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博士論文

Congruence properties of Apéry numbers , binomial coefficients and Fourier coefficients of certain η-products

（アペリー数、二項係数とあるエータ積の
フーリェ係数の合同の性質について）

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Contents

Congruence properties of Apéry numbers , binomial coefficients and Fourier coefficients of certain η-products

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§1. Introduction.

Let, for any $n\geq 0$,

$$a(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \quad , \quad u(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \quad .$$

R.Apéry's proof of the irrationality of $\zeta(2)$ and $\zeta(3)$ made use of these numbers, respectively (see van-der-Poorten [23]). So we call these numbers *Apéry numbers*. The first few values are given by $a(0)=1, a(1)=3, a(2)=19, a(3)=147, a(4)=1251$ and $u(0)=1, u(1)=5, u(2)=73, u(3)=1445, u(4)=33001$.

So far, many properties of $a(n)$ and $u(n)$ were discovered by several people. Chowla-Cowles-Cowles[7] first considered congruences for $u(n)$, and some elementary congruences were proved by Gessel[11], Mimura[22] and Beukers[4].

Moreover, these numbers are concerned with the theory of differential equations, algebraic geometry, automorphic forms and formal groups. Stienstra-Beukers[24] showed that Apéry numbers were related to Picard-Fuchs equations associated to certain algebraic variety(see Beukers-Peters[6], too), and they proved some congruences using the theory of formal groups. Recently, Koike[20] showed some relations between Apéry numbers and hypergeometric series over finite fields.

At first, in Section 2, we will collect the results for the Apéry numbers in Beukers [2],[5] by way of preparation.

In Section 3, we shall study about *super congruences* for the Apéry numbers. These are congruences modulo $p^r (r>1)$ which we can not prove using the usual method in the theory of formal groups. We shall prove the following congruences conjectured by Beukers[5]. Let $p \geq 3$ be a prime, and write

$$\sum_{n=1}^{\infty} \xi_n q^n = q \prod_{n=0}^{\infty} (1-q^{2n})^4 (1-q^{4n})^4 .$$

If $u(\frac{p-1}{2}) \not\equiv 0 \bmod p$ then

$$u(\tfrac{p-1}{2}) \equiv \xi_p \mod p^2 .$$

And, let $p \geq 5$ be a prime, and write

$$\sum_{n=1}^{\infty} \alpha_n q^n = q \prod_{n=0}^{\infty} (1-q^{4n})^6 .$$

Then

$$a(\tfrac{p-1}{2}) \equiv \alpha_p \mod p^2 .$$

For the more general statements see Theorem 3 and Theorem 4 of this paper. The most general statements conjectured by Beukers are still open. Our method is applicable to the mod p^2 determination of other numbers such as $v(n) = \sum_{k=0}^{\infty} \binom{n}{k}^3 (-1)^n$.

In Section 4, we shall study about the congruences between Fourier coefficients of certain modular forms and binomial coefficients $\binom{2f}{f}$ where $f = \frac{p-l}{k}$ is a integer, l and k are positive integers with $(k,l)=1$ and p is a prime $p \equiv l \bmod k$. The main result is the following congruence (see Theorem 6 of this paper). Let k and l be the above and put $m = 4l/k$. Write

$$\sum_{n=1}^{\infty} \gamma_n^{(k,l)} q^n = \eta(k\tau)^2 \eta(2k\tau)^{1+m} \eta(4k\tau)^{3-3m} \eta(8k\tau)^{2m-2} .$$

where $\eta(\tau) = q^{1/24} \prod_{n=0}^{\infty} (1-q^n)$ is the Dedekind η-function with $q=e^{2\pi i\tau}$and $Im\ \tau>0$. Then

$$\binom{2f}{f} \equiv (-1)^f \gamma_p^{(k,l)} \quad mod\ p \quad .$$

The numbers $\binom{2f}{f}$ are related to formal groups as the special case of the congruences of Atkin- Swinnerton-Dyer type. Some modular forms which appear in this section are non- holomorphic, so we can not use the theory of Hecke operators and we do not know about the properties of the coefficients $\gamma_n^{(k,l)}$. But we prove the new congruences of the Fourier coefficients of certain modular forms in Corollaries 1 and 2. For example,

$$l\ \gamma_p^{(k,l)} \equiv -2(2l+k)\ \gamma_p^{(k,k+l)} \ mod\ p\ .$$

In Section 5, we shall prove the following congruences of $u(\frac{p-l}{k})$ applying to arguments in Section 4. Let k, l be positive integers with $(k,l)=1$ and write

$$\sum_{n=1}^{\infty} \xi_n^{(k,l)} q^n = \eta(k\tau)^{m-2}\eta(2k\tau)^{10-m}\eta(3k\tau)^{6-m}\eta(6k\tau)^{m-6}$$
$$- 9\, \eta(k\tau)^{m-6}\eta(2k\tau)^{6-m}\eta(3k\tau)^{10-m}\eta(6k\tau)^{m-2} .$$

with $m=12l/k$. Then ,for any prime $p \equiv l \bmod k$,

$$u(\frac{p-l}{k}) \equiv \xi_p^{(k,l)} \mod p$$

(see Theorem 8). But, we do not know the details of the properties of $\xi_p^{(k,l)}$.

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§2. Some facts.

In this section, we mainly describe the results obtained by Beukers[2],[3] and [5] by way of preparation of Sections 3,4 and 5. We may state about the numbers $u(n)$ as we can take the same method for the numbers $a(n)$.

Let

$$\mathfrak{U}(t) = \sum_{n=0}^{\infty} u(n)t^n$$

be the generating function of $u(n)$. The function $\mathfrak{U}(t)$ is the holomorphic solution around $t=0$ of the 3rd order linear differential equation

$$(2\text{-}1)\qquad (t^4 - 34t^3 + t^2)\frac{d^3y}{dt^3} + (2t^3 - 153t^2 + 3t)\frac{d^2y}{dt^2} + (7t^2 - 112t + 1)\frac{dy}{dt} + (t - 5)\,y = 0\ ,$$

because the numbers $u(n)$ satisfy the recurrence

$$(n+1)^3u(n+1) - (34n^3+51n^2+27n+5)u(n) + n^3u(n-1) = 0\ .$$

Let $y_0=\mathfrak{U}(t)$, y_1 and y_2 be solutions of (2-1). Then we see

$$(2\text{-}2)\qquad y_0 = \Phi_0^2\ ,\ y_1 = \Phi_0\Phi_1\ ,\ y_2 = \Phi_1^2\ .$$

where Φ_0 and Φ_1 are some solutions of the differential equation

$$(t^3-34t^2+1)\frac{d^2\Phi}{dt^2} + (2t^2-51t+1)\frac{d\Phi}{dt} + \frac{1}{4}(t-10)\Phi = 0\ .$$

By transformations $t = \dfrac{x(1-9x)}{1-x}$ and $\varphi = \sqrt{1-x}\,\Phi$, we have

$$(2\text{-}3)\qquad x(x-1)(9x-1)\frac{d^2\varphi}{dx^2} + (27x^2-20x-1)\frac{d\varphi}{dx} + 3(3x-1)\varphi = 0\ .$$

This is the Picard-Fuchs equation associated to the family of the elliptic curves

$$Y^2 + (1+x)XY - x(x-1)Y = X^3 - x(x-1)X^2 \quad . \tag{2-4}$$

Beukers and Stienstra[24] studied about the relations between the Picard-Fuchs equations and the modular forms.

