Parafermionic current algebras of torus/orbifold models

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(Degree)
博士（学術）

(Date of Degree)
1990-03-31

(Date of Publication)
2008-06-16

(Resource Type)
doctoral thesis

(Report Number)
甲0888

(URL)
https://hdl.handle.net/20.500.14094/D1000888

DOCTORAL THESIS

Parafermionic Current Algebras of Torus/Orbifold Models

(トーラス／オービフォルド模型のパラフェルミオン的カレント代数)

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January 1990

ACKNOWLEDGMENTS

I would like to express my gratitude to Professor Masakuni Ida for valuable suggestions, instructions and careful reading of this manuscript. I am so much obliged to Professor Tetsuro Kitazoe and Dr. Makoto Sakamoto for helpful suggestions and continuous encouragement. I am grateful to Dr. Kazuhiro Kimura and Dr. Masayoshi Tabuse, my collaborators, for many useful discussions. I would like to thank many of my colleagues at Kobe University.

ABSTRACT

We study the quantum field theory of bosons on the torus and the orbifold. When we quantize a closed string on the torus, we consider interactions of strings in the covering space of the torus. As a result we can give cocycle factors in vertex operators. When a torus is in special moduli, there are some chiral algebraic structures in the physical spectrum and the representation space is finite. Then such torus models are equivalent to some rational conformal field theories. We call them rational torus models. We discuss the condition to formulate orbifold models consistently from torus models. This condition restricts the constant antisymmetric tensor field B. We show that there are parafermionic current algebras in Z_N orbifold models. Some applications of Z_N parafermions on orbifold models are also given.

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1. INTRODUCTION

The string theory [1] was first suggested as the dual resonance model to describe interactions of hadrons and hadronic resonances with high spins. The description of the interactions appears in beautiful Veneziano amplitudes. However, once it was recognized that string theories are consistent only in 26 dimensional spacetime for bosonic strings and in ten-dimensional spacetime for superstrings, then the string theory as a theory of strong interactions of hadrons died in 1970's.

After 1973 or 1974, instead of the string theory, quantum chromodynamics based on the color gauge symmetry of $SU(3)$ has become the orthodox theory of strong interactions. By combining it with the Weinberg and Salam theory based on $SU(2) \times U(1)$ gauge symmetry of weak interactions, the standard model with Higgs mechanism was constructed. We agree with the idea that the standard model is a good theory which describes well the phenomena of the world at energy scale up to ~ 100**GeV**.

But we think now that the standard model does not play a role as a unified theory of matter and all the forces containing gravity. It is because the standard model has problems of the hierarchy, Higgs scalars, many 'elementary' particles and too many parameters, etc. In this circumstance, research of the unified theory has begun.

Up to early 1980's, there had been two important developments in unified theories and they were crucial for reviving the string theory. One is that the only framework containing scalar fields and allowing the hierarchy of the coupling constants is the supersymmetry [2], which is the symmetry to interchange fermionic and bosonic degrees of freedom. The other is an idea of the Kaluza-Klein theory [3], that is, high dimensional gravity coupled to Yang-Mills theory, which allows the extra dimensions for the physical degrees of freedom as appeared previously in the string theory.

In 1984, Green and Schwarz [4] discovered the cancellation of anomalies in the ten-dimensional superstring theory with $E_8 \times E_8'$ gauge symmetry. After that, the string theory is considered as the most promising theory for unification of matter and all the forces containing gravity.

It is recognized that the quantum mechanical determination of the physical degrees of freedom may reveal the connection of the spectrum of matter and the geometrical structure of the world itself. In fact, our world is of maximal four-dimensional spacetime. From this, we have to have extra degrees of freedom other than spacetime in string theories and to study them in order to describe the whole geometry of our world and to get the spectrum of matter.

The string theory is based on an idea that there is a one-dimensional object as the fundamental unit of matter, which distributes in the physical degrees of freedom. String theories have an infinite number of coordinates corresponding to the Fourier modes of the string distribution. Therefore, string theories provide us with an infinite number of massive states.

It is remarkable that the spectrum is completely determined through the first quantization if we adopt some geometry of the world. For example, the spectra and gauge symmetries of heterotic string theories [5] on Z_3 orbifold are determined and classified into several kinds of models [6]. We have no freedom to introduce new particles by hands in string theories.

The interaction of strings is smoother than that of points because it can be described as a manifold with boundaries, while the interaction of points can not. Therefore, it follows that string theories are free of ultraviolet divergences. We can say that if we consider Riemann surfaces with boundaries which correspond to incoming and outgoing particles, then we have already introduced interactions. This can be done by Polyakov's path-integral method on Riemann surfaces [7]. Then multiloop amplitudes correspond to Riemann surfaces of higher genus.

Is there a necessity to do more than to determine the spectrum and interactions? An answer is probably to say that one needs to study field theories in order to reveal the structure of vacua and to find a true vacuum of string theories. No string field theories, however, have been successful in this respect up to now.

At present, we have no string theories other than the following ones. If we work in the ten-dimensional heterotic string theory, we see that massless modes consist of the $E_8 \times E_8'$ super-Yang-Mills theory and the ten-dimensional $N = 1$ supergravity. The four-dimensional string theory is formulated through the compactified internal space, Calabi-Yau manifolds [8], orbifolds [9], etc. The variety of internal spaces is just that of four-dimensional string theories.

Conformal field theory [10] is the main language to describe superstring theories. When a one-dimensional object, as we call it string, moves in the physical degrees of freedom of the spacetime, the gauge symmetry and the internal space, the trace of its motion becomes a connected strip, called world sheet. This world sheet is parametrized by two variables, σ as string distribution, τ as the development of time. The trace of the motion in the physical degrees of freedom is the map to target spaces from the world sheet. To quantize the physical degrees of freedom, we treat the map as the field on the world sheet. This is the first quantization of the string theory.

The physical contents of the superstring theory does not depend on the reparametrization, $(\tau,\sigma) \rightarrow (\tau'(\tau,\sigma),\sigma'(\tau,\sigma))$, and the two-dimensional gravity. This implies constraints that two-dimensional energy-momentum tensor, which is the generator of the two-dimensional reparametrization, is zero on the physical states. These constraints satisfy the Virasoro algebra. The consistency of the constraints means that the total conformal charge of matter and ghost must vanish.

In the four-dimensional superstring theory, the physical degrees of freedom other than four-dimensional spacetime are described as a compact internal space. The geometry of the internal space determines the spectra of matter in the world. Many models of the internal space are constructed by using torus models [11], minimal $N=2$ superconformal models [12], (super) coset constructions by using $N=1$ super-Kac-Moody algebras [13], Landau-Ginzburg action [14] and possible orbifold models in each case.

Almost all models except torus models are expressed by nontrivial current algebras and their representations. Bosonic degrees of freedom of the maximal commutative subalgebra of the current algebra, which is $U(1)$ current part called the maximal torus, can be described by using free scalar bosons. The remaining degrees are called parafermionic current algebras. We know such parafermionic current algebras derived from level $k>1$ Kac-Moody algebras and $c>1$, $N=2$ superconformal models.

In this paper, we consider orbifold models defined through torus models. It is because we think that orbifold models contain parafermionic current algebras and sometimes become completely equivalent to the parafermionic system without $U(1)$ current algebras.

A torus associated with some root system has automorphisms with the Weyl group whose elements are monomials of the Weyl reflections. We consider the orbifold defined as the quotient of the torus by some of the elements of the Weyl group. In the torus theory, there are $U(1)$ current states in the physical spectrum. It is, however, possible to exclude those states at all by choosing the elements of the Weyl group properly. (For example, we can choose all the elements.) In that case, we say that the orbifold model becomes the system of some parafermionic current algebras.

We will discuss Z_N orbifold model so that there are Z_N parafermionic current algebras. For the existence of Z_N parafermionic current algebras, the representation space must be finite even in the torus model through which we define the orbifold model. Therefore, we first discuss the condition of the finiteness of the representation space of general torus models in this paper.

The representation space of torus models consists of the $U(1)$ current modules beginning at the primary states defined by the vertex operators. If all the vertex operators are not contractible to a finite number of vertex operators, the representation space of the theory appears as the sum of an infinite number of the sectors $\mathcal{H}_{(i,\bar{i})}$:

$$\mathcal{H} = \sum_{(i,\bar{i})} \mathcal{H}_{(i,\bar{i})} \otimes j \otimes \bar{j}. \tag{1.1}$$

This happens in the case when the square of the radius of the torus is irrational.

For the contractibility of the representation space, it is a necessary condition that nontrivial chiral algebraic structures $\{J\}$ exist. By the nontrivial chiral algebraic structure we mean the chiral vertex operator in the physical spectrum. We show that chiral algebraic structures have integral conformal dimensions. The existence of the chiral vertex operators in the representation space restricts strongly the radius of the torus and the antisymmetric tensor field B. For instance, the square of the radius must be a rational number.

We call the model the rational torus model if the representation space is finite at last. In this case, we can rewrite the equation (1.1) by the module of the chiral algebraic structures:

$$\mathcal{H} = \sum_{(i,\bar{i})} \mathcal{H}_{(i,\bar{i})} \otimes \{j, J\} \otimes \{\bar{j}, \bar{J}\}. \tag{1.2}$$

We say that the representation space becomes finite through the enhancement of the symmetry of the model. We understand this well in the Frenkel-Kac construction of the simply-laced level 1 Kac-Moody algebras in torus models.

