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KNOTTED 2-SPHERES AND TORI IN THE 4-SPHERE

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Knotted 2-spheres and tori in the 4-sphere

(4次元球面内の2次元球面および輪環面)

by

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Table of Contents

Preface

This thesis is written under the subject "Knotted 2-spheres and tori in the 4-sphere" to be submitted for the degree of Doctor of Philosophy at Kobe University. An *n-knot* is a pair (S^{n+2}, S^n) determined by either a smooth or piecewise-linear locally flat embedding of an n-sphere S^n in the $(n+2)$-sphere S^{n+2}. 1-knot theory, what is called classical knot theory, has been studied extensively. 2-knot theory has been also studied by many people and many results are known, though some fundamental properties still remain unsolved. 2-knot theory may be extended to the knot theory of surfaces in the 4-sphere. Although it is expected that the knot theory of surfaces in the 4-sphere is more complicated than that of 2-spheres, few examples and constructions are known.

In this thesis, we shall study knotted 2-spheres and tori in the 4-sphere which are obtained from classical knots and links having cyclic periods by *symmetry-spinning*. The process of *spinning* was first defined by Artin [Ar] in 1925. It is the most geometrically appealing way to construct a 2-knot from a classical knot. There are several descriptions of the Artin spin. We may describe it as follows. Let (S^3, K) be a 1-knot. Choose a small 3-ball B_- which meets K in an arc K_- such that (B_-, K_-) is homeomorphic to the standard ball pair. Removing $Int(B_-, K_-)$ from S^3 gives a knotted ball pair (B_+, K_+). Then the *Artin spin* of K is the 2-knot

$$(S^4, \sigma K) = \partial(B_+, K_+) \times B^2 \cup_\partial (B_+, K_+) \times \partial B^2.$$

Fox (cf. [Fo]) considered a modification which combines the spinning process with a simultaneous rotation of K_+ about its axis. This operation has come to be known as *twist-spinning*. Let m be an integer. Take an unknotted arc A in B_+ with $\partial A = \partial K_+$.

The axis A is oriented so that the orientations induced by A and K_+ on their common boundary coincide. Let $f_\theta : B_+ \to B_+$ be the rotation through θ radians about A in the positive meridian direction. Then

$$(S^4, \tau^m K) = \partial(B_+, K_+) \times B^2 \cup_\partial \left(B_+ \times \partial B^2, \bigcup_{0 \le \theta \le 2\pi} (f_{m\theta}(K_+) \times \{\theta\})\right)$$

is a 2-knot, called the *m-twist-spin* of K. Note that the 0-twist-spin is the Artin spin. Twist spins have been studied by Zeeman [Ze], and he proved the remarkable result that every m-twist-spin ($m \ne 0$) is fibered. The knot k is fibered if the exterior $X(k)$ is a fiber bundle over the circle, that is, $X(k)$ is the mapping torus of a homeomorphism of the fiber. More precisely, the main result of [Ze] asserted that the m-twist-spin of a 1-knot K is a fibered 2-knot in S^4 with closed fiber $\Sigma_m(K)$, the m-fold cyclic branched covering space of S^3 over K, and with closed monodromy the canonical generator of the group Z_m of cyclic branched covering transformations. In particular, if $m = 1$ we have $\Sigma_1(K) = S^3$ and the 1-twist-spin is always trivial, since it bounds a 3-ball. (The process of twist-spinning generalizes to higher dimensions and Zeeman proved a corresponding fibration theorem, too.)

In [Fo] Fox introduced another variation of the spinning process, called *roll-spinning*, but he only gave a picture of rolling the figure-eight knot and showed that the roll spin of the figure-eight cannot be obtained from the figure-eight by twist-spinning. (But his knot was later shown to be obtained from the trefoil by twist-spinning [N-T].) To be exact, Fox's roll-spun figure-eight is the symmetry-spun figure-eight using period 2 in terms of Litherland, as stated below.

Subsequently, Litherland [Li 1] gave a reformulation of twist-spinning and a precise definition of roll-spinning, and further, he introduced a general process, which is called *deform-spinning*. The notion of deforming the knotted arc K_+ during the spinning process may be expressed as follows. Let $f_\theta : B_+ \to B_+$ ($\theta \in I$) be an isotopy rel ∂B_+ such that $f_1(K_+) = K_+$. Such an isotopy is also called a *deformation* of (B_+, K_+). Then the 2-knot

$$\partial(B_+, K_+) \times B^2 \cup_\partial \left(B_+ \times \partial B^2, \bigcup_{\theta \in I} (f_\theta(K_+) \times \{\theta\})\right)$$

is the *deform spin* of K corresponding to $f = \{f_\theta\}$, where a circle is identified with R/Z. He considered "untwisted" deformations and showed that, provided that one combines the untwisted deformation with a twist, the resulting knot is again fibered. *Symmetry-spinning* is an example of deform-spinning given by Litherland. It can be defined for 1-knots having cyclic periods, and it may be considered as a generalization of roll-spinning. Roughly speaking, the deformation of the knotted ball pair corresponding to symmetry-spinnig is derived from the periodic homeomorphism which acts on (S^3, K).

For other formulations and extensions, see [G-K], [Pl], [Mo].

Imitating the construction of deform-spun 2-knots we can form the deform-spun torus, however few results are known.

Let (S^3, K) be a 1-knot. Removing a small 3-ball disjoint from K gives a pair (B^3, K). Then the knotted torus

$$(S^4, F(K)) = (B^3, K) \times \partial B^2 \cup_\partial S^2 \times B^2$$

is called the *spun torus* of K. Livingston [Lv 1] proved that if K is nontrivial, the spun torus of K is irreducible, that is, it is not equivalent to the connected sum of any 2-knot and the standardly embedded torus in S^4. In fact, Livingston used the spun tori to construct knotted surfaces of arbitrary genus in S^4 which cannot be written as the connected sum of a knotted surface of lower genus and an unknotted surface. Boyle [Bo] studied a method of obtaining knotted surfaces in S^4 by attaching 1-handles to a given knotted surface. This method was studied by Hosokawa and Kawauchi [H-K] before. Boyle defined the spun torus and the twist-spun torus in terms of 1-handles, and showed that the nontrivial spun torus is irreducible while the twist-spun tori are not necessarily irreducible; e.g. the 2-twist-spun torus of a 2-bridge knot K is a connected sum of the 2-twist-spun 2-knot of K and the unknotted torus.

For other works on knotted surfaces, refer to [As 1], [Li 2], [Lv 2], [P-R], [Km].

Recall that a knot is fibered if its exterior is a fiber bundle over the circle. It is known that a fibered knot has nice properties from a geometric and algebraic viewpoint, and so fibered knots give a very important special class of knots. In particular, the fiber of a fibered 2-knot is a 3-manifold. It makes the study of fibered 2-knots of great interest from the viewpoint of 3-manifold theory as well as 2-knot theory.

In 1986, when I was in the first year of the graduate school, I studied fibered 2-knots in S^4 (or more generally, a homology 4-sphere) with fiber a punctured lens space or a punctured connected sum of lens spaces. A lens space is a 3-manifold having a Heegaard decomposition of genus one. There are two reasons for selecting these fibered 2-knots. One is that any 2-knot with Seifert surface a punctured connected sum of lens spaces is determined by its exterior [Gl]. The other is that the diffeotopy groups of all lens spaces were computed by Bonahon [Bn] and Hodgson-Rubinstein [H-R]. I proved that any fibered 2-knot with fiber a punctured lens space $L(p,q)^\circ$ is the 2-twist-spin of the 2-bridge knot

$S(p,q)$, and in general, any fibered 2-knot with fiber a punctured connected sum of r-copies of $L(p,q)$ and cyclic monodromy is the r-cable knot about the 2-twist-spin of $S(p,q)$. These results were arranged for my first paper [Te 3].

In January, 1987, following advice of my supervisor Professor Fujitsugu Hosokawa, I began to study Fox's paper [Fo] and Litherland's paper [Li 1], which dealt with deform-spinning. Litherland showed that if a 1-knot K is a torus knot then the deformation group is generated by the twist-spinning deformation. In other words, we can obtain only twist spins from a torus knot by deform-spinning. Using the moving picture method, I showed that any symmetry spin of a torus knot is a certain twist spin [Te 1], [Te 2].

The moving picture method is a classical and fundamental method to study 2-knots, however its effect is not made clear sufficiently. In [Ka 3], Kanenobu remarked that the 3-twist-spun trefoil, Fox's roll-spun figure-eight, and the knot "K_1", which is constructed by Kanenobu in [Ka 4] using the moving picture as an example of 2-knot whose knot group has an element of order 4, have isomorphic knot groups, and asked whether they are equivalent. I gave an affirmative answer by using the moving picture method in cooperation with Nakanishi [Te 1], [Te 2], [N-T].

A *prism manifold* is a Seifert fibered manifold with orbit-manifold S^2 and with three exceptional fibers of index corresponding to the triple $(2,2,\alpha)$ with $\alpha > 1$ (cf. [Ja], [Or]). Broadly speaking, a prism manifold is obtained from a twisted I-bundle over the Klein bottle by gluing a solid torus along its boundary. Asano [As 2] and Rubinstein [Ru] independently determined the diffeotopy groups of prism manifolds. Not all punctured prism manifolds can be embedded in S^4 as fibers of fibered 2-knots. A punctured prism manifold can be embedded in a homology 4-sphere as a fiber of a fibered 2-knot if and only if $\alpha = 2$ [Te 2], [Yo]. There is an important infinite series of prism manifolds M_d, whose fundamental groups are $Q(8) \times Z_d$, where $Q(8)$ is the quaternion group of order 8 and d is an odd integer. We may express M_d as the Seifert fibered manifold with invariants $\{\pm(d-3)/2; (o_1, 0); (2,1), (2,1), (2,1)\}$ (cf. [Or]). Morichi [Mr] realized a smoothly embedding of every M_d° in S^4. He has used an idea due to Hosokawa and Suzuki [H-S]. Thus there exists a 2-knot in S^4 which admits M_d° as a Seifert surface. Does there exist a fibered 2-knot in S^4 (with the standard smooth structure) whose fiber is M_d°? It may be impossible to decide whether Morichi's 2-knot is fibered, because his construction relies heavily on a moving pictorial description.

The existence of such a fibered 2-knot is full of meaning. Hillman [Hi 1] determined all the 2-knot groups with finite commutator subgroup. Let π be a 2-knot group with commutator subgroup π' finite. Then $\pi' \cong G \times Z_d$ where $G \cong \{1\}, Q(8)$,the generalized binary tetrahedral group $T(k)$ or the binary icosahedral group I^*, and $(d, 2|G|) = 1$. When $G = \{1\}$ all the groups are realized by the 2-twist-spins of certain 2-bridge knots. The commutator subgroup of the 3-twist-spin of the trefoil is $Q(8)$. (In fact, its fiber is M_1°.) Yoshikawa [Yo] has shown that the direct products of $T(k)$ and I^* with cyclic groups are realized by the 2-twist-spins of certain pretzel knots, however for the remaining groups $Q(8) \times Z_d$ $(d > 1)$ he only got fibered 2-knots in homotopy 4-spheres. These groups cannot be realized by twist spins (cf. [Hi 2]).

On September 9, 1987, Professor Taizo Kanenobu told me that he obtained a fibered 2-knot in S^4 with fiber M_5° from the figure-eight by symmetry-spinning at the R.I.M.S.

in Kyoto University. Kanenobu has used the surgery description technique due to Rolfsen [Ro] to identify the fiber, and showed that there exists a fibered 2-knot in S^4 with fiber M_5° or M_{11}° [Ka 1]. His idea motivated me to try it independently, and I obtained the same result (cf. [Te 2]). In my Master Thesis [Te 2], I had expected that we can obtain a fibered 2-knot with fiber $M_{13}^\circ, M_{19}^\circ$ by symmetry-spinning 7_3 knot, 8_4 knot, respectively. It seemed to be difficult for me to identify the fiber, because of its very complicated surgery description.

On February 21, 1988, I was successful in deciding the fiber. The result was for my expectation. Reconsidering Litherland's proof [Li 1], I became aware that one can make use of a tower of branched coverings to decide a fiber. These results were arranged for my second paper [Te 4].

Since then, my attempt to get a fibered 2-knot with fiber M_d° for other value of d failed many times. Litherland defined the process of symmetry-spinning for 1-knots having cyclic periods, but he studied only the case of a single cyclic period. I have studied the case of two cyclic periods. On May 14, 1989, I got a fibered 2-knot with fiber M_3°, M_{21}° or M_{27}° by symmetry-spinning certain pretzel knots [Te 5].

I might expect that there exists a fibered 2-knot with fiber M_d° for any other value of d, and so that the remaining groups $Q(8) \times Z_d$ are also realizable by smooth (fibered) 2-knots in the standard S^4, but I have been unable to prove this at present.

When a deformation is untwisted in terms of [Li 1], the resulting deform spin is not always fibered. Roughly speaking, an untwisted deformation of (S^3, K) preserves a Seifert surface for K. Therefore an untwisted deform spin of K has a nice Seifert (hyper)surface derived from a Seifert surface for K. Of course, if a 1-knot K is fibered, then any untwisted deform spin of K is always fibered.

In the autumn of 1988, I studied untwisted deform spins of 2-bridge knots of genus one by symmetry-spinning of period 2, and computed Alexander modules by constructing the infinite cyclic covering spaces. Before long I got Kanenobu's preprint [Ka 2], which dealt with untwisted deform spins, in particular symmetry spins of 2-bridge knots of genus one and pretzel knots. In [Ka 2] Kanenobu had shown that in general symmetry spins of 2-bridge knots of genus one are not fibered, in fact the commutator subgroups of the knot groups are nontrivial free products with amalgamation by constructing the infinite cyclic covering spaces.

On January 17, 1989, I began to study symmetry spins in the case of two cyclic periods. Soon I gained the existence of "untwisted" unknotting deformations for a certain class of knots. The 1-twist-spin of any knot is trivial, but twist-spinning is not untwisted. When the deform spin of a 1-knot K corresponding to a deformation γ is trivial, we shall call γ an *unknotting deformation* of K. It was unknown whether there exists an untwisted unknotting deformation. In general a symmetry spin obtained by using two cyclic periods may also be non-fibered. If so, we can obtain a non-fibered embedding of a punctured Brieskorn manifold $\Sigma(2,s,k)^{\circ}$ in S^4 for odd integers s and k with $(s,k)=1$; e.g. the punctured Poincaré homology 3-sphere $\Sigma(2,3,5)^{\circ}$.

Since the summer of 1989, I have studied the construction of orientable or nonorientable surfaces in S^4 by deform-spinning. It is easy to construct tori whose knot groups are infinite cyclic by symmetry-spinning periodic knots. Hosokawa and Kawauchi [H-K] conjecture that any knotted surface in S^4 with infinite cyclic knot group is necessarily unknotted. A *knotted surface* is a smoothly embedded closed connected oriented surface in S^4. A knotted surface is *unknotted* if it bounds a handlebody. I was successful in showing that all the symmetry-spun tori whose knot groups are infinite cyclic are unknotted [Te 6]. In general a symmetry-spun torus of a periodic knot K is equivalent to a spun torus of a factor knot of K. The proof is based on Dehn surgery on twins in S^4 (cf. [Mo]). Other deformations may be used to construct tori with infinite cyclic knot group, however it remains to be seen that the tori are indeed unknotted.

I also note that deform-spinning may be used to provide remarkable examples of knotted Klein bottles in S^4.

In Chapter I we give the definitions of deform spins of knots following Litherland [Li 1] and we show how to identify the closed fibers of symmetry spins. In particular, we investigate symmetry spins of some classes of knots and give remarkable examples of fibered 2-knots having Seifert fibered manifolds as closed fibers. We also exhibit the first examples of "untwisted" unknotting deformations.

In Chapter II we define symmetry-spun tori and show that any symmetry-spun torus of K is equivalent to a spun torus of a factor knot of K.

Finally, we describe deform-spun knots using the moving picture method in Appendix.