Proposition 1.(Beukers and Stienstra) *Let $f(x)$ be a holomorphic solution of* (2-3) *around* $x=0$ *with* $f(0)=1$ *and put*

$$x(\tau) = \eta(\tau)^4\eta(2\tau)^{-8}\eta(3\tau)^{-4}\eta(6\tau)^8 \quad .$$

where $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ *is the Dedekind* η*-function with* $q=e^{2\pi i\tau}$ *and* $Im(\tau)>0$. *Then*

$$f(x(\tau)) = 1 + 3\sum_{k=1}^{\infty}\frac{\chi(k)q^k}{1-q^k} = E_1(\tau,\chi) \ ,$$

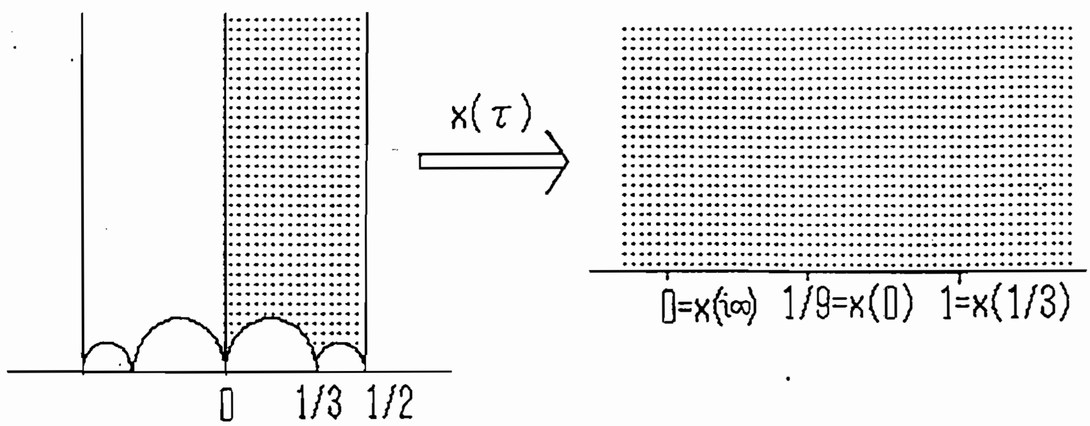
where $E_1(\tau,\chi)$ *denotes the Eisenstein series of weight* 1 *and* $\chi(k)$ *is the Diriclet charcter of modulo 6 with* $\chi(-1)=-1$.

We give a sketch of the proof of Proposition 1. Elliptic curves (2-4) are the Tate normal forms with a point(0,0) of order 6, and they are parametrized by the modular curve $\mathbb{H}/\Gamma_1(6)$ where

$$\Gamma_1(6) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod 6 \right\} \quad .$$

The function $x(\tau)$ is the generator of the function fields on $\Gamma_1(6)$

and maps the shaded open area in the picture below univalently onto the upper half plane and satisfies $x(i\infty)=0$, $x(0)=1/9$, $x(1/3)=1$, $x(1/2)=\infty$.



Now, put $\omega_1(\tau) = E_1(\tau,\chi)$ and $\omega_2(\tau) = \tau E_1(\tau,\chi)$. We can consider ω_1 and ω_2 as multivalued function on the x-plane via the mapping $\tau \longrightarrow x(\tau)$. We denote them by $\omega_1(x)$ and $\omega_2(x)$. After an analytic continuation along a closed path γ in $\mathbb{C}-\{0,1/9,1\}$ corresponding to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(6)$, ω_1 and ω_2 are changed by the transformation

(2-5) $$\begin{pmatrix} \omega_2(x) \\ \omega_1(x) \end{pmatrix} \xrightarrow{\ \gamma\ } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2(x) \\ \omega_1(x) \end{pmatrix} .$$

Now $\omega_1(x)$ and $\omega_2(x)$ satisfy the equation

(2-6) $$\begin{vmatrix} \omega_1 & \omega_1' \\ \omega_2 & \omega_2' \end{vmatrix} F'' - \begin{vmatrix} \omega_1 & \omega_1'' \\ \omega_2 & \omega_2'' \end{vmatrix} F' + \begin{vmatrix} \omega_1' & \omega_1'' \\ \omega_2' & \omega_2'' \end{vmatrix} F = 0$$

It is straightforward to see that

$$\begin{vmatrix} \omega_1 & \omega_1' \\ \omega_2 & \omega_2' \end{vmatrix} = \frac{(\omega_1)^2}{(dx/d\tau)} , \qquad \begin{vmatrix} \omega_1 & \omega_1'' \\ \omega_2 & \omega_2'' \end{vmatrix} = \frac{d}{dx}\begin{vmatrix} \omega_1 & \omega_1' \\ \omega_2 & \omega_2' \end{vmatrix}$$

and

$$\begin{vmatrix} \omega_1' & \omega_1'' \\ \omega_2' & \omega_2'' \end{vmatrix} = \left(\frac{dx}{d\tau}\right)^{-3}\{\ 2(d\omega_1/d\tau)^2 - \omega_1(d/d\tau)^2\omega_1\}\ ,$$

and these determinants are rational functions of x by (2-5).

Here we can check that $x^{-1}dx/d\tau$ is a modular form of weight 2 for $\Gamma_1(6)$ and

$$\frac{(\omega_1)^2}{(dx/d\tau)} = x^{-1}\ \frac{(\omega_1)^2}{x^{-1}dx/d\tau}$$

has simple poles at $\tau = i\infty$, 0 and 1/3, and a zero of order 3 at $\tau=1/2$.

Hence, we obtain

$$\begin{vmatrix} \omega_1 & \omega_1' \\ \omega_2 & \omega_2' \end{vmatrix} = \frac{(\omega_1)^2}{dx/d\tau} = \frac{c_1}{x(x-1)(9x-1)}$$

for some constant c_1, and

$$\begin{vmatrix} \omega_1' & \omega_1'' \\ \omega_2' & \omega_2'' \end{vmatrix} = \frac{c_2\ x + c_3}{x^2(x-1)^2(9x-1)^2}$$

in the same way. We can determine the constants c_1, c_2 and c_3 by comparing with

$$\omega(x) = 1 + 3x + 15x^2 + 93x^3 + 639x^4 + \cdots\ .$$

Hence we see that (2-3) equals (2-6). See Stienstra- Beukers[24], Beukers[2],[5] and Stiller[25]. □

The following proposition is the direct consequence of the above (see Beukers[5]).

Proposition 2.(Beukers) *Let*

$$t(\tau) = \eta(\tau)^{12}\eta(2\tau)^{-12}\eta(3\tau)^{-12}\eta(6\tau)^{12} .$$

Then

$$\mathfrak{U}(t(\tau)) = \frac{1}{24}\{2E_2(2\tau) - 3E_2(3\tau) - 5E_2(\tau) + 30E_2(6\tau)\}$$

where $E_2(\tau) = 1 + 24 \sum_{k=1}^{\infty} \sigma_1(k)q^k$ *is the Eisenstein series of weight* 2 *with* $\sigma_1(k) = \sum_{d|k} d$.

Moreover, let $\lambda(\tau) = \sqrt{t(2\tau)}$. *Then we have*

(2-7) $\qquad \mathfrak{U}(\lambda^2)d\lambda = \{\eta(2\tau)^4\eta(4\tau)^4 - 9\eta(6\tau)^4\eta(12\tau)^4\}dq/q$.

The following lemma is convenient for the proof of the congruences that is related to the theory of formal groups(see Beukers[5] and Stienstra-Beukers[24]).

