Automorphic forms and their applications to number theory

秋山, 茂樹

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神戸大学大学院自然科学研究科

秋山　茂樹

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（保型形式とその数論への応用）

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Preface

The triangular functions sin(x), cos(x) have a period 2π. These functions can be seen as the mappings from the real torus $\mathbb{R}/(2\pi\mathbb{Z})$, which is the division of $\mathbb{R}$ by the discrete subgroup $2\pi\mathbb{Z}$ in $\mathrm{Aut}(\mathbb{R})$; the group of real analytic automorphisms of $\mathbb{R}$. Automorphic functions are the holomorphic mappings from $\mathbb{H}$ to $\mathbb{C}$ with periods Γ, where $\mathbb{H}$ is the complex upper half plane and Γ is the discrete subgroup of $\mathrm{Aut}(\mathbb{H})=PSL(2,\mathbb{R})$; the group of complex analytic automorphisms of $\mathbb{H}$. In the triangular functions case, the discrete group $2\pi\mathbb{Z}$ is commutative. But in the automorphic functions case, the group Γ is essentially non-commutative.

Automorphic forms are the generalization of automorphic functions, which contain automorphic functions and their derivatives and more. The history of automorphic forms is deeply connected with that of number theory.

In Chapter 1, we deal with some special application of automorphic forms. Let us consider a certain family of discrete subgroups Γ which contains fuchsian triangle groups. The n-th coefficient of the Fourier expansion at cusps or elliptic fixed points has the form;

$$a_n = b_n \cdot r^n$$

where b_n is a rational number and r is a complex constant. Using these facts, we can construct an invariant quantity with respect to inclusion relations of groups. In the case of fuchsian triangle groups, the quantity can be written explicitly by the gamma function. See [1].

In Chapter 2, we treat the dimension formula of automorphic forms of weight one. In case of weight ≥ 2, the dimension formula was written explicitly, using the geometric data of Γ. But in our case, we cannot get expressions like these up to now. In the work of Hiramatsu [7], he gave the dimension of weight one for cocompact group Γ, using the residue of the Selberg type zeta function. The essential tool to derive this formula is the Selberg trace formula for a kernel function:

$$k_s(z,\phi,z',\phi') = \left| \frac{\sqrt{y\ y'}}{(z-\bar{z}')/2i} \right|^s \frac{\sqrt{y\ y'}}{(z-\bar{z}')/2i} \quad e^{-i(\phi - \phi')},$$

with $\mathrm{Re}(s)>1$.

In this section, we devote to the case that Γ is not cocompact, and $\Gamma \not\ni \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and derive the dimension formula of the same type. See [6]. In the case that $\Gamma \ni \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, see [8].

In Chapter 3, we deal with the Selberg trace formula for odd weight. See [2]. First we rewrite the Selberg trace formula in this case, concentrating on the difference between the contribution of regular cusps and that of irregular cusps. Such a difference is already known even in the case of the dimension formula for weight ≥ 3. So we are interested in writing this difference clearly in the general case.

Second we improve the dimension formula of weight one. In Chapter 2, we gave the dimension formula of weight one, using the residue of the Selberg type zeta function. But this formula is unsatisfactory because the zeta function has no functional equation. In this section we gave the dimension formula of weight one in general situation, using more natural zeta function which has a functional equation. The main result is

$$\dim S_1(\Gamma,\chi) + \dim S_1(\Gamma,\bar{\chi}) = \operatorname*{ord}_{s=1/2} Z_\Gamma^*(s,\chi),$$

where "ord" denote the order of zeros, and $Z_\Gamma^*(s,\chi)$ is the Selberg zeta function and χ is a finite dimensional unitary representation of Γ.

Chapter 1.

On the Fourier coefficients of automorphic forms of triangle groups

§ 1-0. Introduction

In this chapter, we want to construct a certain analytic invariant quantity with respect to the inclusion relations of the special discontinuous groups.

Denote $J_d(z)$ the absolute invariant of the Hecke group $\mathfrak{G}_d$. Then J_d has the following Fourier expansion at $i\infty$:

$$J_d(z)=\sum_{n=-1}^{\infty} a_n r^n q^n ,$$

where $a_n \in \mathbb{Q}$, $r \in \mathbb{R}$ and $q=\exp\left(\frac{\pi i z}{\cos(\pi/d)}\right)$.

The value r is algebraic if and only if d=3,4,6 and ∞ ([11][16]). These results can be extended to the case of fuchsian triangle groups and the expansion at an elliptic fixed point ([17][18]).

Now we consider the ratio of the value r's when there is an inclusion relation of groups. In § 1-1. we will show using purely algebraic method that the ratio is algebraic and etc. In the remaining section we put into concrete this result in the case of triangle groups. Especially in this case, some power of the ratio belongs to the imaginary quadratic field.

§ 1-1. Notation and results

Let q be an indeterminate, and $\mathbb{K}$ be some subfield of the complex number field $\mathbb{C}$. The *quasi* $\mathbb{K}$*-rational power series of style* r is the formal power series of the form

$$\sum_{n\geq \ell} a_n r^n q^n \quad (a_n \in \mathbb{K},\ r \in \mathbb{C}^*= \mathbb{C}-\{0\} ,\ \ell \in \mathbb{Z}) .$$

The *quasi* $\mathbb{K}$*-rational vector space of style* r is the vector space over $\mathbb{C}$ spanned by these series. The style r of a power series is determined up to an equivalence relation ;

$$(r_1/r_2)^{\sigma} \in \mathbb{K} \qquad \text{-------(1)}$$

for some $\sigma \in \mathbb{Z}$.

Let $\mathbb{F}$ be a subfield of $\mathbb{C}$ containing $\mathbb{K}$, and $c_1, c_2 \cdots, c_t$ be the complex numbers. We say that $\{c_1, c_2 \cdots, c_t\}$ is $\mathbb{F}$-independent over $\mathbb{K}$ if the property (P) is satisfied for all $d_i \in \mathbb{K}$ $(i=1,\cdots,t)$.

(P) $\sum_{i=1}^{t} d_i c_i \in \mathbb{F}$ then $d_i = 0$ for $i=1,\cdots,t$.

We can now state the main theorem.

Theorem 1

Let $\mathbb{V}$ *be the quasi* $\mathbb{K}$*-rational vector space of style* r_1 *,and* f *be an element of* $\mathbb{V}$. *Suppose that* f *is the quasi* $\mathbb{K}$*-rational infinite power series of style* r_2 . *Then the ratio of the styles* $\gamma = r_1/r_2$ *is algebraic over* $\mathbb{K}$, *and* f *is a linear combination of the basis of quasi* $\mathbb{K}$*-rational power series of style* r_1 *over* $\mathbb{K}(\gamma)$.

Moreover there are distinct non negative integers $l_0(=0), l_1, \cdots \cdots, l_m$ $(0 \le m \le \dim \mathbb{V})$, *and infinite numbers of* n *such that* $\{\gamma^{n-l_0}, \gamma^{n-l_1}, \cdots\cdots, \gamma^{n-l_m}\}$ *is not* $\mathbb{K}$*-independent.*

We can take the value m not larger than the maximum number of power series in the quasi $\mathbb{K}$-rational basis whose leading coefficient a_ℓ is 0 . In § 1-4. we will consider automorphic forms which has real axis as a natural boundary. In this case, the condition of *infinite* series is naturally satisfied. The conclusion of this theorem is rather complicated, but if the following conjecture holds, we can rewrite the theorem in a better style.

Conjecture 1

Assume that $\gamma \in \mathbb{C}$ *has the last properties of Theorem* 1 *then* γ^t *is an algebraic number of degree* m+1 *over* $\mathbb{K}$ *for some natural number* t. *Exchanging indices, we have* $l_i = l \cdot i \ (i=0,\cdots,m)$.

The style r for a quasi $\mathbb{K}$-rational power series is determined by the equivalence relation (1). The style of the quasi $\mathbb{K}$-rational *vector space* is determined by the following theorem.

Theorem 2

Let $\mathbb{V}$ *be the quasi* $\mathbb{K}$*-rational vector space whose style is taken in two ways as* r_1, r_2. *If* $\mathbb{V}$ *has at least one infinite power series, then there exists some natural number* s *such that*

$$(r_1/r_2)^{s} \in \mathbb{K} .$$

Choose basis of $\mathbb{V}$ *of the form*

$$\sum a_{n,k}\, r_1^n\, q^n \quad (k=1,2,\cdots,s;\ s=\dim \mathbb{V})$$

If the vector $(a_{n,1}, a_{n,2}, \cdots\cdots, a_{n,s})$ *is non zero for all* n, *then the number* s *can be taken not larger than* $\dim \mathbb{V}$.

§ 1-2. The proof of Theorem 1.