We will state the consistency condition of orbifold models. It restricts the antisymmetric tensor B as

$$[B, W]\Lambda \in \Lambda^*, \tag{1.3}$$

where W is the discrete rotation defining orbifold and Λ is the lattice to define the torus and Λ^* is its dual so that $\Lambda \cdot \Lambda^* \in Z$. The examples given in this paper satisfy this condition.

In order to continue the discussion towards the Z_N orbifold models, we concentrate on the models of A_{N-1} torus in this paper. The other torus models associated with the root systems of B_N, C_N, D_N, E_N, F_4 and G_2 are left to study. When there are chiral algebraic structures with conformal dimension 2, which is the same as the energy-momentum tensor, we can construct the energy-momentum tensor of the coset type as a subalgebra. If we take the radius $R = 1$ and the antisymmetric tensor field $B = 0$, then such chiral algebraic structures exist. There are parafermionic current algebras, and the representation space is finite in this case. We introduce chiral $Z_N \times Z_N$ charges $(k, \bar{k})$ of the state when the state has left-moving $U(1)$ charge belonging to $\{\lambda_k + \Lambda\}$ lattice and right-moving charge belonging to $\{\lambda_{\bar{k}} + \Lambda\}$ lattice. Parafermionic current is of charges $(2k,0)$, $(0,2k)$, $k = 1, 2, \cdots, N-1$. The left and right Z_N charges are always equal for all the physical states of the model.

We discuss the Z_N orbifold defined by a cyclic subgroup $\{1, W, W^2 \cdots, W^{N-1}\}$ derived from an element W of the order N in the Weyl group. (This is an abelian orbifold.) The representation space of the orbifold model consists of the Z_N-invariant states of the torus model. Parafermionic currents are projected into Z_N eigenstates of the action W. They are not necessarily Z_N-invariant because they are not physical operators. Next we incorporate the twisted sector in which bosons are in the twisted boundary conditions. Then we construct parafermionic current in each twisted sector.

To understand the orbifold model as the system of parafermionic current algebras may cause the request for the reconstruction of the well-known current algebras such as Kac-Moody algebras and $N = 2$ superconformal models. The reconstruction is not always possible. It may be possible to reconstruct only for the system whose total conformal charge is integral because we have only orbifold models with integral conformal charge. When we do it, we add some $U(1)$ currents, or geometrically say torus, and introduce an abelian embedding as the discrete shift and the connection through the antisymmetric tensor field B.

Some applications as examples of the reconstruction are given in chapter 4. The level 3 $\widehat{SU(3)}$ Kac-Moody algebra is constructed by the $A_2 \times A_2'/Z_3$ orbifold model [15]. The $N = 2$, $c = 3$ superconformal model is constructed by the orbifold model, $A_2 \times A_1/Z_3$. If we consider the orbifold model of $\{A_2 \times A_1\}^3/Z_3$, it may be possible to construct the internal space model of the 4-dimensional heterotic string model with the $E_6 \times E_8'$ gauge symmetry. Such a consideration should be done in the future. We discuss, however, simply the $N = 2$ superconformal algebra in the orbifold model.

In the course of the classification of conformal field theories, the method used in this paper is one of the approaches to the conformal field theories with integral conformal charges. It must be remarked that we discuss only symmetric models because we are not ready to develop asymmetric models.

The paper is organized as follows. In chapter 2 we will quantize the closed bosonic string on the torus and discuss rational torus models. In chapter 3 we discuss parafermionic current algebras in Z_N orbifold models. In chapter 4 we give some applications. Chapter 5 is devoted to summary and discussion.

2. TORUS MODELS

§1 Quantization of Closed Bosonic String on Torus

We quantize closed bosonic string on torus/orbifold correctly. A correct quantization will give cocycle factors in vertex operators naturally.

The action principle for free bosons of coordinate fields of the bosonic closed string on torus is

$$S = \int d\tau \, \frac{1}{4\pi} \int_{s-\pi}^{s+\pi} d\sigma \left\{ \eta^{ab} \partial_a X^I \partial_b X^I + \epsilon^{ab} B^{IJ} \partial_a X^I \partial_b X^J \right\}, \tag{2.1}$$

where s is a parameter which indicates the starting point in σ. B^{IJ} is a constant background antisymmetric tensor field. We introduce N-dimensional torus $\mathcal{R}^N/\Lambda$ through the equivalence relation in the position

$$X \sim X + 2\pi\Lambda, \tag{2.2}$$

where Λ is the N-dimensional lattice.

The equation of motion, $(\partial_\tau + \partial_\sigma)(\partial_\tau - \partial_\sigma)X(\tau,\sigma) = 0$, implies that bosons separate to left- and right-moving parts,

$$X(z,\bar{z}) = \frac{1}{\sqrt{2}}\{\varphi(z) + \bar{\varphi}(\bar{z})\}, \tag{2.3}$$

where $z = e^{i(\tau+\sigma)}$ and $\bar{z} = e^{i(\tau-\sigma)}$. We parametrize the fields $\varphi(z)$ and $\bar{\varphi}(\bar{z})$:

$$\varphi(z) = q_L + p_L(\tau+\sigma) + i\sum_{n\neq 0} \frac{\alpha_n}{n} e^{in(\tau+\sigma)} \tag{2.4}$$

and

$$\bar{\varphi}(\bar{z}) = q_R - p_R(\tau - \sigma) + i \sum_{n \neq 0} \frac{\bar{\alpha}_n}{n} e^{in(\tau - \sigma)}. \tag{2.5}$$

The oscillating parts are quantized as:

$$[\alpha^I_m, \alpha^J_n] = m\delta^{IJ}\delta_{n+m=0}, \quad [\bar{\alpha}^I_m, \bar{\alpha}^J_n] = m\delta^{IJ}\delta_{n+m=0}, \quad [\alpha^I_m, \bar{\alpha}^J_n] = 0. \tag{2.6}$$

This means the correlations of $< \varphi(z)\varphi(w) >\sim -log(z - w)$ for $|z| > |w|$ and $< \bar{\varphi}(\bar{z})\bar{\varphi}(\bar{w}) >\sim -log(\bar{z} - \bar{w})$ for $|\bar{z}| > |\bar{w}|$ if we work in the euclidean spacetime through the Wick rotation, $\tau \to -i\tau$.

We must carefully quantize zero modes. The center-of-mass coordinate is given by

$$\hat{X} = \frac{1}{2\pi} \int_{s-\pi}^{s+\pi} d\sigma X(\tau, \sigma)|_{\tau=0} = \frac{1}{\sqrt{2}}\{q_L + q_R + s(p_L + p_R)\}. \tag{2.7}$$

The total momentum is canonically defined as

$$\hat{P} = \int_{s-\pi}^{s+\pi} d\sigma \frac{\delta S}{\delta \dot{X}} = \frac{1}{\sqrt{2}}\{p_L - p_R + B(p_L + p_R)\}, \tag{2.8}$$

where $(Bp_R)^I = B^{IJ}p^J_R$. By the torus condition, wave functions of the string state must be invariant under the translation $X \to X + 2\pi\Lambda$. Therefore we can conclude $\hat{P} \in \Lambda^*$ dual to Λ.

By the closed string condition on the torus, $X(\sigma + 2\pi) = X(\sigma) + 2\pi\Lambda$, we define the winding number operator $\hat{L}$ by the parameters as

$$\hat{L} = \frac{1}{\sqrt{2}}(p_L + p_R) \tag{2.9}$$

and $\hat{L} \in \Lambda$. We will define the coordinate operator conjugate to the winding number $\hat{L}$. Let us first assume the commutation relations

$$[q_L^I, p_L^J] = i\delta^{IJ}, \qquad [q_R^I, p_R^J] = -i\delta^{IJ},$$

$$[q_L^I, p_R^J] = [q_R^I, p_L^J] = 0, \tag{2.10}$$

$$[p_L^I, p_R^J] = [p_R^I, p_R^J] = [p_L^I, p_L^J] = 0$$

and $[q_L^I, q_L^J], [q_R^I, q_R^J]$ and $[q_L^I, q_R^J]$ are left undetermined. These commutation relations cause no inconsistency in the quantization in the chiral theory. With the definitions (2.8) and (2.9), if we write the wave function of the torus model roughly as

$$exp[ip_L\varphi(z)]exp[-ip_R\bar{\varphi}(\bar{z})]$$

$$= exp\left[\frac{i}{\sqrt{2}}(p+l-Bl)\varphi(z)\right] exp\left[\frac{i}{\sqrt{2}}(p-l-Bl)\bar{\varphi}(\bar{z})\right]$$

$$= exp[ipX(z,\bar{z})]exp\left[il\left\{\frac{\varphi(z)-\bar{\varphi}(\bar{z})}{\sqrt{2}} + BX(z,\bar{z})\right\}\right], \tag{2.11}$$

then we can define the field $Q(z,\bar{z})$ conjugate to $\hat{L}$ by

$$Q(z,\bar{z}) = \frac{1}{\sqrt{2}}[\varphi(z) - \bar{\varphi}(\bar{z})] + BX(z,\bar{z}). \tag{2.12}$$

From this, we define the zero mode $\hat{Q}$ conjugate to $\hat{L}$ as

$$\hat{Q} = \frac{1}{2\pi}\int_{s-\pi}^{s+\pi} d\sigma Q(\tau,\sigma)|_{\tau=0} = \frac{1}{\sqrt{2}}\{q_L - q_R + s(p_L - p_R)\} + B\hat{X}$$

$$= \frac{1}{\sqrt{2}}(q_L - q_R) + B\hat{X} + s(\hat{P} - B\hat{L}). \tag{2.13}$$

This operator satisfies the commutation relation $[\hat{Q}^I, \hat{L}^J] = i\delta^{IJ}$.

