanish when the gauge group is $E_8 \times E_8$ besides SO(32). This statement suggests that a consistent $E_8 \times E_8$ superstring theory can be also formulated. A new type of superstring theory was found by Gross et al[6]. Their theory, called the heterotic string, has gauge group $E_8 \times E_8$ or $Spin(32)/Z_2$ and consists of only closed strings. Hence, at present, we know that five superstring theories (SO(32)SST I, SST IIa, SST IIb and $E_8 \times E_8$ or $Spin(32)/Z_2$ heterotic strings) are anomaly-free and finite at one loop (perhaps all order).

Note that, in the point particle theory, quantum gravity has nonrenomalizable divergences. On the other hand, superstring theories contain gravity and are finite. Thus they seem to be consistent quantum theories including gravity.

Since the superstring theory is 10-dimensional (bosonic one is 26-dimensional), to obtain the effective 4-dimensional theory, extra 6 (or 22) spatial dimensions should be compactified[#]. To solve this compactification problem, we must consider the string

Recently, the four-dimensional string theory is also constructed [7].

theory in a curved background. The string propagation in background fields can be described by a two-dimensional nonlinear σ-model[8]. On the other hand, the condition for consistent quantization of the string theory is the Weyl (conformal) invariance on the two-dimensional string world-sheet. Therefore, the Weyl invariance of the quantum σ-model is the consistency condition of the string theory in background fields. Since the Weyl anomaly of the σ-model depends on the background fields, the Weyl invariance of the σ-model restricts the background configurations and this condition seems to be equivalent to the equations of motion for background fields obtained from the string (tree-level) effective action[8]. Recently, it was pointed out that the string loop effects contribute to the Weyl anomaly and that the background field equation is modified by the string loops[9,10].

On the other hand, in the string theory, the vanishing one-point amplitude $\langle\langle V\rangle\rangle=0$ is the condition for a classical vacuum solution (or, at the quantum level, an extremum of the effective potential) by analogy with field theory. Thus it is natural to anticipate that the Weyl invariance condition is equivalent to $\langle\langle V\rangle\rangle=0$ including the string loop. At the string tree level, this equivalence is plausible[11], but at the string loop level, this equivalence is not confirmed explicitly.

In this thesis, we calculate $\langle\langle V\rangle\rangle$ using the Polyakov's path integral [12-14] and show that $\langle\langle V\rangle\rangle=0$ is equivalent to the Weyl invariance condition of the σ-model to the string one-loop order. Next, we discuss the higher dimensional cosmology based on the string-loop corrected equation of motion. Since the string vacuum energy appears as a one-loop correction to the equations of motion for the background fields, we can expect the cosmological evolution different from an ordinary Kaluza-Klein cosmology[15-17].

In this thesis we consider only the closed bosonic string theory.

This thesis is organized as follows. In sect. 2, we briefly review Polyakov's path integral formulation of the closed bosonic string. In sect. 3, we consider the Weyl anomaly in the bosonic nonlinear σ-model with the metric and dilaton fields and obtain the background field equations. In sect. 4, we investigate effects of string loops on the Weyl anomalies and get the string-loop corrected equation. In sect. 5, we calculate the one-point amplitude using the Polyakov's path integral and show that the vanishing of this amplitude provides the same equation as the string-loop corrected one. In sect. 6, we discuss the cosmological evolution by using the string-loop corrected equation of motion. Finally sect. 7 gives conclusions and discussions.

2. Polyakov String

In this section, we briefly review Polyakov's path integral formulation of the closed bosonic string[12-14].

The basic object in a string theory is a one-dimensional curve, called a string, whose evolution sweeps out a two-dimensional surface (or world-sheet) in space-time. The classical Nambu-Goto action is the area spanned by such a surface:

$$A = \int d^2z \sqrt{\det(h_{ab})}, \tag{2.1}$$

where $h_{ab} = \frac{\partial X^\mu}{\partial z^a}\frac{\partial X_\mu}{\partial z^b}$ is the induced metric on the surface:

$$ds^2 = dX^\mu dX_\mu = \frac{\partial X^\mu \partial X_\mu}{\partial z^a \partial z^b} dz^a dz^b .$$

This action is a non-linear function of the coordinates of the string and this non-linearity leads to difficulties in quantization.

Polyakov's prescription for the quantum theory of the bosonic string is to start instead from the classical action# [12]

$$S_0 = \frac{1}{2}\int_M d^2z \sqrt{g}\, g^{ab}\, \partial_a X^\mu \partial_b X_\mu . \tag{2.2}$$

We set the string tension $T=1/2\pi\alpha'$ equal to unity in this section.

Here M is a two-dimensional compact surface. z^a, a=1,2, are the world-sheet coordinates on M. $X^\mu(z)$ is an embedding of M into space-time or space-time coordinate: E={X: M → space-time}. We shall assume that space-time is flat and Euclidean (R^d). g_{ab} is the world-sheet metric on M: m={g: metric on M}.

The variation of the action S_0 with respect to X^μ and g_{ab} gives the classical equations of motion

$$\Delta X^\mu = -\frac{1}{\sqrt{g}}\partial_a(\sqrt{g}\, g^{ab}\, \partial_b X^\mu) = 0 \ , \tag{2.3}$$

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2} g_{ab} g^{cd} \partial_c X^\mu \partial_d X_\mu = 0, \tag{2.4}$$

where T_{ab} is the energy-momentum tensor. From (2.4), g_{ab} is conformally equivalent to the metric h_{ab} induced by R^d. Therefore (2.3) is reduced to the equation for a surface of minimal area

$$\partial_a(\sqrt{h}\, h_{ab}\, \partial_b X^\mu) = 0, \tag{2.5}$$

and the action S_0 is just the Nambu-Goto action A.

The action S_0 is invariant under:

(i) The group of reparametrizations or diffeomorphisms of the world-sheet M: Diff(M)

$$z^a \rightarrow z'^a(z),$$

$$g_{ab}(z) \rightarrow \frac{\partial z'^c}{\partial z^a}\frac{\partial z'^d}{\partial z^b} \quad g_{cd}(z').$$

(ii) The group of Weyl or conformal rescaling of the metric: Conf(M)

$$z^a \rightarrow z^a,$$

$$g_{ab}(z) \rightarrow e^{2\tau(z)} g_{ab}(z).$$

(iii) The group of Poincaré translations $X^\mu \rightarrow a^\mu{}_\nu X^\nu + X_0^\mu$.

As a result of the local Weyl invariance, the trace of T_{ab} is identically equal to zero whether or not the equation of motion hold, and the classical equation can determine the metric only up to a conformal factor.

In the quantum theory of Polyakov string, we integrate functionally over space-time coordinates X^μ and over metric g_{ab}. In general renormalization is needed and the action should be chosen to be a most general renormalizable one with couplings of non-negative dimension, and consistent with (i) and (iii):

$$S[X,g]= \frac{1}{2}\int_M d^2z \sqrt{g}\, g^{ab}\, \partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi}\int_M d^2z \sqrt{g}\, R + \mu^2\int_M d^2z \sqrt{g}, \quad (2.6)$$

where $\frac{1}{4\pi}\int_M d^2z \sqrt{g}\, R = \chi(M)=2-2h$ is the Euler number of M. h is the number of handles on the surface, which is called the genus of the surface. In general, for any value of μ^2 the Weyl invariance cannot be maintained due to the Weyl anomalies.

The Polyakov partition function is defined by

$$Z = \sum_{\text{topologies}} \int_{\mathrm{m}\times E} [dg][dX]\ e^{-S[X,g]} \quad . \tag{2.7}$$

However, in quatization of the string theory, the classical invariances must be maintained and then this integral overcounts physically equivalent configurations related by the group of diffeomorphism and by the group of Weyl rescaling. Thus we must identify equivalent configurations and count each one just once. In other words, we should integrate not over $\mathrm{m}\times E$ but over the quotient space $\mathrm{m}\times E/\mathrm{Diff}(M)\times \mathrm{Conf}(M)$. When all anomalies vanish, the precise definition for the Polyakov partition function can be given by

$$Z = \sum_{\text{topologies}} \int_{\mathrm{m}\times E} [dg][dX] \frac{1}{\mathrm{Vol}_g(\mathrm{Diff})\mathrm{Vol}_g(\mathrm{Conf})}\ e^{-S[X,g]}, \tag{2.8}$$

where $\mathrm{Vol}_g(\mathrm{Diff})$ and $\mathrm{Vol}_g(\mathrm{Conf})$ are the volume of Diff(M) and Conf(M) through g_{ab}, respectively.

To get the measure [dX], we first define the metric (or the norm) for deformations δX^μ:

$$\|\delta X^\mu\|^2 = \int_M d^2z\ \sqrt{g}\ \delta X^\mu \delta X_\mu \quad . \tag{2.9}$$

The measure is defined by requiring

$$\int [d\delta X]\ e^{-\frac{1}{2}\|\delta X^{\mu}\|^{2}} = 1 \quad . \tag{2.10}$$

Similarly, to obtain the measure [dg] on $T_g(m)$, the tangent space to m at the point g, we define the metric

$$\begin{aligned} \|\delta g\|^{2}_{g} &= \int_{M} d^2 z \sqrt{g}\,(G^{abcd} + u g^{ab} g^{cd})\, \delta g_{ab} \delta g_{cd} \\ &\equiv \langle \delta g, \delta g \rangle_{g} \ , \end{aligned} \tag{2.11}$$

where u is an arbitrary positive real number and G^{abcd} is the projector onto the space of symmetric traceless tensors:

$$G^{abcd} = \frac{1}{2}(g^{ac}g^{bd} + g^{ad}g^{bc} - g^{ab}g^{cd}). \tag{2.12}$$

This suggests that one performs an orthogonal decomposition on δg:

$$\delta g_{ab} = \delta h_{ab} + 2 g_{ab}(\delta\tau), \tag{2.13}$$

where δh_{ab} is the symmetric traceless part and $\delta\tau$ is the trace part. Inserting (2.13) into (2.11), the metric is reduced to

$$\|\delta g\|^{2} = \int_{M} d^2 z \sqrt{g}\, G^{abcd}(\delta h_{ab})(\delta h_{cd}) + 16\, u \int_{M} d^2 z \sqrt{g}(\delta\tau)^{2}. \tag{2.14}$$

Thus the measure [dg] is given by

$$[dg]=[dh][d\tau] \quad . \tag{2.15}$$

Since the metrics (2.9) and (2.11) are invariant under diffeomorphism, but not invariant under Weyl rescaling of g, the measures [dX] and [dg] are also not invariant under the Weyl transformation. This is the origin of the Weyl anomalies.

Let $\hat{g}$ be an admissible metric on M, then for a conformal factor σ the metric

$$g = \hat{g}\ e^{2\sigma} \tag{2.16}$$

is an admissible metric on M. If $\partial M=0$, we can choose $\hat{g}$ to be a constant curvature metric. (See appendix A.) Thus we analyze the effect of gauge transformations on a surface determined by the gauge fixing condition (2.16). Under a diffeomorphism with infinitesimal generator δV_a connected to the identity $Diff_0(M)$, the change in the metric is given by the Lie derivative:

$$\delta g_{ab}\Big|_{g=\hat{g}e^{2\sigma}} = \nabla_a(\delta V_b) + \nabla_b(\delta V_a), \tag{2.17}$$

where ∇_a denotes the covariant derivative with respect to $g=\hat{g}e^{2\sigma}$. The change in the metric by changing the conformal factor σ:

$$\delta g_{ab} = 2(\delta\sigma)g_{ab} = 2(\delta\sigma)\hat{g}_{ab}e^{2\sigma}. \qquad (2.18)$$

Under orthogonal decomposition (2.13), we obtain

$$\delta g_{ab} = \delta h_{ab} + 2g_{ab}(\delta\tau) , \qquad (2.19)$$

where

$$\delta h_{ab} = 2G_{ab}{}^{cd}\nabla_c(\delta V_d) \equiv (P_1\delta V)_{ab}, \qquad (2.20)$$

$$2\delta\tau = 2\delta\sigma + g^{ab}\nabla_a(\delta V_b) \qquad (2.21)$$

and the operator P_1 maps vectors into symmetric traceless 2-tensors. The change of the variables from h and τ to V and σ is

$$[dh][d\tau] = \left|\frac{\partial(h,\tau)}{\partial(V,\sigma)}\right| [dV][d\sigma]. \qquad (2.22)$$

The above Jacobian is written as

$$\left|\frac{\partial(h,\tau)}{\partial(V,\sigma)}\right| = \begin{vmatrix} P_1 & 0 \\ * & 1 \end{vmatrix} = \det P_1 = [\det P_1^\dagger P_1]^{1/2}, \qquad (2.23)$$

where the operator $P_1^\dagger$ is the adjoint of P_1, i.e.,it maps symmetric traceless tensors into vectors.

A vector δV satisfying $P_1(\delta V)=0$ is called a conformal Killing vector (CKV). From (2.20) and (2.21), a diffeomorphism generated by CKV is equivalent to a change in the conformal factor. Since each deformation of the metric is only counted once, such diffeomorphism must be omitted. Thus infinitesimal generator δV_a limits to $\delta V_a{}^{\perp}$, which is orthogonal to CKV, and the correct Faddeev-Popov determinant is $\det' P_1^{\dagger} P_1$, where the prime denotes the omission of the zero eigenvalues.