Let

$$\sum_{n\geq \ell} a_{n,k}\, r_1^n\, q^n \quad (k=1,2,\cdots,s=\dim \mathbb{V})$$

be the basis of $\mathbb{V}$, where $a_{n,k} \in \mathbb{K}$, $r_1 \in \mathbb{C}^*$ and $\ell \in \mathbb{Z}$. By the assumption, we have

$$
\begin{aligned}
f &= \sum_{n\geq \ell} c_n \, r_2^n \, q^n \\
&= \sum_{k=1}^{s} d_k \Big(\sum_{n\geq \ell} a_{n,k} \, r_1^n \, q^n \Big) \quad \in \mathbb{V} \quad .
\end{aligned}
$$

So

$$
c_n \, r_2^n = r_1^n \sum_{k=1}^{s} d_k \, a_{n,k} \qquad (n \geq \ell). \qquad \text{----------(2)}
$$

Put $\gamma = r_1/r_2$, $\mathbb{D} = (d_1, d_2, \cdots\cdots\cdots, d_s)$ and

$$
a_n = {}^t(a_{n,1}, a_{n,2}, \cdots\cdots\cdots, a_{n,s}).
$$

Then (2) is written in the form

$$
c_n = \gamma^n \, \mathbb{D}\cdot a_n \qquad \text{-----------(3)}
$$

So

$$
c_n = \gamma^n \, (\mathbb{D}\cdot P)\cdot(P^{-1}\cdot a_n)
$$

for $P \in GL_s(\mathbb{K})$. We can change basis of $\mathbb{V}$ by this method in order to get the assertion. At first we say that d_i $(i=1,\cdots,s)$ can be taken in $\mathbb{K}(\gamma)$. Assume $d_1 \notin \mathbb{K}(\gamma)$. If $d_1, d_2, \cdots\cdots, d_t$ are $\mathbb{K}(\gamma)$-independent and $d_1, d_2, \cdots\cdots, d_{t+1}$ are not $\mathbb{K}(\gamma)$-independent, then we may replace d_{t+1} with

$$
d_{t+1} + \sum_{i=1}^{t} h_i d_i \qquad (h_i \in \mathbb{K}).
$$

Thus we are able to think that d_{t+1} belongs to $\mathbb{K}(\gamma)$ from the start. Repeating this argument we get

$$
\begin{cases}
d_1, d_2, \cdots\cdots\cdots, d_t & \text{are } \mathbb{K}(\gamma)\text{-independent ;} \\
d_{t+1}, \cdots\cdots\cdots, d_s & \text{belong to } \mathbb{K}(\gamma), \text{ where } t \geq 1 \ .
\end{cases}
$$

From (2) we have

$$
\frac{c_n}{\gamma^n} - \sum_{k=t+1}^{s} d_k \, a_{n,k} = \sum_{k=1}^{t} d_k \, a_{n,k} \quad \in \mathbb{K}(\gamma) \ .
$$

Thus $a_{n,k} = 0$ for $k=1,\cdots\cdots,t$. This is a contradiction. So we get $d_k \in \mathbb{K}(\gamma)$ for $k=1,2,\cdots,s$. Using similar arguments we can assume

$$\begin{cases} \gamma^{\ell} d_1, \gamma^{\ell} d_2, \cdots\cdots\cdots, \gamma^{\ell} d_t \text{ are } \mathbb{K}\text{-independent ;} \\ d_{t+1} = d_{t+2} = \cdots\cdots\cdots = d_s = 1/\gamma^{\ell} . \end{cases}$$

Without loss of generality, we can assume $t < s$. Define

$$T_n = \left\{ (g_1, g_2, \cdots\cdots, g_t, g^*) \in \mathbb{K}^{t+1} \;\middle|\; g^* \gamma^{n-\ell} + \sum_{k=1}^{t} \gamma^n d_k g_k \in \mathbb{K} \right\}$$

and

$$S_{n,e} = \left\{ (g_1, g_2, \cdots\cdots, g_{t-e+1}) \in \mathbb{K}^{t-e+1} \;\middle|\; (g_1, g_2, \cdots\cdots, g_t, g^*) \in T_n \right\} .$$

We define $n_1, n_2, \cdots$ by induction. By the definition we know

$$S_{\ell,1} = \{0\} .$$

Let n_1 be the smallest number of n such that $S_{n,1} \neq \{0\}$. We may assume $g_t \neq 0$ so that we can replace d_t by

$$d_t + g_t^{-1} \sum_{k=1}^{t-1} d_k g_k + g_t^{-1} g^* \gamma^{-\ell} ,$$

and multiply some element of $\mathbb{K}^*$: we can put $d_t = 1/\gamma^{n_1}$. If $n_1, n_2, \cdots\cdots, n_w$ are defined and $d_t = 1/\gamma^{n_1}, d_{t-1} = 1/\gamma^{n_2}, \cdots, d_{t-w+1} = 1/\gamma^{n_w}$, then we may assume $S_{n,w} = \{0\}$ for $n = \ell, \ell+1, \cdots\cdots, n_w - 1$, and $S_{n_w, w+1} = \{0\}$. Since we have chosen the basis of $\mathbb{V}$, there is a number n such that $S_{n,w+1} \neq \{0\}$ if $w+1 \leq t$. Let n_{w+1} be the smallest number of these. Then we may put $d_{t-w} = 1/\gamma^{n_{w+1}}$, according to the same argument. Thus we may consider that $d_1 = 1/\gamma^{n_t}, d_2 = 1/\gamma^{n_{t-1}}, \cdots\cdots\cdots, d_t = 1/\gamma^{n_1}$. There are infinite numbers of n such that $n > n_w$ and $a_n \neq 0$, because f is an infinite power series. This concludes the proof.

§ 1-3. The proof of Theorem 2

Let

$$\sum_{n\geq\ell} a_{n,k}\, r_1^n\, q^n\ , \quad \sum_{n\geq\ell} b_{n,k}\, r_2^n\, q^n \quad (k=1,2,\cdots,s=\dim \mathbb{V})$$

be two quasi $\mathbb{K}$-rational basis of $\mathbb{V}$ whose styles r_1, r_2 respectively.

Put

$$\sum_{n\geq\ell} b_{n,k}\, r_2^n\, q^n = \sum_{k=1}^{s} d_{j,k} \sum_{n\geq\ell} a_{n,k}\, r_1^n\, q^n\ ,$$

$$\gamma = r_1/r_2\ ,\ \mathbb{D} = (\ d_{j,k}\ )\ ,$$

$$a_n = {}^t(\ a_{n,1},\ a_{n,2}\ ,\ \cdots\cdots\cdots\cdots, a_{n,s}\ )\ ,$$

$$b_n = {}^t(\ b_{n,1},\ b_{n,2}\ ,\ \cdots\cdots\cdots\cdots, b_{n,s}\ )\ .$$

Then

$$b_n = \gamma^n\, \mathbb{D}\cdot a_n\ . \qquad \text{-----------(4)}$$

So

$$P\ b_n = \gamma^n\ (P\cdot\mathbb{D}\cdot Q)\cdot(Q^{-1}a_n\ ) \qquad P,Q \in GL_s(\mathbb{K}).$$

In this way we will change basis. Next lemma is well known (see [14] page 81).

<u>Lemma</u>

Let $\sum_{n\geq\ell} \xi_{n,k}\, q^n$ $(k=1,\cdots,s)$ *be linearly independent formal power series over* $\mathbb{C}$. *Put* $\Xi_n = {}^t(\ \xi_{n,1}\ , \xi_{n,2}\ ,\cdots\cdots, \xi_{n,s}\ )$, *then the vector space spanned by all* Ξ_n $(n=1,2,\cdots\cdots)$ *has rank* s.

Take $n_1, n_2, \cdots\cdots, n_s$ ($n_i \geq \ell$) such that $a_{n_1}, a_{n_2}, \cdots\cdots, a_{n_s}$ are linearly independent over $\mathbb{C}$. Then from (4) we get

$$\mathbb{D}\cdot(\ a_{n_1}, a_{n_2}, \cdots\cdots, a_{n_s}\ ) = (\ \gamma^{-n_1}\, b_{n_1}, \gamma^{-n_2}\, b_{n_2}, \cdots\cdots, \gamma^{-n_s}\, b_{n_s}\ )\ .$$

Put

$P^{-1} = (b_{n_1}, b_{n_2}, \cdots\cdots, b_{n_s})$, $Q = (a_{n_1}, a_{n_2}, \cdots\cdots, a_{n_s})$.

Then

$$P\cdot D\cdot Q = \begin{bmatrix} \gamma^{-n_1} & & & \\ & \gamma^{-n_2} & & \\ & & \ddots & \\ & & & \gamma^{-n_s} \end{bmatrix} .$$

Since there are at least one infinite power series, there exists n such that $n > n_s$ and $\gamma^{n-n_k} \in \mathbb{K}$ for some k $(k=1,\cdots,s)$. This assures the first assertion of Theorem 2.

