Universal Pretzel links and unknotting tunnels

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Universal Pretzel Links and Unknotting Tunnels

（普遍プレッツェル絡み目と結び目解消トンネル）

BY

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Introduction and Preliminaries

Topology is the branch of geometry which studies the property of figures under arbitrary continuous transformations. Just as ordinary geometry considers two figures to be the same if each can be carried into the other by a rigid motion, topology considers two figures to be the same if each can be mapped onto the other by a one-to-one continuous function.

Knot Theory is a basic and important field of low dimensional combinatorial topology. A classical knot is an embedding of a circle S^1 into the three-sphere S^3. Generally, a smoothly embedding of S^n into S^{n+2} has been studied in "higher dimensional knot theory", but this thesis is concerned with classical knots. And a link is an embedding of a disjoint union of circles S_i^1, $(1 \leq i \leq \mu)$, into the three-sphere S^3. Two knots are equivalent if they are ambient isotopic. A knot K may be a representative of a class of equivalent knots or the class itself. If two knots K and K' are equivalent, we shall say they are same. A three-manifold is defined to be a topological space which is locally homeomorphic to the Euclidean three-space.

There are very close relations between knots and three-manifolds. In 1960's, it was discovered independently by Lickorish and by Wallace that every closed, orientable, connected three-manifold may be obtained by surgery on a link in the three-sphere.

In 1920, Alexander stated, and proved rather sketchily, that every connected closed orientable three-manifold may be constucted as a branched covering of three-sphere. Since then, this has been sharpened so that one may require that the associated unbranched covering is at most three-fold and that the downstairs branching set is connected, i.e. a knot. This improvement is due independently to Hilden and Montesinos, using differrent methods. In 1982, Thurston introduced the notion of a universal link and gave an example of a universal link in [18]. A link L in the three-sphere is said to be *universal* if every closed orientable three-manifold can be represented as a covering of the three-sphere branched over L.

Since then, Hilden, Lozano and Montesinos gave a necessary and sufficient condition for a two-bridge to be universal.

In Sections 1 and 2, we shall give a necessary and sufficient condition for a pretzel link to be unversal. In Section 1, we will give a necessary and sufficient condition for a chain, that is a special type of a pretzel link, and in Section 2, for a pretzel link.

Let K be a knot in the three-sphere, and $E(K)$ the exterior of K. A Heegaard decomposition of $E(K)$ is the union of a handlebody H_g of genus g with $(g-1)$ 2-handles which are attached to H_g along curves on ∂H_g. The Heegaard genus of K is the minimal genus of Heegaard decompositions of $E(K)$. Two Heegaard decompositions of $E(K)$ are called homeomorphic if there exists a homeomorphism of $E(K)$ sending the handlebody of one of the decompositions to that of the other one. We call τ an *unknotting tunnel* if τ is a properly embedded arc in $E(K)$ such that $cl(E(K) - N(\tau))$ is a genus two handlebody. Here $N(\tau)$ denotes a regular neighbourhood of τ. The concept of an unknotting tunnel is closely related to a genus two Heegaared decomposition of $E(K)$. Boileau, Rost, and Zieschang completely classified unknotting tunnels for torus knots using the results on Nielsen equivalence classes of generator systems for torus knot groups. Bleiler and Moriah applied their method to distinguish the upper and lower tunnels for two-bridge knots, and Kobayashi found other unknotting tunnels for two-bridge knots, and classified them up to homeomorphism. And Sakuma classified these unknotting tunnels using double coset of knot group.

In Section 3, we shall give an alternative proof of Sakuma's result using dihedral coverings. Let $p : \widetilde{E(K)} \longrightarrow E(K)$ be covering space. If two unknotting tunnels τ_1 and τ_2 are isotopic, $p^{-1}(\tau_1)$ must be isotopic to $p^{-1}(\tau_2)$ in $\widetilde{E(K)}$. So, if we want to show that τ_1 and τ_2 are not isotopic, it is sufficient to show that $p^{-1}(\tau_1)$ is not isotopic to $p^{-1}(\tau_2)$ in $\widetilde{E(K)}$.

Throughout the paper, we work in the smooth or p.l. category.

Acknowledgment

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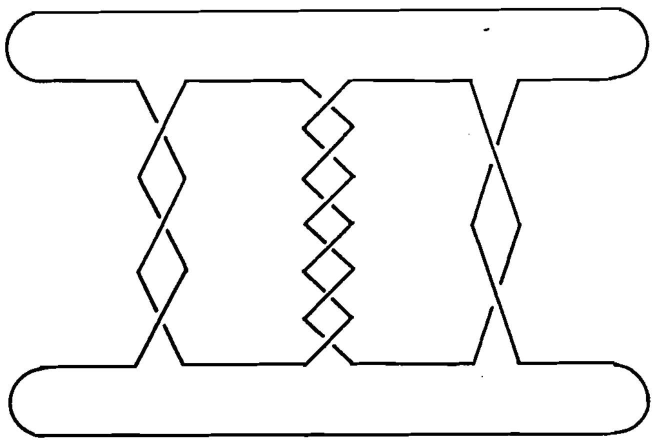
1. Universal chains

1.1. Definition. A link L in S^3 is said to be *universal* if every closed orientable 3-manifold can be represented as a covering of S^3 branched over L.

1.2. Definition. A *pretzel link* is a link consisting of 2-strand braids with q_1-,q_2-, ...,q_m-half twists, which we denote by $p(q_1, q_2, ..., q_m)$. We assume that $q_i \neq 0$ for $i = 1, 2, ..., m$. For example, $p(3, 6, -2)$ is shown in Figure 1.1.

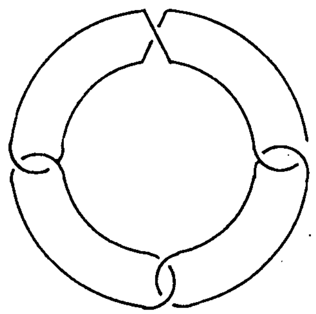


$p(3,6,-2)$

Figure 1.1.

If $q_i = \pm 1$, then $p(q_1, q_2, ..., q_m)$ is equivalent to $p(q_i, q_1, ..., q_{i-1}, q_{i+1}, ..., q_m)$. So we can deform $p(q_1, q_2, ..., q_m)$ into $p(\varepsilon, \varepsilon, ..., \varepsilon, p_1, p_2, ..., p_n)$, where $\varepsilon = \pm 1$ and $| p_i |> 1$ $(i = 1, 2, ..., n)$. We denote this pretzel link by $p(-\varepsilon b; p_1, p_2, ..., p_n)$, where b is the number of ε. If $b \neq 0$ and $p_i = -2\varepsilon$, then $p(-\varepsilon b; p_1, p_2, ..., p_n)$ is equivalent to $p(-\varepsilon(b-1); p_1, p_2, ..., -p_i, ..., p_n)$. So if $b \neq 0$, we can assume that every p_i is not -2ε.

1.3. Definition. A *chain* $C(\alpha, \beta)$ is a pretzel knot or link of type $P(\varepsilon, \varepsilon, ..., \varepsilon, 2, 2, ..., 2)$ where $\varepsilon = \pm 1$, the number of 2 is α, and the sum of ε is β. (For example $C(3, -1)$ and $C(4, -2)$ are illustrated in Figure 1.2.)



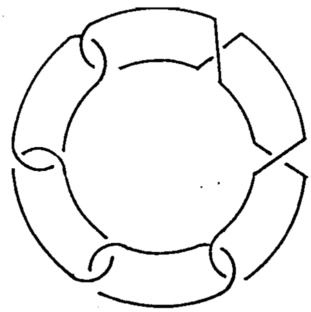


Figure 1.2

1.4. Theorem.[16] *A chain $C(\alpha,\beta)$ is universal if and only if $\alpha \neq 0$ and $(\alpha,\beta) \neq (1,-2)$, $(1,-1)$, $(1,0)$, $(1,1)$, $(2,-2)$, $(2,-1)$, $(2,0)$, $(3,-2)$, $(3,-1)$ or $(4,-2)$.*

Proof of Theorem. The "only if" part is clear. In fact, for $\alpha = 0$ or $(\alpha,\beta) = (1,-2), (1,-1), (1,0), (1,1), (2,-2), (2,-1), (2,0), (3,-2), (3,-1)$ or $(4,-2)$, we can easily see that $C(\alpha,\beta)$ is a union of fibers of a graph-manifold structure on S^3. Hence, in these cases, $C(\alpha,\beta)$ is not universal. The rest of the paper is devoted to the proof of the "if" part.

1.5. Lemma. *A chain $C(\alpha,\beta)$ is universal, if $\alpha - \beta \equiv 0$ (mod 3) and $\alpha \geq 5$.*

Proof. The link illustrated in Figure 1.3 is universal (see [4, Fig. 9]).
In the ball B in Figure 1.3, we consider the operations as illustrated in Figure 1.4. This operation keeps the universality of links by an argument in [4, pp. 20-21]. Do this operation successively, and we get the chains with $\alpha - \beta \equiv 0$ (mod 3), and $\alpha \geq 5$. This completes the proof.

1.6. Lemma. *A chain $C(\alpha,\beta)$ is universal, if $\alpha \geq 5$.*

Proof. The chain $C(\alpha,\beta)$, where $\alpha \geq 5$, is depicted in Figure 1.5, with an assignment of permutations to the components C_1 and C_2.

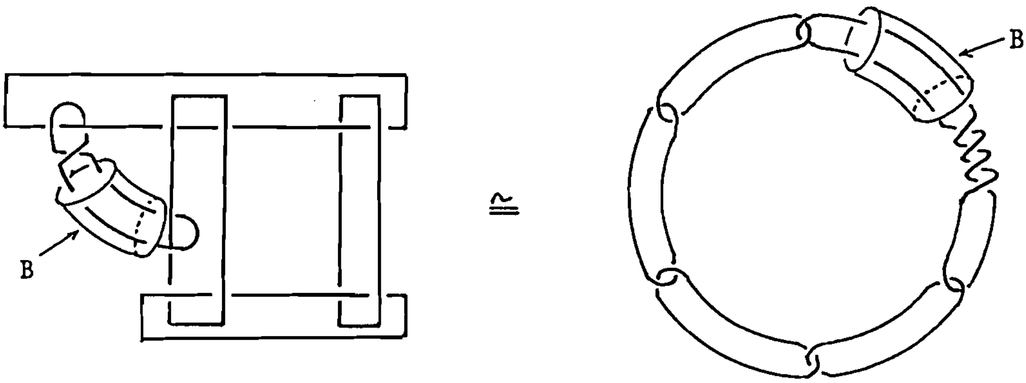


Figure 1.3

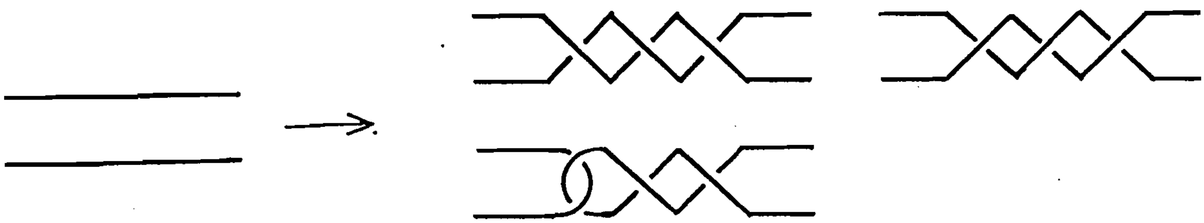


Figure 1.4.

The corresponding dihedral covering is also S^3 and the lift of $C(\alpha,\beta)$ is described in Figure 1.6.

This link contains a sublink which is equivalent to the chain $C(\alpha',\beta')$ where $\alpha' = 3(\alpha-4)+4 = 3\alpha-8$ and $\beta' = 3\beta+4$. Note that $\alpha'-\beta' = 3(\alpha-\beta-4) \equiv 0 \pmod 3$ and $\alpha' \geq 5$ since $\alpha \geq 5$. Thus $C(\alpha',\beta')$ is universal by Lemma 1.5, and hence $C(\alpha,\beta)$ is universal.

1.7. Lemma. *A chain $C(1,\beta)$ is universal, if $\beta \neq 0, \pm 1, -2$.*

Proof. The chain $C(1,\beta)$, where $\beta \neq 0, \pm 1, -2$ is a hyperbolic 2-bridge knot, so it is universal by [6].

1.8. Lemma. *A chain $C(2,\beta)$ is universal, if $\beta \neq 0, -1, -2$.*

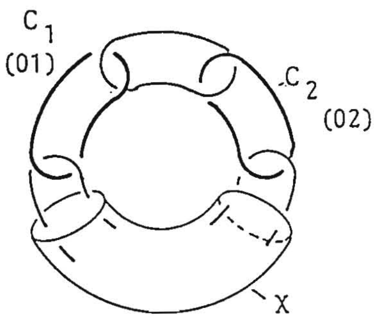


Figure 1.5

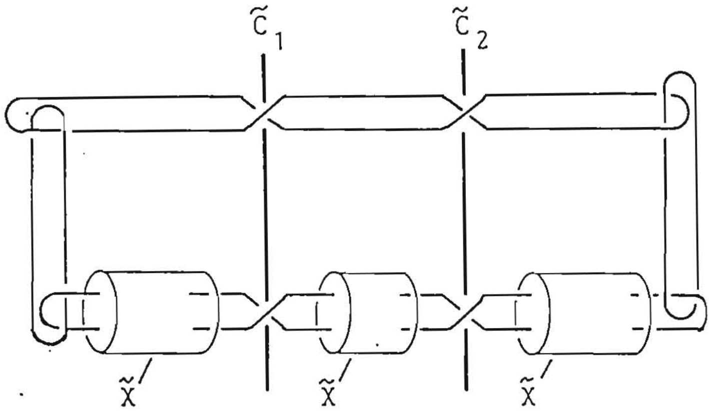


Figure 1.6

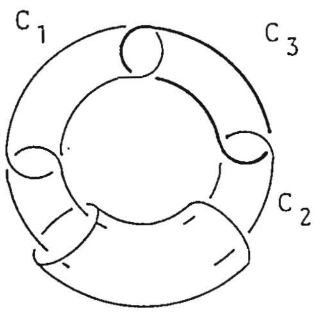


Figure 1.7

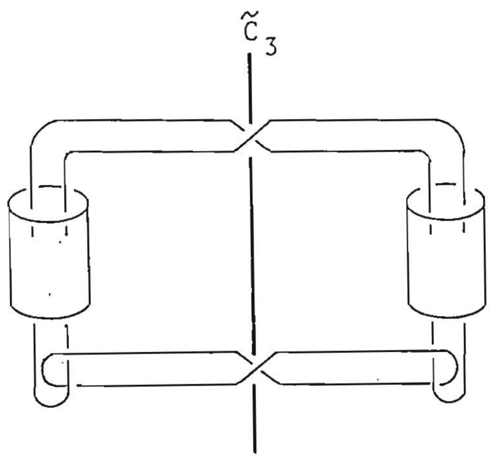


Figure 1.8

Proof. The chain $C(2,\beta)$, where $\beta \neq 0, -1, -2$ is a hyperbolic 2-bridge link, so it is universal by [6].

1.9. Lemma. *A chain* $C(3,\beta)$ *is universal, if* $\beta \neq -2, -1$.

Proof. The preimage of $C_1 \cup C_2$ under the 2-fold cyclic covering of S^3 branched over C_3 , shown in Figure 1.7, is a chain $C(2, 2\beta + 2)$ as shown in Figure 1.8. If $\beta \neq -2, -1$, then $C(2, 2\beta + 2)$ is universal. Thus $C(3,\beta)$ is universal by [6], if $\beta \neq -1, -2$.