The wave function of the string state with the momentum p and the winding number l on the torus must include the factor of

$$exp(ip\hat{X})exp(il\hat{Q}). \tag{2.14}$$

In order to reveal these factors in vertex operators, we have to reparametrize the field operators as

$$X(z,\bar{z}) = \hat{X} + (\hat{P} - B\hat{L})\tau + \hat{L}(\sigma - s) + Oscillators,$$

$$Q(z,\bar{z}) = \hat{Q} + (\hat{L} + B\hat{P} - BB\hat{L})\tau + \hat{P}(\sigma - s) + Oscillators. \tag{2.15}$$

We see that these operators do not depend on σ but on $(\sigma - s)$. From this, we understand that for any s, the string is described so that the starting point is

$$X(s-\pi)|_{\tau=0} = \hat{X} - \pi\hat{L}. \tag{2.16}$$

Now we see that the configurations of the string states are independent of s. This implies that the quantization of the zero modes $\hat{X}, \hat{P}, \hat{Q}$ and $\hat{L}$ does not depend on s. Therefore, the complete wave functions are in the form

$$\Psi(p,l) = exp\{ipX(z,\bar{z})\}exp\{ilQ(z,\bar{z})\}. \tag{2.17}$$

We discuss a correct quantization of the remaining commutation relations. With the definitions of the zero modes, we introduce the canonical commutation relations

$$[\hat{X}^I, \hat{P}^J] = i\delta^{IJ}, \quad [\hat{Q}^I, \hat{L}^J] = i\delta^{IJ},$$

$$[\hat{P}^I, \hat{L}^J] = 0, \qquad [\hat{X}^I, \hat{L}^J] = 0, \tag{2.18}$$

$$[\hat{Q}^I, \hat{P}^J] = 0, \qquad [\hat{X}^I, \hat{Q}^J] = T^{IJ}.$$

We here introduce a tensor T^{IJ} formally which is usually taken to be zero. First we redefine the zero modes $\hat{q}_L$ and $\hat{q}_R$ independent of s:

$$\hat{q}_L = \frac{\hat{X} + \hat{Q} - B\hat{X}}{\sqrt{2}}, \qquad \hat{q}_R = \frac{\hat{X} - \hat{Q} + B\hat{X}}{\sqrt{2}}. \tag{2.19}$$

We want to discuss the commutation relations of these operators. By using these relations(2.19) and (2.18), we have now

$$2[\hat{q}_L^I, \hat{q}_L^J] = B^{JK}T^{KI} - B^{IK}T^{KJ} + T^{IJ} - T^{JI},$$

$$2[\hat{q}_R^I, \hat{q}_R^J] = B^{JK}T^{KI} - B^{IK}T^{KJ} - T^{IJ} + T^{JI}, \tag{2.20}$$

$$2[\hat{q}_L^I, \hat{q}_R^J] = -B^{JK}T^{KI} + B^{IK}T^{KJ} - T^{IJ} - T^{JI}.$$

We will determine the form of the tensor T^{IJ}.

If we suppose the interaction of the three strings in the covering space where the configurations of the string states are presented, it is very natural for us to have an idea that starting point of the string configuration does not move when the two strings link up to become the other string. We accept the idea and use it as a principle for the quantization.

Let us consider the string state of the winding number $\hat{L} = 0$ and at the position of the center of mass coordinate $\hat{X} = x$. If another string state of $\hat{L} = l$ and $\hat{X} = x + \pi l$ connects with the state at the position x, the linked string state is not of $\hat{X} = x + x + \pi l$ but of $\hat{X} = x + \pi l$. We can represent it in the level of the first quantization. It follows that when we operate $e^{il\hat{Q}}$ to the state $|\hat{X} = x, \hat{L} = 0>$, we get a state $|\hat{X} = x + \pi l, \hat{L} = l>$.

We can explain it in another way. The starting point of the string state $\hat{X} - \pi\hat{L}$ does not move when the operator $e^{il\hat{Q}}$ acts to the state. We conclude that if $\hat{L}$ increases by l, then $\hat{X}$ must also increase by πl. As a result we have

$$T^{IJ} = -i\pi\delta^{IJ}. \tag{2.21}$$

Finally we conclude that

$$\begin{aligned} &[\hat{q}_L^I, \hat{q}_L^J] = i\pi B^{IJ}, \quad [\hat{q}_R^I, \hat{q}_R^J] = i\pi B^{IJ}, \\ &[\hat{q}_L^I, \hat{q}_R^J] = -i\pi(B^{IJ} - \delta^{IJ}). \end{aligned} \tag{2.22}$$

These nontrivial commutation relations are the natural origin of cocycle factors in the vertex operators. We think that this is a physical interpretation of cocycle factors which is already discussed by Sakamoto [16].

In the twisted sector of orbifold case, the "zero modes" are constrained by the fixed point equation

$$\hat{P} = \frac{p_L - p_R}{\sqrt{2}} + B\hat{L} = 0, \tag{2.23}$$

$$\hat{X}^I_{f.p.} = (W^l)^{IJ}\hat{X}^J_{f.p.} + 2\pi\hat{L}^I \tag{2.24}$$

in W^l-sector. The "zero modes" are expressed in the very complicated forms containing the oscillating modes and the zero modes in the twisted sector. Anyway, the "zero modes" are given and we will quantize them. Using eq.(2.24) and the commutation relation $[\hat{Q}^I, \hat{L}^J] = i\delta^{IJ}$, we derive

$$[\hat{X}^I, \hat{Q}^J] = -i2\pi\left(\frac{1}{1-W^l}\right)^{IJ}. \tag{2.25}$$

This is equivalent to the commutation relations:

$$[\hat{q}^I_L, \hat{q}^J_L] = i\pi\left\{(1-B)^{JM}\left(\frac{1}{1-W^l}\right)^{MI} - (1-B)^{IM}\left(\frac{1}{1-W^l}\right)^{MJ}\right\},$$

$$[\hat{q}^I_R, \hat{q}^J_R] = i\pi\left\{-(1+B)^{JM}\left(\frac{1}{1-W^l}\right)^{MI} + (1+B)^{IM}\left(\frac{1}{1-W^l}\right)^{MJ}\right\},$$

$$[\hat{q}^I_L, \hat{q}^J_R] = i\pi\left\{(1+B)^{JM}\left(\frac{1}{1-W^l}\right)^{MI} + (1-B)^{IM}\left(\frac{1}{1-W^l}\right)^{MJ}\right\},$$

$$[\hat{q}^I_L, p^J_L] = \frac{i}{2}(1-B)^{JI}, \qquad [\hat{q}^I_L, p^J_R] = \frac{i}{2}(1+B)^{JI},$$

$$[\hat{q}^I_R, p^J_R] = \frac{i}{2}(1+B)^{JI}, \qquad [\hat{q}^I_R, p^J_L] = \frac{i}{2}(1-B)^{JI}. \tag{2.26}$$

This result has been already discussed by Itoh *et al.* [17].

Correct cocycle factors are realized naturally in vertex operators by the relations eq.(2.22) and eq.(2.26) in untwisted- and twisted-sectors, respectively.

§2 Rational Torus

We discuss wave functions on the torus and the condition of the rational torus. The representation space of the torus model consists of vertex operators and their module of the $U(1)$ currents in the model. The $U(1)$ currents are themselves physical and chiral operators. We call a physical and chiral operator a chiral algebraic structure. Therefore, the number of vertex operators, which are representatives of the equivalence classes of the multiplicatives of the $U(1)$ currents, is just the size of the representation space if there is not a chiral algebraic structure other than $U(1)$ currents.

In general torus models, vertex operators are made from the products of the fundamental vertex operators such as

$$\Psi(p = v_i^*, l = 0), \qquad \Psi(p = 0, l = v_i), \tag{2.27}$$

where v_i^* and v_i are the basis vectors of the lattices Λ^* and Λ, respectively, and the vectors satisfy $v_i^* \cdot v_j = \delta_{ij}$, $i, j = 1, \cdots, N$. In a sense, this is the integrability property of the torus model.

However, we want to search models with the integrability as a conformal field theory. A conformal field theory is factorizable into the holomorphic and antiholomorphic sectors. The integrability as a conformal field theory means the integrability in each sector. Then it is a necessary condition that the nontrivial chiral algebraic structures, that is, chiral vertex operators in the physical spectrum, exist.