Put

$$\mathbb{U} = \{ \Xi \in \mathbb{C}^s \mid D\cdot\Xi \in \mathbb{K}^s \},$$

where $\Xi = {}^t(\xi_1, \xi_2, \cdots, \xi_s)$. $\mathbb{U}$ is the vector space over $\mathbb{K}$. We define the linear map φ by

$$\varphi : \mathbb{U} \longrightarrow \mathbb{K}^s \qquad \xi \longrightarrow D\cdot\xi .$$

As φ is injective, we get $\dim_{\mathbb{K}} \mathbb{U} \leq s$. Each $\gamma^n a_n$ belongs to $\mathbb{U}$. So $\gamma^{\ell} a_{\ell}, \gamma^{\ell+1} a_{\ell+1}, \cdots\cdots\cdots, \gamma^{\ell+s} a_{\ell+s}$ are linearly dependent over $\mathbb{K}$. There exist $(k_0, k_1, \cdots\cdots, k_s) \in \mathbb{K}^{s+1} - \{0\}$ such that

$$\sum_{j=0}^{s} k_j \gamma^{\ell+j} a_{\ell+j} = 0 .$$

We can find j $(j=0,\cdots,s)$ such that $k_j \neq 0$, then choose i $(i=1,\cdots,s)$ such that $a_{\ell+j,i} \neq 0$. Then

$$\sum_{j=0}^{s} k_j \gamma^j a_{\ell+j,i} = 0$$

gives the non-trivial algebraic relation whose degree is not larger than s . This proves the second statement of Theorem 2.1

§ 1-4. The ratio of styles in the case of fuchsian triangle groups

In this section we treat the special case of fuchsian triangle groups. For the precise notation, we refer to [18]. Let $\Delta = \Delta(p,q,r)$ be the triangle group whose signature is (p,q,r). If $1/p+1/q+1/r < 1$ then this group is realized and acts on the complex upper half plane $\mathbb{H}$ discontinuously. The fundamental domain of Δ is ABCD where ABC is the hyperbolic triangle, and ADC is the reflexion with respect to the geodesic AC . Denote $A_\Delta^{k,\nu}$ the space of holomorphic automorphic forms of Δ and of weight k, multiplier ν .

Take $f \in A_\Delta^{k,\nu}$ then

f is expanded at the elliptic point A of order p :

$$f(z) = (z-\bar{A})^{-k} \sum_{n\geq 0} a_n \left(\frac{z-A}{z-\bar{A}}\right)^n .$$

Ignoring $(z-\bar{A})^{-k}$, we know that $A_\Delta^{k,\nu}$ is the quasi rational vector space . The style of $A_\Delta^{k,\nu}$ depend only on the vertex A and Δ. Choosing good fundamental domain as Th 2 in [18], we can write down this style value:

$$\gamma(p;q,r) = \frac{\Gamma\left(1+\frac{1}{p}\right)\Gamma\left(\frac{1}{2}\{1-\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\}\right)\Gamma\left(\frac{1}{2}\{1-\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\}\right)}{\Gamma\left(1-\frac{1}{p}\right)\Gamma\left(\frac{1}{2}\{1+\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\}\right)\Gamma\left(\frac{1}{2}\{1+\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\}\right)} \, s ,$$

where $s^2 = \dfrac{\cos\left(\varepsilon - \frac{\pi}{p}\right)\cos\left(\varepsilon - \frac{\pi}{q}\right)}{\cos(\varepsilon)\cos\left(\varepsilon - \frac{\pi}{r}\right)}$, $\varepsilon = \dfrac{\pi}{2}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)$.

We can easily check that

$$\gamma(p;q,r) = \gamma(p;r,q) .$$

Assume $\Delta_1=\Delta_1(p_1,q_1,r_1) \subset \Delta_2 = \Delta_2(p_2,q_2,r_2)$ and $A_1B_1C_1D_1$ be the fundamental domain of Δ_1 which is suitably located in the sense of Th 2 of [18]. That is to say, $A_1=\sqrt{-1}$ and $B_1= t\sqrt{-1}$ $(t > 1)$. We can't always assume that $A_2B_2C_2D_2$ is suitably located. Let ϕ be the natural covering map from $\mathbb{H}/\Delta_1$ to $\mathbb{H}/\Delta_2$, and assume $\phi(A_1)=A_2$. Of course $p_1 | p_2$. Denote θ $(0 \leq \theta \leq \pi)$ the angle of $B_2A_1B_1$. All inclusion relations of triangle groups are classified in [15]. So we can calculate the value θ in a straight forward way. After tedious calculations we know that

$$\cos(2\, p_2\theta) \in \mathbb{Q}$$

for all inclusion relations. For example, in the case of $\Delta_1(5,4,4) \subset \Delta_2(5,2,4)$, we get $\theta = \frac{\pi}{10}$. When we regard this relation as $\Delta_1(4,4,5) \subset \Delta_2(4,5,2)$, we get $\cos(4\,\theta)= \frac{3}{5}$. The rotation at A_1 and of angle θ causes small change of the style. Using the relation of [18] page 4, we see that the style is multiplied by $e^{\sqrt{-1}\,\theta}$. In all cases, the value $e^{\sqrt{-1}\,\theta}$ is algebraic. From Theorem 1, we see that the ratio $\Upsilon(p_1;q_1,r_1)/\Upsilon(p_2;q_2,r_2)$ is algebraic, because $A^{k,\nu}_{\Delta_1} \supset A^{k,\nu}_{\Delta_2}$ and $A^{k,\nu}_{\Delta_2}$ contains elements other than constant functions for sufficiently large k. If the conjecture of § 1-1. is true, then some power of the ratio $\Upsilon(p_1;q_1,r_1)/\Upsilon(p_2;q_2,r_2)$ is of degree at most 7, because

$$\dim A^{k,\nu}_{\Delta_1} \leq \dim A^{k,\nu}_{\Delta_2} + 3 \leq 6 \quad .$$

Thus we are interested in calculating these ratios of the styles.

<u>Theorem 3</u>

Let Δ_1, Δ_2 be fuchsian triangle groups and $\Delta_1 \subset \Delta_2$. Then the ratio $\Upsilon(p_1;q_1,r_1)/\Upsilon(p_2;q_2,r_2)$ is given by the following table.

(I) *Normal case*

$$\frac{\Gamma(p;p,p)}{\Gamma(p;3,3)} = 3^{-3/2p} \qquad \frac{\Gamma(p\ ;p,p)}{\Gamma(2p;2,3)} = 2^{1/p}\ 3^{-3/2p}$$

$$\frac{\Gamma(p\ ;q,q)}{\Gamma(2p;2,q)} = 2^{1/p} \qquad \frac{\Gamma(q;\ q,p)}{\Gamma(q;2,2p)} = 2^{-2/q}$$

(II) *Non-normal case*

$$\frac{\Gamma(7;7,7)}{\Gamma(7;2,3)} = 2^{-6/7}\ 3^{-3/7}$$

$$\frac{\Gamma(7;7,2)}{\Gamma(7;2,3)} = 2^{-1/2}\ 3^{-3/7} \qquad \frac{\Gamma(2;7,7)}{\Gamma(2;3,7)} = 3^{-1/2}\ 7^{-1/4}$$

$$\frac{\Gamma(7;3,3)}{\Gamma(7;2,3)} = 2^{-6/7}\ 3^{-3/14} \qquad \frac{\Gamma(3;3,7)}{\Gamma(3;2,7)} = 2^{-1}\ 7^{-1/6}$$

$$\frac{\Gamma(8;8,4)}{\Gamma(8;2,3)} = 2^{-1/2}\ 3^{-3/8} \qquad \frac{\Gamma(4;8,8)}{\Gamma(8;2,3)} = 3^{-3/8}$$

$$\frac{\Gamma(8;8,3)}{\Gamma(8;2,3)} = 2^{-1/4}\ 3^{-1/2} \qquad \frac{\Gamma(3;8,8)}{\Gamma(3;2,8)} = 2^{-3/2}$$

$$\frac{\Gamma(9;9,9)}{\Gamma(9;2,3)} = 2^{-2/3}\ 3^{-1/6} \qquad \frac{\Gamma(9;9,9)}{\Gamma(3;2,9)} = 2^{-1}\ 3^{-5/6}$$

$$\frac{\Gamma(5;4,4)}{\Gamma(5;2,4)} = 2^{-1} \qquad \frac{\Gamma(4;4,5)}{\Gamma(4;2,5)} = 2^{-1/2}\ 5^{-1/4}$$

$$\frac{\Gamma(4p;4p,p)}{\Gamma(4p;\ 2,3)} = 2^{-1/2p}\ 3^{-3/4p} \qquad \frac{\Gamma(p;4p,4p)}{\Gamma(4p;\ 2,3)} = 2^{5/2p}\ 3^{-3/4p}$$

$$\frac{\Gamma(2p;2p,p)}{\Gamma(2p;\ 2,4)} = 2^{-2/p} \qquad \frac{\Gamma(p;2p,2p)}{\Gamma(2p;\ 2,4)} = 1$$

$$\frac{\Gamma(3p;\ 3,p)}{\Gamma(3p;\ 2,3)} = 2^{-2/p} \qquad \frac{\Gamma(p;\ 3,3p)}{\Gamma(3p;\ 2,3)} = 2^{-2/p}\ 3^{2/p}$$

$$\frac{\Gamma(3;3p,p)}{\Gamma(3;2,3p)} = 2^{-1}$$

$$\frac{\Gamma(2p;\ 2,p)}{\Gamma(2p;\ 2,3)} = 3^{-3/2p} \qquad \frac{\Gamma(p;2,2p)}{\Gamma(2p;2,3)} = 2^{3/p}\ 3^{-3/2p}$$

$$\frac{\Gamma(2;2p,p)}{\Gamma(2;2p,3)} = 3^{-1/2}$$

Corollary

Let Δ_1, Δ_2 *be fuchsian triangle groups and* $\Delta_1 \subset \Delta_2$. *We have*

$$\left(\Gamma(p_1;q_1,r_1)/\Gamma(p_2;q_2,r_2)\right)^{2p_2} \in \mathbb{Q} \quad .$$

Prime factors which appear in the numerator and the denominator are the prime factors of $q_1r_1q_2r_2$.