Lemma 1. *Let* p *be a prime and*

$$\omega(t) = \sum_{n=1}^{\infty} b_n t^{n-1}dt$$

be a differential form with $b_n \in \mathbb{Z}_p$. *Let* $t(u) = \sum_{n=1}^{\infty} c_n u^n$ *with* $c_n \in \mathbb{Z}_p$, c_1 *is a p-adic unit , and suppose*

$$\omega(t(u)) = \sum_{n=1}^{\infty} d_n u^{n-1}du \qquad .$$

Then (2-8) *is equivarent to* (2-9) *for* $m, r \in \mathbb{N}$ *and* $\alpha_p, \beta_p \in \mathbb{Z}_p$, $p|\beta_p$:

(2-8) $$b(mp^r) - \alpha_p b(mp^{r-1}) + \beta_p b(mp^{r-2}) \equiv 0 \mod p^r \quad .$$

(2-9) $$d(mp^r) - \alpha_p d(mp^{r-1}) + \beta_p d(mp^{r-2}) \equiv 0 \mod p^r \quad .$$

Proof. Note that the congruences(2-8) are equivalent to

$$\omega(t) - \frac{\alpha_p}{p}\,\omega(t^p) + \frac{\beta_p}{p^2}\,\omega(t^{p^2}) = dF_1(t) \quad , \quad F_1(t) \in \mathbb{Z}_p[[t]] \ .$$

Since

$$t(u)^{np} = t(u^p)^n + np\ G_n(u) \qquad , \qquad G_n(u) \in \mathbb{Z}_p[[u]]$$

and

$$\frac{1}{p}\omega(t(u)^p) = \sum_{n=1}^{\infty} \frac{b_n}{np}\, d(t^{pn}) \quad ,$$

we see

$$\frac{1}{p}\omega(t(u)^p) = \sum_{n=1}^{\infty} \frac{b_n}{np}\, d(t(u^p)^n) + b_n dG_n(u)$$

$$= \frac{1}{p}\omega(t(u^p)) + dF_2(u) \quad , \quad F_2(u) \in \mathbb{Z}[[u]] \ .$$

Similarly

$$\frac{1}{p}\omega(t(u)^{p^2}) = \frac{1}{p}\omega(t(u^{p^2})) + dF_3(u) \quad , \quad F_3(u) \in \mathbb{Z}_p[[u]].$$

Hence (2-9) implies

$$\omega(t(u)) - \frac{\alpha_p}{p}\,\omega(t(u^p)) + \frac{\beta_p}{p^2}\,\omega(t(u^{p^2})) = dF_4(u), \quad F_4(u) \in \mathbb{Z}_p[[u]] \ .$$

Conversely, since c_1 is a p-adic unit, we can write

$$u(t) = \sum_{n=1}^{\infty} \tilde{c}_n\, t^n \quad , \quad \tilde{c}_n \in \mathbb{Z}_p \ .$$

Thus we have completed the proof. □

Now, since $\eta(2\tau)^4\eta(4\tau)^4 = \sum_{n=1}^{\infty} \xi_n q^n = q\prod_{n=0}^{\infty}(1-q^{2n})^4(1-q^{4n})^4$ is an unique cusp form of weight 4 for $\Gamma_0(8)$, its corresponding Dirichlet series has Euler product

$$\sum_{n=1}^{\infty}\frac{\xi_n}{n^s} = \prod_{p:\mathrm{odd}}\left(1 - \xi_p\, p^{-s} + p^{3-2s}\right)^{-1} .$$

Let

$$\eta(2\tau)^4\eta(4\tau)^4 - 9\eta(6\tau)^4\eta(12\tau)^4 = \sum_{n=1}^{\infty}\xi(n)q^n .$$

Then

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\xi(n)}{n^s} &= \sum_{n=1}^{\infty}\frac{\xi_n}{n^s} - 3^{2-s}\sum_{n=1}^{\infty}\frac{\xi_n}{n^s}\\ &= (1-3^{2-s})\prod_{p:\mathrm{odd}}\left(1 - \xi_p\, p^{-s} + p^{3-2s}\right)^{-1} .\end{aligned}$$

Hence, for all odd prime p,

$$\xi(mp^r) - \xi_p\,\xi(mp^{r-1}) + p^3\,\xi(mp^{r-2}) \equiv 0 \mod p^r .$$

Combining Lemma 1 and Proposition 2 (2-7), we can obtain the following theorem (see Beukers[5]).

Theorem 1. (Beukers) *Let $p\geq 3$ be a prime, and write*

$$\sum_{n=1}^{\infty}\xi_n q^n = q\prod_{n=0}^{\infty}(1-q^{2n})^4(1-q^{4n})^4 . \tag{2-10}$$

Let $m, r\in\mathbb{N}$, m odd, then we have

$$u\left(\frac{mp^r-1}{2}\right) - \xi_p u\left(\frac{mp^{r-1}-1}{2}\right) + p^3u\left(\frac{mp^{r-2}-1}{2}\right) \equiv 0 \mod p^r . \tag{2-11}$$

In the case of the numbers $a(n)$, the generating function

$$\mathcal{A}(t) = \sum_{n=0}^{\infty} a(n) t^n$$

is a holomorphic solution of the Picard-Fuchs equation

$$t(t^2-11t-1)\frac{d^2F}{dt^2} + (3t^2-22t-1)\frac{dF}{dt} + (t-3)F = 0$$

associated to the family of elliptic curves

$$Y^2 = X^3 + (t^2+6t+1)X^2 + 8t(t+1)X + 16t^2. \tag{2-12}$$

Therefore, we can prove the following theorem in the same way. See Beukers[2] and Stienstra-Beukers[24].

Theorem 2. (Beukers and Stienstra) *Let $p \geq 3$ be a prime, and write*

$$\sum_{n=1}^{\infty} \alpha_n q^n = q \prod_{n=0}^{\infty} (1-q^{4n})^6 \quad . \tag{2-13}$$

Let $m, r \in \mathbb{N}$, m odd, then we have

$$a(\frac{mp^r-1}{2}) - \alpha_p a(\frac{mp^{r-1}-1}{2}) + (-1)^{\frac{p-1}{2}} p^2 a(\frac{mp^{r-2}-1}{2}) \equiv 0 \mod p^r \quad . \tag{2-14}$$

§3. Super Congruence for the Apéry Numbers.

Let $\{w(n)\}_{n=1}^{\infty}$ be a sequence of rational or p-adic integers. We will consider the congruences

$$w(mp^{r}) \equiv a\ w(mp^{r-1}) \qquad \bmod p^{\kappa r}$$

where κ, m and r are positive integers and a is a p-adic integer. If $\kappa=1$ then these congruences arise from the theory of formal groups (see Hazewinkel[13], Stienstra-Beukers[24]). In the cases of $\kappa>1$, we call these congruences *super congruences* (see Coster[10]). In this section, we will treat the super congruences for the Apéry numbers $a(n)$ and $u(n)$, i.e., we shall prove that the congruences in Theorem 1 and Theorem 2 hold mod $p^{\kappa r}$ in the case of $\kappa=2>1$ and $r=1$.

