Stochastic partial differential equations and stochastic controls

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STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

AND STOCHASTIC CONTROLS

確率偏微分方程式と確率制御

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Contents

INTRODUCTION

Stochastic partial differential equations appear in many areas, for example filtering theory of diffusion processes, statistical hydrodynamics, population genetics, control theory, etc. These equations describe the evolutions in time of processes with values in function spaces. For linear problems, the typical example is the Zakai equation, the solution of which being an unnormalized conditional density of diffusion process, and this equation is investigated by several authors, cf. Krylov, Kunita, Pardoux, Rozovskii and Shimizu (see [11] — [16], [22], [23], [25], [26]). Equations of population genetics and Navier-Stokes equation with random external forces are the important examples of non-linear problems. The former is studied by Dawson [3], Fleming [5] and others, and for the latter, see Krylov & Rozovskii [13].

The purpose of this paper is the study of stochastic partial differential equations and their applications to stochastic controls.

In Chapter 1 and 2 , we are concerned with control problems of systems governed by stochastic partial differential equations. Let W(t) be a d'-dimensional standard Brownian motion defined on a probability space ($\Omega, \mathcal{F}, \mathcal{F}_t, P$) and U(t) an admissible control, namely, a process with values in Γ, where Γ is a convex and compact subset of $\mathbb{R}^L$, called a control region. We consider the following stochastic partial differential equation.

$$
(0.1)\begin{cases} dq(t,x) = \sum_{i,j=0}^{d} \frac{\partial}{\partial x_i}\Big(a^{ij}(x,y+W(t),U(t))\frac{\partial}{\partial x_j} q(t,x) \\ \qquad\qquad\qquad\qquad + f^{i}(x,y+W(t),U(t))\Big)dt \\ \quad + \sum_{k=1}^{d'}\Big(\sum_{i=0}^{d} b^{i}_{k}(x,y+W(t))\frac{\partial}{\partial x_i} q(x,t) + g_k(x,y+W(t))\Big)\, dW^{k}(t). \end{cases}
$$

A solution $q(t) = q(t,U)$ of (0.1) is sought in the space of Sobolev type $H^1(\mathbb{R}^d)$. Define a criterion $J(U)$ by

(0.2) $J(U) = E[\ F(\ q(\cdot,U)\ ) + G(\ q(T,U)\ )\ ]$

where F and G are real valued functions on $L^2(0,T;L^2(\mathbb{R}^d))$ and $L^2(\mathbb{R}^d)$ respectively. The problem is to minimize a criterion $J(U)$ by choosing a suitable admissible control.

In Chapter 1, assuming that

$\left(a^{ij}(x,y,u) - \frac{3}{2}\sum_{k=1}^{d'} b^{i}{}_{k}(x,y)b^{j}{}_{k}(x,y) \right)_{i,j=1,\cdots,d}$ is uniformly positive definite and some regulality conditions on the coefficients, we show the continuity of solutions $q(\cdot,U)$ on U as $[\ w\text{-}L^2(0,T;L^2(\mathbb{R}^d))\ ]$-random variables, where $[\ w\text{-}X\ ]$ denotes the space X carrying the weak topology. Then we can prove the existence of optimal control. Moreover, we apply our results to stochastic control with partial observation.

Chapter 2 is the extension of Chapter 1. In this chapter, an admissible control $U(t)$ is replaced by an admissible relaxed control $\mu(t,du)$, namely, a process with values in the space of probability measures on Γ, and coefficients a^{ij} and f^{i} are replaced by the following $\tilde{a}^{ij}$ and $\tilde{f}^{i}$ respectively,

$$\tilde{a}^{ij}(t,x,y+W(t),\mu) = \int_{\Gamma} a^{ij}(x,y+W(t),u)\mu(t,du)$$

and

$$\tilde{f}^{i}(t,x,y+W(t),\mu) = \int_{\Gamma} f^{i}(x,y+W(t),u)\mu(t,du).$$

Assuming that $\left(a^{ij}(x,y,u) - \frac{3}{2} \sum_{k=1}^{d'} b^i{}_k(x,y) b^j{}_k(x,y) \right)_{i,j=1,\cdots,d}$ is non-negative definite and some regularity conditions on the coefficients, we prove the continuous dependence of solutions $q(\cdot,\mu)$ on relaxed control μ as [$s\text{-}L^2(0,T;L^2(\mathbb{R}^d))$]-random variable, where [s-X] denotes the space X carrying the strong topology. Moreover, the existence of optimal relaxed control, the Bellman principle and some other properties is proved.

In Chapter 3, we are concerned with the Caucy problem for non-linear stochastic partial differential equations. The main aim of this chapter is to show the existence of solutions for the following equations.

$$(0.3) \quad du(t) = (Au(t) + F(u(t)))dt + G(u(t))dW(t),$$

where A is a second-order elliptic differential operator, F and G are continuous operators from $L^2(\mathbb{R}^d)$ to itself and $W(t)$ is a one dimensional Brownian motion.

When F and G satisfy the Lipschitz condition and A is uniformly elliptic, Pardoux [23] and Walsh [29] proved the existence and uniqueness of the solutions for (0.3) by Picard's method of successive approximation. But, if F and G are merely continuous, Picard's method is not effective. To overcome this difficulty, we approximate the equation (0.3) by Cauchy polygon. Using this approximate sequence, we show the existence of solutions. Furthermore, we show a sort of stability on the perturbation of coefficients.

CHAPTER 1

On the existence of optimal control for controlled stochastic partial differential equations

§1 Introduction

In this chapter we are concerned with stochastic control problems of the following kind. Let $Y(t)$ be a d'-dimentional Brownian motion defined on a probability space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ and $u(t)$ an admissible control. We consider the Cauchy problem of stochastic partial differential equations (SPDE in short)

$$(1.1)\quad \begin{cases} dp(t,x) = L(Y(t),u(t))p(t,x)dt + M(Y(t))p(t,x)dY(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}^d,\ t > 0 \\ p(0,x) = \phi(x) \end{cases}$$

where $L(y,u)$ is the 2nd order elliptic differential operator and $M(y)$ the 1st order differential operator.

By a solution $p(t) = p^u(t)$, we mean H^1-valued $\mathcal{F}_t$-adapted process which satisfies

$$(p(t),\eta) = (\phi,\eta) + \int_0^t \langle L(Y(s),u(s))p(s)\ ,\ \eta \rangle ds$$

$$+ \int_0^t (M(Y(s))p(s)\ ,\ \eta)\ dY(s), \qquad t \geq 0$$

for any smooth η where $\langle\cdot,\cdot\rangle$ is the pairing between H^{-1} and

H^1 and $(\cdot,\cdot)$ is $L^2(\mathbb{R}^d)$ inner product (see [11] & [22]).

The SPDE (1.1) is related to the filtering, stochastic control with partial obserbation, population genetics etc. and investigated by Pardoux, Krylov & Rozovskii and Rozovskii & Shimizu, etc.

The purpose of this paper is to prove the existence of optimal controls for the following problem. Define a criterion $J(u)$ by

$$(1.2) \quad J(u) = E[\ F(p^u) + G(p^u(T))\ ]$$

where F and G are real valued functions on $L^2(0,T;L^2(\mathbb{R}^d))$ and $L^2(\mathbb{R}^d)$ respectively. Now we want to minimize $J(u)$ by a suitable choice of an admissible process u .

In §2 we will recall some known results in our convenient way and formulate our problem precisely. In §3 we will prove that the solution p^u depends on u continuously which derives the existence of optimal control [Theorem 3.2]. In §4 we apply our results to stochastic control with partial observation, where an observation noise may depend on a state noise.

§2 Notation and preliminaries

We assume the following conditions (A.1) ~ (A.3).

$$(A.1) \qquad \begin{aligned} b &: \mathbb{R}^d \times \mathbb{R}^{d'} \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^L \\ \sigma &: \mathbb{R}^d \times \mathbb{R}^{d'} \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^{d'} \\ a &: \mathbb{R}^d \times \mathbb{R}^{d'} \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d \\ h &: \mathbb{R}^d \times \mathbb{R}^{d'} \longrightarrow \mathbb{R}^{d'} \end{aligned}$$

are bounded and continuous and a is symmetric.

(A.2) There exists $\delta > 0$ such that

$2a(x,y) - 3\sigma(x,y)\sigma^*(x,y) \geq \delta I$ for any $(x,y) \in \mathbb{R}^d \times \mathbb{R}^{d'}$

where σ^* is the transposed matrix of σ.

(A.3) $a(\cdot,y)$, $\sigma(\cdot,y)$ are $C^{\hat{m}+1}$- class in $x \in \mathbb{R}^d$,

$h(\cdot,y)$, $b(\cdot,y)$ are $C^{\hat{m}}$ - class in $x \in \mathbb{R}^d$,

and their derivatives are bounded and continuous in $(x,y) \in \mathbb{R}^d \times \mathbb{R}^{d'}$, where $\hat{m} = \max\{2,m\}$ and m is a given nonnegative integer.

Let Γ be a convex and compact subset of $\mathbb{R}^L$.

Definition 2.1 $\mathcal{A} = (\Omega,\mathcal{F},P,Y,u)$ is called an admissible system, if $(\Omega,\mathcal{F},P)$ is a probability space and u is a Γ-valued measurable process and Y is a d'-dimensional $(\mathcal{F}_t)$-Brownian motion on $(\Omega,\mathcal{F},P)$, where $\mathcal{F}_t = \sigma\{ Y(s), \int_0^s u(\tau)\, d\tau \ ; \ s \leq t \}$.

$\mathfrak{A}$ denotes the totality of admissible systems.

For $\mathcal{A} \in \mathfrak{A}$, $\pi^{\mathcal{A}}$ denotes the image measure of (Y,u) on $C(0,T;\mathbb{R}^{d'}) \times L^2(0,T;\Gamma)$.

Endowing the uniform topology on $C(0,T;\mathbb{R}^{d'})$ and the weak topology on $L^2(0,T;\Gamma)$,we have

Lemma 2.1 $\{ \pi^{\mathcal{A}} \ ; \ \mathcal{A} \in \mathfrak{A} \}$ is compact under the Prokhorov metric. (See Fleming & Pardoux [7] Lemma 2.3)

Define $L(y,u) \in \mathscr{L}(H^1, H^{-1})$, $M^k(y) \in \mathscr{L}(H^1, L^2(\mathbb{R}^d))$ $(k = 1,\cdots,d',\ y \in \mathbb{R}^{d'}, u \in \Gamma)$ by

$$(2.1)\quad \langle L(y,u)p\ ,\ q \rangle = -\sum_{i,j=1}^{d} \left(a_{ij}(\cdot,y)\frac{\partial p}{\partial x_i}\ ,\ \frac{\partial q}{\partial x_j} \right) + \sum_{j=1}^{d} \left(\tilde{b}_j(\cdot,y,u)p\ ,\ \frac{\partial q}{\partial x_j} \right)$$

$$(2.2)\quad (M^k(y)p\ ,\eta) = -\sum_{i=1}^{d} \left(\sigma_{ik}(\cdot,y)\frac{\partial p}{\partial x_i}\ ,\ \eta \right) + \left(\tilde{h}_k(\cdot,y)p\ ,\ \eta \right)$$

for p, $q \in H^1$ and $\eta \in L^2(\mathbb{R}^d)$, where $(\cdot,\cdot)$ = the inner product in $L^2(\mathbb{R}^d)$, $\langle\cdot,\cdot\rangle$ = the duality pairing between H^{-1} and H^1 and

$$\tilde{b}_j(x,y,u) = \sum_{\ell=1}^{L} b_{j\ell}(x,y)u_\ell - \sum_{i=1}^{d}\frac{\partial a_{ij}}{\partial x_i}(x,y)$$

$$\tilde{h}_k(x,y) = h_k(x,y) - \sum_{i=1}^{d}\frac{\partial \sigma_{ik}}{\partial x_i}(x,y)$$

By (A.1) ~ (A.3), there exists $\alpha > 0$ and $\lambda \in \mathbb{R}$ such that

$$(2.3)\quad -2\langle L(y,u)p\ ,p \rangle + \lambda\|p\|_0^2 \geq \alpha\|p\|_1^2 + 3\sum_{k=1}^{d'}\|M^k(y)p\|_0^2$$

for any $p \in H^1$, $y \in \mathbb{R}^d$, $u \in \Gamma$

where $\|\cdot\|_\ell$ = the H^ℓ- norm $(\ell = 0,\pm1,\cdots)$

(for the proof, see §2 of Krylov & Rozovskii [11]).

(2.3) is called the coercivity condition.

For an admissible system $\mathcal{A} = (\Omega,\mathcal{F},P,Y,u)$, putting $L^{\mathcal{A}}(t) = L(Y(t),u(t))$ and $M^{\mathcal{A}k}(t) = M^k(Y(t))$, we consider the Cauchy problem of SPDE on $(\Omega,\mathcal{F},P)$,

$$(2.4) \quad \begin{cases} dp(t) = L^{\mathcal{A}}(t)p(t)dt + M^{\mathcal{A}}(t)p(t)dY(t) & t > 0 \\ p(0) = \phi \in H^{\hat{m}} \end{cases}$$

where $M^{\mathcal{A}}(t) = (M^{\mathcal{A}1}(t),\cdots,M^{\mathcal{A}d'}(t))$.

Definition 2.2 By a solution of SPDE (2.4), we mean an H^1-valued $\mathcal{F}_t$- adapted process $p(t)$ defined on $(\Omega,\mathcal{F},P)$ such that

(1) $E[\int_0^T \|p(t)\|_1^2 \, dt] < \infty$.

(2) for any $\eta \in H^1$ and $t \in [0,T]$

$$(2.5) \quad (p(t) , \eta) = (\phi , \eta) + \int_0^t \langle L^{\mathcal{A}}(s)p(s) , \eta \rangle ds + \int_0^t (M^{\mathcal{A}}(s)p(s) , \eta) dY(s)$$

holds.

By the coercivity condition (2.3), we have the following proposition. (See [12], [22])

Proposition 2.1 For each $\mathscr{A} \in \mathfrak{U}$, the equation (2.4) has a unique solution $p = p^{\mathscr{A}}$ which satisfies

$$(2.7) \quad p \in L^2(\ (0,T)\times\Omega\ ;\ H^{\hat{m}+1}\ ) \cap L^2(\ \Omega\ ;\ C(0,T;H^{\hat{m}}\ )\ )$$

and

$$(2.8) \quad \|p(t)\|_0^2 = \|\phi\|_0^2 + 2\int_0^t \langle\ L^{\mathscr{A}}(s)p(s)\ ,\ p(s)\ \rangle\, ds$$

$$+ 2\int_0^t (\ M^{\mathscr{A}}(s)p(s)\ ,\ p(s)\ )\, dY(s) + \int_0^t \|M^{\mathscr{A}}p(s)\|_0^2\, ds$$

The solution $p = p^{\mathscr{A}}$ of the SPDE (2.4) is called the response for $\mathscr{A}$.

Remark 2.1 We can apply the results of Pardoux [22] also to the triplet (V,H,V^*) , where $V = H^{\ell+1}$, $H = H^{\ell}$ and $V^* = H^{\ell-1}$ ($\ell = 0,1,\cdots,\hat{m}$). Define $\tilde{L}(y,u) \in \mathscr{L}(\ H^{\ell+1},H^{\ell-1}\ )$, $\tilde{M}(y) \in \mathscr{L}(\ H^{\ell+1},H^{\ell}\ )$ similarly to $L(y,u)$, $M(y)$, where we replace $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$ by " $\langle\cdot,\cdot\rangle_{\ell}$ = the duality pairing between $H^{\ell-1}$ and $H^{\ell+1}$ " and " $(\cdot,\cdot)_{\ell}$ = the inner product in H^{ℓ}" respectively in (2.1), (2.2). Then the coercivity condition holds. (In (2.3), $\|\cdot\|_0$ and $\|\cdot\|_1$ are replaced by $\|\cdot\|_{\ell}$ and $\|\cdot\|_{\ell+1}$ respectively.) Appealing to Krylov & Rozovskii [11], the solution p of (2.4) turns out a unique solution of SPDE (2.9)

$$
(2.9)\quad \begin{cases} dp(t) = \tilde{L}(Y(t),u(t))p(t)dt + \tilde{M}(Y(t))p(t)dY(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad t > 0 \\ p(0) = \phi \end{cases}
$$

Moreover $p(t)$ satisfies similar equality to (2.8). (i.e. "0" is replaced by "ℓ".)

Let $F : L^2(0,T;H^{m+1}) \longrightarrow \mathbb{R}$ and $G : H^m \longrightarrow \mathbb{R}$ be weakly continuous functions.

For $\mathscr{A} \in \mathfrak{A}$, we define the pay-off function $J(\mathscr{A})$ by

$$
(2.10)\quad J(\mathscr{A}) = E[\ F(p^{\mathscr{A}}) + G(p^{\mathscr{A}}(T))\ ]
$$

We want to minimize its value by a suitable choice of $\mathscr{A} \in \mathfrak{A}$.

§3 Existence of optimal control

First of all we will prove that the solution $p^{\mathscr{A}}$ of (2.4) depends on $\mathscr{A}$ continuously .

Theorem 3.1 If $\pi^{\mathscr{A}(n)} \longrightarrow \pi^{\mathscr{A}}$ in law , then

(3.1) $p^{\mathscr{A}(n)} \longrightarrow p^{\mathscr{A}}$ in law as $L^2(0,T\ ;\ H^{m+1})$- random variable

and

(3.2) $p^{\mathscr{A}(n)}(T) \longrightarrow p^{\mathscr{A}}(T)$ in law as H^m- random variable ,

where we endow the weak topologies on $L^2(0,T\ ;\ H^{m+1})$ and H^m.

For the proof we need the following two lemmas.

Lemma 3.1 There exists a constant $K > 0$ such that

$$(3.3) \quad E\{ \int_0^T \|p^{\mathscr{A}}(t)\|_{\ell+1}^2 \, dt \} \le K\|\phi\|_{\ell}^2$$

$$(3.4) \quad E\{ \sup_{0\le t\le T} \|p^{\mathscr{A}}(t)\|_{\ell}^2 \} \le K\|\phi\|_{\ell}^2$$

$$(3.5) \quad E\{ \int_0^T \|p^{\mathscr{A}}(t)\|_{\ell}^4 \, dt \} \le K\|\phi\|_{\ell}^4$$

for any $\mathscr{A} \in \mathfrak{U}$. $(\ \ell = 0,1,\cdots,\hat{m}\ )$

According to [17] we introduce the spaces $\mathscr{H}_\gamma(D)$ and $\mathscr{H}_\gamma(T,D)$ as follows. Set $\hat{\psi}(\cdot,x)$ = the Fourier transformation in t of $\psi(\cdot,x)$, $\|\cdot\|_{2,D}$ = the $H^2(D)$-norm and $\|\cdot\|_*$ = the norm of the dual space $\left(H^2(D) \right)^*$, where we identify $H^1(D)$ with its dual space.

$$\mathscr{H}_\gamma(D)$$

$$= \left\{ \psi \in L^2(-\infty, \infty ; H^2(D)) \ ; \ \int_{-\infty}^{\infty} |\tau|^{2\gamma}\|\hat{\psi}(\tau)\|_*^2 \, d\tau < \infty \right\}$$

where

$$\|\psi\|_{\mathscr{H}_\gamma(D)} = \left\{ \int_{-\infty}^{\infty}\|\psi(t)\|_{2,D}^2 \, dt + \int_{-\infty}^{\infty} |\tau|^{2\gamma}\|\hat{\psi}(\tau)\|_*^2 \, d\tau \right\}^{1/2}$$

$$\mathscr{H}_\gamma(T,D) = \left\{ \psi|_{[0,T]} \ ; \ \psi \in \mathscr{H}_\gamma(D) \right\}$$

where

$$\|\psi\|_{\mathcal{H}_\gamma(T,D)} = \inf\left\{ \|\varphi\|_{\mathcal{H}_\gamma(D)} \;;\; \varphi(t) = \psi(t) \text{ a.e. on } [0,T] \right\}$$

Remark 3.1 If D is a bounded and open subset of $\mathbb{R}^d$ with a smooth boundary , then , by the compactness lemma ([17] p60) the imbedding : $\mathcal{H}_\gamma(T,D) \longrightarrow L^2(0,T\;;\;H^1(D))$ is compact.

Lemma 3.2 Let $0 < \gamma < 1/4$, then for each $\mathcal{A} \in \mathfrak{U}$,

$$p^{\mathcal{A}} \in \mathcal{H}_\gamma(T,D) \quad \text{a.s.}$$

and there exists $K > 0$ such that

$$(3.6) \qquad E[\, \|p^{\mathcal{A}}\|^2_{\mathcal{H}_\gamma(T,D)} \,] \le K\|\phi\|_2^2 \qquad \forall\, \mathcal{A} \in \mathfrak{U}.$$

Proof of Lemma 3.1 (3.3) and (3.4) are easy variants of Corollary 2.2 of Krylov & Rozovskii [11]. Now we will show (3.5). Since the response p is the solution of (2.9), using Itô's formula, we get

$$(3.7) \quad \|p(t)\|_\ell^4 = \|\phi\|_\ell^4$$
$$+ 4\int_0^t \|p(s)\|_\ell^2 \;\langle\, \tilde{L}(s)p(s)\;,\; p(s)\,\rangle_\ell \, ds$$
$$+ 2\int_0^t \|p(s)\|_\ell^2 \;\|\tilde{M}(s)p(s)\|_\ell^2 \, ds$$
$$+ 4\sum_{k=1}^{d'} \int_0^t (\, \tilde{M}^k(s)p(s)\;,\; p(s)\,)_\ell^2 \, ds$$

$$+ 4\int_0^t \|p(t)\|_\ell^2 \, (\tilde{M}(s)p(s) \, , \, p(s))_\ell \, dY(s)$$

where $\tilde{L}(t) = \tilde{L}(Y(t),u(t))$ and $\tilde{M}(t) = \tilde{M}(Y(t))$.

Hence , using the coercivity condition, we have

(3.8) $E[\|p(t)\|_\ell^4] - \|\phi\|_\ell^4$

$$= 2E[\int_0^t \|p(s)\|_\ell^2 \, \{ 2\langle \tilde{L}(s)p \, , \, p \rangle_\ell + \|\tilde{M}(s)p\|_\ell^2 \} \, ds]$$

$$+ 4E[\int_0^t \sum_{k=1}^{d'} (\tilde{M}(s)p \, , \, p)_\ell^2 \, ds]$$

$$\leq 2E[\int_0^t \|p(s)\|_\ell^2 \, \{ \lambda'\|p(s)\|_\ell^2 - \alpha'\|p(s)\|_{\ell+1}^2 \} \, ds]$$

$$\leq 2\lambda' E[\int_0^t \|p(s)\|_\ell^4 \, ds]$$

So the Gronwall's inequality derives (3.5).

Proof of Lemma 3.2 For the convenience, we extend $p(t)$ on $(-\infty , \infty)$ in the following way

$$\begin{aligned} p(t) &= p(t) \quad , \ t \in [0 , T] \\ &= 0 \qquad\ , \ t \in (-\infty , \infty) \setminus [0 , T] \end{aligned}$$

Since $p(t)$ is a solution of (2.9), applying Itô's formula, we obtain

$$(3.9)\quad 2\pi i\tau(\,\hat{p}(\tau)\,,\,\eta\,)_2$$

$$= (\,\phi\,,\,\eta\,)_2 - (\,p(T)\,,\,\eta\,)_2 \exp\{-2\pi i\tau T\}$$

$$+ \langle\,\widehat{Lp}(\tau)\,,\,\eta\,\rangle_2 + \int_0^T \exp\{-2\pi i\tau t\}\,(\,\tilde{M}(t)p\,,\,\eta\,)_2\, dY(t)$$

for any $\eta \in H^3$.