Remark

Consider the case $\Delta(5,4,4) \subset \Delta(5,2,4)$. As the elliptic point of order 4 of $\Delta(5,4,4)$ and the elliptic point of order 2 of $\Delta(5,2,4)$ are *not* identified by the covering map ϕ, it seems that we can't get the assertion of the corollary when we calculate $\Gamma(4;4,5)/\Gamma(2;4,5)$. (The value becomes $\pi^{-3/2}\,\Gamma(1/4)\Gamma(1/40)\Gamma(9/40)$ up to algebraic factor.) So we can get informations not only of the inclusion relation but also of the covering surface from this corollary.

Let us conclude this chapter with next fascinating conjecture.

Conjecture 2

Let $\Gamma_i\,(i=1,2)$ *be the fuchsian triangle groups and* $r_i\,(i=1,2)$ *be the corresponding styles. When* $\gamma = r_1/r_2$ *is algebraic, then* Γ_1 *and* Γ_2 *are commensurable.*

There are no counter example for this conjecture up to now. Unfortunately we know few about the transcendency of Γ-value, this conjecture seems far out of our reach.

Chapter 2

On some dimension formula for automorphic forms of weight one

§ 2-0. Introduction

In this chapter, we give a certain dimension formula for automorphic forms of weight one. In the case of weight $m \geq 2$, we can compute the dimension by the Riemann-Roch theorem for algebraic function of one variable. But we can get no informations of weight one by this way, because we cannot make good use of the duality of $m \leftrightarrow 2-m$ in the Riemann-Roch theorem.

In his paper [12], Selberg has introduced the celebrated trace formula. He also calculated the dimension for $m \geq 2$, using this formula. So it is an interesting problem to apply his formula for the case of weight one. In [7], Hiramatsu gave the dimension formula for weight one by this method for the cocompact group Γ. The dimension was expressed not by the geometric data of Γ but by the residue of the Selberg type zeta function.

In this chapter, we treat the case of the cofinite group Γ not containing $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, which has parabolic elements and give the formula of the same type. In this case, we must subtract the effect of the continuous spectra. For this purpose, we define the Maass-Eisenstein series which attaches to each cusp and modify the kernel function.

Unfortunately our dimension formula is not computable, because the analytic continuation of this zeta function is given by the same trace formula. The essential reason why we cannot get the effective dimension formula of weight one by this way is that the Selberg trace formula also have the duality of $m \leftrightarrow 2-m$. The explanation of this situation will be given in Chapter 3.

§ 2-1. Notations

Let Γ be the fuchsian group of the first kind not containing the element $\begin{pmatrix}-1 & \\ & -1\end{pmatrix}$, and suppose Γ has a non-compact fundamental domain in the upper half plane $\mathbb{H}$. Let T be the real torus $\mathbb{R}/\mathbb{Z}$, and $\hat{\mathbb{H}} = \mathbb{H} \times T$. Denote by $\mathcal{L}^2(\Gamma\backslash\hat{\mathbb{H}})$ the space of functions $f(z,\phi)$ on $\hat{\mathbb{H}}$ satisfying:

(1) $f(z,\phi)$ is a measurable function on $\mathbb{H}$

(2) $f(\gamma\cdot(z,\phi))=f(z,\phi)$ for $\gamma \in \Gamma$

(3) $\int_{\Gamma\backslash\hat{\mathbb{H}}} |f(z,\phi)|^2 \, dz \, d\phi < \infty$,

where $dz = y^{-2}\, dx\, dy$.

Put $$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y \frac{\partial}{\partial x}\frac{\partial}{\partial \phi} .$$

Spectral decomposition of $\mathcal{L}^2(\Gamma\backslash\hat{\mathbb{H}})$ with respect to Δ can be given in the form

$$\mathcal{L}^2(\Gamma\backslash\hat{\mathbb{H}}) = \bigoplus_{\lambda} \mathcal{L}_0^2(\Gamma\backslash\hat{\mathbb{H}},\lambda) \oplus \mathcal{L}_{sp}^2(\Gamma\backslash\hat{\mathbb{H}}) \oplus \mathcal{L}_{cont}^2(\Gamma\backslash\hat{\mathbb{H}})$$

where $\mathcal{L}_0^2(\Gamma\backslash\hat{\mathbb{H}},\lambda)$ is the space of Maass cusp forms, $\mathcal{L}_{sp}^2(\Gamma\backslash\hat{\mathbb{H}})$ is the discrete part of the orthogonal complement of $\mathcal{L}_0^2(\Gamma\backslash\hat{\mathbb{H}},\lambda)$ and $\mathcal{L}_{cont}^2(\Gamma\backslash\hat{\mathbb{H}})$ is the continuous spectrum.

We denote by $\mathcal{L}(m,\lambda)$ the set of functions $f(z,\phi)$ satisfying

(4) $f(z,\phi) \in \mathcal{L}^2(\Gamma\backslash\hat{\mathbb{H}})$

(5) $\Delta f(z,\phi) = \lambda f(z,\phi)$

$$(6) \quad \frac{\partial}{\partial \phi} f(z,\phi) = - m\sqrt{-1}\ f(z,\phi)$$

To obtain the dimension of holomorphic automorphic forms of weight one, we note

$$\sqrt{y}\ \exp(-\sqrt{-1}\ \phi)\ S_1(\Gamma) = \mathscr{L}(1,\ -1/4),$$

where $S_m(\Gamma)$ is the space of holomorphic automorphic forms of weight m. This relation is the special case of Hejhal [5] vol II, p.383. (See also [8].)

§ 2-2. The definition of Eisenstein series

We consider an invariant integral operator on the space $\mathscr{L}(m,\lambda)$ defined by a point—pair invariant kernel

$$k(z,\phi,z',\phi') = \left| \frac{\sqrt{y\ y'}}{(z-\bar{z}')/2i} \right|^{\delta} \frac{\sqrt{y\ y'}}{(z-\bar{z}')/2i} \quad e^{-i(\phi - \phi')}\ .$$

where $\delta > 1$. Then the operator k vanishes on $\mathscr{L}(m,\lambda)$ for all $m \neq 1$. It is easy to see that the integral

$$\int_{\Gamma\backslash\hat{\mathbb{H}}} \sum_{\gamma\in\Gamma} k(z,\phi,\gamma(z',\phi'))\ dz\ d\phi$$

is uniformly bounded at a neighborhood of each irregular cusps of Γ. We also see that by the Riemann-Roch theorem, the number of regular cusps is even. In the following we assume that κ_1,κ_2 is a maximal set of regular cusps of Γ which are not equivalent with respect to Γ.

Let Γ_i be the stabilizer of κ_i in Γ, and fix elements $\sigma_i \in SL(2,\mathbb{R})$ so that $\sigma_i^{-1}\Gamma_i\sigma_i$ is equal to the group $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix} : m \in \mathbb{Z} \right\}$. Then the Eisenstein series attached to the regular cusp κ_i is defined by

$$(7)\quad E_i(z,\phi;s) = \sum_{\substack{\sigma\in\Gamma_i\backslash\Gamma \\ \sigma_i^{-1}\sigma = \begin{pmatrix} * & * \\ c & d\end{pmatrix}}} \frac{y^s}{|cz+d|^{2s}}\, e^{-\sqrt{-1}(\phi+\ \mathrm{Arg}(cz+d))},$$

where i=1,2, and Re(s) > 1. It is easy to check that

$$(8)\quad E_i(\gamma(z,\phi);s) = E_i(z,\phi;s) \quad \text{for } \gamma \in \Gamma;$$

$$(9)\quad \Delta\, E_i(z,\phi,s) = s(s-1)\, E_i(z,\phi,s);$$

$$(10)\quad \frac{\partial}{\partial\,\phi} E_i(z,\phi;s) = -\sqrt{-1}\, E_i(z,\phi;s).$$

By the above (8), $E_i(z.\phi;s)$ has the Fourier-Bessel expansion at κ_i in the form

$$E_i(\sigma_j(z,\phi);s) = \sum_{k=-\infty}^{\infty} a_{ij,k}(y,\phi;s)\ e^{2\pi\sqrt{-1}kx}.$$

The constant term $a_{ij,0}(y,\phi;s)$ is given by

$$e^{\sqrt{-1}\phi} a_{ij,0}(y,\phi;s) = a_{ij,0}(y;s)$$

$$= \delta_{ij} y^s + \varphi_{ij}(s) y^{1-s},$$

where δ_{ij}=1 or 0 according to i=j or not, and

$$\varphi_{ij}(s) = -\sqrt{-1\pi}\ \frac{\Gamma(s)}{\Gamma(s+1/2)} \sum_{c\neq 0} \frac{(\mathrm{sgn}\ c) N_{ij}(c)}{|c|^{2s}}$$

with $N_{ij}(c) = \#\{\ 0 \le d < |c| : \begin{pmatrix} * & * \\ c & d\end{pmatrix} \in \sigma_i^{-1}\Gamma\,\sigma_j\ \}$. Define 2×2 matrix $\Phi(s)$ by $(\varphi_{ij}(s))$. Considering the involution $\gamma \to \gamma^{-1}$, it is easy to see that $\Phi(s)$ is an alternative matrix. This matrix is called the Eisenstein matrix.