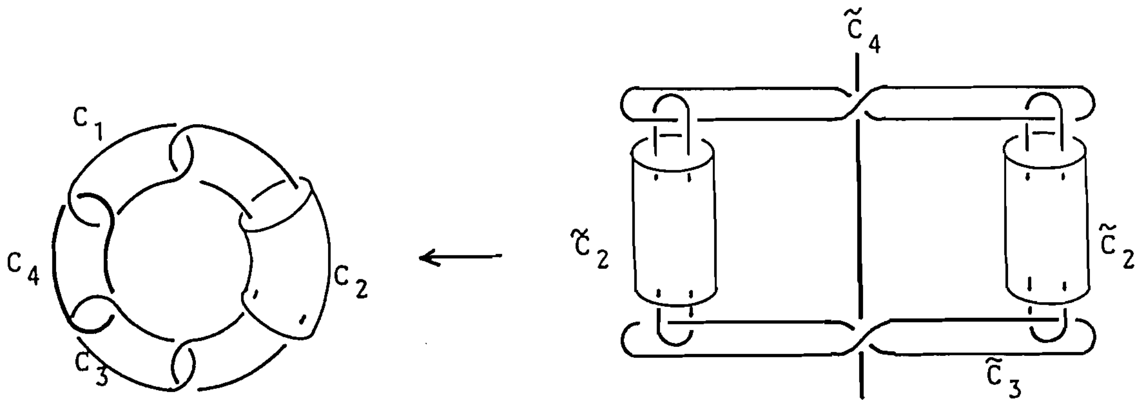


Figure 1.9

1.10. Lemma. *A chain $C(4,\beta)$ is universal, if $\beta \neq -2$.*

Proof. The proof is divided into the following three steps.

Step 1. If $\beta \neq -2$, then there is a covering $p_1 : S^3 \longrightarrow S^3$ branched over $C(4,\beta)$, such that $p_1^{-1}(C(4,\beta))$ contains a sublink L_1 which is equivalent to $C(4,\beta_1)$ with $\beta_1 = 2 \cdot 3^n \cdot (\beta+2) - 2$ for some $n \geq 3$.

Step 2. There is a covering $p_2 : S^3 \longrightarrow S^3$ branched over L_1, such that $p_2^{-1}(L_1)$ contains a sublink L_2 which is equivalent to $P(-2 \cdot 3^n \cdot (\beta+2), -2 \cdot 3^n \cdot (\beta+2), ..., -2 \cdot 3^n \cdot (\beta+2))$ where the number of $-2 \cdot 3^n \cdot (\beta+2)$ is eight.

Step 3. There is a covering $p_3 : S^3 \longrightarrow S^3$ branched over L_2, such that $p_3^{-1}(L_2)$ contains a sublink which is equivalent to $C(8,0)$ or $C(8,-8)$. Since $C(8,0)$ and $C(8,-8)$ are universal by Lemma 1.5, $C(4,\beta)$ is universal.

Proof of Step 1. Let C_i $(1 \leq i \leq 4)$ be the components of $C(4,\beta)$ as illustrated in Figure 1.9. Then the preimage of $C_1 \cup C_2 \cup C_3$ under the 2-fold cyclic covering of S^3 branched over C_4 is a chain $C(4, 2(\beta+1))$. (See Figure 1.9.)

Let C_i' $(1 \leq i \leq 4)$ be the components of $C(4, 2(\beta+1))$ as illustrated in Figure 1.10, and consider the irregular 3-fold covering of S^3 branched along $C_1' \cup C_3'$ (see Figure 1.11).

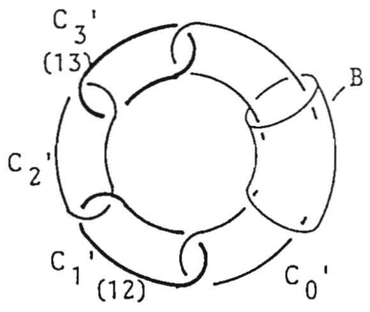


Figure 1.10

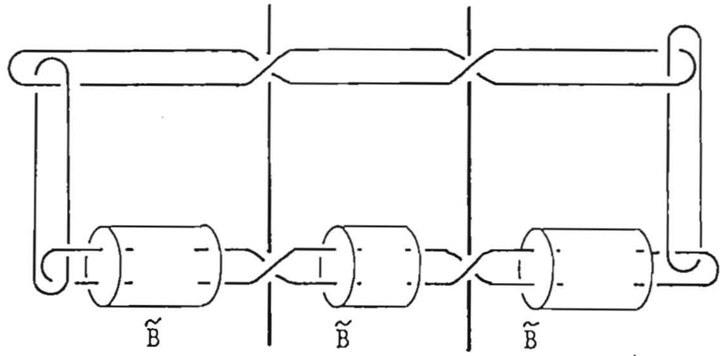


Figure 1.11

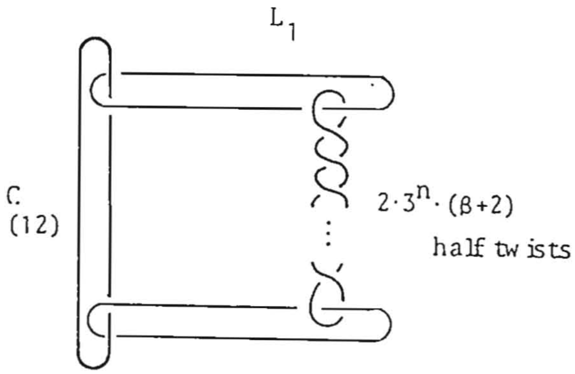


Figure 1.12

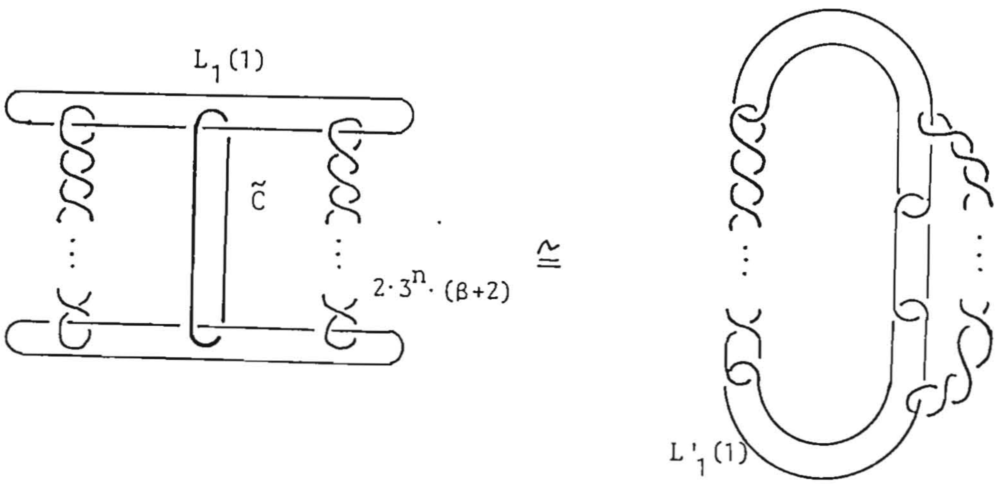


Figure 1.13

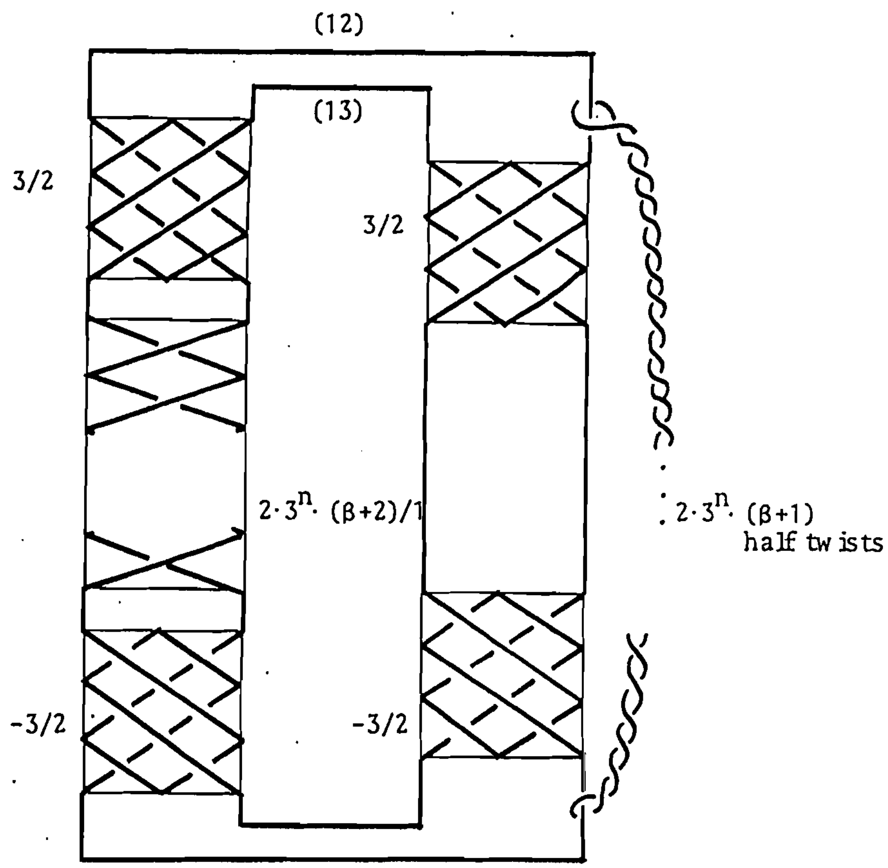


Figure 1.14

Then the inverse image of $C_2' \cup C_4'$ under this covering is a chain $C(4, 3 \cdot 2(\beta + 2) - 2)$. Repeat the above operations (n + 1)-times ($n \geq 3$), then we obtain the desired branched covering p_1 and the sublink L_1 of $p_1^{-1}(C(4, 2(\beta + 1))$.

Proof of Step 2. Let C be a component of $L_1 = C(4, 2 \cdot 3^n \cdot (\beta + 2) - 2)$, and consider the double cover of S^3 branched over C (see Figure 1.12). Then the inverse image $L_1(1)$ of L_1 under this covering is illustrated in Figure 1.13.

$L_1(1)$ contains a sublink $L_1'(1)$ which is equivalent to $C(4, 2 \cdot 3^n \cdot (\beta + 2) - 2)$ as illustrated in Figure 1.13. Consider the irregular 3-fold covering of S^3 branched over $L_1'(1)$ whose monodromy is given by Figure 1.14. The corresponding covering space is S^3 by [8], and

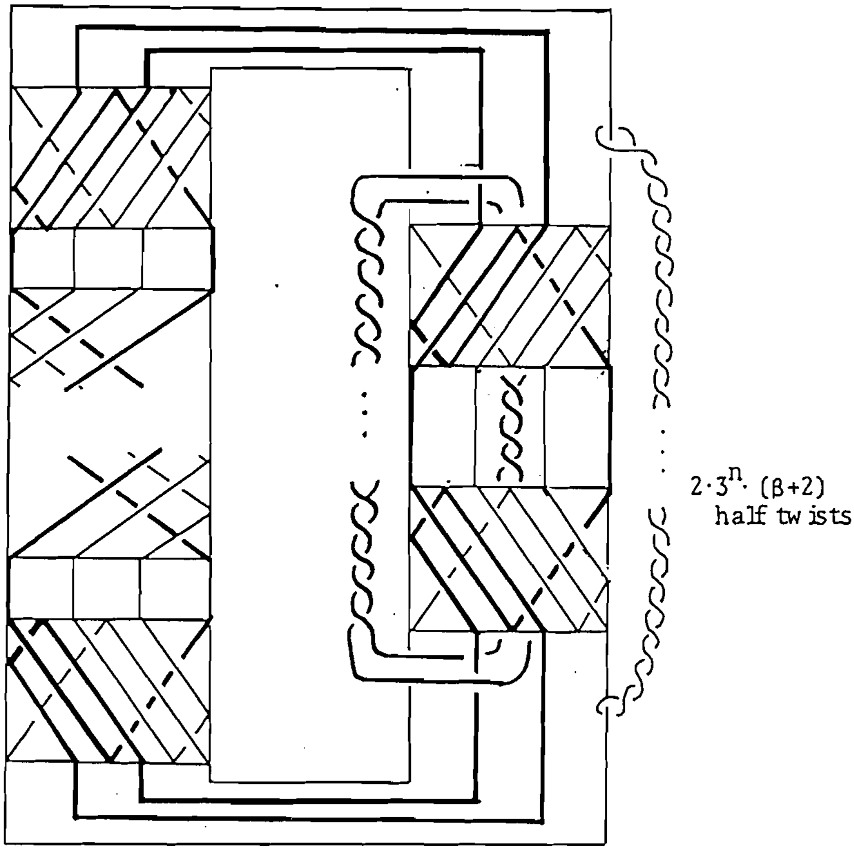


Figure 1.15.

the inverse image of $L_1(1)$ is shown in Figure 1.15 (cf.[7, Figure 10. 6, pp. 499-455]). Let $L_1(2)$ be the sublink of the inverse image of $L_1(1)$ under this covering depicted by the bold line in Figure 1.15. Then it is equivalent to the link illustrated in Figure 1.16. Repeating the same procedure, we obtain the link as illustrated in Figure 1.17 and repeating the same procedure once more, we obtain the link as illustrated in Figure 1.18, which contains the desired sublink $L_2 = P(-2 \cdot 3^n \cdot (\beta - 2), ..., -2 \cdot 3^n \cdot (\beta - 2))$.

Proof of Step 3. First, we prove the following lemma.

1.11. Lemma. *There exists a covering* $f : S^3 \longrightarrow S^3$ *branched*

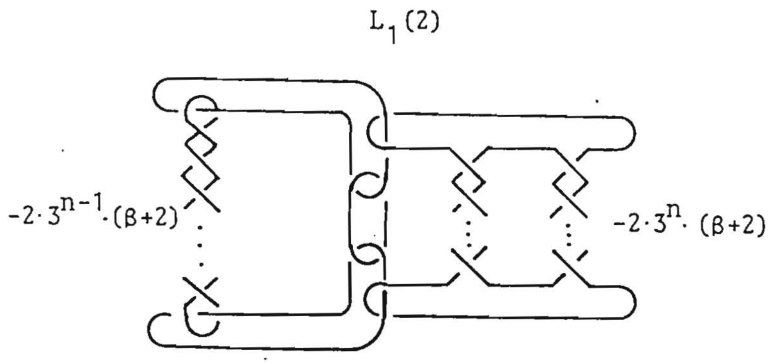


Figure 1.16

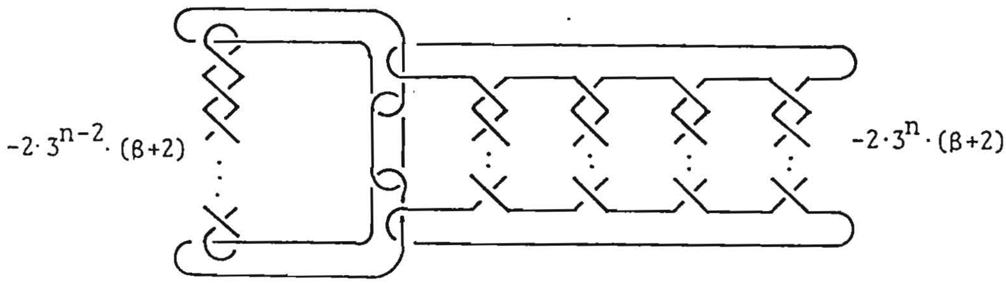


Figure 1.17

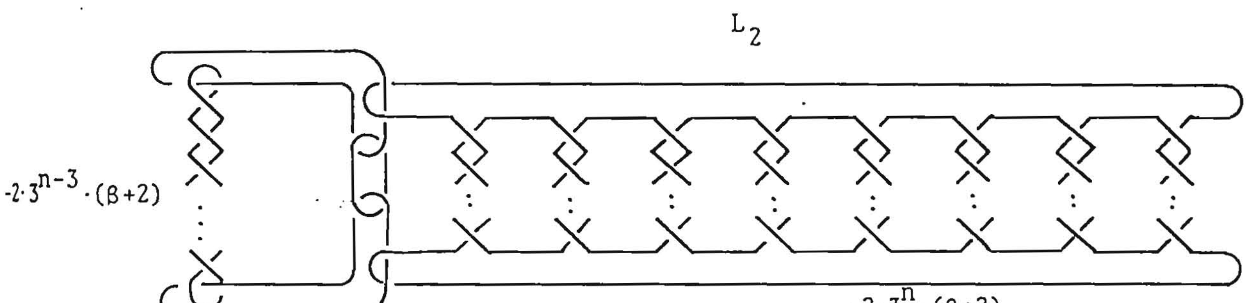


Figure 1.18

over $P(\alpha a_1, \alpha a_2, ..., \alpha a_n)$ $(\alpha > 1, odd)$ *such that* $f^{-1}(P(\alpha a_1, \alpha a_2, ..., \alpha a_n)) \supset P(a_1, a_2, ..., a_n)$.