Theorem 3. *Let $p\geq 5$ be a prime and $m\in\mathbb{N}$, m odd, and write*

$$\sum_{n=1}^{\infty} \alpha_n q^n = q \prod_{n=0}^{\infty} (1-q^{4n})^6 .$$

Then we have

$$a(\tfrac{mp-1}{2}) - \alpha_p\ a(\tfrac{m-1}{2}) \equiv 0 \qquad \bmod p^2 .$$

Theorem 4. *Let $p\geq 3$ be a prime and $m\in\mathbb{N}$, m odd, and write*

$$\sum_{n=1}^{\infty} \xi_n q^n = q \prod_{n=0}^{\infty} (1-q^{2n})^4(1-q^{4n})^4 .$$

If $u(\frac{p-1}{2})\not\equiv 0 \bmod p$ then

$$u(\tfrac{mp-1}{2}) - \xi_p\, u(\tfrac{m-1}{2}) \equiv 0 \qquad \bmod p^2 \quad .$$

F.Beukers informed me that Theorem 3 is proved by L.Van Hamme[12] in the cases of $p\equiv 1 \bmod 4$ using properties of the p-adic gamma function. We prove the general case involving $p\equiv 3 \bmod 4$ by entirely different method.

In Theorem 4, $u(\frac{p-1}{2}) \equiv 0 \bmod p$ for p=11, 3137 if p<100000. But these cases hold, too.

However, in the cases of $r>2$, these super congruences are still open.

3-1. Congruences of a(n).

The numbers $a(n)$ satisfy the recurrence

(3-1) $\qquad (n+1)^2 a(n+1) = (11n^2+11n+3)a(n) + n^2 a(n-1) \qquad n\geq 1$.

We know the following result. Let p be an odd prime, and $m\geq 0$, then

(3-2) $\qquad a(mp) \equiv a(m) \quad \bmod p^2$,

(3-3) $\qquad a(p-1) \equiv 1 \quad \bmod p^2$.

By (3-1),(3-2) and (3-3), we have $a(p-2) \equiv -3+5p \bmod p^2$ and $a(p+1) \equiv 9+15p \bmod p^2$.

Proposition 3. *Let $m\geq 0$, $n\geq 0$ and $m+n=p-1$. Then*

$$a(m) \equiv (-1)^m a(n) \mod p .$$

Proof. We proceed by induction on m to show that $a(m)\equiv(-1)^m a(p-m-1)$ mod p. From the above result, $a(0)\equiv a(p-1)\equiv 1$ mod p and $a(1)\equiv -a(p-2)\equiv 3$ mod p. Let $0<m<p-1$. From the recurrence (3-1),

$$\begin{aligned}
(m+1)^2 a(m+1) &= (11m^2+11m+3)a(m) + m^2 a(m-1) \\
&\equiv \{11(p-m)^2-11(p-m)+3\}a(m) + (p-m)^2 a(m-1) \\
&\equiv \begin{cases} -\{11(p-m)^2-11(p-m)+3\}a(p-m-1) + (p-m)^2 a(p-m) & \text{if } m : \text{odd} \\ \{11(p-m)^2-11(p-m)+3\}a(p-m-1) - (p-m)^2 a(p-m) & \text{if } m : \text{even} \end{cases} \\
&\equiv \begin{cases} (m+1)^2 a(p-m-2) & \text{if } m : \text{odd} \\ -(m+1)^2 a(p-m-2) & \text{if } m : \text{even} \end{cases} \mod p . \quad \square
\end{aligned}$$

Proposition 4. *For all primes p, $n\geq 0$ and $0\leq m\leq p-1$, we have*

$$a(np+m) \equiv a(m)a(n) \mod p .$$

Proof. We shall need Lucas' congruence

$$\binom{a+pb}{c+pd} \equiv \binom{a}{c}\binom{b}{d} \mod p$$

for $0\leq a, c<p$, and

$$\binom{(a+pb)+(c+pd)}{c+pd} \equiv \binom{a+c}{c}\binom{b+d}{d} \mod p .$$

Then for $0\leq m<p$ we have

$$a(m+pn) = \sum_{k=0}^{m+pn} \binom{m+pn}{k}^2 \binom{m+pn+k}{k}$$

$$= \sum_{i=0}^{p-1} \sum_{j=0}^{n} \binom{m+pn}{i+pj}^2 \binom{m+pn+i+pj}{i+pj}$$

$$\equiv \sum_{i=0}^{p-1} \sum_{j=0}^{n} \binom{m}{i}^2 \binom{n}{j}^2 \binom{m+i}{i} \binom{n+j}{j} \quad \bmod p$$

$$= \{ \sum_{i=0}^{m} \binom{m}{i}^2 \binom{m+i}{i} \} \{ \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{j} \}$$

$$= a(m)a(n) \quad . \quad \square$$

3-2. Congruences of b(n).

Let $b(0)=0$ and, for any $n\geq 1$,

$$b(n) = \sum_{k=1}^{n} \binom{n}{k}^2 \binom{n+k}{k} \left[\frac{2}{n-k+1} + \cdots + \frac{2}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+k} \right] .$$

These numbers are (differential) of $a(n)$ and they take important parts in the congruence of mod p^2 as shown in Gessel[11,Theorem 4].

Proposition 5. *The numbers $b(n)$ satisfy the recurrence*

$$(3\text{-}4) \qquad (n+1)^2 b(n+1) = (11n^2+11n+3)b(n) + n^2 b(n-1)$$
$$- 2(n+1)a(n+1) + 11(2n+1)a(n) + 2na(n-1) ,$$

and for all primes $p\geq 3$, $n\geq 0$ and $0\leq m\leq p-1$, we have

$$a(np+m) \equiv \{a(m)+pnb(m)\}a(n) \quad \bmod p^2 \quad .$$

Proof. Let

$$B_{n,k} = (k^2+3(2n+1)k-11n^2-9n-2) \binom{n}{k}^2 \binom{n+k}{k} H_{n,k} + (6k-22n-9) \binom{n}{k}^2 \binom{n+k}{k},$$

and $$H_{n,k} = \frac{2}{n-k+1} + \cdots + \frac{2}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+k} \quad ,$$

then we have

$$B_{n,k}-B_{n,k-1}=(n+1)^2\binom{n+1}{k}^2\binom{n+1+k}{k}H_{n+1,k}-(11n^2+11n+3)\binom{n}{k}^2\binom{n+k}{k}H_{n,k}$$
$$-n^2\binom{n-1}{k}^2\binom{n-1+k}{k}H_{n-1,k}+2(n+1)\binom{n+1}{k}^2\binom{n+1+k}{k}$$
$$-11(2n+1)\binom{n}{k}^2\binom{n+k}{k}-2n\binom{n-1}{k}^2\binom{n-1+k}{k}.$$

Taking summation from 1 to $n+1$ on k, recurrence(3-4) follows.

Next, we see that by Proposition 4 for fixed n and p, there exist numbers $\tilde{b}(k)$, with $\tilde{b}(0)=0$, such that

$$\text{(3-5)} \qquad a(k+pn) \equiv a(k)a(n) + p\,\tilde{b}(k) \mod p^2 ,$$

for $0 \le k < p$. Let us write the recurrence(3-1) in the form

$$\sum_{i=0}^{2} r_i(n)a(n-i) = 0 .$$

Note that this congruence holds for $n \ge 1$ if $a(-1)$ assigned any arbitrary value. Substituting $k+pn$ for n, and using (3-5) and Taylor's expansion, we have

$$\begin{aligned} 0 &= \sum_{i=0}^{2} r_i(k+pn)a(k+pn-i) \\ &\equiv \sum_{i=0}^{2}\{r_i(k) + p\,n\,r_i'(k)\}\{a(k-i)a(n) + p\,\tilde{b}(k-i)\} \mod p^2 \\ &\equiv p\sum_{i=0}^{2}\{r_i(k)\tilde{b}(k-i) + n\,r_i'(k)a(k-i)a(n)\} \quad \mod p^2 \end{aligned}$$

for $0<k<p$. Multiplying (3-4) by $na(n)$, we see

$$\sum_{i=0}^{2}\{r_i(k)nb(k-i)a(n) + n\,r_i'(k)a(k-i)a(n)\} = 0$$

with $b(0)=0$. Then since $r_0(k)=k^2$ is not divisible by p for $0<k<p$, we have $\tilde{b}(k)\equiv nb(k)a(n)$ mod p for $0\le k<p$. □