We now have the physical wave functions Ψ on torus with the momentum $p \in \Lambda^*$ and the winding number $l \in \Lambda$:

$$\Psi(p,l) = exp\{i(pX(z,\bar{z}) + lQ(z,\bar{z}))\}$$

$$= exp\left\{i\frac{p+l-Bl}{\sqrt{2}}\varphi(z)\right\} exp\left\{i\frac{p-l-Bl}{\sqrt{2}}\bar{\varphi}(\bar{z})\right\}, \qquad (2.28)$$

which is invariant under the shift $X \to X + 2\pi\Lambda$ and $Q \to Q + 2\pi\Lambda^*$. We call these wave functions Ψ vertex operators in conformal field theory. The set of $\{\Psi\}$ is the representation space of the conformal field theory of bosons on torus.

If $p - l - Bl = 0$ for a $\Psi(p,l)$, then $\Psi(p,l)$ is independent of $\bar{z}$ and we call it chiral algebra $J(z)$:

$$J(z) = exp(i\sqrt{2}l\varphi(z)). \qquad (2.29)$$

This current has positive integral conformal dimension

$$d = l \cdot l = l \cdot (p - Bl) = l \cdot p \in Z_+. \qquad (2.30)$$

Similarly, $\bar{J}(\bar{z})$'s are defined. It is well known that a simply-laced level 1 Kac-Moody algebra is realized in the case of $d = 1$ as Frenkel-Kac construction [18].

If we introduce equivalence relations of $J(z) \sim 1$ and $\bar{J}(\bar{z}) \sim 1$, then we can classify $\{\Psi\}$ into the equivalence classes. We can take a representative of each equivalence class so that it has the smallest conformal dimension. Then we express the set of the representatives as $\{\Psi\}/\{J, \bar{J}\}$. Since there are the physical currents $j = i\partial\varphi(z)$, $\bar{j} = i\bar{\partial}\bar{\varphi}(\bar{z})$ in torus models, which are not always physical in orbifold model, we get $\{\Psi/J\} = \{\Psi\}/\{J,\, \bar{J},\, j,\, \bar{j}\}$. The existence of nontrivial chiral algebras is a necessary condition for the finiteness of the representation space.

We consider torus $\mathcal{R}^N/\Lambda$. Λ is the direct sum of simple Lie lattice Λ_i with a radius R_i, $\Lambda = \oplus\Lambda_i R_i$. Then the equation $p - l - Bl = 0$ is expressed as

$$m_i = \sum_{j=1}^{N}[R_iR_j(\alpha_i \cdot \alpha_j) + B_{ij}]n_j, \tag{2.31}$$

where $p = \lambda_i m_i R_i^{-1} \in \Lambda^*$, $l = \alpha_i n_i R_i \in \Lambda$, $B^{IJ} = B_{ij}R_i^{-1}R_j^{-1}\lambda_i^I\lambda_j^J$ and α_i is a simple root and λ_i is a fundamental weight of the weight lattice and $n_i, m_i \in Z$. These equations are constraints on B_{ij}. The equation (2.30) becomes

$$d = \sum_{(i,j)}(\alpha_i \cdot \alpha_j)n_iR_in_jR_j = \sum_i m_in_i. \tag{2.32}$$

We must solve these equations.

§3 A_{N-1} type Rational Torus

We are now at the stage to study a special case, A_{N-1} torus. We consider the simplest case, $R = 1$ and $B = 0$. Then a wave function of the model is

$$\Psi(p = m_i\lambda_i, l = n_i\alpha_i) = exp[i(pX(z,\bar{z}) + lQ(z,\bar{z}))]$$

$$= exp\left[i\frac{m_i\lambda_i + n_i\alpha_i}{\sqrt{2}}\varphi(z)\right] exp\left[i\frac{m_i\lambda_i - n_i\alpha_i}{\sqrt{2}}\bar{\varphi}(\bar{z})\right]. \tag{2.33}$$

In this case, we have chiral algebraic structures

$$J^\alpha(z) = exp(i\sqrt{2}\alpha \cdot \varphi(z)) \tag{2.34}$$

and antichiral algebraic structures

$$\bar{J}^{\alpha}(\bar{z}) = exp(i\sqrt{2}\alpha \cdot \bar{\varphi}(\bar{z})) \tag{2.35}$$

for all the root vectors α.

Finiteness of this model is stated in the following. For A_{N-1} case, the dual lattice Λ^* is made of the composition of root lattice Λ:

$$\Lambda^* = \{0+\Lambda\} \cup \{\lambda_1+\Lambda\} \cup \{\lambda_2+\Lambda\} \cup \cdots \cup \{\lambda_k+\Lambda\} \cup \cdots \cup \{\lambda_{N-1}+\Lambda\}. \tag{2.36}$$

We assign Z_N charge k to each sublattice $\{\lambda_k+\Lambda\}$.

Let a, b are the vectors in the lattice $\{\lambda_k+\Lambda\}$. If we take $p = a+b$, $l = a-b \in \Lambda$, then

$$\Psi(p,l) = exp\left[i\frac{2a}{\sqrt{2}}\varphi\right] exp\left[i\frac{2b}{\sqrt{2}}\bar{\varphi}\right] \tag{2.37}$$

is primary up to J and $\bar{J}$. Let us introduce nonlocal current algebra as

$$\psi_L^a(z) = exp\left[i\frac{2a}{\sqrt{2}}\varphi(z)\right], \qquad \psi_R^b(\bar{z}) = exp\left[i\frac{2b}{\sqrt{2}}\bar{\varphi}(\bar{z})\right], \tag{2.38}$$

which have Z_N charge $(2k,0)$ and $(0,2k)$ and conformal dimension $(\frac{k(N-k)}{N},0)$ and $(0,\frac{k(N-k)}{N})$, respectively. This wave function $\Psi(p,l)$ corresponds to $\epsilon^{(a,b)}_{(2k,2k)}$ state in the Z_N parafermionic system [19]. This is a excited state of SL_2 invariant vacuum by parafermionic current ψ_L^a and ψ_R^b.

If we take $p = a$, $l = a-b \in \Lambda$, then

$$\Psi(p,l) = exp\left[i\frac{2(a-b)+b}{\sqrt{2}}\varphi\right] exp\left[i\frac{b}{\sqrt{2}}\bar{\varphi}\right]$$

$$\sim J(a-b)\Psi(b,0). \tag{2.39}$$

$\Psi(b,0)$ is primary up to J and $\bar{J}$. This wave function has Z_N charge (k,k) and conformal dimension $(\frac{k(N-k)}{4N}, \frac{k(N-k)}{4N})$. This corresponds to spin state $\sigma^{(b,b)}_{(k,k)}$ in the parafermionic system.

If we take $p = l_1 - l_2 \in \Lambda$, $l = l_2 \in \Lambda$, then

$$\Psi(p,l) = exp\left[i\frac{l_1}{\sqrt{2}}\varphi\right] exp\left[i\frac{-2l_2+l_1}{\sqrt{2}}\bar{\varphi}\right]$$

$$\sim \bar{J}(-l_2)\Psi(l_1,0). \tag{2.40}$$

$\Psi(l_1,0)$ is primary up to J and $\bar{J}$. This wave function has Z_N charge $(0,0)$, Z_N neutral, and conformal dimension $(\frac{l_1^2}{4}, \frac{l_1^2}{4})$. This corresponds to Z_N neutral state, $\epsilon^{(l_1,l_1)}_{(0,0)}$, in the parafermionic system.

To summarize these three statements, we conclude that there are no other types of wave functions in the theory.

3. ORBIFOLD MODELS OF A_{N-1}/Z_N

§1 Orbifold Models

We consider the orbifold which are described as the quotient of the torus by the discrete group $\{W\}$, as $X \sim WX$. $\{W\}$ is a subgroup of the Weyl group of the torus. The Weyl group is generated by Weyl reflections. The Weyl reflection Γ_i is the reflection with respect to the hyperplane orthogonal to the simple root α_i:

$$\Gamma_i(x) = x - \frac{2(\alpha_i \cdot x)}{\alpha_i^2}\alpha_i. \tag{3.1}$$

We here discuss the representation space of orbifold models. Now we must take account of the orbit of vertex operators or string states with respect to W.

Let us suppose that there is a string state with the momentum p and the winding number l. It is the vertex operator:

$$\Psi(p,l) = exp\left[i\frac{p+l-Bl}{\sqrt{2}}\varphi\right] exp\left[i\frac{p-l-Bl}{\sqrt{2}}\bar{\varphi}\right]. \tag{3.2}$$

If we act W on the state as $(\varphi,\bar{\varphi}) \to (W\varphi, W\bar{\varphi})$, then we will have a vertex operator:

$$\begin{aligned} W\Psi(p,l) &= exp\left[i\frac{p+l-Bl}{\sqrt{2}}(W\varphi)\right] exp\left[i\frac{p-l-Bl}{\sqrt{2}}(W\bar{\varphi})\right] \\ &= exp\left[i\frac{W^{-1}p+W^{-1}l-W^{-1}Bl}{\sqrt{2}}\varphi\right] \\ &\quad\times exp\left[i\frac{W^{-1}p-W^{-1}l-W^{-1}Bl}{\sqrt{2}}\bar{\varphi}\right] \qquad (3.3) \\ &= \Psi(W^{-1}p-[W^{-1},B]l, W^{-1}l) = \Psi(p',l'). \end{aligned}$$

We see that the W transformation of $(\varphi, \bar{\varphi})$ is equivalent to the transformation [20] of the vectors (p, l) as

$$\begin{cases} p \to p' = W^{-1}p - [W^{-1}, B]l \\ \\ l \to l' = W^{-1}l. \end{cases} \tag{3.4}$$

The orbifold model is consistent if and only if

$$[W, B]l \in \Lambda^{*} \tag{3.5}$$

for all $l \in \Lambda$ and W, because every W-orbit must belong to the spectrum of the torus model, $p' \in \Lambda^{*}$.