§ 2-3. The Selberg trace formula

First we define the compact part of $E_i(z,\phi;s)$ by

$$E_i^Y(z,\phi;s) = \begin{cases} E_i(z,\phi;s) - a_{ij,0}(\mathrm{Im}(\sigma_i z),\phi;s) & \text{if } \mathrm{Im}(\sigma_i z) > Y \\ E_i(z,\phi;s) & \text{otherwise,} \end{cases}$$

where Y denotes a sufficiently large number. Then, the following Maass-Selberg relation of our case can be obtained in a similar way to the proof of Theorem 2.3.2. in Kubota [10]:

$$(11)\quad \frac{1}{2\pi}(E_i^Y(z,\phi;s), E_i^Y(z,\phi;\overline{s'})) = \frac{Y^{s+s'-1} - \varphi_{ij}(s)\overline{\varphi_{ij}(\overline{s'})}Y^{-s-s'+1}}{s+s'-1},$$

where $i \neq j$. Using this, we see that the Eisenstein matrix $\Phi(s)$ converges to a unitary matrix when s tends to the point $s_0 = 1/2 + \sqrt{-1}r_0$.

Therefore we have

$$\Phi(s_0)\Phi(1-s_0) = \Phi(s_0)\Phi(\overline{s_0}) = -\Phi(s_0)\overline{\Phi(s_0)} = \Phi(s_0)\,{}^t\overline{\Phi(s_0)} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix};$$

and hence each $E_i(z,\phi;s)$ has a meromorphic continuation to the whole s-plane, and the column vector $\mathcal{E}(z,\phi:s) = {}^t(E_1, E_2)$ satisfies the functional equation

$$\mathcal{E}(z,\phi;s) = \Phi(s)\,\mathcal{E}(z,\phi;1-s).$$

Since Γ is a cofinite group, the integral operator defined by $k(z,\phi,z',\phi')$ is not always compact. To subtract the effect of the continuous spectrum, we put

$$H(z,\phi,z',\phi') = \frac{1}{8\pi^2}\sum_{i=1}^{2}\int_{-\infty}^{\infty} h(r)E_i(z,\phi;s)\overline{E_i(z',\phi';s)}\,dr,$$

where $s = 1/2 + \sqrt{-1}r$. Here $h(r)$ denotes the eigenvalue of $k(z,\phi.z',\phi')$ in $\mathcal{L}(1,\lambda)$ which is given by

$$h(r) = h(r,\delta) = 2^{2+\delta}\,\pi\,B(1/2,(1+\delta)/2)\,B(\delta/2+\sqrt{-1}r, \delta/2-\sqrt{-1}r)$$

with $\lambda = s(s-1) = -r^2 - 1/4$, and $B(x,y)$ is the beta function. Define

$$K(z,\phi,z',\phi') = \sum_{\gamma\in\Gamma} k(z,\phi,\gamma(z',\phi'))$$

and

$$K^*(z,\phi,z',\phi') = K(z,\phi,z',\phi') - H(z,\phi,z',\phi').$$

Calculating the asymptotic behavior at each cusps κ_i, we see that the integral operator defined by K^* is compact.

Considering traces of K^* on $\mathcal{L}_0^2(\Gamma\backslash\hat{\mathbb{H}})$, we obtain the following trace formula. (See Selberg [13].)

$$(12) \qquad \sum_{j=1}^{\infty} h(\lambda_j) = \int_{\Gamma\backslash\hat{\mathbb{H}}} K^*(z,\phi,z,\phi)\, dz\, d\phi$$

$$= \int_{\Gamma\backslash\hat{\mathbb{H}}} \Big(\sum_{\gamma\in\Gamma} k(z,\phi,\gamma(z,\phi)) - H(z,\phi,z,\phi)\Big) dz\, d\phi$$

$$= \lim_{Y\to\infty} \sum_{[\alpha]} \int_{\Gamma(\alpha)\backslash\hat{\mathbb{H}}^*} \sum_{\gamma\in\Gamma(\alpha)} k(z,\phi,\gamma(z,\phi))\, dz\, d\phi$$

$$- \int_{\Gamma\backslash\hat{\mathbb{H}}^*} H(z,\phi,z,\phi)\, dz\, d\phi,$$

where each of λ_j denotes an eigenvalue corresponding to an orthonormal basis $\{f_j\}$ for $\mathcal{L}_0(\Gamma\backslash\hat{\mathbb{H}})$ and the last summation is taken over conjugacy classes $[\alpha]$ of Γ. Here we denote by $\Gamma(\alpha)$ the centralizer of α in Γ and $\hat{\mathbb{H}}^* = \hat{\mathbb{H}} - \bigcup_{i=1}^{2} \bigcup_{\gamma\in\Gamma} \gamma\,\sigma_i\{\, z \in \mathbb{H} : \mathrm{Im}(z) > Y \,\}$.

So far, the positive quantity δ is restricted to the condition $\delta > 1$. To get informations of the spectra of Δ, we treat δ as a complex valuable from now. The analytic continuation of the right hand side of (12) is given by the explicit calculation in § 2-4. For the continuation of the left hand side of (12), we should notice that $h(r)$ is rapidly decreasing when $|\mathrm{Re}(r)| \to \infty$.

To get the dimension formula of weight one, we want the multiplicity of the eigenvalue $\lambda = -1/4$. When $\lambda = -1/4$, we have $r = 0$ and

$$\operatorname*{Res}_{\delta=0} h(0,\delta) = 16\pi^2.$$

If $\lambda \neq -1/4$ the function $h(r,\delta)$ is holomorphic at $\delta = 0$. So we have

$$\dim S_1(\Gamma) = \frac{1}{16\pi^2} \operatorname*{Res}_{\delta=0} \int_{\Gamma\backslash\hat{\mathbb{H}}} K^*(z,\phi,z,\phi)\, dz\, d\phi.$$

§ 2-4. A formula for the dimension d_1

Throughout this section, we neglect the $o(1)$ term with respect to $Y \to \infty$. We put

$$\sum_{[\alpha]} \int_{\Gamma(\alpha)\backslash\hat{\mathbb{H}}^*} \sum_{\gamma\in\Gamma(\alpha)} k(z,\phi,\gamma(z,\phi))\, dz\, d\phi - \int_{\Gamma\backslash\hat{\mathbb{H}}^*} H(z,\phi,z,\phi)\, dz\, d\phi,$$

$$= J(id) + J(R) + J(H) + J(\infty),$$

where $J(id), J(R), J(H)$, and $J(\infty)$ denote respectively the identity component, the elliptic component, the hyperbolic component, and the parabolic component of the traces.

(I) The identity component

$$J(id) = \int_{\Gamma\backslash\hat{\mathbb{H}}} dz\, d\phi = 2\pi \operatorname{vol}(\Gamma\backslash\mathbb{H}).$$

(II) The elliptic component

$$J(R) = \sum_{R} \Bigg(\frac{2^{\delta}\pi\, B(1/2,(1+\delta)/2)}{\#\Gamma(R) \sin\theta}$$

$$\times \int_{-\infty}^{\infty} B(\delta/2+\sqrt{-1}r, \delta/2-\sqrt{-1}r) \frac{\sinh r(\pi-2\theta)}{\sinh \pi r}\, dr$$

$$+ \frac{2^{\delta}\pi\sqrt{-1}}{\#\Gamma(R)\sin\theta} B(1/2,(1+\delta)/2)\, B(\delta/2,\delta/2)\Bigg),$$

where R is taken over the representatives of elliptic conjugacy classes of Γ. Suppose that R is conjugate to $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ in $SL_2(\mathbb{R})$. From this formula, we have

$$\text{(13)} \qquad \lim_{\delta\to 0} \delta\, J(R) = \sum_{R} \frac{4\pi^2\sqrt{-1}}{\#\Gamma(R)\sin\theta} .$$

Considering the involution $R \to R^{-1}$, we see that the right hand side of (13) vanishes.