There are deformations of the metric which are not given by (2.19). Such deformation is called Teichmüller one of the metric. We have the orthogonal decomposition of $T_g(m)$[13,14]

$$T_g(m) = T_g{}^{(Conf)} \oplus T_g{}^{(Diff0\perp)} \oplus T_g{}^{(Teich)}, \qquad (2.24)$$

where $T_g{}^{(Conf)} = \{2\delta\sigma g_{ab}\}$, $T_g{}^{(Diff0\perp)} = \{\text{Image } P_1\}$ and $T_g{}^{(Teich)} = \{\ker P_1^{\dagger}\}$ (see fig.1 and 2). $\ker P_1^{\dagger}$ is the kernel of $P_1^{\dagger}$.

Let S be a gauge slice within m transversal to the orbits of $Diff_0(M) \times Conf(M)$, where $\{t^r\}$ is a set of coordinates and $\{\chi^{(r)}\}$ a set of tangent vectors for the slice S. The deformation of the metric on the slice S is given by

$$\delta g_{ab} = \chi^{(r)}{}_{ab} \delta t^r \qquad (2.25)$$

and we let Λ_g be the orthogonal projection on $T_g{}^{(Teich)}$. Then the deformation of the metric is decomposed into (see fig.3)

$$\delta g_{ab} = 2\delta\sigma g_{ab} + (P_1 \delta V)_{ab} + (\Lambda_g \chi^{(r)})_{ab} \delta t^r. \qquad (2.26)$$

Inserting (2.26) into (2.14), we get

$$\|\delta g_{ab}\|^2_g = \|\delta\sigma\|^2_g + \|P_1\delta V^\perp\|^2_g + \langle\Lambda_g\chi^{(r)},\Lambda_g\chi^{(s)}\rangle\delta t^r\delta t^s$$

$$= \|\delta\sigma\|^2_g + \|P_1\delta V^\perp\|^2_g \qquad (2.27)$$

$$+\langle\Lambda_g\chi^{(r)},\psi^{(r')}\rangle\langle\psi^{(r')},\psi^{(s')}\rangle^{-1}\langle\psi^{(s')},\Lambda_g\chi^{(s)}\rangle\delta t^r\delta t^s ,$$

where $\psi^{(r)}$ is a basis for ker $P_1^\dagger$. Thus the measure [dg] is expressed by

$$[dg] = [d\sigma][\det{}'P_1^\dagger P_1]^{1/2}[dV^\perp]\frac{\det\langle\Lambda\chi,\psi\rangle_g}{[\det\langle\psi,\psi\rangle_g]^{1/2}}[dt]$$

$$= [d\sigma][dV^\perp][dt][\det{}'P_1^\dagger P_1]^{1/2}\ \frac{\det\langle\chi,\psi\rangle_g}{[\det\langle\psi,\psi\rangle_g]^{1/2}} , \qquad (2.28)$$

where $\det\langle\Lambda\chi,\psi\rangle_g=\det\langle\chi,\psi\rangle_g$ since $\Lambda_g^\dagger=\Lambda_g$ and $\Lambda_g\Big|_{\ker P_1^\dagger} = 1$.

We can rewrite the volume of $\mathrm{Diff}_0(M)$ as

$$\mathrm{Vol}_g(\mathrm{Diff}_0) = \mathrm{Vol}_g(\mathrm{Diff}_0{}^\perp)\ \mathrm{Vol}_g(\mathrm{CKV}), \qquad (2.29)$$

$$\mathrm{Vol}_g(\mathrm{CKV}) = [\det\langle\phi,\phi\rangle_g]^{1/2}\prod_i d\alpha^i , \qquad (2.30)$$

where ϕ^i is a basis for ker P_1 and $d\alpha^i$ is an appropriate parameter and

$$\frac{\mathrm{Diff}(M)}{\mathrm{Diff}_0(M)} = \text{Mapping Class Group of } M \ . \qquad (2.31)$$

The functional integral over X is reduced to

$$\int_E [dX]\ e^{-S_0[X,g]} = \left(\frac{2\pi}{\int d^2z\sqrt{g}}\ \det{}'\Delta_g \right)^{-d/2}, \tag{2.32}$$

where Δ_g is the laplacian on M with metric g, and we remove the zero modes corresponding to translations $X \to X + X_0$ (this leads to an overall factor of the volume of space-time, which we drop). Thus the Polyakov partition function is

$$Z = \sum_{\text{topologies}} \int \frac{[d\sigma]}{\mathrm{Vol}_g(\mathrm{Conf})} \frac{[dV^\perp]}{\mathrm{Vol}_g(\mathrm{Diff})} [dt][\det{}'P_1^\dagger P_1]^{1/2}$$

$$\times \frac{\det\langle\chi,\psi\rangle_g}{[\det\langle\psi,\psi\rangle_g]^{1/2}} \left(\frac{2\pi}{\int d^2z\sqrt{g}}\ \det{}'\Delta_g \right)^{-d/2}$$

$$= \sum_{\text{topologies}} \frac{1}{|\mathrm{MCG}|} \int \frac{[d\sigma]}{\mathrm{Vol}_g(\mathrm{Conf})} [dt] \left(\frac{\det{}'P_1^\dagger P_1}{\det\langle\psi,\psi\rangle_g \det\langle\phi,\phi\rangle_g} \right)^{1/2}$$

$$\times \left(\frac{2\pi}{\int d^2z\sqrt{g}}\ \det{}'\Delta_g \right)^{-d/2} \frac{\det\langle\chi,\psi\rangle_g}{\prod d\alpha^i}, \tag{2.33}$$

where |MCG| means the number of elements in the mapping class group of M.

We now analyze the behavior of

$$\left(\frac{\det{}'P_1^\dagger P_1}{\det\langle\psi,\psi\rangle_g \det\langle\phi,\phi\rangle_g} \right)^{1/2} \left(\frac{2\pi}{\int d^2z\sqrt{g}}\ \det{}'\Delta_g \right)^{-d/2} \tag{2.34}$$

under Weyl transformation. (Since the term $\det\langle\chi,\psi\rangle$ is Weyl invariant, we will omit this term.) Using the heat kernel of the determinant:

$$\ln \det{}'H = -\int_{\varepsilon}^{\infty} \frac{dt}{t} \; \mathrm{Tr}' e^{-tH} \quad , \tag{2.35}$$

where ε is an ultraviolet cutoff, we can evaluate the variation of (2.34) under infinitesimal Weyl rescaling. The results are[12,13]

$$\delta \ln \left[\left(\frac{\det{}' P_1^{\dagger} P_1}{\det\langle\psi,\psi\rangle_g \det\langle\phi,\phi\rangle_g} \right)^{1/2} \left(\frac{2\pi}{\int d^2z\sqrt{g}} \; \det{}'\Delta_g \right)^{-d/2} \right] \tag{2.36}$$

$$= -\frac{1}{24\pi}(26-d)\int d^2z \sqrt{g}\, R\, \delta\sigma - \frac{1}{2\pi\varepsilon}(1-d/2)\int d^2z \sqrt{g}\, \delta\sigma \; .$$

The partition function becomes

$$Z = \sum_{\text{topologies}} \frac{1}{|\mathrm{MCG}|} \int \frac{[d\sigma]}{\mathrm{Vol}_{\hat{g}}(\mathrm{Conf})} [dt] \left(\frac{\det{}'\hat{P}_1^{\dagger}\hat{P}_1}{\det\langle\psi,\psi\rangle_{\hat{g}} \det\langle\phi,\phi\rangle_{\hat{g}}} \right)^{1/2} \tag{2.37}$$

$$\times \left(\frac{2\pi}{\int d^2z\sqrt{\hat{g}}} \; \det{}'\Delta_{\hat{g}} \right)^{-d/2} \frac{\det\langle\chi,\psi\rangle_{\hat{g}}}{\prod d\alpha^i} \; e^{-S_{conf}} \quad ,$$

where

$$S_{conf} = \frac{26-d}{24\pi}\left[\int d^2z \sqrt{\hat{g}}\, \hat{R}\, \sigma + \int d^2z \sqrt{\hat{g}}\, \hat{g}^{ab}\partial_a\sigma\partial_b\sigma \right] \tag{2.38}$$

$$+\left[\frac{1}{4\pi\varepsilon}(1-d/2)+\mu^2 \right]\int d^2z \sqrt{\hat{g}}\, e^{2\sigma} \; .$$

This shows that in d=26 (critical dimension) the theory can be made Weyl invariant by choosing μ^2 appropriately.
Then the partition function is reduced to

$$Z = \sum_{\text{topologies}} \frac{1}{|MCG|} \int [dt] \frac{1}{\mathrm{Vol}_{\hat{g}}(CKV)} \frac{\det\langle\chi,\psi\rangle_{\hat{g}}}{[\det\langle\psi,\psi\rangle_{\hat{g}}]^{1/2}} [\det{}'\hat{P}_1^{\dagger}\hat{P}_1]^{1/2} \tag{2.39}$$

$$\times \left(\frac{2\pi}{\int d^2z\sqrt{\hat{g}}} \det{}'\Delta_{\hat{g}} \right)^{-13} ,$$

where $\mathrm{Vol}_{\hat{g}}(CKV)=[\det\langle\phi,\phi\rangle_{\hat{g}}]^{1/2}\prod d\alpha^i$. The quotient space $m/\mathrm{Diff}_0(M)\times\mathrm{Conf}(M)$ is known as Teichmüller space T and T/MCG is the moduli space. Hence, if the integrand is invariant under the transformation of the mapping class group (or the modular group), we obtain the final expression:

$$Z = \sum_{\text{topologies}} \int [dt]_{\text{moduli}} \frac{1}{\mathrm{Vol}_{\hat{g}}(CKV)} \frac{\det\langle\chi,\psi\rangle_{\hat{g}}}{[\det\langle\psi,\psi\rangle_{\hat{g}}]^{1/2}} [\det{}'\hat{P}_1^{\dagger}\hat{P}_1]^{1/2} \tag{2.40}$$

$$\times \left(\frac{2\pi}{\int d^2z\sqrt{\hat{g}}} \det{}'\Delta_{\hat{g}} \right)^{-13} .$$

The n-point scattering amplitude is defined by[12-14]

$$\langle V(k_1)\ldots.V(k_n)\rangle$$

$$= \sum_{\text{topologies}} \int_{m\times E} [dg][dX] \frac{V(k_1)\ldots.V(k_n)}{\mathrm{Vol}_g(\mathrm{Diff})\mathrm{Vol}_g(\mathrm{Conf})} e^{-S[X,g]} . \tag{2.41}$$

Here $V(k)$ is the vertex operator for an on-shell physical particle state with momentum k. It must obey the following covariance properties [18]:

(i) Space-time translation invariance.

This requires that $V(k)$ must be the form

$$V(k) = \int d^2z\, e^{ik\cdot X(z)} U(z,k) \qquad (2.42)$$

with U a function of the derivative of $X^\mu(z)$.

(ii) Space-time Lorentz invariance.

This requires that the space-time indices $\mu,\nu,\ldots$ of the derivatives $\partial X^\mu/\partial z^a \partial z^b \ldots$ in U must be contracted with a real polarization tensor $e_{\mu\nu\ldots}(k)$, which transforms according to a real representation of the little group of k_μ.

(iii) World-sheet reparametrization invariance.

The derivatives of $X^\mu(z)$ in $U(z,k)$ must be covariant ones $X^\mu{}_{;a;b\ldots}$. The a, b indices in these covariant derivative must be contracted with g^{ab} and a factor $\sqrt{g}$ is required for the volume element.

(iv) Weyl invariance

The vertex operators must be invariant under Weyl rescaling after inclusion of all Weyl anomalies.

We choose conformal coordinates on the world-sheet so that the metric is $g_{ab}=e^{2\sigma}\delta_{ab}$, and take a complex basis $(z,\bar{z})$. Condition (iii) requires that U behave under $z \to z'(z)$ as a tensor of type (1,1), where z' is an analytic function of z, not $\bar{z}$. In general a tensor t of type (p,q) transform according to

$$t(z,\bar{z}) \to t'(z',\bar{z}') = \left(\frac{dz'}{dz}\right)^{-p}\left(\frac{d\bar{z}'}{d\bar{z}}\right)^{-q} t(z,\bar{z}) \quad . \qquad (2.43)$$

Then V(k) is invariant under $z \to z'$. $\partial X^{\mu}/\partial z$ is a tensor of type (1,0) and $g_{z\bar{z}} = \frac{1}{2} e^{2\sigma}$ is a tensor of type (1,1). Thus

$$U(z,\bar{z},k) = e^{-(N-1)2\sigma} e_{\mu..\nu..\lambda..\bar{\mu}..}(k)\left(\frac{\partial X^{\mu}}{\partial z}\right)\cdots\left(\frac{D^2X^{\nu}}{Dz^2}\right)\cdots\left(\frac{D^3X^{\lambda}}{Dz^3}\right)\cdots$$
$$\cdots\left(\frac{\partial X^{\bar{\mu}}}{\partial \bar{z}}\right)\cdots, \qquad (2.44)$$

where N, the total number of z derivatives, is equal to the total number of $\bar{z}$ derivatives, because the a,b,... indices are contracted with g^{ab}. Condition (iv) requires that V be independent of σ. V(k) has σ dependence in (2.44) and also one arising from Weyl anomalies in the path integral over X(z). Possible sources of conformal anomalies are

(a) Contractions of X in exp(ik·X).