Full contents of the representation space of orbifold are obtained through the projection with respect to all the elements of $\{W\}$, P_N^W of all wave functions in torus model, $\{\Psi/J\}$:

$$P_N^W = \frac{1}{N}\sum_{i=0}^{N-1} W^i, \tag{3.6}$$

where N is the order of the element W, $W^N = 1$. We consider the abelian orbifold in this paper so that $\{W\}$ has an element W and its multiples. This $\{W\}$ is abelian and called a cyclic group.

§2 Parafermionic Current Algebra of A_{N-1}/Z_N Orbifold: Untwisted Sector

Since we have investigated the representation space of the A_{N-1} torus models in the last chapter, we are ready to discuss Z_N orbifold models. Let us start with an explicit definition of Z_N orbifold. The torus A_{N-1} is defined by the relation

$$X \sim X + 2\pi\Lambda, \tag{3.7}$$

where Λ is the lattice generated by the simple root system of A_{N-1}, $\alpha_1, \cdots, \alpha_{N-1}$. An element of Weyl group $W = \Gamma_1\Gamma_2\cdots\Gamma_{N-1}$ transposes vectors as

$$W : \alpha_i \rightarrow \alpha_{i+1} \qquad (\alpha_N = \alpha_0 = -\alpha_1 - \cdots - \alpha_{N-1}), \tag{3.8}$$

so that W is the discrete rotation of order N. Then we have Z_N orbifold defined as the quotient of the element W. The wave function of the string living in the Z_N orbifold must be invariant under this rotation.

In the untwisted sector of Z_N orbifold, we apply the same quantization as in the torus model. Full contents of the representation space of Z_N orbifold are given by Z_N projection P_N of all wave functions in torus model, $\{\Psi/J\}$:

$$P_N = \frac{1}{N}\sum_{i=0}^{N-1} W^i, \tag{3.9}$$

where $W^N = 1$. We have now only the projected chiral algebraic structures. Therefore, $U(1)$ currents are excluded out of the structures.

To show parafermionic structure, we introduce Z_N projection of nonlocal parafermionic currents and $U(1)$ currents. Nonlocal operators admit Z_N phase. Z_N projection into Z_N eigenstates is defined as

$$P_N^j = \frac{1}{N}\sum_{l=0}^{N-1} \theta^{-lj} W^l, \quad \theta = e^{\frac{2\pi i}{N}}. \tag{3.10}$$

By these projections, we have parafermionic currents from the vertex operators of eq. (2.38):

$$\psi_k^j = P_N^j \psi(\lambda_k), \qquad W\psi_k^j = \theta^j \psi_k^j \tag{3.11}$$

and projected coordinates:

$$Y^j = P_N^j(\lambda_1 \cdot \varphi), \qquad WY^j = \theta^j Y^j. \tag{3.12}$$

We have no Z_N invariant $U(1)$ current because trivially $Y^0 = 0$. Every parafermionic current ψ_k^j has its energy-momentum tensor T_j which is introduced by the operator product expansion

$$\psi_k^j(z)(\psi_k^j)^\dagger(w) = \psi_k^j(z)\psi_{N-k}^{-j}(w)$$

$$= \frac{1}{(z-w)^{2d_k}} \left\{ 1 + (z-w)^2 \frac{2\Delta}{c} T_j(w) + o(z-w)^3 \right\}, \tag{3.13}$$

where Δ, equal to $d_k = \frac{k(N-k)}{N}$, is the conformal dimension of ψ measured by energy-momentum tensor T_j and c is the conformal charge of the Virasoro algebra of T_j. We have an explicit form of the energy-momentum tensor

$$T_j(z) = \frac{1}{N+2} \left\{ (i\partial\varphi)^2 + \sum_{\alpha>0} \theta^j P_N J(\alpha) + \theta^{-j} P_N J(-\alpha) \right\}. \tag{3.14}$$

For each j, energy-momentum tensor satisfies the Virasoro algebra of central charge $c = \frac{2N-2}{N+2}$. This means that parafermionic current projected into every direction and every phase is just like a system of Z_N parafermionic algebra. But totally the system is of the energy-momentum tensor $\frac{1}{2}(i\partial\varphi)^2$ with conformal charge $c_{total} = N - 1$. The most of parafermionic current algebras are not closed except for Z_N invariant sector, $\{\psi^0_{k=1,\cdots,N-1}, T_0\}$.

We can explicitly calculate the structure constants of the Z_N parafermionic current algebra by using our bosonic representation. That will coincide with the result given by Zamolodchikov and Fateev [19]. We have already verified the coincidence in the Z_2 and Z_3 cases [21].

§3 Parafermionic Current Algebra of A_{N-1}/Z_N Orbifold: Twisted Sector

We now discuss the twisted sectors in Z_N orbifold. A closed string around the fixed point on orbifold in W^l-sector ($l = 1, \cdots, N-1$) satisfies the boundary condition

$$X(\sigma + 2\pi) = W^l X(\sigma) + 2\pi\Lambda \tag{3.15}$$

on torus T^{N-1}. Then $Y^j(e^{2\pi i}) \sim W^l Y^j(z) = \theta^{lj} Y^j(z)$. They are expanded in this sector as

$$\begin{aligned} Y^j(z) &= y_j + p_j(\tau + \sigma) + i\sum_n \frac{\alpha_{n-\frac{lj}{N}}}{n - \frac{lj}{N}} z^{-n+\frac{lj}{N}}, \\ \bar{Y}^j(\bar{z}) &= \bar{y}_j - p_j(\tau - \sigma) + i\sum_n \frac{\beta_{n+\frac{lj}{N}}}{n + \frac{lj}{N}} \bar{z}^{-n-\frac{lj}{N}}. \end{aligned} \tag{3.16}$$

The oscillation parts are quantized as

$$[\alpha_{m-\frac{lj}{N}}, \alpha_{n-\frac{l(N-j)}{N}}] = \left(m - \frac{lj}{N}\right)\delta_{m+n=l}, \tag{3.17}$$

$$[\beta_{m-\frac{lj}{N}}, \beta_{n-\frac{l(N-j)}{N}}] = \left(m - \frac{lj}{N}\right)\delta_{m+n=l}. \tag{3.18}$$

Then we get vacuum energy of the primary states σ_l, which is the twisted boson's vacuum,

$$v_l = \sum_{j=1}^{N-1} \frac{1}{4} \left[\frac{lj}{N}\right] \left(1 - \left[\frac{lj}{N}\right]\right), \tag{3.19}$$

where $[x]$ means subtraction of the integral parts of x.

In the twisted sector, the wave functions are of almost the same form as in the untwisted sector. But the absence of the zero mode momentum causes a little modification of vertex operators as $\Psi(p,l) \rightarrow \tilde{\Psi}(p,l) = z^{-h}\bar{z}^{-\bar{h}} exp\{ip \cdot X(z,\bar{z})\} exp\{il \cdot Q(z,\bar{z})\}$, where h and $\bar{h}$ are the conformal dimensions of the operators.

Chiral algebraic structures in the twisted sector,

$$\begin{aligned} \tilde{J}(\alpha) &= z^{-2} exp(i\sqrt{2}\alpha \cdot \varphi) \\ &= \tilde{\Psi}(p = \alpha, l = \alpha) \end{aligned} \tag{3.20}$$

cause discrete translation of the fixed points, $X \rightarrow X + x_{f.p.}$, where $x_{f.p.}$ is a solution of the fixed point equation, $x_{f.p.} = W^l x_{f.p.} + 2\pi l$.

We have the twisted parafermionic current

$$\tilde{\psi}^j_k = \frac{1}{z^{d_k}} P^j_N exp(i\sqrt{2}\lambda_k \cdot \varphi(z)). \tag{3.21}$$

This produces energy-momentum tensor in the twisted W^l sector:

$$\tilde{T}_j = \frac{1}{N+2} \left\{ (i\partial\varphi)^2 + \frac{2v_l}{z^2} + \sum_{\alpha>0} P_N \theta^j \tilde{J}(\alpha)\epsilon(j,l) + P_N \theta^{-j} \tilde{J}(-\alpha)\epsilon(j,l)^{-1} \right\}, \tag{3.22}$$

where $\epsilon(j,l)$ is the phase factor generated by the twisted oscillators. The vacuum state of the twisted boson is at the fixed point $x_{f.p.}$. Then we have

$$< \tilde{J}(\alpha) >\text{vacuum}= \frac{exp(i\sqrt{2}\alpha \cdot x_{f.p.})}{z^2}. \tag{3.23}$$

Therefore the conformal dimension, measured by $\tilde{T}_j$, of the vacuum state is

$$(\tilde{T}_j)^0 = \frac{1}{N+2}\left\{2v_l + \sum_{\alpha>0} P_N \theta^j exp(i\sqrt{2}\alpha \cdot x_{f.p.})\epsilon(j,l) \right.$$
$$\left. + P_N \theta^{-j} exp(i\sqrt{2}\alpha \cdot x_{f.p.})\epsilon(j,l)^{-1}\right\}. \tag{3.24}$$

From this expression, we conclude that the primary states, as the parafermionic current algebra in each projected sector, have different conformal dimensions in every fixed point. It follows that the spin structures of the parafermionic currents are different in each fixed point. The total vacuum energy is calculated as

$$\frac{N+2}{2}\cdot\frac{1}{2N}\sum_{j=0}^{N-1}(\tilde{T}_j)^0 = v_l \tag{3.25}$$

and this coincides with boson's vacuum energy as we expected. We conclude that in the twisted sector, the spin structure of the every parafermionic current algebra is not equal and is very complicated.