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Winter, 1990 Masakazu Teragaito

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Chapter I. Symmetry-spun 2-knots

Let π be the commutator subgroup of the knot group of a knot in the 4-sphere S^4. In [Hi 1] it is shown that if π is finite, then $\pi = G \times Z_d$ where $G = \{1\}$, the quaternion group $Q(8)$, the binary icosahedral group I^* or the generalized binary tetrahedral group $T(k)$ and d is an odd integer which is relatively prime to the order of G. Yoshikawa [Yo] has shown that these groups can be realized as the commutator subgroups of the knot groups of knots in S^4 except $Q(8) \times Z_d, d > 1$. Actually these knots were constructed by twist-spinning certain 2-bridge knots and pretzel knots. The exceptional groups were realized only as the commutator subgroups of knot groups of knots in homotopy 4-spheres. Note that $Q(8) \times Z_d$ is isomorphic to the fundamental group of a prism manifold M_d, that is, the Seifert fibered manifold with invariants $\{b; (o_1, 0); (2,1), (2,1), (2,1)\}, d = |2b+3|$ (cf. [Ja], [Or]). Since then, by using deform-spinning introduced by Litherland [Li 1], Kanenobu [Ka 1] showed that for $d = 5, 11$ (equivalently $b = -4, 4$) there is a fibered 2-knot in S^4 whose fiber is the punctured prism manifold M_d°; thus for these values of d, the groups $Q(8) \times Z_d$ are realized as the commutator subgroups of knot groups of knots in S^4. Kanenobu has used the surgery description method (cf. [Ro]) for genus one 2-bridge knots to identify fibers.

In this chapter we shall show that other five values can be realized.

Theorem 3.5. *There exists a fibered 2-knot in S^4 whose fiber is a punctured prism manifold M_d° with fundamental group isomorphic to $Q(8) \times Z_d$ for $d = 3, 5, 11, 13, 19, 21, 27$ (equivalently $b = 0, -4, 4, -8, 8, -12, 12$).*

Our examples will be constructed by symmetry-spinning certain 2-bridge knots and pretzel knots. It should be noted that a fibered 2-knot with fiber M_d° $(d > 1)$ cannot be constructed by twist-spinning (cf. [Hi 2]). It is unknown whether there exists such a fibered 2-knot in S^4 for any other value of d.

We also consider 2-knots arising from 1-knots with two cyclic periods by untwisted deform-spinning.

In Appendix we shall consider deform-spun 2-knots from a view of moving picture.

We shall work in the piecewise-linear category. All manifolds will be oriented and all submanifolds are assumed to be locally-flat. A circle is identified with the quotient space R/Z. The unit interval $[0,1]$ is denoted by I.

1. Deform-spun knots

Let (S^3, K) be a knot and $K \times D^2$ be a tubular neighbourhood of K. Let $X(K) = cl(S^3 - K \times D^2)$ be the exterior of K. We always assume that $K \times v$ $(v \in \partial D^2)$ is null-homologous in $X(K)$. Let $\mathcal{H}(K)$ be the group of self-homeomorphism g of (S^3, K) with $g|_{K\times D^2} = id$, and $\mathcal{D}(K)$ be $\mathcal{H}(K)$ modulo isotopy rel $K \times D^2$. $\mathcal{D}(K)$ is called the *deformation group* of K. It makes no matter the choice of tubular neighbourhoods. We call elements of $\mathcal{D}(K)$ *deformations* of K. It is well-known that there is a map $p : X(K) \to \partial D^2$ such that $p|_{\partial X(K)} : \partial X(K) = K \times \partial D^2 \to \partial D^2$ is the projection (cf. [K-W], [Ro]). By a *projection* for K, we shall mean a pair $(p, K \times D^2)$. A deformation $\gamma \in \mathcal{D}(K)$ is *untwisted* if there is a projection $(p, K \times D^2)$ and a *compatible representative* g of γ, that is, $g|_{K\times D^2} = id$ and $p(g|_{X(K)}) = p$.

Fix a point x on K. Take a ball neighbourhood K_- of x in K, and set $B_- = K_- \times D^2$. Then (B_-, K_-) is a standard ball pair. Let (B_+, K_+) be the complementary ball pair. For

$g \in \mathcal{H}(K)$, construct $\partial(B_+, K_+) \times B^2 \cup_\partial (B_+, K_+) \times_g \partial B^2$, where

$$(B_+, K_+) \times_g \partial B^2 = (B_+ \times I, K_+ \times I) \Big/ \Big((x,0) \sim \big(g(x), 1\big) \text{ for all } x \in B_+ \Big).$$

This is a locally-flat sphere pair depending only on the class γ of g [Li 1:Lemma 1.2]. We denote by γK this 2-knot in S^4, and call the *deform-spun knot* of K corresponding to γ or, simply, γ-*spin* of K.

Example 1.1. *Twist-spinning.* Let (S^3, K) be a knot with exterior $X(K) = cl(S^3 - K \times D^2)$. Take a collar $\partial X(K) \times I$ of $\partial X(K)$ in $X(K)$ such that $\partial X(K)$ is identified with $\partial X(K) \times \{0\}$. Let $t : (S^3, K) \to (S^3, K)$ be the homeomorphism defined by

$$t(x, \theta, \phi) = (x, \theta + \phi, \phi) \qquad \text{for} \quad (x, \theta, \phi) \in K \times \partial D^2 \times I,$$

$$t(y) = y \qquad \text{for} \quad y \notin \partial X(K) \times I.$$

Let τ be the class of t in $\mathcal{D}(K)$. It is clear that τ is not untwisted. For an integer m, the τ^m-spin of K is called the *m-twist-spin* of K. The 0-twist-spin is just the Artin spin. This definition corresponds to Zeeman's original construction.

Example 1.2. *Symmetry-spinning.* Let (S^3, K) be a knot. Suppose that K has cyclic period $n \neq 0$, that is, there is an orientation-preserving periodic homeomorphism g on S^3 of period n which preserves K and its orientation, and $Fix(g) \cong S^1$. In this case, $J = Fix(g)$ is an unknot disjoint from K. Let $q : S^3 \to S^3/g \quad (\cong S^3)$ be the quotient map and write $\overline{K} = q(K)$ and $\overline{J} = q(K)$. There is a projection $(\overline{p}, \overline{K} \times D^2)$ for $\overline{K}$ such that $\overline{K} \times D^2$ is a tubular neighbourhood of $\overline{K}$ disjoint from $\overline{J}$. Then $q^{-1}(\overline{K} \times D^2)$ is a g-invariant tubular neighbourhood $K \times D^2$ of K such that $q(x, v) = (nx, v)$ for $x \in K, v \in D^2$. We recall that $K \times v \quad (v \in \partial D^2)$ is null-homologous in $X(K) = cl(S^3 - K \times D^2)$. It is clear that $(\overline{p}q, K \times D^2)$ is a projection for K. Since $j = lk(K, J)$ is coprime to n, we can choose an integer k such that $jk \equiv 1 \pmod{n}$. It follows that $g|_{K\times D^2}$ is given by $(x, v) \to (x+k/n, v)$. As in Example 1.1, we take a collar $\partial X(K)\times I$ of $\partial X(K) = K \times \partial D^2$ which is disjoint from J, and define a homeomorphism $s_{n,k} : (S^3, K) \to (S^3, K)$ as follows;

$$s_{n,k}(x, \theta, \phi) = (x - k(1-\phi)/n, \theta, \phi) \quad \text{for} \quad (x, \theta, \phi) \in K \times \partial D^2 \times I,$$

$$s_{n,k}(x, v) = (x - k/n, v) \quad \text{for} \quad (x, v) \in K \times D^2,$$

$$s_{n,k}(y) = y \quad \text{for} \quad y \in X(K) - \partial X(K) \times I.$$

Then $s_{n,k}g|_{K\times D^2} = id$, $s_{n,k}g|_{cl(X(K)-\partial X(K)\times I)} = g$ and $\overline{p}q(s_{n,k}g|_{X(K)}) = \overline{p}q$. Let $\zeta_{n,k}$ be the class of $s_{n,k}g$ in $\mathcal{D}(K)$. It is now evident that $\zeta_{n,k}$ is untwisted. This may be regarded as a generalization of roll-spinning introduced by Fox [Fo]. It can also be described in terms of the moving picture method (see Appendix).

Example 1.3. *Symmetry-spinning again.* Let (S^3, K) be a knot and suppose that there are orientation-preserving periodic homeomorphisms g_i $(i = 1, 2)$ on (S^3, K) of order n_i such that $g_1g_2 = g_2g_1$, $(n_1, n_2) = 1$, and $J_1 \cup J_2$ is the Hopf link with $lk(J_1, J_2) = 1$, where $J_i = Fix(g_i)$. Let $n = n_1n_2$ and $g = g_1g_2$. Let $q : S^3 \to S^3/g$ be the quotient map and write $\overline{K} = q(K)$ and $\overline{J}_i = q(J_i)$. The map q is the $Z_{n_1} \oplus Z_{n_2}$-branched cover branched over $\overline{J}_1 \cup \overline{J}_2$, corresponding to $Ker[\pi_1(S^3 - \overline{J}_1 \cup \overline{J}_2) \to H_1(S^3 - \overline{J}_1 \cup \overline{J}_2) \to Z_{n_1} \oplus Z_{n_2}]$, where the first map is the Hurewicz homomorphism and the second sends a meridian t_1 (t_2 resp.) of $\overline{J}_1$ ($\overline{J}_2$ resp.) to $(1, 0)$ ($(0, 1)$ resp.) $\in Z_{n_1} \oplus Z_{n_2}$. There is a projection $(\overline{p}, \overline{K}\times D^2)$ for $\overline{K}$ such that $\overline{K}\times D^2$ is disjoint from $\overline{J}_i$. Then $q^{-1}(\overline{K}\times D^2)$ is a g-invariant tubular neighbourhood $K\times D^2$ of K such that $q(x, v) = (nx, v)$ for $x \in K, v \in D^2$. Since $j_i = lk(K, J_i)$ is coprime to n_i, we can choose an integer k_i such that $j_ik_i \equiv 1 \pmod{n_i}$. As in Example 1.2, $g_i|_{K\times D^2}$ is given by $(x, v) \to (x + k_i/n_i, v)$. Hence we have

$g|_{K\times D^2} : (x, v) \to (x + k/n, v), k = k_2n_1 + k_1n_2$. Let $\omega_{n,k}$ be the class of $s_{n,k}g$ in $\mathcal{D}(K)$. It is obvious that $\omega_{n,k}$ is untwisted.

2. The fiber

Zeeman [Ze] has shown that if $m \neq 0$ then $\tau^m K$ is fibered, and the closed fiber is the m-fold cyclic branched covering space of S^3 branched over K. In particular, $\tau^1 K$ is always unknotted. For deform-spun knots, Litherland proved the corresponding fibering theorem.

Theorem 2.1. [Li 1:Theorem 2.4] *Let (S^3, K) be a knot and $\gamma \in \mathcal{D}(K)$ an untwisted deformation. For any non-zero integer m, $\tau^m\gamma K$ is fibered.*

In fact, Litherland gave the closed fiber and the characteristic homeomorphism of $\tau^m\gamma K$. But it is sufficient to consider the case where $\gamma = \zeta_{n,k}$ or $\omega_{n,k}$ for our purpose.

Let a, b be coprime integers with $b \neq 0$. Let $\Phi : K \times \partial D^2 \to K \times \partial D^2$ be a homeomorphism $(x, \theta) \to (x + b\theta, a\theta)$. By $S^3(K, a/b)$ we shall mean the manifold obtained from S^3 by removing $K \times D^2$ and sewing it back using Φ. Let K^* be the image of $K \times \{0\}$ under this surgery. Moreover for any integers c, d with $d \neq 0$, choose coprime integers a, b such that $a/b = c/d$, and let $S^3(K, c/d) = S^3(K, a/b)$.

Proposition 2.2. [Li 1:Proposition 5.4] *Let (S^3, K) be a knot having cyclic period n. Let $g, \overline{K}, \overline{J}, k$ and $\zeta_{n,k}$ be as in Example 1.2. For an integer $m > 0$, let M be the mn-fold cyclic branched covering space of $S^3(\overline{K}, m/k)$ branched over $\overline{K}^* \cup \overline{J}$, corresponding to $Ker[\pi_1(S^3 - \overline{K} \cup \overline{J}) \to Z < t_0 > \times Z < t_1 > \to Z_{mn} < t >]$. Here t_0 (t_1 resp.) corresponds to a meridian of $\overline{K}$ ($\overline{J}$ resp.) and the last homomorphism sends t_0 to t, and t_1 to t^{-m}. Then the fiber of $\tau^m\zeta_{n,k}K$ is M°.*

Note that the projection $M \to S^3(\overline{K}, m/k)$ is n to 1 over $\overline{K}^*$, and m to 1 over $\overline{J}$.

Proposition 2.3. *Let (S^3, K) be a knot having the property as described in Example 1.3. Let $g, \overline{K}, \overline{J}_i, k_i \quad (i = 1, 2), k = k_2 n_1 + k_1 n_2$ and n be as before. For $m > 0$, let M be the mn-fold cyclic branched covering space of $S^3(\overline{K}, m/k)$ branched over $\overline{K}^* \cup \overline{J}_1 \cup \overline{J}_2$, corresponding to $Ker[\pi_1(S^3 - \overline{K} \cup \overline{J}_1 \cup \overline{J}_2) \to Z < t_0 > \times Z < t_1 > \times Z < t_2 > \to Z_{mn} < t >]$. Here $t_0 \quad (t_1, t_2$ resp.) corresponds to a meridian of $\overline{K} \quad (\overline{J}_1, \overline{J}_2$ resp.) and the last homomorphism sends t_0 to t, and $t_1 t_2$ to t^{-m}. Then the fiber of $\tau^m \omega_{n,k} K$ is M°.*

Note that the projection $M \to S^3(\overline{K}, m/k)$ is n to 1 over $\overline{K}^*$, mn_2 to 1 over $\overline{J}_1$, mn_1 to 1 over $\overline{J}_2$. This is a generalization of Proposition 2.2, and can be proved similarly. We shall present here the sketch of proofs and how to identify the manifold M.

Sketch of proofs. In [Li 1], it is shown that the closed fiber is $M = K \times D^2 \cup_\beta \{(y, \phi) \in X(K) \times_{s_{n,k} g} S^1 | p(y) = m\phi\}$, where $\beta : K \times \partial D^2 \to \{(y, \phi) \in \partial X(K) \times_{s_{n,k} g} S^1 | p(y) = m\phi\}$ is given by $(x, \phi) \to ((x, m\phi), \phi)$, and $p = \overline{p} q$. Then g acts on M naturally, since $pg = p$. Let $M_1 = M/g$. It is easy to see that M_1 is obtained from $\Sigma_m(\overline{K})$, the m-fold cyclic branched covering space of S^3 over $\overline{K}$, by performing $1/k$-sugery (with respect to the induced framing) along the lift of $\overline{K}$. Thus M_1 is the m-fold cyclic branched covering space of $S^3(\overline{K}, m/k)$ over $\overline{K}^*$. These observations imply that M is as described in Propositions.

In fact, given such a knot K, we can construct M as follows.

First, for $\tau^m \zeta_{n,k}$-spins, take $\Sigma_m(\overline{K})$ and let $\tilde{J}$ be the lift of $\overline{J}$ which is not necessarily connected. Let M_1 be the manifold obtained from $\Sigma_m(\overline{K})$ by performing $1/k$-surgery along the lift of $\overline{K}$, and let $\tilde{J}^*$ be the image of $\tilde{J}$. Finally, take the n-fold cyclic branched covering space of M_1 over $\tilde{J}^*$, we get M.

Secondly, for $\tau^m\omega_{n,k}$-spins, take $\Sigma_m(\overline{K})$ and let $\widetilde{J}_i$ be the lift of $\overline{J}_i$ $(i = 1, 2)$. M_1 is the same, and let $\widetilde{J}_i^*$ be the image of $\widetilde{J}_i$ under this surgery. In this case, take the $Z_{n_1} \oplus Z_{n_2}$-branched covering space of M_1 over $\widetilde{J}_1^* \cup \widetilde{J}_2^*$, and so we get M. In particular, if $\overline{K}$ is unknotted, then $\Sigma_m(\overline{K})$ and M_1 are homeomorphic to S^3. Actually we shall deal with only this case.

3. 2-bridge knots and pretzel knots

Let $C(r, s)$ be the 2-bridge knot $S(4rs + 2r + 1, 2r)$ (Schubert's notation) for $r \neq 0$ and $s \geq 0$. Note that $C(r, 0)$ is the torus knot of type $(2, 2r + 1)$. Then $C(r, s)$ has a symmetry g of order 2 with $Fix(g) = J$, shown in Figure 1. Here $2s + 1$ indicates the number of left-half twists, and $2r$ the number of half twists, which are right-handed if $r \geq 0$, left-handed if $r < 0$: C(2,1) in illustration.