Proposition 6. *Let* $m\ge 0$, $n\ge 0$ *and* $m+n=p-1$. *Then*

$$b(m) \equiv (-1)^{m-1}b(n) \mod p \ .$$

Proof. From the congruence(3-2),(3-3) and Proposition 5, $b(0)\equiv -b(p-1)\equiv 0$ mod p. And by the definition of $b(n)$, $\mathrm{ord}_p b(p)\ge 0$. Then $b(1)\equiv b(p-2)\equiv 5$ mod p by the recurrence(3-4). By induction on m, similarly in Proposition 3, we can prove it. □

Theorem 5. *Let* $m\ge 0$, $n\ge 0$ *and* $m+n=p-1$. *Then*

$$a(m) \equiv (-1)^{m}\{ a(n) - pb(n) \} \mod p^2 \ .$$

Proof. It is clear from (3-2),(3-3) and Proposition 6 in the case of $m=0,1$. From the recurrences(3-1),(3-4) and the congruence

$$\begin{aligned}(m+1)^2a(m+1) \equiv \{11(p-m)^2-11(p-m)+3\}a(m) + (p-m)^2a(m-1)\\ - 11p\{2(p-m)-1\}a(m) - 2p(p-m)a(m-1) \qquad \mod p^2,\end{aligned}$$

it can be also shown by inductive method. □

3-3. Congruences of c(n).

If $p\equiv 3$ mod 4, we can not obtain the congruence of $b(\frac{p-1}{2})$ from

Proposition 6. Therefore we prepare the numbers $c(n)$.

Let, for all odd numbers $n \geq 1$,

$$c(n) = \sum_{k=1}^{n} \binom{n}{k}^3 (-1)^k \left[\frac{3}{n-k+1} + \cdots + \frac{3}{n} \right].$$

Let p be an odd prime. From the congruences $\binom{\frac{p-1}{2}+k}{k} \equiv (-1)^k \binom{\frac{p-1}{2}}{k}$ mod p

and $$\frac{1}{\frac{p-1}{2}-k+1} + \cdots + \frac{1}{\frac{p-1}{2}} + \frac{1}{\frac{p+1}{2}} + \cdots + \frac{1}{\frac{p-1}{2}+k} \equiv 0 \mod p$$

where $1 \leq k \leq \frac{p-1}{2}$, we have $3b(\frac{p-1}{2}) \equiv c(\frac{p-1}{2})$ mod p if $p \equiv 3$ mod 4.

Proposition 7. *The numbers $c(n)$ satisfy the recurrence*

$$n^2 c(n) = -3\{9(n-1)^2-1\}c(n-2) \tag{3-6}$$

for all odd numbers $n \geq 3$.

Proof. Let

$$f_n(k) = 2(14n^2+n-1) - 3(26n^2-n-3)k/n + 3(29n^2-3)k^2/n^2$$
$$-3(15n^2+2n-1)k^3/n^3 + 3(3n+1)k^4/n^3,$$

$$g_n(k) = 2(28n+1) - 3(26n^2+3)k/n^2 + 18k^2/n^3$$
$$+3(15n^2+14n-3)k^3/n^4 - 9(2n+1)k^4/n^4,$$

and $C_{n,k} = \frac{3}{n-k+1} + \cdots + \frac{3}{n}$.

Then we have

$$(n+1)^2 \binom{n+1}{k}^3 C_{n+1,k} + 3(9n^2-1)\binom{n-1}{k}^3 C_{n-1,k}$$
$$+ 2(n+1)\binom{n+1}{k}^3 + 54n\binom{n-1}{k}^3$$

$$= f_n(k)\binom{n}{k}^3 c_{n,k} + f_n(k-1)\binom{n}{k-1}^3 c_{n,k-1} + g_n(k)\binom{n}{k}^3 + g_n(k-1)\binom{n}{k-1}^3 .$$

We multiply both sides by $(-1)^k$. Taking summation from 1 to $n+1$ on k,

$$(3\text{-}7)\qquad (n+1)^2 c(n+1) + 3(9n^2-1)c(n-1) + 2(n+1)\sum_{k=0}^{n+1}\binom{n+1}{k}^3(-1)^k + 54n\sum_{k=0}^{n-1}\binom{n-1}{k}^3(-1)^k = 0 .$$

If $n\equiv 0 \bmod 2$, two latter summations are equal to 0. □

The numbers $c(n)$ satisfy the recurrence(3-7) if $n\equiv 1 \bmod 2$.

Proposition 8. *Let* $p\equiv 3 \bmod 4$ *be a prime, then we have*

$$c\left(\frac{p-1}{2}\right) \equiv 0 \mod p .$$

Proof. It is trivial if $p=3$. If $p\equiv 7 \bmod 12$ then $\frac{p+2}{3}$ is odd. By (3-6), we have

$$\left(\frac{p+2}{3}\right)^2 c\left(\frac{p+2}{3}\right) + 3\left\{9\left(\frac{p-1}{3}\right)^2-1\right\}c\left(\frac{p-4}{3}\right) = 0 .$$

Then $c(\frac{p+2}{3})\equiv 0 \bmod p$. Hence, $c(n)\equiv 0 \bmod p$ for $\frac{p+2}{3}\le n\le p-2$ and n odd. If $p\equiv 11 \bmod 12$ then $\frac{p+4}{3}$ is odd. Therefore it can be proved in the same way. □

3-4. Proof of Theorem 3.

Beukers and Stienstra showed that the generating function of $a(n)$ is a holomorphic solution of the Picard-Fuchs equation associated to the family of elliptic curves(2-12). From this argument and the ζ-function of a certain K3-surface, they proved Theorem 2 (see Beukers[2] and Stienstra-Beukers[24]). Moreover, we know that the right hand side of (2-13) is equal to $\eta(4z)^6$ with $q=e^{2\pi iz}$, $Im(z)>0$ (where $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind η-function). From the Jacobi-Macdonald formula, we see

$$\alpha_p = \begin{cases} 4a^2-2p & \text{if } p\equiv 1 \bmod 4 \text{ and } p=a^2+b^2, \quad a\equiv 1 \bmod 2 \\ 0 & \text{if } p\equiv 3 \bmod 4 . \end{cases}$$

Hence if $p\equiv 1 \bmod 4$ then $\alpha_p \not\equiv 0 \bmod p$. According to Theorem 2, if $m=1$ and $r=1$ then $a(\frac{p-1}{2})\equiv\alpha_p\not\equiv 0 \bmod p$.

Let us prove Theorem 3 using congruences of $a(n)$, $b(n)$, c(n), and Theorem 2.

If $p\equiv 1 \bmod 4$ then $\frac{p-1}{2}$ is even. From Proposition 6, $b(\frac{p-1}{2})\equiv -b(\frac{p-1}{2}) \bmod p$. Hence $b(\frac{p-1}{2}) \equiv 0 \bmod p$. Then $a(\frac{mp^2-1}{2})\equiv a(\frac{mp-1}{2})a(\frac{p-1}{2}) \bmod p^2$ and $a(\frac{mp-1}{2})\equiv a(\frac{m-1}{2})a(\frac{p-1}{2}) \bmod p^2$. Putting $r=2$ in Theorem 2, $a(\frac{mp^2-1}{2})\equiv\alpha_p a(\frac{mp-1}{2}) \bmod p^2$. Since $a(\frac{p-1}{2})\not\equiv 0 \bmod p$, it is reduced to $a(\frac{mp-1}{2}) \equiv \alpha_p a(\frac{m-1}{2}) \bmod p^2$.