Proof of Lemma 1.11.. The covering of S^3 branched over n-component trivial link whose monodromy is given by Figure 1.19 is homeomorphic to S^3, and the branched line with branched index q is shown in Figure 1.20.

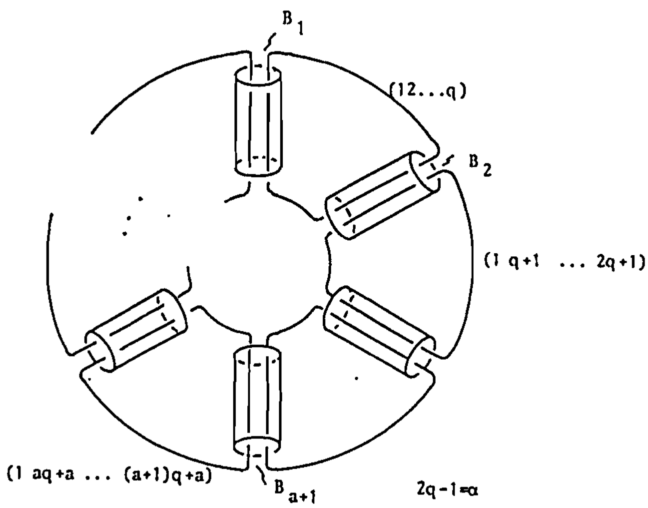
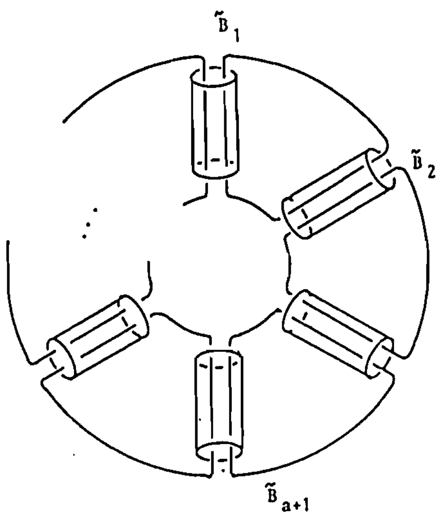


Figure 1.19



Figue 1.20

By αa_i-half twists at each ball B_i , this n-component trivial link can be deformed to $P(\alpha a_1, \alpha a_2, ..., \alpha a_n)$. To visualize the branch set upstairs, we look the covering of B_i. Each of these twists lifts to one half twist (cf. [14, p. 318]).

Thus the branch line, whose branch index q is $P(a_1, a_2, ..., a_n)$, completing the proof.

Now, for $\beta + 2 = 2^r \cdot s$ $(s :$ odd), there is a branched covering $p_2 : S^3 \longrightarrow S^3$ branched over $P(2^{r+1} \cdot 3^n \cdot s, ..., 2^{r+1} \cdot 3^n \cdot s)$ such that its preimage contains a link which is homeomorphic to

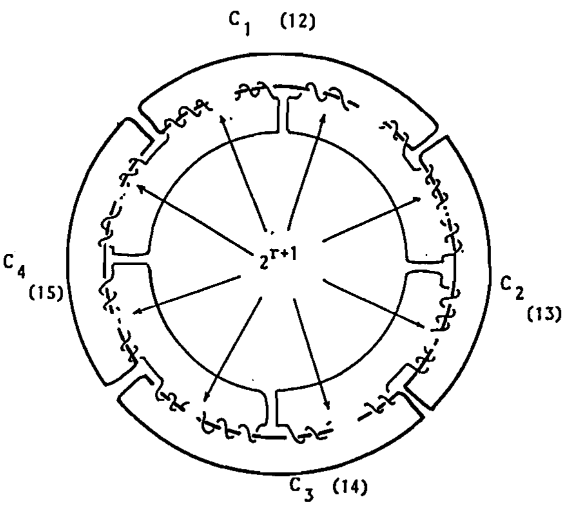


Figure 1.21

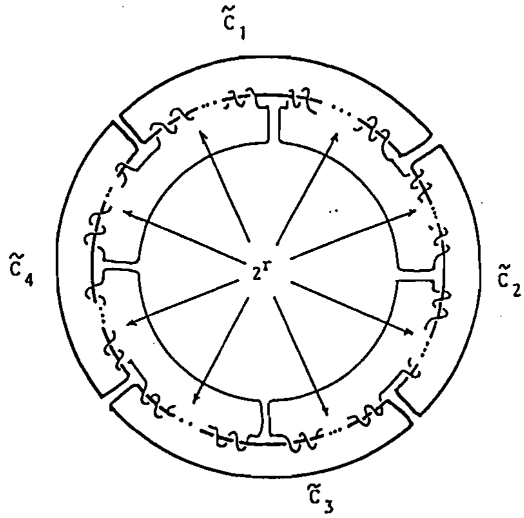


Figure 1.22

$P(2^{r+1}, 2^{r+1}, ..., 2^{r+1})$ by Lemma 1.11. If $r = 0$, the proof is complete. If $r \neq 0$, we consider the covering whose monodomy is given by Figure 1.21. Then it is homeomorphic to S^3, and the preimage of the branch lines is illustrated in Figure 1.22. This contains $P(2^r, 2^r, ..., 2^r)$ as a sublink. So, repeating this procedure r-times, we obtain $P(2, 2, ..., 2) \cong C(8, 0)$ as the preimage of a covering.

By combining Lemma 1.5 through Lemma 1.10, we complete the proof of Theorem.

2 Universal pretzel links

A. Introduction and Main Theorem.

2.1. In this section we consider the following question *"which pretzel links are universal ?"* We remark that a link, being a union of fibers of a graph-manifold structure on S^3 is not universal. That is because the branched covering space over such a link is a graph-manifold.

2.2. We consider only for the case that $p(-\varepsilon b; p_1, p_2, ..., p_n)$ has two components or more, and so only the following two cases occur :

(I) At least two p_i's are even.
(II) All of the p_i's are odd, and $n + b$ is even. (In this case, the number of components is two.)

We say that $p(-\varepsilon b; p_1, p_2, ..., p_n)$ is type (I) (or type (II), resp.), if it is of the case (I) (or (II), resp.).

2.3. Theorem. *For the pretzel link p of type (I), p is univeral if and only if p is none of the following :*

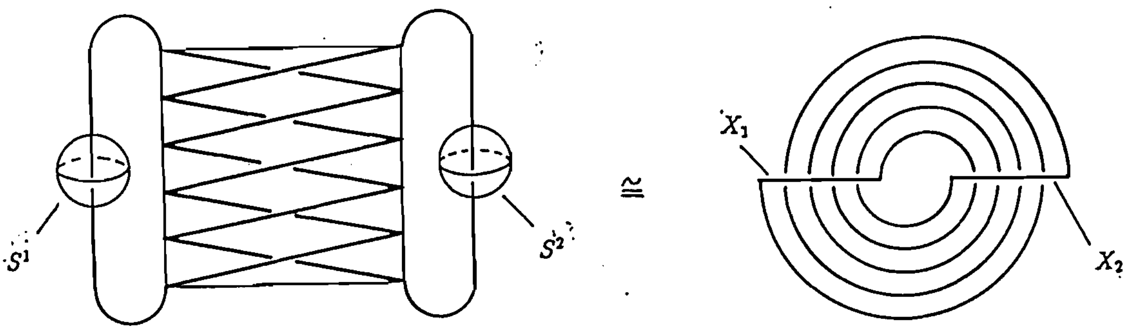
$p(2s, 2t)$, $p(2, -2, s)$, $p(0; -2, 3, 4)$, $p(0; 2, -3, -4)$ $p(0; 3, 6, -2)$, $p(0; -3, -6, 2)$, $p(0; 4, 4, -2)$, $p(0; -4, -4, 2)$, $p(0; 2, 2, -2, -2)$, where $s, t \in \mathbf{Z}\backslash\{0\}$.

As a consequence of Theorem 2.3, we have :

2.4. Theorem. *For a pretzel link p of type (I), p is universal if and only if p is not a union of fibers of any graph-manifold structure on S^3.*

B. Preliminaries.

We represent the 2-bridge torus knot $T(2, a)$ as in Figure 2.1.



$T(2,5)$.

Figure 2.1

2.5. Lemma. *For an odd integer a, the following branched covering space of S^3 branched over $T(2,a)$ is S^3.*

The monodromy map $\phi : \pi_1(S^3 \backslash T(2,a)) \longrightarrow S_{|a|}$, from the knot group of $T(2,a)$ to the symmetry group of $|\ a\ |$ indices, is defined by

$\phi(x_1) = (1\ 2\ ...q)$

$\phi(x_2) = (1\ q+1\ q+2\ ...2q-1)$

where $q = (|\ a\ | +1)/2$ and x_1, x_2 are the meridians as in Figure 2.1.

We denote this covering by $f : S^3 \longrightarrow S^3$.

Proof. The branched covering space of S^3 branched over a 2-component trivial link, associated with the monodromy as in Figure 2.2, is S^3. And the preimage of this link is a $2q$-component trivial link. Now we consider the 2-disk D as shown in Figure 2.2. The $|\ a\ |$-fold branched covering space of D associated with ϕ is also a 2-disk $\tilde{D}$. By a-half twists at D, this trivial link can be deformed into the torus knot $T(2,a)$. And the a-half twists at D are lifted to one-half twist at $\tilde{D}$ (cf. [9], [14, p.317]).

Hence the covering space is S^3. And the preimage of $T(2,a)$ is the torus link $T(2q, \varepsilon q) = T(|\ a\ | +1, \varepsilon \frac{|a|+1}{2}) = \frac{|a|+1}{2} T(2,\varepsilon)$ where

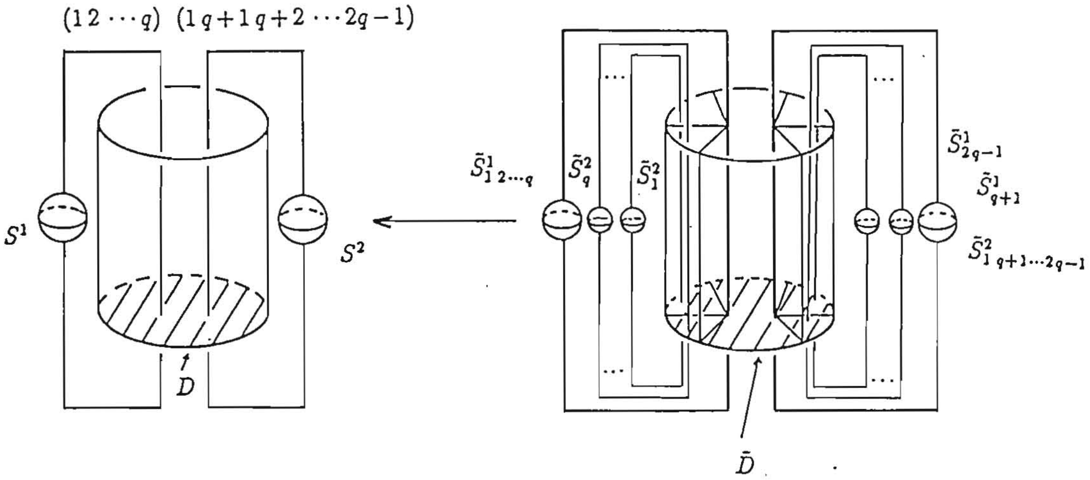


Figure 2.2

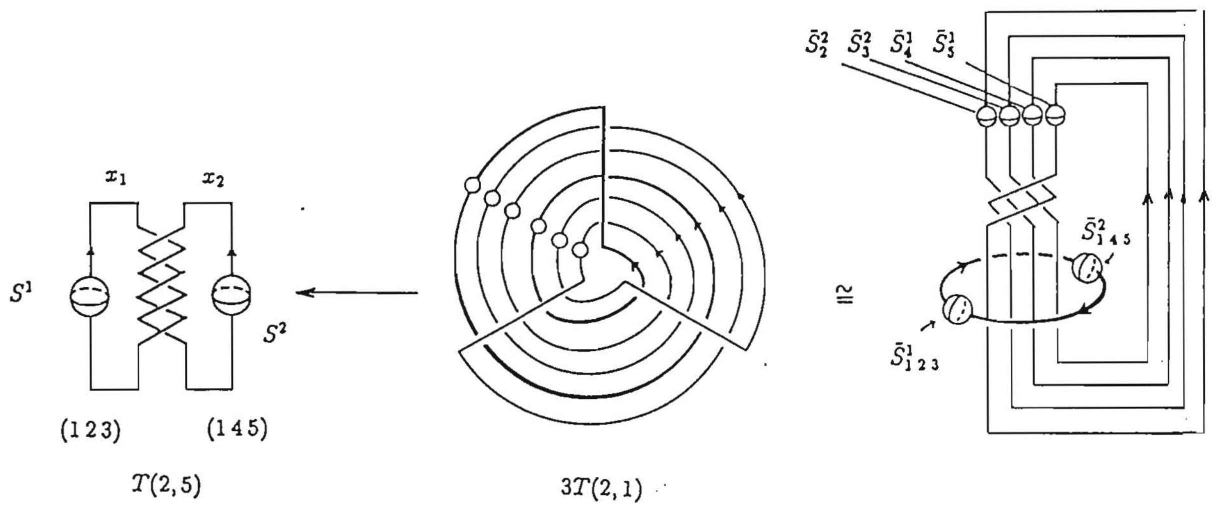


Figure 2.3

$\varepsilon = 1$ if $a > 0$ or $\varepsilon = -1$ if $a < 0$. (In Figure 2.3, a = 5.) In Figures 2.2 and 2.3, the bold lines have branched index q. This completes the proof of Lemma 2.5.

2.6. The line with branch index q is a component of $\frac{|a|+1}{2}T(2, \varepsilon)$, and so it is a trivial knot. Then the n-fold cyclic branched covering branched space over this knot is S^3. The preimage of the other components is $\frac{|a|-1}{2}T(2,\varepsilon n)$.

We denote this covering by $g : S^3 \longrightarrow S^3$.

The monodromy of the covering $f \circ g : S^3 \longrightarrow S^3$ branched over $T(2,a)$, is definded by

$x_1 \longmapsto (1_1\ 2_1 \cdots q_1\ 1_2\ 2_2\ \cdots q_2\ \cdots\cdots 1_n\ 2_n\ \cdots q_n)$,
$x_2 \longmapsto (1_1\ (q+1)_1\ (q+2)_1\ \cdots (2q-1)_1\ 1_2\ (q+1)_2\ (q+2)_2\ \cdots (2q-1)_2\ \cdots\cdots 1_n\ (q+1)_n\ (q+2)_n\ \cdots (2q-1)_n)$.

We call this covering the (C_1)-covering. Unless confusion, we may use a simple form like as

$x_1 \longmapsto (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)\ (=\ (1_1\ 2_1\ 3_1\ 1_2\ 2_2\ 3_2\ 1_3\ 2_2\ 3_3))$,
$x_2 \longmapsto (1\ 10\ 11\ 4\ 12\ 13\ 7\ 14\ 15)\ (=\ (1_1\ 4_1\ 5_1\ 1_2\ 4_2\ 5_2\ 1_3\ 4_3\ 5_3))$.

2.7. Remark. The length of the cyclic permutation associated with this monodromy is $n \times q$.

2.8. We consider the following branched covering space of S^3 branched over $T(2,a_1)\sharp T(2,a_2)\sharp \cdots \sharp T(2,a_m)$, where a_i is odd for $i = 1,2,\ldots,m$.