(b) Contractions of X in the covariant derivatives with X in exp(ik·X).

(c) Contractions of X in the covariant derivatives with each other.

The σ dependence of (a) is given by

$$\exp(ik\cdot X) = \exp(-k_\mu k^\mu \sigma/4\pi):\exp(ik\cdot X): \quad , \qquad (2.45)$$

where : : indicates that the contractions of X within the function inside : : are to be dropped in the path integral. The cancellation of σ dependence of (2.44) and (2.45) gives the mass-shell condition

$$m^2 \equiv -k_\mu k^\mu = 8\pi(N-1) \quad , \qquad N=0,1,2,\ldots, \qquad (2.46)$$

The σ dependence of (b) is eliminated by the transverse conditions

$$k^\mu e_{\mu\nu\ldots} = 0. \qquad (2.47)$$

and the σ dependence of (c) is absent if $e_{\mu\nu\ldots}$ satisfies the traceless conditions

$$\eta^{\mu\nu} e_{\mu\nu\ldots} = 0. \qquad (2.48)$$

In this way we obtain vertex operators for physical particles as follows:

$$m^2=-8\pi \qquad V(k) = \int d^2z \sqrt{g}\, e^{ik\cdot X(z)}, \quad \text{(vertex of tachyon)} \qquad (2.49a)$$

$$m^2 = 0 \qquad V(k) = \int d^2z \sqrt{g}\, g^{ab}\, \partial_a X^\mu \partial_b X^\nu e^{ik\cdot X(z)}\, e_{\mu\nu}(k) \;, \qquad (2.49b)$$

(vertex of massless particle)

and so on. With all anomalies canceled, the amplitude can also reduce to finite dimensional integrals over moduli space by factoring out the volume of the diffeomorphism and the conformal groups.

We here consider the n-point tachyon amplitude. In this case, the X integration can be performed by completing the square, so that

$$\langle V(k_1)\ldots V(k_n)\rangle$$

$$= \sum_{\text{topologies}} \int_{\text{moduli}} [dt]_i \frac{[d\sigma]}{\mathrm{Vol}_g(\mathrm{Conf})} \frac{1}{\mathrm{Vol}_{\hat{g}}(\mathrm{CKV})} \frac{\det\langle\chi,\psi\rangle_{\hat{g}}}{[\det\langle\psi,\psi\rangle_{\hat{g}}]^{1/2}} [\det{}'\hat{P}_1^\dagger \hat{P}_1]^{1/2}$$

$$\times \left(\frac{2\pi}{\int d^2z\sqrt{\hat{g}}} \det{}'\Delta_{\hat{g}} \right)^{-d/2} e^{-S_{conf}} \qquad (2.50)$$

$$\times \left(\prod_{i=1}^{n} \int d^2z \sqrt{g} \right) e^{-\frac{1}{2}\sum_{ij} k_i^\mu k_{j\mu} G(z_i,z_j)} \;,$$

$$\times (2\pi)^d \delta(k_1+\ldots+k_n)$$

where $G(z_i,z_j)$ is a Green's function for the laplacian Δ_g. For $z_i \neq z_j$, $G(z_i,z_j)$ is independent of the conformal factor

$$G_{e^{2\sigma}\hat{g}}(z_i,z_j) = G_{\hat{g}}(z_i,z_j) \;, \qquad z_i \neq z_j \;. \qquad (2.51)$$

On the other hand, for $z_i=z_j$, $G(z_i,z_j)$ depends on the conformal factor

$$G_{e^{2\sigma}\hat{g}}(z_i,z_j)=G_{\hat{g}}(z_i,z_j)+\frac{1}{4\pi}2\sigma(z_i)\ , \qquad z_i=z_j \quad . \tag{2.52}$$

However

$$\int d^2z\ \sqrt{g}\ \exp(-1/2\ k_i{}^2 G_g(z_i,z_i)) \tag{2.53}$$

is independent of σ for $k^2=8\pi$, and in d=26 the amplitude becomes an integral over moduli space:

$$\langle V(k_1)\ldots V(k_n)\rangle$$

$$= \sum_{\text{topologies}}\int [dt]_{\text{moduli}}\ \frac{1}{\text{Vol}\hat{g}(\text{CKV})}\ \frac{\det\langle\chi,\psi\rangle\hat{g}}{[\det\langle\psi,\psi\rangle\hat{g}]^{1/2}}[\det{}'\hat{P}_1^{\dagger}\hat{P}_1]^{1/2}$$

$$\times\ \left(\frac{2\pi}{\int d^2z\sqrt{\hat{g}}}\ \det{}'\Delta\hat{g}\right)^{-13} \tag{2.54}$$

$$\times\ \left(\prod_{i=1}^{n}\int d^2z\ \sqrt{\hat{g}}\ \right)\ e^{-\frac{1}{2}\sum_{ij}k_i^{\mu}\ k_{j\mu}\ G_{\hat{g}}(z_i,z_j)} \quad .$$

$$\times(2\pi)^{26}\delta(k_1+\ldots+k_n) \quad .$$

3. Weyl Anomaly in the Bosonic σ-Model and String Equations of Motion

The string propagation in a non-trivial background of massless condensates (graviton, dilaton, etc.) can be described by the two-dimensional nonlinear σ-model[8]. Thus we believe the Weyl invariance of this σ-model is a necessary condition for consistent quantization of the string theory on a background and that this condition is equivalent to the equations of motion for the massless background fields in the string effective action[8,19,20]. The Weyl invariance implies the vanishing of the trace of the energy-momentum tensor or the absence of the Weyl anomaly for the two-dimensional σ-model. In this section we consider the Weyl anomaly in the bosonic σ-model in a background metric and with dilaton couplings.

The bare action of the renormalized bosonic σ-model in curved $d=2+\varepsilon$ dimensional space is[8,19,20]

$$S_0= \frac{1}{4\pi\alpha'} \int d^dz [\sqrt{g}\, g^{ab}\partial_a X^\mu \partial_b X^\nu G_{0\mu\nu}(X) + \alpha' \sqrt{g}\, \bar{R}\, \Phi_0(X)] \quad .(3.1)$$

Here $G_{0\mu\nu}(X)$ and $\Phi_0(X)$ are metric and the dilaton field, respectively. $\bar{R} = \frac{1}{d-1} R^{(d)}$, where $R^{(d)}$ is the scalar curvature of g_{ab}. Subscript "0" indicates bare quantities. The dilaton term explicitly breaks the classical Weyl invariance but is required by renormalization. Therefore this term is introduced at $O(\alpha')$.

We will use dimensional regularization and choose the renormalized couplings to be dimensionless. The bare couplings have the mass dimension ε=d-2. In the minimal subtraction scheme, the renormalized metric and dilaton are defined by

$$G_{0\mu\nu} = \mu^{\varepsilon}\left(G_{\mu\nu} + \sum_{n=1}^{\infty} \frac{1}{\varepsilon^{n}} T^{G}_{n\mu\nu}(G) \right) , \tag{3.2}$$

$$\Phi_{0} = Z_1\Phi + Z_2 = \mu^{\varepsilon}\left(\Phi + \sum_{n=1}^{\infty} \frac{1}{\varepsilon^{n}} T^{\Phi}_{n}(G,\Phi)\right) , \tag{3.3}$$

$$T^{\Phi}_{n} = \xi_{n}(G)\Phi + \kappa_{n}(G) ,$$

where μ is the renormalization scale. Z_1 is the renormalization operator for a scalar coupling and Z_2 is a function depending on G, which is the additive renormalization for Φ since a divergence proportional to $R^{(d)}$ arises from the first term of the action. The renormalization group β-functions are given by

$$\hat{\beta}^{G}_{\mu\nu} = -\varepsilon G_{\mu\nu} + \beta^{G}_{\mu\nu}, \qquad \beta^{G}_{\mu\nu} = -T^{G}_{1\mu\nu} + G_{\rho\sigma} \frac{\partial}{\partial G_{\rho\sigma}} T^{G}_{1\mu\nu} , \tag{3.4}$$

$$\hat{\beta}^{\Phi} = -\varepsilon\Phi + \beta^{\Phi}, \quad \beta^{\Phi} = -\gamma\Phi + \omega ,$$

$$\gamma = -G_{\mu\nu} \frac{\partial}{\partial G_{\mu\nu}} \xi_{1}, \quad \omega = -\kappa_{1} + G_{\mu\nu} \frac{\partial}{\partial G_{\mu\nu}} \kappa_{1} . \tag{3.5}$$

The energy-momentum tensor is defined by

$$T_{ab} = \frac{2}{\sqrt{g}} \frac{\delta S_{0}}{\delta g^{ab}} . \tag{3.6}$$

Its trace, the Weyl anomaly, is found under $\delta g_{ab} = 2\sigma g_{ab}$, $\delta(\sqrt{g}R) = \sqrt{g}(\varepsilon R\sigma - 2\nabla^{2}\sigma)$,

$$\sqrt{g}\,T^a{}_a = \frac{1}{4\pi\alpha'}\sqrt{g}\,g^{ab}\partial_a X^\mu \partial_b X^\nu(-\varepsilon G_{0\,\mu\nu}) + \frac{1}{4\pi}\sqrt{g}\,\bar{R}(-\varepsilon\Phi_0)$$

$$+ \frac{1}{2\pi}\,\partial_a(\sqrt{g}\,g^{ab}\partial_b\Phi_0). \qquad (3.7)$$

Using $\partial_a\Phi_0=\partial_a X^\mu\partial_\mu\Phi_0$ and the equation of motion

$$D_0{}^a\partial_a X^\mu = \frac{1}{2}\alpha'\,\bar{R}\,D_0{}^\mu\Phi_0 \quad , \qquad (3.8)$$

we obtain

$$\partial_a(\sqrt{g}\,g^{ab}\partial_b\Phi_0) = \sqrt{g}\,g^{ab}\partial_a X^\mu\partial_b X^\nu D_{0\,\mu}\partial_\nu\Phi_0 + \frac{\alpha'}{2}\sqrt{g}\,\bar{R}\,D_0{}^\mu\Phi_0\partial_\mu\Phi_0 \;.$$

$$(3.9)$$

Hence

$$\sqrt{g}\,T^a{}_a = \frac{1}{4\pi\alpha'}\sqrt{g}\,g^{ab}\partial_a X^\mu\partial_b X^\nu(-\varepsilon G_{0\,\mu\nu}+2\alpha' D_{0\,\mu}\partial_\nu\Phi_0)$$

$$(3.10)$$

$$+ \frac{1}{4\pi}\sqrt{g}\,\bar{R}(-\varepsilon\Phi_0+\alpha' D_0{}^\mu\Phi_0\partial_\mu\Phi_0).$$

Next we consider the renormalization of the composite operators[20]. Let the action (3.1) be denoted by

$$S_0 = \int d^dz\, A_0{}^i\cdot\phi_{0\,i} \quad , \qquad (3.11)$$

where the composite operators $A_0{}^i$ represent

$$A_G^{\mu\nu} = \frac{1}{4\pi\alpha'} \sqrt{g}\, g^{ab}\partial_a X^\mu \partial_b X^\nu \delta^D(y-x), \qquad (3.12)$$

$$A_\Phi = \frac{1}{4\pi} \sqrt{g}\, \bar{R}\, \delta^D(y-x) \quad . \qquad (3.13)$$

and $\phi_{0\,i}=\{G_{0\,\mu\nu},\Phi_0\}$. The dot denotes the scalar product: $f\cdot h=\int d^D y f(y)h(y)$. The renormalized operators (or the normal products [21]) $[A_i]$ are defined by

$$\int d^d z[A_i]= \frac{\delta S_0}{\delta\phi^i} \quad . \qquad (3.14)$$

In general,

$$[A_i]=A_{0\,j}Z^j{}_i \quad , \quad Z^j{}_i= \mu^\varepsilon(\delta^j{}_i + \sum_n \frac{1}{\varepsilon^n} X_n{}^j{}_i(\phi) \ ) \ . \qquad (3.15)$$

On the other hand, from (3.14) and (3.1) we find

$$\int d^d z[A_i]F^i= \frac{\delta S_0}{\delta\phi^i}F^i= \int d^2 z A_{0\,j} \frac{\partial\phi_0{}^j}{\partial\phi^i}F^i \ . \qquad (3.16)$$

and its local expression:

$$[A_i]F^i= A_{0\,j} \frac{\partial\phi_0{}^j}{\partial\phi^i}F^i + \partial_a(\Omega^a{}_i F^i), \qquad (3.17)$$

where $F^i(y)$ are arbitrary functions. The total derivative term can be rewritten as

$$\partial_a(\Omega^a{}_i F^i) = A_{0i}\Lambda^i{}_j F^j,\quad \Lambda^i{}_j = \mu^{\varepsilon} \sum_n \frac{1}{\varepsilon^n} Q_n{}^i{}_j(\phi), \qquad (3.18)$$

since A_{0i} form the full set of dimension 2 operators, and

$$Z^i{}_j = \frac{\partial \phi_0{}^i}{\partial \phi^j} + \Lambda^i{}_j,\quad X_n{}^i{}_j = \frac{\partial T_n{}^i}{\partial \phi^j} + Q_n{}^i{}_j. \qquad (3.19)$$

Thus if we write the trace of the energy-momentum tensor as

$$\begin{aligned} \sqrt{g}T^a{}_a &= A_{0i}\psi^i, \\ \psi^i &= -\varepsilon\phi_0{}^i + \lambda^i{}_j\phi_0{}^j \\ &= \mu^{\varepsilon}(-\varepsilon\phi^i - T_1{}^i(\phi) + \lambda^i(\phi) + O(1/\varepsilon)), \end{aligned} \qquad (3.20)$$

where $\lambda^i = \lambda^i{}_j(\phi)\phi^j$, $\phi_0{}^i = \mu^{\varepsilon}(\phi^i + \Sigma \frac{1}{\varepsilon^n} T_n{}^i(\phi))$, we obtain

$$\begin{aligned} A_{0i}\psi^i &= [A_i]Z^{-1\,i}{}_j\psi^j, \\ Z^{-1\,i}{}_j\psi^j &= -\varepsilon\phi^i - T_1{}^i(\phi) + \lambda^i(\phi) + \left(\frac{\partial T_1{}^i}{\partial \phi^j}\phi^j + Q_1{}^i{}_j\phi^j\right) + O(1/\varepsilon), \\ \sqrt{g}T^a{}_a &= [A_i]\hat{\bar{\beta}}^i = [A_i]\{-\hat{\beta}^i + \lambda^i + Q_1{}^i{}_j\phi^j\}. \end{aligned} \qquad (3.21)$$

Here all the pole terms ($\sim 1/\varepsilon^n$, $n \geq 1$) must be canceled in (3.21) because $T^a{}_a$ is a finite operator.