If $p\equiv 3 \bmod 4$ and $p\neq 3$ then $a(\frac{p-1}{2}) \equiv \frac{p}{2}b(\frac{p-1}{2}) \equiv \frac{p}{6}c(\frac{p-1}{2}) \bmod p^2$ by Theorem 5. From Proposition 8, We have $a(\frac{p-1}{2}) \equiv 0 \bmod p^2$. Hence $a(\frac{mp-1}{2})\equiv a(\frac{p-1}{2})a(\frac{m-1}{2})\equiv 0 \bmod p^2$. Thus we have completed the proof.

3-5. Proof of Theorem 4.

The proof of super congruences for the numbers $u(n)$ is easy using Gessel's result in the same way.

Proposition 9 (Gessel) . *Let* $d(0)=0$ *and*

$$d(n) = 2(2n+1)\sum_{k=1}^{n}\binom{n}{k}^2\binom{n+k}{k}^2\left\{\sum_{i=1}^{k}\frac{1}{(n-i+1)(n+i)}\right\}, \quad n\geq 1 .$$

Then for any prime p , *and* $0\leq k<p$, *we have*

$$u(k+pn) \equiv \{ u(k) + pnd(k) \} u(n) \qquad \bmod p^2 .$$

Proof. The congruence can be proved in similar method of the proof of Proposition 4 of this paper. See Gessl[11]. □

By the explicit formula of $d(n)$, we have $d(\frac{p-1}{2}) \equiv 0 \mod p$. Then it follows that

$$u(\tfrac{p^2-1}{2}) \equiv \{ u(\tfrac{p-1}{2}) \}^2 \qquad \bmod p^2 .$$

Hence by puting $r=2$ and $m=1$ in Theorem 1, we have

$$u(\tfrac{p^2-1}{2}) \equiv \xi_p\, u(\tfrac{p-1}{2}) \qquad \bmod p^2 .$$

Thus

$$\{ u(\tfrac{p-1}{2}) \}^2 \equiv \xi_p\, u(\tfrac{p-1}{2}) \qquad \bmod p^2 .$$

Now since $u(\frac{p-1}{2}) \not\equiv 0 \mod p$, it is reduced to $u(\frac{p-1}{2}) \equiv \xi_p \mod p^2$.

Hence we have completed the proof of Theorem 4 .

3-6. Applications to other numbers.

Above method is applicable to other numbers which satisfy the relations such as (2-11) and (2-14), and we can use the mod p^2 determinations of the certain numbers. For example. Let, for any $n \geq 0$,

$$v(n) = (-1)^n \sum_{k=0}^{n} \binom{n}{k}^3 .$$

F.Beukers and J.Stienstra[24] showed the following congruence. Let $p \geq 3$, and write

$$\sum_{n=1}^{\infty} \gamma_n q^n = q \prod_{n=1}^{\infty} (1-q^n)^2 (1-q^{2n})(1-q^{4n})(1-q^{8n})^2 .$$

Then, for $m, r \in \mathrm{N}$, m odd,

$$v\left(\frac{mp^r-1}{2}\right) - \gamma_p v\left(\frac{mp^{r-1}-1}{2}\right) + \left(\frac{-2}{p}\right) p^2 v\left(\frac{mp^{r-2}-1}{2}\right) \equiv 0 \mod p^r ,$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi-Legendre symbol.

The numbers $\tilde{v}(n)$ which are (differential) of $v(n)$ can be formulated to

$$\tilde{v}(n) = 3(-1)^n \sum_{k=1}^{n} \binom{n}{k}^3 \left[\frac{1}{n-k+1} + \cdots + \frac{1}{n} \right] .$$

And for all primes $p \geq 3$, $n \geq 0$ and $0 \leq m \leq p-1$, we have

$$v(np+m) \equiv \{ v(m) + pn\tilde{v}(m) \} v(n) \mod p^2 .$$

Then $v(\frac{p-1}{2})$ of mod p^2 is determined by our method if $\left(\frac{-2}{p}\right)=1$, that is

$$v\left(\frac{p-1}{2}\right) \equiv \gamma_p + \frac{p}{2}\tilde{v}\left(\frac{p-1}{2}\right) \mod p^2 .$$

§4. Congruences of binomial coefficients $\binom{2f}{f}$.

Let k and l be positive integers with $(k,l)=1$. Let p be a prime, $p \equiv l \bmod k$ and the integer f is defined by $p=kf+l$. We consider the congruences modulo p of binomial coefficients of the form $\binom{2f}{f}$. In the classical results, for $k=4$ and $l=1$, Gauss proved that

$$\binom{2f}{f} \equiv 2a \bmod p \quad ,$$

where $p=a^2+b^2=4f+1$ and $a\equiv 1 \bmod 4$. For $k=3$ and $l=1$, Jacobi proved that

$$\binom{2f}{f} \equiv -a \bmod p \quad ,$$

where $4p=a^2+27b^2$ and $a\equiv 1 \bmod 3$. Moreover, the number $2a$ (resp. $-a$) can be regarded as the p-th Fourier coefficient of the cusp form of CM-type associated with the Hecke character of $\mathbb{Q}(\sqrt{-1})$ (resp. $\mathbb{Q}(\sqrt{-3})$). In the recent results, for $l=1$ and $k\leq 24$, these were studied by Hudson and Williams [15] using Jacobi sums.

In this section, we shall prove the congruence properties between binomial coefficients $\binom{2f}{f}$ and Fourier coefficients of certain η-products :

Theorem 6. *Let k and l be the above and put $m = 4l/k$. Write*

$$\sum_{n=1}^{\infty} \gamma_n^{(k,l)} q^n = \eta(k\tau)^2 \eta(2k\tau)^{1+m} \eta(4k\tau)^{3-3m} \eta(8k\tau)^{2m-2} \quad .$$

where $\eta(\tau)=q^{1/24} \prod_{n=0}^{\infty} (1-q^n)$ is the Dedekind η-function with $q=e^{2\pi i\tau}$ and

$Im\ \tau>0$. *Then , for* $p \equiv l \bmod k$ *and* $p=kf+l$,

$$\binom{2f}{f} \equiv (-1)^f \gamma_p^{(k,l)} \mod p .$$

For some k and l, η-products in Theorem 6 are non-holomorphic automorphic forms of weight 2, so they were not very studied for details. But we can obtain the congruence relations like Corollary 1 for the family of these functions .

4-1. Proof of Theorem 6.

We consider the generating function $F(t)=\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}t^n$.

Since the numbers $(-1)^n\binom{2n}{n}$ satisfy the recurrence

$$(4\text{-}1) \qquad (n+1)(-1)^{n+1}\binom{2(n+1)}{n+1} = -(2n+1)(-1)^n\binom{2n}{n} \quad , \qquad n\geq 0 ,$$

we have

$$F(t) = (1+4t)^{-1/2} .$$

Proposition 10. *Let* k *and* l *be positive integers with* $(k,l)=1$ *and* $m = 4l/k$. *Write*

$$(4\text{-}2) \qquad \lambda(\tau) = \left(\eta(2k\tau)\eta(4k\tau)^{-3}\eta(8k\tau)^2 \right)^{4/k} = \sum_{n=1}^{\infty} A_n q^n \quad (A_1=1) .$$

Then

$$(4\text{-}3) \qquad F(\lambda^k)d(\lambda^l) = l\{\eta(k\tau)^2\eta(2k\tau)^{m+1}\ \eta(4k\tau)^{3-3m}\eta(8k\tau)^{2m-2}\}\ \frac{dq}{q} \quad .$$

Remark 1. We may use the branch of k-th roots $x^{1/k}$ so that it takes positive real values on the positive real axis, i.e., the leading coefficients $\gamma_l^{(k,l)}$ and A_1 in the η-product of Theorem 6 and Proposition 10 are equal to 1 respectively.