In Figure 2.4, S^i $(i = 0,1,\ldots,m)$ is a 2-sphere and S^i $(i = 1,2,\ldots,m-1)$ divides the factors of $T(2,a_i)$'s. And $x_0,x_1,\ldots,x_m$ are the meridians as in Figure 2.4. Let $q_i = (|\ a_i\ |+1)/2$ and q the least common multiple of $q_1,\ q_2,\ldots,q_m$. Now we define the permutations corresponding to $x_0,x_1,\ldots,x_m$ as follows :

$x_i \longmapsto (i_1,i_2,\ldots,i_q)$ for $i = 0,1,\ldots,m$,
where we identify $i_{1+kq_{i+1}}$ with $(i+1)_{1+kq_{i+1}}$ for $0 \leq i \leq m-1$, $0 \leq k < q/q_{i+1}$.

We call this covering the (C_2)-covering for $T(2,a_1)\sharp \cdots \sharp T(2,a_m)$.

2.9. Example. $T(2,5)\sharp T(2,3)\sharp T(2,3)$.

The monodromy of the (C_2)-covering is defined by

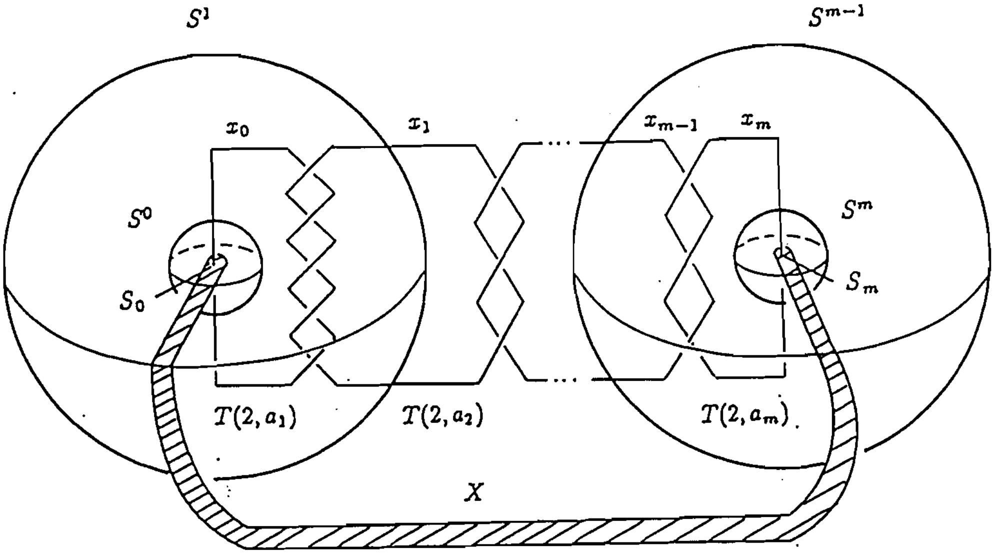


Figure 2.4

$$
\begin{aligned}
x_0 &\longmapsto (0_1\ 0_2\ 0_3\ 0_4\ 0_5\ 0_6),\\
x_1 &\longmapsto (1_1\ 1_2\ 1_3\ 1_4\ 1_5\ 1_6),\\
x_2 &\longmapsto (2_1\ 2_2\ 2_3\ 2_4\ 2_5\ 2_6),\\
x_3 &\longmapsto (3_1\ 3_2\ 3_3\ 3_4\ 3_5\ 3_6),
\end{aligned}
\quad \text{where} \quad
\begin{pmatrix} 0_1 = 1_1 = 2_1 = 3_1 \\ 0_4 = 1_4 \\ 1_3 = 2_3 = 3_3 \\ 1_5 = 2_5 = 3_5 \end{pmatrix}.
$$

or

$$
\begin{aligned}
x_0 &\longmapsto (1\ 2\ 3\ 4\ 5\ 6),\\
x_1 &\longmapsto (1\ 7\ 8\ 4\ 9\ 10),\\
x_2 &\longmapsto (1\ 11\ 8\ 12\ 9\ 13),\\
x_3 &\longmapsto (1\ 14\ 8\ 15\ 9\ 16).
\end{aligned}
$$

2.10. Lemma. *For $T(2,a_1)\sharp T(2a,a_2)\sharp\cdots\sharp T(2,a_m)$, the (C_2)-covering space is S^3.*

Proof. For each factor $T(2,a_i)$, this covering is the (C_1)-covering. Hence, we can assume that the corresponding monodromy is

$$
\begin{aligned}
x_{i-1} &\longmapsto (1_1\ 2_1\ \cdots (q_i)_1\ 1_2\ 2_2\ \cdots (q_i)_2\ \cdots\cdots 1_r\ 2_r\ \cdots (q_i)_r),\\
x_i &\longmapsto (1_1\ (q_i+1)_1\ \cdots (2q_i-1)_1\ 1_2\ (q_i+1)_2\ \cdots (2q_i-1)_2\ \cdots\cdots\\
&\quad 1_r\ (q_i+1)_r\ \cdots (2q_i-1)_r),
\end{aligned}
$$

where $r = q/q_i$.

From 2.6 and Figure 2.3, the preimage is as in Figure 2.5. In Figure 2.5, $\tilde{S}^i_{12\cdots q}$ is the lift of S^i, corresponding to the letter $1, 2, ..., q$ in the permutation of the monodromy.

Making the connect sum of $T(2, a_i)$ and $T(2, a_{i+1})$ at S^i, is lifted to the following operation in the covering space :

(1) We attach these the (C_1)-covering spaces of $T(2, a_i)$ and $T(2, a_{i+1})$ at $\tilde{S}^i_{1_1\ (q_i+1)_1\ (q_i+2)_1 \cdots\cdots (2q_i-1)_r}$.

(2) At the others, $\tilde{S}^i_j$ (unbranched), we attach a copy of the 3-ball bounded by S^i in S^3 which does not contain $T(2, a_i)$.

So, the covering space is S^3. We can perfoem this operation for $T(2, a_1)\sharp \cdots \sharp T(2, a_{i-1})$ and $T(2, a_i)$ $(i = 2, 3, ..., n)$. Then, the covering space of the (C_2)-covering is S^3. This completes the proof of Lemma 2.10.

2.11. Remark. The knot with branch index q is trivial, because it is a connected sum of trivial knots.

2.12. Example. The preimage of $T(2,5)\sharp T(2,3)\sharp T(2,3)$ under the (C_2)-covering.

The monodromy is given in Example 2.9. Figure 2.6 (a_1) ((a_2), (a_3), resp.) shows the preimage of $T(2, a_1) = T(2, 5)$ ($T(2, a_2) = T(2, 3), T(2, a_3) = T(2, 3)$, resp.) for the (C_2)-covering. The preimage of $T(2,5)\sharp T(2,3)\sharp T(2,3)$ is shown in Figure 2.7.

2.13. Preparation I. We consider the preimage of the 2-disk X as in Figure 2.4 under the (C_2)-covering. Let $X \cap \{T(2, a_1)\sharp T(2, a_2)\sharp \cdots \sharp T(2, a_m)\} = \{S_0, S_m\}$. The preimage of X is the covering space of the disk X branched over $\{S_0, S_1\}$. And {the letter in the permutation corresponding to S_0 } $\cap$ {the letter in the permutation corresponding to S_m } $= \{0_1 (= m_1)\}$. So the covering space of X is also a 2-disk $\tilde{X}$. Let X intersect each S^i in the equator. See Figure 2.4.

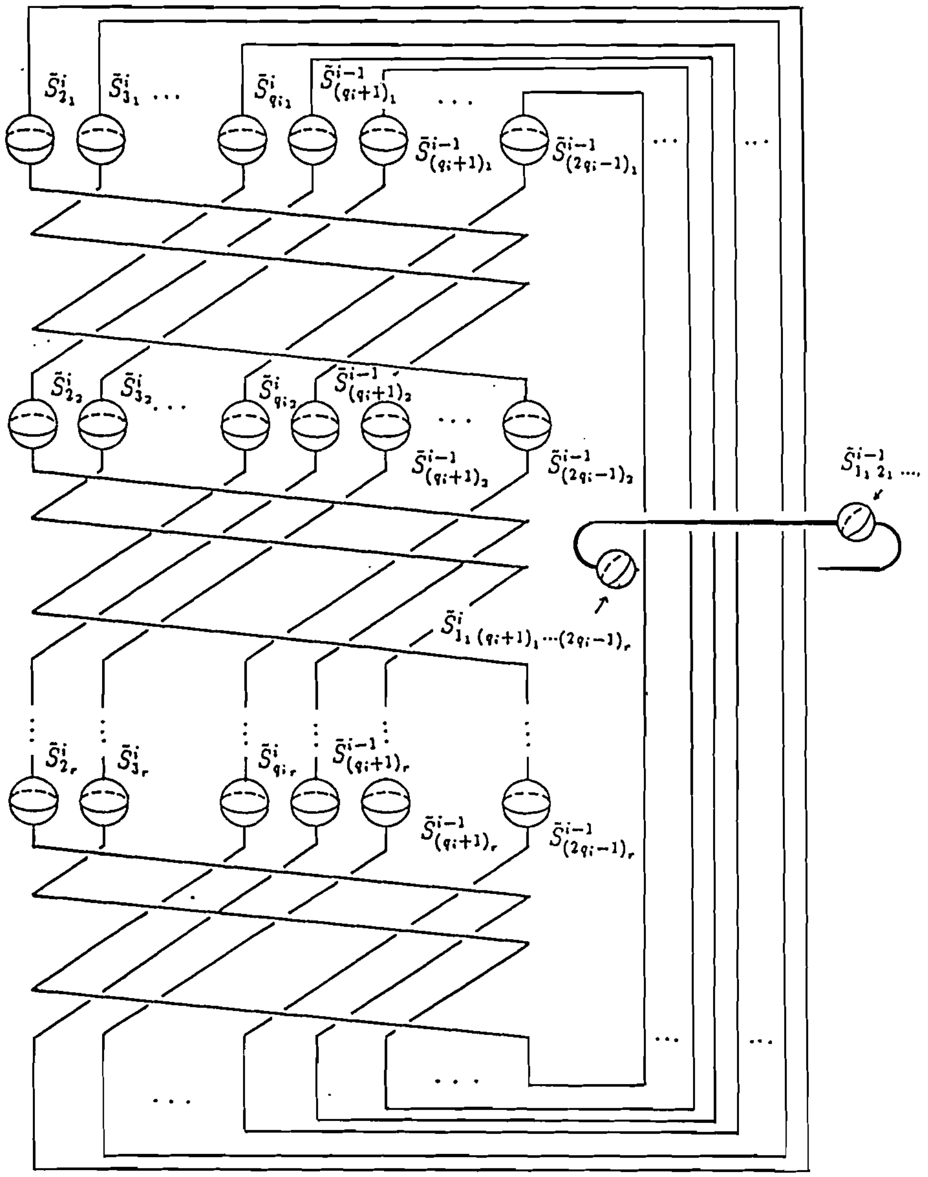


Figure 2.5

2.14. The relation between the preimage of X and the trivial knot with branch index q.

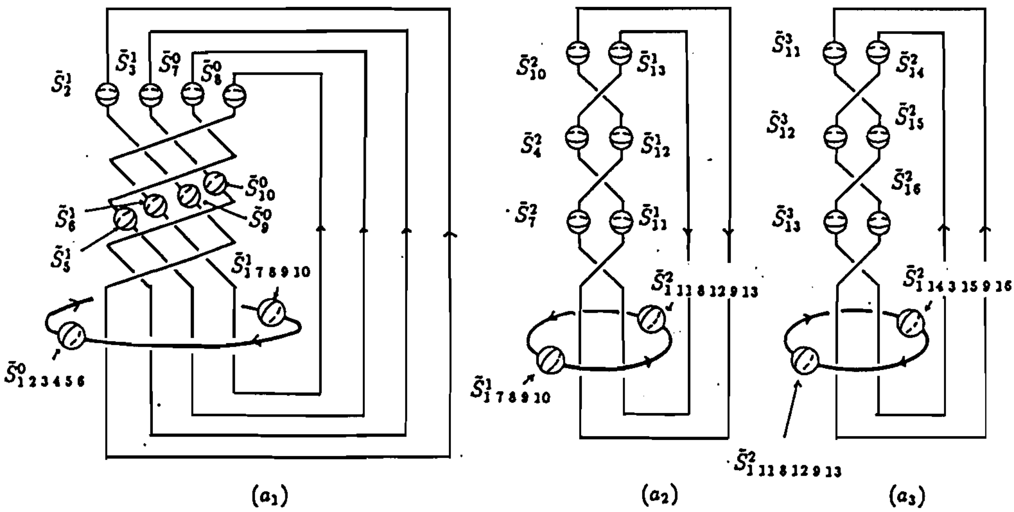


Figure 2.6

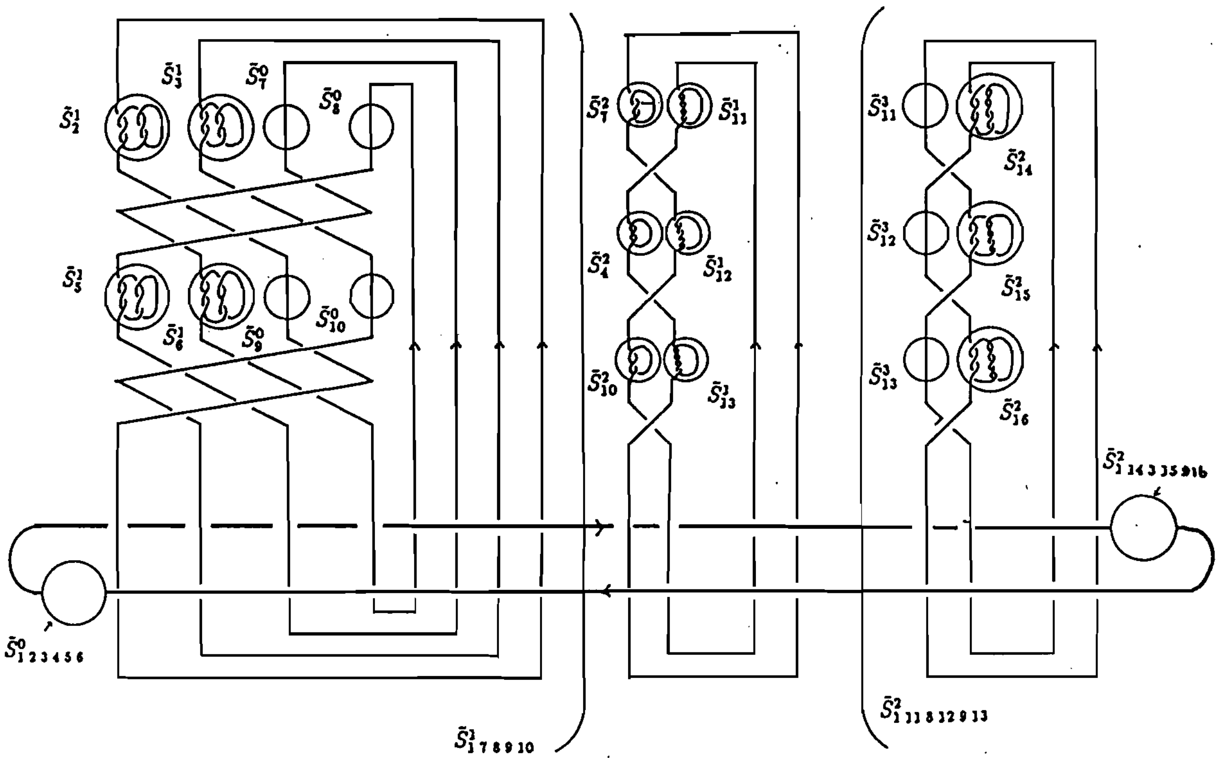


Figure 2.7

For each $T(2, a_i)$ and X, the preimage of $f : S^3 \longrightarrow S^3$ in Lemma 2.5 is shown in Figure 2.8. We notice the trivial knot with branch index q as in Figure 2.9. In Figure 2.9, the twist of the band "1" depends only on the sign of a_i. (In Figure 2.9, $a_i > 0$.) The preimage of $\tilde{X}$ under the r-fold cyclic covering $g : S^3 \longrightarrow S^3$ is shown in Figure 2.10. Then we perform this operation for each $T(2, a_i)$ and attach them at the lifts of S^i.

2.15. Example. For $T(2,5)\sharp T(2,3)\sharp T(2,3)$, Figure 2.11 (a_1) ((a_2), (a_3), resp.) shows the preimage of X for the (C_2)-covering of $T(2, a_1)$ ($T(2, a_2)$, $T(2, a_3)$, resp.), and the knot with branch index q. Figure 2.12 shows the preimage of X and the knot with branch index q.