Let $\{\,\eta_k\,\}_{k\geq 1}$ be an orthonormal basis in H^3 . Using (3.3), (3.4) and (3.9), we have

$$(3.10)\quad 4\pi^2\tau^2 E[\,\|\hat{p}(\tau)\|_1^2\,]$$

$$= 4\pi^2\tau^2 \sum_{k=1}^{\infty} E\{\,|(\,\hat{p}(\tau)\,,\,\eta_k\,)_2\,|^2\,\}$$

$$\leq K_1\|\phi\|_2^2 + K_2 E[\,\|\,\widehat{Lp}(\tau)\,\|_1^2\,]$$

Let $0 < \gamma < 1/4$ and $0 < \kappa < 3/2$, then

$$(3.11)\quad \int_{-\infty}^{\infty} E\{\,|\tau|^{2\gamma}\|\hat{p}(\tau)\|_*^2\,\}\, d\tau$$

$$\leq \int_{|\tau|\leq 1} E[\,\|\hat{p}(\tau)\|_1^2\,]\, d\tau + \int_{|\tau|\geq 1} E[\,\frac{2|\tau|^2}{1+|\tau|^\kappa}\,\|\hat{p}(\tau)\|_1^2\,]\, d\tau$$

$$\leq K_3 \Big\{\, E[\,\int_{-\infty}^{\infty}\|p(t)\|_1^2\, dt\,] + \int_{-\infty}^{\infty}\frac{d\tau}{1+|\tau|^\kappa}\,\|\phi\|_1^2$$

$$+ E[\,\int_{-\infty}^{\infty}\|\,\tilde{L}(t)p\,\|_1^2\, dt\,]\,\Big\}$$

$$\leq K_4\,\|\phi\|_2^2$$

This concludes the lemma.

Remark 3.2 (3.5) implies the uniform integrability of

$$\int_0^T \| p^{\mathscr{A}}(t) \|_{\ell}^2 \, dt \ , \ \mathscr{A} \in \mathfrak{A} \ .$$

Remark 3.3 We define the metric d on $H = L^2(0,T;H^{m+1}(\mathbb{R}^d))$ by

$$d(p,q) = \sum_{k=1}^{\infty} \frac{1}{2^k} \min(|(e_k \ , \ p - q)| \ , \ 1) \qquad p,q \in H$$

where $(\cdot,\cdot)$ is the inner product on H and $\{ e_k \}_{k=1}^{\infty}$ is the orthonormal basis on H. Then Lemma 3.1 and Prokhorov's theorem imply that the totality of image measure $p^{\mathscr{A}}$ ($\mathscr{A} \in \mathfrak{A}$) is relatively compact as a set of measures on the metric space (H , d).

On the other hand, on each bounded set of H the weak topology is metrizable by the metric d . Therefore, for any weakly closed set F of H , $F \cap \{ q \in H \ ; \ \|q\| \le r \}$ ($r > 0$) is closed with respect to the metric d .

Under this observation, $\{ p^{\mathscr{A}} \ ; \ \mathscr{A} \in \mathfrak{A} \}$ is relatively compact as a set of measures on H associated with the weak topology.

Proof of Theorem 3.1 Let D_k ($k = 1,2,\cdots$) be bounded and open subsets of $\mathbb{R}^d$ with smooth boundary , $\bar{D}_k \subset D_{k+1}$ and $\bigcup_{k=1}^{\infty} D_k = \mathbb{R}^d$. For an admissible system $\mathscr{A} = (\Omega,\mathscr{F},P,Y,u)$,

$$\mu^{\mathscr{A}} = \text{the image measure of } (Y,u,p^{\mathscr{A}}) \text{ on } S,$$

$$\mu_k^{\mathscr{A}} = \text{the image measure of } (Y,u,p^{\mathscr{A}}) \text{ on } S_k$$

where

$$S = C(0,T;\mathbb{R}^{d'})\times L^2(0,T;\Gamma)\times L^2(0,T;H^{m+1}(\mathbb{R}^d)),$$

and

$$S_k = C(0,T;\mathbb{R}^{d'})\times L^2(0,T;\Gamma)\times L^2(0,T;H^1(D_k))$$

endowing the weak topology on $L^2(0,T;H^{m+1}(\mathbb{R}^d))$ and the strong topology on $L^2(0,T;H^1(D_k))$. By the compactness of $\{ \pi^{\mathcal{A}} ; \mathcal{A} \in \mathfrak{A} \}$ and Remark 3.3, $\mathfrak{P} = \{ \mu^{\mathcal{A}} ; \mathcal{A} \in \mathfrak{A} \}$ is relatively compact. Moreover, by Lemma 3.2 and Remark 3.1 , $\mathfrak{P}_k = \{ \mu_k^{\mathcal{A}} ; \mathcal{A} \in \mathfrak{A} \}$ is relatively compact.

Hence there exist a subsequence $\{ \mathcal{A}(n') \}_{n'}$, a probability μ on S and a probability μ_k on S_k ($k = 1,2,\cdots$) such that

(3.12) $\mu^{\mathcal{A}(n')} \longrightarrow \mu$ in law as $n' \longrightarrow \infty$

and

(3.13) $\mu_k^{\mathcal{A}(n')} \longrightarrow \mu_k$ in law as $n' \longrightarrow \infty$.

By Skorohod's theorem, we can construct the S_k- valued random variables $(Y_{n'},u_{n'},p_{n'})$, (Y,u,p), $n' = 1,2,\cdots$, on a probability space $(\Omega,\mathcal{F},P)$ such that

(3.14) the law of $(Y_{n'},u_{n'},p_{n'}) = \mu_k^{\mathcal{A}(n')}$, $n' = 1,2,\cdots$,

(3.15) the law of $(Y,u,p) = \mu_k$

and

(3.16) $(Y_{n'},u_{n'},p_{n'}) \longrightarrow (Y,u,p)$ almost surely ($n' \longrightarrow \infty$)

as S_k- valued random variables.

Now we will prove the following lemma.

Lemma 3.3 Let $\psi : [0,T] \longrightarrow \mathbb{R}$ be an absolutely continuous function with $\psi' \in L^2(0,T)$ and $\psi(T) = 0$ and $\eta \in C_0^\infty(\mathbb{R}^d)$ with $\mathrm{supp}(\eta) \subset D_k$,then (Y,u,p) of (3.16) satisfies

$$(3.17)\quad (\phi,\eta)\psi(0) + \int_0^T \psi'(t)(\, p(t)\, ,\, \eta\,)\, dt$$

$$+ \int_0^T \psi(t)\langle\, L(Y(t),u(t))p\, ,\, \eta\, \rangle\, dt + \int_0^T \psi(t)(\, M(Y(t))p\, ,\, \eta\,)\, dY(t) = 0$$

Proof Since $p_{n'}$ is the solution of the SPDE (2.4) for $(Y_{n'},u_{n'})$, using Itô's formula to (2.5), we get

$$(3.17)_{n'}\quad (\phi,\eta)\psi(0) + \int_0^T \psi'(t)(\, p_{n'}(t)\, ,\, \eta\,)\, dt$$

$$+ \int_0^T \psi(t)\langle\, L(Y_{n'}(t),u_{n'}(t))p_{n'}\, ,\, \eta\, \rangle\, dt$$

$$+ \int_0^T \psi(t)(\, M(Y_{n'}(t))p_{n'}\, ,\, \eta\,)\, dY_{n'}(t) = 0$$

By Remark 3.2 and (3.16), we get

$$(3.18)\quad E[\int_0^T \|p_{n'}(t) - p(t)\|_{1,D_k}^2 dt\,] \longrightarrow 0 \quad (\, n' \longrightarrow \infty\,)$$

Recalling "$\mathrm{supp}(\eta) \subset D_k$", we obtain

$$(3.19)\quad \int_0^T \psi(t)\langle\, L(Y_{n'}(t),u_{n'}(t))p_{n'}\, ,\, \eta\, \rangle\, dt$$

$$\longrightarrow \int_0^T \psi(t)\langle\, L(Y(t),u(t))p\, ,\, \eta\, \rangle\, dt \quad \text{in } L^2(\Omega).$$

$$(3.20)\quad \psi(t)(\, p_{n'}(t),\eta\,) \longrightarrow \psi(t)(\, p(t),\eta\,) \quad \text{in } L^2(\, [0,T]\times\Omega\,)$$

and

(3.21) $\psi(t)(\ M(Y_{n'}(t))p_{n'}\ ,\ \eta) \longrightarrow \psi(t)(\ M(Y(t))p\ ,\ \eta\ )$

in $L^2(\ [0,T]\times\Omega\ )$

For the proof of (3.19), putting

$$q_{n'}(t) = \psi(t)(\ b_{i\ell}(\cdot,Y_{n'}(t))p_{n'}(t)\ ,\ \eta\ )$$

$$q(t) = \psi(t)(\ b_{i\ell}(\cdot,Y(t))p(t)\ ,\ \eta\ )$$

and $u(t) = (\ u^1(t),\cdots,u^L(t)\ )$, we have

$$(3.22)\quad \int_0^T \psi(t)(\ b_{i\ell}(\cdot,Y_{n'}(t))p_{n'}(t)\ ,\ \eta\ )\ u^{\ell}_{n'}(t)\ dt - \int_0^T \psi(t)(\ b_{i\ell}(\cdot,Y(t))p(t)\ ,\ \eta\ )\ u^{\ell}(t)\ dt$$

$$= \int_0^T u^{\ell}_{n'}(t)(\ q_{n'}(t) - q(t))\ dt + \int_0^T (\ u^{\ell}_{n'}(t) - u^{\ell}(t)\ )q(t)\ dt$$

By (3.18), the 1st term of the right hand side of (3.22) converges to 0 in $L^2(\Omega)$. By Remark 3.2 and (3.16), we get

$$(3.23)\quad E\left[\ \left\{ \int_0^T (\ u^{\ell}_{n'}(t) - u^{\ell}(t)\ )q(t)\ dt \right\}^2 \right] \longrightarrow 0$$

This implies (3.19). (3.20) and (3.21) can be proved similarly. Moreover, combining (3.21) with (3.16), we get

$$(3.24)\quad \int_0^T \psi(t)(\ M(Y_{n'}(t))p_{n'}\ ,\ \eta\ )\ dY_{n'}(t) \longrightarrow \int_0^T \psi(t)(\ M(Y(t))p\ ,\ \eta\ )\ dY(t)\quad \text{in}\ L^2(\Omega)$$

Hence, by taking limit of $(3.17)_{n'}$, we obtain (3.17).

Let $i_k : S \longrightarrow S_k$ be the canonical injection. Then by the definition

(3.25) $\quad i_k(\mu^{\mathscr{A}(n')}) = \mu_k^{\mathscr{A}(n')}$ and $i_k(\mu) = \mu_k$

Let $(\tilde{Y},\tilde{u},\tilde{p})$ be S - valued random variable whose law $= \mu$. Then (3.25) implies that the law of $(\tilde{Y},\tilde{u},\tilde{p}|_{D_k}) = \mu_k$.

Hence, by Lemma 3.3, $(\tilde{Y},\tilde{u},\tilde{p}|_{D_k})$ satisfies the equation (3.17). Noting that $\mathrm{supp}(\eta) \subset D_k$, we obtain

(3.26)
$$(\phi,\eta)\psi(0) + \int_0^T \psi'(t)(\tilde{p}(t),\eta)\, dt + \int_0^T \psi(t)\langle L(\tilde{Y}(t),\tilde{u}(t))\tilde{p} , \eta \rangle\, dt + \int_0^T \psi(t)(M(\tilde{Y}(t))\tilde{p} , \eta)\, d\tilde{Y}(t) = 0$$

Since k is arbitrary, (3.26) holds for any $\eta \in C_0^\infty(\mathbb{R}^d)$.

By the same argument as Theorem 1.3 in [22], $\tilde{p}$ becomes a solution of SPDE (2.4) for $(\tilde{Y},\tilde{u})$. Since the law of $(\tilde{Y},\tilde{u}) = \pi^{\mathscr{A}}$, we get

(3.27) $\quad \mu$ = the law of $(\tilde{Y},\tilde{u},\tilde{p}) = \mu^{\mathscr{A}}$

This means that any convergent subsequence of $\{ \mu^{\mathscr{A}(n')} \}$ converges to $\mu^{\mathscr{A}}$. Hence the original sequence $\{ \mu^{\mathscr{A}(n)} \}$ converges to $\mu^{\mathscr{A}}$. So we get (3.1). Next we consider the law of $(Y,u,p^{\mathscr{A}},p^{\mathscr{A}}(T))$ then by the similar argument we can prove (3.2).

Theorem 3.2 If F and G are bounded from below, then there exists an optimal admissible system $\hat{\mathcal{A}} \in \mathfrak{U}$ that is

(3.28) $\inf \{ J(\mathcal{A}) ; \mathcal{A} \in \mathfrak{U} \} = J(\hat{\mathcal{A}})$.

Proof By theorem 3.1,

$$J_n(\mathcal{A}) = E[\min\{ F(p^{\mathcal{A}}), n \} + \min\{ G(p^{\mathcal{A}}(T)), n \}]$$

is continuous on $\mathfrak{U}$. Since $J(\mathcal{A})$ is the limit function of non-decreasing sequence $\{ J_n(\mathcal{A}) \}_{n=1}^{\infty}$, it is lower-semicontinuous on $\mathfrak{U}$. This concludes the theorem.

§4 Optimal control for partially observed diffusions

In this section we will apply theorem 3.2 to the stochastic control problems for partially observed diffusions where an observation noise may depend on a state noise.

We assume the following conditions (A.4) ~ (A.6).

(A.4) $\hat{\sigma} : \mathbb{R}^d \times \mathbb{R}^{d'} \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ is bounded and continuous.

(A.5) There exists $\delta > 0$ such that

$$\hat{\sigma}(x,y)\hat{\sigma}^*(x,y) - 2\sigma(x,y)\sigma^*(x,y) \geq \delta I \quad \text{for } \forall\ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d$$

(A.6) $\hat{\sigma}(\cdot,y)$ is C^3-class in $x \in \mathbb{R}^d$ and all derivatives are bounded and contiuous in $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$.

Put $a(x,y) = (\hat{\sigma}(x,y)\hat{\sigma}^*(x,y) + \sigma(x,y)\sigma^*(x,y))/2$, then $a(x,y)$ and $\sigma(x,y)$ satisfy (A.2).

Now we will consider the optimal control problems of the following kind. Let X(t) denote the state process being controlled, Y(t) the observation process and u(t) the control process. The state and observation processes are governed by the stochstic differential equations

$$
(4.1) \quad \begin{cases} dX(t) = b(X(t),Y(t))u(t)dt + \hat{\sigma}(X(t),Y(t))d\hat{W}(t) \\ \qquad\qquad + \sigma(X(t),Y(t))dW(t) \\ X(0) = \xi \end{cases}
$$

and

$$
(4.2) \quad \begin{cases} dY(t) = h(X(t))dt + dW(t) \\ Y(0) = 0 \end{cases}
$$

where $\hat{W}$ and W are independent Brownian motions with values in $\mathbb{R}^d$ and $\mathbb{R}^{d'}$ respectively on a probability space $(\Omega,\mathcal{F},\hat{P})$.

The problem is to minimize a criterion of the form

$$
(4.3) \quad J(u) = \hat{E}[\int_0^T f(X(t)) dt + g(X(T))]
$$

In the customary version of stochastic control under partial observation, u(t) is a function of the observation process Y(s), $s \le t$. Instead of discussing the problem of this type , we treat some wider class of admissible controls inspired by

Fleming & Pardoux [7].

Let

$$(4.4) \qquad \rho(t) = \exp\{ \int_0^t h(X(s))\, dY(s) - \frac{1}{2}\int_0^t |h(X(s))|^2\, ds \}$$

Then $\hat{W}$ and Y become independent Brownian motions under a new probability P defined by

$$(4.5) \qquad dP = \rho(T)^{-1} d\hat{P}$$

and $X(t)$ becomes a solution of the following SDE

$$(4.6) \begin{cases} dX(t) = \{ b(X(t),Y(t))u(t) - \sigma(X(t),Y(t))h(X(t)) \}\, dt \\ \qquad\qquad + \hat{\sigma}(X(t),Y(t))d\hat{W}(t) + \sigma(X(t),Y(t))dY(t) \\ X(0) = \xi \end{cases}$$

Suppose ξ has a probability density $\phi \in H^2(\mathbb{R}^d)$.

Definition 4.1 $\mathcal{A} = (\Omega,\mathcal{F},P,\hat{W},Y,u,\xi)$ is called an admissible system, if

(1) $(\Omega,\mathcal{F},P)$ is a probability space

(2) u is Γ- valued measurable process

(3) Y is a d'- dimensional $(\mathcal{F}_t)$ Brownian motion where

$$\mathcal{F}_t = \sigma\{ Y(s), \int_0^s u(\tau)\, d\tau \ ; s \le t \}$$

(4) $\hat{W}$ is a d- dimensional Brownian motion

(5) ξ is a d- dimensional random variable and its distribution has the density ϕ

(6) ξ , $\hat{W}$ and (Y,u) are independent with respect to P .

For an admissible system $\mathscr{A}$, the solution $X(t) = X^{\mathscr{A}}(t)$ of the SDE (4.6) is called the response for $\mathscr{A}$. Putting $d\hat{P} = \rho(T)dP$, we define the pay-off function by

$$(4.7) \quad J(\mathscr{A}) = \hat{E}\left[\int_0^T f(X^{\mathscr{A}}(t))\, dt + g(X^{\mathscr{A}}(T)) \right]$$

where f, $g \in L^2(\mathbb{R}^d)$ and non-negative.

By the similar argument as Rozovskii [25], we obtain the following.

Proposition 4.1 Let $p^{\mathscr{A}}$ be a solution of the SPDE (2.4) for an admissible system $\mathscr{A}$, then $p^{\mathscr{A}}(t)$ is the unnormalized conditional density of $X^{\mathscr{A}}(t)$ with respect to $\mathscr{F}_t$. Namely, for every $\varphi \in L^{\infty}(\mathbb{R}^d)$, $t \in [0,T]$

$$(4.8) \quad E[\varphi(X^{\mathscr{A}}(t))\rho(t) \mid \mathscr{F}_t] = (\varphi , p^{\mathscr{A}}(t)) \quad \text{P-a,s.}$$

holds, where $(\cdot,\cdot)$ is the inner product in $L^2(\mathbb{R}^d)$.

Using (4.8), we get

$$(4.9) \quad J(\mathscr{A}) = E\left[\int_0^T (f , p^{\mathscr{A}}(t))\, dt + (g , p^{\mathscr{A}}(T)) \right]$$

Since $(f , p^{\mathscr{A}}(t))$ and $(g , p^{\mathscr{A}}(T))$ are non-negative, Theorem 3.2 assures the existence of an optimal admissible system. Namely,

Theorem 4.1 There exists an optimal admissible system $\tilde{\mathscr{A}}$, that is

$$\inf_{\mathscr{A}\,:\,\text{ad.sys.}} J(\mathscr{A}) = J(\tilde{\mathscr{A}}) \qquad (4.10)$$

CHAPTER 2

Optimal controls for stochastic partial differential equations

1 Introduction

In this chapter we are concerned with control problems of systems governed by the following stochastic partial differential equations (SPDE in short)

$$
(1.1)\begin{cases} dq(t,x)= \sum_{i,j=0}^{d} \frac{\partial}{\partial x_i}\left(a^{ij}(x,y+W(t),U(t))\frac{\partial}{\partial x_j} q(t,x) \right. \\ \qquad\qquad \left. + f^i(x,y+W(t),U(t)) \right)dt \\ \quad + \sum_{k=1}^{d'} \left(\sum_{i=0}^{d} b_k^i(x,y+W(t))\frac{\partial}{\partial x_i} q(x,t) + g_k(x,y+W(t))\right) dW^k(t) \end{cases}
$$

where $W = (W^1,\cdots,W^{d'})$ is a d'- dimensional standard Wiener process and $U(t)$ an admissible control, $0 \le t \le T$, with T fixed.

The problem is to minimize a given criterion by choosing a suitable admissible control. Namely, we treat stochastic optimal controls for distributed parameter systems. The SPDE (1.1) describes intuitively a physical object governed by a partial differential equation with random perturbation, which has been investigated from various viewpoints (cf Y.Fujita [8], N.V.Krylov & B.Rozovskii [11,14,15], H.Kunita [16], E.Pardoux [22], J.B.Walsh [29]). But other important example is the Zakai equation for controlled partially observed diffusions (cf [2], [4], [7], [18], [24]). In this case, inhomogeneous terms f^i and g_k are zero and $b^i{}_k$ arises from the correlation between system and observation noises. Moreover the Wiener process W is the obserbation process and the coefficients a^{ij} and $b^{i}{}_{k}$ depend on W (cf [7], [24]).

The main aim of this chapter is to show the existence of an optimal relaxed control for systems governed by the SPDE (1.1) under the ellipticity condition (see (A.2)); in particular we assume that $\left(a^{ij}(x,y,u) - \frac{3}{2} \sum_{k=1}^{d'} b^{i}{}_{k}(x,y) b^{j}{}_{k}(x,y) \right)_{i,j=1,\cdots,d}$ is non-negative definite and some regularity conditions on the coefficients. In particular, if $b^{i}{}_{k} = 0$ for $i = 1,\cdots,d$, $k = 1,\cdots,d'$, then the matrix $(a^{ij}(x,y,u))_{i,j=1,\cdots,d}$ may be degenerate.

Let Γ be a compact convex subset of $\mathbb{R}^{L}$. We call it a control region. Λ denotes the set of all measures on $[0,T] \times \Gamma$, such that $\lambda([0,t] \times \Gamma) = t$ for any $t \in [0,T]$. The relaxed control, which is introduced in [4] and [6], is a Λ-valued random variable (see Definition 2.1) and acts linearly on coefficients. Thus a relaxed control μ has a density μ', namely $\mu(dt,du) = \mu'(t,du)dt$, and when we apply a relaxed control μ, the coefficients a^{ij} and f^{i} are replaced by the following $\tilde{a}^{ij}$ and $\tilde{f}^{i}$ respectively,

$$\tilde{a}^{ij}(t,x,y + W(t),\mu) = \int_{\Gamma} a^{ij}(x,y + W(t),u)\mu'(t,du)$$

and

$$\tilde{f}^{i}(t,x,y + W(t),\mu) = \int_{\Gamma} f^{i}(x,y + W(t),u)\mu'(t,du).$$

Moreover, the system moves according to the following SPDE (1.2)

$$(1.2)\begin{cases} dq(t,x) = \displaystyle\sum_{i,j=0}^{d} \frac{\partial}{\partial x_i} \left(\tilde{a}^{ij}(x,y + W(t),\mu)\frac{\partial}{\partial x_j} q(t,x) \right. \\ \qquad\qquad\qquad\qquad \left. + \tilde{f}^{i}(x,y + W(t),\mu) \right) dt \\ \quad + \displaystyle\sum_{k=1}^{d'} \left(\sum_{i=0}^{d} b_{k}^{i}(x,y + W(t))\frac{\partial}{\partial x_i} q(x,t) + g_{k}(x,y + W(t)) \right) dW^{k}(t). \end{cases}$$

Now Λ bocomes a compact metric space, by being endowed with the weak convergence topology, and the set of all relaxed controls turns out to be a compact metric space by being endowed with the Prohorov metric. Consequently, for our aim, it is enough to show that the solution of the SPDE (1.2) depends on the relaxed control continuously. But this is a difficult problem. We overcome this obstacle, by using a method similar to that used by N. Nagase [18] and the evaluations for SPDE given by N.V. Krylov & B. Rozovskii [14]. By this means, we can prove the existence of an optimal relaxed control. Moreover, by applying our existence theorems to the Zakai equation, we can obtain an optimal control for partially observed diffusions with correlated noise (see Section 7). This result is new, and is a generalization of [2,4,6,7 and 18].