Proof. First we prove the case of k=4 and l=1. We consider the following congruence modular subgroup

$$\Gamma_0(8) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \bmod 8 \right\} .$$

It has no elliptic elements , and a set of representatives of inequivalent cusps is $\{ i\infty , 0 , \frac{1}{4} , \frac{1}{2} \}$. $\mathbb{H}^*/ \Gamma_0(8)$ is a curve of genus 0. Putting

$$t(\tau) = \eta(2\tau)^4 \eta(4\tau)^{-12} \eta(8\tau)^8 ,$$

it is a modular function with respect to $\Gamma_0(8)$, and the values at the cusps are given by $t(i\infty)=0$ (simple), $t(0)=\frac{1}{4}$, $t(\frac{1}{4})=\infty$ (simple) , and $t(\frac{1}{2})=-\frac{1}{4}$. Hence $t(\tau)$ generates the function field of modular functions with respect to $\Gamma_0(8)$. Therefore we see that $F^2(t(\tau))$ $= \frac{1}{1+4t(\tau)}$ has a simple pole at $\tau=\frac{1}{2}$ and a simple zero at $\tau=\frac{1}{4}$. $M_k(\Gamma_0(8))$ (resp. $S_k(\Gamma_0(8))$) denotes the space of modular forms (resp. cusp forms) of weight k. It is not hard to check that $t^{-1}\frac{dt}{d\tau}$ is in $M_2(\Gamma_0(8))$ and it has a simple zero at $\tau=0$, $\frac{1}{2}$. Hence the function

(4-4) $$\Psi(\tau) = (\frac{1}{2\pi i})^4 F^4(t(\tau))(t^{-1}\frac{dt}{d\tau})^4 \ t(\tau)$$
$$= q - 8q^2 + 12q^3 - 64q^4 + 210q^5 - 96q^6 + \cdots\cdots$$

is an element of $S_8(\Gamma_0(8))$. We choose

$$\eta(\tau)^8\eta(2\tau)^8 = q - 8q^2 + 12q^3 - 64q^4 + 210q^5 - \cdots\cdots$$

as another form (this is an old form) in $S_8(\Gamma_0(8))$. Since

dim $S_8(\Gamma_0(8)) = 5$, comparing with the coefficients , we have

(4-5) $$\Psi(\tau) = \eta(\tau)^8 \ \eta(2\tau)^8 \quad .$$

Taking 4-th roots with Remark 1 and replacing τ by 4τ , we have

(4-6) $$F(\lambda^4)d\lambda = \eta(4\tau)^2\eta(8\tau)^2 \ dq/q \quad .$$

In the general case, from (4-4) and (4-5) , we see

$$\Psi_{k,l}(\tau) = (\frac{1}{2\pi i})^k \ F(t(\tau))^k(t^{-1}\frac{dt}{d\tau})^k \ t(\tau)^l$$
$$= \eta(\tau)^{2k}\eta(2\tau)^{4l+k}\eta(4\tau)^{3k-12l}\eta(8\tau)^{8l-2k} \quad .$$

Hence our proposition follows from taking k-th roots and replacing τ by $k\tau$. □

Remark 2. When k=4 and l=1 , since the function

$$\sum_{n=1}^{\infty} \gamma_n q^n = \eta(4\tau)^2\eta(8\tau)^2$$

is the unique cusp form in $S_2(\Gamma_0(32))$, applying Beukers[5,Prop.3] to (4-3), for any $m,r\in\mathbb{N}$, $m\equiv1$ mod 4 and any prime $p\equiv1$ mod 4 , we have

$$\binom{(mp^r-1)/2}{(mp^r-1)/4}(-1)^{(mp^r-1)/4} - \gamma_p \binom{(mp^{r-1}-1)/2}{(mp^{r-1}-1)/4}(-1)^{(mp^{r-1}-1)/4}$$

$$+ \; p \quad \binom{(mp^{r-2}-1)/2}{(mp^{r-2}-1)/4}(-1)^{(mp^{r-2}-1)/4} \equiv 0 \quad \bmod p^r .$$

These congruences are quite Atkin-Swinnerton-Dyer type associated to the elliptic curve: $y^2 = x^3 + 2x$ (see Atkin-Swinnerton-Dyer[1]).

In our case, we can not use directly the method of Beukers[5] or Stienstra-Beukers[24,Th.A9] because the non-holomorphy of η-products of the right hand of Proposition obstructs that we apply the theory of Hecke operators to them. But the following lemma is useful.

Lemma 2. *Let p be a prime and*

$$\omega(t) = \sum_{n=1}^{\infty} b_n t^{n-1} dt$$

be a differential form with $b_n \in \mathbb{Z}_p$. Let $t(u) = \sum_{n=1}^{\infty} c_n u^n$ with $c_n \in \mathbb{Z}_p$, c_1 is a p-adic unit , and suppose

$$\omega(t(u)) = \sum_{n=1}^{\infty} d_n u^{n-1} du \qquad .$$

Then $d_p \equiv c_1 \, b_p \mod p$.

Proof. It is clear that

$$\omega(t) - b_p t^{p-1} dt = t^p G_1(t) dt + dG_2(t) \quad , \quad G_1(t), G_2(t) \in \mathbb{Z}_p[[t]] \; .$$

It is straightforward to see that

$$t^{p-1} dt = c_1^p \; u^{p-1} du + u^p \; G_3(u) du \quad , \quad G_3(u) \in \mathbb{Z}_p[[u]] \quad .$$

Then we can write

$$\omega(t(u)) - b_p c_1^p u^{p-1} du = u^p G_4(u) du + dG_5(u) \quad , \quad G_4(u), G_5(u) \in \mathbb{Z}_p[[u]].$$

Hence

$$d_p - b_p c_1^p \equiv d_p - b_p c_1 \equiv 0 \mod p \ . \qquad \square$$

Now, (4-2) and (4-3) satisfy the condition of Lemma 2 because the denominators of the coefficients of q-expansion do not divide p . Comparing with the equation

$$\frac{1}{l} F(\lambda^k) d(\lambda^l) = \sum_{n=1}^{\infty} (-1)^n \binom{2n}{n} \lambda^{kn+l-1} d\lambda = \sum_{n=0}^{\infty} \gamma_n^{(k,l)} q^{n-1} dq \ ,$$

we have proof of our Theorem 6.

The following corollary is obtained by applying the consequence of our theorem to the recurrence (4-1) .

Corollary 1. *Let k , l and $\gamma_n^{(k,l)}$ be the above . Then , for $p \equiv l \mod k$,*

$$l \ \gamma_p^{(k,l)} \equiv -2(2l+k) \ \gamma_p^{(k,k+l)} \mod p \ .$$

4-2. Examples.

Let k=4 and l=3 . Then

$$\sum_{n=1}^{\infty} \gamma_n^{(4,3)} q^n = \eta(4\tau)^2 \eta(8\tau)^4 \eta(16\tau)^{-6} \eta(32\tau)^4 \quad .$$

$$= q^3 - 2\, q^7 - 5\, q^{11} + 10\, q^{15} + 13\, q^{19} + \cdots\, .$$

If $p=11$ then $\binom{2f}{f} = \binom{4}{2} = 6 \equiv -2 = \gamma_{11}^{(4,3)} \mod 11$.