Figure 1

Lemma 3.1. *Let $C(r, s), g$ and J be as above, and let $k = sign(r)$. Then the closed fiber of $\tau^m\zeta_{2,k}C(r, s)$ is given as follows;*

(1) For $m = |2r|, \sharp^m L(2s + 1, s)$.

(2) For $m = |2r| - 1$,

the Seifert fibered manifold $\{-4r; (o_1, 0); (s + 1, 1), \dots m \dots, (s + 1, 1)\}$ $(r > 0)$,

the Seifert fibered manifold $\{-4r; (o_1, 0); (s, 1), \dots m \dots, (s, 1)\}$ $(r < 0, s \neq 0)$,

$\sharp^{m-1} S^2 \times S^1$ $(r < 0, s = 0)$.

(3) For $m = |2r| + 1$,

the Seifert fibered manifold $\{4r; (o_1, 0); (s, 1), \dots m \dots, (s, 1)\}$ $(r > 0, s \neq 0)$, *the Seifert fibered manifold* $\{4r; (o_1, 0); (s+1, 1), \ldots m \ldots, (s+1, 1)\}$ $(r < 0)$,

$\sharp^{m-1} S^2 \times S^1$ $(r > 0, s = 0)$.

Proof. We shall follow the procedure given in Section 2 in determining the closed fiber. Let $q : S^3 \to S^3/g$ be the quotient map. Then $\overline{C}(r, s) = q(C(r, s))$ and $\overline{J} = q(J)$ are unknotted (Figure 2).

Figure 2

Let $\widetilde{C}$ and $\widetilde{J}$ be the lifts of $\overline{C}$ and $\overline{J}$ in the m-fold cyclic branched covering space of S^3 over $\overline{C}$, which is homeomorphic to S^3. Trivialize $1/k$-surgery by $(-k)$-twist (cf. [Ro:Ch 9]), and we have a link $\widetilde{J}^*$ in S^3 again. To prove Lemma, it is therefore sufficient to identify the link $\widetilde{J}^*$ in each case.

(1) $m = |2r|$. Begin with the diagram of $\widetilde{J}^*$ in Figure 3(i), and deform the arc connecting bottom terminals of the first tangle to the dotted position. Repeating this gives a diagram in Figure 3(ii), which shows that $\widetilde{J}^*$ is $\sharp^m S(2s+1, s)$.

Figure 3

(2) $m = |2r| - 1$. Begin with the diagram in Figure 4(i), and deform the arc connecting the right-bottom terminal of the first tangle with the left-bottom one of the second to the dotted position. Repeating this, and after a slight move, gives a diagram in Figure 4(ii), which shows that $\widetilde{J}^*$ is the Montesinos link (cf. [B-Z])

$$\mathcal{M}(-4r; (s+1, 1), \ldots m \ldots, (s+1, 1)) \quad \text{if } r > 0,$$

$$\mathcal{M}(-4r; (s, 1), \ldots m \ldots, (s, 1)) \quad \text{if } r < 0, s \neq 0,$$

$$\text{the trivial } m\text{-component link} \quad \text{if } r < 0, s = 0.$$

Figure 4

(3) $m = |2r| + 1$. Similarly (Figure 5(i),(ii)), we have that $\widetilde{J}^*$ is

$$\begin{array}{rl} \mathcal{M}(4r; (s,1), \ldots m \ldots, (s,1)) & \text{if} \quad r > 0, s \neq 0, \\ \mathcal{M}(4r; (s+1,1), \ldots m \ldots, (s+1,1)) & \text{if} \quad r < 0, \\ \text{the trivial } m\text{-component link} & \text{if} \quad r > 0, s = 0. \end{array}$$

Figure 5

Finally, we take the 2-fold cyclic branched covering space over $\widetilde{J}^*$.

Example 3.2. (Compare [Ka 1:Examples 1,2,3])

(1) The knot $C(r,0)$ is the torus knot $T_{2,2r+1}$. In the deformation group $\mathcal{D}(T_{2,2r+1})$, we have $\tau^{|2r|}\zeta_{2,k} = \tau^{-k}, \tau^{|2r|-1}\zeta_{2,k} = \tau^{-1-k}, \tau^{|2r|+1}\zeta_{2,k} = \tau^{1-k}$, where $k = sign(r)$ (cf. [Li1:Corollary 6.5], [Te 1], [Te 2]). These equations are consistent with the results of Lemma 3.1. In fact, $T_{2,2r+1}$ is fibered, and its genus is equal to r if $r > 0$, $-r-1$ if $r < 0$. Hence, the τ^0-spin of $T_{2,2r+1}$, the Artin spin, is a fibered knot whose fiber is $punc(\sharp^{2r} S^2 \times S^1)$ if $r > 0$, $punc(\sharp^{-2r-2} S^2 \times S^1)$ if $r < 0$.

(2) The knot $C(1,s)$ is the $(-s-1)$-twist knot. (We shall adopt the convention that (-1)-twist knot is the right-handed trefoil.) By Lemma 3.1, the closed fibers of $\tau^1\zeta_{2,1}C(1,s)$ and $\tau^2\zeta_{2,1}C(1,s)$ are $L(4s+3, s+1), L(2s+1, s)\sharp L(2s+1, s)$, respectively. Considering the characteristic maps, these are the 2-twist-spin of $S(4s+3, s+1)$ and the 2-cable about the 2-twist-spin of $S(2s+1, s)$ [Te 3]. For the s-twist knot $C(-1, s)$, similar results arise.

(3) The closed fiber of $\tau^{2r+1}\zeta_{2,1}C(r,1)$ $(r > 0)$ is $L(6r+1, -1)$. Hence, it is the 2-twist-spin of $S(6r+1, -1)$.

Let $P(r,s)$ be the pretzel knot as illustrated in Figure 6, where s is an odd integer, and $2r+1$ indicates the number of half twists (left-handed if $r \geq 0$, right-handed if $r < 0$). Note that $P(0,s)$ and $P(-1,s)$ are torus knots of type $(2,s),(2,-s)$, respectively.

Figure 6

It is clear that $P(r,s)$ has two symmetries g_1 of order s, and g_2 of order 2 such that $g_1 g_2 = g_2 g_1$. Let $J_i = Fix(g_i)$ $(i = 1,2)$, and orient them such that $lk(P(r,s),J_1) = 2, lk(P(r,s),J_2) = (-1)^r s, lk(J_1,J_2) = 1$. Thus the knot $P(r,s)$ has the property as described in Example 1.3. By considering a suitable power of g_1, we may assume $k = \pm 1$, and consider these cases.

Lemma 3.3. *Let $P(r,s)$ be as above. Then the closed fiber of $\tau^1 \omega_{2s,k} P(r,s), k = \pm 1$, is given as follows;*

(1) the Seifert fibered manifold $\{0;(o_1,0);(r,1),\ldots s\ldots,(r,1)\}$ $(k=1, r \neq 0)$,

(2) the Seifert fibered manifold $\{0;(o_1,0);(r+1,1),\ldots s\ldots,(r+1,1)\}$ $(k=-1, r \neq -1)$,

(3) $\sharp^{s-1} S^2 \times S^1$ $(k = 1, r = 0,$ *or* $k = -1, r = -1)$.

Proof. The proof is similar to that of Lemma 3.1. Let $q : S^3 \to S^3/g$ be the quotient map, where $g = g_1 g_2$. Let $\overline{P}(r,s) = q(P(r,s)), \overline{J}_i = q(J_i)$ $(i = 1,2)$. Note that $\overline{P}(r,s)$ is unknotted (Figure 6). Since we consider the 1-twist-spinning, M_1 (see Section 2) is obtained from S^3 by performing $1/k$-surgery along $\overline{P}(r,s)$. Hence M_1 is homeomorphic to S^3. Trivialize the surgery by $(-k)$-twist. Let J_i^* be the image of $\overline{J}_i$ under $(-k)$-twist (i=1,2) (Figure 7).

Figure 7

Finally, we must take the $Z_s \oplus Z_2$-branched covering space of M_1 over $J_1^* \cup J_2^*$, corresponding to $Ker[\pi_1(M_1 - J_1^* \cup J_2^*) \to Z < t_1 > \times Z < t_2 > \to Z_s \oplus Z_2]$, where the last homomorphism sends a meridian t_1 (t_2 resp.) of J_1^* (J_2^* resp.) to $(1,0)$ ($(0,1)$ resp.). Take the s-fold cyclic branched covering space over J_1^*, and identify the lift $\widetilde{J}_2^*$ of J_2^*. By taking the 2-fold branched covering over $\widetilde{J}_2^*$, the results follow immediately.

Let $Q(r,s)$ be the pretzel knot as illustrated in Figure 8, where s is an odd integer, $2r+1$ indicates the number of half twists (left-handed if $r \geq 0$, right-handed if $r < 0$).

Figure 8

Then $Q(r,s)$ has two symmetries g_1 of order s, and g_2 of order 2, and has the property as described in Example 1.3. We may assume $k = 1$, and consider this case.

Lemma 3.4. *Let $Q(r,s)$ be as above. Then the closed fiber of $\tau^1\omega_{2s,1}Q(r,s)$ is given as follows;*

(1) the Seifert fibered manifold $\{-4s; (o_1, 0); (r+1, 1), \ldots s \ldots, (r+1,1)\}$ $(r \neq -1)$,

(2) $\sharp^{s-1} S^2 \times S^1$ $(r = -1)$.

Proof. We can determine the closed fiber in the same way as the proof of Lemma 3.3. See Figure 9.

Figure 9

Now we shall prove the main theorem of this chapter, which gives a partial answer to the problem of Hillman and Yoshikawa.

Theorem 3.5. *There exists a fibered 2-knot in S^4 whose fiber is a punctured prism manifold M_d° with fundamental group isomorphic to $Q(8)\times Z_d$ for $d = 3, 5, 11, 13, 19, 21, 27$.*

Proof. In Lemma 3.1(2), we set $(r, s) = (2, 1)$ or $(-2, 2)$. Then the closed fiber is the prism manifolds M_{13}, M_{19} respectively. In Lemma 3.1(3), we set $(r, s) = (-1, 1), (1, 2)$, and then we get M_5, M_{11} respectively. Let $(r, s) = (2, 3)$ in Lemma 3.3(1), or $(1, 3)$ in Lemma 3.3(2). Then in either case we get M_3. Finally, let $(r, s) = (1, 3), (-3, 3)$ in Lemma 3.4(1). Then we obtain M_{21}, M_{27} respectively.

4. Untwisted deform-spun knots

Let (S^3, K) be a knot with projection $(p, K \times D^2)$ (see Section 1). If $\theta \in \partial D^2$ is a regular value, then $F^\theta = p^{-1}(\theta)$ is a compact, codimension 1 submanifold of the exterior $X(K)$ and $\partial F^\theta = K \times \{\theta\}$. That is, F^θ is a *Seifert surface* for K. Let $\gamma \in \mathcal{D}(K)$ be an untwisted deformation with compatible representative $g : (S^3, K) \to (S^3, K)$. That is, $g|_{K\times D^2} = id$ and $p(g|_{X(K)}) = p$. Then for each $F^\theta, g(F^\theta) = F^\theta$. The exterior $X(\gamma K)$ of the γ-spin of K is the space $X(K) \times_g \partial B^2 \cup K_- \times \partial D^2 \times B^2$. If F^θ is a Seifert surface for K, then the space $F^\theta \times_g \partial B^2 \cup K_- \times \{\theta\} \times B^2$ gives a Seifert surface for γK, which is denoted by γF^θ.

Lemma 4.1. *Let (S^3, K_i) be a knot with projection $(p_i, K_i \times D^2)$ $(i = 1, 2)$. Let $F_i = p_i^{-1}(\theta)$ be a Seifert surface for K_i. Let $\gamma_i \in \mathcal{D}(K_i)$ be an untwisted deformation with compatible representative g_i. If there exists a homeomorphism $h : F_1 \to F_2$ such that $hg_1 = g_2h$, then γ_1K_1 and γ_2K_2 have homeomorphic Seifert surfaces γ_1F_1 and γ_2F_2.*

The proof is straightforward, so we omit it.

Let $P(r,s)$ be the pretzel knot stated in Section 3. As shown in Figure 10, $P(r,s)$ has a Seifert surface $F(r,s)$ of genus $(s-1)/2$, which is invariant under g_i $(i=1,2)$ and $J_1 \cap F(r,s) = \{2 \text{ points}\}$, $J_2 \cap F(r,s) = \{s \text{ points}\}$. Note that $F(0,s)$ $(F(-1,s)$ resp.) is a fiber surface for $P(0,s)$ $(P(-1,s)$ resp.), which is the $(2,s)$ $((2,-s)$ resp.) -torus knot.

Figure 10

Theorem 4.2. *The $\omega_{2s,k}$-spin of $P(r,s)$ has a Seifert surface homeomorphic to the punctured Brieskorn 3-manifold $\Sigma(2,s,k)^\circ$.*

Proof. By Lemma 4.1, $\omega_{2s,k}P(r,s)$ and $\omega_{2s,k}P(0,s)$ have homeomorphic Seifert surfaces. It is therefore sufficient to show that $\omega_{2s,k}F(0,s)$ is homeomorphic to $\Sigma(2,s,k)^\circ$. The map $s_{2s,1}(g_1 g_2)$ is just the monodromy map on the fiber surface $F(0,s)$ (cf. [Mi 2:Section 9], [Kf:Chapter 19]). It follows that $\omega_{2s,k}F(0,s)$ is the punctured one of the k-fold cyclic branched covering of $P(0,s)$. This completes the proof.

Remark 4.3. The k-fold cyclic branched covering of the $(2,s)$-torus knot is $\Sigma(2,s,k)$ (cf. [Mi 1]). Hence the k-twist-spin of the $(2,s)$-torus knot is the fibered 2-knot whose fiber is $\Sigma(2,s,k)^\circ$. The knot $P(r,s)$ is a torus knot if and only if $r=0,-1$. We might expect that any nontrivial $\omega_{2s,k}$-spin of a non-torus knot $P(r,s)$ is non-fibered, but I have been unable to prove this. In fact, Kanenobu [Ka 2] has observed that if $P(r,s)$ is non-torus and if $s \not| r$ then the $\omega_{2s,1}^2$-spin is non-fibered with Seifert surface $\Sigma(2,2,s)^\circ = L(s,1)^\circ$.

Corollary 4.4. *If $k=\pm 1$, then the $\omega_{2s,k}$-spin of $P(r,s)$ is unknotted, that is, it bounds a 3-cell.*

If $P(r,s)$ is a torus knot, then $\omega_{2s,k} = \tau^{\pm k}$ in $\mathcal{D}(P(r,s))$ [Li 1:Corollary 6.5]. But if $P(r,s)$ is non-torus, the untwisted deformation $\omega_{2s,k}$ is not contained in the subgroup $< \tau >$ of $\mathcal{D}(P(r,s))$ generated by τ [Li 1:Corollary 6.3]. Thus Corollary 4.4 means the existence of "untwisted unknotting deformations" for certain knots.

Chapter II. Symmetry-spun tori

In this chapter we will study knotted tori in S^4 which are obtained from classical links or knots having cyclic periods by symmetry-spinning. By spinning and twist-spinning a knot in a manner similar to the classical methods of Artin and Zeeman, we get the corresponding knotted tori. It has been studied by several authors [As 1], [Bo], [Lv 1], [Lv 2]. It is easy to construct many examples of tori in S^4 having infinite cyclic knot groups by symmetry-spinning. For example, let K be a 2-bridge knot in S^3. Then K has cyclic period 2. Removing an invariant 3-ball disjoint from K gives a pair (B^3, K). Think of S^4 as $B^3 \times S^1 \cup S^2 \times D^2$. If during the rotation B^3 through the factor S^1 its cyclic period acts on K, then K sweeps out a knotted torus $T(K)$ which is called the *symmetry-spun torus* of K. In this case, $\pi_1(S^4 - T(K))$ is infinite cyclic, so it may be unknotted. There is a conjecture that a surface in S^4 with infinite cyclic knot group is necessarily unknotted, that is, it bounds a handlebody [H-K]. In fact, in Section 3 we will prove that any symmetry-spun obtained from a periodic link or knot is equivalent to the spun of its factor link or knot. In the case that K is a 2-bridge knot, its factor knot is trivial. Hence $T(K)$ is indeed unknotted.