In Section 2, we will introduce several metric spaces which are appropriate to our control problems and define a relaxed systems in wider sense as a generalization of an admissible control. In Section 3, we study the way in which the solution depends on the initial data and the relaxed system. In particular, we will prove the continuous dependence of the solution on the relaxed system, when we endow with the weak convergence topology on the space of image measures of relaxed systems [Theorems 3.1 and 3.2]. Section 4 is concerned with existence theorems [Theorems 4.1 and 4.2]. In Section 5, we will construct an approximate optimal control which is adapted to a Wiener process. Since the Wiener process in the Zakai equation is nothing but the observation process, we have an approximate optimal control, which is a function of the observed data, for partially observed diffusions. The Bellman principle will be proved in Section 6 and some applications will be discussed in

Section 7.

2 Preliminaries

Let us define the operators L and $M = (M_1, \cdots, M_{d'})$ by

$$(2.1)\quad L(y,u)\psi(x) = \sum_{i,j=0}^{d} \partial_i (a^{ij}(x,y,u)\partial_j\psi(x) + f^i(x,y,u))$$

and

$$(2.2)\quad M_k(y)\psi(x) = \sum_{i=0}^{d} b^i{}_k(x,y)\partial_i\psi(x) + g_k(x,y)$$

$$\text{for } x \in \mathbb{R}^d ,\ y \in \mathbb{R}^{d'} ,\ u \in \Gamma$$

respectively, where ∂_0 = identity and $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1,\cdots,d$ and a^{ij}, f^i, $b^i{}_k$ and g_k are bounded and uniformly continuous.

We denote by L^2_r , $r \geq 0$, the space of real valued Borel functions on $\mathbb{R}^d$ with the norm defined by :

$$\|f\|_{0,r} = \left(\int_{\mathbb{R}^d} |(1 + |x|^2)^{r/2} f(x)|^2 \, dx \right)^{1/2}$$

Let H^m_r be the subspace of L^2_r consisting of functions whose generalized derivatives up to the order m belong to L^2_r.
Clearly H^m_r becomes a Hilbert space with the inner product

$$(f,g)_{m,r} = \sum_{|\alpha|\leq m} \frac{|\alpha|!}{\alpha^1!\cdots\alpha^d!} \int_{\mathbb{R}^d} (1 + |x|^2)^r D^\alpha f(x) D^\alpha g(x) \, dx,$$

where $\alpha = (\alpha^1, \cdots ,\alpha^d)$ is a multi-index with non-negative integer α^i, $|\alpha| = \alpha^1 + \cdots + \alpha^d$ and $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha^1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha^d}$.
Let us set $\|f\|^2_{m,r} = (f,f)_{m,r}$ and, for $r = 0$, $L^2_0 = L^2$, $H^m_0 = H^m$, $(\cdot,\cdot)_{m,0} = (\cdot,\cdot)_m$ and $\|\cdot\|_{m,0} = \|\cdot\|_m$, for simplicity, if no confusion occurs.

Now we introduce the following conditions.

(A.1) $D^{\alpha}a^{ij}$, $D^{\alpha}b^{i}_{k}$ ($0 \le |\alpha| \le m+1$, $i,j = 0,1,\cdots,d$, $k = 1,\cdots,d'$) are bounded and uniformly continuous,

(A.2) ellipticity condition : $a^{ij} = a^{ji}$, $i,j = 1,\cdots,d$, and $\left(a^{ij} - \frac{3}{2} b^{i}\cdot b^{j} \right)_{i,j = 1,\cdots,d}$ is a non-negative definite matrix , where $b^{i} = (b^{i}_{1}, \cdots, b^{i}_{d'})$ and " $\cdot$ " means the inner product in $\mathbb{R}^{d'}$.

(A.3) $f^{i}(\cdot,y,u)$, $g_{k}(\cdot,y) \in H^{m+1}$; $i = 0,\cdots,d$, $k = 1,\cdots,d'$, and their H^{m+1}- norms are bounded in $(y,u) \in \mathbb{R}^{d'} \times \Gamma$.

$(A.4)_{\ell,r}$ $f^{i}(\cdot,y,u)$, $g_{k}(\cdot,y) \in H^{\ell+1}_{r}$ and their $H^{\ell+1}_{r}$- norms are bounded in y and u.

$(A.4)_{\ell}$ For some $r > 0$, $(A.4)_{\ell,r}$ holds.

Hereafter we always assume (A.1) ~ (A.3) and, for simplicity, we say

(2.3) $| D^{\alpha}a^{ij}(x,y,u) | \le K$, $| D^{\alpha}b^{i}_{k}(x,y) | \le K$,

$$\| f^{i}(\cdot,y,u) \|_{m+1} \le K , \quad \| g_{k}(\cdot,y) \|_{m+1} \le K .$$

To study relaxed systems (in wider sense), we need the following spaces.

By Λ we denote the set of all measures λ on $[0,T] \times \Gamma$ such that

(2.4) $\lambda([0,s] \times \Gamma) = s$, for $s \le T$.

Endowing with the weak convergence topology, we have the following proposition,

Proposition 2.1 Λ is a compact metric space.

Proof By applying the Prohorov metric, Λ becomes a separable metric space. Suppose $\lambda_n \in \Lambda$ tends to λ weakly as $n \longrightarrow \infty$. Then $\lambda_n(\cdot \times \Gamma) \longrightarrow \lambda(\cdot \times \Gamma)$ weakly as a measure on $[0,T]$. Since $\lambda_n(\cdot \times \Gamma)$ is Lebesgue measure by (2.4), $\lambda(\cdot \times \Gamma)$ also satisfies (2.4). Since Λ is tight , by virtue of compactness of $[0, T] \times \Gamma$, this completes the proof. □

Let us set $B(\Gamma)$ = Borel field on Γ, $\sigma_t(\Lambda)$ = the σ-field generated by $\{ \lambda([0,s] \times A) ; s \le t, A \in B(\Gamma) \}$ and $\sigma(\Lambda) = \sigma_T(\Lambda)$. Let $\mathcal{P} = \mathcal{P}(\Lambda)$ be the space of probabilities on $(\Lambda, \sigma(\Lambda))$, endowed with the weak convergence topology. Then Prohorov's theorem asserts,

Proposition 2.2 $\mathcal{P}$ is a compact metric space.

By virtue of (2.4), λ has a $\sigma_t(\Lambda)$-adapted kernel λ', namely, $\lambda(dt,du) = \lambda'(t,du)dt$, and $\lambda'(t,\cdot)$ is a probability on Γ for almost all t. Moreover, if λ^* is a kernel of λ, then $\lambda'(t,\cdot) = \lambda^*(t,\cdot)$ for almost all t. Let us set

$$\tilde{h}(t,x,y,\lambda) = \int_\Gamma h(x,y,u)\lambda'(t,du) \quad \text{for} \quad h = a^{ij} \text{ and } f^i$$

and

$$(2.5) \quad \tilde{L}(t,y,\lambda)\psi(x) = \int_\Gamma L(y,u)\psi(x)\,\lambda'(t,du)$$

$$= \sum_{i,j=0}^{d} \partial_i \left(\tilde{a}^{ij}(t,x,y,\lambda)\partial_j\psi(x) + \tilde{f}^i(t,x,y,\lambda) \right)$$

Now we introduce a relaxed system, according to [4] & [6].

Definition 2.1 $\mathcal{R} = (\ \Omega\ ,\ \mathcal{F}\ ,\ \mathcal{F}_t\ ,\ P\ ,\ W\ ,\ \mu\ )$ is called a relaxed system, if

(2.6) $(\ \Omega,\mathcal{F},\mathcal{F}_t,P\ )$ is a probability space with filteration $\mathcal{F}_t$;

(2.7) W is an $\mathcal{F}_t$-adapted d'-dimensional Wiener process with $W(0) = 0$;

and

(2.8) μ is an $\mathcal{F}_t$-adapted Λ-valued random variable (Λ - r.v. in short). Namely, $\mu(\ B_1\times B_2\ )$ is $\mathcal{F}_t$-measurable whenever $B_1 \in B[\ 0\ ,\ t\ ]$ and $B_2 \in \sigma(\Gamma)$(= topological σ-field on Γ).

For simplicity, we put $\mathcal{R} =(\ W\ ,\ \mu\ )$, if no confusion occurs, and sometimes we call μ a relaxed control.

$\mathcal{A} = (\ \Omega\ ,\ \mathcal{F}\ ,\ \mathcal{F}_t\ ,\ P\ ,\ W\ ,\ U\ )$ is an admissible system , if (2.8) is replaced by (2.9) below.

(2.9) U is a Γ-valued $\mathcal{F}_t$-adapted process.

Remark Since $U(t)$ is regarded as $\mu'(t,\cdot) = \delta_{U(t)}$, where δ_x means δ-measure at x , $\mathcal{A}$ is also a relaxed system.

$\mathfrak{R}$ and $\mathfrak{U}$ denote the totalities of relaxed and admissible systems respectively. Let $\pi(\mathcal{R})$ be the image measure of (W, μ) on $C(0,T;\mathbb{R}^d) \times \Lambda$. Again endowing with the weak convergence topology on the space $\Pi = \{\ \pi(\mathcal{R})\ ;\ \mathcal{R} \in \mathfrak{R}\ \}$, we have the following proposition.

Proposition 2.3 Π is a compact metric space.

Proof is easy, since W is a Wiener process and Λ is a compact metric space. □

Definition 2.2 We say $\mathcal{R}_n$ converges to $\mathcal{R}$, (put $\mathcal{R}_n \longrightarrow \mathcal{R}$) if, $\pi(\mathcal{R}_n) \longrightarrow \pi(\mathcal{R})$ weakly.

Consider the SPDE (2.10) for $\mathcal{R} = (\Omega, \mathcal{F}, \mathcal{F}_t, P, W, \mu)$,

$$(2.10)\quad \begin{cases} dq(t) = \tilde{L}(t, y + W(t), \mu)q(t)dt \\ \qquad\qquad\qquad + M(y + W(t))q(t)dW(t) \\ q(0) = \phi \quad (\in H^m) \end{cases}$$

An H^1-valued $\mathcal{F}_t$-adapted process $q = q(\cdot, \phi, \mathcal{R})$ is called a solution of (2.10) (or a response for $\mathcal{R}$), if (2.11) and (2.12) hold.

$$(2.11)\quad E\left\{ \int_0^T \| q(t) \|_{1,0}^2 \, dt \right\} < \infty$$

and, for any $\eta \in C_0^\infty$ (smooth function on $\mathbb{R}^d$ with compact support) and almost all t,

$$(2,12)\quad (q(t), \eta) = (\phi, \eta) + \int_0^t \langle \tilde{L}(s, y + W(s), \mu)q(s), \eta \rangle \, ds + \int_0^t (M(y + W(s))q(s), \eta) \, dW(s), \text{ w.p. } 1$$

hoids, where $(\cdot,\cdot) = L^2(\mathbb{R}^d)$-inner product and $\langle\ ,\ \rangle$ = duality pairing between H^{-1} and H^1 under $H^0 = (H^0)^*$ (= dual space of H^0), namely

$\langle \tilde{L}(s, y + W(s), \mu)q(s), \eta \rangle$

$$= \sum_{i,j=0}^{d} (-1)^{|i|} (\tilde{a}^{ij}(s,\cdot,y + W(s),\mu)\partial_j q(s) \ , \ \partial_i \eta),$$

where $|i| = 0$ (for $i = 0$), $= 1$ (for $i = 1,\cdots,d$).

Clearly (2.12) does not depend on any special choice of derivative μ'. The SPDE (2.10) can be regarded as an H^{-1}-valued SDE. (See K. Itô [10] for the general theory of Hilbert space valued SDE.)

According to [14] and [15], we see the following theorem.

Theorem 2.1 (Krylov & Rozovskii) (I) Suppose the conditions (A.1) ~ (A.3). Then, the SPDE (2.10) has a unique solution $q \in L^2([0,T] \times \Omega \ ; \ H^m) \cap L^2(\Omega \ ; \ C(0,T;H^{m-1}))$.

$q(t)$ is a Borel function of $\{ \phi \ , \ W(s) \ s \le t \ , \ \mu([0,s]\times B) \ s \le t \ \ B \in \sigma(\Gamma) \}$ and there exists a constant N, depending only on T and K in (2.3), such that

$$(2.13) \quad E\{ \sup_{t \le T} \| q(t) \|_{\ell,0}^2 \} \le N \{ \| \phi \|_{\ell,0}^2 + \sup_{y,u} \sum_{i=0}^{d} \| \partial_i f^i(\cdot,y,u) \|_{\ell,0}^2 + \sup_{y} \| g(\cdot,y) \|_{\ell+1,0}^2 \} \ , \quad \ell = 0,1,\cdots,m$$

(II) Besides (A.1) ~ (A.3) , we assume $(A.4)_{\ell,r}$ and $\phi \in H_r^\ell$,

Then the following evaluation holds.

$$(2.14) \quad E\{ \sup_{t \le T} \| q(t) \|_{\ell,r}^2 \} \le N' \{ \| \phi \|_{\ell,r}^2 + \sup_{y,u} \sum_{i=0}^{d} \| \partial_i f^i(\cdot,y,u) \|_{\ell,r}^2 + \sup_{y} \| g(\cdot,y) \|_{\ell+1,r}^2 \}$$

where $N' = N'(T,K,r)$

(III) Suppose $F^i : [0,T] \times \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R}^1$, $i = 0,1,\cdots d$,

and $G_k : [0,T] \times \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R}^1$, $k = 1,\cdots,d$, are $\mathcal{F}_t$-adapted and

$$E\{ \int_0^T \| F^i(t) \|^2_{m+1,0}\, dt \} < \infty , \quad E\{ \int_0^T \| G_k(t) \|^2_{m+1,0}\, dt \} < \infty .$$

Let ξ be a solution of the following SPDE;

$$(2.15)\begin{cases} d\xi(t) = \sum_{i,j=0}^{d} \partial_i (\tilde{a}^{ij}(t,y + W(t),\mu)\partial_j\xi(t) + F^i(t))dt \\ \qquad + (\sum_{i=0}^{d} b^i(t,y + W(t))\partial_i\xi(t) + G(t))dW(t) \\ \xi(0) = \varphi \in H^m \end{cases}$$

Then, ξ satisfies the following evaluation (2.16)

$$(2.16)\quad E\{ \sup_{t\le T} \| \xi(t) \|^2_{\ell,0} \} \le N(\| \varphi \|^2_{\ell,0} + E\{ \int_0^T (\sum_{i=0}^{d} \| \partial_i F^i(t) \|^2_{\ell,0} + \sum_{k=1}^{d'} \| G_k(t) \|^2_{\ell+1,0})dt \}$$

$(\ell = 0,1,\cdots,m)$ where $N = N(T,K)$.

Remark Krylov & Rozovskii proved Theorem 2.1, replacing (A.2) by a weaker condition (A.2′).

(A.2′) $a^{ij} = a^{ji}$, $i,j = 1,\cdots,d$ and $\left(a^{ij} - \frac{1}{2} b^i\cdot b^j \right)_{i,j = 1,\cdots,d}$ is a non-negative definite matrix .

But we state all of our theorems under the condition (A.2), since we need (A.2) for Proposition 3.1 etc.

3 Continuous dependence of $q(,\phi,y,\mathfrak{R})$ on ϕ , y , $\mathfrak{R}$.

Since we are mainly concerned with the probability law $\pi(\mathfrak{R})$, we may assume the following canonical form , if necessary:

$\Omega = C(0,T;\ \mathbb{R}^{d'}) \times \Lambda$, $\mathcal{F} = \sigma(\Omega)$ = the topological σ-field on Ω

W = the first coordinate function on Ω , $W(t,\omega) = W(\omega)(t)$

μ = the second coordinate function on Ω ,

$\mu(B,\omega) = \mu(\omega)(B), \; B \in \sigma([0,T] \times \Gamma)$

$\mathcal{F}_t = \sigma\{ W(s) , s \le t , \mu(B_1 \times B_2) , B_1 \in B[0,t] , B_2 \in \sigma(\Gamma) \}$

$P = \pi(\mathcal{R})$.

First we see the following lemma, which is crucial to the SPDE with ellipticity condition (A.2). So it will be proved in the Appendix, according to [14].

Lemma (special case of Lemma 2.1 of [14])

For any $t \in [0,T]$, $y \in \mathbb{R}^{d'}$ and $\lambda \in \Lambda$, put $a^{ij}(\cdot) = a^{ij}(t,\cdot,y,\lambda)$ and $b^i(\cdot) = b^i(\cdot,y)$, for simplicity. Under the conditions (A.1) and (A.2), there exists a constant N, depending only on K in (2.3), T, and ℓ ($= 0,1,\cdots,m$), such that

$$(*) \quad \int \sum_{|\gamma|\le\ell} \left(2 \sum_{i,j=0}^{d} (-1)^{|i|} D^\gamma \partial_i u \, D^\gamma (a^{ij}\partial_j u + \hat{f}^i) + 3 \left| D^\gamma (\sum_{i=0}^{d} b^i \partial_i u + \hat{g}) \right|^2 \right) dx$$

$$\le N \{ \|u\|_\ell^2 + \sum_{i=0}^{d} \|\partial_i \hat{f}^i\|_\ell^2 + \sum_{k=0}^{d'} \|\hat{g}_k\|_{\ell+1}^2 \}$$

for any fixed three functions u, $\hat{f}^i$, $\hat{g}_k \in H^{\ell+1}$ and $\hat{g} = (\hat{g}_1,\cdots,\hat{g}_{d'})$.

Remark (1) When we take $\mathcal{F}^i(\cdot,y,\mu)$ and $g(\cdot,y)$ of (2.1) as $\hat{f}^i$ and $\hat{g}$ respectively, (*) turns out to be the following form:

$$2\langle \tilde{L}(t,y,\mu)u , u \rangle_\ell + 3 | M(y)u |_\ell^2$$
$$\le N \{ \|u\|_\ell^2 + \sum_{i=0}^{d} \|\partial_i \mathcal{F}^i(\cdot,y,\mu)\|_\ell^2 + \sum_{k=0}^{d'} \|g_k(\cdot,y)\|_{\ell+1}^2 \}$$

(2) [14] says that Lemma holds under the conditions (A.1) and (A.2′), if we replace "3" of the integrand of the left hand side with "1". So a stronger condition (A.2) yields a stronger evaluation

(*), which is necessary for Proposition 3.1.

Proposition 3.1 There is a constant $C = C(T,K,\ell)$ such that

$$(3.1)\quad \sup_{t\le T} E\{ \| q(t) \|_\ell^4 \} \le C\{ \| \phi \|_\ell^4 + \sup_{y,u} \sum_{i=0}^{d} \| \partial_i f^i(\cdot,y,u) \|_\ell^4 + \sup_y \| g(\cdot, y) \|_{\ell+1}^4 \},$$

$$\ell = 0,1,\cdots,m-1.$$

Proof For simplicity, we put $\mathcal{F}(t) = \mathcal{F}(t,y + W(t),\mu)$ $g(t) = g(y + W(t))$, $\tilde{L}(t) = \tilde{L}(t,y + W(t),\mu)$, $M(t) = M(y + W(t))$ and $\langle\ ,\ \rangle_\ell$ = duality pairing between $H^{\ell-1}$ and $H^{\ell+1}$ under $H^\ell = (H^\ell)^*$.

Then q satisfies

$$(3.2)\quad (q(t) , \eta)_\ell = (\phi , \eta)_\ell + \int_0^t \langle \tilde{L}(s)q(s) , \eta \rangle_\ell \, ds + \int_0^t (M(s)q(s) , \eta)_\ell \, dW(s)$$

$$\text{for } \eta \in H^{\ell+1} , t \le T .$$

So Ito's formula derives

$$(3.3)\quad \| q(t) \|_\ell^2 = \| \phi \|_\ell^2 + 2\int_0^t \langle \tilde{L}(s)q(s) , q(s) \rangle_\ell \, ds + \int_0^t \| M(s)q(s) \|_\ell^2 \, ds + 2\int_0^t (M(s)q(s) , q(s))_\ell \, dW(s).$$

Thus we see

$$(3.4)\quad E[\| q(t) \|_\ell^4] - \| \phi \|_\ell^4$$

$$= 2 E\{ \int_0^t \| q(s) \|_\ell^2 \{ 2 \langle \tilde{L}(s)q(s) , q(s) \rangle_\ell + \| M(s)q(s) \|_\ell^2 \} ds \}$$

$$+ 4 E\{ \int_0^t (M(s)q(s) , q(s))_\ell^2 ds \}$$

$$\le 2 E\{ \int_0^t \| q(s) \|_\ell^2 \{ 2 \langle \tilde{L}(s)q(s) , q(s) \rangle_\ell + 3 \| M(s)q(s) \|_\ell^2 \} ds \}$$

$$\le C_1 E[\int_0^t \|q(s)\|_\ell^2 (\|q(s)\|_\ell^2 + \sum_{i=0}^d \|\partial_i \mathcal{F}^i(s)\|_\ell^2 + \|g(s)\|_{\ell+1}^2)ds]$$

appealing to Lemma. Hence we have

$$(3.5) \quad E[\| q(t) \|_\ell^4] \le C_2 \Big(E[\int_0^t \| q(s) \|_\ell^4 ds] + \| \phi \|_\ell^4 + E[\int_0^t (\sum_{i=1}^d \| \partial_i \mathcal{F}^i(s) \|_\ell^4 + \| g(s) \|_{\ell+1}^4) ds] \Big).$$

So Gronwall's inequality completes the proof. □

Now we will study continuous dependence of $q(\cdot,\phi,y,\mathfrak{R})$ on $\mathfrak{R}$. For the following theorem 3.1, we endow with the weak topology on $L^2(0,T;H^m)$ and H^{m-1}. Later Theorem 3.2 is concerned with strong topology on these spaces.

From now on, we always assume $m \ge 3$.

Theorem 3.1 Suppose $\mathfrak{R}_n \longrightarrow \mathfrak{R}$. Then, for $\phi \in H^m$ and $y \in \mathbb{R}^d$, we have

$$(3.6) \quad (W_n,\mu_n,q(\cdot,\phi,y,\mathfrak{R}_n)) \longrightarrow (W,\mu,q(\cdot,\phi,y,\mathfrak{R})) \text{ in law}$$

as $C(0,T; \mathbb{R}^{d'}) \times \Lambda \times [w - L^2(0,T; H^m)]$- r.v.

$$(3.7) \quad (W_n,\mu_n,q(t,\phi,y,\mathfrak{R}_n)) \longrightarrow (W,\mu,q(t,\phi,y,\mathfrak{R})) \text{ in law}$$

as $C(0,T; \mathbb{R}^{d'}) \times \Lambda \times [w - H^{m-1}]$ - r.v.

where " w - X " denotes the space X carrying the weak topology.

Proof This theorem is an extension of Theorem 3.1 in [18] to the elliptic case (A.2) and we can apply the same method as [18], using the evaluation (*). First we introduce two spaces $\mathcal{H}_\gamma(D)$ and $\mathcal{H}_\gamma(D,T)$. Let D be a bounded open set of $\mathbb{R}^d$, with smooth boundary. Define $\mathcal{H}_\gamma(D)$ and $\mathcal{H}_\gamma(T,D)$ as follows (cf [13]).

(3.8) $$\mathcal{H}_\gamma(D) = \{ \varphi \in L^2(-\infty,\infty;H^{m-1}(D)) \;;\; \int_{-\infty}^{\infty} |\tau|^{2\gamma} \|\hat{\varphi}(\tau)\|_*^2 \, d\tau < \infty \}$$

with the norm

(3.9) $$\| \varphi \|^2_{\mathcal{H}_\gamma(D)} = \int_{-\infty}^{\infty} \|\varphi(t)\|^2_{H^{m-1}(D)} dt + \int_{-\infty}^{\infty} |\tau|^{2\gamma} \|\hat{\varphi}(\tau)\|_*^2 \, d\tau$$

where, for simplicity, we put $\hat{\varphi}(\tau) = \int_{-\infty}^{\infty} \exp(-2\pi i\tau t)\varphi(t)\, dt$ in this proof and $\|\cdot\|_*$ = norm of $(H^{m-1}(D))^*$ $\Big($ = dual space of $H^{m-1}(D)$ under $H^{m-2}(D) = (H^{m-2}(D))^*$ $\Big)$

and

(3.10) $$\mathcal{H}_\gamma(T,D) = \{ \varphi|_{[0,T]} \;;\; \varphi \in \mathcal{H}_\gamma(D) \}$$

with the norm

(3.11) $$\| \varphi \|_{\mathcal{H}_\gamma(T,D)} = \inf\{ \| \psi \|_{\mathcal{H}_\gamma(D)} \;;\; \varphi = \psi \text{ a.e. on } [0,T] \}$$

respectively.