Therefore we get the Weyl anomaly

$$4\pi\alpha'\sqrt{g}\,T^a{}_a = [\sqrt{g}\,g^{ab}\partial_a X^\mu \partial_b X^\nu\ \hat{\bar{\beta}}^G_{\mu\nu}(X)] + [\alpha'\sqrt{g}\ \bar{R}\ \hat{\bar{\beta}}^\Phi(X)],$$

$$\hat{\bar{\beta}}^G_{\mu\nu} = \hat{\beta}^G_{\mu\nu} + 2\alpha' D_\mu \partial_\nu \Phi + D_{(\mu} W_{\nu)}\ , \tag{3.23}$$

$$\hat{\bar{\beta}}^\Phi = \hat{\beta}^\Phi + \alpha' D^\mu \Phi \partial_\mu \Phi + D^\mu \Phi\ W_\mu\ , \tag{3.24}$$

where the W_μ-terms are due to the total derivative term in (3.17) [20].

Note that the operator of the trace of the 2-dimensional energy-momentum tensor is expressed by finite composite operators multiplied by the Weyl anomaly coefficients $\bar{\beta}$, which are in general different from the ordinary renormalization group $\bar{\beta}$-functions:

$$\bar{\beta}^G_{\mu\nu} = \beta^G_{\mu\nu} + 2\alpha' D_\mu \partial_\nu \Phi + D_{(\mu} W_{\nu)}, \tag{3.25}$$

$$\bar{\beta}^\Phi = \beta^\Phi + \alpha' D^\mu \Phi \partial_\mu \Phi + D^\mu \Phi\ W_\mu\ . \tag{3.26}$$

The global scale anomaly is expressed by

$$\int d^d z\sqrt{g}\ T^a{}_a = \frac{1}{4\pi\alpha'}\int d^d z\{[\sqrt{g}\,g^{ab}\partial_a X^\mu \partial_b X^\nu\ \hat{\beta}^G_{\mu\nu}(X)] + [\alpha'\sqrt{g}\ \bar{R}\ \hat{\beta}^\Phi(X)]\},$$
(3.27)

since the total derivative terms drop out by the integration over z.

Using the normal coordinate expansions (see ref. 22 and appendix B), we find the Weyl anomaly coefficients up to 1-loop (i.e. $O(\alpha')$):

$$\bar{\beta}^{G}_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha' D_{\mu}\partial_{\nu}\Phi + O(\alpha'^{2}) , \tag{3.28}$$

$$\bar{\beta}^{\Phi} = \frac{1}{6}(D-26) - \frac{1}{2}\alpha' D^{2}\Phi + \alpha' D^{\mu}\Phi\partial_{\mu}\Phi + O(\alpha'^{2}) . \tag{3.29}$$

Here we include the contribution of the reparametrization ghost in the constant terms in (3.29) [12].

The Weyl invariance conditions, $\bar{\beta}^{G}_{\mu\nu} = \bar{\beta}^{\Phi} = 0$ are equivalent to the equation of motion from the (tree level) effective action (D=26)

$$I = c \int d^{D}y \sqrt{G}\, e^{-2\Phi}\{ \alpha' (R + 4\, \partial^{\mu}\Phi\partial_{\mu}\Phi) + O(\alpha'^{2}) \}. \tag{3.30}$$

Calculating the expectation value of $T^{a}{}_{a}$ by expanding it near a classical solution $\bar{X}$, we obtain[20]

$$\begin{aligned}\langle T^{a}{}_{a}\rangle &= \frac{2}{\sqrt{g}} g^{ab}\frac{\delta W}{\delta g^{ab}} \\ &= \frac{1}{4\pi\alpha'}\langle g^{ab}\partial_{a}X^{\mu}\partial_{b}X^{\nu}\ \bar{\beta}^{G}_{\mu\nu}(X)\rangle + \frac{1}{4\pi}\langle \bar{R}\ \bar{\beta}^{\Phi}(X)\rangle \\ &= \frac{1}{4\pi\alpha'}\ g^{ab}\partial_{a}X^{\mu}\partial_{b}X^{\nu}\ \tilde{\beta}^{G}_{\mu\nu}(\bar{X}) + \frac{1}{4\pi}\ \bar{R}\ \tilde{\beta}^{\Phi}(\bar{X})\rangle \\ &\qquad + \text{non-local terms},\end{aligned} \tag{3.31}$$

$$\tilde{\beta}^{G}_{\mu\nu} = \bar{\beta}_{\mu\nu} + \dots , \tag{3.32}$$

$$\tilde{\beta}^{\Phi} = \bar{\beta}^{\Phi} - \frac{1}{4}\bar{\beta}_{\mu\nu}G^{\mu\nu} + \ldots$$

$$= \frac{1}{6}(D-26) - \frac{1}{4}\alpha'(R+4D^2\Phi-4\alpha'\partial^{\mu}\Phi\partial_{\mu}\Phi) + O(\alpha'^2), \qquad (3.33)$$

where $Z=e^{-W}=\int[dX]e^{-S_0}$, $\langle\ldots\rangle=\frac{1}{Z}\int[dX]e^{-S_0}\ldots$ and we use

$$\langle\, \partial X^{\mu}\partial X^{\nu}\rangle = \frac{1}{4}\ \alpha' G^{\mu\nu}\ \bar{R} + \ldots$$

Note that the equations, $\bar{\beta}^{G}_{\mu\nu}=0$, $\tilde{\beta}^{\Phi}=0$ is equivalent to $\bar{\beta}^{G}_{\mu\nu}=0$, $\bar{\beta}^{\Phi}=0$ and thus to $\delta I/\delta G_{\mu\nu}=0$, $\delta I/\delta\Phi=0$.

4. String Loop Corrections to β-Functions

In the previous section, we have considered the Weyl invariance conditions of two-dimensional σ-model and found that these conditions are equivalent to the equations of motion from the string effective action at the string tree-level.

The next step is to investigate the effects of string loops (i.e. higher genus Riemann surfaces) on the Weyl invariance. Naively, the β-functions at string tree level cannot be modified by string loops because they are related to ultraviolet divergences or short-distance behavior of the two-dimensional theory and are independent of the world-sheet topologies. Recently, however, it was pointed out that the divergences in the integration of moduli parameters of Riemann surfaces, which come from boundaries of the parameter space where handles shrink to zero size, are responsible for the Weyl symmetry breaking[9,10]. Hence string loop effects (i.e. small handles) can contribute to the β-function. In this section we discuss string one-loop corrections to β-functions for the closed bosonic string.

Consider the nonlinear σ-model describing the closed bosonic string in a metric and a dilaton background fields $G_{\mu\nu}(X)$ and $\Phi(X)$. The action is

$$S = \frac{1}{4\pi\alpha'}\int d^2z[\sqrt{g}\, g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + \alpha'\sqrt{g}\,\bar{R}\,\Phi(X)] \ . \quad (4.1)$$

Since the two-dimensional σ-model is ultraviolet divergent, counterterms must be added to S_0 to be finite. The counterterm action is

$$\delta S = \frac{\ln \kappa}{2\pi} \int d^2z[\sqrt{g}\, g^{ab}\partial_a X^\mu \partial_b X^\nu \delta G_{\mu\nu}(X) + \alpha'\sqrt{g}\, \bar{R}\, \delta\Phi(X)], \quad (4.2)$$

where κ is the world-sheet cutoff and δG and $\delta\Phi$ are function of the background fields, which define the renormalization group β-functions. (See sect. 3 and appendix B.)

On the other hand, string 1-loop (torus T^2) amplitude is divergent when a handle shrinks to zero. In this limit the string 1-loop amplitude becomes the product of a string tree (S^2) amplitude, a zero momentum dilaton propagator and 1-loop (T^2) dilaton tadpole[9,10]. The zero momentum dilaton propagator, which is a source of divergence, is given by

$$\Delta = \frac{1}{p^2}\bigg|_{p^2=0} = \int_a^1 dx\, x^{p^2-1}\bigg|_{\substack{p^2=0\\ a\to 0}} = \int_a^1 \frac{dx}{x}\bigg|_{a\to 0} = -\ln a\,\bigg|_{a\to 0}. \quad (4.3)$$

From the viewpoint of the σ-model on S^2, this divergence is interpreted as the insertion of the vertex operator for the emission of a zero momentum dilaton:

$$\frac{\ln a}{2\pi}\, g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}\, g^2 J_c\ , \quad (4.4)$$

where a is the size of a handle, g is the string coupling constant and J_c is the dilaton tadpole amplitude. Thus we add a new counterterm

$$\delta S^{loop} = \frac{\ln a}{2\pi} \int d^2z[\ \sqrt{g}\ g^{ab}\partial_a X^{\mu}\partial_b X^{\nu}\eta_{\mu\nu}\ g^2 J_c\ ] \tag{4.5}$$

to the σ-model action in order to eliminate this divergence.

For $a \gg \kappa$, the β-function of the σ-model on T^2 will coincide with those on S^2. However, if $a \sim \kappa$, the small handle affects the short-distance property of the theory. Hence we can choose $a=\kappa$ and from the counterterms $\delta S+\delta S^{loop}$ we obtain the string-loop corrected β-function $\hat{\beta}$ for the background metric:

$$\hat{\beta}^{G}_{\mu\nu} = \beta^{G}_{\mu\nu} - g^2 J_c\ \eta_{\mu\nu} \quad . \tag{4.6}$$

The vanishing of the string-loop corrected β-function, $\hat{\beta}=0$, is believed to be equivalent to the string-loop corrected equation of motion, which is derived from the loop corrected effective action to $O(\alpha')$ $(D=26)$

$$I = c \int d^D y\ \sqrt{G}\ e^{-2\Phi}\{\ \alpha'(\ R + 4\ \partial^{\mu}\Phi\partial_{\mu}\Phi\ ) + 2g^2 J_c\ \}. \tag{4.7}$$

This action is just the Einstein one with the dilaton field and the string one-loop cosmological constant in the lowest order of α'. In sect. 6, we will consider higher dimensional cosmology based on this action.

5. One-Point Amplitude and String-Loop Corrected Equation of Motion [23]

In string theory, the vanishing of the one-point amplitude of a vertex operator V

$$\langle\langle V \rangle\rangle = 0 \qquad (5.1)$$

must be the condition for a classical vacuum solution or, at the quantum level, a minimum of the effective potential by analogy with field theory. On the other hand, world-sheet Weyl invariance or vanishing β-function ($\beta = 0$) is needed if string theory is to make sense [8]. In order to have any sensible physical interpretation, $\beta = 0$ must coincide with the equations of motion. Really, it is well understood that, at tree level in string theory, Eq.(5.1) is a consequence of world-sheet Weyl invariance [11]. This implies a self-consistency between the background fields and the dynamics of the string.

Recently, Fischler and Susskind [9] showed that the cosmological constant of closed bosonic string theory appears as a one-loop correction to the β-function for the background metric field. At present, many authors have successively investigated string loop corrections to the β-function [10], believing that the vanishing of some corrected β-function $\hat{\beta}$ gives the equation of motion. In analogy with the tree level case, it is natural to anticipate that $\hat{\beta} = 0$ is equivalent to $\langle\langle V \rangle\rangle = 0$. But this statement has not been confirmed explicitly.

In this section we examine directly $\langle\langle V \rangle\rangle$ to one-loop order (i.e. a torus correction) in the case of closed bosonic string theory supposing that the Weyl invariance holds in the theory and show that $\langle\langle V \rangle\rangle = 0$ provides the same string-loop corrected equation of motion as that obtained from $\hat{\beta} = 0$.