1.Preliminaries

We will work in the smooth category. All manifolds will be oriented, and all submanifolds are assumed to be locally-flat. The circle is taken to be the quotient space $S^1 = R^1/(\theta \sim \theta + 2\pi \text{ for all } \theta \in R^1)$. We will write $\theta \in S^1$. $(a, b, \ldots)$ stands for the g.c.d. of the integers $a, b, \ldots$. A *knotted surface* is a pair (S^4, F), where F is a closed oriented (and connected or not) surface in S^4. Two knotted surfaces (S^4, F) and (S^4, G) are *equivalent* if there exists a diffeomorphism $f : (S^4, F) \to (S^4, G)$ preserving the orientations of S^4 and F. Then we use the notation $(S^4, F) \cong (S^4, G)$. Changing one or both of the orientations, we have three new knotted surfaces $(S^4, -F), -(S^4, -F)$ and $-(S^4, F)$. (Note that we are considering a surface itself, not its embedding map.) A knotted surface (S^4, F) is called (-)*amphicheiral* if $(S^4, F) \cong -(S^4, F)$.

Let U be a standardly embedded torus in S^4 and let $D^2 \times U$ be a tubular neighbourhood of U in S^4. We can assume that its framing is canonical, that is, the homomorphism induced by the inclusion map $H_1(0 \times U; Z) \to H_1(p \times U; Z) \to H_1(S^4 - U; Z)$ where $p \in \partial D^2$, is zero (cf. [Li 2:Lemma 1]). Let $\bar{l} = \partial D^2 \times 0 \times 0, \bar{s} = 0 \times S^1 \times 0, \bar{r} = 0 \times 0 \times S^1$ be curves on $\partial D^2 \times U = \partial D^2 \times S^1 \times S^1$.

Let $E^4 = cl(S^4 - D^2 \times U)$, which is a (trivial) twin (see [Iw 1:Lemma 2.1], [Mo]). Let l, s, r be canonical curves on ∂E^4, which are identified with $\bar{l}, \bar{s}, \bar{r}$, respectively under the natural identification map $i : \partial D^2 \times U \to \partial E^4$. Then l, s, r represent a basis of $H_1(\partial E^4; Z)$.

Let $f : \partial E^4 \to \partial E^4$ be a diffeomorphism with $f_\sharp [l \quad s \quad r] = [l \quad s \quad r] A^f$, where $A^f \in GL(3, Z) \cong \pi_0 Diff(\partial E^4)$. Then f can be extended to a diffeomorphism $\widetilde{f} : E^4 \to E^4$ if and only if $A^f \in H$, where

$$H = \left\{ \begin{bmatrix} \pm 1 & 0 & 0 \\ * & \alpha & \gamma \\ * & \beta & \delta \end{bmatrix} \in GL(3, Z) \middle| \quad \alpha + \beta + \gamma + \delta \equiv 0 \pmod 2 \right\}$$

(see [Mo:Theorem 5.3]).

Let $\sigma : \partial E^4 \to \partial E^4$ be a diffeomorphism of matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Then $E^4 \cup_{\sigma i} D^2 \times S^1 \times S^1$ is diffeomorphic to S^4 (cf. [Go], [Mo]).

2. Symmetry-spun tori

Let K_1 be a knot in $D^2 \times S^1$, which may be geometrically inessential. (We should exclude cases where K_1 bounds a disk in $D^2 \times S^1$ and K_1 is ambient isotopic to the core.) Let $p_a : D^2 \times S^1 \to D^2 \times S^1$ be the a-fold cyclic cover given by $(x, \theta) \to (x, a\theta)$ for $a \in Z \backslash \{0\}$. Let $r_\phi : D^2 \times S^1 \to D^2 \times S^1$ be the rotation map given by $(x, \theta) \to (x, \theta + \phi)$ for $\phi \in S^1$. Let $K_a = p_a^{-1}(K_1) \subset D^2 \times S^1$, which may be a link, and K_a is given the orientation induced by K_1 if $a > 0$, and is given the opposite orientation if $a < 0$. Then K_a is invariant under the rotation $r_{2\pi/a}$. Note that the pairs $(D^2 \times S^1, K_a)$ and $(D^2 \times S^1, K_{-a})$ are diffeomorphic by a diffeomorphism $(x, \theta) \to (x, -\theta)$, which is orientation-reversing both on $D^2 \times S^1$ and K_a.

Lemma 2.1. *Let $\rho : D^2 \times S^1 \to D^2$ be the projection map. Then $\rho\big(K_a \cap (D^2 \times \theta)\big) = \rho\big(K_b \cap (D^2 \times a\theta/b)\big)$ for $a, b \in Z\backslash\{0\}, \theta \in S^1$.*

Proof. Since the covering $p_a : D^2 \times S^1 \to D^2 \times S^1$ is induced by a covering $S^1 \to S^1$ given by $\theta \to a\theta$, $\rho\big(K_a \cap (D^2 \times \theta)\big) = \rho\big(K_1 \cap (D^2 \times a\theta)\big) = \rho\big(K_b \cap (D^2 \times a\theta/b)\big)$.

Definition 2.2. Let $a, b \in Z$ with $b \neq 0$. We define a surface $T^a(K_b)$ in $D^2 \times S^1 \times S^1$, which satisfies

$$T^a(K_b) \cap (D^2 \times S^1 \times \theta) = i_\theta r_{a\theta/b}(K_b).$$

Then we get two knotted surfaces in S^4, called the *symmetry-spun tori* of K_b, identifying ∂E^4 and $\partial D^2 \times S^1 \times S^1$ using the natural identification i and the twisted identification σi (see Section 1). We denote $\big(S^4, T^a(K_b)\big)$, $\big(S^4, \widetilde{T}^a(K_b)\big)$, respectively.

Each connected component of $T^a(K_b)$ is a torus. It is easy to see that K_b has (b, w) components and $T^a(K_b)$ has (a, b, w) components, where w is the winding number of K_1

in $D^2 \times S^1$. Note that $T^0(K_b)$ is just a *spun torus* of K_b (cf. [Bo], [Lv 1]).

Lemma 2.3. *If* $a \equiv b \pmod{2c}$, *then* $(S^4, T^a(K_c)) \cong (S^4, T^b(K_c))$, $(S^4, \widetilde{T}^a(K_c)) \cong (S^4, \widetilde{T}^b(K_c))$.

Proof. Define $f : D^2 \times S^1 \times S^1 \to D^2 \times S^1 \times S^1$ by $(x, \phi, \theta) \to (x, \phi + 2\theta, \theta)$. Then

$$f(T^a(K_c)) \cap (D^2 \times S^1 \times \theta) = f i_\theta r_{a\theta/c}(K_c) = i_\theta r_{(a+2c)\theta/c}(K_c),$$

since $f i_\theta = i_\theta r_{2\theta}$ on $D^2 \times S^1$. The diffeomorphism $\tau = i(f|_{\partial D^2 \times S^1 \times S^1})i^{-1} : \partial E^4 \to \partial E^4$ has a matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, so τ extends to E^4 (see Section 1). Hence $(S^4, T^a(K_c)) \cong (S^4, T^{a+2c}(K_c))$, which gives the first equivalence. Since $\tau\sigma = \sigma\tau$ on ∂E^4, the second follows.

Lemma 2.4. $(S^4, T^a(K_b)) \cong (S^4, \widetilde{T}^{a\pm b}(K_b))$.

Proof. Define $f : D^2 \times S^1 \times S^1 \to D^2 \times S^1 \times S^1$ by $(x, \phi, \theta) \to (x, \phi - \theta, \theta)$. Then

$$f(T^a(K_b)) \cap (D^2 \times S^1 \times \theta) = f i_\theta r_{a\theta/b}(K_b) = i_\theta r_{(a-b)\theta/b}(K_b).$$

The diffeomorphism $\tau = i(f|_{\partial D^2 \times S^1 \times S^1})i^{-1} : \partial E^4 \to \partial E^4$ has a matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, so $\sigma^{-1} = \tau$. From this the result follows.

Lemma 2.5.

(1) $(S^4, T^a(-K_b)) \cong (S^4, -T^a(K_b))$, *where* $-K_b$ *is obtained from* K_b *by reversing its orientation.*

(2) $(S^4, T^{-a}(K_b)) \cong -(S^4, T^a(K_b))$.

(3) $(S^4, T^a(K_{-b})) \cong -(S^4, T^a(K_b))$.

And also, the corresponding equivalences on the twisted cases hold.

Proof. It is easy to see (1). For (2) define $f : D^2 \times S^1 \times S^1 \to D^2 \times S^1 \times S^1$ by $(x, \phi, \theta) \to (x, \phi, -\theta)$. Then

$$f(T^{-a}(K_b)) \cap (D^2 \times S^1 \times \theta) = f i_{-\theta} r_{(-a)(-\theta)/b}(K_b) = i_\theta r_{a\theta/b}(K_b),$$

and f induces an orientation-reversing diffeomorphism on $T^{-a}(K_b)$. The diffeomorphism $\tau = i(f|_{\partial D^2 \times S^1 \times S^1})i^{-1} : \partial E^4 \to \partial E^4$ has a matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, so τ extends to E^4. Hence $(S^4, T^{-a}(K_b)) \cong -(S^4, T^a(K_b))$.

For (3) define $g : D^2 \times S^1 \to D^2 \times S^1$ by $(x, \phi) \to (x, -\phi)$, and let $h = g \times id : D^2 \times S^1 \times S^1 \to D^2 \times S^1 \times S^1$. Then

$$\begin{aligned} h(T^a(K_{-b})) \cap (D^2 \times S^1 \times \theta) &= h i_\theta r_{a\theta/(-b)}(K_{-b}) = i_\theta g r_{a\theta/(-b)}(K_{-b}) \\ &= i_\theta r_{a\theta/b} g(K_{-b}) = i_\theta r_{a\theta/b}(-K_b). \end{aligned}$$

Since $i(h|_{\partial D^2 \times S^1 \times S^1})i^{-1}$ extends to E^4, we have the equivalence

$$(S^4, T^a(K_{-b})) \cong (-S^4, T^a(-K_b)) \cong -(S^4, T^a(K_b)).$$

For the twisted cases, by Lemma 2.4 and the above equivalences

$$(S^4, \widetilde{T}^{-a}(K_b)) \cong (S^4, T^{-a+b}(K_b)) \cong -(S^4, T^{a-b}(K_b)) \cong -(S^4, \widetilde{T}^a(K_b)),$$

$$(S^4, \widetilde{T}^a(K_{-b})) \cong (S^4, T^{a-b}(K_{-b})) \cong -(S^4, T^{a-b}(K_b)) \cong -(S^4, \widetilde{T}^a(K_b)).$$

Corollary 2.6. *Spun tori $(S^4, T^0(K_b)), (S^4, \widetilde{T}^0(K_b))$ are (-)amphicheiral.*

It is known that every ribbon 2-knot is (-)amphicheiral and in particular every spun 2-knot is so (cf. [Go], [Su]). In the case $a = 0$, $T^0(K_b)$ is obtained by spinning K_b, so it has a symmetric normal form. Hence $T^0(K_b)$ is a ribbon surface (cf. [K-S-S]).

Corollary 2.7. *Let* $a \equiv 0 \pmod{b}$. *Then*

$$(S^4, T^a(K_b)) \cong \begin{cases} (S^4, T^0(K_b)) & \text{if } a/b \equiv 0 \pmod{2}, \\ (S^4, \widetilde{T}^0(K_b)) & \text{if } a/b \equiv 1 \pmod{2}. \end{cases}$$

Proof. This is an immediate consequence of Lemmas 2.3 and 2.4.

Remark 2.8. It is clear that $\pi_1(S^4 - T^0(K_b)) \cong \pi_1(S^4 - \widetilde{T}^0(K_b))$. If K is a nontrivial knot or a link with no separated trivial component, then the exteriors of spun tori $(S^4, T^0(K))$ and $(S^4, \widetilde{T}^0(K))$ have different diffeomorphism types. This can be proved using the Z_2–intersection number (see [Iw 2:Lemma 2.8], [Lv 1]).

Lemma 2.9. $(S^4, T^a(K_b)) \cong (S^4, T^b(K_{-a}))$ *for* $a, b \in Z \backslash \{0\}$.

Proof. Define an orientation-preserving diffeomorphism $f : D^2 \times S^1 \times S^1 \to D^2 \times S^1 \times S^1$ by $(x, \phi, \theta) \to (x, \theta, -\phi)$. By Lemma 2.1,

$$\begin{aligned} T^a(K_b) \cap (D^2 \times (-\phi) \times S^1) &= \bigcup_{\theta \in S^1} (T^a(K_b) \cap (D^2 \times (-\phi) \times \theta)) \\ &= \bigcup_{\theta \in S^1} i_\theta|_{D^2 \times (-\phi)}(r_{a\theta/b}(K_b) \cap (D^2 \times (-\phi))) \\ &= \bigcup_{\theta \in S^1} i_\theta|_{D^2 \times (-\phi)}\Big(\rho(K_b \cap (D^2 \times (-\phi - a\theta/b))), -\phi\Big) \\ &= \bigcup_{\theta \in S^1} i_\theta|_{D^2 \times (-\phi)}\Big(\rho(K_{-a} \cap (D^2 \times (b\phi/a + \theta))), -\phi\Big) \\ &= \bigcup_{\theta \in S^1} i_\theta|_{D^2 \times (-\phi)}\Big(\rho(r_{b\phi/(-a)}(K_{-a}) \cap (D^2 \times \theta)), -\phi\Big). \end{aligned}$$

Hence

$$\begin{aligned} f(T^a(K_b)) \cap (D^2 \times S^1 \times \phi) &= i_\phi\Big(\bigcup_{\theta \in S^1} \Big(\rho(r_{b\phi/(-a)}(K_{-a}) \cap (D^2 \times \theta)), \theta\Big)\Big) \\ &= i_\phi r_{b\phi/(-a)}(K_{-a}), \end{aligned}$$

and f induces an orientation-preserving diffeomorphism on $T^a(K_b)$. The diffeomorphism $\tau = i(f|_{\partial D^2 \times S^1 \times S^1})i^{-1}$ has a matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$, so τ extends to E^4. Therefore $(S^4, T^a(K_b)) \cong (S^4, T^b(K_{-a}))$.

Thus we have $(S^4, T^a(K_b)) \cong -(S^4, T^b(K_a))$ by Lemmas 2.4 and 2.9. Note that a similar equivalence on the twisted case does not hold, because the above equivalence permutes the cores of the twin E^4 (see [Mo]).

Take a symmetric Wirtinger presentation of a periodic link or knot K_b. By van Kampen Theorem we can see that $\pi_1(S^4 - T^a(K_b)) \cong \pi_1(S^4 - T^0(K_d)) \cong \pi_1(S^3 - K_d)$, where $d = (a, b)$.

Theorem 2.10. *For $a, b \in Z\backslash\{0\}$, let $a = 2^p a', b = 2^q b'$ with $p, q \geq 0$ and a', b' odd. Then*

$$(S^4, T^a(K_b)) \cong \begin{cases} (S^4, T^0(K_d)) & \text{if } \; p \neq q, \\ (S^4, \widetilde{T}^0(K_d)) & \text{if } \; p = q, \end{cases}$$

where $d = (a, b) > 0$.

We will give a proof of Theorem 2.10 in the next section.

Corollary 2.11. *Let K_1 be a trivial knot. Let $a, b \in Z$ with $b \neq 0$. If $(a, b) = 1$, then $(S^4, T^a(K_b))$ and $(S^4, \widetilde{T}^a(K_b))$ are unknotted, that is, these bound solid tori in S^4.*

Example 2.12. Let K_b be the pretzel link (or knot) $p(n, \ldots b \ldots, n)$ in S^3. It is clear that K_b has cyclic period b. Let $g : (S^3, K_b) \to (S^3, K_b)$ be the diffeomorphism of period b. Let z be a point on $Fix(g)$ and B_0 an invariant regular neighbourhood of z disjoint from K_b. Removing $IntB_0$ from S^3 gives the pair (B^3, K_b). We may assume that $g|_{\partial B^3} = id$. Choose an integer a with $(a, b) = 1$. Then

$$(S^4, T^a(K_b)) \cong \partial B^3 \times D^2 \cup_\partial (B^3, K_b) \times_{g^a} \partial D^2,$$

$$(S^4, \widetilde{T}^a(K_b)) \cong \partial B^3 \times D^2 \cup_\tau (B^3, K_b) \times_{g^a} \partial D^2,$$

where

$$(B^3, K_b) \times_{g^a} \partial D^2 = (B^3, K_b) \times [0, 1] \Big/ \Big((x, 0) \sim (g^a(x), 1) \quad \text{for all } x \in B^3 \Big),$$

and $\tau : S^2 \times S^1 \to S^2 \times S^1$ represents the nontrivial element of $\pi_1(SO(3))$ (cf. [Gl]). By Corollary 2.11 symmetry-spun tori $(S^4, T^a(K_b))$ and $(S^4, \widetilde{T}^a(K_b))$ are unknotted.