(Ⅲ) The hyperbolic component

$$J(H) = 2^{s+2}\pi\, B(1/2,(s+1)/2)$$

$$\times \sum_{T}\sum_{k=1}^{\infty} \frac{(\mathrm{sgn}\ \mathrm{tr}T)^k \log N\{T\}}{N\{T\}^{k/2} - N\{T\}^{-k/2}} (N\{T\}^{k/2} + N\{T\}^{-k/2})^{-s},$$

where the summation about T is taken over primitive hyperbolic conjugacy classes of Γ. Here we put a, a^{-1} are the eigenvalue of T such that $|a| > 1$ and $N\{T\} = a^2$.

The results of [7] which correspond to (Ⅱ),(Ⅲ) contain minor errors which were corrected by the author himself.

(Ⅳ) The parabolic component

We have

$$J(\infty) = \lim_{Y\to\infty}\left(\sum_{i=1}^{2} \int_{\Gamma_i\backslash\mathbb{H}^*} \sum_{\gamma\in\Gamma} k(z,\phi,\gamma(z,\phi))dz\, d\phi - \int_{\Gamma\backslash\mathbb{H}^*} H(z,\phi,z,\phi)\, dz\, d\phi \right).$$

For the first half of $J(\infty)$, we have

$$\int_{\Gamma_i\backslash\mathbb{H}^*} \sum_{\substack{\gamma\in\Gamma_i \\ \gamma\neq 1}} k(z,\phi,\gamma(z,\phi))\, dz\, d\phi =$$

$$= \int_0^Y \int_0^1 \sum_{n\neq 0} k(z,\phi,\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\cdot(z,\phi))y^{-2}\, dx\, dy\, d\phi$$

$$= 2\pi \sum_{n\neq 0} \frac{1}{n} \int_{n/Y}^{(\mathrm{sgn}\ n)\infty} \left|\frac{2}{2+\sqrt{-1}t}\right|^{\delta} \frac{2}{2+\sqrt{-1}t}\, dt$$

$$= 4\pi \sum_{n\geq 1} \frac{1}{n} \int_{n/Y}^{\infty} \left(\frac{4}{4+t^2}\right)^{\delta/2+1} dt$$

$$= 4\pi\left((C + \log Y)\, B(1/2,(1+\delta)/2) + \int_0^{\infty} \log(t)\left(\frac{4}{4+t^2}\right)^{\delta/2+1} dt\right).$$

Here C denotes the Euler constant. To derive the last formula, we use the Euler-Maclaurin summation formula. See Kubota [10] p.103 ~ p.104 . We notice that the first part of $J(\infty)$ can be written in the form

$$4\pi\, B(1/2,(\delta+1)/2)\log Y + \alpha(\delta),$$

where $\lim_{\delta\to 0} \delta\, \alpha(\delta) = 0$.

For the second half of $J(\infty)$, we employ (11). Then we have

$$\frac{1}{8\pi^2} \int_{\Gamma\backslash\hat{H}^*} \int_{-\infty}^{\infty} h(r)E_i(z,\phi;1/2+\sqrt{-1}r)\overline{E_i(z,\phi;1/2+\sqrt{-1}r)}\, dr\, dz\, d\phi$$

$$= \frac{1}{8\pi^2} \lim_{t\to 1/2} \int_{\Gamma\backslash\hat{H}} \int_{-\infty}^{\infty} h(r)E_i^Y(z,\phi;t+\sqrt{-1}r)\overline{E_i^Y(z,\phi;t+\sqrt{-1}r)}\, dr dz d\phi + o(1)$$

$$= \frac{1}{4\pi} \lim_{t\to 1/2} \int_{-\infty}^{\infty} h(r)\, \frac{Y^{2t-1} - \varphi_{ij}(t+\sqrt{-1}r)\overline{\varphi_{ij}(t+\sqrt{-1}r)}Y^{1-2t}}{2t-1}\, dr + o(1)$$

$$= 4\pi\, B(1/2,(\delta+1)/2)\log Y - \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r)\varphi_{ij}'(1/2+\sqrt{-1}r)\varphi_{ij}(1/2-\sqrt{-1}r)dr$$

$$+ o(1),$$

as $Y \to \infty$ and $t \to 1/2$, where $j \neq i$.

We note that the Eisenstein matrix $\Phi(s)$ is unitary and thus $\det \Phi(s)$ is not zero on the line $\mathrm{Re}(s)=1/2$. So we have

$$\varphi_{12}(1/2)\varphi_{21}(1/2) \neq 0.$$

From the analytic continuation of this part, we see that the function

$$\int_{-\infty}^{\infty} h(r,\delta)\left(\varphi_{12}'(1/2+\sqrt{-1}r)\varphi_{12}(1/2-\sqrt{-1}r)+\varphi_{21}'(1/2+\sqrt{-1}r)\varphi_{21}(1/2-\sqrt{-1}r)\right)dr$$

is holomorphic at $\delta=0$.

Now we can state the main theorem.

Theorem

Let Γ be the fuchsian group of the first kind not containing the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, *and suppose that the number of regular cusps of Γ is two. Let d_1 be the dimension of the space of cusp forms of weight one with respect to Γ. Then d_1 is given by*

$$d_1 = \frac{1}{4} \operatorname*{Res}_{s=0} \zeta^*(s),$$

where

$$\zeta^*(s) = \sum_{T} \sum_{k=1}^{\infty} \frac{(\mathrm{sgn}\ \mathrm{tr}T)^k \log N\{T\}}{N\{T\}^{k/2} - N\{T\}^{-k/2}} (N\{T\}^{k/2} + N\{T\}^{-k/2})^{-s}.$$

Here the summation with respect to T is taken over primitive hyperbolic conjugacy classes, and $N\{T\}= a^2$, *where* a, a^{-1} *are the eigenvalue of T satisfying* $|a| > 1$.

Remark. Let Γ be a general fuchsian group of the first kind not containing the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then, using the properties of the Eisenstein series defined at each regular cusp of Γ, we can reprove that the number of regular cusps of Γ is even. We can also prove that in the same way as in the above case, the contribution from parabolic conjugacy classes to d_1 vanishes.

Chapter 3

Selberg trace formula for odd weight

§ 3-0. Introduction

Let Γ be a fuchsian group of the first kind which does not contain the element $-I=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$. The first aim of this chapter is to rewrite the Selberg trace formula for odd weight and the group Γ for the general kernel function, in a form which makes clear the difference between the contribution of regular cusps and of irregular cusps. Let us explain this difference by an example.

Put $\Gamma_1 = \Gamma_1(4) = \left\{ \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv c+1 \equiv 1 \pmod 4 \right\}$. Then Γ_1 is generated freely by $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}$. Denote by Γ_2 the group generated by $\begin{pmatrix}-1 & -1\\ 0 & -1\end{pmatrix}$ and $\begin{pmatrix}-1 & 0\\ -4 & -1\end{pmatrix}$. Although the action of Γ_1 and that of Γ_2 on $\mathbb{H}$ are equivalent, we have

$$\dim S_3(\Gamma_1) = 0,$$
$$\dim S_3(\Gamma_2) = 1,$$

by the classical dimension formula. We can easily see that $\{ 0, 1/2, \infty \}$ are the Γ_i-inequivalent cusps ($i=1,2$). For Γ_1, we see that $\{ 0, \infty \}$ are regular and $\{ 1/2 \}$ is irregular. But for Γ_2, all cusps are irregular. These facts causes the difference of the dimension. So it is an interesting problem to write down the difference in the general case.

Our second aim is to improve the dimension formula for weight one. In Chapter 2, we studied a dimension formula for weight one, using special kernel function

$$h(r) = 2^{2+s}\, \pi\, B(1/2, (1+s)/2)\, B(s/2+\sqrt{-1}r, s/2-\sqrt{-1}r).$$

We note that the corresponding Selberg type zeta function does not have a functional equation. In this chapter, we apply the Selberg kernel

$$h(r) = \frac{1}{r^2+(s-1/2)^2} - \frac{1}{r^2 + \beta^2} \qquad (\beta>>0)$$

to the above trace formula. Then we can give a dimension formula of weight one, using more natural "zeta" function which has a functional equation of type $s \leftrightarrow 1-s$.