Now we divide the proof into three steps. The 1st step is the preliminary lemma, which is useful for proving the compactness of space of solutions.

Lemma 3.1 For any fixed $\gamma \in (0 , 1/4)$,

(3.12) $$q(\;,\phi,y,\mathcal{R}) \in \mathcal{H}_\gamma(T,D) , \quad \text{w.p. } 1$$

holds, and there is a constant $K_1 = K_1(T,K)$, such that

$$(3.13)\quad E\{ \| q(\ ,\phi,y,\mathfrak{R}) \|^2_{\mathcal{H}_\gamma(T,D)} \} \le K_1 I_m(\phi,f,g), \quad \text{for } \forall\, \mathfrak{R} \in \Re,$$

where

$$(3.14)\quad I_m(\phi,f,g) = \| \phi \|_m^2 + \sup_{y,u} \sum_{i=0}^{d} \| \partial_i f^i(\cdot,y,u) \|_m^2 + \sup_{y} \| g(\cdot,y) \|^2_{m+1}.$$

Proof Put

$$h(t) = \begin{cases} h(t,\mathfrak{R}) , & t \in [0,T] \\ 0 , & t \notin [0,T], \end{cases}$$

for $h(t,\mathfrak{R}) = q(t,\phi,y,\mathfrak{R})$, $f(\cdot,y + W(t),\mu)$ and $g(\cdot,y + W(t))$. $\tilde{L}(t)$ and $M(t)$, $t \in (-\infty,\infty)$, are defined in the same way as (2.5) and (2.2), respectively. Since q is a solution, the following equality (3.15) holds, for any $\eta \in H^m$,

$$(3.15)\quad (q(t) , \eta)_{m-1} = (\phi , \eta)_{m-1} + \int_0^t \langle \tilde{L}(s)q(s) , \eta \rangle_{m-1} ds + \int_0^t (M(s)q(s) , \eta)_{m-1} dW(s).$$

Therefore we have

$$(3.16)\quad 2\pi i\tau (\hat{q}(\tau),\eta)_{m-1} = \int_{-\infty}^{\infty} (- \frac{d}{dt} \exp(-2\pi i\tau t))(q(t),\eta)_{m-1} dt$$

$$= (\phi,\eta)_{m-1} - \exp(-2\pi i\tau T)(q(T),\eta)_{m-1} + \langle \widehat{Lq}(\tau),\eta \rangle_{m-1} + \int_0^T \exp(-2\pi i\tau t)(M(t)q(t),\eta)_{m-1} dW(t).$$

Let $\eta_j \in C_0^\infty(\mathbb{R}^d)$, $j = 1,2,\cdots$ be a complete orthonormal system of H^m. Then we get

$$(3.17)\quad 4\pi^2\tau^2 E[\| \hat{q}(\tau) \|^2_{m-2}] = 4\pi^2\tau^2 \sum_{j=1}^{\infty} E\{ |(\hat{q}(\tau),\eta_j)_{m-1}|^2 \}$$

$$\le C_1 \Big(\|\phi\|^2_{m-2} + E\{ \| q(T) \|^2_{m-2} \} + E\{ \| \widehat{Lq}(\tau) \|^2_{m-2} \}$$

$$+ \sum_{k=1}^{d'} E \int_0^T \| M_k(t)q(t) \|_{m-2}^2 dt \Big).$$

Since $\|\cdot\|_* \le \|\cdot\|_{m-2} \le \|\cdot\|_{m-1}$, we see

$$\tau^2 E[\, \| \hat{q}(\tau) \|_*^2 \,] \le C_2 \Big(\|\phi\|_{m-1}^2 + E[\, \| q(T) \|_{m-1}^2 \,] + E[\int_0^T (\| q(t) \|_m^2 + \| g(t) \|_{m-1}^2)\, dt \,] + E[\, \| \widehat{\tilde{L}q}(\tau) \|_{m-2}^2 \,] \Big)$$

$$\le C_3 \{ I_m(\phi, f, g) + E[\, \|\widehat{\tilde{L}q}(\tau)\|_{m-2}^2 \,] \}.$$

Hence for any fixed $\kappa \in (1, 3/2)$,

$$\int_{-\infty}^{\infty} E[\, |\tau|^{2\gamma} \| \hat{q}(\tau) \|_*^2 \,]\, d\tau$$

$$\le \int_{|\tau| \le 1} E[\, \| \hat{q}(\tau) \|_*^2 \,]\, d\tau + \int_{|\tau| > 1} E[\, \frac{2\, |\tau|^2}{1 + |\tau|^{\kappa}} \| \hat{q}(\tau) \|_*^2]\, d\tau$$

$$\le C_4 \Big(\int_{-\infty}^{\infty} E[\, \| \hat{q}(\tau) \|_{m-2}^2\, d\tau + I_m(\phi, y, \mathfrak{R}) \int_{-\infty}^{\infty} \frac{2}{1 + |\tau|^{\kappa}}\, d\tau + \int_{-\infty}^{\infty} E[\, \| \widehat{\tilde{L}q}(\tau) \|_{m-2}^2\, d\tau \Big)$$

$$\le C_5 \Big(\int_{-\infty}^{\infty} E(\| q(t) \|_m^2 + \| \tilde{L}q(t) \|_{m-2}^2)\, dt + I_m(\phi, f, g) \Big)$$

$$\le C_6\, I_m(\phi, f, g)$$

where $C_i = C_i(T, K)$. From this we get

$$(3.18) \quad E[\, \| q \|_{\mathcal{H}_\gamma(D)}^2 \,] \le K_1\, I_m(\phi, f, g),$$

and complete the proof of Lemma 3.1. □

2nd step. Let D_k ($k = 1, 2, \cdots$) be a bounded and open subset of $\mathbb{R}^d$ with smooth boundary, $\overline{D}_k \subset D_{k+1}$ and $\bigcup_{k=1}^{\infty} D_k = \mathbb{R}^d$. Define a metric d by

$$d(p,q) = \sum_{k=1}^{\infty} \frac{1}{2^k} \min\left(1 \,,\, \{ \int_0^T \| p(t) - q(t) \|_{H^{m-2}(D_k)}^2 dt \}^{1/2} \right)$$

for p , $q \in L^2(0,T;H^{m-2})$. $\mathscr{W}^{m-2}(0,T)$ denotes the completion of $L^2(0,T;H^{m-2})$ w.r.t. the metric d . Put $S_1 = C(0,T;\mathbb{R}^{d'}) \times \Lambda \times \mathscr{W}^{m-2}(0,T)$ and $S_2 = C(0,T;\mathbb{R}^{d'}) \times \Lambda \times [\, w - L^2(0,T;H^{m-2})\,]$.

For $\mathscr{R} = (W,\mu)$, $m_1(\mathscr{R})$ and $m_2(\mathscr{R})$ denote the image measures of $(W,\mu,q(\cdot,\phi,y,\mathscr{R}))$ on S_1 and S_2 respectively.

$B_r = \{ q \in L^2(0,T;H^{m-2}) \;;\; \|q\|_{\mathscr{H}_\gamma(T,D_k)} \le (2^k r)^{1/2} \;,\; k = 1,2,\cdots \}$ is compact in $\mathscr{W}^{m-2}(0,T)$, because the injection $\mathscr{H}_\gamma(T,D_k) \longrightarrow L^2(0,T;H^{m-2}(D_k))$ is a compact operator (cf. [17]).

On the other hand, Lemma 3.1 asserts

$$P(\, q(\cdot,\phi,y,\mathscr{R}) \notin B_r \,) \le K_1 I_m(\phi,f,g)/r.$$

Hence, $\{ m_1(\mathscr{R}) \,,\, \mathscr{R} \in \mathfrak{R} \}$ is relatively compact by Proposition 2.3. Moreover, $\{ m_2(\mathscr{R}) \,,\, \mathscr{R} \in \mathfrak{R} \}$ is also relatively compact by (2.13) and Remark 3.3 in [18].

3rd step. Suppose $\mathscr{R}_n \longrightarrow \mathscr{R}$. Then we can choose a subsequence $\{n_j\}$, such that $m_1(\mathscr{R}_{n_j})$ and $m_2(\mathscr{R}_{n_j})$ converge to some probability measures m_1 and m_2 respectively. So their marginal distributions on $C(0,T;\mathbb{R}^{d'}) \times \Lambda$ coincide with $\pi(\mathscr{R})$ and $i_k(\, m_1(\mathscr{R}_n)\,) = j_k(\, m_2(\mathscr{R}_n)\,)$ and $i_k(\, m_1\,) = j_k(\, m_2\,)$ ($k = 1,2,\cdots$), where

$i_k : S_1 \longrightarrow C(0,T;\mathbb{R}^{d'}) \times \Lambda \times L^2(0,T;H^{m-2}(D_k))$ and
$j_k : S_2 \longrightarrow C(0,T;\mathbb{R}^{d'}) \times \Lambda \times L^2(0,T;H^{m-2}(D_k))$ are the canonical injections.

$$m_1(\, C(0,T;\mathbb{R}^{d'}) \times \Lambda \times L^2(0,T;H^{m-2})\,) = 1$$

holds by (2.13).

Endowing with the metric d , we can apply Skorohod's theorem.

Hence, there exist S_1-valued random variables ($\hat{W}_{n_j}, \hat{\mu}_{n_j}, \hat{q}_{n_j}$) and ($\hat{W}, \hat{\mu}, \hat{q}$) on a suitable probability space ($\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}$), such that

(3.19) the law of ($\hat{W}_{n_j}, \hat{\mu}_{n_j}, \hat{q}_{n_j}$) = $m_1(\mathcal{R}_{n_j})$

The law of ($\hat{W}, \hat{\mu}, \hat{q}$) = m_1 (= limit measure of $m_1(\mathcal{R}_{n_j})$),

(3.20) with probability 1,

(I) $\hat{W}_{n_j} \longrightarrow \hat{W}$ uniformly on $[0,T]$

(II) $\hat{\mu}_{n_j} \longrightarrow \hat{\mu}$ weakly

(III) $\hat{q}_{n_j} \longrightarrow \hat{q}$ in $\mathcal{W}^{m-2}(0,T)$.

Moreover, since (3.1) implies the uniform integrability, we have

(IV) $\hat{q}_{n_j}\big|_{D_k} \longrightarrow \hat{q}\big|_{D_k}$ in $L^2([0,T]\times\Omega; H^{m-2}(D_k))$

for $k = 1,2,\cdots$.

Hence, from (I) and (II) , we see, for $\forall\, x \in \mathbb{R}^d$

(3.21)
$$\int_0^T \psi(t)\tilde{a}(t,x,y + \hat{W}_{n_j}(t),\hat{\mu}_{n_j})dt$$
$$= \int_0^T \psi(t) \int_\Gamma a(x,y + \hat{W}_{n_j}(t),u)\ \hat{\mu}'_{n_j}(t,du)\ dt$$
$$\xrightarrow[n_j \to \infty]{} \int_0^T \psi(t) \int_\Gamma a(x,y + \hat{W}(t),u)\ \hat{\mu}'(t,du)\ dt$$
$$= \int_0^T \psi(t)\tilde{a}(t,x,y + \hat{W}(t),\hat{\mu})\ dt$$

for any bounded continuous function ψ on $[0,T]$. Namely, we have

(3.22)
$$\tilde{a}(\cdot,x,y + \hat{W}_{n_j},\hat{\mu}_{n_j}) \longrightarrow \tilde{a}(\cdot, x, y + \hat{W}, \hat{\mu})$$

in [$w - L^2(0,T)$].

Since $\tilde{q}_{n_j}$ is a response for $\mathcal{R}_{n_j}$ = ($\hat{W}_{n_j}$, $\hat{\mu}_{n_j}$), we see, for any bounded absolutely continuous function ψ with $\psi' \in L^2(0,T)$ and

$\psi(T) = 0$, and $\eta \in C_0^\infty$,

$$(3.23)\quad \int_0^T \psi(t)\, d(\hat{q}_{n_j}(t), \eta)$$

$$= -\psi(0)(\phi, \eta) - \int_0^T (\hat{q}_{n_j}(t), \eta)\psi'(t)\, dt$$

$$= \int_0^T \langle \tilde{L}(t, y + \hat{W}_{n_j}(t), \hat{\mu}_{n_j})\hat{q}_{n_j}(t), \eta \rangle \psi(t)\, dt$$

$$+ \int_0^T \psi(t)\, (M(y + \hat{W}_{n_j}(t))\hat{q}_{n_j}(t), \eta)\, d\hat{W}_{n_j}(t).$$

Hence, we get, as $n_j \longrightarrow \infty$

$$(3.24)\quad - \int_0^T (\hat{q}(t), \eta)\psi'(t)\, dt$$

$$= \psi(0)(\phi, \eta) + \int_0^T \langle \tilde{L}(t, y + \hat{W}(t), \hat{\mu})\hat{q}(t), \eta \rangle \psi(t)\, dt$$

$$+ \int_0^T \psi(t)(M(y + \hat{W}(t))\hat{q}(t), \eta)\, d\hat{W}(t)$$

whenever supp $\eta \subset D_k$ for some k.

(3.24) yields that $\hat{q}$ is a response for $(\hat{W}, \hat{\mu})$. Since $\pi(\hat{W}, \hat{\mu}) = \pi(\mathfrak{R})$, we obtain

m_1 = the law of $(\hat{W}, \hat{\mu}, \hat{q}) = m_1(\mathfrak{R})$ and also $m_2 = m_2(\mathfrak{R})$.

This fact concludes (3.6).

In the same way we can prove (3.7). □

Now we will deal with $L^2(0,T;H^{m-2})$ and H^{m-2} instead of $[\ w - L^2(0,T;H^{m-2})\ ]$ and $[\ w - H^{m-2}\ ]$. Put $\Phi_r = H^m \cap H_r^{m-2}$, $r > 0$, with the norm $|\cdot|_r = \|\cdot\|_m + \|\cdot\|_{m-2,r}$. By applying [15], we evaluate $q(t,x)$ for large $|x|$.

Theorem 3.2 Suppose $(A.4)_{m-2,r}$ besides (A.1) ~ (A.3).

Then for $\phi \in \Phi_r$, we have

(3.25) $q(\cdot,\phi,y, \mathfrak{R}_n) \longrightarrow q(\cdot,\phi,y, \mathfrak{R})$
in law as $L^2(0,T;H^{m-2})$-r.v.,

and for any fixed t

(3.26) $q(t,\phi,y, \mathfrak{R}_n) \longrightarrow q(t,\phi,y, \mathfrak{R})$
in law as H^{m-2}- r.v.,

whenever $\mathfrak{R}_n \longrightarrow \mathfrak{R}$.

Proof By theorem 2.1, there exists a constant C depending only on T, K, r and ϕ such that

(3.27) $E[\int_{\mathbb{R}^d} (1 + |x|^2)^r \{D^\alpha q(t,x,\phi,y,\mathfrak{R})\}^2 \, dx] \le C$

for all t, α, $0 \le t \le T$, $0 \le |\alpha| \le m-2$, and $\mathfrak{R} \in \mathfrak{R}$.

Hence, we have

(3.28) $E[\int_{|x|>\rho} D^\alpha q(t,x)^2 \, dx] \le \dfrac{C}{(1 + \rho^2)^r}$.

By virtue of Skorohod's theorem, there exist $L^2(0,T;H^{m-2})$-valued random variables $\hat{q}_n$ and $\hat{q}$ on a suitable probability space ($\hat{\Omega},\hat{\mathcal{F}},\hat{P}$), such that

(3.29) $\hat{q}_n$ and $\hat{q}$ have the same laws as $q(\ ,\phi,y,\mathfrak{R}_n)$ and $q(\ ,\phi,y,\mathfrak{R})$ respectively,

and with probability 1

(3.30) $\hat{q}_n \longrightarrow \hat{q}$ in $L^2(0,T;H^{m-2}(D))$

for any bounded subset D of $\mathbb{R}^d$.

On the other hand , we see from (3.1)

(3.31) $E[\left(\int_D \{D^\alpha q(t,x)\}^2 \, dx \right)^2] \le E[\| q(t) \|_{m-2}^4] \le C'$
for $0 \le |\alpha| \le m-2$,

where C' is independent from D, t and $\mathfrak{R}$.

Since this implies the uniform integrability, we get

$$(3.32)\quad E\Big[\int_0^T \int_D \sum_{|\alpha|\le m-2} \{ D^\alpha \hat{q}_n(t,x) - D^\alpha \hat{q}(t,x) \}^2 \, dx \, dt \Big] \longrightarrow 0.$$

Combining (3.32) with (3.28), we obtain

$$E\Big[\int_0^T \| \hat{q}_n(t) - \hat{q}(t) \|_{m-2}^2 \, dt \Big] \longrightarrow 0.$$

This concludes (3.25).

For the proof of (3.26), we can apply the same argument. □

Putting

$$(3.33)\qquad \Phi = \bigcup_{r>0} \Phi_r ,$$

we see

Corollary 3.1 Suppose $\mathfrak{R}_n = (W_n , \mu_n)$ tends to $\mathfrak{R} = (W , \mu)$. Then, under the conditions (A.1) ~ (A.3), $(A.4)_{m-2}$ and $\phi \in \Phi$, there exist $\hat{\mathfrak{R}}_n = (\hat{W}_n , \hat{\mu}_n)$ and $\hat{\mathfrak{R}} = (\hat{W} , \hat{\mu})$, on a suitable probability space such that

(I) $\pi(W_n , \mu_n) = \pi(\hat{W}_n , \hat{\mu}_n)$, $\pi(W , \mu) = \pi(\hat{W} , \hat{\mu})$

and with probability 1,

(II) $\hat{W}_n \longrightarrow \hat{W}$ uniformly on $[0,T]$

(III) $\hat{\mu}_n \longrightarrow \hat{\mu}$ weakly

(IV) $\hat{q}_n \longrightarrow \hat{q}$ in $L^2(0,T;H^{m-2})$

(V) $\hat{q}_n(t) \longrightarrow \hat{q}(t)$ in H^{m-2}

where $\hat{q}_n$ and $\hat{q}$ are responses for $\hat{\mathfrak{R}}_n$ and $\hat{\mathfrak{R}}$ respectively.

Next we will study the dependence of q on the initial (ϕ, y).

Theorem 3.3

$$(3.34)\quad E[\sup_{t\le T} \| q(t,\phi,y,\mathfrak{R}) - q(t,\psi,y,\mathfrak{R}) \|_\ell^2] \le N \| \phi - \psi \|_\ell^2$$

($\ell = 0,1,\cdots,m$), where N is the constant of (2.13).

$$(3.35)\quad E[\sup_{t\le T} \| q(t,\phi,y_1,\mathfrak{R}) - q(t,\phi,y_2,\mathfrak{R}) \|_\ell^2]$$
$$\le N_1 (1 + I_{\ell+2}(\phi,f,g)) | y_1 - y_2 |^2)$$

($\ell = 0,1,\cdots,m-2$), where $N_1 = N_1(T,K)$.

Proof Put $p = q(\cdot,\phi,y,\mathfrak{R}) - q(\cdot,\psi,y,\mathfrak{R})$. Then p satisfies the following SPDE

$$(3.36)\quad \begin{cases} dp(t) = \sum_{i,j=0}^{d} \partial_i (\tilde{a}^{ij}(t,y + W(t),\mu)\partial_j p(t)dt \\ \qquad + \sum_{i=0}^{d} b^i(t,y + W(t))\partial_i p(t)\, dW(t) \\ p(0) = \phi - \psi \end{cases}$$

Therefore (2.13) derives (3.34).

Put $\xi = q_1 - q_2$ where $q_i = q(\cdot,\phi,y_i,\mathfrak{R})$. Then we have

$$(3.37)\quad \begin{cases} d\xi(t) = \{\tilde{L}_1(t)\xi(t) + (\tilde{L}_1(t) - \tilde{L}_2(t))q_2(t)\}dt \\ \qquad + \{M_1(t)\xi(t) + (M_1(t) - M_2(t))q_2(t)\}dW(t) \\ \xi(0) = 0 \end{cases}$$

where $\tilde{L}_i(t) = \tilde{L}(t,y_i + W(t),\mu)$ and $M_i(t) = M(t,y_i + W(t))$, $i = 1,2$. So (2.16) asserts

$$(3.38)\quad E[\sup_{t\le T} \| \xi(t) \|_\ell^2] \le N\, E[\int_0^T \|(\tilde{L}_i(t) - \tilde{L}_2(t))q_2(t)\|_\ell^2\, dt$$
$$+ \int_0^T \|(M_1(t) - M_2(t))q_2(t)\|_{\ell+1}^2\, dt].$$

Thus we see, from (A.1),

$$(3.39)\quad E[\sup_{t\le T} \| \xi(t) \|_\ell^2] \le N_1|y_1 - y_2|^2 (1 + I_{\ell+2}(\phi,f,g))$$

($\ell = 0,1,\cdots,m-2$), where $N_1 = N_1(T,K)$. □

Corollary 3.2 There is a constant $N_2 = N_2(T,K)$ such that

$$(3.40)\quad E[\sup_{t\le T} \| q(t,\phi_1,y_1,\mathfrak{R}) - q(t,\phi_2,y_2,\mathfrak{R}) \|_\ell^2]$$
$$\le N_2\Big(|y_1 - y_2|^2(1 + \min(\|\phi_1\|_{\ell+2}^2, \|\phi_2\|_{\ell+2}^2) + \sup_y \| g(\cdot,y) \|_{\ell+3}^2$$
$$+ \sup_{y,u} \sum_{i=0}^{d} \| \partial_i f(\cdot,y,u) \|_{\ell+2}^2) + \| \phi_1 - \phi_2 \|_\ell^2 \Big)$$

$$\ell = 0,1,\cdots,m-2$$

4 Optimal relaxed systems

Let $F : L^2(0,T;\ H^{m-2}) \longrightarrow \mathbb{R}^1$ and $G : H^{m-2} \longrightarrow \mathbb{R}^1$ be uniformly continuous with linear growth , namely

(4.1) for any $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that

$$| F(\psi_1) - F(\psi_2) | < \varepsilon \quad \text{if } \| \psi_1 - \psi_2 \|_{L^2(0,T;\ H^{m-2})} < \delta$$
$$| G(\varphi_1) - G(\varphi_2) | < \varepsilon \quad \text{if } \| \varphi_1 - \varphi_2 \|_{m-2} < \delta$$

and there is $\alpha > 0$ such that

$$(4.2)\quad |F(\psi)| \le \alpha(1 + \|\psi\|_{L^2(0,T;\ H^{m-2})})$$
$$|G(\varphi)| \le \alpha(1 + \|\varphi\|_{m-2}).$$

For $\mathfrak{R} \in \mathfrak{R}$, we will define the pay-off function J and the value function V by

$$(4.3)\quad J(\phi,y,\mathfrak{R}) = E[F(q(\cdot,\phi,y,\mathfrak{R})) + G(q(T,\phi,y,\mathfrak{R}))]$$

and

$$V(\phi,y) = \inf_{\mathfrak{R} \in \mathfrak{R}} J(\phi,y,\mathfrak{R})$$

respectively. Then Theorem 3.2 and Proposition 2.3 assert the existence of an optimal relaxed system. Now we have

Theorem 4.1 Under the conditions (A.1) ~ (A.3) and $(A.4)_{m-2}$, there exists an optimal relaxed system $\mathcal{R}^* = \mathcal{R}^*(\phi, y)$ for $\phi \in \Phi$ (see (3.33)), namely,

(4.4) $V(\phi, y) = J(\phi, y, \mathcal{R}^*)$

holds. Moreover, for any $r > 0$, we can choose $\mathcal{R}^*(\phi, y)$, so that $\pi(\mathcal{R}^*(\phi, y))$ is a Borel map from $\Phi_r \times \mathbb{R}^d$ into $\mathcal{P}(C[0,T] \times \Lambda)$.