2.16. Generally, from Example 2.15, the preimage $\tilde{X}$ of X, which is branched, and the trivial knot with branch index q are indicated in Figure 2.13. The twists of the preimage $\tilde{X}$ depend on the sum of signs of a_i's.

For the knot and X as in Figure 2.14, we consider the (C_2)-covering similarly. The preimage is like as in Figure 2.15.

2.17. Preparation II. Let the permutation corresponding to the meridian x_1 for the (C_2)-covering of $T(2, a_1)\sharp T(2, a_2)\sharp \cdots \sharp T(2, a_m)$ be $(1_1\ 2_1\ \cdots (q_1)\ 1_2\ 2_2 \cdots (q_1)_2\ \cdots\cdots 1_r\ 2_r\ \cdots (q_1)_r)$. Now we consider the lifts of X corresponding to 2_1 and $(q_1)_r$. We denote these lifts by $\tilde{X}_{2_1}$ and $\tilde{X}_{(q_1)_r}$.

Remark. 2_1 and $(q_1)_r$ are not the letters contained in the permutation of x_0 and x_2. Hence $g \circ f \mid \tilde{X}_{2_1} : \tilde{X}_{2_1} \longrightarrow X$ and $g \circ f \mid \tilde{X}_{(q_1)_r} : X_{(q_1)_r} \longrightarrow X$ are homeomorphisms. If the permutation corresponding to x_1 is $(1\ 2)$ then $2_1 = (q_1)_r = 2$.

From Figure 2.8, the preimage of $f : S^3 \longrightarrow S^3$ branched over $T(2, a_1)$ is shown in Figure 2.16. And the preimage of $g : S^3 \longrightarrow S^3$ is shown in Figure 2.17. We perform the same operation for $T(2, a_2)$. Then the preimages $\tilde{X}_{2_1}$ and $\tilde{X}_{(q_1)_r}$ are shown in Figure 2.18. Figure 2.19 shows the subset of the preimage under the 2-fold branched covering branched over l in Figure 2.18.