§ 3-1. Notation

Let $\mathbb{H}$ be the complex upper half plane and $T=\mathbb{R}/(2\pi)$. We put $\tilde{\mathbb{H}}= \mathbb{H} \times T$, $G=SL(2,\mathbb{R})$ and $\tilde{G}= G \times T$. Then $\tilde{G}$ acts transitively on $\tilde{\mathbb{H}}$ in the following way:

$$(g, \alpha)\cdot(z, \phi)=(g\cdot z , \phi+ \arg j(g,z)-\alpha) \qquad (g, \alpha)\in\tilde{G} ,(z, \phi)\in\tilde{\mathbb{H}} ,$$

where $g= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G$, $g\cdot z= \dfrac{az +b}{cz +d} \in\mathbb{H}$ and $j(g,z)= cz +d$. With the involution $\xi(z,\phi)=(- \overline{z} ,- \phi)$, the triple $(G,\mathbb{H},\xi)$ is the weakly symmetric Riemannian space. The ring of $\tilde{G}$ invariant differential operators on this space is generated by Δ and $\dfrac{\partial}{\partial \phi}$ where

$$\Delta= y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + y \frac{\partial}{\partial x} \frac{\partial}{\partial \phi} .$$

Let Γ be the discrete subgroup of G not containing $-I$. We identify G with $G \times \{0\}$, and Γ with $\Gamma \times \{0\}$. Take a unitary representation χ of Γ of degree ν ($< \infty$) . Let $\kappa_1,\kappa_2,\cdots,\kappa_\omega$ be the complete representatives of Γ-inequivalent cusps of $\Gamma\backslash\mathbb{H}$. Γ_i denotes the stabilizer of κ_i, and $\Gamma_i^0 =\Gamma_i\cap \ker \chi$. We will consider the cusp form of $\Gamma\backslash\mathbb{H}$, so take χ under the condition $[\Gamma_i,\Gamma_i^0] < \infty$ for $i=1,2,\cdots,\omega$. Take $\sigma_i\in G$ such that $\sigma_i\infty =\kappa_i$, satisfying the following condition:

$$\begin{cases} \text{If } \kappa_i \text{ is regular then } \Gamma_\infty = \sigma_i^{-1}\Gamma_i\sigma_i \text{ is generated by } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \\ \text{If } \kappa_i \text{ is irregular then } \Gamma_\infty = \sigma_i^{-1}\Gamma_i\sigma_i \text{ is generated by } \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}. \end{cases}$$

We consider $\mathbb{C}^\nu$ valued, square integrable functions on $\hat{\mathbb{H}}$ satisfying

$$f(\gamma\cdot(z,\phi)) = \chi(\gamma) f((z,\phi)),$$

for any $\gamma\in\Gamma$, and denote the space which consists of these functions by $L^2_\chi(\Gamma,\hat{\mathbb{H}})$. The Selberg eigenspace is a subspace of $L^2_\chi(\Gamma,\hat{\mathbb{H}})$ defined by two additional conditions:

$$\begin{cases} 1) & \dfrac{\partial}{\partial\phi} f = -\sqrt{-1}m\ f\ , \\ 2) & \Delta f = \lambda f\ . \end{cases}$$

$\mathcal{L}_\chi(m,\lambda)$ denotes this space. We assume the eigen values λ are numbered in the following way

$$\frac{|m|}{2}\left(\frac{|m|}{2} - 1\right) \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots\cdots.$$

For the convenience, we set $\lambda_n = -\left(r_n^2 + \frac{1}{4}\right) = s_n(s_n - 1)$ and $s_n = \frac{1}{2} + \sqrt{-1}r_n$.

To describe the continuous spectrum, we define the Eisenstein series which attaches to κ_i $(i=1,\cdots,\omega)$:

$$E_i(z,\ \phi\ ;s) = \sum_{\sigma\in\ \Gamma_i\setminus\Gamma} \operatorname{Im}(\ \sigma_i^{-1}\sigma\ z)^s \exp\left(-m\sqrt{-1}\ \sigma_i^{-1}\sigma\ \phi\right)\chi^{-1}(\sigma)\ P_i$$

$$= \sum_{\sigma} \frac{y^s}{|cz+d|^{2s}} \exp\left(-m\sqrt{-1}\ (\ \phi + \arg\ (cz+d))\right)\chi^{-1}(\sigma_i\sigma)\ P_i.$$

In the last summation, σ is taken over all representatives of $\Gamma_\infty\setminus\sigma_i^{-1}\Gamma$, and $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. P_i is defined by

$$P_i = \begin{cases} \dfrac{1}{r_i} \displaystyle\sum_{g\in\Gamma_i/\Gamma_i^0} \chi(g) & (\text{ if } \kappa_i \text{ is regular });\\ \dfrac{1}{2r_i} \displaystyle\sum_{i=1}^{2r_i} (-1)^i\chi(\eta^i) & (\text{ if } \kappa_i \text{ is irregular }), \end{cases}$$

where $r_i = [\ \Gamma_i : \Gamma_i^0\ ]$ and $\eta\in\Gamma_i$ is chosen so that η mod $(\Gamma_i^0)^2$ should be a generator of $\Gamma_i/(\Gamma_i^0)^2$. These Eisenstein series are meromorphically continued to the whole s-plane and satisfies an analogous functional equation with Chapter 2 (cf.[6][9]). We denote by $\Phi_m(s)$ the constant term matrix of these Eisenstein series.

§ 3-2. The Selberg trace formula for odd weight

First of all we rewrite the Selberg trace formula in our case. The calculation was done by Hejhal in [5] vol II, but we do this by our formulation of the Eisenstein series, using the Euler-Maclaurin summation formula.

Theorem 1 (**Selberg trace formula for odd weight**).

Let N *be a non negative integer and* m= 2N+1. *We assume that* h(r) *is an analytic function in the region* $|\mathrm{Im}(r)| \leq \max(N, 1/2)+\delta$ *satisfying following two conditions:*

1) $h(r)=h(-r)$;

2) *There exists a sufficiently large number* M *such as*

$$h(r) \leq M\ |1+\mathrm{Re}(r)|^{-2-\delta},$$

where δ *is some positive real number. Put*

$$g(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\ e^{-\sqrt{-1}\pi r u}\ dr,$$

then the following formula holds:

$$\sum_{n=1}^{\infty} h(r_n)=\frac{\nu\ \mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi}\left[\ \int_{-\infty}^{\infty} rh(r)\coth(\pi r)\ dr + 2\sum_{k=0}^{N} k\ h(\sqrt{-1}\ k)\ \right]$$

$$+\sum_{\{T\}\in\mathrm{hyperbolic}} \frac{\mathrm{Tr}(\chi(T))\ \mathrm{sgn}(T)\ \ln N\{T_0\}}{N\{T\}^{1/2} - N\{T\}^{-1/2}}\ g(\ln N\{T\})$$

$$+\sum_{\{R\}\in\mathrm{elliptic}} \frac{\mathrm{Tr}(\chi(R))}{4\ {}^{\#}\Gamma(R)\ \sin\theta}\left[\ \int_{-\infty}^{\infty} h(r)\ \frac{\sinh\ (\pi-2\theta)r}{\sinh\ \pi r}\ dr\right.$$

$$-\sqrt{-1}\, h(0) + 2\sqrt{-1} \sum_{k=0}^{N} e^{2\sqrt{-1}k\theta}\, h(\sqrt{-1}k) \Big]$$

$$- g(0) \sum_{\substack{\alpha_{ij} \neq 0 \\ \text{regular}}} \ln |1-e^{2\pi i \alpha_{ij}}| + \sum_{\substack{\alpha_{ij} \neq 0 \\ \text{regular}}} \left(\frac{1}{2} - \alpha_{ij}\right)\left(\sum_{k=0}^{N} h(\sqrt{-1}k) - \frac{h(0)}{2}\right)$$

$$- g(0) \sum_{\substack{\alpha_{ij} \neq -1/2 \\ \text{irregular}}} \ln |1+e^{2\pi i \alpha_{ij}}| - \sum_{\substack{\alpha_{ij} \neq -1/2 \\ \text{irregular}}} \alpha_{ij}\left(\sum_{k=0}^{N} h(\sqrt{-1}k) - \frac{h(0)}{2}\right)$$

$$+ \frac{\iota}{4} h(0) - \frac{h(0)}{4} \mathrm{Tr}\left(\Phi_m(\tfrac{1}{2})\right)$$

$$+ \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\,\mathrm{Tr}\left(\Phi'_m(\tfrac{1}{2} + \sqrt{-1}r)\,\Phi_m(\tfrac{1}{2} - \sqrt{-1}r)\right)dr - \iota\, g(0) \ln 2$$

$$- \frac{\iota}{2\pi}\int_{-\infty}^{\infty} h(r)\psi(1+\sqrt{-1}r)\,dr + \iota \int_{0}^{\infty} g(u)\, \frac{1-\cosh(\frac{mu}{2})}{2\sinh(u/2)}\, du .$$

Here the notation is as follows. We denote by "~" the conjugation in SL(2,ℝ) and by { } its conjugate class. Take λ such that $T \sim \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$, where $|\lambda| > 1$; $\Gamma(T)$ denotes the centralizer of T in Γ and $^{\#}\Gamma(T)$ is the order of this group. T_0 is a generator of $\Gamma(T)$, $N\{T\} = \lambda^2$ and sgn T= sgn λ. $R \sim \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$; T_i is a generator of Γ_i, the stabilizer group of cusp κ_i.

$$\chi(T_i) \sim \begin{pmatrix} \exp(2\pi\sqrt{-1}\alpha_{i1}) & & \\ & \ddots & \\ & & \exp(2\pi\sqrt{-1}\alpha_{i\nu}) \end{pmatrix} .$$

We determine α_{ij} so that

$$\begin{cases} \alpha_{ij} \in [\,0\,,\,1) & \text{if } \kappa_i \text{ is regular} \\ \alpha_{ij} \in [\,-1/2\,,\,1/2\,) & \text{if } \kappa_i \text{ is irregular .} \end{cases}$$

ι=A+B where A denotes the number of pairs (i,j) such that $\alpha_{ij}=0$, where i moves in the range that κ_i is regular and $j = 1, \cdots\cdots, \nu$. And B denotes the number of pairs (i,j) such that $\alpha_{ij} = -\frac{1}{2}$, where i moves in the range that κ_i is irregular and $j = 1, \cdots\cdots, \nu$. $\psi(s) = \frac{\Gamma'}{\Gamma}(s)$ is the digamma function.