3. Proof of Theorem 2.10

In this section, we prove Theorem 2.10. To do this, we use the generalized Euclidean algorithm.

For $a, b \in Z\backslash\{0\}$ with $|b| \geq 2$, let $x_{-1} = a, x_0 = b$. Then

$$x_{-1} = x_0 y_1 + x_1, \qquad \text{where} \quad y_1 \quad \text{even and} \quad 2 \leq |x_1| \leq |x_0|.$$

If $|x_1| < |x_0|$, then

$$x_0 = x_1 y_2 + x_2, \qquad \text{where} \quad y_2 \quad \text{even and} \quad 2 \leq |x_2| \leq |x_1|.$$

If $|x_2| < |x_1|$, then proceed to the next step. Repeating this, we have that there exists an integer k such that

$$x_{k-1} = x_k y_{k+1} + x_{k+1}, \qquad \text{where} \quad |x_i| \geq 2 \quad (0 \leq i \leq k),$$

and one of the following cases holds,

(i-1) $x_{k+1} = 0$, y_{k+1} is even,

(i-2) $x_{k+1} = 0$, y_{k+1} is odd,

(ii-1) $|x_{k+1}| = 1, y_{k+1}$ is even, x_k is even,

(ii-2) $|x_{k+1}| = 1, y_{k+1}$ is even, x_k is odd.

As in the (usual) Euclidean algorithm, we can prove that $(a, b) = |x_k| \geq 2$ (Cases i-1,i-2), $(a, b) = |x_{k+1}|$ (Cases ii-1,ii-2). In particular, if $(a, b) = 1$ then neither (i-1) nor (i-2) occurs.

Lemma 3.1. *Let $a, b \in Z\backslash\{0\}$ with $|b| \geq 2$. Let $a = 2^p a', b = 2^q b'$ with $p, q \geq 0$ and a', b' odd. In the generalized Euclidean algorithm, we have the following.*

(i) Let $(a, b) \neq 1$. If $p \neq q$, then (i-1) occurs, otherwise (i-2).

(ii) Let $(a, b) = 1$. If $p \neq q$, then (ii-1) occurs, otherwise (ii-2).

Proof. Suppose that $(a, b) = 1$. Then as stated before, either (ii-1) or (ii-2) occurs. If x_k is even, than either a or b is even, so $p \neq q$. Assume that x_k is odd. Since x_{k-1} is odd, both a and b are odd, so $p = q = 0$. Thus in the case $(a, b) = 1$, if $p \neq q$ then (ii-1) occurs, otherwise (ii-2).

Next suppose that $d = (a, b) \geq 2$. Let $a = a''d, b = b''d$. If $|b''| = 1$, then $a = \pm ba''$. Hence if a'' is even, then (i-1) occurs, otherwise (i-2). So if $p \neq q$ then (i-1) occurs, otherwise (i-2). If $|b''| \geq 2$, then we apply the generalized Euclidean algorithm to the pair $\{a'', b''\}$. Since $(a'', b'') = 1$, either (ii-1) or (ii-2) occurs. Multiplying the equations by d, we have

$$a = by_1 + dx_1, \ldots, dx_{k-1} = dx_k y_{k+1} \pm d.$$

Since $d \geq 2$, we must proceed to the next step $dx_k = (\pm d)(\mp x_k)$. Hence if x_k is even, then (i-1) occurs, otherwise (i-2). But if x_k is even, then either a'' or b'' is even, so $p \neq q$. If x_k is odd, then both a'' and b'' are odd, so $p = q$. Thus in the case $(a, b) \geq 2$, if $p \neq q$ then (i-1) occurs, otherwise (i-2). The proof is complete.

Proof of Theorem 2.10. If $|b| = 1$, then the result is a consequence of Corollary 2.7. Suppose that $|b| \geq 2$. We apply the generalized Euclidean algorithm. Then by Lemma 2.3,

$$(S^4, T^{x_{-1}}(K_{x_0})) \cong (S^4, T^{x_1}(K_{x_0})),$$

and by Lemmas 2.4 and 2.9,

$$(S^4, T^{x_1}(K_{x_0})) \cong -(S^4, T^{x_0}(K_{x_1})).$$

Hence

$$(S^4, T^{x-1}(K_{x_0})) \cong -(S^4, T^{x_0}(K_{x_1})).$$

Repeating this, we have

$$(S^4, T^{x-1}(K_{x_0})) \cong \pm(S^4, T^{x_{k-1}}(K_{x_k})).$$

In each case, we have the following;

(i-1) $(S^4, T^{x_{k-1}}(K_{x_k})) \cong (S^4, T^{x_{k+1}}(K_{x_k})) \cong \pm(S^4, T^0(K_d))$.

(i-2) $(S^4, T^{x_{k-1}}(K_{x_k})) \cong (S^4, T^{x_k}(K_{x_k})) \cong \pm(S^4, \widetilde{T}^0(K_d))$.

(ii-1) $(S^4, T^{x_{k-1}}(K_{x_k})) \cong (S^4, T^{x_{k+1}}(K_{x_k})) \cong \pm(S^4, T^1(K_{x_k}))$

$\cong \pm(S^4, T^{x_k}(K_1)) \cong \pm(S^4, T^0(K_1))$.

(ii-2) $(S^4, T^{x_{k-1}}(K_{x_k})) \cong \pm(S^4, T^{x_k}(K_1)) \cong \pm(S^4, \widetilde{T}^0(K_1))$.

Thus $(S^4, T^a(K_b)) \cong \pm(S^4, T^0(K_d))$ or $\pm(S^4, \widetilde{T}^0(K_d))$. By Corollary 2.6,

$$(S^4, T^a(K_b)) \cong (S^4, T^0(K_d)) \text{ or } (S^4, \widetilde{T}^0(K_d)).$$

The result follows from Lemma 3.1.

Appendix. From the viewpoint of moving picture

We have the so-called *moving picture method* to describe a 2-manifold in R^4, but it is still difficult to determine whether the given two knotted surfaces, in particular 2-knots, are equivalent or not by using the moving picture method. For brevity we shall consider only 2-knots. By virtue of [K-S-S], we can assume that a 2-knot has only elementary critical points, and furthermore that all maximum-disks (all upper saddle-bands, all lower saddle-bands, or all minimum-disks, resp.) have the same level. Therefore, we can describe a 2-knot into one picture consisting of an equatorial cross section, upper saddle-bands and lower saddle-bands. We shall call such a picture a moving picture of a 2-knot. To be exact, we can define a moving picture as follows. Here, $R^3[t]$ denotes the hyperplane $R^3 \times \{t\}$ in $R^4 = R^3 \times R^1$.

Let K be a 2-knot in R^4. A *moving picture* of K is a system (k, U, L) such that

(1) k is a classical knot in R^3;

(2) U is a set of images of embeddings $u_i : I \times I \to R^3$ such that $U \cap k = \{u_i(I \times \{0,1\})\}$ and $(k - (U \cap k)) \cup \{u_i(\{0,1\} \times I)\}$ is a trivial link O_u in R^3, which bounds a set of disks D_u in R^3;

(3) L is a set of images of embeddings $l_i : I \times I \to R^3$ such that $L \cap k = \{l_i(I \times \{0,1\})\}$ and $(k - (L \cap k)) \cup \{l_i(\{0,1\} \times I)\}$ is a trivial link O_l in R^3, which bounds a set of disks D_l inR^3;

(4)

$$(K \cap R^3[t], R^3[t]) = \begin{cases} (D_u, R^3) & \text{for } t = 2 \\ (O_u, R^3) & \text{for } 2 > t > 1 \\ (k \cup U, R^3) & \text{for } t = 1 \\ (k, R^3) & \text{for } 1 > t > -1 \\ (k \cup L, R^3) & \text{for } t = -1 \\ (O_l, R^3) & \text{for } -1 > t > -2 \\ (D_l, R^3) & \text{for } t = -2 \\ (\text{an empty set}, R^3) & \text{otherwise} \end{cases}$$

We call U *upper saddle-bands* and L *lower saddle-bands.* There are two elementary moves on a moving picture (cf. [Su]).

Firstly, we can exchange the levels of saddle-bands. That is, let down the upper saddle-band into the level of lower saddle-bands and pull up the lower saddle-band into the level of upper saddle-bands. If necessary, we may slide the roots of bands by an ambient isotopy to keep the number of components of the equatorial cross section.

Secondly, we consider an image b of an embedding $n : I \times I \to R^3[2]$ (or $R^3[-2]$) such that $b \subset D_u$ (or D_l) and $b \cap K \subset \partial D_u$ (or ∂D_l). Then we move b into the level of upper (or lower) saddle-bands, and we get a new band. Conversely, if an upper (or lower) saddle-band can move in $R^3[2]$ (or $R^3[-2]$) with avoiding D_U (or D_l), we can eliminate such a band.

We shall show how to describe a deform-spun knot by using the moving picture method. Let K be a knot in R^3. First, we consider the Artin spin of K. Let K_+ be an associated knotted arc in the half-space $R^3_+ = \{(x, y, z, 0) \in R^4 | z \geq 0\}$ such that $\partial K_+ \subset A = \{(x, y, 0, 0) \in R^4\}$. Spinning R^3_+ about the axis A, and R^3_+ sweeps out R^4, and simultaneously K_+ sweeps out a 2-sphere $\sigma K \subset R^4$, which is just the Artin spin (or the 0-twist-spin) of K. Thus $\sigma K = \{(x, y, z\cos\theta, z\sin\theta) \in R^4 | (x, y, z, 0) \in K_+, 0 \leq \theta \leq 2\pi\}$. We can see that $\sigma K \cap R^3[0] = K \sharp rK$, where rK denotes the mirror image of K, and each minimum (or maximum) of K_+ with respect to the third coordinate induces a critical point of σK with respect to the fourth coordinate. Then we can deform σK by an ambient isotopy of R^4 such that each critical point of σK corresponds to a maximum-disk, a minimum-disk, or a saddle-band. Moreover, we can deform σK so that all maximum-disks (minimum-disks, upper saddle-bands, or lower saddle-bands, resp.) have the same fourth level. So it is easy to get a moving picture of σK. For example, a moving picture in Figure 11 corresponds to the spin of the trefoil knot with one minimum, where the band with label u (l resp.) means the upper (lower resp.) saddle-band.

Figure 11

Next, we consider a deform-spun knot. Let K be a knot in R^3 and K_+ an associated knotted arc in R^3_+. Take a 3-ball B_0 in $Int R^3_+$ which contains a knotted part of K_+ (Figure 12).

Figure 12

Let $g_\theta : R^3_+ \to R^3_+ \quad (\theta \in I)$ be an ambient isotopy such that g_θ fixes $cl(R^3_+ - B_0)$ and $g_1(K_+) = K_+$. During the rotation of R^3_+ about the axis A, deform R^3_+ by $g = \{g_\theta\}$. Since K_+ returns to its original position, K_+ sweeps out a 2-knot $K(g)$, which is the deform-spun knot corresponding to g. Thus $K(g) = \big\{ \big(g_\theta(x), g_\theta(y), g_\theta(z)\cos 2\pi\theta, g_\theta(z)\sin 2\pi\theta\big) \in R^4 \big| (x,y,z,0) \in K_+, \theta \in I \big\}$. See [Li 1:Section 1]. We may assume that g deforms R^3_+ within a final sufficiently short time. Then for a suitable small number $\varepsilon > 0$, $\big(K(g) \cap R^3[t], R^3[t]\big) = \big(\sigma K \cap R^3[t], R^3[t]\big), t \geq 0, t \leq -\varepsilon$, where σK is the Artin spin of K. Moreover we may assume that $K(g)$ has no critical point with respect to $t \quad (-\varepsilon \leq t \leq 0)$.

Thus we have a moving picture $(K\sharp rK, U, L^g)$ of $K(g)$ which is obtained from the moving picture $(K\sharp rK, U, L)$ of σK by deforming the lower saddle-bands L using g.

Examples. (1) The 2-knot $k \subset R^4$ shown in Figure 13 is the 1-twist-spin of the trefoil, where u (l resp.) indicates the upper (lower resp.) saddle-band of k.

Figure 13

If we change the levels of the bands u and l, then we have an equivalent 2-knot $\widetilde{k}$ given in Figure 14, whose equatorial cross section is unknotted.

Figure 14

In fact, the unknottedness of $\widetilde{k}$ can be checked by deforming the moving picture $(O, \widetilde{u}, \widetilde{l})$ using a suitable level preserving, vertical-line preserving isotopy of R^4. Refer to [Na].

(2) Let $P(r, 3)$ be the pretzel knot described in Section 3 of Chapter I. We consider the $\omega_{6,1}$-spin of $P(r, 3)$, denoted by T for short. A moving picture of T is given in Figure 15.

Figure 15

By Corollary 4.4 of Chapter I, T is unknotted. Indeed, we visualize the equivalence of T and a trivial 2-knot as follows. In the moving picture of T in Figure 15, we slide the root of l_1 as in Figure 16 (i) and exchange the levels of saddle-bands. Then we have Figure 16 (ii), which is equivalent to Figure 16 (iii).

Figure 16

References

[Ar] E.Artin: *Zur Isotopie zweidimensionaler Flächen im* R_4,Abh. Math. Sem. Univ. Hamburg,**4** (1925), 174-177.

[As 1] K.Asano: *A note on surfaces in 4-spheres*, Math. Sem. Notes Kobe Univ.,**4** (1976), 195-198.

[As 2] K.Asano: *Homeomorphisms of prism manifolds*, Yokohama Math. J., **26** (1978), 19-25.

[Bn] F.Bonahon: *Diffeotopies des espaces lenticulaires*, Topology, **22** (1983), 305-314.

[Bo] J.Boyle: *Classifying 1-handles attached to knotted surfaces*, Trans. Amer. Math. Soc., **306** (1988), 475-487.

[B-Z] G.Burde and H.Zieschang: Knots, de Gruyter Studies in Math. **5**, Walter de Gruyter, Berlin-New York, 1985.

[Fo] R.H.Fox: *Rolling*, Bull. Amer. Math. Soc., **72** (1966), 162-164.

[Gl] H.Gluck: *The embedding of two-spheres in the four-sphere*, Trans. Amer. Math. Soc., **104** (1962), 308-333.

[G-K] D.L.Goldsmith and L.H.Kauffman: *Twist spinning revisted*, Trans. Amer. Math. Soc., **239** (1978), 229-251.

[Go] C.McA.Gordon: *A note on spun knots*, Proc. Amer. Math. Soc., **58** (1976), 361-362.

[Hi 1] J.A.Hillman: *High dimensional knot groups which are not two-knot groups*, Bull. Austral. Math. Soc., **16** (1977), 449-462.

[Hi 2] J.A.Hillman: 2-Knots and their Groups, Austral. Math. Soc. Lecture Series **5**, Cambridge Univ. Press, Cambridge, 1989.

[H-R] C.Hodgson and J.H.Rubinstein: *Involutions and isotopies of lens spaces*, Lecture Notes in Math., **1144** (1983), 60-96, Springer-Verlag.

[H-K] F.Hosokawa and A.Kawauchi: *Proposals for unknotted surfaces in four-spaces*, Osaka J. Math., **16** (1979), 233-248.

[H-S] F.Hosokawa and S.Suzuki: *On punctured lens spaces in 4-space*, Math. Sem. Notes Kobe Univ., **10** (1982), 323-344.

[Iw 1] Z.Iwase: *Good torus fibrations with twin singular fibers*, Japan J. Math., **10** (2) (1984), 321-352.

[Iw 2] Z.Iwase: *Dehn-surgery along a torus T^2-knot*, Pacific J. Math., **133** (2) (1988),289-299.

[Ja] W.Jaco: Lectures on three manifold topology, Conference board of Math. Science, Regional Conference Series in Math., **43**, Amer. Math. Soc., 1980.

[Km] S.Kamada: *On projective planes in 4-sphere obtained by deform-spinnings*, preprint, Osaka City University (1990).

[Ka 1] T.Kanenobu: *Deforming twist spun 2-bridge knots of genus one*, Proc. Japan Acad. Ser. A **64** (1988), 98-101.