By using the Polyakov's path integral (refs. 12-14 and 24, see also sect.2 and 3), one-point massless particle amplitude for the closed bosonic string (d=26) propagating in a background metric is given by

$$\langle\langle V(p) \rangle\rangle = \sum_{\text{topologies}} \frac{g^{-\chi}}{\mathrm{Vol(CKV)}} \int [\mathrm{dModuli}](\det{}' P_1^\dagger P_1)^{1/2}$$

$$\times \int [dX]\, e^{-S}\, V(p) \quad , \qquad (5.2)$$

with the action

$$S = \frac{1}{2\pi\alpha'} \int d^2z \sqrt{g}\, g^{ab}\, \frac{1}{2}\, \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) \quad . \qquad (5.3)$$

Here $G_{\mu\nu}(X)$, g, χ and $V(p)$ are the background metric, the coupling constant, the Euler number of the world-sheet and the vertex operator for massless particles, respectively. The integral over the zero mode gives a factor of $(2\pi)^{26}\delta(p)$ and the vertex operators are defined for on-shell physical states, so that one-point amplitude must be evaluated at zero momentum. We will write $\langle\langle V(p=0) \rangle\rangle = \langle\langle V(0) \rangle\rangle$ from now on.

At the string tree level, the requirement of the Weyl invariance determines $G_{0\mu\nu}$ (classical vacuum solution) for the background metric and it guarantees

$$\langle\langle\, V(0\ ;\ G_0)\,\rangle\rangle_{S^2} = 0 \qquad , \tag{5.4}$$

because of the existence of conformal Killing vector (CKV) on S^2. However, at the string one-loop level, we generally see

$$\langle\langle\, V(0\ ;\ G_0)\,\rangle\rangle_{T^2} \neq 0 \qquad . \tag{5.5}$$

This means that $G_{0\mu\nu}$ is not the true vacuum solution. Therefore it should be necessarily modified to $G_{0\mu\nu} + \Delta G_{1\mu\nu}$ so that, up to the string one-loop level,

$$\langle\langle\, V(0)\,\rangle\rangle = \langle\langle\, V(0\ ;G_0+\Delta G_1)\rangle\rangle_{S^2} + \langle\langle\, V(0\ ;\ G_0\ )\rangle\rangle_{T^2} = 0\ . \tag{5.6}$$

Repeating this manipulation to higher order may lead to the string coupling perturbative expansion of the background solution.

In the following discussion of $\langle\langle V(0)\rangle\rangle$, we consider the flat space-time to be the classical solution#. Then, the metric $G_{\mu\nu}$ =

This is due to the reason that we consider $\langle\langle V(0)\rangle\rangle_{T^2}$ in the flat space-time. As concerns the following evaluation of $\langle\langle V(0)\rangle\rangle_{S^2}$, we need not limit the classical solution to the flat one.

$\eta_{\mu\nu} + h_{\mu\nu}$ and $h_{\mu\nu} \sim O(g^2)$ to one-loop order. Furthermore we take flat coordinates, i.e. $g_{ab} = \delta_{ab}$, on the world-sheet, assuming that the Weyl anomaly cancels#. To evaluate the quantum corrections in the σ model [22], we expand $X^{\mu}(z)$ in terms of the Riemann normal coordinate $\xi^{\mu}(z)$ around a fixed point $X_0^{\mu}(z)$ and make ξ^{μ} a dimensionless field by the replacement $\xi^{\mu} \to (2\pi\alpha')^{1/2}\xi^{\mu}$.

Now we calculate $\langle\langle V(0) \rangle\rangle_{S^2}$ to leading order in $h_{\mu\nu}$ and α':

$$\begin{aligned}\langle\langle V(0)\rangle\rangle_{S^2} &= \frac{g^{-2}}{\mathrm{Vol(CKV)}}(\det{}' P_1^{\dagger}P_1)^{1/2}\int[d\xi]\; e^{-(S_0+S_{int})}(V_0+\Delta V) \\ &= c_1\frac{g^{-2}}{\mathrm{Vol(CKV)}}\Bigg\{\int d^2z d^2z' \langle V_0(z,0)\cdot\frac{2\pi\alpha'}{3}R_{\mu\rho\nu\sigma}(X_0) \\ &\qquad \times \partial\xi^{\mu}\bar{\partial}\xi^{\nu}\xi^{\rho}\xi^{\sigma}(z')\rangle_0 \\ &\qquad + \int d^2z\,\langle\Delta V(z,0)\rangle_0\Bigg\}, \end{aligned} \tag{5.7}$$

In general, the world-sheet has the conformal flat coordinates and there should exist the dilaton term in the action. But, focusing our attention on the Fischler-Susskind procedure only for the background metric, we here neglect dilaton contributions for simplicity.

where $c_1 = (\det{}' P_1^\dagger P_1)^{1/2} (\det{}'\Delta)^{-13}$,

$$S_0 = \int d^2z \, \partial\xi^\mu \bar{\partial}\xi^\nu G_{\mu\nu}(X_0),$$

$$S_{int} = \int d^2z \, \left(- \frac{2\pi\alpha'}{3} R_{\mu\rho\nu\sigma}(X_0) \, \partial\xi^\mu \bar{\partial}\xi^\nu \xi^\rho \xi^\sigma + \cdots \right).$$

$\langle(\cdots)\rangle_0$ stands for

$$\langle(\cdots)\rangle_0 = \frac{\int [d\xi] \, e^{-S_0}(\cdots)}{\int [d\xi] \, e^{-S_0}} .$$

$V(p)$ is expanded as

$$V(p) = \frac{1}{2\pi\alpha'} \int d^2z \, e_{\mu\nu}(p) \, \partial_a X^\mu \partial^a X^\nu \, e^{ip\cdot X}$$

$$= \int d^2z \, e_{\mu\nu}(p) \, : \partial_a\xi^\mu \partial^a\xi^\nu e^{ip\cdot X_0} e^{i(2\pi\alpha')^{1/2} p\cdot\xi}$$

$$\times \left(1 - \frac{i}{2} 2\pi\alpha' p_\rho \Gamma^\rho_{\sigma\lambda} \xi^\sigma \xi^\lambda - \cdots \right):$$

$$= \int d^2z \, e_{\mu\nu}(p): \partial_a\xi^\mu \partial^a\xi^\nu e^{ip\cdot X_0} \left(1 + i(2\pi\alpha')^{1/2} p\cdot\xi \right.$$

$$\left. - 2\pi\alpha'(p\cdot\xi)^2 - \frac{i}{2} 2\pi\alpha' p_\rho \Gamma^\rho_{\sigma\lambda} \xi^\sigma \xi^\lambda - \cdots \right): \qquad (5.8)$$

by the normal coordinate expansion, so we get

$$V_0(p) = \int d^2z \; e_{\mu\nu} \; : \partial\xi^\mu \bar{\partial}\xi^\nu e^{ip\cdot X_0} :$$

and $\Delta V(p)$ is the next-order term of the remaining part in Eq.(5.8). Here we make $V(p)$ renormalized by taking normal ordering under the condition:

$$(p^2 + i\Gamma^\rho_{\sigma\sigma} \, p_\rho) \, e_{\mu\nu}(p) = 0 \quad ,$$

$$p^\mu \, e_{\mu\nu}(p) = 0 \quad .$$

On the first term in Eq.(5.7), the contraction $\langle \xi^\mu \, \xi^\nu \rangle$ gives a logarithmic divergence, and the contractions $\langle \partial\xi^\mu \bar{\partial}\xi^\nu \rangle$ and $\langle \partial\xi^\mu \xi^\nu \rangle$ do not contribute in the dimensional regularization. By introducing a short-distance cutoff κ on the world-sheet, the logarithmic divergence is expressed as

$$\langle \xi^\mu(z)\xi^\nu(z') \rangle_{z'\to z} = - \frac{1}{4\pi} \, \eta^{\mu\nu} \, \log \kappa \quad .$$

Hence the first term is reduced to

$$\frac{c_1 \, g^{-2}}{\mathrm{Vol(CKV)}} \int d^2z d^2z' \, \frac{2\pi\alpha'}{3} \, e_{\lambda\kappa} R_{\mu\rho\nu\sigma}(X_0)$$

$$\times \langle \partial\xi^{\lambda}(z)\partial\xi^{\mu}(z')\rangle\langle\bar{\partial}\xi^{\kappa}(z)\bar{\partial}\xi^{\nu}(z')\rangle\langle\xi^{\rho}(z')\xi^{\sigma}(z'')\rangle_{z''\to z'}$$

$$= \frac{-c_1\ g^{-2}\int d^2z d^2z'\ \frac{2\pi\alpha'}{3}\left(\frac{1}{4\pi}\right)^3 e^{\mu\nu}\ R_{\mu\nu}(X_0)\frac{1}{|z-z'|^4}\ \log\kappa}{\mathrm{Vol(CKV)}} . \quad (5.9)$$

Owing to the conformal transformation SL(2,C), the volume Vol(CKV) of the group generated by the conformal Killing vector is rewritten as

$$\mathrm{Vol(CKV)} = \int \frac{d^2z_1 d^2z_2 d^2z_3}{|z_1-z_2|^2|z_2-z_3|^2|z_3-z_1|^2}$$

$$= |a-b|^2\int\frac{d^2z_1}{|z_1-a|^2|z_1-b|^2}\int\frac{d^2z_2d^2z_3}{|z_2-z_3|^4} .$$

Taking into account this form in Eq.(5.9), we get

$$\frac{-c_1\ g^{-2}\frac{2\pi\alpha'}{3}\left(\frac{1}{4\pi}\right)^3 e^{\mu\nu}\ R_{\mu\nu}\ (X_0)\ \log\kappa}{|a-b|^2\int\frac{d^2z}{|z-a|^2|z-b|^2}} . \quad (5.10)$$

The denominator of Eq.(5.10) gives a logarithmic divergence when z approaches to a or b. So identifying this divergence with $\log\kappa$ in the numerator, we finally obtain as the first term

$$c\ g^{-2}\ e^{\mu\nu}\ R_{\mu\nu}(X_0) , \quad (5.11)$$

where $c = \alpha' c_1 / 3(4\pi)^3$. On the second term in Eq.(5.7), ΔV, which is the composite operator of ξ, is a correction term to V_0 coming from the curved space-time. Since it should be evaluated at zero momentum, $\langle \Delta V \rangle_0$ becomes zero and, therefore, the second term vanishes.

As we have seen, $\langle\langle V(0) \rangle\rangle_{S^2}$ turns out to be finite through dividing by Vol(CKV). This is a delightful result, considering that $\langle\langle V(0) \rangle\rangle_{T^2}$ will be finite except for the contribution of the tachyon mode. In fact we know that $\langle\langle V(0) \rangle\rangle_{T^2}$ in flat space-time [24] is

$$\langle\langle V(0) \rangle\rangle_{T^2} = \frac{1}{\mathrm{Vol(CKV)}} \int [d\mathrm{Moduli}](\det{}' P_1^\dagger P_1)^{1/2} \left(\frac{2\pi}{\int d^2z} \det{}' \Delta\right)^{-13}$$

$$\times\ e_{\mu\nu} \int d^2z \, \langle \partial X^\mu(z) \bar{\partial}' X^\nu(z') \rangle_{z' \to z} \qquad (5.12)$$

Here, after renormalization, $\langle \partial \xi^\mu(z) \bar{\partial}' \xi^\nu(z') \rangle_{z' \to z} = - \eta^{\mu\nu} \frac{1}{\tau_2}$, which does not depend on z, and $\int d^2z = \tau_2$. Whence Eq.(5.12) becomes

$$- \int \frac{d^2\tau}{4\pi\tau_2{}^2} (2\pi\tau_2)^{-12} \, e^{4\pi\tau_2} \, \left| \prod_n (1 - e^{2\pi i n \tau}) \right|^{-48} e_{\mu\nu} \, \eta^{\mu\nu}$$

$$\equiv - J_c \, e_{\mu\nu} \, \eta^{\mu\nu} \qquad . \qquad (5.13)$$

Thus, imposing the vanishing of one-point amplitude up to string one-loop order

$$\langle\langle V(0) \rangle\rangle = c\, g^{-2}\, e^{\mu\nu}\, R_{\mu\nu}(X_0) - J_c\, e^{\mu\nu}\, \eta_{\mu\nu} = 0 \quad , \quad (5.14)$$

we get the loop corrected equation of motion

$$c\, R_{\mu\nu} = g^2\, J_c\, \eta_{\mu\nu} \quad . \quad (5.15)$$

This result agrees with the condition of being conformal anomaly free, i.e. $\hat{\beta} = 0$ to one-loop order [9,10].

In considering string loop corrections, there exist various kinds of divergences, which arise from integrations over distinct boundary regions of moduli space and may contribute to $\hat{\beta}$ [10]. To achieve the vanishing $\hat{\beta}$ function for massless fields, it is necessary to cancel divergences due to dilaton tadpole against σ-model divergences. In the N-point amplitude, the former can be interpreted to arise from the graph of a dilaton emitted from S^2 and absorbed by the vacuum and the latter from that of a dilaton emitted from S^2 and coupled by the massless background fields. That is, when N points all coalesce,

$$\langle\langle V_1 \cdots V_N \rangle\rangle_{T^2;\eta_{\mu\nu}} \rightarrow \langle\langle V_1 \cdots V_N\, V_{dil}(0) \rangle\rangle_{S^2;\eta_{\mu\nu}}$$

$$\times \log \kappa\, \langle\langle V_{dil}(0) \rangle\rangle_{T^2;\eta_{\mu\nu}} \quad ,$$

$$\langle\langle V_1\cdots V_N \rangle\rangle_{S^2;\eta_{\mu\nu}+h_{\mu\nu}} \to \langle\langle V_1\cdots V_N\ V_{dil}(0) \rangle\rangle_{S^2;\eta_{\mu\nu}}$$

$$\times \log \kappa \langle\langle V_{dil}(0) \rangle\rangle_{S^2;\eta_{\mu\nu}+h_{\mu\nu}} .$$

Both divergences are due to the massless dilaton propagator at zero momentum. Then $\langle\langle V\rangle\rangle$ can be regarded as the coefficient of these logarithmically divergent terms. This strongly suggests the equivalence between $\langle\langle V\rangle\rangle=0$ and $\hat{\beta}=0$.