§ 3-3. The Selberg Zeta Function for Odd Weight

Put

$$Z_\Gamma^*(s,\chi) = \prod_\alpha \prod_{n=0}^{\infty} \det \left(E_\nu - \operatorname{sgn}(P_\alpha)\, \chi(P_\alpha)\, N\{P_\alpha\}^{-s-n} \right)$$

where the first product $\prod$ is taken over all primitive hyperbolic conjugate classes $\{P_\alpha\}$ of Γ , and E_ν is the $\nu\times\nu$ unit matrix. Now we can write down the functional equation. We put

$$\xi_\Gamma^*(s,\chi) = \frac{d}{ds} \log Z_\Gamma^*(s,\chi)$$

$$= \sum_\alpha \sum_{k=1}^{\infty} \frac{\operatorname{Tr}(\chi(P_\alpha^k))\ \operatorname{sgn}(P_\alpha^k)\ \ln N\{P_\alpha\}}{N\{P_\alpha\}^{k/2} - N\{P_\alpha\}^{-k/2}} N\{P_\alpha\}^{-(s-1/2)k}$$

Theorem 2 (Functional Equation). *We have*

$$\xi_\Gamma^*(s,\chi) + \xi_\Gamma^*(1-s,\chi) = -\nu \operatorname{vol}(\Gamma\backslash\mathbb{H})(s-1/2)\cot(\pi(s-1/2))$$

$$- \pi \sum_{\{R\}} \frac{\operatorname{Tr}(\chi(R))}{{}^{\#}\Gamma(R)\sin\theta} \frac{\sin((\pi-2\theta)(s-1/2))}{\sin(\pi(s-1/2))}$$

$$+ 2 \sum_{\substack{\alpha_{ij}\neq 0 \\ \text{regular}}} \ln\left|1 - e^{2\pi\sqrt{-1}\alpha_{ij}}\right| + 2 \sum_{\substack{\alpha_{ij}\neq -1/2 \\ \text{irregular}}} \ln\left|1 + e^{2\pi\sqrt{-1}\alpha_{ij}}\right|$$

$$- \frac{\varphi'}{\varphi}(s) - \frac{\iota}{2}\Big(\xi_m(s) + \xi_m(1-s)\Big) + 2\,\iota\,\ln 2$$

where $\xi_m(s) = \psi(s+m/2) + \psi(s-m/2) - 2\psi(s) - 2\psi(s+1/2)$, $\varphi(s) = \det \Phi_m(s)$.

proof.

Apply Th. 1 to the kernel function

$$h(r) = \frac{1}{r^2+(s-1/2)^2} - \frac{1}{r^2 + \beta^2},$$

where β is a sufficiently large positive number. Then the analytic continuation of the right hand side can be done except the hyperbolic component by the precise argument of complex integration of each component. Analytic continuation of the left hand side due to the fact

$$\sum_{\lambda} |r|^{-2-\delta} < \infty,$$

where δ is an arbitrary positive number.

§ 3-4. A dimension formula of the space of cusp forms of weight one

First we consider the space $\mathscr{G}_m(\Gamma,\chi)$ which consists of $\mathbb{C}^{\nu}$ valued holomorphic functions satisfying :

$$\begin{cases} 1)\ F|[T]_m = \chi(T)\ F \qquad \text{for } T \in \Gamma; \\ 2)\ \int_{\Gamma\backslash\mathbb{H}} {}^tF(z)\ \overline{F(z)}\ y^m\ dz < \infty , \end{cases}$$

where $F|[T]_m = F(T\cdot z)\ j(T,z)^{-m}$. The connection of this space $\mathscr{G}_m(\Gamma,\chi)$ and the Selberg eigenspace is given by the next lemma.

Lemma 1.

$$\mathscr{L}_{\chi}\left(m+2,\frac{m}{2}\left(1 + \frac{m}{2}\right)\right) = y^{(m+2)/2} \exp\left(-\sqrt{-1}(m+2)\phi\right)\ \mathscr{G}_{m+2}(\Gamma,\chi).$$

$$\mathscr{L}_{\chi}\left(m,\frac{m}{2}\left(1 + \frac{m}{2}\right)\right) = y^{-m/2} \exp\left(-\sqrt{-1}m\ \phi\right)\ \overline{\mathscr{G}_{-m}(\Gamma,\overline{\chi})}.$$

Lemma 2.

Suppose $\lambda \neq \frac{m}{2}\left(1+\frac{m}{2}\right)$, *then* $\dim \mathcal{L}_\chi(m,\lambda) = \dim \mathcal{L}_\chi(m+2,\lambda)$.

Using these two lemmas, we can calculate the difference between the dimension of $\mathcal{S}_m(\Gamma,\chi)$ with that of $\mathcal{S}_{2-m}(\Gamma,\chi)$, and induce the explicit dimension formula for $m \geq 2$. In the case of weight one, we have

Theorem 3.

$$\dim \mathcal{S}_1(\Gamma,\chi) - \dim \mathcal{S}_1(\Gamma,\bar{\chi})$$

$$= \sqrt{-1}\sum_{\{R\}} \frac{\mathrm{Tr}(\chi(R))}{2\ {}^{\#}\Gamma(R)\ \sin\theta}$$

$$+ \sum_{\substack{\alpha_{ij}\neq 0\\ \text{regular}}} \left(\frac{1}{2} - \alpha_{ij}\right) - \sum_{\substack{\alpha_{ij}\neq -1/2\\ \text{irregular}}} \alpha_{ij} - \frac{1}{2}\mathrm{Tr}\left(\Phi_1\left(\frac{1}{2}\right)\right).$$

Now we treat the trace formula in a different way. Assume that $h(r)=h(r,s)$ is a meromorphic function of r and s, and the trace formula is analytically continued to the whole s-plane. Let $h(r,s)$ has a pole $s=m/2$ when $r=\sqrt{-\lambda-1/4}$ and $\lambda = \frac{m}{2}\left(\frac{m}{2}-1\right)$, and $h(r,s)$ is holomorphic at $s=m/2$ whenever $r\neq\sqrt{-\lambda-1/4}$. This situation can be realized by various functions of r and s. Especially we can take the Selberg kernel $\frac{1}{r^2+(s-1/2)^2} - \frac{1}{r^2+\beta^2}$, where $\beta >> 0$. Let us compare the residues at $s=m/2$ of both sides. If $m \geq 3$, we get the analogous formula of Theorem 3. In this case, the hyperbolic contribution vanishes because the Selberg zeta function is holomorphic at $s=m/2$. But if $m=1$, we have

Theorem 4.

$$\dim \mathcal{S}_1(\Gamma,\chi) = \frac{1}{2} \operatorname*{ord}_{s=1/2} Z_\Gamma^*(s,\chi) + \sqrt{-1} \sum_{\{R\}} \frac{\mathrm{Tr}(\chi(R))}{4\ {}^\#\Gamma(R) \sin\theta}$$

$$+ \frac{1}{2} \sum_{\substack{\alpha_{ij}\neq 0 \\ \text{regular}}} \left(\frac{1}{2} - \alpha_{ij}\right) - \frac{1}{2} \sum_{\substack{\alpha_{ij}\neq -1/2 \\ \text{irregular}}} \alpha_{ij} - \frac{1}{4}\mathrm{Tr}\left(\Phi_1(\frac{1}{2})\right)$$

where "ord" denotes the order of zeros.

By Theorem 3 and 4, we get

Theorem 4'.

$$\dim \mathcal{S}_1(\Gamma,\chi) + \dim \mathcal{S}_1(\Gamma,\overline{\chi}) = \operatorname*{ord}_{s=1/2} Z_\Gamma^*(s,\chi).$$

This result is the good explanation why the residue of the Selberg type zeta function appear in the dimension formula of [6],[7] [8] and [9]. Comparing trace formulas of different odd weight, we easily get the following.

Theorem 5.

For $N \geq 1$, *we have*

$$\dim \mathcal{S}_1(\Gamma,\chi) = \dim \mathcal{L}_\chi(2N+1, -\frac{1}{4}) + \frac{1}{2}\mathrm{Tr}\ \Phi_1\left(\frac{1}{2}\right).$$

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