[Ka 2] T.Kanenobu: *Untwisted deform-spun knots: Examples of symmetry-spun 2-knots*, in Transformation Groups (K.Kawakubo (ed.)), Lecture Notes in Math., **1375** (1990), 145-167, Springer-Verlag.

[Ka 3] T.Kanenobu: *Groups of higher dimensional satellite knots*, J. Pure Appl. Algebra, **28** (1983), 179-188.

[Ka 4] T.Kanenobu: *2-knot groups with elements of finite order*, Math. Sem. Notes Kobe

Univ., **8** (1980), 557-560.

[Kf] L.H.Kauffman: On Knots, Annals of Math. Studies **115**, Princeton Univ. Press, Princeton, 1987.

[K-S-S] A.Kawauchi, T.Shibuya and S.Suzuki: *Descriptions on surfaces in four-space, I. Normal forms*, Math. Sem. Notes Kobe Univ., **10** (1982), 75-125; *ibid, II. Singularities and cross-sectional links*, ibid. **11** (1983),31-69.

[K-W] M.Kervaire and C.Weber: *A survey of multidimensional knots,* Lecture Notes in Math., **685** (1978), 61-134, Springer-Verlag.

[Li 1] R.A.Litherland: *Deforming twist-spun knots*, Trans. Amer. Math. Soc., **250** (1979), 311-331.

[Li 2] R.A.Litherland: *The second homology of the knot group of a knotted surface*, Quart. J. Math. Oxford (2), **32** (1981), 425-434.

[Lv 1] C.Livingston: *Stably irreducible surfaces in* S^4, Pacific J. Math., **116** (1) (1985), 77-84.

[Lv 2] C.Livingston: *Indecomposable surfaces in 4-space*, Pacific J. Math., **132** (2) (1988), 371-378.

[Mi 1] J.W.Milnor: *On the 3-dimensional Brieskorn manifolds* $M(p,q,r)$, in Knot Groups and 3-Manifolds (L.Neuwirth (ed.)), Annals of Math. Studies **84**, Princeton Univ. Press, Princeton, 1975, 175-225.

[Mi 2] J.W.Milnor: Singular Points of Complex Hypersurfaces, Annals of Math. Studies **61**, Princeton Univ. Press, Princeton, 1968.

[Mo] J.M.Montesinos: *On twins in the four-sphere I*, Quart. J. Math. Oxford (2), **34** (1983), 171-199; *ibid II*, ibid, **35** (1984), 73-83.

[Mr] K.Morichi: *On punctured prism manifolds in 4-space*, Master Thesis, Kobe University, 1985.

[Na] Y.Nakanishi: *A note on the Zeeman Theorem*, preprint, Kobe University (1990).

[N-T] Y.Nakanishi and M.Teragaito: *2-knots from a view of moving picture*, Kobe J. Math., **7** (1990) (to appear).

[Or] P.Orlik: Seifert manifolds, Lecture Notes in Math., **291**, Springer-Verlag, 1972.

[Pl] S.P.Plotnick: *Fibred knots in S^4 - twisting, spinning, rolling, surgery, and branching*, in Four-Manifold Theory (C.McA.Gordon and R.C.Kirby (ed.)), Contemporary Math., **35**, Amer. Math. Soc., Providence (1984), 437-459.

[P-R] T.M.Price and D.M.Roseman: *Embeddings of the projective plane in four space*, preprint.

[Ro] D.Rolfsen: Knots and Links, Math. Lecture Series **7**, Publish or Perish Inc., Berkeley, 1976.

[Ru] J.H.Rubinstein: *On 3-manifolds that have finite fundamental group and contain Klein bottles*, Trans. Amer. Math. Soc., **251** (1979), 129-137.

[Su] S.Suzuki: *Knotting problems of 2-spheres in 4-sphere*, Math. Sem. Notes Kobe Univ., **4** (1976), 241-371.

[Te 1] M.Teragaito: *Twisting and rolling*, Proc. of a symp. held at the R.I.M.S., Kyoto Univ., **636** (1988), 153-169.

[Te 2] M.Teragaito: *On fibered knots*, Master Thesis, Kobe University, 1988.

[Te 3] M.Teragaito: *Fibered 2-knots and lens spaces*, Osaka J. Math., **26** (1989), 57-63; Addendum ibid., 953.

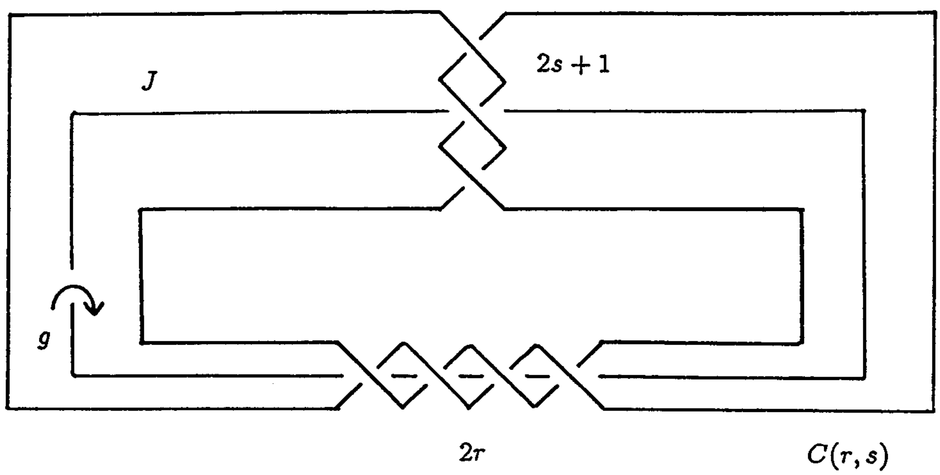
[Te 4] M.Teragaito: *Twisting symmetry-spins of 2-bridge knots*, Kobe J. Math., **6** (1989), 117-126.

[Te 5] M.Teragaito: *Twisting symmetry-spins of pretzel knots*, Proc. Japan Acad. Ser. A **66** (1990), 179-183.

[Te 6] M.Teragaito: *Symmetry-spun tori in the four-sphere*, The Proceedings of the International Conference on Knot Theory and Related Topics, 1990 (to appear).

[Yo] K.Yoshikawa: *On 2-knot groups with the finite commutator subgroup*, Math. Sem. Notes Kobe Univ., **8** (1980), 321-330.

[Ze] E.C.Zeeman: *Twisting spun knots*, Trans. Amer. Math. Soc., **115** (1965), 471-495.



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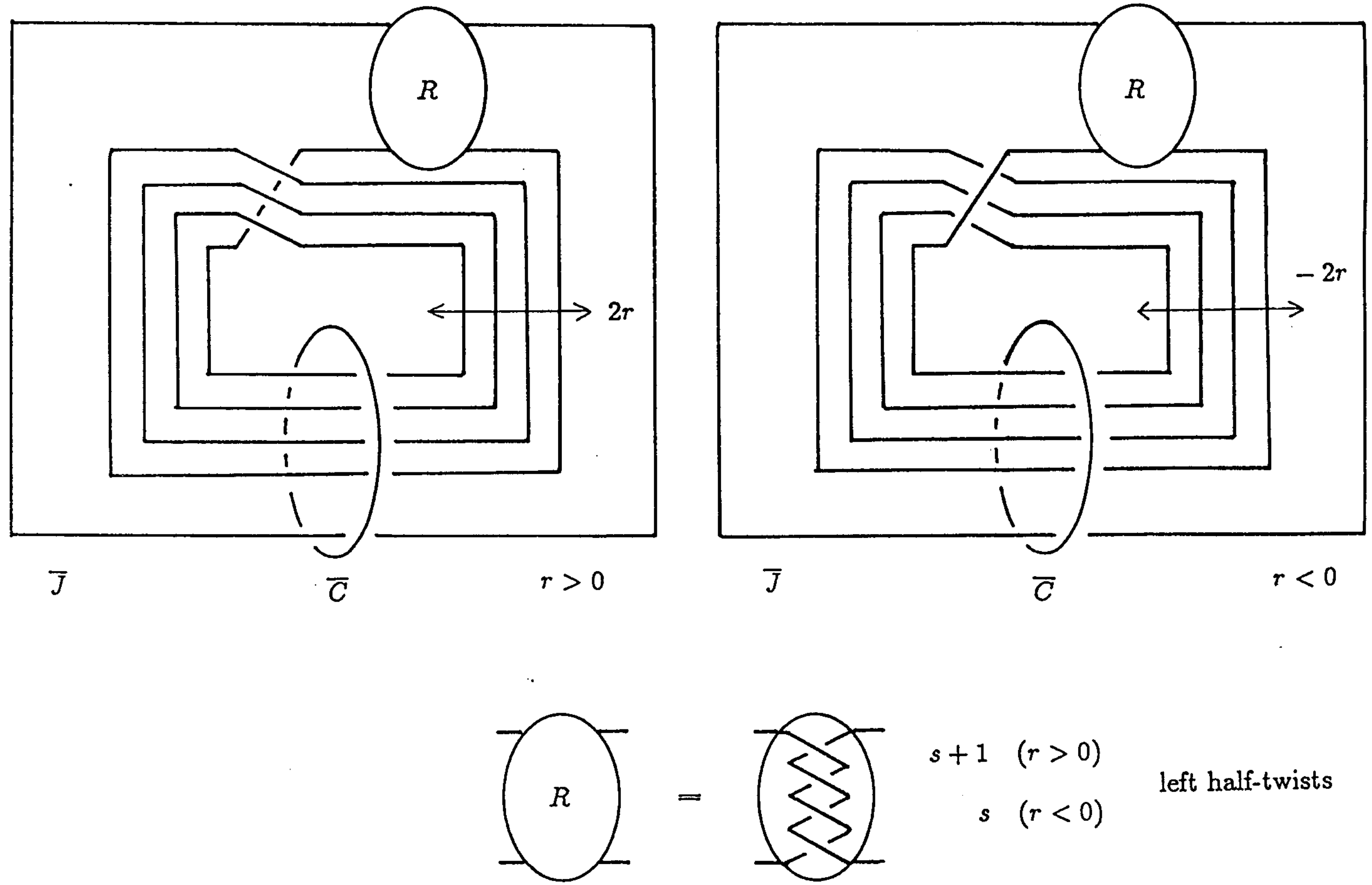

Fig.2

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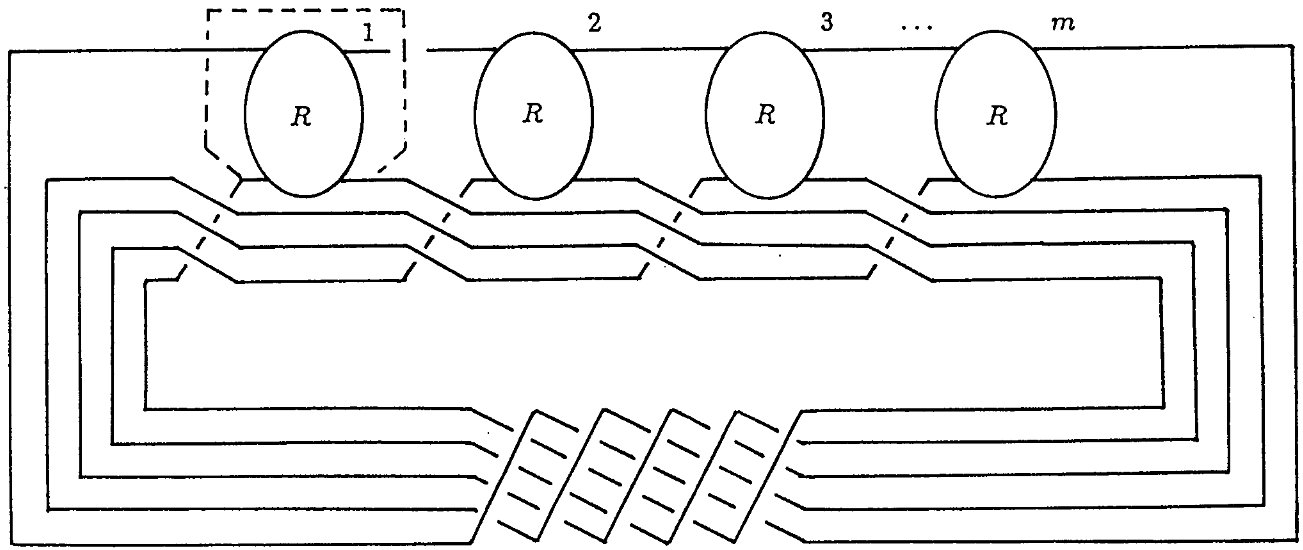

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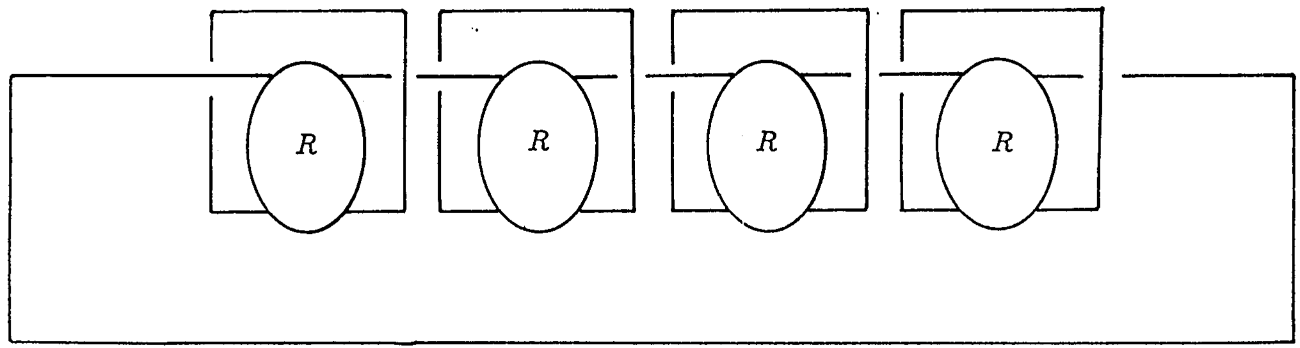

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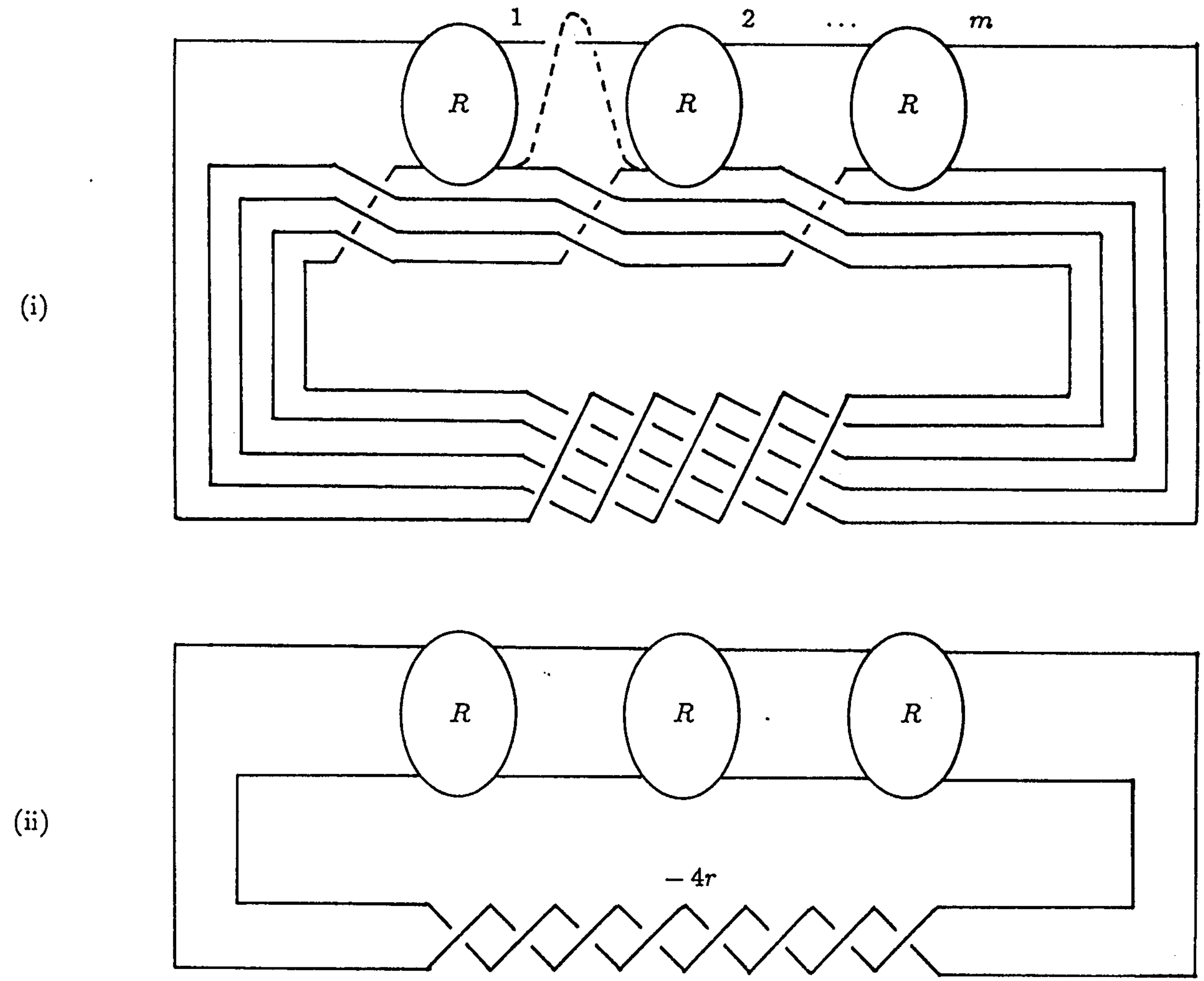