4. APPLICATIONS

In the last chapter, we have discussed orbifold models, which is defined as the quotient of the torus by the discrete rotation and in which there are no $U(1)$ chiral algebraic structures. We have investigated purely parafermionic system so far. In this chapter, we will discuss orbifold models with some $U(1)$ chiral algebraic structures. We consider the models of the type $(Parafermion) \otimes (Torus)$.

It is well known that with Z_k parafermionic current algebra, which was introduced by Zamolodchikov and Fateev [19], and a single boson on S^1 , one can construct level k $\widehat{SU(2)}$ Kac-Moody algebra or level k, $N = 2$ minimal superconformal model with conformal charge $c = \frac{3k}{k+2}$. However, we want to ask questions, why the current algebraic structures, e.g. Kac-Moody algebra, exist and how the radius of torus is determined. We think that it is not easy to answer in the algebraic approach to the geometry of models because the question may be asked at the final stage of the investigation of the models in the approach. Conversely we start from the geometry, called orbifold, and then we investigate algebraic structures of the model in this paper. Therefore, we might be able to answer the questions in our approach.

First we will give the realization of level 3 $\widehat{SU(3)}$ Kac-Moody algebra [15]. Next we will construct the $N = 2$ superconformal algebra.

§1 Level 3 $\widehat{SU(3)}$ Kac-Moody Algebra

Now we will construct level 3 $\widehat{SU(3)}$ Kac-Moody algebra by an orbifold model $A_2 \times A_2'/Z_3$ with the conformal charge 4.

We define the torus $A_2 \times A_2'$ by $\mathcal{R}^4/\Lambda$. Λ is the direct sum of the lattices, $\Lambda = R\Lambda_{A_2} \oplus r\Lambda_{A_2'}$. We define Z_3 orbifold by the Z_3 operation, the order 3 discrete rotation W, $W^3 = 1$, in the torus A_2 and the order 3 discrete shift V, $3V \in \Lambda_{A_2'}$, in the torus A_2', as

$$(X, X') \sim (WX + 2\pi R\Lambda_{A_2}, X' + 2\pi rV), \tag{4.1}$$

where X and X' are the coordinates on the covering spaces of the torus A_2 and A_2', respectively.

A wave function on torus is

$$\Psi(p,l) = e^{ipX}e^{ilQ} = exp(i\frac{p+l-Bl}{\sqrt{2}}\varphi)exp(i\frac{p-l-Bl}{\sqrt{2}}\bar{\varphi}). \tag{4.2}$$

We here introduce the notation of vectors of the lattice Λ and Λ^* as

$$p = p_{A_2} + p_{A_2'} \quad \begin{cases} p_{A_2} = R^{-1}(m_1\lambda_1 + m_2\lambda_2) \\ p_{A_2'} = r^{-1}(m_3\lambda_3 + m_4\lambda_4), \end{cases} \tag{4.3}$$

$$l = l_{A_2} + l_{A_2'} \quad \begin{cases} l_{A_2} = R(n_1\alpha_1 + n_2\alpha_2) \\ l_{A_2'} = r(n_3\alpha_3 + n_4\alpha_4), \end{cases} \tag{4.4}$$

where all n_i and m_i are integers. λ_1 and λ_2 are the fundamental weight of $\Lambda^*_{A_2}$ and α_1 and α_2 are the simple roots of Λ_{A_2}. λ_3 and λ_4 are the fundamental weight of $\Lambda^*_{A_2'}$ and α_3 and α_4 are the simple roots of $\Lambda_{A_2'}$.

If Kac-Moody algebra exists in this Z_3-orbifold model, there must be a chiral algebraic structure with dimension 1. We require the chirality condition $p - l - Bl = 0$ and $l^2 = 1$. Since we attempt to realize the level 3 algebra with the structure of (parafermion)×(vertex on torus), the chiral currents are made of A_2 part with dimension $l_{A_2}^2 = 2/3$ and A_2' part with dimension $l_{A_2'}^2 = 1/3$. Then we have the equations

$$1 = l \cdot p = \sum_{i=1,2,3,4} m_i n_i, \tag{4.5}$$

$$\frac{2}{3} = l_{A_2}^2 = 2R^2(n_1^2 + n_2^2 - n_1 n_2), \tag{4.6}$$

$$\frac{1}{3} = l_{A_2'}^2 = 2r^2(n_3^2 + n_4^2 - n_3 n_4), \tag{4.7}$$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \left[\begin{pmatrix} 2R^2 & -R^2 & 0 & 0 \\ -R^2 & 2R^2 & 0 & 0 \\ 0 & 0 & 2r^2 & -r^2 \\ 0 & 0 & -r^2 & 2r^2 \end{pmatrix} + \begin{pmatrix} 0 & B_{12} & B_{13} & B_{14} \\ B_{21} & 0 & B_{23} & B_{24} \\ B_{31} & B_{32} & 0 & B_{34} \\ B_{41} & B_{42} & B_{43} & 0 \end{pmatrix} \right] \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix}. \tag{4.8}$$

We take account of the consistency condition (3.5) of the orbifold model. In this case, $[W, B]l \in \Lambda^*$ implies that $3B_{A_2A_2'} \in Z$. If we assume that a chiral algebraic structure for $\vec{n} = (1\ 0\ 1\ 0)$ and $\vec{m} = (1\ 0\ 0\ 0)$ and W-orbits of it exist, then we find a solution,

$$R^2 = \frac{1}{3}, \quad r^2 = \frac{1}{6} \tag{4.9}$$

and

$$B_{12} = 0, \quad B_{34} = -\frac{1}{2}, \quad B_{13} = B_{14} = B_{23} = B_{24} = \frac{1}{3}. \tag{4.10}$$

We have the general forms of the spectrum of the torus model in the untwisted sector:

$$\begin{aligned} \sqrt{2}p_L &= p + l - Bl \\ &= \sqrt{3}\{(m_1 - \tfrac{n_3+n_4}{3} + \tfrac{2n_1-n_2}{3})\lambda_1 + (m_2 - \tfrac{n_3+n_4}{3} + \tfrac{2n_2-n_1}{3})\lambda_2\} \\ &\quad +\sqrt{6}\{(m_3 + \tfrac{n_1+n_2}{3} + \tfrac{n_3+n_4}{3})\lambda_3 + (m_4 + \tfrac{n_1+n_2}{3} + \tfrac{n_4-2n_3}{3})\lambda_4\}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} \sqrt{2}p_R &= p - l - Bl \\ &= \sqrt{3}\{(m_1 - \tfrac{n_3+n_4}{3} - \tfrac{2n_1-n_2}{3})\lambda_1 + (m_2 - \tfrac{n_3+n_4}{3} - \tfrac{2n_2-n_1}{3})\lambda_2\} \\ &\quad +\sqrt{6}\{(m_3 + \tfrac{n_1+n_2}{3} - \tfrac{n_3+n_4}{3})\lambda_3 + (m_4 + \tfrac{n_1+n_2}{3} - \tfrac{n_4-2n_3}{3})\lambda_4\}. \end{aligned}$$

General wave functions on the orbifold are obtained by Z_3 projection P_3 of torus model. P_3 is defined by the operation (W, V). We can find Kac-Moody algebra in $\{\Psi\}$. Finally level 3 $\widehat{SU(3)}$ Kac-Moody algebra is of the form

$$
\begin{aligned}
E^{\alpha_3} &= P_3 exp\left\{ i(\sqrt{\frac{2}{3}}\alpha_1 + \sqrt{\frac{1}{3}}\alpha_3)\varphi \right\}, \\
E^{\alpha_4} &= P_3 exp\left\{ i(\sqrt{\frac{2}{3}}\alpha_1 + \sqrt{\frac{1}{3}}\alpha_4)\varphi \right\}, \qquad (4.12) \\
H_i &= \sqrt{3} i\partial\varphi_i, \qquad i = 3, 4.
\end{aligned}
$$

We remark that the step operators are just in the form of (parafermion)×(vertex on torus) that has been discussed by Gepner [22]. By eq.(4.11), all the wave functions in the untwisted sector are in the representations of level 3 $\widehat{SU(3)}$ Kac-Moody algebra:

	Repr.	dimensions
ω^0-sector;	1	0
		1/2
		1
		1.

(4.13)

Since now we do not discuss only a subalgebra but the total algebra, we have the energy-momentum tensor of the Virasoro form $T = \frac{1}{2}j_{A_2}^2 + \frac{1}{2}j_{A_2'}^2$ in the untwisted sector, where $j_{A_2} = i\partial\varphi_{A_2}$ and $j_{A_2'} = i\partial\varphi_{A_2'}$. Therefore the conformal dimension of the primary state is derived as $(\frac{p_L^2}{2}, \frac{p_R^2}{2})$.