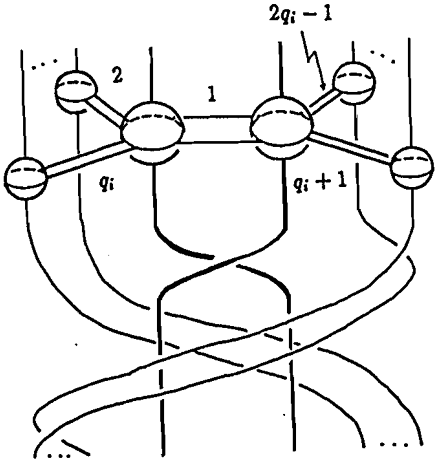


Figure 2.8

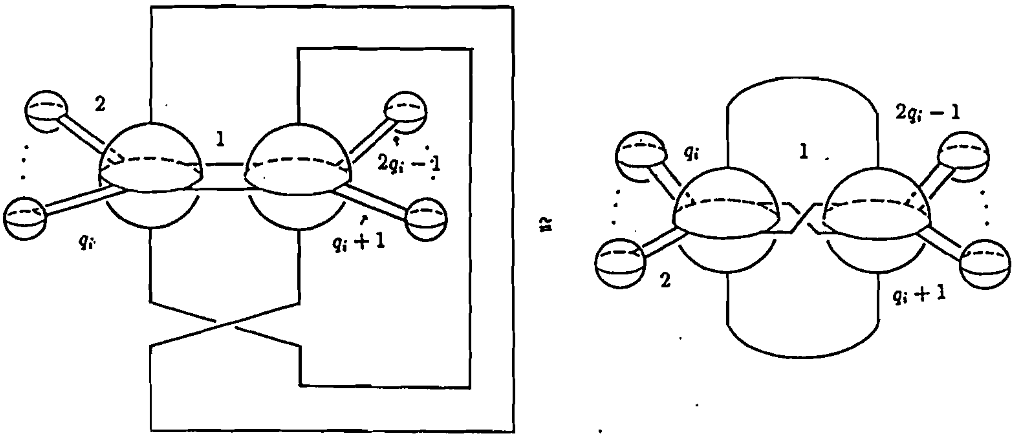


Figure 2.9

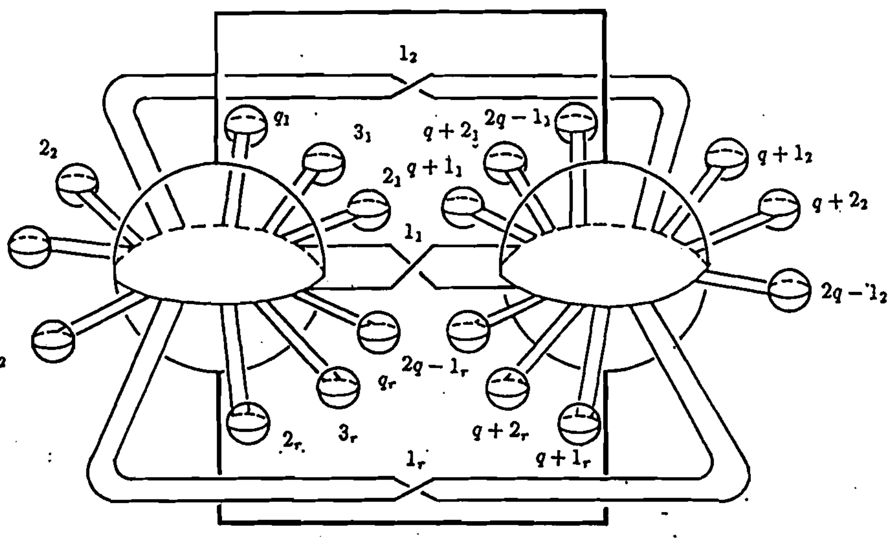


Figure 2.10

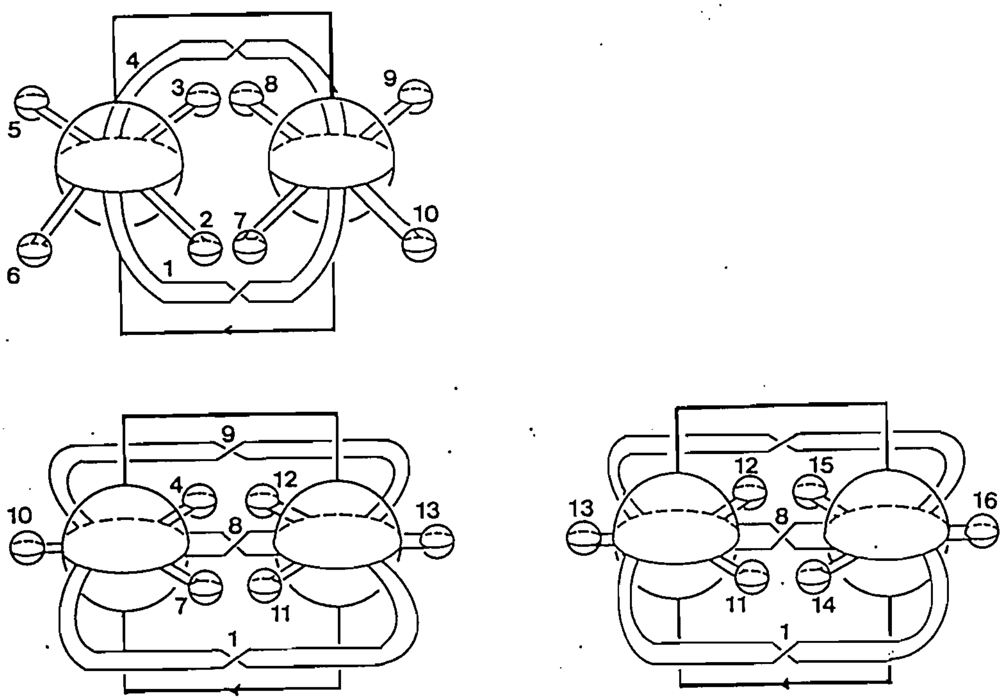


Figure 2.11

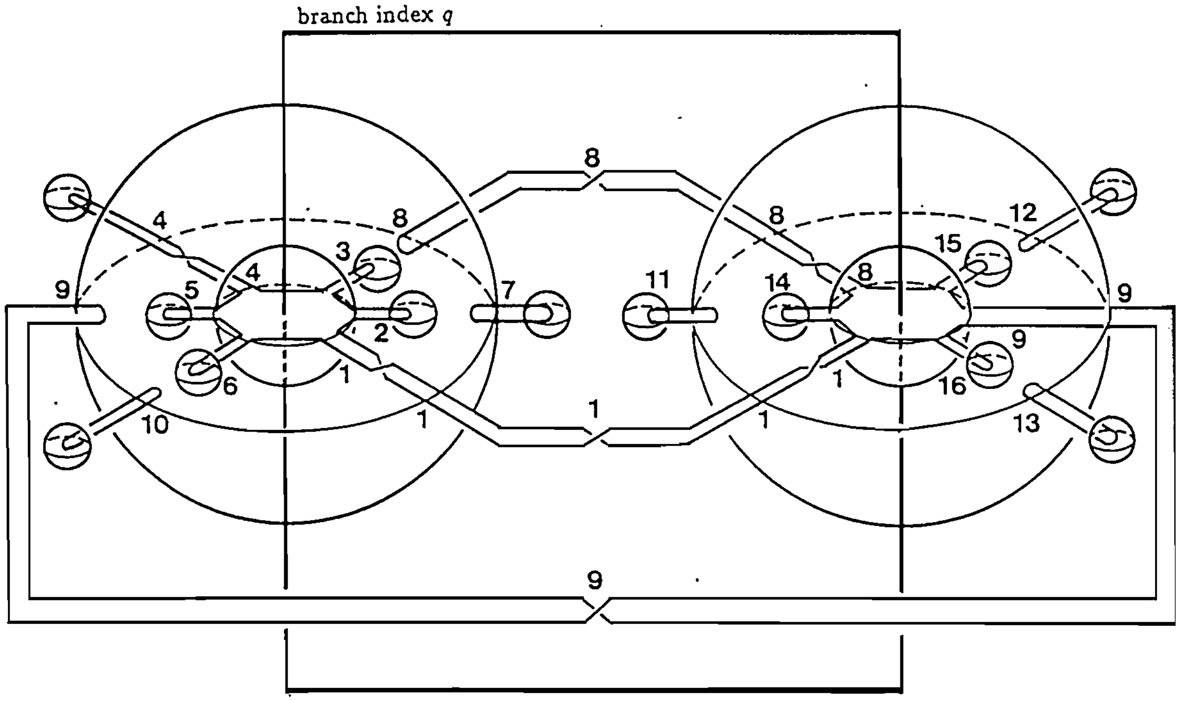


Figure 2.12

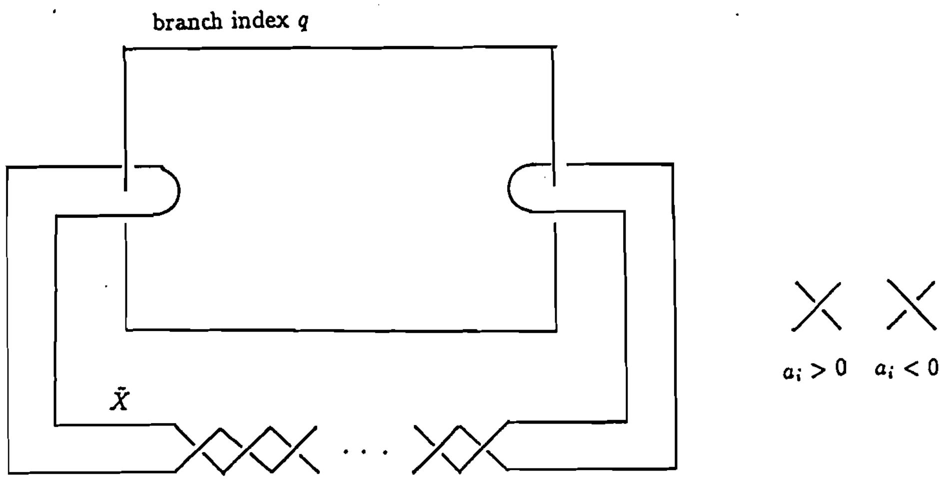


Figure 2.13

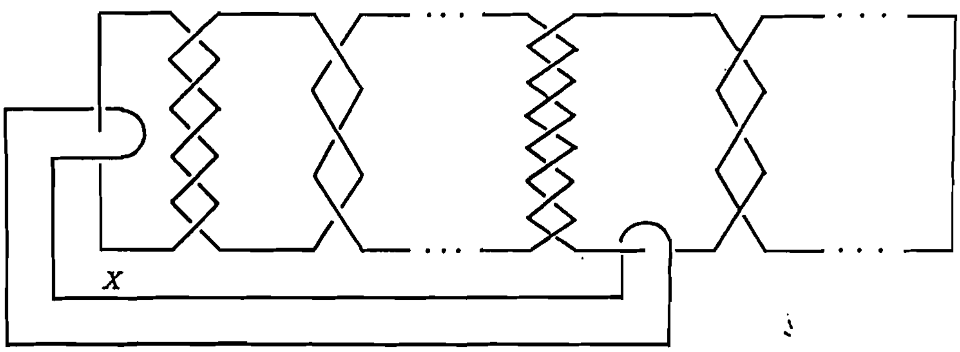


Figure 2.14

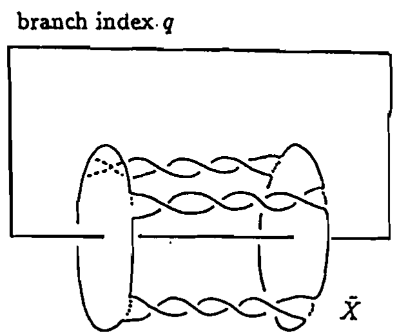


Figure 2.15

We rewrite the bold lines in Figure 2.19 as in Figure 2.20. In

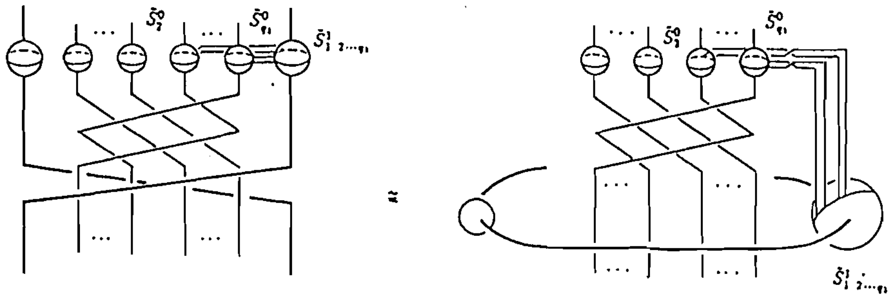


Figure 2.16

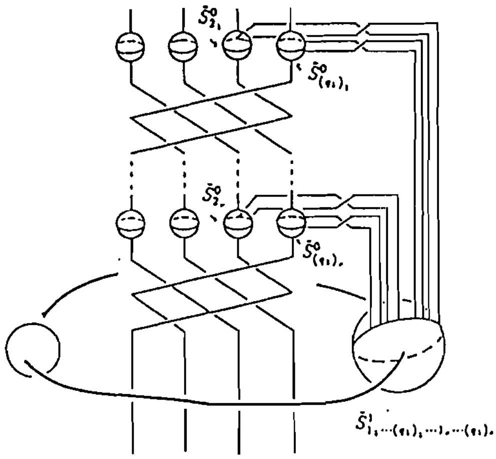


Figure 2.17

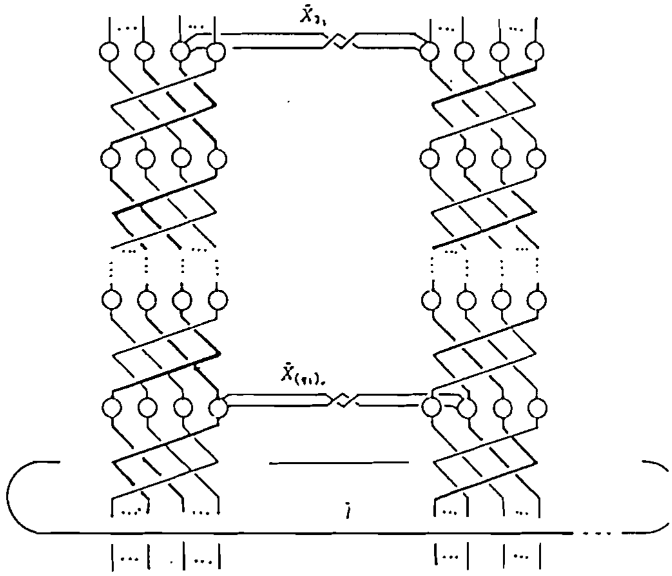


Figure 2.18

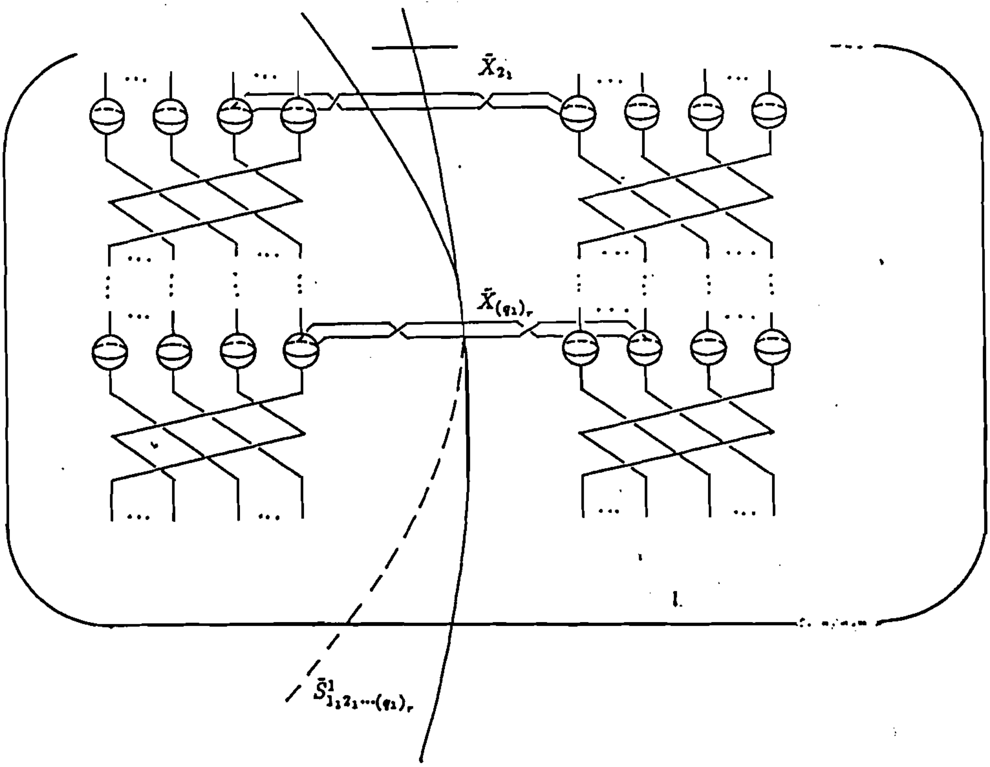


Figure 2.19

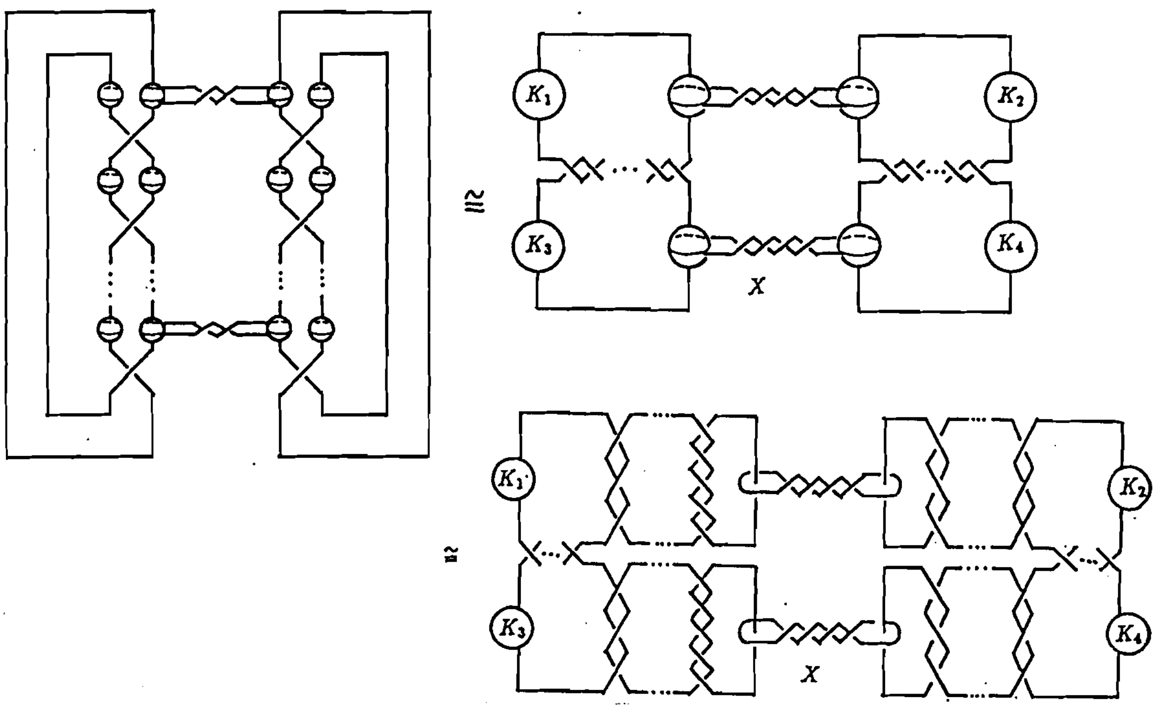


Figure 2.20

Figure 2.20 K_i $(1 \leq i \leq 4)$ is the connected sum of 2-bridge torus knots.

Remark. In Figure 2.20, $a_1, a_2 > 0$, but even in the other cases we can consider similar figures.

C. Proof of Theorem 2.3

In order to show a link L is universal, it is sufficient to find a branched covering space $p : S^3 \longrightarrow S^3$ branched over L such that $p^{-1}(L)$ contains a universal link.

We denote the α-component pretzel link $p(-\varepsilon b; q_1, q_2, ..., q_m)$ by $l_1 \cup l_2 \cup \cdots \cup l_\alpha$. In this case, the pretzel link is type (I) then each l_i is a trivial knot, a 2-bridge torus knot, or a connected sum of 2-bridge torus knots.

We will give the proof on the number of components, $\alpha \geq 5, \alpha = 4, 3$, or 2. The pretzel links in the list of Theorem 2.3 are not universal. Because these links are unions of fibers of graph-manifold structures on S^3. See Figure 2.21.

2.18. Theorem. *Let α be the number of components of $p(-\varepsilon b; q_1, q_2, ..., q_m)$ and $P = p(-\varepsilon b; q_1, q_2, ..., q_m)\sharp(\sharp_{i=1}^{n} T(2, a_i))$, where $n \in N \cup \{0\}$ and a_i is odd for $i = 1, 2, ..., n$.*

If $\alpha \geq 5$, then P is universal.

2.19. Proposition (*cf* Th.1.4). *Let $p = p(-\beta; 2, 2, ..., 2)$ where $\beta \in Z$, $\alpha = \sharp 2 =$ the number of components of p. If $\alpha \geq 5$, then p is universal.*

2.20. Proof of Theorem 2.18. Let P be as in Figure 2.22. In Figure 2.22, R is a tangle containg a connected sum of 2-bride link facters as in Figure 2.22.

(Step I) We divide the Step (I) into two cases.

(Case 1) l_1 is a trivial knot.
(Case 2) Otherwise.

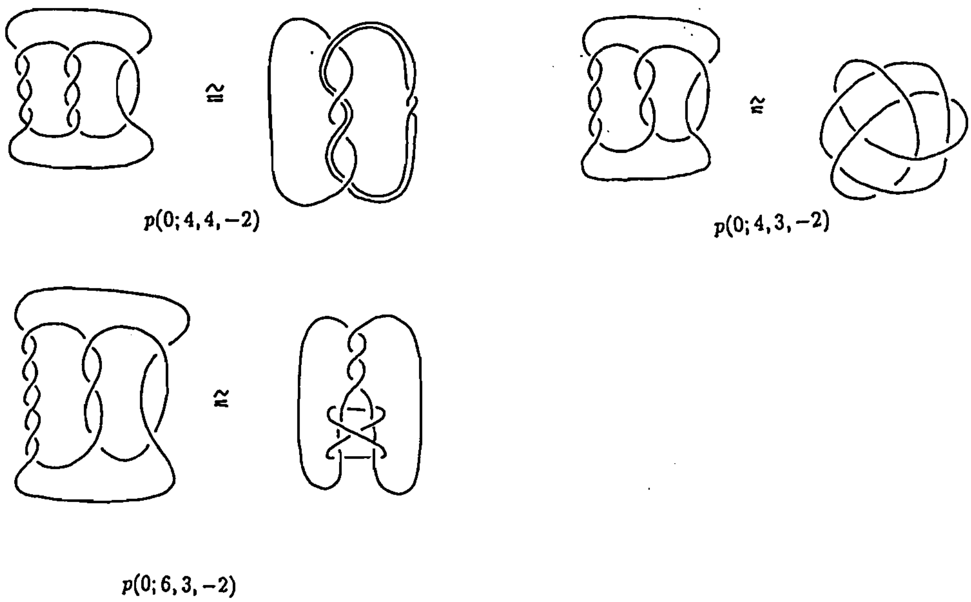


Figure 2.21

(Case 1) Let $A = | lk(l_1, l_\alpha) |, B = | lk(l_1, l_2) |$ and C the least common multiple of A and B. We consider the C-fold cyclic covering $j : S^3 \longrightarrow S^3$ branched over l_1. See Figure 2.23, where $A = 3, B = 2$, and $C = 6$.

Let $\tilde{l}_i$ $(i = 1, 2, ..., \alpha)$ be the component of $j^{-1}(l_1)$ as in Figure 2.23. Note that R has at least four components. Figure 2.24 shows $\tilde{l}_1 \cup \tilde{l}_2 \cup \cdots \cup \tilde{l}_\alpha$. Then this operation changes each component into one of the following :

$| lk(\tilde{l}_1, \tilde{l}_\alpha) | = | lk(\tilde{l}_1, \tilde{l}_2) | = 1$,

$\tilde{l}_1$ is a trivial knot,

$\tilde{l}_\alpha$ is a connected sum of l_α and a connected sum of 2-bridge torus knots (possibilty empty), which is the component contained in R,

$\tilde{l}_2$ is a connected sum of l_2 and a connected sum of 2-bridge torus knots (possibilty empty), which is the component contained in R,

$\tilde{l}_i = l_i$ otherwise.

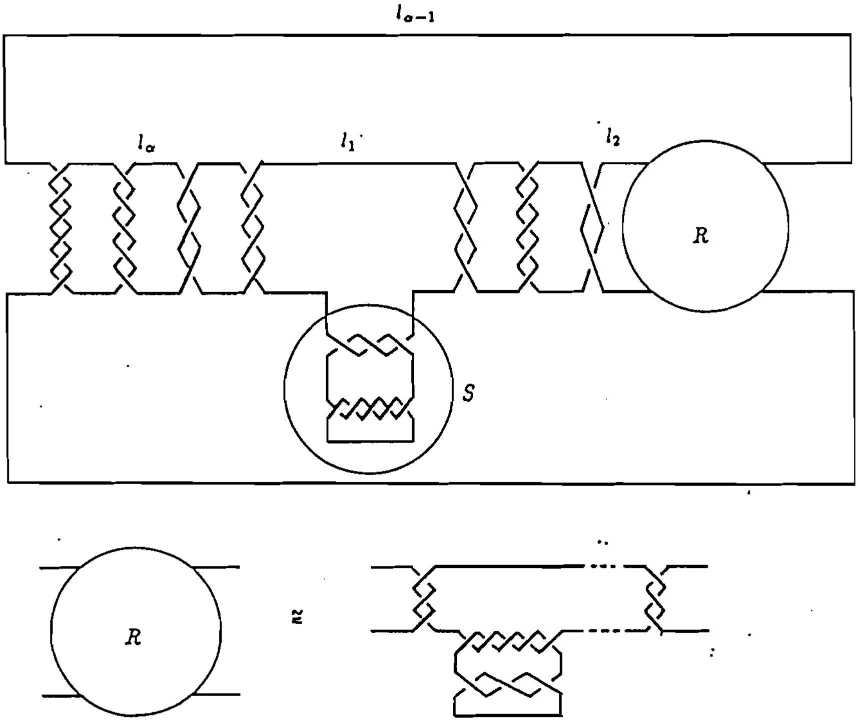


Figure 2.22

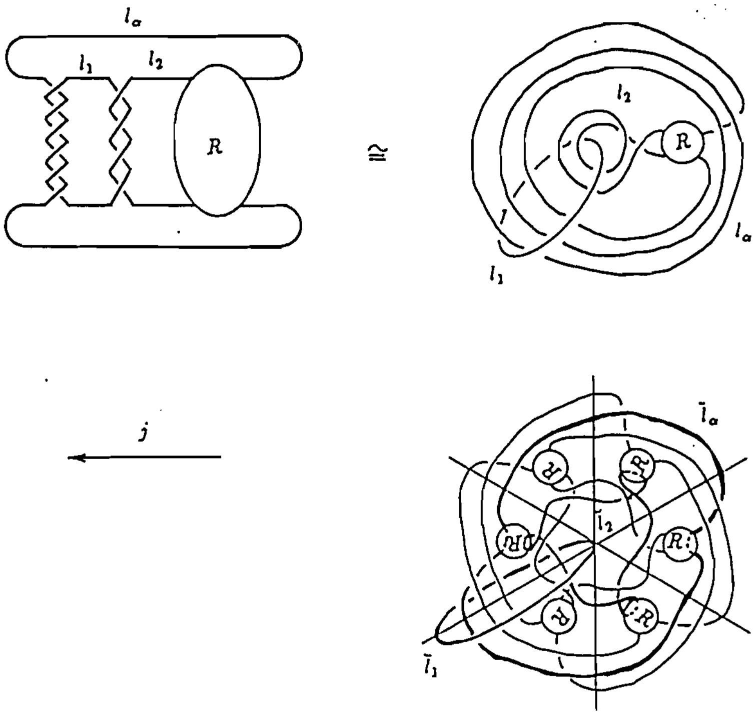


Figure 2.23

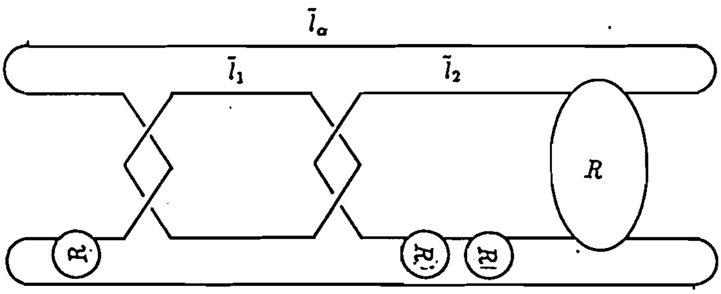


Figure 2.24

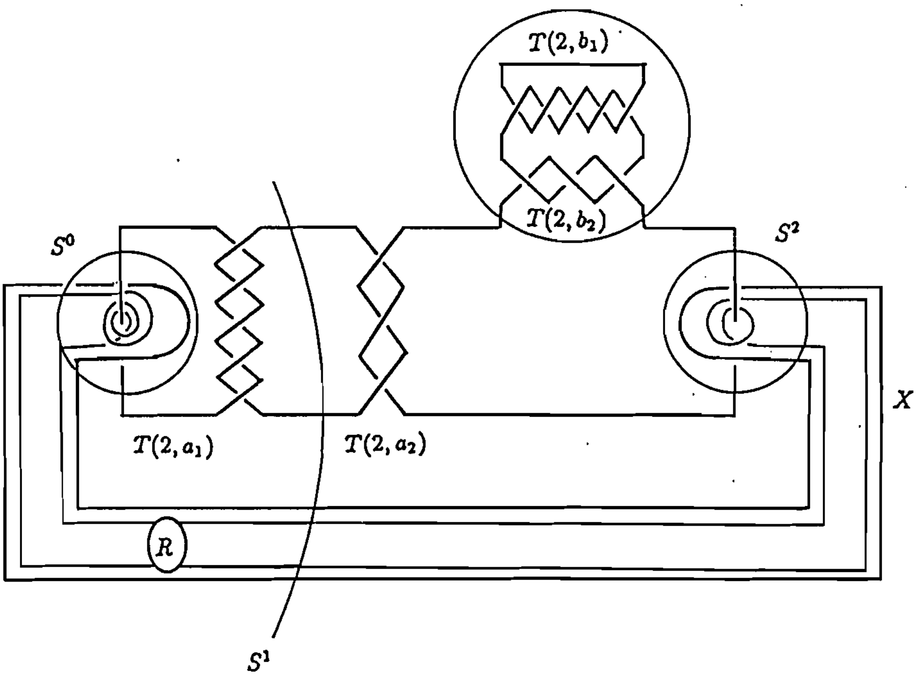


Figure 2.25

To simplify we denote $\tilde{l}_1 \cup \tilde{l}_2 \cup \cdots \cup \tilde{l}_\alpha$ by $l_1 \cup l_2 \cup \cdots \cup l_\alpha$.