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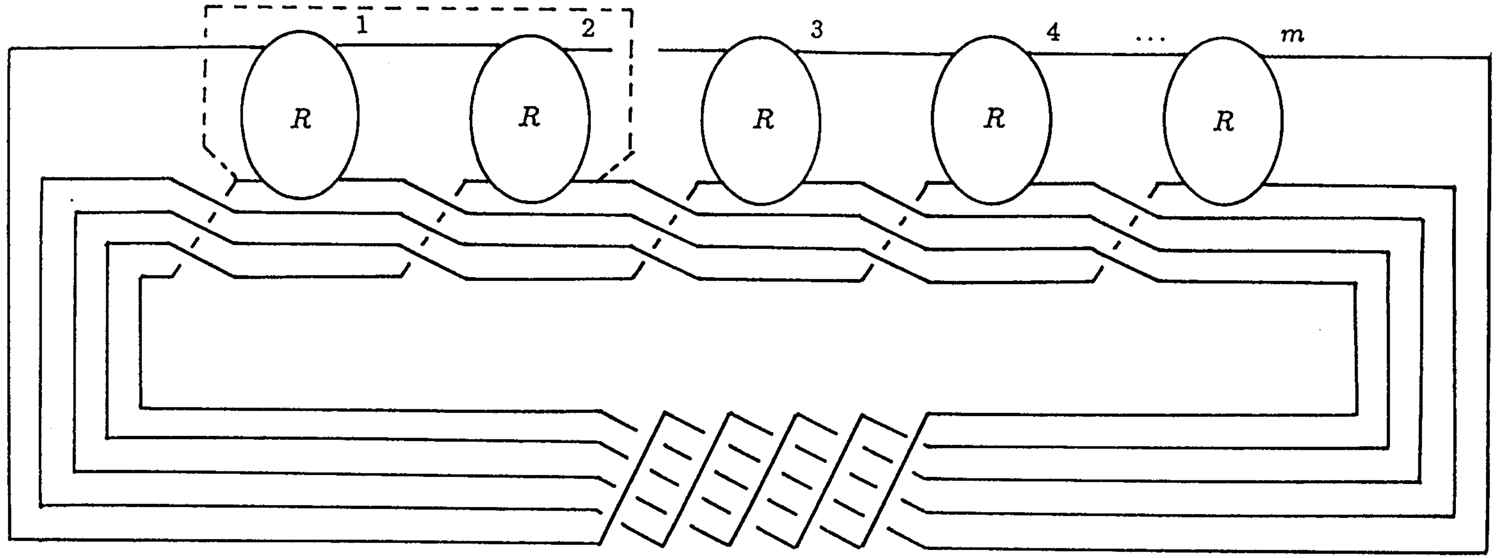

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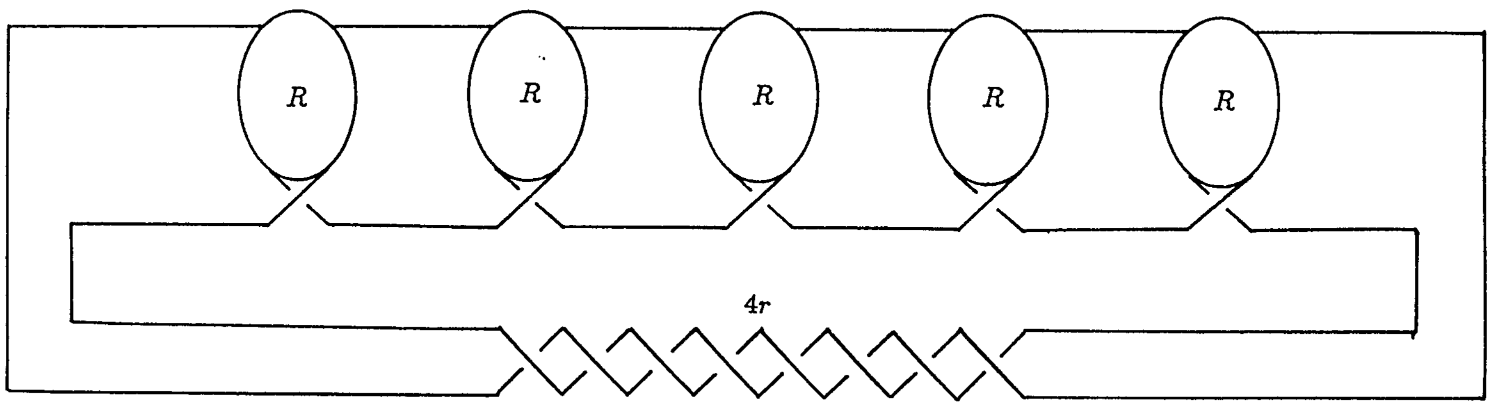

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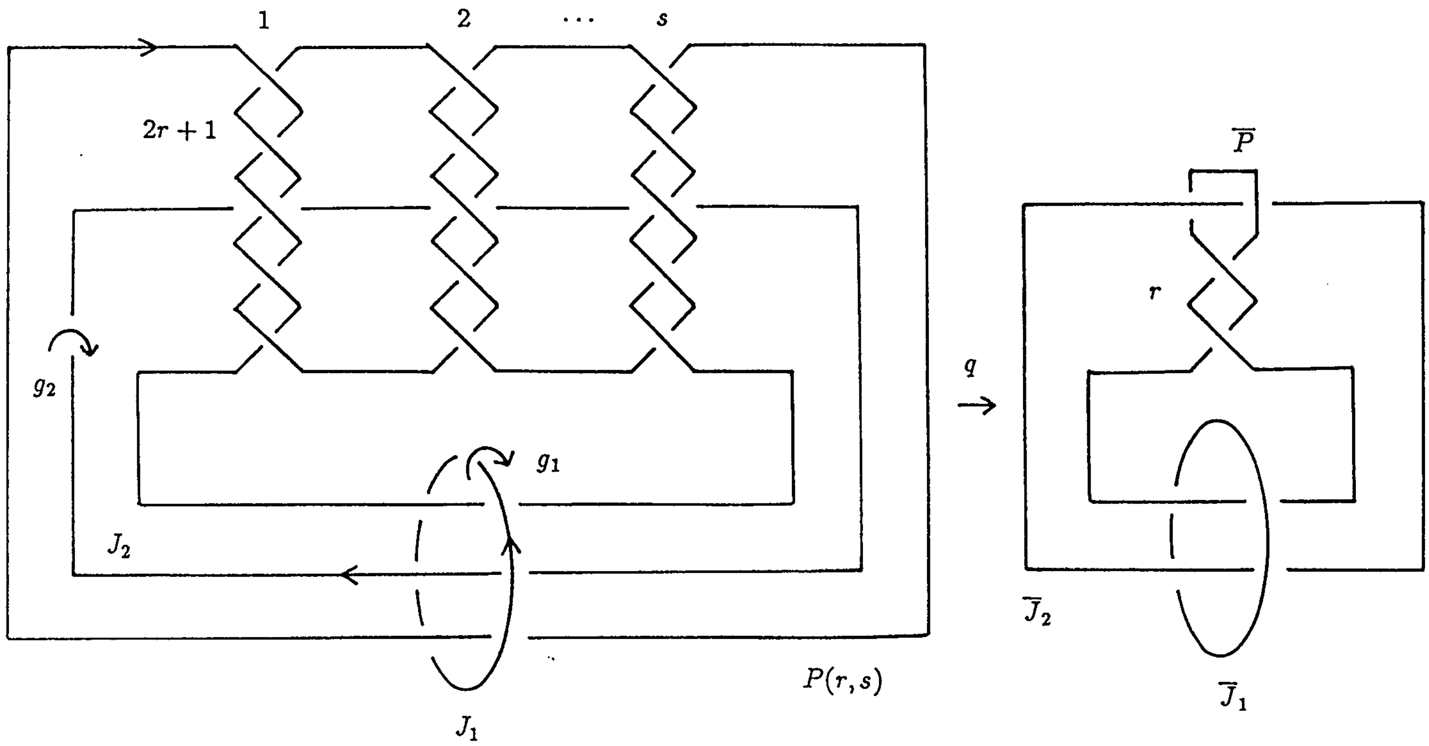

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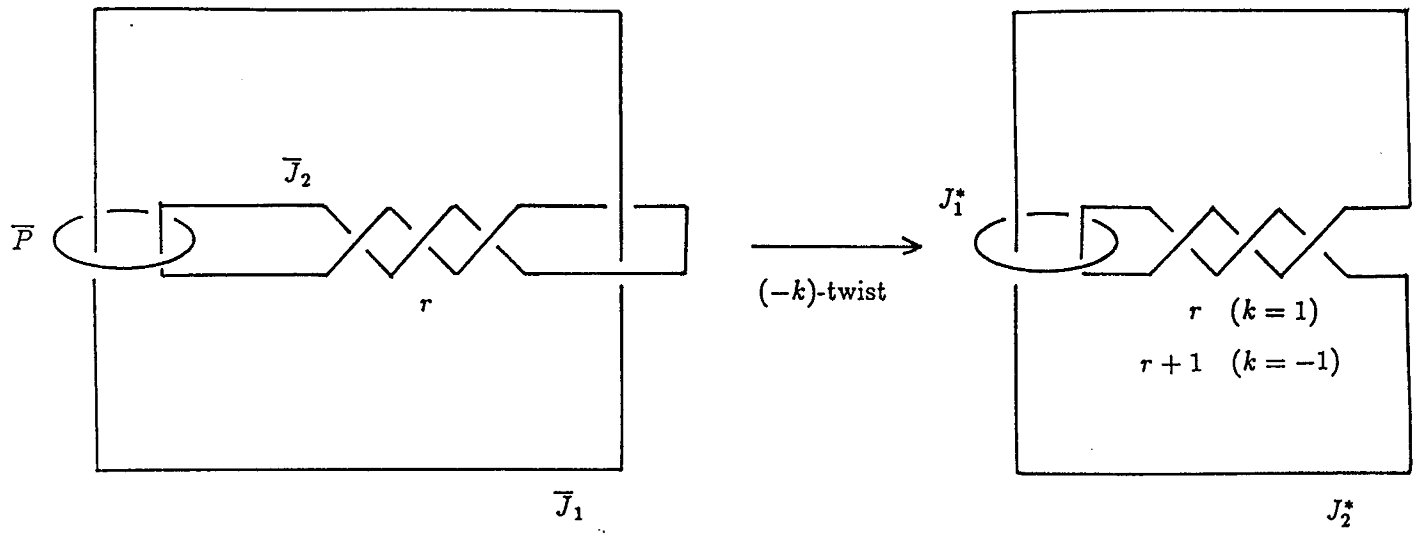

$\overline{P}$
$\overline{J}_2$
r
$\overline{J}_1$
$(-k)$-twist
J_1^*
$r \quad (k=1)$
$r+1 \quad (k=-1)$
J_2^*

Fig.7

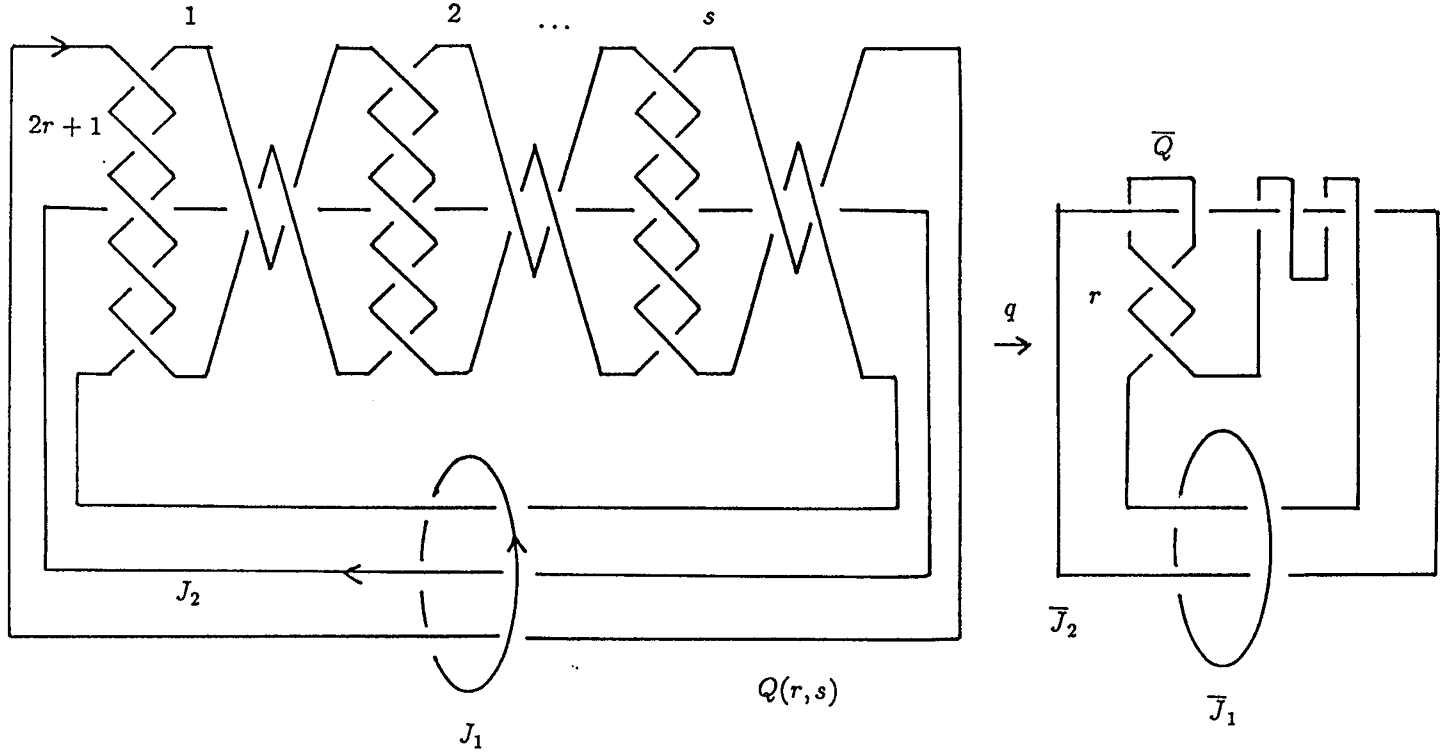
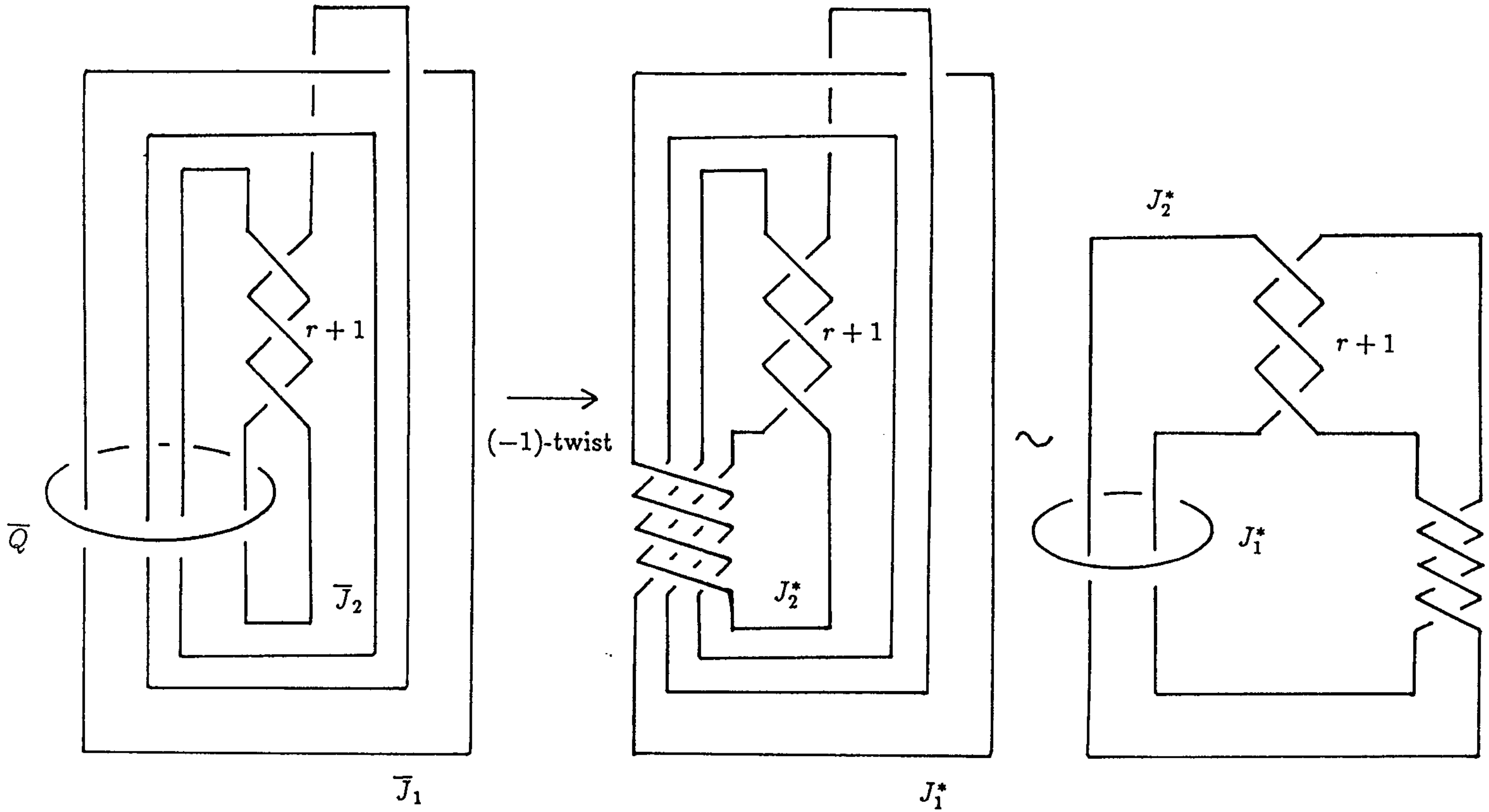