Proof Suppose $\mathcal{R}_n \longrightarrow \mathcal{R}$. Putting $q_n = q(\cdot, \phi, y, \mathcal{R}_n)$ and $q = q(\cdot, \phi, y, \mathcal{R})$, $F(q_n)$ and $G(q_n(T))$ converge to $F(q)$ and $G(q(T))$ in law respectively. On the other hand, (3.1) derives

$$\sup_n E[F(q_n)^2] \le C_1(1 + \sup_n E[\int_0^T \| q_n(t) \|_{m-2}^2 dt]) < \infty$$

Thus, the uniform integrability asserts

" $E[F(q_n)] \longrightarrow E[F(q)]$ ".

In the same way we can prove

" $E[G(q_n(T))] \longrightarrow E[G(q(T)]$ "

Hence, $J(\phi, y, \mathcal{R})$ is continuous in $\mathcal{R}$. Thus, Proposition 2.3 concludes (4.4).

For the proof of the latter half, we apply the same arguments as [28, Chap. 12]. Putting

(4.5) $\mathfrak{X}(\phi, y) = \{ \pi(\mathcal{R}) \ ; \ V(\phi, y) = J(\phi, y, \mathcal{R}) \}$,

we show the following lemma.

Lemma 4.1 $\mathfrak{X}(\phi, y)$ is non-empty and compact .

Proof $\mathfrak{X}(\phi, y)$ is non-empty by (4.4). So we will prove the closedness of $\mathfrak{X}(\phi, y)$. Suppose $\pi(\mathcal{R}_n) \in \mathfrak{X}(\phi, y)$ and converges to

$\pi(\mathfrak{R})$ weakly. Then $J(\phi,y,\mathfrak{R}_n) \longrightarrow J(\phi,y,\mathfrak{R})$.

Hence $J(\phi,y,\mathfrak{R}) = V(\phi,y)$, namely, $\pi(\mathfrak{R}) \in \mathfrak{X}(\phi,y)$. □

Let $\phi_n \longrightarrow \phi$ in Φ_r and $y_n \longrightarrow y$. Suppose $\pi(\mathfrak{R}_n) \in \mathfrak{X}(\phi_n,y_n)$ and $\pi(\mathfrak{R}_n) \longrightarrow \pi(\mathfrak{R})$ weakly. Then we will show $\pi(\mathfrak{R}) \in \mathfrak{X}(\phi,y)$, which completes the proof.

(4.6) $| J(\phi_n,y_n,\mathfrak{R}_n) - J(\phi,y,\mathfrak{R}) |$

$$\le | J(\phi_n,y_n,\mathfrak{R}_n) - J(\phi,y,\mathfrak{R}_n) | + | J(\phi,y,\mathfrak{R}_n) - J(\phi,y,\mathfrak{R}) |$$

We see, from (3.1), (3.40) and (4.1), the following:

(4.7) 1st term of the right hand side of (4.6)

$$\le 2\varepsilon + E[\, | F(q(\phi_n,y_n,\mathfrak{R}_n)) - F(q(\phi,y,\mathfrak{R}_n)) | \,;\, A_n]$$
$$+ E[\, | G(q(T,\phi_n,y_n,\mathfrak{R}_n)) - G(q(T,\phi,y,\mathfrak{R}_n)) | \,;\, B_n]$$
$$\le 2\varepsilon + C_1(1 + \|\phi_n\|_{m-2} + \|\phi\|_{m-2})\{|y_n - y|(1 + \|\phi\|_m) + \|\phi_n - \phi\|_{m-2}\}/\delta$$

with C_1 independent from ε and n , where

$$A_n = \{ \| q(\phi_n,y_n,\mathfrak{R}_n) - q(\phi,y,\mathfrak{R}_n) \|_{L^2(0,T;\, H^{m-2})} > \delta \}$$

and

$$B_n = \{ \| q(T,\phi_n,y_n,\mathfrak{R}_n) - q(T,\phi,y,\mathfrak{R}_n) \|_{m-2} > \delta \} .$$

Since $J(\phi,y,\mathfrak{R})$ is continuous in $\mathfrak{R}$, (4.6) and (4.7) yield

(4.8) $J(\phi_n,y_n,\mathfrak{R}_n) \longrightarrow J(\phi,y,\mathfrak{R})$

Using " $| V(\phi_n,y_n) - V(\phi,y) | \le \sup_{\mathfrak{R} \in \mathfrak{R}} | J(\phi_n,y_n,\mathfrak{R}) - J(\phi,y,\mathfrak{R}) |$ ",

(4.7) derives

(4.9) $V(\phi_n,y_n) \longrightarrow V(\phi,y)$

Thus, we have

$$J(\phi,y,\mathfrak{R}) = \lim_{n \to \infty} J(\phi_n,y_n,\mathfrak{R}_n) = \lim_{n \to \infty} V(\phi_n,y_n) = V(\phi,y)$$

Namely, $\pi(\mathfrak{R}) \in \mathfrak{X}(\phi,y)$.

Therefore we can take a Borel selector S_r of $\mathfrak{X}(\phi,y)$, i.e.

$S_r : \Phi_r \times \mathbb{R}^{d'} \longrightarrow \mathcal{P}(C[0,T] \times \Lambda)$, Borel map, such that $S_r(\phi,y) \in \mathfrak{X}(\phi,y)$ ([28] Chap.12).

So $S_r(\phi,y) = \pi(\mathcal{R}^*(\phi,y))$ holds. This completes the proof of Theorem 4.1. □

Since a relaxed control turns out to be an admissible control under the Roxin condition, we can get an optimal admissible control. Now we introduce the convexity condition for coefficients of (2.1). Put $c(y,u) = (a^{ij}(\cdot,y,u) , f^{i}(\cdot,y,u) ; i,j = 0,\cdots,d)$ and $C(y,\Gamma) = \{ c(y,u) ; u \in \Gamma \}$.

Convexity condition (Roxin condition) For any $y \in \mathbb{R}^{d'}$, $C(y,\Gamma)$ is a convex subset of $C(\mathbb{R}^d ; \mathbb{R}^{(d+1)(d+2)})$.

Endowing with the compact uniform topology on $C(\mathbb{R}^d ; \mathbb{R}^{(d+1)(d+2)})$ we have

Proposition 4.2 Under the convexity condition $C(y,\Gamma)$ is compact and convex.

Proof $c(y,\cdot)$ is continuous in Γ. Since Γ is compact, $C(y,\Gamma)$ is compact. □

Let us set $\tilde{c}(\cdot,y,\nu) = \int_\Gamma c(\cdot,y,u)\ \nu(du)$ for $\nu \in \mathcal{P}(\Gamma)$, namely, $\tilde{c}(\cdot,y,\nu) = (\tilde{a}(\cdot,y,\nu) , \tilde{f}(\cdot,y,\nu))$.
Putting $\Gamma(y,\nu) = \{ u \in \Gamma ; \tilde{c}(\cdot,y,\nu) = c(\cdot,y,u) \}$, we see

Proposition 4.3 $\Gamma(y,\nu)$ is non-empty and compact.

Proof Since $C(y,\Gamma)$ is convex and compact, $\tilde{c}(\cdot,y,\nu) \in C(y,\Gamma)$. So $\Gamma(y,\nu) \neq \phi$. Now we will show that $\Gamma(y,\nu)$ is closed. Suppose $u_n \in \Gamma(y,\nu)$ and $u_n \longrightarrow u$. Then $c(\cdot,y,u_n) \longrightarrow c(\cdot,y,u)$. Thus $c(\cdot,y,u) = \tilde{c}(\cdot,y,\nu)$. This completes the proof.

Again appealing to [28, Chap. 12], we see

Proposition 4.4 There exists a Borel selector $\tilde{S}$ of $\Gamma(y,\nu)$, i.e. $\tilde{S} : \mathbb{R}^{d'} \times \mathcal{P}(\Gamma) \longrightarrow \Gamma$ Borel map , such that $\tilde{S}(y,\nu) \in \Gamma(y,\nu)$.

Proof Suppose $\nu_n \longrightarrow \nu$ weakly and $y_n \longrightarrow y$. Then

(4.10) $| \tilde{c}(x,y_n,\nu_n) - \tilde{c}(x,y,\nu) |$

$$\leq \int_\Gamma | c(x,y_n,u) - c(x,y,u) | \, d\nu_n + | \tilde{c}(x,y,\nu_n) - \tilde{c}(x,y,\nu) |$$

$$\leq \sup_{x,u} | c(x,y_n,u) - c(x,y,u) | + | \tilde{c}(x,y,\nu_n) - \tilde{c}(x,y,\nu) |$$

holds. By the uniform continuity of c , the first term tends to 0 as $n \longrightarrow \infty$. The second term also tends to 0 by the assumption " $\nu_n \longrightarrow \nu$ weakly ". Hence, as $n \longrightarrow \infty$, $| \tilde{c}(\cdot,y,\nu_n) - \tilde{c}(\cdot,y,\nu) | \longrightarrow 0$ uniformly in any compact set of $\mathbb{R}^d$, by virtue of uniform continuity of c . This derives

(4.11) $\tilde{c}(\cdot,y_n,\nu_n) \longrightarrow \tilde{c}(\cdot,y,\nu)$, as $n \longrightarrow \infty$,

Suppose $u_n \in \Gamma(y_n , \nu_n)$ tends to u. Since $c(\cdot,y_n,u_n) \longrightarrow c(\cdot,y,u)$, (4.11) yields " $u \in \Gamma(y,\nu)$ ". This concludes Proposition 4.4. □

For $\mathcal{R} = (\Omega, \mathcal{F},\mathcal{F}_t,P,W,\mu)$, we define an $\mathcal{F}_t$-adapted process U by

(4.12) $U(t) = \tilde{S}(y + W(t), \mu'(t))$.

Then we have

(4.13) $\tilde{c}(x,y + W(t),\mu'(t)) = c(x,y + W(t),U(t))$

and

(4.14) $\tilde{L}(t,y + W(t),\mu) = L(y + W(t),U(t))$.

Hence, $q = q(\ ,\phi,y,\mathcal{R})$ satisfies

$$(4.15) \begin{cases} dq(t) = L(y + W(t),U(t))q(t)dt + M(y + W(t))q(t)dW(t) \\ q(0) = \phi. \end{cases}$$

Since (4.15) has a unique solution, q turns out to be the response for the admissible system $\mathcal{A} = (\Omega,\mathcal{F},\mathcal{F}_t,P,W,U)$.

Although an admissible system can be regarded as a relaxed system, we denote the pay-off function by $J(\phi,y,\mathcal{A})$, stressing an admissible system $\mathcal{A}$. Recalling Theorem 4.1, we get

Theorem 4.2 Supposing (A.1) ~ (A.3), $(A.4)_{m-1}$ and the convexity condition, there is an optimal admissible system $\mathcal{A}^*$, for $\phi \in \Phi$, such that

(4.16) $V(\phi,y) = \inf_{\mathcal{A} \in \mathfrak{U}} J(\phi,y,\mathcal{A}) = J(\phi,y,\mathcal{A}^*)$.

Proof Put $U^*(t) = \tilde{S}(y + W^*(t),\mu^{*\prime}(t))$ for an optimal relaxed system $\mathcal{R}^* = (\Omega,\mathcal{F},\mathcal{F}_t,P^*,W^*,\mu^*)$. Then $\mathcal{A}^* = (\Omega,\mathcal{F},\mathcal{F}_t,P^*,W^*,U^*)$ satisfies

(4.17) $V(\phi,y) = J(\phi,y,\mathcal{R}^*) = J(\phi,y,\mathcal{A}^*) \geq \inf_{\mathcal{A} \in \mathfrak{U}} J(\phi,y,\mathcal{A})$

Since " $V(\phi,y) \leq \inf_{\mathcal{A} \in \mathfrak{U}} J(\phi,y,\mathcal{A})$" , (4.17) derives (4.16). □

For Sections 5 and 6, we will introduce a subsidiary relaxed system. $\mathcal{R} = (W, \mu)$ is called a constant relaxed system , if $\mu'(t, d\mathcal{U}, \omega) = \nu(d\mathcal{U})$ for any t and ω . In this case, we will call μ a constant relaxed control ν and denote $\mathcal{R} = (W, \nu)$. Stressing the terminal time T, we put

(4.18) $\mathcal{J}(T,\phi,y,\nu) = J(T,\phi,y,\mu)$, if $\mu' = \nu$ ($\in \mathcal{P}(\Gamma)$)

$$v(T,\phi,y) = \inf_{\nu \in \mathcal{P}(\Gamma)} \mathcal{J}(T,\phi,y,\nu)$$

$$\mathfrak{I}(T,\phi,y) = \{ \nu \in \mathcal{P}(\Gamma) \ ; \ v(T,\phi,y) = \mathcal{J}(T,\phi,y,\nu) \}.$$

Appealing to the fact " $\mathcal{R}_n = (W_n, \nu_n)$ converges to $\mathcal{R} = (W, \nu)$ iff $\nu_n \longrightarrow \nu$ weakly " , we get

Theorem 4.3 Under the conditions (A.1) ~ (A.3) and $(A.4)_{m-2}$, $\mathfrak{I}(T,\phi,y)$ is non-empty and compact. Moreover, there is a Borel selector $\mathcal{G}_{T,r}$ of $\mathfrak{I}(T,\phi,y)$, for $(\phi, y) \in \Phi_r \times \mathbb{R}^{d'}$.

We consider the following usual pay-off function for the Bellman principle.

Let $h : H^{m-2} \times \mathbb{R}^{d'} \longrightarrow \mathbb{R}^1$ be quadratic growth and satisfy (4.19), namely,

$$| h(\phi, y) | \le C (1 + \|\phi\|_{m-2}^2 + |y|^2)$$

and

(4.19) $| h(\phi_1, y_1) - h(\phi_2, y_2) |$

$$\le C\Big((\|\phi_1\|_{m-2} + \|\phi_2\|_{m-2}) \|\phi_1 - \phi_2\|_{m-2} + (|y_1| + |y_2|) |y_1 - y_2| \Big)$$

By $\mathcal{G}$ we denote the set of functions $g : H^m \times \mathbb{R}^{d'} \longrightarrow \mathbb{R}^1$, which satisfy (4.20) and (4.21) below,

(4.20) $| g(\phi, y) | \le C_g(1 + \|\phi\|_{m-2}^2 + |y|^2)$

and

(4.21) for any ε , $b > 0$, there is $\delta = \delta(\varepsilon, b, g) > 0$ such that , for $(\phi_i, y_i) \in B_b \left(= \{ (\phi, y) \in H^m \times \mathbb{R}^{d'} \; ; \; \|\phi\|_{m-2} < b \, , \; |y| < b \} \right)$

$$| g(\phi_1, y_1) - g(\phi_2, y_2) | < \varepsilon$$

holds, whenever $\|\phi_1 - \phi_2\|_{m-2} < \delta$ and $|y_1 - y_2| < \delta$.

Define J and V by (4.22) and (4.23) respectively,

$$(4.22) \quad J(t,\phi,y,\mathfrak{R},g) = E[\int_0^t h(q(s), y + W(s))ds + g(q(t), y + W(t))]$$

where $q = q(\; ,\phi,y,\mathfrak{R})$, and

$$(4.23) \quad V(t,\phi,y,g) = \inf_{\mathfrak{R} \in \mathfrak{R}} J(t,\phi,y,\mathfrak{R},g)$$

For a constant relaxed system, we define $\mathcal{J}$ and v in the same way.

Proposition 4.5 $J(t,\cdot,\cdot,\mathfrak{R},g)$, $V(t,\cdot,\cdot,g)$, $\mathcal{J}(t,\cdot,\cdot,\mathfrak{R},g)$ and $v(t,\cdot,\cdot,g)$ belong to $\mathcal{G}$, whenever $g \in \mathcal{G}$.

Proof From (2.13), we see

$$(4.24) \quad | J(t,\phi,y,\mathfrak{R},g) | \le E[C\int_0^t \{1 + \|q(s)\|_{m-2}^2 + |y + W(s)|^2\} + C_g\{1 + \|q(t)\|_{m-2}^2 + |y + W(t)|^2 \}]$$

$$\le C(t,g)(1 + \|\phi\|_{m-2}^2 + |y|^2)$$

where $C(t,g)$ is independent of $\mathfrak{R}$. So J, V, $\mathcal{J}$ and v also satisfy the quadratic growth condition (4.20).

Recalling Corollary 3.2, we will show (4.21).

Put $q_i = q(\; ,\phi_i,y_i,\mathfrak{R})$ for $(\phi_i, y_i) \in B_b$. Then we have

$$(4.25) \quad E[\int_0^t | h(q_1(s), y_1 + W(s)) - h(q_2(s), y_2 + W(s)) | ds]$$

$$\le C \; E[\int_0^t \sum_{i=1}^{2} \|q_i(s)\|_{m-2}\|q_1(s) - q_2(s)\|_{m-2} + \sum_{i=1}^{2} |y_i + W(s)||y_1 - y_2| ds]$$

$$\le C\Big(\sum_{i=1}^{2} \{ E[\int_0^t \|q_i(s)\|_{m-2}^2 ds] \}^{1/2} \{ E[\int_0^t \|q_1(s) - q_2(s)\|_{m-2}^2 ds] \}^{1/2}$$

$$+\sum_{i=1}^{2}(|y_i| + \sqrt{t})|y_1 - y_2| \Big)$$

$$\le C_1(t)\ (1 + b)(|y_1 - y_2|(1 + b) + \|\phi_1 - \phi_2\|_{m-2})$$

$$\le C_1(t)\ (1 + b)^2(|y_1 - y_2| + \|\phi_1 - \phi_2\|_{m-2})$$

(4.26) $E[\ |\ g(q_i(t), y_i + W(t))\ |\ ;\ \|q_i(t)\|_{m-2} > n\ ]$

$$\le C_g\ E[\ 1 + \|q_i(t)\|^2_{m-2} + |y_i + W(t)|^2\ ;\ \|q_i(t)\|_{m-2} > n\ ]$$

$$\le C_g\Big(E[\ 1 + \|q_i(t)\|^4_{m-2} + |y_i + W(t)|^4]\Big)^{1/2}\ \Big(E[\|q_i(t)\|^2_{m-2}\ /n^2]\Big)^{1/2}$$

$$\le C_2(g,t)(\ 1 + \|\phi_i\|^2_{m-2} + |y_i|^2\ )\ (\ 1 + \|\phi_i\|_{m-2} + |y_i|\ )/n$$

$$\le C_3(g,t)\ (\ 1 + b\ )^3/n$$

(4.27) $E[\ |\ g(q_i(t), y_i + W(t))\ |\ ;\ |y_i + W(t)| > n\ ]$

$$\le C_4(g,t)(\ 1 + \|\phi_i\|^2 + |y_i|^2\ )(\ |y_i| + \sqrt{t}\ )/n$$

$$\le C_5(g,t)(\ 1 + b\ )^3/n$$

Taking a large enough $n = n(\varepsilon, b, t, g)$ such that

(4.28) $(\ C_3(g,t) + C_5(g,t)\ )(1 + b)^3 < \varepsilon n/4,$

we get

(4.29) $|\ J(t,\phi_1,y_1,\mathfrak{R},g) - J(t,\phi_2,y_2,\mathfrak{R},g)\ |$

$$\le E[C\int_0^t \sum_{i=1}^{2} \|q_i(s)\|_{m-2}\|q_1(s) - q_2(s)\|_{m-2} + \sum_{i=1}^{2} |y_i + W(s)||y_1 - y_2|\ ds]$$

$$+ E[\ |\ g(q_1(t), y_1 + W(t)) - g(q_2(t), y_2 + W(t))\ |\ ;\ \|q_i(t)\|_{m-2} < n,$$

$$|y_i + W(t)| < n\ ,\ i = 1,2\ ]\ + \varepsilon.$$

From the continuity condition (4.21) for g, we see

(4.30) the middle term of the right hand side of (4.29)

$$< \varepsilon + 2C_g(\ 1 + 2n^2\ )\ P\{\ \|q_1(t) - q_2(t)\|_{m-2} > \delta(\varepsilon,n,g)\ \}$$

$$< \varepsilon + 2C_g(\ 1 + 2n^2\ )\ E[\ \|q_1(t) - q_2(t)\|^2_{m-2}/\ \delta^2(\varepsilon,n,g)\ ],$$

whenever $|y_1 - y_2| < \delta(\varepsilon,n,g)$.

Using (3.48), (4.29) and (4.30), we can choose a positive constant $\gamma = \gamma(t,\varepsilon,b,g)$, independent from $\mathfrak{R}$, such that

(4.31) $|\ J(t,\phi_1,y_1,\mathfrak{R},g) - J(t,\phi_2,y_2,\mathfrak{R},g)\ | < \varepsilon,$

whenever $\|\phi_1 - \phi_2\|_{m-2} < \delta$ and $|y_1 - y_2| < \delta$.

Since " $| V(t,\phi_1,y_1,g) - V(t,\phi_2,y_2,g) |$

$$\leq \sup_{\mathcal{R} \in \mathfrak{R}} | J(t,\phi_1,y_1,\mathcal{R},g) - J(t,\phi_2,y_2,\mathcal{R},g) | ".$$

we can complete the proof. □

Now, applying arguments similar to (4.6) ~ (4.9), we get the following theorem.

Theorem 4.4 Under the conditions (A.1) ~ (A.3) and $(A.4)_{m-2}$, there exists an optimal relaxed system $\mathcal{R}^*(\phi,y)$, such that $\pi(\mathcal{R}^*(\phi,y))$ is Borel measurable w.r.t. $(\phi,y) \in \Phi_r \times \mathbb{R}^{d'}$, i.e.

$$J(t,\phi,y,\mathcal{R}^*(\phi,y),g) = \inf_{\mathcal{R} \in \mathfrak{R}} J(t,\phi,y,\mathcal{R},g)$$

Example quadratic loss.

I) Put $h(\phi,y) = \|\phi\|^2 (= \|\phi\|_0^2)$ and $g = 0$. Then h satisfies (4.19). So there exists an optimal relaxed system $\mathcal{R}^* = \mathcal{R}^*(\phi,y)$, i.e.

$$\min_{\mathcal{R} \in \mathfrak{R}} E\left[\int_0^T \| q(t,\phi,y,\mathcal{R}) \|^2 dt \right] = E\left[\int_0^T \| q(t,\phi,y,\mathcal{R}^*) \|^2 dt \right]$$

II) Put $h = 0$ and $g(\phi) = \|\phi\|^2$. Then $g \in \mathcal{G}$. So there exists an optimal relaxed system $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}(\phi,y)$, i.e.

$$\min_{\mathcal{R} \in \mathfrak{R}} E[\| q(T,\phi,y,\mathcal{R}) \|^2] = E[\| q(T,\phi,y,\tilde{\mathcal{R}}) \|^2]$$

5 Approximation

In this section, we will show that there exists an approximate optimal control, which is adapted to a Wiener process.