Throughout this section, we considered the flat space-time to be the classical solution. If the classical solution is a curved space-time, $G_{\mu\nu}=G_{0\mu\nu}+\Delta G_{1\mu\nu}$ and $\Delta G_{1\mu\nu} \sim O(g^2)$ to one-loop order and we will obtain the one-loop corrected equation of motion

$$c\, R_{\mu\nu} = g^2\, J_c'\, G_{\mu\nu} \qquad ,$$

where J_c' is the one-loop cosmological constant in the curved space-time.

6. Higher Dimensional Cosmology with String Vacuum Energy [25]

String theories are consistently formulated in higher dimensional space-time, so that we are obliged to have the important problem of how to explain the large separation between the scale of our three- dimensional space and that of the extra one, as in ordinary Kaluza-Klein cosmology [15-17]. In order to approach this problem, many authors have investigated the (string tree level) effective Lagrangian obtained in the field theory limit of string theories, especially, the ten-dimensional supergravity derived from superstring theory [26]. Then, there arise the curvature squared (and higher-order) and the dilaton terms. These new terms lead to the different scenario from ordinary Kaluza-Klein cosmology. However, such a treatment does not seem to reflect soundly the characteristic of string theories because the contribution of the string vacuum energy to the energy-momentum tensor is not considered. In fact, the winding-up of closed strings around tori is closely connected with stability of the extra space and, in sect. 4 and 5, we have found that the string vacuum energy (the cosmological constant) of closed bosonic string theory appears as a one-loop correction to the equation of motion for the background metric field. Even in the superstring case, we can not ignore this effect if supersymmetry is broken for some reason [27].

In this section, we investigate the cosmological evolution in the closed bosonic string theory with the one-loop vacuum energy, for the winding effect is determined mainly by bosonic sectors, i.e. string coordinates of the string theory. Our analysis is carried out based on $M_d \times [T^1]^D$ where M_d is d-dimensional maximally symmetric space-time , $[T^1]^D$ is $T^1 \times \ldots \times T^1$ (D-times) and d+D=26, since compactifications on flat tori satisfy the consistency for the Weyl invariance in the string world-sheet. As we are interested in the era after the Planck time, we take here the Einstein equations to describe the evolution of the universe #.

Now we assume that (d+D)-dimensional metric is the generalized Robertson-Walker form

$$g_{MN} = \begin{pmatrix} -1 & & \\ & R^2(t)\bar{g}_{mn}(x) & \\ & & r_i{}^2(t)\bar{g}_{ij}(y) \end{pmatrix} \quad ; \qquad (6.1)$$

$$M,N=0,1,\ldots,d+D-1;\ m,n=1,2,\ldots,d-1;\ i,j=1,2,\ldots,D.$$

Here $\bar{g}_{mn}(x)$ is the metric of (d-1)-dimensional space M_{d-1}, $\bar{g}_{ij}(y)$ $(=\delta_{ij})$ is the metric of $[T^1]^D$ and $R(t)$ $(r_i(t))$ is the

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We here assume that the Planck length $\sim \sqrt{\alpha'}$, where α' is the string slope (the inverse of string tension). In sect. 4 and 5, we obtain the string-loop corrected equation of motion and the string effective action at the lowest order of α'. (See eqs.(4.6), (4.7) and (5.15).) Thus in the low-energy region the gravitational equations can be described by the Einstein ones and the string vacuum energy contributes to the energy-momentum tensor. The dilaton field is assumed to be constant.

time-dependent scale factor of M_{d-1} (i-th T^1 of $[T^1]^D$). The energy-momentum tensor is taken as

$$T_{MN} = \begin{pmatrix} \rho & & \\ & p g_{mn} & \\ & & p_i g_{ij} \end{pmatrix} , \tag{6.2}$$

where ρ is the energy density and p (p_i) is the pressure in M_{d-1} (i-th T^1). Then, from the Einstein equation: $R_{MN}-\frac{1}{2}g_{MN}R=8\pi G T_{MN}$, we have

$$(d-1)\left(\frac{\ddot{R}}{R}\right)+\sum_{j=1}^{D}\left(\frac{\ddot{r}_j}{r_j}\right)= 8\pi G\left(\frac{-T^L{}_L}{d+D-2} -\rho\right) , \tag{6.3a}$$

$$\frac{d}{dt}\left(\frac{\dot{R}}{R}\right)+ (d-1)\left(\frac{\dot{R}}{R}\right)^2+\left(\frac{\dot{R}}{R}\right)\sum_{j=1}^{D}\left(\frac{\dot{r}_j}{r_j}\right) + \frac{k}{R^2}= 8\pi G\left(\frac{-T^L{}_L}{d+D-2} + p\right), \tag{6.3b}$$

$$\frac{d}{dt}\left(\frac{\dot{r}_i}{r_i}\right)+ (d-1)\left(\frac{\dot{R}}{R}\right)\left(\frac{\dot{r}_i}{r_i}\right)+\left(\frac{\dot{r}_i}{r_i}\right)\sum_{j=1}^{D}\left(\frac{\dot{r}_j}{r_j}\right)= 8\pi G\left(\frac{-T^L{}_L}{d+D-2} + p_i\right), \tag{6.3c}$$

where $T^L{}_L=-\rho+(d-1)p+\sum_{j=1}^{D}p_j$ is the trace of the energy-momentum tensor, G is the $(d+D)$-dimensional gravitational constant and k is the curvature constant of M_{d-1}. ρ, p and p_i can be determined from the free energy F as follows:

$$\rho = \frac{1}{\Omega_{d-1}\Omega_D}\left[- T^2\frac{\partial}{\partial T}(F/T) \right] , \tag{6.4a}$$

$$p = -\frac{1}{d-1}\frac{1}{\Omega_{d-1}\Omega_D}R\frac{\partial F}{\partial R} , \tag{6.4b}$$

$$p_i = -\frac{1}{\Omega_{d-1}\Omega_D} r_i\frac{\partial F}{\partial r_i} , \tag{6.4c}$$

where Ω_{d-1} (Ω_D) is the volume of M_{d-1} ($[T^1]^D$):

$$\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)} R^{d-1} \; , \; \Omega_D = \prod_{i=1}^{D} (2\pi r_i). \tag{6.5}$$

The free energy in the one-loop approximation is generally given by

$$\beta F = \frac{1}{2} \mathrm{Tr} \ln H ,$$

where $\beta = 1/T$ and H is the hamiltonian of the bosonic string. The exact form of this free energy is very complicated, but at low temperature ($T < 1/\sqrt{\alpha'}$), F can be reduced to [16,17]

F $\simeq$ (string vacuum energy at zero temperature)
+(free energy of massless free gas in thermal equilibrium).

We first give the vacuum energy of the closed bosonic string theory at zero temperature. d-dimensional mass of the string on $M_d \times [T^1]^D$ is given by

$$\frac{\alpha'}{2}(\mathrm{mass})^2 = N + \tilde{N} - 2 + \frac{1}{2} \sum_{i=1}^{D} \left(\frac{m_i^2}{b_i^2} + \ell_i^2 b_i^2 \right) ; \tag{6.6}$$

$$b_i = r_i/\sqrt{\alpha'} \; ; \quad m_i, \ell_i = 0, \pm 1, \pm 2, \ldots \quad .$$

m_i and ℓ_i are the discrete momentum quantum number and the winding number on i-th T^1, respectively. N and $\tilde{N}$ are the number operators of right- and left-movers with the constraint

$$N - \tilde{N} = \sum_{i=1}^{D} m_i \ell_i \quad . \tag{6.7}$$

Then the vacuum energy in the one-loop approximation is

$$\begin{aligned} V_{st} &= \frac{1}{2} \mathrm{Tr} \ln \left(\frac{\alpha'}{2} p_d{}^2 + \frac{\alpha'}{2} (\mathrm{mass})^2 \right) \\ &= \frac{1}{2} \mathrm{Tr} \int_0^1 dx \left[x^{\frac{\alpha'}{2} p_d{}^2 + \frac{\alpha'}{2} (\mathrm{mass})^2 - 1} \right] / \ln x \ , \end{aligned} \tag{6.8}$$

where the trace means the integral over d-dimensional momenta p_d and the sum over discrete momenta, winding numbers and oscillator modes. Taking the modular invariance of this vacuum energy into account and subtracting tachyon contributions, we get the vacuum energy as follows

$$\begin{aligned} V_{st} = \frac{-\pi \Omega_{d-1}}{(2\pi\alpha')^{d/2}} & \int_F d^2\tau \, (2\pi\tau_2)^{-(d+2)/2} e^{4\pi\tau_2} [| f(e^{2\pi i\tau}) |^{-2(d+D-2)} - 1] \\ & \times \prod_{i=1}^{D} \sum_{m_i, \ell_i = -\infty}^{\infty} e^{-2\pi i \tau_1 m_i \ell_i} \, e^{-\pi\tau_2 (m_i{}^2/b_i{}^2 + \ell_i{}^2 b_i{}^2)} \quad ; \end{aligned} \tag{6.9}$$

$$F: \ -1/2 \le \tau_1 \le 1/2 \ , \ \tau_2 > 0 \ , \ |\tau| \ge 1 \quad ,$$

where $z = xe^{i\sigma'} = e^{2\pi i\tau}$, $\tau = \tau_1 + i\tau_2$ and

$$f(z) = \prod_{n=1}^{\infty} (1 - z^n) \ .$$

Details are given in refs.[28,29]. We note that V_{st} is invariant under $b_i \to b_i^{-1}$. In the limit in which all of b_i's $\to \infty$, the vacuum energy becomes

$$V_{st} \longrightarrow \frac{-\pi\Omega_{d-1}}{(2\pi\alpha')^{d/2}(2\pi)^{D/2}} \prod_{i=1}^{D}(2\pi b_i)\int_F d^2\tau \ (2\pi\tau_2)^{-(d+D+2)/2} e^{4\pi\tau_2} \times[|f(e^{2\pi i\tau})|^{-2(d+D-2)} -1]$$

$$\sim 2.1\times10^{-8} \frac{\Omega_{d-1}}{(2\pi\alpha')^{d/2}(2\pi)^{D/2}} \prod_{i=1}^{D}(2\pi b_i) >0 . \qquad (6.10)$$

In this limit, the vacuum energy becomes eq.(6.10) with b_i^{-1} instead of b_i. Since V_{st} is invariant under $b_i \to b_i^{-1}$ and both $V_{st}(b_i \to \infty)$ and $V_{st}(b_i \to 0)$ are positive, V_{st} has a minimum at the position all of b_i's are equal to 1. In the case of general torus compactification, the vacuum energy can be obtained only by replacing $\frac{m_i^2}{b_i^2} + \ell_i^2 b_i^2$ with $\frac{m_i}{b_i} g^*_{ij} \frac{m_j}{b_j} + (\ell_i b_i) g_{ij} (\ell_j b_j)$ in eq.(6.9). Here g_{ij} is a torus metric and g^*_{ij} is the dual metric. Then the vacuum energy remains invariant under $b_i g_{ij} b_j \to \frac{1}{b_i} g^*_{ij} \frac{1}{b_j}$ (the generalization of $b_i \to b_i^{-1}$) and its minimum value comes to depend on g_{ij}.

At zero temperature, expressing $V_{st}=\Omega_{d-1}\tilde{V}_{st}$, the R.H.S. of eqs.(6.3a,b,c) are reduced to

$$\frac{8\pi G}{d+D-2} \quad \frac{1}{\Omega_D} \left[-(D-2)\tilde{V}_{st} + D\, r_i \frac{\partial \tilde{V}_{st}}{\partial r_i} \right] \qquad , (6.3a,b)'$$

$$\frac{8\pi G}{d+D-2} \quad \frac{1}{\Omega_D} \left[d\ \tilde{V}_{st} - (d-2)\ r_i \frac{\partial \tilde{V}_{st}}{\partial r_i} \right] \quad . \qquad (6.3c)'$$

Eq.(6.3c)′ seems to allow r_i to stop expanding when it becomes negative. For the L.H.S. of eq.(6.3c) becomes $\ddot{r}_i/r_i$ when $\dot{r}_i=0$ and the condition of a maximum is $\dot{r}_i=0$ and $\ddot{r}_i<0$. If we consider that r_i should become constant ($\equiv r_{i(0)}$) and 4-dimensional cosmological constant should become zero at the final stage of the universe, both eqs.(6.3a,b)′ and (6.3c)′ must be equal to zero at $r_i=r_{i(0)}$. Namely

$$\left.\frac{\partial \tilde{V}_{st}}{\partial r_i}\right|_{r_i=r_{i(0)}} = 0 \qquad , \qquad (6.11a)$$

$$\left.\tilde{V}_{st}\right|_{r_i=r_{i(0)}} = 0 \qquad . \qquad (6.11b)$$

As eq.(6.11a) is realized only at $r_i=\sqrt{\alpha'}$, it is necessary that eq.(6.11b) is also given at $r_i=\sqrt{\alpha'}$. Therefore we wish to choose such a metric g_{ij} as will allow the minimum value to be (nearly) zero. But at present, we don't yet know the values of V_{st} with respect to various tori, so we here regard $V_{st}(b_i)$ - $V_{st}(1)$ as the vacuum energy $V_{st}^{(r)}(b_i)$ for such a metric g_{ij}. Then we get $V_{st}^{(r)}(1)=0$ #.