We next incorporate the twisted sector. In the twisted sector, we have the energy momentum tensor

$$
T = \frac{1}{2}j_{A_2}^2 + \frac{1}{9z^2} + \frac{1}{2}j_{A_2'}^2. \qquad (4.14)
$$

A shift vector V, which generates a shift in $X_{3,4}$-space, can be taken as

$$rV = \frac{1}{3\sqrt{6}}(\alpha_4 - \alpha_3) = \frac{1}{\sqrt{6}}(\lambda_4 - \lambda_3) \tag{4.15}$$

so that V is compatible with Z_3 invariance of Kac-Moody algebra. With this choice, we have the physical wave functions $\Psi(p, kV + l)$ in W^k-sector. $\Psi(p, kV + l)$ belongs to the state of $SU(3)_L \times SU(3)_R$ charge $(H, \bar{H}) \in (k\lambda_4 + \Lambda_{root}, k\lambda_4 + \Lambda_{root})$, where H is left-moving $SU(3)$ charge defined in eq.(4.12) and $\bar{H}$ is similarly defined.

We attempt to explain fixed points. There are three fixed points O, A and B on orbifold in which $X_{fp} \in 2\pi R\Lambda_{root}$, $2\pi R(\Lambda_{root} + \lambda_1)$, $2\pi R(\Lambda_{root} + 2\lambda_1)$, respectively. Since we have quantized as $[X^I, Q^J] = -i2\pi \left(\frac{1}{1-W}\right)^{IJ}$ in the twisted sector, a wave function $\Psi(p, l)$ causes a shift

$$X \rightarrow X + x_{fp}(l) \tag{4.16}$$

of the state, where $x_{fp}(l)$ is a fixed point associated with l, a solution of the constraint eq.(2.24). Therefore, the step operators, E^{α_3}, E^{α_4} and $E^{-\alpha_3-\alpha_4}$, which have $l_{A_2} \in (3\Lambda_{root} + \alpha_1)$, translate fixed points as O $\rightarrow$ A, A $\rightarrow$ B and B $\rightarrow$ O and their conjugates translate fixed points in the opposite direction.

We have now obtained all the other representations of level 3 $\widehat{SU(3)}$ Kac-Moody algebra:

	Repr.	dimensions
ω^1-sector;		2/9
		5/9
		8/9,
ω^2-sector;		2/9
		5/9
		8/9.

(4.17)

These dimensions agree with the formula

$$h(\lambda) = \frac{C(\lambda)}{x\psi^2 + C_2}, \tag{4.18}$$

where $C(\lambda)$ is the quadratic Casimir of the representation and x is the level and ψ is the highest root and C_2 is the quadratic Casimir of the adjoint representation.

§2 N=2 Superconformal Algebra

In this section we construct the $N = 2$ superconformal algebra by using the Z_3 orbifold models.

Supercurrents G^+ and G^- of the $N = 2$ superconformal algebra are the nonlocal operator with conformal dimension 3/2. Therefore G^+ itself does not belong to the physical spectrum. We will see that $G^+\bar{G}^+$ appears in the physical spectrum. It is necessary that the chiral algebraic structure corresponding to $(G^+)^2$ exist in the physical spectrum as we see it in the realization of the $N = 2$, $c = 1$ superconformal model by a single free boson. By looking at the operator product expansion among $U(1)$ current I and G^+

$$I(z)G^+(w) = \frac{1}{z-w}G^+(w), \tag{4.19}$$

we see that G^+ has the $U(1)$ charge $Q = 1$. The operator product expansion

$$I(z)I(w) = \frac{c}{3(z-w)^2} \tag{4.20}$$

implies that we can write $U(1)$ current by a free boson as

$$I = i\sqrt{\frac{c}{3}}\partial\varphi. \tag{4.21}$$

By using this definition and eq. (4.19), we can also separate the part of G^+ depending on φ as

$$G^+ = \tilde{G}^+ exp\left(i\sqrt{\frac{3}{c}}\varphi\right). \tag{4.22}$$

If we write $\tilde{G}^+$ by the vertex operator in therms of other free bosons as

$$\tilde{G}^+ = exp\left(iv\tilde{\varphi}\right), \tag{4.23}$$

it follows that $(G^+)^2$ has conformal dimension 6 because G^+ has 3/2.

Let us make $c = 3$, $N = 2$ superconformal model by orbifold models.

First we construct the torus model which is defined by the lattice $\Lambda = RA_d \oplus rA_1$. We construct the chiral algebraic structure with conformal dimension 6, which corresponds to $(G^+)^2$. We require that the wave function $\Psi(p, l)$ with

$$p = p_{A_2} + p_{A_1} \quad \begin{cases} p_{A_2} = R^{-1}(m_1\lambda_1 + m_2\lambda_2) \\ p_{A_1} = r^{-1}m_3\lambda_3, \end{cases} \tag{4.24}$$

$$l = l_{A_2} + l_{A_1} \quad \begin{cases} l_{A_2} = R(n_1\alpha_1 + n_2\alpha_2) \\ l_{A_1} = rn_3\alpha_3 \end{cases} \tag{4.25}$$

enjoys the chirality condition $p - l - Bl = 0$ and

$$6 = p \cdot l = \sum_{i=1,2,3} m_i n_i, \tag{4.26}$$

$$\frac{1}{2}\left(2\sqrt{\frac{3}{c}}\right)^2 = \frac{6}{c} = l_{A_1}^2 = 2r^2 n_3^2, \tag{4.27}$$

$$6 - \frac{6}{c} = l_{A_2}^2 = 2R^2(n_1^2 + n_2^2 - n_1 n_2). \tag{4.28}$$

Further, if we assume that such a chiral algebraic structure exists in the case of $\vec{n} = (1\ 0\ 1)$, we have $R^2 = 2$ and $r^2 = 1$.

At last we can find a solution

$$B_{13} = B_{23} = 2, \quad otherwise = 0. \tag{4.29}$$

This matrix B satisfies $[B, W]\Lambda \in \Lambda^*$, which is the consistency condition for orbifold models. Chirality condition $p - l - Bl = 0$ reads

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 2 \\ -2 & 4 & 2 \\ -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}. \tag{4.30}$$

Now we have a set of chiral algebraic structures in the torus model,

$$\vec{n} = \begin{cases} (1\,0\,1) \\ (0\,1\,1) \\ (-1\ -1\,1) \end{cases} \tag{4.31}$$

and their conjugates. From these operators we can construct the chiral algebraic structures

$$\vec{n} = \begin{cases} (1\,2\,0) \\ (2\,1\,0) \\ (1\ -1\,0), \end{cases} \tag{4.32}$$

which act in the A_2 sector only and $\vec{n} = (0\ 0\ 3)$ in the A_1 sector.

We have the general forms of the spectrum of the torus model which give the spectrum in the untwisted sector of the orbifold model:

$$\begin{aligned}
\sqrt{2}p_L &= p + l - Bl \\
&= \tfrac{1}{\sqrt{2}}\{(m_1 + 4n_1 - 2n_2 - 2n_3)\lambda_1 + (m_2 - 2n_1 + 4n_2 - 2n_3)\lambda_2\} \\
&\quad +(m_3 + 2n_1 + 2n_2 + 2n_3)\lambda_3, \\
\sqrt{2}p_R &= p - l - Bl \\
&= \tfrac{1}{\sqrt{2}}\{(m_1 - 4n_1 + 2n_2 - 2n_3)\lambda_1 + (m_2 + 2n_1 - 4n_2 - 2n_3)\lambda_2\} \\
&\quad +(m_3 + 2n_1 + 2n_2 - 2n_3)\lambda_3,
\end{aligned} \tag{4.33}$$

General wave functions on the orbifold are obtained by Z_3 projection P_3 of torus model. P_3 is defined by the operation $(W, V = 0)$. In this spectrum we find the operator $\Psi = G(\alpha)\bar{G}(\alpha)$:

$$G(\alpha) = exp\left\{i(\alpha + \frac{\alpha_3}{\sqrt{2}})\varphi\right\}, \quad \bar{G}(\alpha) = exp\left\{i(\alpha + \frac{\alpha_3}{\sqrt{2}})\bar{\varphi}\right\}. \tag{4.34}$$

for every $\alpha \in \Lambda_{A_2}$. Every $G(\alpha)$ has dimension 3/2. We construct Z_3 invariant current as

$$G^+ = \sqrt{\frac{2}{3}}\left[G(\alpha_1) + G(\alpha_2) + G(-\alpha_1 - \alpha_2)\right] \tag{4.35}$$

and its conjugate

$$G^- = \sqrt{\frac{2}{3}}\left[G(-\alpha_1) + G(-\alpha_2) + G(\alpha_1 + \alpha_2)\right]. \tag{4.36}$$

We must remark that m_1 and m_2 in eq.(4.33) have to be even integers so that the boundary condition of the supercurrent G^+ is well defined. Otherwise the boundary conditions of $G(\alpha_1)$, $G(\alpha_2)$ and $G(-\alpha_1-\alpha_2)$ are different. This is a problem for the representation. If it is solved, then the boundary conditions of G^+ is determined by m_3. We confirm that chiral algebraic structure appears as $(G^+)^2$

$$G^+(z)G^+(w) = \frac{2}{3}\{J(-\alpha_1)+J(-\alpha_2)+J(\alpha_1+\alpha_2)\}(w)+O(z-w). \qquad (4.37)$$