Fig.8



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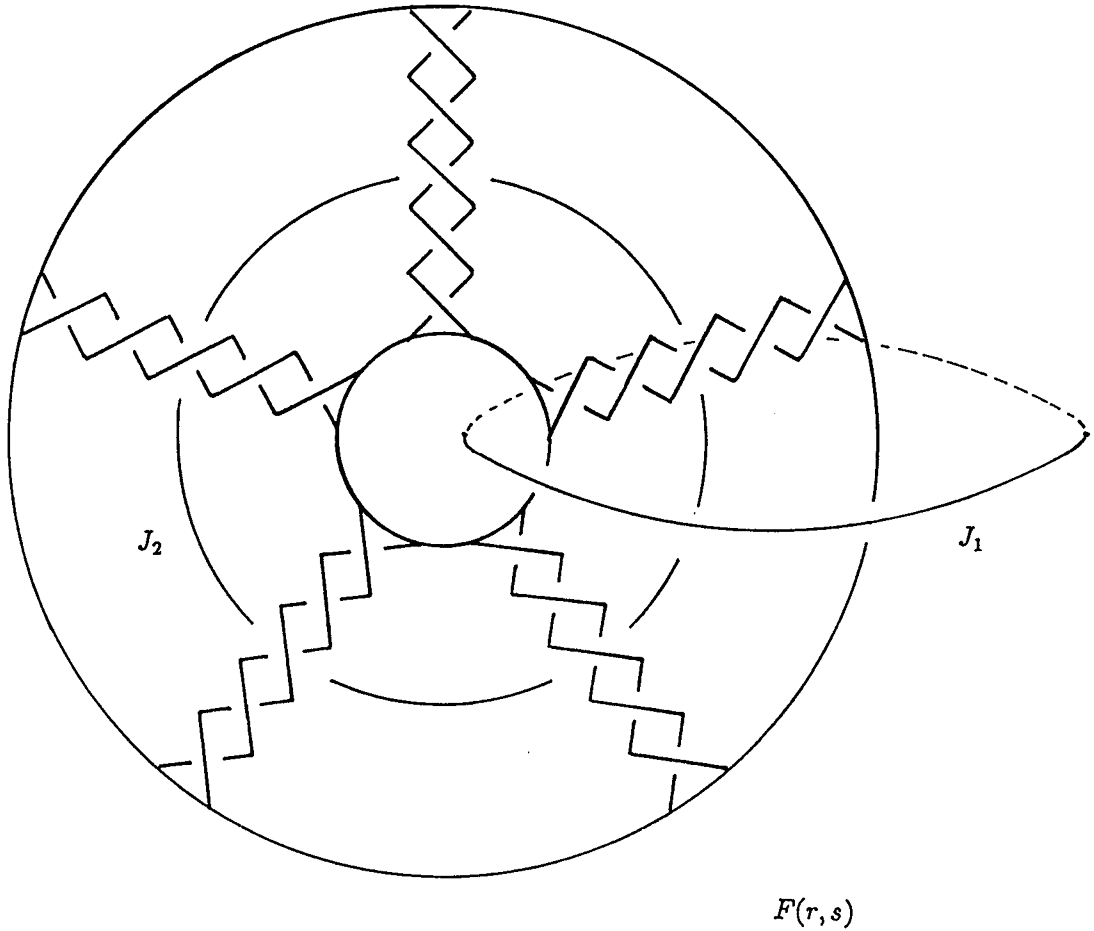

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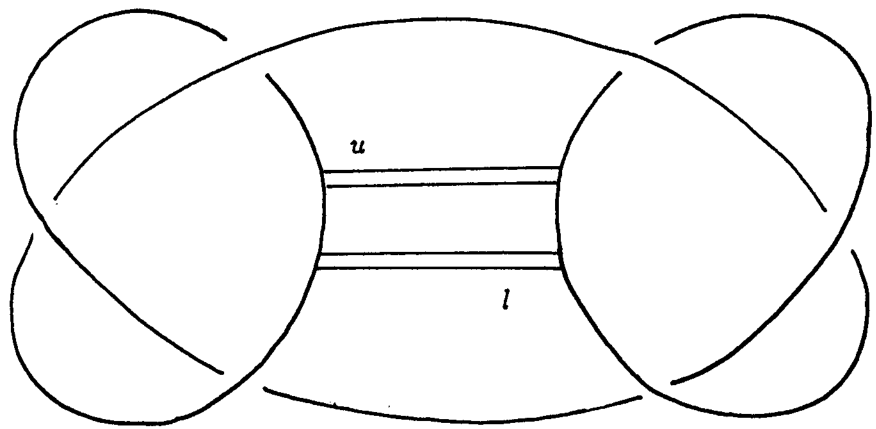

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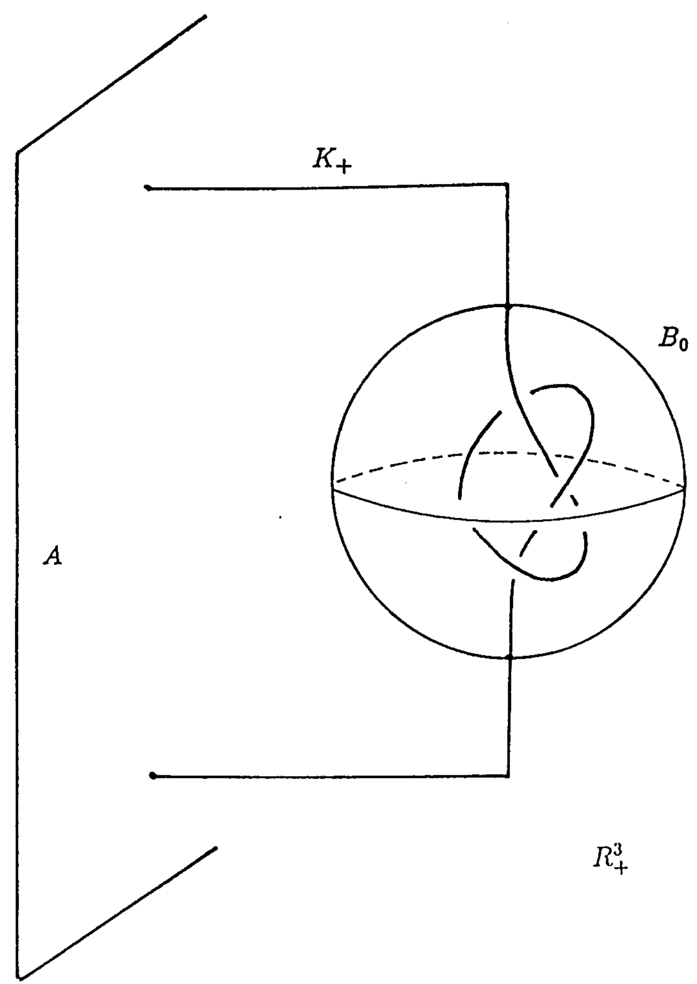

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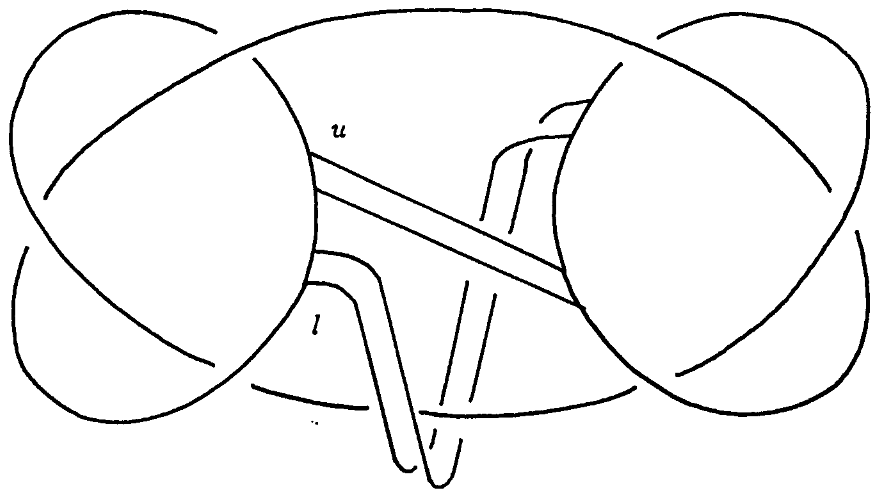

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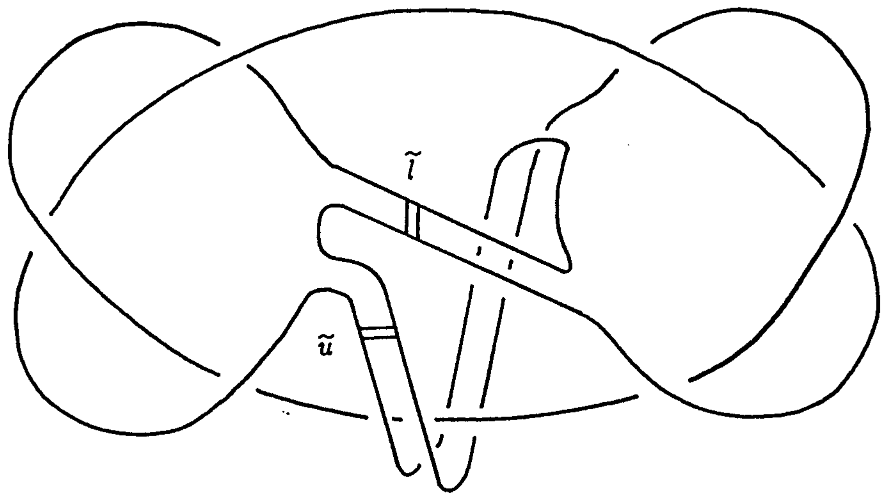

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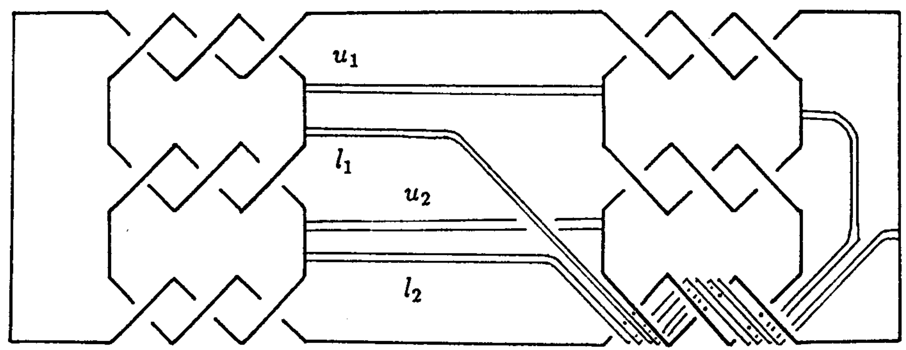

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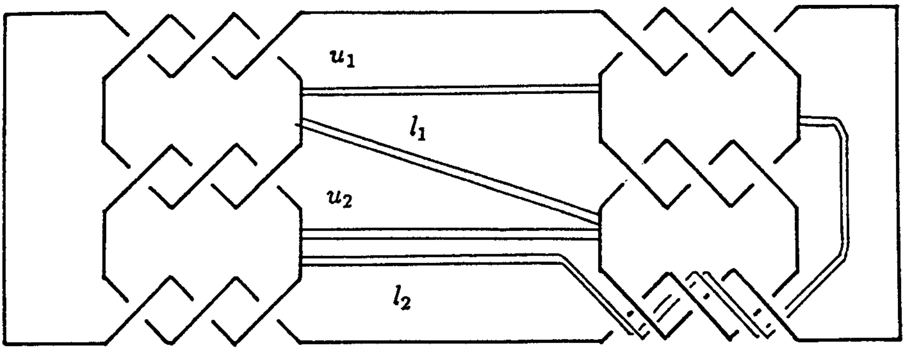

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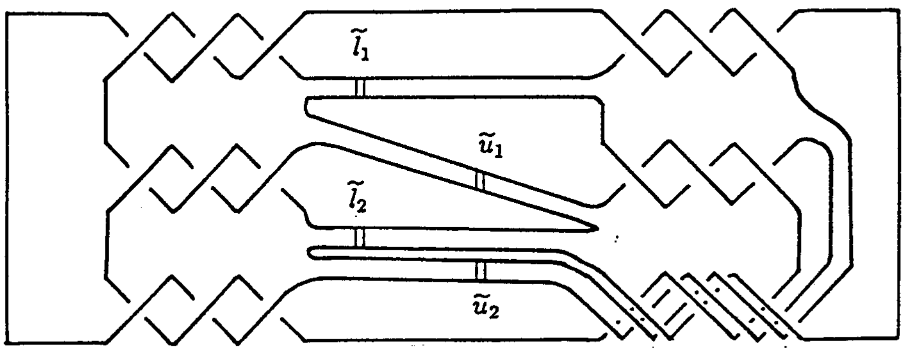

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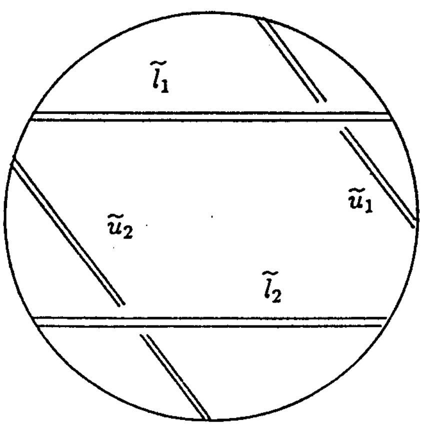

Fig.16