(Case 2) We consider the (C_2)-covering branched over l_2. Let q be the length of the permutation. To consider the preimage of $l_2 \cup l_3 \cup \cdots \cup l_\alpha$, we can regared this link is contained in the 3-ball $X \times \mathrm{I}$ where $\mathrm{I} = [0,1]$. See Figure2.25.

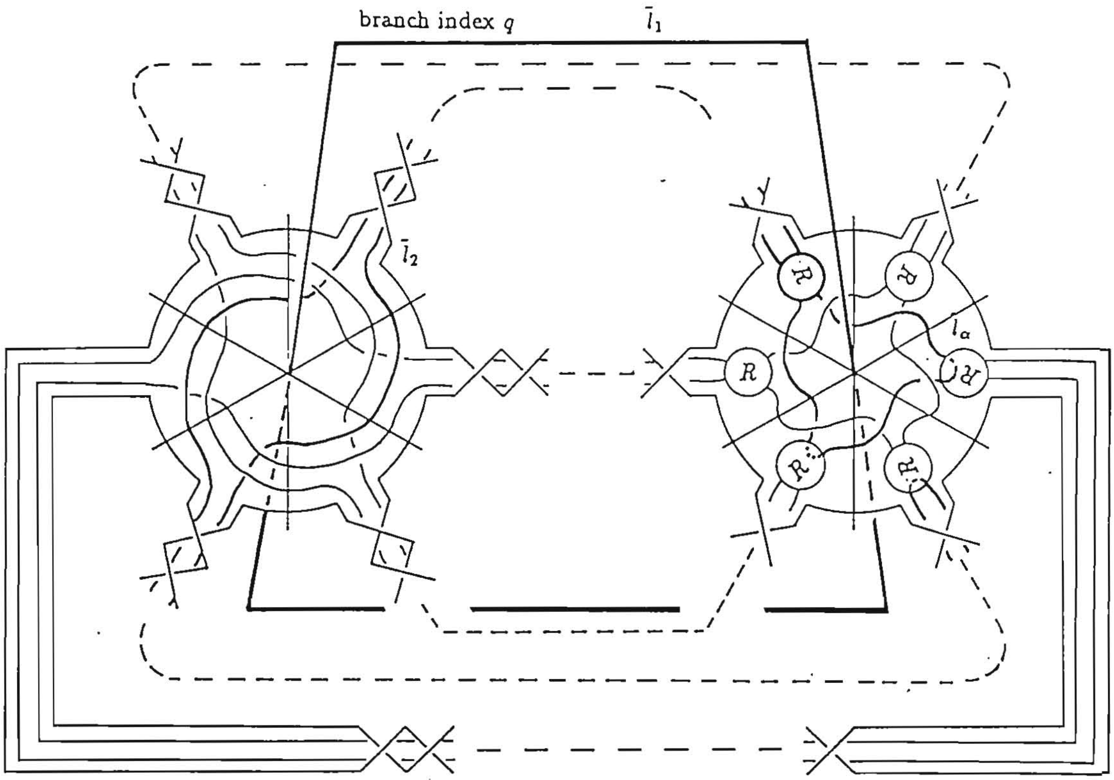


Figure 2.26

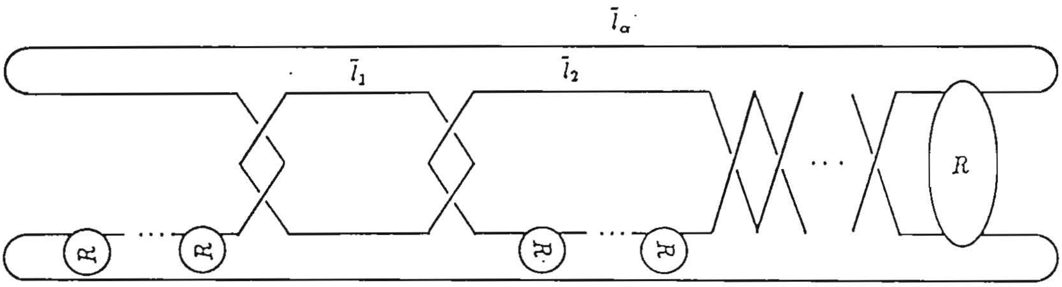


Figure 2.27

From Figures 2.14 and 2.15, the trivial knot with branch index q

and the preimage $\tilde{X}$ of X can be regarded as the same as in Figure 2.15.

Let $A = | lk(l_1, l_\alpha) |$, $B = | lk(l_1, l_2) |$, C the least common multiple of A and B. Figure 2.26 shows the preimage of $\tilde{X}$ in the C-fold cyclic covering space branched over the trivial knot of Figure 2.26. We rewrite the bold lines in Figure 2.26 as in Figure 2.27.

Now $\tilde{l}_i$ is the preimage of l_i which is shown in Figure 2.27. Then we have a new link $\tilde{l}_1 \cup \tilde{l}_2 \cup \cdots \cup \tilde{l}_\alpha$ such that

$| lk(\tilde{l}_1, \tilde{l}_\alpha) | = | lk(\tilde{l}_1, \tilde{l}_2) | = 1$,

$\tilde{l}_1$ is a trivial knot,

$\tilde{l}_2$ ($\tilde{l}_\alpha$ resp.) is the connected sum of l_2 (l_α resp.) and the connected sum of 2-bridge torus knots (possibly empty) which is contained in the tangle R,

$\tilde{l}_i = l_i$ otherwise.

The twists of $\tilde{l}_2$ in Figure 2.27 are the result of the twists on X. For convenience, we use the same simbole $l_1 \cup \cdots \cup l_\alpha$ instead of $\tilde{l}_1 \cup \cdots \cup \tilde{l}_\alpha$.

(Step II) We divide the Step (II) into two cases.

(Case 1) l_2 is trivial.
(Case 2) Otherwise.

(Case 1) We perform the same operation of (Case 1) in (Step I). Since l_1 is trivial and $| lk(l_1, l_\alpha) | = | lk(l_1, l_2) | = 1$, we have a new link $\tilde{l}_1 \cup \cdots \cup \tilde{l}_\alpha$ such that

$\tilde{l}_1$ and $\tilde{l}_2$ are trivial knots,

$| lk(\tilde{l}_1, \tilde{l}_\alpha) | = | lk(\tilde{l}_1, \tilde{l}_2) | = | lk(\tilde{l}_2, \tilde{l}_3) | = 1$,

$\tilde{l}_3 = l_3 \sharp \{T(2, a_1) \sharp \cdots \sharp T(2, a_s)\}$ where $T(2, a_1) \sharp \cdots \sharp T(2, a_s)$ (possibly empty) $\subset R$,

$\tilde{l}_i = l_i$ otherwise.

For convenience, we use the same simbole $l_1 \cup \cdots \cup l_\alpha$ instead of $\tilde{l}_1 \cup \cdots \cup \tilde{l}_\alpha$.

(Case 2) We perform the operation of (Case 2) in (Step I). Since l_1 is trivial and $| lk(l_1, l_\alpha) |=| lk(l_1, l_2) |= 1$, we have a new link $\tilde{l}_1 \cup \cdots \cup \tilde{l}_\alpha$ such that

$\tilde{l}_1$ and $\tilde{l}_2$ are trivial knots,
$| lk(\tilde{l}_1, \tilde{l}_\alpha) |=| lk(\tilde{l}_1, \tilde{l}_2) |=| lk(\tilde{l}_2, \tilde{l}_3) |= 1$,
$\tilde{l}_3 = l_3 \sharp \{T(2, b_1) \sharp \cdots \sharp T(2, b_s)\}$, where $T(2, b_1) \sharp \cdots \sharp T(2, b_s)$ (possibly empty) $\subset R$,
$\tilde{l}_i = l_i$ otherwise.

For convenience, we use the same simbole $l_1 \cup \cdots \cup l_\alpha$ instead of $\tilde{l}_1 \cup \cdots \cup \tilde{l}_\alpha$.

We perform this operation in order on the number of the index of l_i. Finally, we perform this operation at l_α. Since l_1 and $l_{\alpha-1}$ are trivial and $| lk(l_{\alpha-1}, l_\alpha) |= | lk(l_\alpha, l_1) |= 1$, then the connected sum does not appear in l_1. We have the following link,

l_i $(i = 1, 2, ..., \alpha)$ is trivial knot,
$| lk(l_i, l_{i+1}) |= 1$ $(i = 1, 2, ..., \alpha$ and $l_{\alpha+1} = l_1)$.

This link is the pretzel link $p(-\beta; 2, 2, ..., 2)$ where $\sharp 2 = \alpha \geq 5$, $\beta \in \mathbf{Z}$. From Proposition 2.19, this link is universal. This completes the proof of Theorem 2.18.

2.21. Corollary. *The pretzel link, which has at least five components, is universal.*

2.22. Proposition. *If* $\alpha = 4$, *all pretzel links are universal, but except* $p(0; 2, 2, -2, -2)$.

Proof. Let $p = p(-b\varepsilon; q_1, q_2, ..., q_n) = l_1 \cup l_2 \cup l_3 \cup l_4$. We divide the proof into three cases.

(Case I) $p \supset T(2, a_1) \sharp T(2, a_2) \sharp \cdots \sharp T(2, a_m)$ (a_i is odd, $| a_i |\geq 3$, for $i = 1, 2, ..., m$, and $m \geq 2$). Then p is universal.

(Case II) $p \supset T(2, a)$ (a is odd and $| a |\geq 3$). Then p is universal.

(Case III) All l_i's $(i = 1, 2, 3, 4)$ are trivial. Then p is universal, but except $p(0; 2, 2, -2, -2)$.

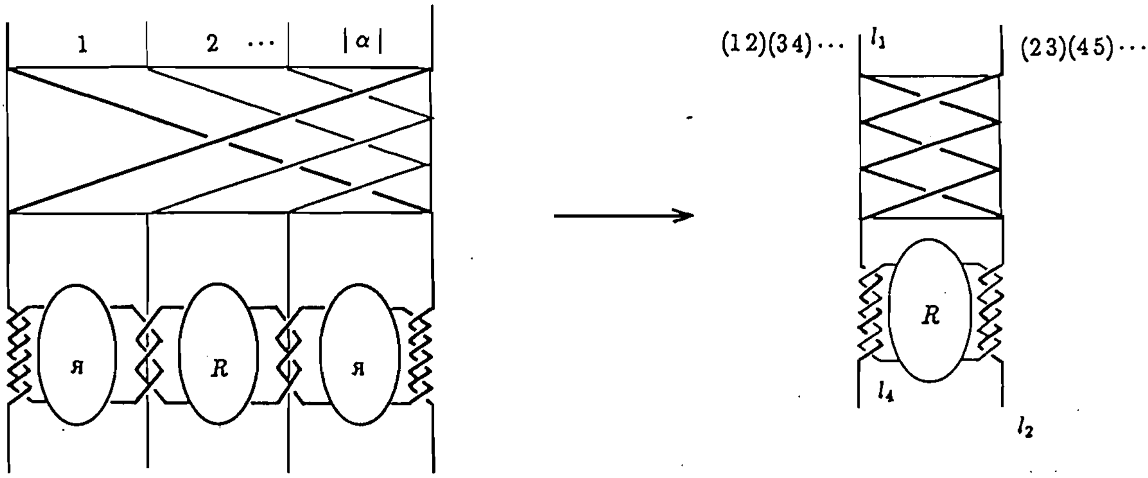


Figure 2.28

Proof of Case (I). Let $l_1 = T(2, a_1)\sharp T(2, a_2)\sharp \cdots \sharp T(2, a_m)$. We consider the (C_2)-covering branched over l_1. We regard that $l_2 \cup l_3 \cup l_4 \subset X \times \mathrm{I}$, see Figure 2.25. From Preparation 2.17 and Figure 2.20, we have (a 10-component pretzel link) $\sharp$ (2-bridge torus knots). From Theorem 2.18, this link is universal, completing the proof.

Proof of Case (II). Let $l_1 = T(2, a)$. Figure 2.28 shows the $|\, a \,|$-fold irregular dihedral branched covering space corresponding to $l_2 = T(2, a)$. See [6] and [7].

The bold line in Figure 2.28 is a pretzel link, which has at least $2 \,|\, a \,| + 2$ components. Since $|\, a \,| \geq 3$, this link is universal, completing the proof.

Proof of Case (III). We divide the proof into two cases.

(Case III-1) $|\, lk(l_i, l_{i+1}) \,| = 1$ $(1 \leq i \leq 4$ and $l_5 = l_1)$.
Then p is universal, but except $p(0; 2, 2, -2, -2)$.

The proof is in Theorem 1.4.

(Case III-2) $|\, lk(l_i, l_{i+1}) \,| \neq 1$ for some $i(1 \leq i \leq 4)$. Then p is universal.

Proof. Since l_i and l_{i+1} are trivial, $l_i \cup l_{i+1} = T(2, a)$ (a is even and $|\, a \,| \geq 4$). We consider the $|\, a \,|$-fold irregular dihedral covering space branched over $T(2,a)$ (cf. Figure 2.28). Then we have a pretzel link with $|a|+2$ components or more. Since $|a|\geq 4$, it is universal by Theorem 2.18. This completes the proof of Proposition 2.22.

2.23. Remark. Suppose that the pretzel link has two (three, resp.) components. If it contains a connected sum of 2-bridge torus knots as a sublink, then we perform the operation of Case (I) in the proof of Proposition 2.22. We have { a six (eight resp.)-component pretzel link } $\sharp$ { 2-bridge torus knots}. Then p is universal.

2.24. Proposition. *If $\alpha = 3$, the pretezl link p is universl, but except*
$p(0;4,4,-2)$, $p(0;-4,-4,2)$, and $p(2,-2s)$ (s is even).

Proof. Let $p = l_1 \cup l_2 \cup l_3$. We divide the proof into three cases.

(Case I) $p \supset T(2,a_1)\sharp T(2,a_2)\sharp\cdots\sharp T(2,a_m)$ (a_i is odd, $|a_i|\geq 3$ and $m\geq 2$). Then p is universl.

(Case II) $p \supset T(2,a)$ (a is odd and $a\geq 3$). Then p is universal.

(Case III) All $l_i,(i=1,2,3)$ are trivial. Then p is universal, but except $p(0;4,4,-2)$, $p(0;-4,-4,2)$, and $p(-2,-2,s)$ (s is even).

Proof of Case (I). See Remark 2.23.

Proof of Case (II). Let $l_1 = T(2,a)$. By the operation in Case(II) in the proof of Propositon 2.22, we have a pretzel link with $|a|+2$ components or more. Since $|a|\geq 3$, this link is universal, completing the proof.

Proof of Case (III). We divide the proof into two cases.

(Case III-1) $|lk(l_i,l_{i+1})|=1$ for $1\leq i\leq 3$. Then p is universal, but except $p(0;2,-2,-2)$ and $p(0;-2,2,2)$.