As a different possibility, we may consider that the string model has other parameters, such as vacuum expectation values of some scalar fields, which adjust themselves to minimize the vacuum energy to zero.

We examine the stability of the extra space numerically by the practical calculation of the vacuum energy. The integration in eq.(6.9) with different b_i's is very hard, so we here take all of b_i's to be equal and set d+D=26. In the R.H.S. of eq.(6.3c) we define the potential v(b) by

$$8\pi G\left(\frac{-T^{L}{}_{L}}{d+D-2} + p_i \right) = -\, 8\pi G\; b\frac{dv(b)}{db} \quad . \qquad (6.12)$$

The b-dependence of v is shown in fig.4 for D=4 case . In other cases, the similar dependence can be found. b_0 is a "critical radius", so that if $b < b_0$, $b \rightarrow 1$ and if $b > b_0$, $b \rightarrow \infty$. The numerical values of b_0 are presented in table 1 for D=1,2,4,8 and 16. It is evident that b_0 becomes smaller as D increases. From the above investigation, we find that there exists the solution that the vacuum energy prevents the extra space from expanding to infinity and contracting to a point, that is, the stable solution.

Next we consider the cosmological evolution at low temperature ($T < 1/\sqrt{\alpha'}$). The free energy is given by

$$F = V_{st}^{(r)} - c_1 \Omega_{d-1} \Omega_D T^{d+D} \quad \text{for } R \sim r, \qquad (6.13)$$

where $c_1 = 576 \frac{\zeta(d+D)}{\pi^{(d+D)/2}} \Gamma\left(\frac{d+D}{2} \right)$, and 576 is the number of massless mode degrees of freedom in the 26-dimensional closed bosonic string theory. Then the R.H.S. of eq.(6.3c) becomes

$$8\pi G\left(\frac{-T^{L}{}_{L}}{d+D-2}+p_i\right)=8\pi G\left(\frac{-T^{L}{}_{L}{}^{(0)}}{d+D-2}+p_i{}^{(0)}+p_i{}^{(th)}\right),\quad(6.14)$$

where $p_i{}^{(0)}$ and $p_i{}^{(th)}$ are derived from $V_{st}^{(r)}$ and $-c_1\Omega_{d-1}\Omega_D T^{d+D}$ in eq.(6.13), respectively, and $T^{L}{}_{L}{}^{(th)}$ vanishes in $T^{L}{}_{L} = T^{L}{}_{L}{}^{(0)}+T^{L}{}_{L}{}^{(th)}$. The thermal part $p_i{}^{(th)}$ is

$$p_i{}^{(th)} = c_1 T^{d+D} > 0 \ . \qquad (6.15)$$

The energy-momentum conservation, that results from the Einstein eqs.(6.3), is equivalent to the entropy conservation. The entropy S is determined from the free energy as

$$S = -\frac{\partial F}{\partial T} = c_1\Omega_{d-1}\Omega_D T^{d+D-1} \ , \qquad (6.16)$$

so the entropy conservation is reduced to

$$S \sim R^{d-1} r^{D} T^{d+D-1} = \text{constant}. \qquad (6.17)$$

In the region of $R \sim r$, this becomes

$$X \equiv rT = \text{constant}. \qquad (6.18)$$

Therefore, due to eqs. (6.12), (6.14) and (6.18), there is a "critical value" X_c for X because if $X > X_c$, the extra space goes on expanding and has no chance of being stable. The values of X_c are given in table 2. By choosing $X < X_c$ in the initial stage, it is possible that r goes to $\sqrt{\alpha'}$ oscillating around the minimum of v(b) with thermal effects and R expands. In later time ($R \gg r$, $r \sim \sqrt{\alpha'}$), the free energy becomes

$$F \simeq V_{st}^{(r)} - c_2 \Omega_{d-1} T^d, \tag{6.19}$$

where c_2 is a positive constant and $V_{st}^{(r)} \sim 0$. In this case we can ignore the effects of the extra space and M_d becomes the d-dimensional Friedmann universe.

In this section, we have found that the winding-up of closed strings around tori has a chance to prevent the extra space from expanding and to realize the d-dimensional Friedmann universe with the compactified extra space ($r_i \sim \sqrt{\alpha'}$) in later time. From the fact that the value of X at the initial stage is related to the entropy, we may also find that the entropy should not be large in order to reach the d-dimensional Friedmann universe.

These results suggest that in the low energy world string cosmology corresponds effectively to the ordinary Kaluza-Klein cosmology. But we note that in Kaluza-Klein cosmology curvature terms cause the split of 3-dimensional space and extra space, while in our string case, the winding-effect of strings around tori guarantees the separation, even though the curvature of torus is zero.

7. Conclusions and Discussions

In quantization of the string theory, the classical symmetries (two-dimensional reparametrization and Weyl invariances) must be maintained. In general, however, the Weyl invariance is broken in quantization. In the flat space-time, the bosonic string theory can be made Weyl invariant, if space-time dimensions are 26 (critical dimension). In the curved space-time, the Weyl invariance of the two-dimensional σ-model, which describe the string propagation in a background, restricts the background configurations. On the other hand, the vanishing one-point amplitude <<V>>=0 is the condition for a classical vacuum solution by analogy with field theory. Thus it is expected that the Weyl invariance condition is equivalent to <<V>>=0 including the string loop correction. But at the string loop level, this equivalence is not verified explicitly. In this thesis, we have calculated <<V>> and shown that <<V>>=0 provides the same equation as that obtained from the Weyl invariance condition to string one-loop order and $O(\alpha')$.

Next we have considered the higher dimensional cosmology based on the string-loop corrected effective action. This action has the string vacuum energy term and the string vacuum energy contains the winding effects of closed strings around tori. Therefore we expect that the extra space can not expand infinitely due to this term. Really, we have found that the string vacuum energy has a chance to prevent the extra compact space from expanding.

Let us consider the N-point amplitude at string one-loop level. There are two type of divergences. One arises when all the N points are close together, the other does when all but one points approach each other. These divergences are responsible for the Weyl symmetry breaking. Hence we must cancel out these divergences. The first type of divergences may be canceled by the σ model divergences. This cancellation is interpreted as $\langle\langle V\rangle\rangle=0$. The second type of divergences may be canceled by modifying the tree level vertex operators in a way which corresponds to mass renormalization [30].

In the future, it is important to investigate the cancellation of various divergences, which appear in $\langle\langle V\rangle\rangle$ at higher loop order, in order to realize $\langle\langle V\rangle\rangle=0$ and to show generally that the condition of the Weyl invariance is the same as one of $\langle\langle V\rangle\rangle=0$ up to higher loop order. If we consider the Weyl invariance seriously, we need to add the dilaton term in the action and to investigate $\langle\langle V\rangle\rangle=0$ in this case.

In this thesis, we have considered only the closed bosonic string theory. It is interesting to study the other string theories, e.g. the open string theory and the superstring theory.

Appendix A. The Riemann Surfaces and the Uniformization Theorem

Let M be a two-dimensional orientable manifold, g_{ab} a given metric on M and $\{U_\alpha\}$ a set of coordinate patches of M. On each patch we can choose conformal Euclidean coordinates:

$$ds^2_{(\alpha)} = e^{2\sigma_{(\alpha)}}\delta_{ab}dz^a_{(\alpha)}dz^b_{(\alpha)} \ . \tag{A.1}$$

In the complex coordinates ($z=z^1+iz^2$, $\bar{z}=z^1-iz^2$), the metric is written as

$$ds^2_{(\alpha)} = e^{2\sigma_{(\alpha)}}dz_{(\alpha)}d\bar{z}_{(\alpha)}. \tag{A.2}$$

Since across coordinate patches $ds^2_{(\alpha)}=ds^2_{(\beta)}$, the coordinate transformation is given by $z_{(\alpha)}=f_{\alpha\beta}(z_{(\beta)})$, where f is a holomorphic function. Hence M acquires a complex structure. Conversely, if we are given a complex structure on M we can consider the conformal class

$$ds^2_{(\alpha)} \propto dz_{(\alpha)}d\bar{z}_{(\alpha)} \tag{A.3}$$

on every coordinate patch. A one-dimensional complex manifold is called a Riemann surface.

The uniformization theorem[31] for the Riemann surfaces states that there are essentially three distinct simply connected

Riemann surfaces up to holomorphic equivalence:

(a) the sphere $C \cup \{\infty\}$

(b) the plane C

(c) the upper half plane H

These are the universal covering spaces $\tilde{M}$ for the compact Riemann surfaces M. That is, any M is the quotient of $\tilde{M}/\Gamma$ where Γ is a discrete subgroup of the group of isometries of M, without fixed points.

For the sphere, the group of automorphisms is $SL(2,C)$. Since any of the transformations in this group has three fixed points, Γ are the trivial group $\{1\}$. Thus

$$M = \tilde{M}/\Gamma = C \cup \{\infty\} \ . \tag{A.4}$$

M is a unique Riemann surface of genus zero. For the plane, the group of automorphisms is $\{z \to az+b\}$. Only translations act without fixed points. Thus

$$M = \tilde{M}/\Gamma = \tilde{M}/\text{lattice group } Z+\tau Z = T \text{ (torus)} \ . \tag{A.5}$$

For the upper half plane H, the group of the automorphisms is $SL(2,R)/\{\pm 1\}$. Γ are the discrete subgroups of it, called Fuchsian groups. $M = \tilde{M}/\Gamma$ is the Riemann surface of genus ≥ 2.

There are constant curvature metrics on $\tilde{M}$:

$$ds^2 = \frac{dzd\bar{z}}{(1+|z|^2)^2} \qquad \text{for } C \cup \{\infty\} ,$$

$$ds^2 = dzd\bar{z} \qquad \text{for } C , \qquad (A.6)$$

$$ds^2 = \frac{dzd\bar{z}}{|\mathrm{Im}\ z|^2} \qquad \text{for } H ,$$

and the curvatures of these metrics are 1,0 and -1, respectively. Since these metrics are invariant under each automorphism, there exists a metric of constant curvature on M.

Appendix B. The Background Field Expansion

To calculate the quantum corrections in the σ-model, we usually use the background field method[22]. In this method, the action is expanded around an arbitrary classical solution of the equation of motion $X_0^\mu(z)$ in powers of a quantum field $\pi^\mu(z)$. But the splitting

$$X^\mu(z) = X_0{}^\mu(z) + \pi^\mu(z) \qquad (B.1)$$

is not covariant, and $\pi^\mu(z)$, which is the difference of two coordinates, is not a vector on the manifold (or the curved space-time). So we express $\pi^\mu(z)$ as a local power series in a new field $\xi^\mu(z)$ which is a contravariant vector on the manifold.

To define the field $\xi^\mu(z)$, consider the two points X^μ and $X^\mu+\pi^\mu$ on the manifold. We assume that these points are close enough that there is a unique geodesic which connects them. This geodesic may be parameterized by $\lambda^\mu(t)$ which satisfies the usual geodesic equation

$$\ddot{\lambda}^\mu + \Gamma^\mu_{\nu\rho}\, \dot{\lambda}^\nu \dot{\lambda}^\rho = 0 \quad , \qquad (B.2)$$

where t is an arc length parameter and we choose it such that $\lambda^\mu(0)=X_0{}^\mu$ and $\lambda^\mu(1)=X_0{}^\mu+\pi^\mu$. then, ξ^μ is defined by the tangent vector to the geodesic at t=0, $\xi^\mu=\dot{\lambda}^\mu(0)$, with length equal to the geodesic distance between X_0^μ and $X_0^\mu+\pi^\mu$. Since $\xi^\mu(z)$ is a contravariant vector, an expansion of the action in terms of it will be covariant.

The geodesic equation (B.2) can be iteratively solved to give

$$\lambda^\mu(t) = X_0^\mu + \xi^\mu t - \frac{1}{2}\Gamma^\mu_{\nu\rho}\xi^\nu\xi^\rho t^2 - \frac{1}{3!}\Gamma^\mu_{\nu\rho\sigma}\xi^\nu\xi^\rho\xi^\sigma t^3 - \ldots, \quad (B.3)$$

where $\Gamma^\mu_{\nu\rho\sigma\ldots\tau} = \nabla_\sigma\ldots\nabla_\tau\Gamma^\mu_{\nu\rho}$ and ∇_σ is a covariant derivative on lower indices only and all quantities are evaluated at X_0^μ, so that

$$\pi^\mu = \xi^\mu - \frac{1}{2}\Gamma^\mu_{\nu\rho}\xi^\nu\xi^\rho - \frac{1}{3!}\Gamma^\mu_{\nu\rho\sigma}\xi^\nu\xi^\rho\xi^\sigma - \ldots. \quad (B.4)$$

ξ^μ is called the Riemann normal coordinate and in this system, the geodesics are expressed as straight lines, i.e. $\Gamma^\mu_{(\nu\rho\sigma\ldots)}=0$.