We call $\mathfrak{R} = (W, \mu)$ a step relaxed system, if $\mu'(t) = \mu'([t/\Delta]\Delta)$ with a positive Δ, where [ ] = Gauss symbol. By $\mathfrak{R}_N$ we denote the totality of step relaxed systems with $\Delta = 2^{-N}$. For μ we define an approximate derivative μ'_n as follows:

$$(5.1) \quad \mu'_n(t,\cdot) = \begin{cases} 2^n \mu([t-2^{-n}, t) \times \cdot) & \text{for } t > 2^{-n} \\ t^{-1}\mu([0,t) \times \cdot) & \text{for } t \le 2^{-n}. \end{cases}$$

Put

$$(5.2) \quad \mu'_{n,k}(t,\cdot) = \mu'_n([2^k t]2^{-k}, \cdot)$$

and $\quad \mu_{n,k}([0,t] \times A) = \int_0^t \mu'_{n,k}(s,A)\, ds.$

Then, for a suitable sequence $k(n)$, $n = 1,2,\cdots$, we have, w.p. 1,

$$(5.3) \quad \mu_{n,k(n)} \longrightarrow \mu \text{ weakly.}$$

Hereafter we consider a pay-off function J as (4.22). Therefore, (5.3) yields

$$(5.4) \quad V(t,\phi,y,g) = \lim_{N \to \infty} \inf_{\mathfrak{R}_N} J(t,\phi,y,\mathfrak{R},g).$$

Putting

$$(5.5) \quad \tilde{\mathfrak{R}}_N = \{ \mathfrak{R} = (W,\mu) \in \mathfrak{R}_N \ ; \ \mu \text{ is } W\text{ - adapted} \},$$

we have

Theorem 5.1 Under the conditions (A.1) ~ (A.3) and $(A.4)_{m-2}$, we have, for $\phi \in \Phi$

$$(5.6) \quad \inf_{\mathfrak{R}_N} J(t,\phi,y,\mathfrak{R},g) = \inf_{\tilde{\mathfrak{R}}_N} J(t,\phi,y,\mathfrak{R},g).$$

Proof Since $\tilde{\mathfrak{R}}_N \subset \mathfrak{R}_N$, it is enough to show

$$(5.7)\quad J(t,\phi,y,\mathcal{R},g) \geq \inf_{\mathcal{R}_N} J(t,\phi,y,\mathcal{R},g) \quad \text{for } \forall\, \mathcal{R} \in \mathcal{R}_N.$$

Putting $\Delta = 2^{-N}$ and $j\Delta < t \leq (j+1)\Delta$, we will evaluate I, defined by (5.8),

$$(5.8)\quad I = E[\int_{j\Delta}^{t} h(q(s),y + W(s))ds + g(q(t),y + W(t)) \mid \mathcal{F}_{j\Delta}]$$

where $q = q(\ ,\phi,y,\mathcal{R})$. Under the conditional probability $P(\cdot \mid \mathcal{F}_{j\Delta})$, $W^j(\cdot) = W(\cdot + j\Delta) - W(j\Delta)$ becomes a new Wiener process which is independent of $\mathcal{F}_{j\Delta}$ and $\mu'(\theta + j\Delta, \cdot) = \mu'(j\Delta,\cdot)$, $0 \leq \theta \leq t - j\Delta$, can be regarded as a constant relaxed control. Moreover, the uniqueness of solution derives

$$(5.9)\quad q(\theta+j\Delta,\phi,y,\mathcal{R}) = q(\theta, q(j\Delta,\phi,y,\mathcal{R}), y + W(j\Delta), \mu'(j\Delta)) \quad \text{for } 0 \leq \theta \leq t - j\Delta,$$

Hence, we see

$$(5.10)\quad I \geq \inf_{\nu \in \mathcal{P}(\Gamma)} \mathcal{J}(t - j\Delta, q(j\Delta,\phi,y,\mathcal{R}), y + W(j\Delta), \nu, g) = v(t - j\Delta, q(j\Delta,\phi,y,\mathcal{R}), y + W(j\Delta), g).$$

Defining $v(s) : \mathcal{G} \longrightarrow \mathcal{G}$ by $v(s,\cdot,g) = v(s)g$, we see from (5.10)

$$(5.11)\quad J(t,\phi,y,\mathcal{R},g) \geq E[\int_0^{j\Delta} h(q(s),y + W(s))\, ds + v(t - j\Delta)g(q(j\Delta), y + W(j\Delta))].$$

By the same argument, we calculate $E[\cdots \mid \mathcal{F}_{(j-1)\Delta}]$ and obtain

$$(5.12)\quad J(t,\phi,y,\mathcal{R},g) \geq E[\int_0^{(j-1)\Delta} h(q(s),y + W(s))\, ds + v(\Delta)v(t - j\Delta)g(q((j-1)\Delta), y + W((j-1)\Delta))].$$

Repeating this evaluation, we get

$$(5.13)\quad J(t,\phi,y,\mathcal{R},g) \geq v^j(\Delta)v(t-j\Delta)g(\phi,y).$$

We assume that $(A.4)_{m-2,r_0}$ holds. Then (2.14) asserts that $q(t,\phi,y,\mathcal{R}) \in \Phi_r$ w.p. 1, whenever $\phi \in \Phi_r$ for $r \leq r_0$. According to Theorem 4.3, we can take a Borel selector $\mathcal{G}_r(t,g)$ of $\mathfrak{X}(t,\phi,y,g) = \{ \pi(\mathfrak{R}) ; v(t,\phi,y.g) = J(t,\phi,y,\mathfrak{R},g) \}$.

Let $(\Omega_i, \mathcal{F}_i, P_i)$, $i = 1, \cdots, (j+1)$ be a probability space and W_i be a Wiener process on it. Let us set

$\Omega = \prod_{i=1}^{j+1} \Omega_i$, $\mathcal{F} = \prod_{i=1}^{j+1} \mathcal{F}_i$, $P = \prod_{i=1}^{j+1} P_i$ and

$$(5.14) \quad W(t) = \begin{cases} W_1(t) & \text{for } 0 \le t < \Delta \\ W_1(\Delta) + W_2(t-\Delta) & \text{for } \Delta \le t < 2\Delta \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \sum_{k=1}^{j} W_k(\Delta) + W_{j+1}(t-j\Delta) & \text{for } j\Delta \le t < (j+1)\Delta. \end{cases}$$

Then W becomes a Wiener process on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, where $\mathcal{F}_t = \sigma_t(W)$. Fix $\nu_1 \in \mathfrak{X}(\Delta, \phi, y, v^{j-1}(\Delta)v(t-j\Delta)g)$ arbitrarily and q_1 denotes the solution of (5.15).

$$(5.15) \quad \begin{cases} dq_1(t) = \tilde{L}(y + W(t), \nu_1)q_1(t)dt + M(y + W(t))q_1(t)dW(t), & \\ q_1(0) = \phi & 0 < t \le \Delta. \end{cases}$$

So q_1 is W - adapted.

Put $\nu_2 = \mathcal{G}_r(\Delta, v^{j-2}(\Delta)v(t-j\Delta)g(q_1(\Delta), y + W(\Delta)))$ and q_2 denotes the solution of (5.16).

$$(5.16) \quad \begin{cases} dq_2(t) = \tilde{L}(y + W(t), \nu_2)q_2(t)dt + M(y + W(t))q_2(t)dW(t), & \\ q_2(\Delta) = q_1(\Delta) & \Delta < t \le 2\Delta. \end{cases}$$

Putting $\nu_3 = \mathcal{G}_r(\Delta, v^{j-2}(\Delta)v(t-j\Delta)g(q_2(2\Delta), y + W(2\Delta)))$, we repeat the same argument. Now define μ' by

$$(5.17) \quad \mu'(t) = \nu_k \quad \text{for } t \in [\,(k-1)\Delta, k\Delta\,).$$

Then μ' is W - adapted and $\mathfrak{R} = (\Omega, \mathcal{F}, \mathcal{F}_t, P, W, \mu) \in \mathfrak{R}_N$.

Moreover, putting $q = q_k$ on $[\,(k-1)\Delta, k\Delta\,)$, we get

$$(5.18) \quad E\left(\int_{j\Delta}^{t} h(q(s), y + W(s))ds + g(q(t), y + W(t)) \;\middle|\; \mathcal{F}_{j\Delta} \right)$$

$$= E\left(\int_{j\Delta}^{t} h(q_j(s), y + W(s))\, ds + g(q_j(t), y + W(t)) \right)$$

$$= v(t - j\Delta) g(q(j\Delta), y + W(j\Delta)),$$

$$E\left(\int_{(j-1)\Delta}^{j\Delta} h(q(s), y + W(s))ds + v(t-j\Delta)g(q(j\Delta), y + W(j\Delta)) \middle| \mathcal{F}_{(j-1)\Delta} \right)$$

$$= v(\Delta)v(t - j\Delta)g(q((j-1)\Delta), y + W((j-1)\Delta)) ,$$

and so on . Thus, we have

$$(5.19)\quad J(t,\phi,y,\mathfrak{R},g) = E[\int_0^t h(q(s),y + W(s))ds + g(q(t),y + W(t))]$$

$$= E[\ E\left(\int_{j\Delta}^{t} h(q(s),y + W(s))ds + g(q(t),y + W(t)) \middle| \mathcal{F}_{j\Delta} \right) + \int_0^{j\Delta} h(q(s),y + W(s))ds\ ]$$

$$= E[\int_0^{j\Delta} h(q(s),y + W(s))ds + v(t-j\Delta)g(q(j\Delta),y + W(j\Delta))$$

$$= E[\int_0^{(j-1)\Delta} h(q(s),y+W(s))ds + v(\Delta)v(t-j\Delta)g(q((j-1)\Delta),y+W((j-1)\Delta))]$$

$$= v^j(\Delta)v(t-j\Delta)g(\phi,y).$$

From (5.13) and (5.19), we can conclude (5.7). □

Recalling (5.4), we obtain

Corollary 5.1 Under the same condition of Theorem,

$$(5.20)\quad V(t,\phi,y,g) = \lim_{N \longrightarrow \infty} \inf_{\mathfrak{R}_N} J(t,\phi,y,\mathfrak{R},g)$$

holds. In the other words, there is an approximate optimal step relaxed system, which is adapted to a Wiener process.

Using Chattering lemma [6] , $\mathfrak{R} \in \mathfrak{R}_N$ can be approximated by admissible controls which are adapted to a Wiener process.

Hence, putting

$\mathfrak{A}_N = \mathfrak{A} \cap \mathfrak{R}_N = \{ \mathscr{A} = (W,U) \ ; \ U \text{ is } W\text{-adapted and } U(t) = U([2^N t]2^{-N}) \}$

and $\mathfrak{A} = \bigcup_{N=1}^{\infty} \mathfrak{A}_N$, we have

Corollary 5.2 Under the same condition, there is an approximate optimal step system $\mathscr{A} \in \mathfrak{A}$.

6 Bellman Principle

Now we are ready to prove the Bellman principle. For $\phi \in \Phi_r$ and $\mathscr{A} = (W,U) \in \mathfrak{A}_N$, we will evaluate (6.1)

(6.1) $J(s + t,\phi,y,\mathscr{A},g)$

$$= E\Big(\int_0^t h(q(\theta),y + W(\theta))d\theta$$

$$+ E[\int_t^{t+s} h(q(\theta),y + W(\theta))d\theta + g(q(t+s),y + W(t+s)) \mid \mathcal{F}_t] \Big).$$

Since $W^t(\cdot) = W(\cdot + t) - W(t)$ is a Wiener process independent from $\mathcal{F}_t$, we see

(6.2) conditional expectation of 2nd term $\geq V(s,q(t),y + W(t),g)$ w.p. 1.

This asserts

(6.3) $J(s + t,\phi,y,\mathscr{A},g) \geq J(t,q(t),y + W(t), \mathscr{A} , V(s,\cdot,g))$

$\geq V(t,q(t),y + W(t),V(s,\cdot,g))$.

Now Corollary 5.2 yields

(6.4) $V(s+t,\phi,y,g) \geq V(t,q(t),y + W(t),V(s,\cdot,g))$.

Next we will show the converse inequality of (6.4), by a standard argument.

Let $\mathcal{G}_r(\phi,y)$ denote a Borel selector of $\mathfrak{X}(\phi,y) = \{ \pi(\mathfrak{R}) \ ; \ V(s,\phi,y,g)$

$= J(s,\phi,y,\mathfrak{R},g) \}$. For any $\mathcal{A} = (\Omega,\mathcal{F},\mathcal{F}_t,P,W,U) \in \mathfrak{A}_N$, we put

$\tilde\Omega = C([0,s];\mathbb{R}^{d'}) \times \Lambda$, $\tilde W$ = first coordinate function

$\tilde\mu$ = 2nd coordinate function , $\tilde{\mathcal{F}} = \sigma(\tilde W,\tilde\mu)$, $\tilde{\mathcal{F}}_\theta = \sigma_\theta(\tilde W,\tilde\mu)$

$\Omega^* = \Omega \times \tilde\Omega$, $\mathcal{F}^* = \mathcal{F} \times \tilde{\mathcal{F}}$.

Define P^* by

(6.5) $P^*((\tilde W,\tilde\mu) \in B \mid \mathcal{F}_t) = \mathcal{G}_r(q(t,\phi,y,\mathcal{A}) , y + W(t))(B)$

namely,

$$P^*((\tilde W,\tilde\mu) \in B,(W,\mu) \in C) = \int_{\{ (W,\mu) \in C \}} \mathcal{G}_r(q(t,\phi,y,\mathcal{A}) , y + W(t))(B)\, dP.$$

Hence, $\tilde W$ is a Wiener process on $(\Omega^*,\mathcal{F}^*,P^*)$, independent from W.

Thus, putting

$$W^*(\theta) = \begin{cases} W(\theta) , & \theta \le t \\ W(t) + \tilde W(\theta-t) , & t \le \theta \le s+t \end{cases}$$

$$\mu^*(\theta,\cdot) = \begin{cases} \delta_{U(\theta)}(\cdot) , & \theta \le t \\ \tilde\mu(\theta-t,\cdot) , & t \le \theta \le s+t \end{cases}$$

$\mathcal{F}_\theta^* = \sigma_\theta(W^*,\mu^*)$,

we see $\mathfrak{R}^* = (W^*,\mu^*) \in \mathfrak{R}$ and its response q^* satisfies

(6.6) $$E[\int_t^{t+s} h(q^*(\theta),y + W^*(\theta))d\theta + g(q^*(t+s),y + W(t+s)) \mid \mathcal{F}_t^*]$$

$$= V(s, q(t,\phi,y,\mathcal{A}) , y + W(t), g)$$

$$= V(s, q(t,\phi,y,\mathfrak{R}^*) , y + W^*(t), g).$$

Therefore,

(6.7) $$J(s+t,\phi,y,\mathfrak{R}^*,g) = J(t,\phi,y,\mathfrak{R}^*,V(s,\cdot,g)) = J(t,\phi,y,\mathcal{A},V(s,\cdot,g))$$

holds. This asserts

$V(s+t,\phi,y,g) \le J(t,\phi,y,\mathcal{A},V(s,\cdot,g))$.

Again, Corollary 5.2 concludes the converse inequality of (6.4).

Thus, we obtain

Theorem 6.1 Under the conditions (A.1) ~ (A.3) and $(A.4)_{m-2}$, we have

(6.8) $V(t,\cdot,g) \in \mathscr{G}$ whenever $g \in \mathscr{G}$,

and the Bellman principle holds, i.e.

(6.9) $V(s+t,\phi,y,g) = V(t,\phi,y,V(s,\cdot,g))$

$$\text{for } \phi \in \Phi \text{ and } g \in \mathscr{G} .$$

Remark The Bellman principle is formulated by some nonlinear group [2].

7 Applications

1) Temperature control. Let us consider a heat systems in a random medium. The field of temperature $q(t,x)$ is governed by the following SPDE,

$$dq(t,x) = (\Delta q(t,x) + f(x,U(t))) \, dt + g(x)dW(t), \quad t > 0, \ x \in \mathbb{R}^d,$$

with the initial data $q(0,x) = \phi(x)$, where Δ is the Laplacian operator for x and W a d-dimensional Wiener process. So the temperature is controlled through the external force $f(x,U(t))$. The problem is to minimize the deviation of temperature distribution from the assigned distribution m at a given time T (cf Y. Sakawa [27]), namely, the pay-off function J is defined by

$$J(U) = E[\int_{\mathbb{R}^d} | q(T,x) - m(x) |^2 \, dx].$$

Hence, Theorem 4.4 concludes the existence of an optimal relaxed control, if f and g satisfy the condition $(A.4)_1$.

2) Nervous system. In Chap. 3 of [29], Walsh deals with the following SPDE as the dynamics of nervous system,

$$dq(t,x) = \left(\frac{\partial^2}{\partial x^2} q(t,x) - q(t,x) \right)dt + \left(q(t,x) - g(x) \right)dW(t),$$

$$0 < x < L,\ t > 0,$$

with Neuman boundary condition, where W is a one-dimensional Wiener process, and also considers the barrier problem.

Since a medical treatment acts an external force, we will here consider the following SPDE as its variant,

$$\begin{cases} dq(t,x) = \left(\frac{\partial^2}{\partial x^2} q(t,x) - q(t,x) + f(x,U(t)) \right)dt \\ \qquad\qquad + \left(q(t,x) - g(x) \right)dW(t), \quad x \in \mathbb{R}^1,\ 0 < t \le T, \\ q(0,x) = \phi(x). \end{cases}$$

Although we want to keep $q(t,x)$ near an assigned level λ at a given spot y, we need some smooth modifications. For given two positive constants b and c, we put

$$p(t) = \frac{1}{2c} \int_{-c}^{c} q(t,y + x)dx$$

and

$$h(x) = \begin{cases} 1 & , \quad x \notin (\lambda - b , \lambda + b) \\ \frac{\lambda - x}{b} & , \quad \lambda - b < x < \lambda \\ \frac{x - \lambda}{b} & , \quad \lambda < x < \lambda + b. \end{cases}$$

Now the problem is to minimize $E[\int_0^T h(p(t)) dt]$ and our theorems are applicable.

3) Stochastic control with partial observation.

Let B and $\hat{B}$ be independent Wiener processes with values in $\mathbb{R}^{d'}$ and $\mathbb{R}^d$ respectively. Suppose that the d-dimensional state process X and the d'-dimensional observation process Y are governed by the following stochastic differential equations (SDE in short) with bounded and smooth coefficients:

$$(7.1)\quad \begin{cases} dX(t) = \gamma(\ X(t),Y(t),U(t)\ )dt + \alpha(\ X(t),Y(t),U(t)\ )d\hat{B}(t) \\ \qquad\qquad\qquad\qquad + b(\ X(t),Y(t)\ )dB(t), \\ X(0) = \xi \qquad\qquad\qquad 0 < t \le T \end{cases}$$

and

$$(7.2)\quad dY(t) = f(\ X(t)\ )\,dt + dB(t)\ ,\ Y(0) = 0,$$

where U is an admissible control. So in our model, the state and observation noises may not be independent.

Let h and G : $\mathbb{R}^d \times \mathbb{R}^{d'} \longrightarrow \mathbb{R}^1$, be bounded and Lipschitz continuous. The problem is to minimize the pay-off function J, defined by

$$(7.3)\quad J(U) = E[\ \int_0^T h(\ X(t),Y(t)\ )\,dt + G(\ X(T),\ Y(T)\ )\ ]$$

by a suitable choice of U.

In the customary version of stochastic control with partial observation , U(t) is a function of the observation process Y(s) , s ≤ t namely, admissible control in the strict sense. Here we treat some wider class of admissible controls, according to [7], as following:

$A = (\ \Omega,\mathcal{F},\mathcal{F}_t,\mathring{P},\hat{B},Y,U)$ is called an admissible control system, if

(I) $(\ \Omega,\mathcal{F},\mathcal{F}_t,\mathring{P}\ )$ is a probability space , with $\mathcal{F}_t = \sigma_t(\ Y,U\ )$

(II) Y is a d'-dimensional $\mathcal{F}_t$- Wiener process

(III) U is Γ-valued process

(IV) $\hat{B}$ is a d-dimensional Wiener process on Ω , independent from (Y,U).

Let ξ be a random variable independent from ($\hat{B}$,Y,U) and ϕ

be its probability density. For an admissible system A, we consider SDE ,

$$(7.4)\quad \begin{cases} dX(t) = (\gamma(X(t),Y(t),U(t)) - b(X(t),Y(t))f(X(t)))dt \\ \qquad\qquad + \alpha(X(t),Y(t),U(t))d\hat{B}(t) + b(X(t),Y(t))dY(t) \\ X(0) = \xi. \end{cases}$$

Put

$$(7.5)\quad \rho(t) = \exp(\int_0^t f(X(t))\, dY(t) - \frac{1}{2} \int_0^t |f(X(t))|^2\, dt)$$

and define a new probability P by

$$(7.6)\quad dP = \rho(T)d\mathring{P}.$$

Then Girsanov's theorem asserts that, under the probability P,

$B(t) = Y(t) - \int_0^t f(X(s))\, ds$, $0 \le t \le T$, turns out to be a Wiener process independent from $\hat{B}$, and (X,Y) satisfies (7.1) .

Moreover, the pay-off function J(U) of (7.3) can be written by

$$J(A) = \mathring{E}[\int_0^T h(X(t),Y(t))\rho(t)\, dt + G(X(T),Y(T))\rho(T)]$$

where $\mathring{E}$ means the expectation w.r.t. $\mathring{P}$.

On the other hand, $A = (\Omega,\mathcal{F},\mathcal{F}_t,\mathring{P},\hat{B},Y,U)$ derives an admissible system $\mathcal{A} = (\Omega,\mathcal{F},\mathcal{F}_t,\mathring{P},Y,U)$, and an admissible system turns out to be an admissible control system, when we add an independent Wiener process $\hat{B}$. For $\mathcal{A} = (\Omega,\mathcal{F},\mathcal{F}_t,\mathring{P},Y,U)$, we consider SPDE,

$$(7.7)\quad \begin{cases} dq(t) = L(Y(t),U(t))q(t)\, dt + M(Y(t))q(t)\, dY(t) \\ q(0) = \phi \quad (\in H^3) \end{cases}$$

where

$$(7.8)\quad L(y,u)q = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} a_{ij}(\cdot,y,u)\frac{\partial}{\partial x_i} q - \sum_{j=1}^{d} \frac{\partial}{\partial x_j} (\tilde{a}_j(\cdot,y,u)q)$$

$$M^k(y)q = - \sum_{i=1}^{d} b_{ik}(\cdot,y)\frac{\partial}{\partial x_i} q + \mathcal{F}_k(\cdot,y)q$$

$$a(x,y,u) = (b(x,y)b^*(x,y) + \alpha(x,y,u)\alpha^*(x,y,u))/2$$

$$\tilde{a}_j(x,y,u) = \gamma_j(x,y,u) - \sum_{i=1}^{d} \frac{\partial a_{ij}}{\partial x_i}(x,y,u)$$

and

$$\tilde{f}_k(x,y) = f_k(x) - \sum_{i=1}^{d} \frac{\partial b_{ik}}{\partial x_i}(x,y).$$

Then, under the conditions (A.1) ~ (A.3), $J(\mathring{A})$ can be represented by

(7.9) $$J(\mathring{A}) = \mathring{E}[\int_0^T (h(\cdot,Y(t)),q(t))dt + (G(\cdot,Y(T)),q(T))] .$$

Now we have the following theorem, appealing to Theorems 3.1 and 4.2.

Theorem 7.1 Suppose (A.1) ~ (A.3), $(a.4)_1$ and the convexity condition for the coefficients of the SPDE (7.7). Then, for $\phi \in \Phi$, there is an optimal admissible control system A^*, namely

(7.10) $$J(A^*) = \inf_A J(A) .$$

Appendix

Let us prove Lemma in Section 3. Here we use the following notations, according to [14]:

For $\alpha = (i_1,\cdots,i_\ell)$, $D^\alpha = \partial_{i_1} \cdots \partial_{i_\ell}$, $|\alpha| = \ell$

$\binom{\alpha}{\gamma} = \binom{i_1}{j_1} \cdots \binom{i_\ell}{j_\ell}$ is the binomial coefficient

(for $\gamma = (j_1,\cdots,j_\ell)$, $0 \le j_k \le i_k$)

$|i| = 0$ for $i = 0$, $= 1$ for $i = 1,\cdots,d$,

$\int\cdots dx$ stands for $\int_{\mathbb{R}^d}\cdots dx$ and hereafter N_1, N_2, $\cdots$ denote constants depending only on K , T and ℓ , and repeated indexes are assumed to be summed from 1 (not 0) to d .

We will estimate the principal part of J defined by (1).