If $p=19$ then $\binom{2f}{f} = \binom{8}{4} = 70 \equiv 13 = \gamma_{19}^{(4,3)} \mod 19$.

This form is the non-holomorphic automorphic form of weight 2 with respect to $\Gamma_0(32)$, but we do not know about the properties of $\gamma_p^{(4,3)}$.

Let $k=5$ and $l=2$. Then

$$\sum_{n=1}^{\infty} \gamma_n^{(5,2)} q^n = \eta(5\tau)^2 \eta(10\tau)^{13/5} \eta(20\tau)^{-9/5} \eta(40\tau)^{6/5} \quad .$$

$$= q^2 - 2\, q^7 - \frac{18}{5}\, q^{12} + \frac{36}{5}\, q^{17} + \frac{122}{25}\, q^{22} - \cdots\, .$$

If $p=7$ then $\binom{2f}{f} = \binom{2}{1} = 2 \equiv -(-2) = (-1)\, \gamma_7^{(5,2)} \mod 7$.

If $p=17$ then $\binom{2f}{f} = \binom{6}{3} = 20 \equiv -(\frac{36}{5}) = (-1)^3 \gamma_{17}^{(5,2)} \mod 17$.

4-3. Applications.

We can try to apply our method to other numbers of which the generating function satisfies the differential equation of the form

$$F(\lambda(\tau)^k)\, d\lambda(\tau) = G(\tau)\, \frac{dq}{q} \quad .$$

and several examples can be seen in Beukers[5] and Stienstra-Beukers [24].

For the numbers $\binom{2n}{n}^2$, $n\ge 0$, Steinstra and Beukers[24] proved that the generating function

$$F_1(t) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 t^n$$

satisfies

$$F_1(\lambda^4)d\lambda = \eta(4\tau)^6 \frac{dq}{q} ,$$

where $\lambda(\tau) = \eta(4\tau)^2\eta(8\tau)^{-6}\eta(16\tau)^4$.

Extending this by the same method , we have

$$F_1(\lambda^k)d(\lambda^l) = l\ \eta(k\tau)^{m+2}\eta(2k\tau)^{6-3m}\eta(4k\tau)^{2m-8} \frac{dq}{q} ,$$

where $\lambda(\tau) = \{ \eta(k\tau)\eta(2k\tau)^{-3}\eta(4k\tau)^2 \}^{8/k}$ and $m = 8l/k$.

Consequently ,

Theorem 7. *Let k, l be positive integers with $(k,l)=1$ and write for $m = 8l/k$,*

$$\sum_{n=1}^{\infty} \alpha_n^{(k,l)} q^n = \eta(k\tau)^{m+2}\eta(2k\tau)^{6-2m}\eta(4k\tau)^{2m-8}$$

Then , for any prime $p \equiv l \bmod k$ and $p=kf+l$.,

$$\binom{2f}{f}^2 \equiv \alpha_p^{(k,l)} \bmod p .$$

Remark 3. If k=4 and l=1 then $\alpha_n^{(4,1)} = \alpha_n$. These are the Fourier coefficients of the cusp form $\eta(4\tau)^6$ of CM-type.

Combining this with Theorem 6, we can obtain the congruences of Fourier coefficients of the automorphic forms of the different weights.

Corollary 2. *Let* k , l , $\gamma_n^{(k,l)}$ and $\alpha_n^{(k,l)}$ *be the above*. *Then, for* $p \equiv l \bmod k$,

$$\alpha_p^{(k,l)} \equiv \{ \gamma_p^{(k,l)} \}^2 \mod p .$$

§5. Congruences of $u(\frac{p-1}{k})$.

Let

$$u(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \qquad , \ n>0$$

be Apéry numbers with the proof of irrationality of $\zeta(3)$.

Beukers[5,Proposition 1] proved that the generating function

$$\mathfrak{U}(t) = \sum_{n=0}^{\infty} u(n) t^n$$

satisfies

$$\mathfrak{U}(\lambda^2) d\lambda = \{ \eta(2\tau)^4 \eta(4\tau)^4 - 9 \eta(6\tau)^4 \eta(12\tau)^4 \} \frac{dq}{q} \quad ,$$

where $\lambda(\tau) = \eta(2\tau)^6 \eta(4\tau)^{-6} \eta(6\tau)^{-6} \eta(12\tau)^6$ (see Proposition 2 of this paper). Extending of this in the same method of Proposition 10, we have

$$\mathfrak{U}(\lambda^k) d(\lambda^l) = l \{ \eta(k\tau)^{m-2} \eta(2k\tau)^{10-m} \eta(3k\tau)^{6-m} \eta(6k\tau)^{m-6}$$
$$- 9 \eta(k\tau)^{m-6} \eta(2k\tau)^{6-m} \eta(3k\tau)^{10-m} \eta(6k\tau)^{m-2} \} \frac{dq}{q} \quad ,$$

where $\lambda(\tau) = \{ \eta(k\tau)\eta(2k\tau)\eta(3k\tau)\eta(6k\tau)\}^{12/k}$ and $m = 12l/k$.

Consequently , by Lemma 2, we have

Theorem 8. *Let k, l be positive integers with $(k,l)=1$ and write for $m = 12l/k$,*

$$\sum_{n=1}^{\infty} \xi_n^{(k,l)} q^n = \eta(k\tau)^{m-2} \eta(2k\tau)^{10-m} \eta(3k\tau)^{6-m} \eta(6k\tau)^{m-6}$$
$$- 9 \eta(k\tau)^{m-6} \eta(2k\tau)^{6-m} \eta(3k\tau)^{10-m} \eta(6k\tau)^{m-2} \quad .$$

Then, for any prime $p \equiv l \bmod k$,

$$u\left(\frac{p-l}{k}\right) \equiv \xi_p^{(k,l)} \quad \bmod p .$$

Since the Apéry numbers $u(n)$ satisfy the recurrence

$$(n+1)^3 u(n+1) - (34n^3+51n^2+27n+5)u(n) + n^3 u(n-1) = 0 , \quad n>1 ,$$

the following corollary is an easy consequence .

Corollary 3. *Let* k, l *and* $\xi_n^{(k,l)}$ *be the above . Then for any prime* $p \equiv l \bmod k$,

$$l^3 \xi_p^{(k,l)} + (k+l)^3 \xi_p^{(k,l+2k)}$$
$$\equiv (34l^3+51l^2k+27lk^2+5k^3) \xi_p^{(k,l+k)} \qquad \bmod p.$$

Example. Let $k=3$ and $l=1$. Then

$$\sum_{n=1}^{\infty} \xi_n^{(3,1)} q^n = \eta(3\tau)^2\eta(6\tau)^6\eta(9\tau)^2\eta(18\tau)^{-2}$$
$$-9\eta(3\tau)^{-2}\eta(6\tau)^2\eta(9\tau)^6\eta(18\tau)^2 .$$
$$= q - 11 q^4 - 25 q^7 + 15 q^{10} + 20 q^{13} + \cdots .$$

If $p=7$ then $u(\frac{7-1}{3}) = u(2) = 73 \equiv -25 = \xi_7^{(3,1)} \bmod 7$.

If $p=13$ then $u(\frac{13-1}{3}) = u(4) = 33001 \equiv 20 = \xi_{13}^{(3,1)} \bmod 13$.

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