The expansions of the background fields in terms of ξ are given by

$$G_{\mu\nu}(X_0+\pi) = G_{\mu\nu}(X_0) - \frac{1}{3}R_{\mu\rho\nu\sigma}(X_0)\xi^\rho\xi^\sigma - \ldots, \quad (B.5)$$

$$\Phi(X_0+\pi) = \Phi(X_0) + D_\mu\Phi(X_0)\xi^\mu + \frac{1}{2}D_\mu D_\nu\Phi(X_0)\xi^\mu\xi^\nu + \ldots, \quad (B.6)$$

$$\partial_a(X_0^\mu + \pi^\mu) = \partial_a X_0^\mu + D_a\xi^\mu + \frac{1}{3}R^\mu_{\nu\rho\sigma}\xi^\nu\xi^\rho\partial_a X_0^\sigma + \ldots. \quad (B.7)$$

Combining these expansions (B.5-7) and making ξ^μ a dimensionless field by the replacement $\xi^\mu \to \sqrt{2\pi\alpha'}\ \xi^\mu$, we obtain the background field expansion of the bosonic σ-model action

$$S[X_0+\pi]=\frac{1}{2\pi\alpha'}\int d^2z\Big[\frac{1}{2}\sqrt{g}\, g^{ab}\partial_a(X_0^\mu+\pi^\mu)\partial_b(X_0^\nu+\pi^\nu)G_{\mu\nu}(X_0+\pi)$$

$$+\frac{1}{2}\alpha'\sqrt{g}\,\bar{R}\,\Phi(X_0+\pi)\Big]$$

$$= S[X_0]+\frac{1}{\sqrt{2\pi\alpha'}}\int d^2z\Big[\sqrt{g}\, g^{ab}\partial_a X_0^\mu D_b\xi^\nu G_{\mu\nu}(X_0)$$

$$+\frac{1}{2}\alpha'\sqrt{g}\,\bar{R}\, D_\mu\Phi(X_0)\xi^\mu\Big]$$

$$+\int d^2z\Big[\frac{1}{2}\sqrt{g}\, g^{ab}D_a\xi^\mu D_b\xi^\nu G_{\mu\nu}(X_0)$$

$$-\frac{1}{2}\sqrt{g}\, g^{ab}\partial_a X_0^\mu\partial_b X_0^\nu R_{\mu\rho\nu\sigma}(X_0)\xi^\rho\xi^\sigma$$

$$+\frac{1}{4}\alpha'\sqrt{g}\,\bar{R}\, D_\mu D_\nu\Phi(X_0)\xi^\mu\xi^\nu\Big] +\dots \qquad \text{(B.8)}$$

The linear terms in ξ^μ vanish if the classical equation of motion is used.

We now study the ultraviolet divergences of this σ-model at the one-loop level. The one-loop divergent diagram is shown in fig.5 and in the dimensional regularization ($d=2+\varepsilon$) the one-loop divergent term of the effective action is given by

$$\Gamma_{\infty}^{(1)}=\frac{-1}{2\pi\varepsilon}\int d^2z\ [\frac{1}{2}\sqrt{g}\ g^{ab}\partial_a X^{\mu}\partial_b X^{\nu}R_{\mu\nu}(X)-\frac{1}{4}\alpha'\sqrt{g}\ \bar{R}\ D^2\Phi(X)]. \quad (B.9)$$

To cancel the one-loop divergences, the counterterm must be added to the classical action:

$$S_{c.t.}=\frac{1}{2\pi\alpha'}\int d^2z[\frac{1}{2}\sqrt{g}\ g^{ab}\partial_a X^{\mu}\partial_b X^{\nu}(\frac{\alpha'}{\varepsilon}R_{\mu\nu})$$

$$+\frac{1}{2}\alpha'\sqrt{g}\ \bar{R}\ (-\frac{\alpha'}{2\varepsilon}D^2\Phi\ )\ ]\ . \qquad (B.10)$$

The bare couplings are then (if the renormalized couplings are chosen to be dimensionless)

$$G_{0\,\mu\nu} = \mu^{\varepsilon}(G_{\mu\nu} +\frac{1}{\varepsilon}\alpha' R_{\mu\nu} +\ldots)\ , \qquad (B.11a)$$

$$\Phi_0 = \mu^{\varepsilon}(\Phi -\frac{1}{\varepsilon}\frac{\alpha'}{2} D^2\Phi +\ldots)\ , \qquad (B.11b)$$

where μ is the renormalization scale. The renormalization group β-functions to one-loop order are given by

$$\beta^{G}_{\mu\nu} = \alpha' R_{\mu\nu}\ , \qquad \beta^{\Phi} = -\frac{\alpha'}{2}D^2\ \Phi\ . \qquad (B.12a,b)$$

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Table Captions

Table 1 Values of critical radius b_0 for D=1,2,4,8 and 16 cases.

Table 2 Values of X_c (critical X) for D=1,2,4,8 and 16 cases.

Figure Captions

Fig.1 The operator P_1 maps vector fields into symmetric traceless 2-tensor ones.

Fig.2 Decomposition of $T_g(m)$.

Fig.3 Decomposition of $T_g(m)$ and the orthogonal projection of the tangent vector $\chi^{(r)}$ onto Ker $P_1^\dagger$.

Fig.4 The potential v(b) as a function of the parameter $b=r/\sqrt{\alpha'}$ for D=4 case. b_0 is a "critical radius".

Fig.5 The one-loop divergent diagram. The single line is the field ξ and the double line denotes a background field operator.

D	b_0
1	14.5
2	4.0
4	2.1
8	1.5
16	1.1

Table 1

D	X_c
1	1.46
2	0.42
4	0.22
8	0.16
16	0.13

Table 2

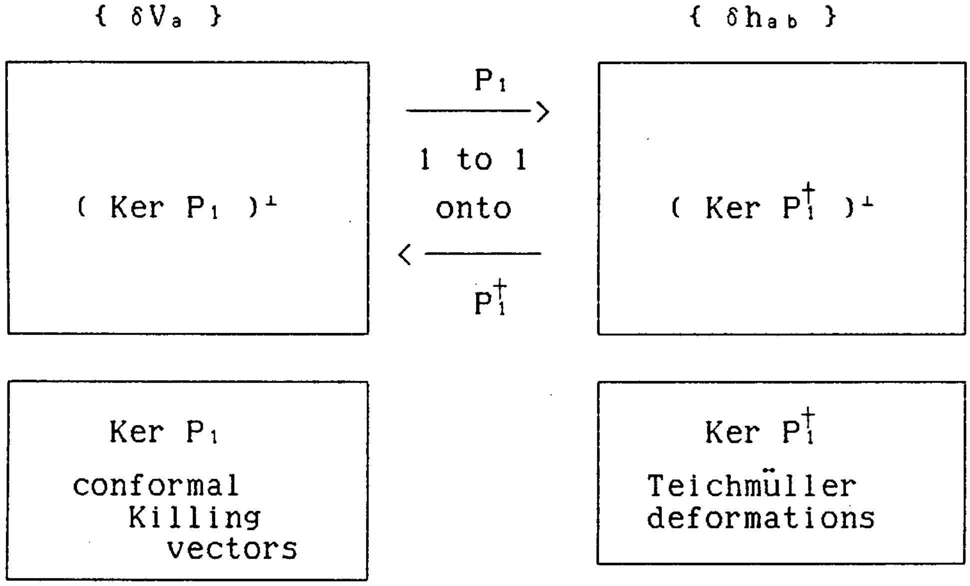

Fig. 1

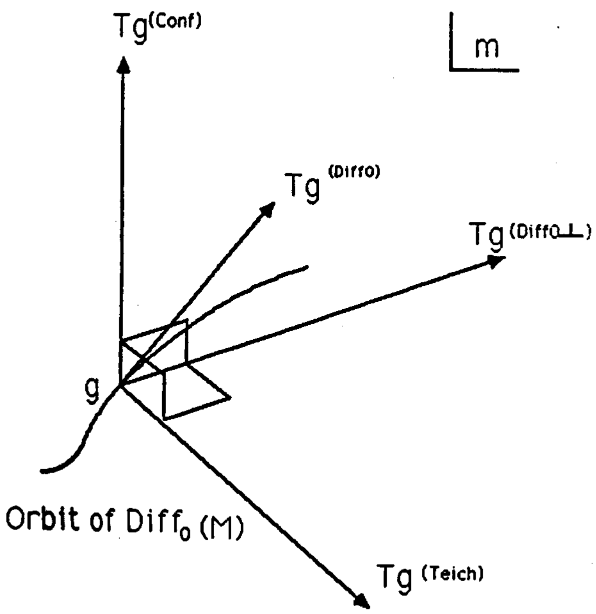

Fig.2

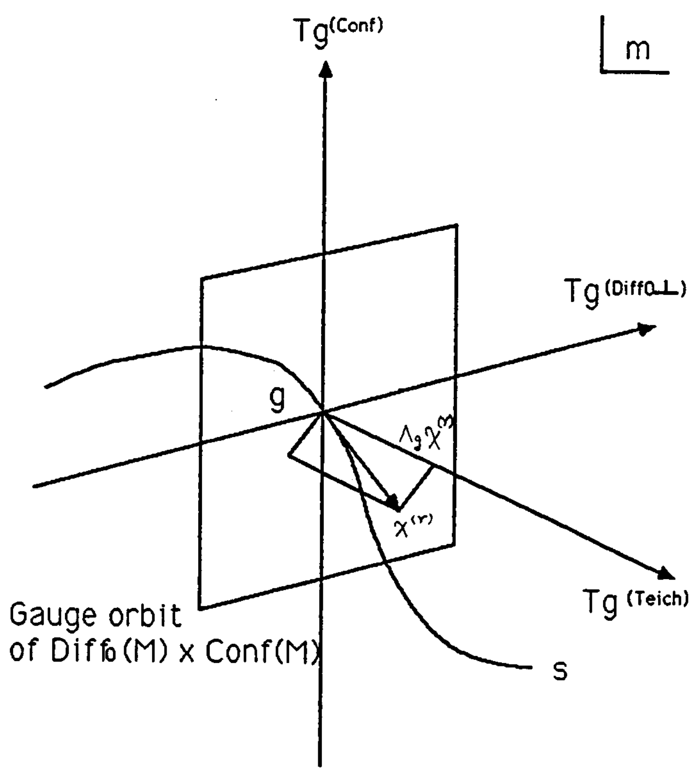

Fig.3

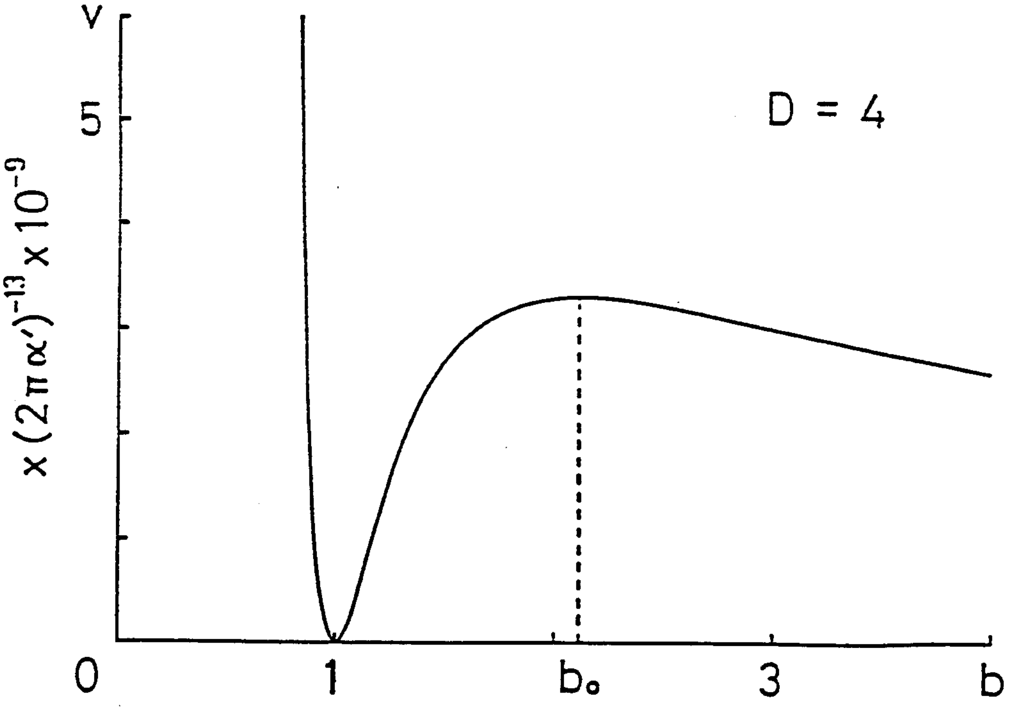

Fig. 4

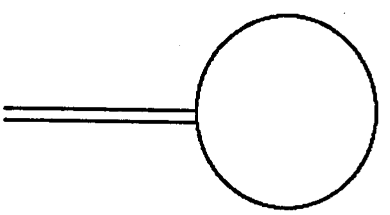

Fig.5