After all we derive $c=3$, $N=2$ superconformal algebra with supercurrents G^+, G^-, $U(1)$ current $I=i\sqrt{\frac{c}{3}}\partial\varphi$ and energy-momentum tensor $T=\frac{1}{2}\sum_{j=1,2,3}(i\partial\varphi_j)^2$.

$$T(z)T(w) = \frac{c}{2(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w}+regular\ term$$

$$I(z)I(w) = \frac{c}{3(z-w)^2}+reg. \qquad (4.38)$$

$$G^+(z)G^-(w) = \frac{2c}{3(z-w)^3}+\frac{2I(w)}{(z-w)^2}+\frac{\partial I(w)+2T(w)}{z-w}+reg.$$

The spectrum in the untwisted sector is given by the eq.(4.33). We find that a state of $\vec{m}=(0\ 0\ \pm 1), \vec{n}=\vec{0}$ is the Ramond vacuum with conformal dimension 1/8 and $U(1)$ charge $Q=\pm 1/2$ and a state of $\vec{m}=(0\ 0\ \pm 2), \vec{n}=\vec{0}$ is the Neveu-Schwarz vacuum with conformal dimension 1/2 and $U(1)$ charge $Q=\pm 1$.

We do not discuss the twisted sector in this paper. We will discuss the details of it in the near future.

In this section, we have simply constructed $c=3$, $N=2$ superconformal algebra by using an orbifold model.

5. SUMMARY AND DISCUSSION

Let us summarize the discussions given in the previous chapters.

We have quantized carefully the closed bosonic string propagating on the torus or the orbifold. We have used the idea that the starting point of the string configuration in the covering space does not move when the other string links up at the endpoint of the previous string. It follows that after linking up two strings to become the third string crooked, the third string is also described as a straight line by the zero mode, up to linear term in σ.

We have discussed the representation space of torus models under this quantization. We concentrate ourselves on torus models with chiral algebraic structures, the rational torus models. Especially, we have studied the A_{N-1} type model that has the discrete Z_N symmetry. As a result we have found that if we take the radius $R = 1$ and the antisymmetric tensor $B = 0$, then the orbifold model is completely described as a system of the Z_N parafermionic current algebras.

We have given a condition to formulate consistently orbifold models from torus models. It is the equation (3.5)

$$[B, W]\Lambda \in \Lambda^*,$$

where Λ is the lattice defining torus and Λ^* is dual to Λ and B is the antisymmetric tensor and W is the discrete rotation. This condition is equivalent to that the W-orbit of the vertex operators is still in the spectrum of the torus model.

As the applications of the parafermionic current algebras in orbifold models, we have shown two examples in which some $U(1)$ currents remain as the chiral algebraic structures. One is the level 3 $\widehat{SU(3)}$ Kac-Moody algebra. This is just the Frenkel-Kac construction when we consider the torus model. In the orbifold model, since the part of the vertex operator is the Z_3 parafermionic current, we have level 3 Kac-Moody algebras. The other one is the $c = 3$, $N = 2$ superconformal model. This is not the case of the Frenkel-Kac construction. But it is a key idea that the chiral algebraic structure with conformal dimension 6 exist. This is well known in the case of a single free boson. Although we have constructed $N = 2$ superconformal algebra, the representation space is not well investigated. We do not understand what the condition, $m_1, m_2 \in 2Z$, which is stated in chapter 4, means. We must study it in the future.

As the superstring theory, it is simple and interesting to construct $c = 9$, $N = 2$ superconformal model by using orbifold models such as $(S^1) \times (Parafermion)$. It remains for us to study other types of Weyl orbifold containing nonabelian orbifold. They should be also investigated in the future.

We have discussed only the models with integral conformal charge. It may be, however, possible to discuss more generally models with rational conformal charges by using torus models defined by the Feigin-Fuchs construction [23] which allows discrete shift only. In such a case and perhaps in any other case, the orbifold is very useful 'geometry' for the study of the conformal models. We think that the method developed in this paper, which connects parafermions and $U(1)$ current sector, may be also useful.

Let us discuss more perspective viewpoint for the string theory.

String theories are probably unified theories containing gravity. Einstein studied four-dimensional geometry and constructed the theory of general relativity as a theory of gravity. After that, he concentrated himself on the unified theory of electromagnetics and gravity. We can say that string theories are theories of geometries for matter and spacetime. Now we are to have a theory of gravity which is quantum mechanically consistent, that is, string theories.

Looking for it, however, is still in the darkness, because there is no principle to investigate the vacuum of string field theories. If this situation does not change, various kinds of string theories will be proposed by many authors. Even though it is so, when we consider realistic models, we should have the vacuum of the string with four-dimensional spacetime and $N = 1$ supersymmetry of spacetime. In such string theories, for example four-dimensional $N = 1$ heterotic string theories, there are quarks and leptons in the 27 dimensional representation of the gauge symmetry E_6.

Future studies will reveal more details in phenomenologies, such as explanation of the standard model, the generation number, Kobayashi-Maskawa matrix and supersymmetry breaking, etc.

To produce varieties of string theories is to produce varieties of internal spaces. It is equally interesting to study algebraic geometries of conformal field theories for internal spaces. However, we can also say that if geometrical interpretations of conformal field theories are left undone, a complete classification of conformal field theories may be enough to study internal spaces.

Conformal field theories themselves are interesting because they describe some exactly solvable models in statistical models on the critical point. On the other hand, statistical models are also exactly solvable on the off-critical point. This gives an insight that some models remain exactly solvable even if one introduces a perturbation which breaks conformal invariance of the theory. In such theories, an infinite number of conserved charges survive even though the conformal invariance is broken. This is a problem of deformations of conformal field theories, which is nothing but introducing interacting potentials, sine-Gordon, hyperbolic sine-Gordon terms, etc. A class of deformations of conformal field theories are known as marginal deformations that do not break conformal invariance. They are deformations of radius of torus models [24] and four-fermi interactions, called Thirring fermions [25,26], etc.

We have discussed bosonic theories with integral conformal charges to describe parafermions. It may be, however, possible to describe almost all of conformal field theories in terms of bosons and the parts of their representations. The classification of conformal field theories will be completed in terms of free bosons in a future. We hope to make some advancement in this direction.

REFERENCES

[1] M.B. Green, J.H. Schwarz and E. Witten, 'Superstring theory' Cambridge University Press (1987).

[2] J. Wess and J. Bagger, 'Supersymmetry and Supergravity' Princeton University Press (1983)

[3] Th. Kaluza, Sitz. Preuss. Akad. Wiss. **K1** (1921) 966.

[4] M.B. Green and J.H. Schwarz, Phys. Lett. **B149** (1984) 117.

[5] D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, Phys. Rev. Lett. **54** (1985) 502.

[6] A. Fujitsu, T. Kitazoe, H. Nishimura and M. Tabuse, to be published in Int. J. Mod. Phys.

[7] A.M. Polyakov, Phys. Lett. **B103** (1981) 207; **B103** (1981) 211.

[8] P. Candelas, G. Horowitz, A. Strominger and E. Witten, Nucl. Phys. **B258** (1985) 46.

[9] L. Dixon, J.A. Harvey, C. Vafa and E. Witten, Nucl. Phys. **B261** (1985) 678; **B274** (1986) 285.

[10] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. **B241** (1984) 333.

[11] K.S. Narain, M.H. Sarmadi and E. Witten, Nucl. Phys. **B279** (1987) 369.

[12] D. Gepner, Nucl. Phys. **B296** (1987) 757.

[13] Y. Kazama and H. Suzuki, Nucl. Phys. **B321** (1989) 232.

[14] B.R. Greene, C. Vafa and N.P. Warner, Nucl. Phys. **B324** (1989) 371.

[15] A. Fujitsu and M. Tabuse, Kobe University preprint KOBE-89-03-revised.

[16] M. Sakamoto, Kobe University preprint KOBE-89-02.

[17] K. Itoh, M. Kato, H. Kunitomo and M. Sakamoto, Nucl. Phys. **B306** (1988) 362.

[18] I.B. Frenkel and V.G. Kac, Inv. Math. **62** (1980) 23.

[19] A.B. Zamolodchikov and V.A. Fateev, Sov. Phys. JETP **62** (1985) 215.

[20] S. Nima, Kyushu University preprint KYUSHU-89-HE-2.

[21] A. Fujitsu and M. Tabuse, Kobe University preprint KOBE-89-09.

[22] D. Gepner, Nucl. Phys. **B290[FS20]** (1987) 10.

[23] B.L. Feigin and D.B. Fuchs, Funct. Anal. **16** (1982) 47.

[24] R. Dijkgraaf, E. Verlinde and H. Verlinde, Comm. Math. Phys. **115** (1988) 649; P. Ginsparg, Nucl. Phys. **B295[FS21]** (1988) 153.

[25] A. Fujitsu and K. Kimura, Int. J. Mod. Phys. **A3** (1988) 2911.

[26] J. Bagger, D. Nemeschanski, N. Seiberg and S. Yankielowicz, Nucl. Phys. **B289** (1987) 53.