For $u \in C_0^\infty(\mathbb{R}^d)$, we put

$$(1)\quad J = \sum_{|\gamma|\le\ell} \int \{ -2\, D^\gamma(a^{ij}\partial_j u)D^\gamma\partial_i u + 3\, D^\gamma(b^i\partial_i u)D^\gamma(b^j\partial_j u) \}\, dx$$

$$= -2 \int \hat{a}^{ij}\, \partial_i u\, \partial_j u\, dx$$

$$+ \sum_{1\le|\gamma|\le\ell} \int \{ -2\, D^\gamma(a^{ij}\partial_j u)D^\gamma\partial_i u + 3\, D^\gamma(b^i\partial_i u)D^\gamma(b^j\partial_j u) \}\, dx,$$

where $\hat{a}^{ij} = a^{ij} - \frac{3}{2}\, b^i\cdot b^j$.

Using integration by part, we get

$$(2)\quad \int -2\, D^\gamma(a^{ij}\partial_j u)D^\gamma\partial_i u\, dx$$

$$= \int -2\, a^{ij}D^\gamma\partial_j u\, D^\gamma\partial_i u\, dx + 2 \sum_{\substack{\alpha+\beta=\gamma\\ |\alpha|=1}} \binom{\gamma}{\alpha} \int D^\alpha a^{ij}\, D^\beta\partial_i\partial_j u\, D^\gamma u\, dx$$

$$+ 2 \sum_{\substack{\alpha+\beta=\gamma\\ |\alpha|\ge1}} \binom{\gamma}{\alpha} \int D^\alpha\partial_i a^{ij}\, D^\beta\partial_j u\, D^\gamma u\, dx$$

$$+ 2 \sum_{\substack{\alpha+\beta=\gamma\\ |\alpha|\ge2}} \binom{\gamma}{\alpha} \int D^\alpha a^{ij}\, D^\beta\partial_i\partial_j u\, D^\gamma u\, dx$$

Appealing to "$|\beta| + 1 \le \ell$ in the 3rd term and $|\beta| + 2 \le \ell$ in the 4th term" ,

$\le$ 1st term + 2nd term + $N_1\, \|u\|_\ell^2$

Since $D^\gamma(\, b^i\partial_i u\,) - b^i D^\gamma\partial_i u$ is independent of the (ℓ+1)-th order derivative of u , we obtain, in the same way as (2),

$$(3)\quad \int 3\, D^\gamma(b^i\partial_i u)\cdot D^\gamma(b^j\partial_j u)\, dx$$

$$= \int 3\, |\, b^i D^\gamma\partial_i u + (\, D^\gamma(b^i\partial_i u) - b^i D^\gamma\partial_i u\,)\, |^2\, dx$$

$$= \int 3\, b^i D^\gamma\partial_i u\cdot b^j D^\gamma\partial_j u\, dx + \int 6 \sum_{\substack{\alpha+\beta=\gamma\\ |\alpha|\ge1}} \binom{\gamma}{\alpha} D^\alpha b^i D^\beta\partial_i u\cdot b^j D^\gamma\partial_j u\, dx$$

$$+ \int 3\, |\, D^\gamma(b^i\partial_i u) - b^i D^\gamma\partial_i u\, |^2\, dx$$

$$\leq \text{1st term} - 6 \sum_{\substack{\alpha+\beta=\gamma \\ |\alpha|=1}} \binom{\gamma}{\alpha} \int (D^{\alpha} b^i)\cdot b^j \, D^{\beta}\partial_i \partial_j u \, D^{\gamma} u \, dx + N_2 \|u\|_{\ell}^2$$

$$= \text{1st term} - 3 \sum_{\substack{\alpha+\beta=\gamma \\ |\alpha|=1}} \binom{\gamma}{\alpha} \int D^{\alpha}(b^i \cdot b^j) \, D^{\beta}\partial_i \partial_j u \, D^{\gamma} u \, dx + N_2 \|u\|_{\ell}^2 .$$

(1),(2) and (3) yield

$$J \leq -2 \sum_{|\gamma|\leq \ell} \int \hat{a}^{ij} \, D^{\gamma}\partial_j u \, D^{\gamma}\partial_i u \, dx$$
$$+ 2 \sum_{1\leq|\gamma|\leq \ell} \sum_{\substack{\alpha+\beta=\gamma \\ |\alpha|=1}} \binom{\gamma}{\alpha} \int D^{\alpha}\hat{a}^{ij} \, D^{\beta}\partial_i \partial_j u \, D^{\gamma} u \, dx + N_3 \|u\|_{\ell}^2 .$$

On the other hand,

$$| D^{\alpha}\hat{a}^{ij} \, D^{\beta}\partial_i \partial_j u |^2 \leq N_4 \, \hat{a}^{ij} \, D^{\beta}\partial_i \partial_k u \, D^{\beta}\partial_j \partial_k u$$
$$\leq N_4 \sum_{|\gamma|\leq \ell} \hat{a}^{ij} \, D^{\gamma}\partial_i u \, D^{\gamma}\partial_j u$$

holds, by virtue of $\hat{a}^{ij} \in C^2(\mathbb{R}^d)$ and matrix $(\hat{a}^{ij}) \geq 0$,

(see Lemma 1.7.1 [21]).

Noting $2\,|ab| \leq \varepsilon^2 |a|^2 + |b|^2/\varepsilon^2$, we get

$$J \leq (N_5 \, \varepsilon^2 - 2) \sum_{|\gamma|\leq \ell} \int \hat{a}^{ij} \, D^{\gamma}\partial_j u \, D^{\gamma}\partial_i u \, dx + (N_3 + N_6/\varepsilon^2) \|u\|_{\ell}^2 .$$

So $J \leq N_7 \|u\|_{\ell}^2$ holds , putting $\varepsilon^2 = 2/N_5$.

Applying the same calculation to the other terms, we can prove Lemma for $u \in C_0^{\infty}(\mathbb{R}^d)$. Since $C_0^{\infty}(\mathbb{R}^d)$ is dense in $H^{\ell+1}$, we can conclude Lemma by the routine method. □

CHAPTER 3

On the Cauchy problem for non-linear stochastic partial differential equations with continuous coefficients

—— Existence Theorem ——

§ 1 Introduction

The subject of this chapter is to show the existence of solutions for the following non-linear stochastic partial differential equation derived by white noise:

$$(1.1) \quad du(t) = (Au(t) + F(u(t)))dt + G(u(t))dW(t),$$

where A is a second-order elliptic differential operator, F and G are continuous operators from $L^2(\mathbb{R}^d)$ to itself and $W(t)$ is a one dimensional Brownian motion.
A solution $u(t)$ of the problem is sought in the space of Sobolev type $H^m(\mathbb{R}^d)$ (for the precise definition of solution, see § 2 Definition 2.1).

When F and G satisfy the Lipschitz condition and A is uniformly elliptic, Pardoux [23] and Walsh [29] proved the existence and uniqueness of the solutions for (1.1) by Picard's method of successive approximation. But, if F and G are merely continuous, Picard's method is not effective. To overcome this difficulty, we approximate the equation (1.1) by Cauchy polygon (see § 3 (3.4)). Moreover, in our problem, the operator A may be degenerate.

This chapter is formulated as follows. In Section 2 we state our problem and recall some results in our convenient way. Section 3 is devoted to the proof of existence theorem [Theorem 3.1]. In Section 4 we prove a sort of stability on the perturbation of coefficient.

§ 2 Preliminaries

Let us define an operator A by

$$(2.1)\quad Au(x) = \sum_{i,j=1}^{d} \partial_i (a_{ij}(x)\partial_j u(x)) + \sum_{i=1}^{d} b_i(x)\partial_i u(x) + c(x)u(x)$$

where $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1,\cdots,d$.

We denote by L^2_r , $r \geq 0$, the space of real valued Borel functions on $\mathbb{R}^d$ with the norm defined by :

$$\|f\|_{0,r} = \left(\int_{\mathbb{R}^d} |(1 + |x|^2)^{r/2} f(x)|^2 \, dx \right)^{1/2}$$

Let H^m_r be the subspace of L^2_r consisting of functions whose generalized derivatives up to the order m belong to L^2_r. Clearly H^m_r becomes a Hilbert space with the inner product

$$(f,g)_{m,r} = \sum_{|\alpha|\leq m} \frac{|\alpha|!}{\alpha^1!\cdots\alpha^d!} \int_{\mathbb{R}^d} (1 + |x|^2)^r D^\alpha f(x) D^\alpha g(x) \, dx,$$

where $\alpha = (\alpha^1, \cdots ,\alpha^d)$ is a multi-index with non-negative integer α^i, $|\alpha| = \alpha^1 + \cdots + \alpha^d$ and $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha^1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha^d}$.

Let us set $\|f\|^2_{m,r} = (f,f)_{m,r}$ and, for $r = 0$, $L^2_0 = L^2$, $H^m_0 = H^m$, $(\cdot,\cdot)_{m,0} = (\cdot,\cdot)_m$ and $\|\cdot\|_{m,0} = \|\cdot\|_m$, for simplicity, if no confusion occurs.

We consider the following equation :

$$(2.2)\quad \begin{cases} du(t) = (A(u(t)) + F(u(t)))dt + G(u(t))dW(t) \\ u(0) = u_0 \qquad\qquad 0 < t \le T \end{cases}$$

where F and G are operators from L^2 to itself and $W(t)$ is a 1-dimensional Brownian motion.

Definition 2.1 By a solution of the equation (2.2), we mean an H^1-valued process $u = (u(t))$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with a reference family $(\mathcal{F}_t)$ such that

(I) there exists a 1-dimensional $(\mathcal{F}_t)$-Brownian motion $W = (W(t))$ with $W(0) = 0$

(II) $u = (u(t))$ is adapted to $(\mathcal{F}_t)$ and

$$E[\int_0^T \|u(t)\|_1^2 \, dt] < \infty$$

(III) for any $\eta \in C_0^\infty(\mathbb{R}^d)$ (C^∞-function on $\mathbb{R}^d$ with compact support) and almost all $t \in [0,T]$,

$$(2.3)\quad (u(t),\eta)_0 = (u_0 ,\eta)_0 + \int_0^t \langle A(u(s)) + F(u(s)),\eta \rangle_0 \, ds + \int_0^t (G(u(s)),\eta)_0 \, dW(s)$$

holds, where $\langle \cdot,\cdot \rangle_\ell$ = the duality pairing between $H^{\ell-1}$ and $H^{\ell+1}$ under $H^\ell = (H^\ell)^*$ (= the dual space of H^ℓ) $\ell = 0,1,2,\cdots$, namely

$$\langle A(u(t)) + F(u(t)), \eta \rangle_\ell$$

$$= - \sum_{i,j=1}^{d} (a_{ij}\partial_j u(t),\partial_i \eta)_\ell + \sum_{i=1}^{d} (b_i \partial_i u(t),\eta)_\ell + (cu(t),\eta)_\ell$$

$+ (F(u(t)),\eta)_\ell .$

To emphasize the particular role of ($\mathcal{F}_t$)-Brownian motion $W = (W(t))$, sometimes we call the pair (W,u) itself a solution of (2.2).

Now we introduce the following conditions.

(A.1) The functions a_{ij}, $\partial_i a_{ij}$, b_i, c, ($i,j = 1,\cdots,d$) and their derivatives up to the order m do not exceed K in absolute value.

(A.2) $a_{ij} = a_{ji}$ ($i,j = 1,\cdots,d$) and $(a_{ij})_{i,j=1,\cdots,d}$ is a non-negative definite matrix.

(A.3) F and G are continuous operators from L^2 to itself with linear growth.

Hereafter we always assume " $m \geq 2$ " .

The following Lemma is proved by Krylov & Rozovskii [14].

Lamma 2.1 (the special case of Lemma 2.1 of [14])

Under the conditions (A.1) and (A.2), there exists a constant λ, depending only on K and m in (A.1), such that

(2.4) $\langle Au,u \rangle_\ell \leq \lambda \|u\|_\ell^2$ for $\forall\, u \in H^{\ell+1}$ ($\ell = 0,1,\cdots,m$).

Now we consider the following equations.

$$(2.5) \begin{cases} du(t) = (A(u(t)) + f(t))dt + g(t)dW(t) \\ u(0) = u_0 \qquad\qquad 0 < t \leq T \end{cases}$$

According to Krylov & Rozovskii [14], we see the following proposition.

Proposition 2.1 (Krylov & Rozovskii)

Let $f, g \in L^2(\Omega \times (0,T) ; H^m)$ be adapted to $(\mathcal{F}_t)$ and $u_0 \in L^2(\Omega ; H^m)$, $\mathcal{F}_0$-measurable. Then (2,5) has a unique solution u, which belongs to $L^2(\Omega \times (0,T) ; H^m) \cap L^2(\Omega ; C(0,T;H^{m-1}))$, and satisfies

$$(2.6) \quad \sup_{0\le t\le T} E[\|u(t)\|_\ell^2] \le e^{CT}\{ E[\|u_0\|_\ell^2] + E[\int_0^T \|f(t)\|_\ell^2 \, dt] + E[\int_0^T \|g(t)\|_\ell^2 \, dt] \} , \ \ell = 0,1,\cdots,m,$$

where C depends only on K in (A.1), and

$$(2.7) \quad \|u(t)\|_\ell^2 = \|u_0\|_\ell^2 + 2\int_0^t \langle Au(s) + f(s), u(s) \rangle_\ell \, ds + 2\int_0^t (g(s), u(s))_\ell dW(s) + \int_0^t \|g(s)\|_\ell^2 \, ds,$$

for $t \in [0,T]$, $\ell = 0,1,\cdots,m-1$.

Moreover the solution u of (2.5) satisfies the following equation:

$$(2.8) \quad (u(t),\eta)_\ell = (u_0,\eta)_\ell + \int_0^t \langle Au(s) + f(s) , \eta \rangle_\ell \, ds + \int_0^t (g(s) , \eta)_\ell \, dW(s),$$

for $t \in [0,T]$, $\eta \in C_0^\infty(\mathbb{R}^d)$, $\ell = 0,1,\cdots,m-1$.

Sketch of proof For $\varepsilon > 0$, define the operator A_ε by (2.1) with a_{ij} replaced by $a_{ij} + \varepsilon\delta_{ij}$. We consider the following equation.

$$(2.9)\quad \begin{cases} du(t) = (A_\varepsilon u(t) + f(t))dt + g(t)dW(t) \\ u(0) = u_0 . \end{cases}$$

By Theorem 1.1 in [11], the equation (2.9) has a unique solution u_ε which belongs to $L^2(\Omega\times(0,T) ; H^{m+1})\cap L^2(\Omega ; C(0,T;H^m))$ and satisfies,

$$(2.10)\quad \|u_\varepsilon(t)\|_m^2 = \|u_0\|_m^2 + 2 \int_0^t \langle A_\varepsilon u_\varepsilon(s) + f(s) , u_\varepsilon(s) \rangle_m ds + 2 \int_0^t (g(s) , u_\varepsilon(s))_m dW(s) + \int_0^t \|g(s)\|_m^2 ds.$$

By Lemma 2.1, there exists a constant λ such that

$$(2.11)\quad \langle A_\varepsilon u , u \rangle_m \le \lambda\|u\|_m^2 \quad \text{for any } u \in H^{m+1} \text{ and } 0 < \varepsilon \le 1.$$

Hence, Gronwall's inequality yields

$$(2.12)\quad \sup_{0\le t\le T} E[\|u_\varepsilon(t)\|_m^2] \le e^{(2\lambda+1)T}\{ E[\|u_0\|_m^2] + E[\int_0^T \|f(t)\|_m^2 dt] + E[\int_0^T \|g(t)\|_m^2 dt] \}, \quad \text{for } 0 < \forall\varepsilon \le 1.$$

So, there exist a subsequence $\{ u_{\varepsilon_n} \}$ and $u \in L^2(\Omega\times(0,T) ; H^m)$ adapted to $(\mathcal{F}_t)$ such that

$$(2.13)\quad u_{\varepsilon_n} \longrightarrow u \quad \text{weakly in } L^2(\Omega\times(0,T) ; H^m) \text{ as } \varepsilon_n \longrightarrow 0.$$

By the same argument as the proof of Theorem 1.3 in [22], we can see that u is a solution of (2.5).

Moreover, using a routine method we can prove the uniqueness of solution of the equation (2.5).

Furthermore, by (2.12) and the uniqueness of solution, for each $t \in [0,T]$, there exists a subsequence $\{ u_{\varepsilon_n}(t) \}$ such that

(2.14) $u_{\varepsilon_n}(t) \longrightarrow u(t)$ weakly in $L^2(\Omega ; H^m)$.

Combining (2.14) with (2.12), we get (2.6).

§ 3 Existence of solutions

Besides (A.1) ~ (A.3), we assume the following conditions.

(A.4) The restrictions of F and G on H^m operate to itself and satisfy the linear growth condition (see (3.2)).

(A.5) For some $r > 0$, the restrictions of F and G on L^2_r operate to itself and satisfy the linear growth condition (see (3.3)).

Namely, there exists a constant L such that

(3.1) $\|H(u)\|_0^2 \le L(1 + \|u\|_0^2)$ for $\forall u \in L^2$ (by (A.3))

(3.2) $\|H(u)\|_m^2 \le L(1 + \|u\|_m^2)$ for $\forall u \in H^m$

(3.3) $\|H(u)\|_{0,r}^2 \le L(1 + \|u\|_{0,r}^2)$ for $\forall u \in L^2_r$

where H = F, G.

(A.6) $u_0 \in H^m \cap L^2_r$, where r is the same number as in (A.5).

Theorem 3.1 Under the conditions (A.1) ~ (A.6), the equation (2.2) has a solution which belongs to $L^2(\Omega \times (0,T) ; H^m) \cap L^2(\Omega ; C(0,T;H^{m-1}))$ and satisfies

(3.4) $$E[\sup_{0\le t\le T} \|u(t)\|_m^2] \le N(1 + \|u_0\|_m^2),$$

where N depends only on K, L in (A.1), (A.5) and T.

Proof of Theorem 3.1 We divide the proof into two steps. In the first step, we construct an approximate sequence of (2.2) and show the preliminary lemmas. Let us define an approximate sequence u_n (n = 1,2,$\cdots$) of (2.2) by

$$(3.5)\quad \begin{cases} du_n(t) = (Au_n(t) + F(u_n(t_k))dt + G(u_n(t_k))dW(t) \\ u_n(0) = u_0 \qquad t \in (t_k , t_{k+1}],\ k = 0,1,\cdots,n-1 \end{cases}$$

where $t_k = kT/n$.

On each small interval (t_k , t_{k+1}], k = 0,1,$\cdots$,n-1, we can apply Proposition 2.1 . Hence we can construct the solution of (3.5) which belongs to $L^2(\Omega \times (0,T) ; H^m)$. For the approximate sequence u_n , n = 1,2,$\cdots$, the following facts hold.

Lemma3.1 There is a constant N, depending only on K, L in (A.1), (A.5) (resp.) and T, such that

$$(3.6)\quad \sup_{0\le t\le T} E[\|u_n(t)\|_m^2] \le N(1 + \|u_0\|_m^2) , \quad n = 1,2,\cdots$$

$$(3.7)\quad \sup_{0\le t\le T} E[\|u_n(t)\|_0^4] \le N(1 + \|u_0\|_0^4) , \quad n = 1,2,\cdots$$

$$(3.8)\quad E[\int_0^T \|u_n(t) - \bar{u}_n(t)\|_0^2\, dt] \le \frac{N(1 + \|u_0\|_m^2)}{n} , \quad n = 1,2,\cdots$$

where $\bar{u}_n(t) = u_n(t_k)$, $t \in [t_k, t_{k+1})$, k = 0,1,$\cdots$,n-1.

Lemma 3.2 There is a constant N, depending only on K, L in (A.1), (A.5) (resp.) and T, such that

$$(3.9)\quad E[\int_{|x| > \rho} |u_n(t,x)|^2\, dx] \le \frac{N(1 + \|u_0\|_{0,r}^2)}{(1 + \rho^2)^r}$$

for any $t \in [0,T]$, n = 1,2,$\cdots$, $\rho > 0$.

First we introduce two spaces $\mathcal{H}_\gamma(D)$ and $\mathcal{H}_\gamma(D,T)$. Let D be a bounded open subset of $\mathbb{R}^d$, with smooth boundary. Define $\mathcal{H}_\gamma(D)$ and $\mathcal{H}_\gamma(T,D)$ as follows (cf Lions [17]).

$$(3.10) \quad \mathcal{H}_\gamma(D) = \{ \varphi \in L^2(-\infty,\infty;H^1(D)) \; ; \; \int_{-\infty}^{\infty} |\tau|^{2\gamma} \|\hat{\varphi}(\tau)\|_*^2 \, d\tau < \infty \}$$

with the norm

$$(3.11) \quad \| \varphi \|^2_{\mathcal{H}_\gamma(D)} = \int_{-\infty}^{\infty} \|\varphi(t)\|^2_{H^1(D)} dt + \int_{-\infty}^{\infty} |\tau|^{2\gamma} \|\hat{\varphi}(\tau)\|_*^2 \, d\tau$$

where, for simplicity, we put $\hat{\varphi}(\tau) = \int_{-\infty}^{\infty} \exp(-2\pi i \tau t)\varphi(t)\, dt$ and $\|\cdot\|_* =$ norm of $(H^1(D))^*$ $\left(= \text{dual space of } H^1(D) \text{ under } H^0(D) = (H^0(D))^* \right)$ and

$$(3.12) \quad \mathcal{H}_\gamma(T,D) = \{ \varphi|_{[0,T]} \; ; \; \varphi \in \mathcal{H}_\gamma(D) \}$$

with the norm

$$(3.13) \quad \| \varphi \|_{\mathcal{H}_\gamma(T,D)} = \inf\{ \| \psi \|_{\mathcal{H}_\gamma(D)} \; ; \; \varphi = \psi \text{ a.e. on } [0,T] \}$$

respectively.

Lemma 3.3 For any fixed $\gamma \in (0,1/4)$,

$$(3.14) \quad u_n \in \mathcal{H}_\gamma(T,D), \text{ w.p. } 1, \quad n = 1,2,\cdots$$

holds and there is a constant N, depending only on K, L in (A.1), (A.5) (resp.) and T, such that

$$(3.15) \quad E[\|u_n\|^2_{\mathcal{H}_\gamma(T,D)}] \le N(1 + \|u_0\|_m^2),$$

for any subset D of $\mathbb{R}^d$ and $n = 1,2,\cdots$.

Proof of Lemma 3.1 Since u_n is the solution of (3.5), Proposition 2.1 derives,

$$(3.16)\quad \sup_{t_k\le t\le t_{k+1}} E[\ \|u_n(t)\|_m^2\ ]$$

$$\le e^{CT/n}\ \{\ E[\ \|u_n(t_k)\|_m^2] + \int_{t_k}^{t_{k+1}} E[(\|F(u_n(t_k))\|_m^2 + \|G(u_n(t_k))\|_m^2)]dt\}$$

$$\le \frac{2LT}{n}e^{CT/n} + (\ 1 + \frac{2LT}{n}\ )e^{CT/n}\ E[\ \|u_n(t_k)\|_m^2],\ (\text{ by } (3.2)\ )$$

where C is independent of k = 0,1,···,n-1 and n = 1,2,···.

Hence we have

$$(3.17)\quad \max_{0\le k\le n-1}\ \sup_{t_k\le t\le t_{k+1}} E[\ \|u_n(t)\|_m^2\ ]$$

$$\le \frac{2LT}{n}\ e^{CT/n}\ \frac{(\ 1 + \frac{2LT}{n}\ )^n\ e^{CT}\ \ - 1}{(\ 1 + \frac{2LT}{n}\ )\ e^{CT/n}\ - 1} + (\ 1 + \frac{2LT}{n}\ )^n\ e^{CT}\|u_0\|_m^2 .$$

Since the right hand side of (3.17) is a convergent sequence of n = 1,2,···, we get (3.6).

Using Itô's formula to (2.7), we have

$$(3.18)\ E[\ \|u_n(t)\|_0^4\ ] = E[\ \|u_n(t_k)\|_0^4\ ]$$

$$+ 4\ E[\int_{t_k}^{t} \|u_n(s)\|_0^2\ \langle\ Au_n(s) + F(u_n(t_k)),u_n(s)\ \rangle_0\ ds\ ]$$

$$+ 2\ E[\int_{t_k}^{t} \|u_n(s)\|_0^2\ \|G(u_n(t_k))\|_0^2\ ds\ ]$$

$$+ 4\ E[\int_{t_k}^{t} (\ G(u_n(t_k)),u_n(s)\ )_0^2\ ds\ ]$$

$$\le E[\ \|u_n(t_k)\|_0^4\ ] + E[\int_{t_k}^{t} \|F(u_n(t_k))\|_0^4\ ds + 3\ E[\int_{t_k}^{t}\|G(u_n(t_k))\|_0^4\ ds]$$

$$+ (\ 4\lambda + 6)E[\int_{t_k}^{t} \|u_n(s)\|_0^4\ ds\ ]$$

for $t_k \le t \le t_{k+1}$, k = 0,1,···,n-1, n = 1,2,···.

So Gronwall's inequality yields

$$(3.19)\quad \sup_{t_k\le t\le t_{k+1}} E[\ \|u_n(t)\|_0^4\ ]$$

$$\le e^{(4\lambda+6)T/n}\{\ E[\ \|u_n(t_k)\|_0^4 + \int_{t_k}^{t_{k+1}} E[\ \|F(u_n(t_k))\|_0^4\ ]ds$$

$$+\ 3\int_{t_k}^{t_{k+1}} E[\ \|G(u_n(t_k))\|_0^4\ ]\ ds\ \ \}$$

$k = 0,1,\cdots,n-1,\ n = 1,2,\ \cdots.$

By virtue of the assumption (3.2), we can apply the similar argument to (3.6) and obtain (3.7).

Put $v_{n,k}(t) = u_n(t + t_k) - u_n(t_k),\ 0 \le t \le T/n$. Then $v_{n,k}$ satisfies the following equation.

$$(3.20)\quad \begin{cases} dv_{n,k}(t) = (\ Av_{n,k}(t) + Au_n(t_k) + F(u_n(t_k))\ )dt \\ \qquad\qquad\qquad\qquad\qquad\qquad + G(u_n(t_k))dW_k(t) \\ v_{n,k}(0) = 0 \end{cases}$$

where $W_k(t) = W(t + t_k) - W(t_k)$. So Proposition 2.1 yields

$$(3.21)\quad \sup_{0\le t\le T/n} E[\ \|v_{n,k}(t)\|_0^2\ ]$$

$$\le e^{CT/n}\int_0^{T/n} E[\ \|Au_n(t_k)\|_0^2 + \|F(u_n(t_k))\|_0^2 + \|G(u_n(t_k))\|_0^2\ ]ds$$

$$\le \frac{N\ e^{CT/n}T}{n}\ \{\ 1 + E[\ \|u_n(t_k)\|_2^2\ \}$$

where N and C are constants independent of k and n. Combining (3.21) with (3.17), we get (3.8). This completes the proof.

Proof of Lemma 3.2 We define the operator $\overline{A}$ by

$$(3.22)\quad \overline{A}u(x) = \sum_{i,j=0}^{d} \partial_i(\ \overline{a}_{ij}(x)\partial_j u(x)\ )$$

where ∂_0 = identity

$$\bar{a}_{ij} = a_{ij} \qquad (i,j = 1,\cdots,d)$$

$$\bar{a}_{0j} = b_j - \sum_{k=1}^{d} a_{jk} R_k \qquad (j = 1,\cdots,d)$$

$$\bar{a}_{i0} = - \sum_{k=1}^{d} a_{ik} R_k \qquad (i = 1,\cdots,d)$$

$$\bar{a}_{00} = c - \sum_{k=1}^{d} b_k R_k + \sum_{k,\ell=1}^{d} a_{k\ell} R_k R_\ell$$

and

$$R_k(x) = \frac{r x_k}{1 + |x|^2} \qquad (k = 1,\cdots,d).$$

Then

$$\langle \bar{A}(Ru), \eta \rangle_0 = \langle A(u), R\eta \rangle_0 \quad \text{for any } u \in H^1 \text{ and } \eta \in C_0^\infty(\mathbb{R}^d),$$

where $R(x) = (1 + |x|^2)^{r/2}$.

Hence $q_n(t) = R\, u_n(t)$ satisfies the following equations.

$$(3.23) \quad \begin{cases} dq_n(t) = (\bar{A}q_n(t) + RF(u_n(t_k)))dt + RG(u_n(t_k))dW(t) \\ q_n(0) = Ru_0 \qquad t \in (t_k , t_{k+1}], \ k = 0,1,\cdots,n-1 \end{cases}$$

(See Krylov & Rozovskii [15] Theorem 2.2).

By virtue of the assumption (3.3), we can repeat the similar argument to (3.16)~(3.17) and obtain

$$(3.24) \quad E[\|q_n(t)\|_0^2] \le N(1 + \|Ru_0\|_0^2)$$

for any $n = 1,2,\cdots,$ and $0 \le t \le T$.

This yields (3.9) and completes the proof.

Proof of Lemma 3.3 Put $f_n(t) = F(\bar{u}_n(t))$ and $g_n(t) = G(\bar{u}_n(t))$.

For the convenience, we extend $u_n(t)$, $f_n(t)$ and $g_n(t)$ on $(-\infty,\infty)$ in the following way,

$$h(t) = h(t) \quad , \ t \in [\ 0\ ,\ T\ ]$$
$$\qquad = 0 \quad , \ t \in (-\infty\ ,\ \infty) \setminus [\ 0\ ,\ T\ ]$$

where $h(t) = u_n(t),\ f_n(t),\ g_n(t)$.

Since u_n is a solution of (3.5), applying Itô's formula to (2.8), we obtain

$$(3.25)\quad 2\pi i\tau\ (\ \hat{u}_n(\tau),\eta\ )_1 = (\ u_0,\eta\ )_1 - (\ u_n(T),\eta\ )_1\exp(-2\pi i\tau T)$$
$$+ \langle\ \widehat{Au}_n(\tau) + \hat{f}_n(\tau),\eta\ \rangle_1 + \int_0^T \exp(-2\pi i\tau t)(\ g_n(t),\eta\ )_1 dW(t),$$

for any $\eta \in C_0^\infty(\ \mathbb{R}^d\ )$.

Let $\eta_j \in C_0^\infty(\ \mathbb{R}^d\ )$, $j = 1,2,\cdots$, be a complete orthonormal system of H^2. Using (3.1) and the similar evaluation to (3.17) in which m is replaced by 0 , we have

$$(3.26)\quad 4\pi^2\tau^2 E[\ \|\hat{u}_n(\tau)\|_0^{\,2}\ ]$$

$$= 4\pi^2\tau^2 \sum_{j=1}^{\infty} E[\ |\ (\ \hat{u}_n(\tau),\eta_j)_1\ |^2\ ]$$

$$\le N_1\{\ \|u_0\|_0^{\,2} + E[\|u_n(T)\|_0^{\,2} + \|\widehat{Au}_n(\tau)\|_0^{\,2} + \|\hat{f}_n(\tau)\|_0^{\,2} + \int_0^T \|g_n(t)\|_0^{\,2} dt]\ \}$$

$$\le N_2\{\ 1 + \|u_0\|_0^{\,2} + E[\ \|\widehat{Au}_n(\tau)\|_0^{\,2} + \|\hat{f}_n(\tau)\|_0^{\,2}\ ]\ \}$$

Hence for any fixed $\kappa \in (\ 1\ ,\ 3/2\ )$,

$$(3.27)\quad \int_{-\infty}^{\infty} E[\ |\tau|^{2\gamma}\|\hat{u}_n(\tau)\|_0^{\,2}\ ]\ d\tau$$

$$\le \int_{|\tau|\le 1} E[\ \|\hat{u}_n(\tau)\|_0^{\,2}\ ]d\tau + \int_{|\tau|>1} E[\ \frac{2|\tau|^2}{1 + |\tau|^\kappa}\ \|\hat{u}_n(\tau)\|_0^{\,2}\ ]\ d\tau$$

$$\le E[\ \int_{-\infty}^{\infty} \|u_n(t)\|_0^{\,2}\ dt\ ] + N_3\ \{\ \int_{-\infty}^{\infty} \frac{d\tau}{1 + |\tau|^\kappa}\ (\ 1 + \|u_0\|_0^{\,2}\ )$$

$$+ \int_{-\infty}^{\infty} E[\ \|Au_n(t)\|_0^2 + \|\hat{f}_n(t)\|_0^2\ ]\, dt\ \}$$

$\leq N_4(\ 1 + \|u_0\|_m^2\ )$ (by (3.6)).

This concludes the Lemma.

Second step: Let $D_k = \{\ x \in \mathbb{R}^d\ ;\ |x| < k\ \}$ ($k = 1,2,\cdots$). Define a metric d by

$$d(p,q) = \sum_{k=1}^{\infty} \frac{1}{2^k} \min(\ 1,\ \{ \int_0^T \|\ p(t) - q(t)\ \|_{L^2(D_k)}^2\ dt\ \}^{1/2}\ )$$

for $p,\ q \in L^2(\ 0,T;L^2(\mathbb{R}^d)\ )$. $\mathscr{W}(0,T)$ denotes the completion of $L^2(\ 0,T;L^2(\mathbb{R}^d)\ )$ with respect to the metric d. Put $S = C(0,T;\mathbb{R}\ )\times\mathscr{W}(0,T)$. $\mu(n)$ ($n = 1,2,\cdots$) denote the image measure of (W,u_n) on S where W is a 1-dimensional Brownian motion appeared in (3.5).

$B_\rho = \{\ q \in \mathscr{W}(0,T)\ ;\ \|q\|_{\mathscr{H}_\gamma(T,D_k)} \leq (\ 2^k\rho\ )^{1/2},\ k = 1,2,\cdots\ \}$ is compact in $\mathscr{W}(0,T)$, because the injection $\mathscr{H}_\gamma(T,D_k) \longrightarrow L^2(\ 0,T;L^2(D_k)\ )$ is a compact operator (cf Lions [17] Chapitre Ⅳ Proposition 4.1).

On the other hand, Lemma 3.3 asserts

$$P(\ u_n \notin B_\rho\ ) \leq \frac{N(\ 1 + \|u_0\|_m^2\ )}{\rho}\ ,\ n = 1,2,\cdots.$$

By Prohorov's theorem, $\{\ \mu(n)\ :\ n = 1,2,\cdots\ \}$ is relatively compact. Hence there is a subsequence $\{n'\}$ and a probability measure μ on S such that $\{\ \mu(n')\ \}_{n'}$ converges weakly to μ. Moreover, by Skorohod's theorem, there exist S-valued random variables ($B_{n'},q_{n'}$) and (B,q) on a suitable probability space ($\hat{\Omega},\hat{\mathscr{F}},\hat{P}$) such that

(3.28) the law of $(B_{n'}, q_{n'}) = \mu(n')$,

the law of $(B, q) = \mu$ (= the limit measure of $\{ \mu(n') \}_{n'}$)

and, with probability 1,

(3.29) $B_{n'} \longrightarrow B$ uniformly on $[0,T]$

(3.30) $q_{n'} \longrightarrow q$ in $\mathscr{V}(0,T)$

that is,

(3.31) $q_{n'} \big|_{D_k} \longrightarrow q \big|_{D_k}$ in $L^2((0,T) \times D_k)$ for $^\forall k = 1,2,\cdots$.

Since (3.7) implies the uniform integrability of

$\{ \int_0^T \| q_{n'}(t) \|_{L^2(D_k)}^2 \, dt \}_{n' \geq 1}$, we have

(3.32) $q_{n'} \big|_{D_k} \longrightarrow q \big|_{D_k}$ in $L^2(\hat{\Omega} \times (0,T) \times D_k)$ for $^\forall k = 1,2,\cdots$.

Hence,

$$(3.33) \quad E[\int_0^T \int_{|x| > \rho} |q(t,x)|^2 \, dx \, dt]$$

$$= \lim_{k \to \infty} \lim_{n' \to \infty} E[\int_0^T \int_{\rho < |x| < k} |q_{n'}(t,x)|^2 \, dx \, dt]$$

$$\leq \frac{TN(1 + \| u_0 \|_{0,r}^2)}{(1 + \rho^2)^r} , \text{ (by (3.9)).}$$

(3.32) and (3.33) yield that $q \in L^2(\hat{\Omega} \times (0,T) \times \mathbb{R}^d)$.

Furthermore, combining (3.9) and (3.33) with (3.32), we have

(3.34) $q_{n'} \longrightarrow q$ in $L^2(\hat{\Omega} \times (0,T) \times \mathbb{R}^d)$.

Moreover, by (3.8), we get

(3.35) $\bar{q}_{n'} \longrightarrow q$ in $L^2(\hat{\Omega} \times (0,T) \times \mathbb{R}^d)$,

where $\bar{q}_{n'}(t) = q_{n'}(t_k)$ if $t \in [t_k , t_{k+1})$, $k = 0,1,\cdots,n-1$.

Hence there exists a subsequence $\{ n'' \}$ of $\{ n' \}$ such that

(3.36) $\bar{q}_{n''}(t) \longrightarrow q(t)$ in $L^2(\mathbb{R}^d)$ for almost all (ω,t).

Since F and G are continuous, we obtain

(3.37) $F(\bar{q}_{n''}(t)) \longrightarrow F(q(t))$ in $L^2(\mathbb{R}^d)$

and

(3.38) $G(\bar{q}_{n''}(t)) \longrightarrow G(q(t))$ in $L^2(\mathbb{R}^d)$

for almost all (ω,t).

By (3.7) and the linear growth condition (3.2), $\{ \|F(\bar{q}_{n''}(t))\|_0^2 \}_{n''}$ and $\{ \|G(\bar{q}_{n''}(t))\|_0^2 \}_{n''}$ are uniformly integrable on $\hat{\Omega} \times (0,T)$. So we get

(3.39) $F(\bar{q}_{n''}(t)) \longrightarrow F(q(t))$ in $L^2(\hat{\Omega} \times (0,T) \times \mathbb{R}^d)$

and

(3.40) $G(\bar{q}_{n''}(t)) \longrightarrow G(q(t))$ in $L^2(\hat{\Omega} \times (0,T) \times \mathbb{R}^d)$.

On the other hand, combining (3.6) and (3.34), we can take a subsequence $\{ n''' \}$ of $\{ n'' \}$ such that

(3.41) $q_{n'''} \longrightarrow q$ weakly in $L^2(\hat{\Omega} \times (0,T) ; H^m)$.

Particulary, we can see that $q \in L^2(\hat{\Omega} \times (0,T) ; H^m)$.

Let φ be an absolutely continuous function from $[0,T]$ into $\mathbb{R}^1$, with $\varphi' \in L^2((0,T))$, $\varphi(T) = 0$ and $\eta \in C_0^\infty(\mathbb{R}^d)$.

Since $(B_{n'''}, q_{n'''})$ is a solution of (3.5), the following equality holds.

$$(3.42)\quad \varphi(0)(u_0, \eta)_0 + \int_0^T \varphi(t)\langle Aq_{n'''}(t) + F(\bar{q}_{n'''}(t)), \eta\rangle_0\, dt$$
$$+ \int_0^T \varphi(t)(G(\bar{q}_{n'''}(t)), \eta)_0\, dB_{n'''}(t)$$
$$+ \int_0^T \varphi'(t)(q_{n'''}(t), \eta)_0\, dt = 0.$$

By (3.29), (3.39), (3.40) and (3.41), we can take the limit in $L^2(\hat{\Omega})$ weakly and obtain

$$(3.43)\quad \varphi(0)(u_0 , \eta)_0 + \int_0^T \varphi(t)\langle Aq(t) + F(q(t)) , \eta \rangle_0 \, dt$$

$$+ \int_0^T \varphi(t)(G(q(t)) , \eta)_0 \, dB(t) + \int_0^T \varphi'(t)(q(t) , \eta)_0 \, dt = 0.$$

By the same argument as the proof of Theorem 1.3 in [22], we see that (B , q) is a solution of (2.2). Since the solution $q \in L^2((0,T)\times\hat{\Omega} ; H^m)$, Remark 1.1 in [14] asserts that $q \in L^2(\hat{\Omega} ; C(0,T;H^{m-1}))$. Moreover, by Theorem 2.2 in [15], we obtain

$$(3.44)\quad E[\sup_{0\le t\le T} \|q(t)\|_m^2]$$

$$\le N_5(\|u_0\|_m^2 + E[\int_0^T (\|F(q(t))\|_m^2 + \|G(q(t))\|_m^2) \, dt]$$

$$\le N_6(1 + \|u_0\|_m^2).$$

This completes the proof.

§ 4 On the convergence of solutions

In this section we will show that the solution of (2.2) has a sort of stability property on the perturbation of coefficients. For $n = 1,2,\cdots$, we suppose that

$$a_{ij}^n , b_i^n , c^n : \mathbb{R}^d \longrightarrow \mathbb{R}^1 \quad (i,j = 1,\cdots,d),$$

$$F_n, G_n : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d),$$

$$u_0^n \in H^m$$

satisfy the conditions (A.1) ~ (A.6) with the same constants as K, L, and r in (A.1), (A.5) (resp.). Define an operator A_n by (2.1) with a_{ij} , b_i , c replaced by a_{ij}^n , b_i^n , c^n respectively.
Now we consider the following stochastic PDE:

$$(4.1)_n \begin{cases} du(t) = (A_n u(t) + F_n(u(t)))dt + G_n(u(t))dW(t) \\ u(0) = u_0^n \qquad 0 < t \le T \end{cases}.$$

Let (W_n , u_n) be the solution of $(4.1)_n$.

Theorem 4.1 Suppose that

(4.2) $a_{ij}^n(x) \longrightarrow a_{ij}(x)$, $b_i^n(x) \longrightarrow b_i(x)$ and $c^n(x) \longrightarrow c(x)$ as $n \longrightarrow \infty$ for $\forall x \in \mathbb{R}^d$, $i,j = 1,\cdots,d$,

(4.3) $F_n(u) \longrightarrow F(u)$ and $G_n(u) \longrightarrow G(u)$ in $L^2(\mathbb{R}^d)$ as $n \longrightarrow \infty$ for $\forall u \in L^2(\mathbb{R}^d)$,

(4.4) $u_0^n \longrightarrow u_0$ weakly in H^m as $n \longrightarrow \infty$, and $\|u_0^n\|_{0,r}$, $n = 1,2,\cdots$, are bounded,

(4.5) $\{ F_n \}$ and $\{ G_n \}$ are equi-uniformly continuous.

Then there exist a subsequence $\{n'\}$ and S ($= C(0,T;\mathbb{R}) \times \mathcal{W}(0,T)$) -valued random variables (W , u) on some probability space such that

(4.6) $(W_n , u_n) \longrightarrow (W , u)$ in law as S - valued random variables.

Moreover the limit (W , u) is a solution of (2.2).

Proof Since the constant N appeared in Lemma 3.1, 3.2 and 3.3 depends only on K, L in (A.1), (A.5) and T, by the similar calculation in § 3, we can obtain

$$(4.7) \quad \sup_{0\le t\le T} E[\|u_n(t)\|_m^2] \le N' (1 + \|u_0^n\|_m^2), \quad n = 1,2,\cdots,$$

$$(4.8) \quad \sup_{0\le t\le T} E[\|u_n(t)\|_0^4] \le N' (1 + \|u_0^n\|_0^4), \quad n = 1,2,\cdots,$$

$$(4.9)\quad E\Big[\int_{|x|>\rho} |u_n(t,x)|^2\, dx \Big] \le \frac{N'(1 + \|u_0{}^n\|_{0,r}^2)}{(1+\rho^2)^r},$$

for any $t \in [0, T]$, $n = 1,2,\cdots$, $\rho > 0$.

and

$$(4.10)\quad E[\, \|u_n\|^2_{\mathcal{H}_\gamma(T,D)} \,] \le N'(1 + \|u_0{}^n\|_m^2),$$

for any bounded subset D of $\mathbb{R}^d$ and $n = 1,2,\cdots$.

Hence, by the same argument as the proof of Theorem 3.1 (see (3.29), (3.34)), we can obtain a subsequence $\{n'\}$ and $\mathcal{S}$ - valued random variables $(B_{n'}, q_{n'})$, (B, q) on a suitable probability space such that

(4.11) the law of $(B_{n'}, q_{n'})$ = the law of $(W_{n'}, u_{n'})$,

(4.12) $B_{n'} \longrightarrow B$ uniformly on $[0, T]$, w. p. 1,

$$(4.13)\quad E\Big[\int_0^T \| q_{n'}(t) - q(t) \|_0^2\, dt \Big] \longrightarrow 0$$

as $n' \longrightarrow \infty$.

So, there exists a subsequence $\{n''\}$ of $\{n'\}$ such that

$$(4.14)\quad E[\, \| q_{n''}(t) - q(t) \|_0^2 \,] \longrightarrow 0 \quad \text{as } n'' \longrightarrow \infty$$

for almost all $t \in [0, T]$.

Fix $t \in [0, T]$ which satisfies (4.14). For each $\varepsilon > 0$, by (4.5), there is a $\delta > 0$ such that

$$(4.15)\quad \| F_n(u) - F_n(v) \|_0 < \varepsilon \quad \text{for } \| u - v \|_0 < \delta \text{ and } n = 1,2,\cdots.$$

Hence

$$(4.16)\quad E[\, \| F_{n''}(q_{n''}(t)) - F_{n''}(q(t)) \|_0^2 \,]$$

$$\le \varepsilon^2 + E[\, \| F_{n''}(q_{n''}(t)) - F_{n''}(q(t)) \|_0^2 : \Lambda_{n''}(\delta) \,], \text{ for } \forall n'',$$

where $\Lambda_{n''}(\delta) = \{ \| q_{n''}(t) - q(t) \|_0 \ge \delta \}$.

From (4.14), we get

$$(4.17)\quad P(\Lambda_{n''}(\delta)) \le \frac{1}{\delta^2} E[\, \| q_{n''}(t) - q(t) \|_0^2 \,] \longrightarrow 0 .$$

Moreover, by the linear growth condition for $F_{n''}$ and (4.8), $\{ \| F_{n''}(q_{n''}(t)) - F_{n''}(q(t)) \|_0^2 \}_{n''}$ is uniformly integrable.

Hence

$$(4.18)\quad E[\| F_{n''}(q_{n''}(t)) - F_{n''}(q(t)) \|_0^2 : \Lambda_{n''}(\delta)] \longrightarrow 0.$$

On the other hand, by (4.3) and the uniform integrability,

$$(4.19)\quad E[\| F_{n''}(q(t)) - F(q(t)) \|_0^2] \longrightarrow 0.$$

Combining (4.19) with (4.16) and (4.18), we get

$$(4.20)\quad E[\| F_{n''}(q_{n''}(t)) - F(q(t)) \|_0^2] \longrightarrow 0 ,$$

for almost all $t \in [0 , T]$.

Furthermore, (4.8) and the linear growth condition yield that $\{ E[\| F_{n''}(q_{n''}(t)) - F(q(t)) \|_0^2] \}_{n''}$ is uniformly integrable on $[0 ,T]$. Hence, we get

$$(4.21)\quad E[\int_0^T \| F_{n''}(q_{n''}(t)) - F(q(t)) \|_0^2 \, dt] \longrightarrow 0 .$$

By the same argument,

$$(4.22)\quad E[\int_0^T \| G_{n''}(q_{n''}(t)) - G(q(t)) \|_0^2 \, dt] \longrightarrow 0.$$

On the other hand, combining (4.7) with (4.13), we can take a subsequence $\{n'''\}$ of $\{n''\}$ such that

$$(4.23)\quad q_{n'''} \longrightarrow q \quad \text{weakly in} \quad L^2(\Omega \times (0,T) ; H^m).$$

Repeating the same argument as (3.42) ~ (3.43), (4.12), (4.21), (4.22) and (4.23) yield that (B , q) is a solution of (2.2). Thus we obtain the subsequence $\{ (W_{n'''} , u_{n'''}) \}_{n'''}$ which converges to a solution (B , q) of (2.2) in law as $\mathcal{S}$ - valued random variable. This completes the proof.

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