Proof. See Theorem 1.4.

(Case III-2) $|lk(l_i,l_{i+1})|\neq 1$ for some i ($i=1,2,3$). Then p is universal, but except $p(2,-2,s)$ (s : even), $p(0;4,4,-2)$ and $p(0;-4,-4,2)$.

Proof. $l_i \cup l_{i+1}$ is equivalent to $T(2,a)$ where a is even and $|a| \geq 4$. Let $l_1 \cup l_2 = l_i \cup l_{i+1}$ and $a_i = |lk(l_i, l_3)|$.

(Case III-2-1) $a_i \equiv 1 \pmod 2$ for $i = 1, 2$. We will show that p is universal, but except $p(2, -2, s)$.

(i) In the case of $a_1 = a_2 = 1$.

We can represent p by $p(-\beta; 2, -2, a)$ ($\beta \in \mathbf{Z}$). Then we have $p(-(|a|\beta - 1) \pm 1; 2, 2)$ as a sublink of the preimage by the $|a|$-fold irregular dihedral covering space branched over $T(2,a)$ (cf. Figure 2.28). Now $p(-\beta'; 2, 2)$ is universal, but except $\beta' = -2, 1$, and 0. If $|a|\beta - 1 \pm 1 = -2, 1$ or 0, then $\beta = 0$, from $|a| \geq 4$. Hence, if $\beta \neq 0$, then this link is universal.

(ii) In the case of $a_1 \neq 1$ or $a_2 \neq 1$.

We can assume that $a_1 \neq 1$. Let x_i ($i = 1, 2$) be the meridian of l_i. Then we consider the $|a|$-fold irregular dihedral covering space branched over $T(2,a)$ with the monodromy defined by

$$x_1 \longmapsto (1\ 2)(3\ 4)...$$
$$x_2 \longmapsto (2\ 3)(4\ 5)...$$

There is a 2-component pretzel link in the preimage (see Figure2.28.). Since a_1 is odd, $a_1 \neq 1$ and $|a| \geq 4$, this pretzel link contains a connected sum of 2-bridge torus knots $T(2,a_1)\sharp \cdots \sharp T(2,a_1)$. From Remark 2.23, p is universal.

(Case III-2-2) $a_1 \equiv 0$ or $a_2 \equiv 0 \pmod 2$. Then p is universal, but except $p(0; 4, 4, -2)$ and $p(0; -4, -4, 2)$.

Proof. We can assume that $a_1 \equiv 0 \pmod 2$. We consider the $|a|$-fold irregular dihedral covering space branched over $T(2,a)$. (See Figure 2.28.) Then we have a new pretzel link. If $|a| \geq 6$ ($a_2 \equiv 0 \pmod 2$ resp.), then the new pretzel link has at least $|a|/2 + 2$ ($|a| + 1$, resp.) components. Thus we have a pretzel link with five components or more. Then it is universal.

We consider the case of $|a| = 4$ and $a_2 \equiv 1 \pmod 2$. The new pretzel link has four-components. If $a_1 \geq 4$ or $a_2 \neq 1$, this pretzel knot is not $p(0;2,2,-2,-2)$, then it is universal. (See Proposition 2.22.)

We consider the case of $a_1 = 2$ and $a_2 = 1$. This pretzel link is equivalent to $p(-\beta;4,4\varepsilon,2)$ or $p(-\beta;-4,4\varepsilon,2)$ ($\varepsilon = \pm 1$). If $p = p(-\beta;4,4\varepsilon,2)$, then we consider the $|\,a\,|$-fold irregular dihedral covering space branched over $T(2,a)$. See Case III-2-1-(ii) in the proof of Proposition 2.24. Then we have a new pretzel link $p(-(4\beta+1+\varepsilon);2,2,2,2)$. This link is universal, if $4\beta+1+\varepsilon \neq -2$. So p is universal, but except $\varepsilon = 1$ and $\beta = -1$. Hence p is universal except $p(0;4,4,-2)$. If $p = p(-\beta;-4,4\varepsilon,2)$, we perform the same operation. Then p is universal, but except $p(0;-4,-4,2)$. This completes the proof of Proposition 2.24.

2.25. Proposition. *If $\alpha = 2$, the pretzel link p is universl, but except $p(0;2s,2t)$ $(s,t \in \mathbf{Z}\backslash\{0\})$, $p(0;3,6,-2)$, $p(0;-3,-6,2)$, $p(0;-2,-3,4)$, $p(0;2,3,-4)$, $p(2,-2,s)$ $(s$: odd).*

Proof. Let $p = l_1 \cup l_2$. We divide the proof into three cases.

(Case I) $p \supset T(2,a_1)\sharp\cdots\sharp T(2,a_m)$ (a_i is odd, $|\,a_i\,| \geq 3$ and $m \geq 2$). Then p is universal.

(Case II) $p \supset T(2,a)$ (a is odd and $|\,a\,| \geq 3$). Then p is universal, but except $p(2,-2,s)$ (s : odd), $p(0;3,6,-2)$, $p(0;-3,-6,2)$, $p(0;3,4,-2)$, $p(0;-3,-4,2)$.

(Case III) Both l_1 and l_2 are trivial. Then p is universal, but except $p(2s,2t)$ $(s,t \in \mathbf{Z}\backslash\{0\})$.

Proof of Case (I). See Remark 2.23.

Proof of Case (II). Let $l_1 = T(2,a)$. From Case (I), we can assume that l_2 is a torus knot or a trivial knot.

Step 1. l_2 is a torus knot. We can denote p by $p(-\beta;2p_1,a,2p_3,p_4)$ ($p_1,p_3 \in \mathbf{Z}\backslash\{0\}$, a, p_4 are odd, and $|\,a\,|$, $|\,p_4\,| \geq 3$).

We consider the $|\,a\,|$-fold dihedral covering space branched over $l = T(2,a)$. See Figure 2.28. If both p_1 and p_3 are even, then we have a $(|\,a\,|+1)$-component pretzel link. If $|\,a\,| = 3$, then we have a 4-component pretzel link and it contains $T(2,p_4)$ ($|\,p_4\,| \geq 3$), and so it is not $p(0;2,2,-2,-2)$. From Proposition 2.22, p is universal. If $|a|\geq 4$, we have a pretzel link which has at least five components. So it is universal.

If p_1 or p_3 is odd, then we have a pretzel link with two components or more (cf. Figure 2.28) and this pretzel link contains $T(2,p_4)\sharp\cdots\sharp T(2,P_4)$. From Theorem 2.18 and Remark 2.23, it is universal.

Step 2. l_2 is a trivial knot. We can denote p by $p(-\beta;2p_1,a,2p_3) = l_1\cup l_2$ (a : odd, p_1, $p_3\in \mathbf{Z}\backslash\{0\}$). We divide the proof into two cases.

(Case II-1) p_1, $p_3\equiv 1 \pmod 2$.
(Case II-1-i) $|p_1|=|p_3|=1$. Then p is universal, but except $p(2,-2,s)$ (s : odd).
(Case II-1-ii) $|p_1|\neq 1$ or $|p_3|\neq 1$. Then p is universal, but except $p(0;3,6,-2)$ and $p(0;-3,-6,2)$.

(Case II-2) $p_1\equiv 0$ or $p_3\equiv 0 \pmod 2$. Then p is universal, but except $p(0;3,4,-2)$ and $p(0;-3,-4,2)$.

Proof of (Case II-1-i). We can denote p by $p(-\beta;2,a,-2)$. We consider the $|a|$-fold dihedral covering space branched over $l_1 = T(2,a)$ (cf. Figure 2.28). Then we have a 2-component pretzel link $p(-|a|\beta-1+a/|a|;2,2)$. If $p(-\beta';2,-2)$ is universal, then $\beta'\neq -2,-1$, and 0 (cf. Theorem1.4). Since $|a|\geq 3$, if $\beta\neq 0$, then $p(-\beta;2,a,-2)$ is universal. We conclude that $p(-\beta;2,a,-2)$ is universal, but except $p(2,s,-2)$ $(=p(2,-2,s))$ (s : odd), completing the proof.

Proof of (Case II-1-ii). We can assume that $|p_1|\neq 1$. We divide the proof into four cases.

(1) $|a|\geq 5$ ———————— Then p is unversal.
(2) $|a|=3$, $|p_3|\neq 1$ ———————— Then p is universal.
(3) $|a|=3$, $|p_3|=1$, $|p_1|\geq 5$ ———————— Then p is universal.
(4) $|a|=3$, $|p_3|=1$, $|p_1|=3$ ———————— Then p is univesal, but except $p(0;3,6,-2)$ and $p(0;-3,-2,6)$.

We consider the $|a|$-fold dihedral covering space branched over $l_1=T(2,a)$. Then we have a 2-component pretzel link.

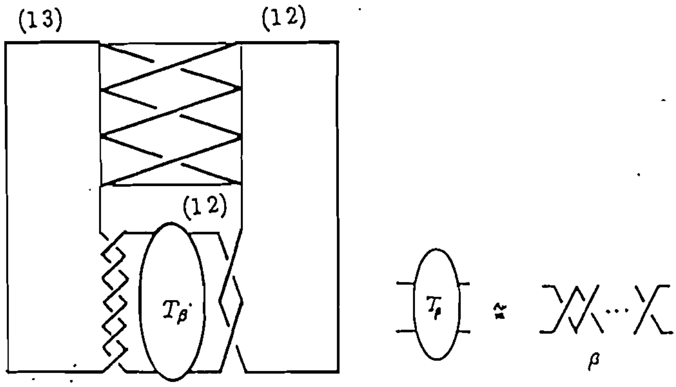


Figure 2.29

(1) Since $|p_1| \neq 1$, the pretzel link contains $T(2,p_1)\sharp \cdots \sharp T(2,p_1)$. From Case (I) in the proof of Proposition 2.25, this pretzel link is universal.

(2) Since $|a| = 3$ and $|p_3| \neq 1$, this pretzel link contains $T(2,p_1)\sharp T(2,p_3)\sharp \cdots$. From Case (I), this link is universal.

(3) We consider the 3 ($=|a|$)-fold dihedral covering branched over $l = T(2,a)$. Then we have $p(-(3\beta + a/|a| + p_3); 2p_1, p_1, 2p_3)$. (cf. Figure 2.28.) Since $|p_1| \geq 5$, this pretzel link is universal from (1).

(4) It is sufficient to consider $p(-\beta; 6,3,2)$ and $p(-\beta; 6,-3,2)$. For $p(-\beta; 6,3,2)$, the 3-fold covering associated with Figure 2.29 is the lens space $L(2,1)$ by [8]. And the 2-fold unbranched covering of $L(2,1)$ is S^3.

First, we consider $p(-\beta; 6,3,2)$. We will show that, in this covering $S^3 \longrightarrow S^3$ the preimage of this pretzel link contains $p(-(6\beta + 4); 2,2,2,2)$. First we consider the link L with the permutation shown in Figure 2.30. If we perform 3 (6 resp.)-half twists at the 3-ball **3** (6 resp.) in Figure 2.30, then we have $p(-\beta; 6,3,2)$. There is a sphere S^2 dividing S^3 into two balls A and B such that A and B are shown in Figure 2.30 (b). The ball B does not contain the permutation (13). Thus the covering space of S^3 branched over L is divided by the preimage of S^2 into two part $\tilde{A}$ and $\tilde{B}$, the lifts of A and B respectively. We note that $\tilde{A}$ and $\tilde{B}$ are tori. Figure

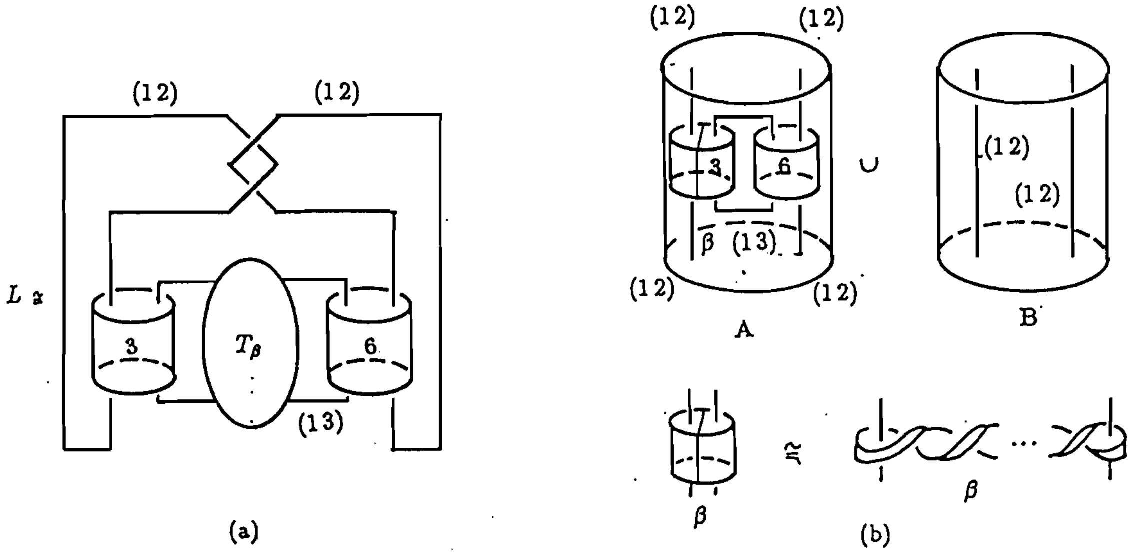


Figure 2.30

2.31 indecates the lift $\tilde{\mathbf{A}}$. In Figure 2.31, the bold line has branched index two, and the others have one.

The link with branch index 1 is contained in $\tilde{\mathbf{A}}$. We denote this link by l. So we can show this link in $L(2,1)$ as Figure 2.32(a). In Figure 2.32, the bold circle denotes the surgery link. By the twist at the ball **3** and **6**, we have the link as in Figure 2.32 (b).(See [14 ; p. 317.].) We denote this link by $\tilde{p}$, which is a sublink of the preimage of $p(-(6\beta+4);2,2,2,2)$. Figure 2.33 shows the preimage of $\tilde{p}$ under the 2-fold unbranched covering $S^3 \longrightarrow L(2,1)$. Now we have $p(-(6\beta+4);2,2,2,2)$. Unless $6\beta+4=-2$, i.e. $\beta=-1$, this pretzel link is universal. (See Proposition 2.22.) So $p(-\beta;6,3,2)$ is universal, but except $p(0;6,3,-2)$ $(=p(-1;6,3,2))$.

For $p(-\beta;6,-3,2)$, we perform the same operation. And we have the pretzel link $p(-(-6\beta+5);2,2,2,2)$. From Proposition 2.22, since $-6\beta+5 \neq -2$, this pretzel link is universal.

Then p is universal, but expect $p(0;6,3,-2)$ and $p(0;-6,-3,2)$, completing the proof.

Proof of (Case II-2). We divide the proof into two cases.

(i) $p_i \equiv 0 \pmod 2$ for $i=1,3$.

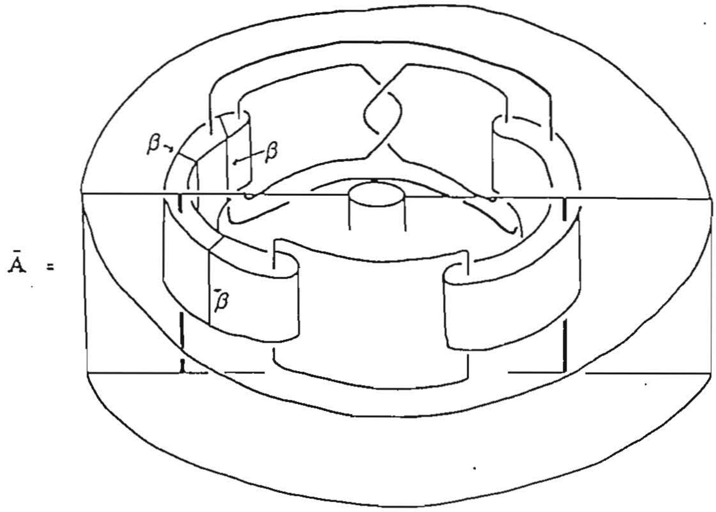


Figure 2.31

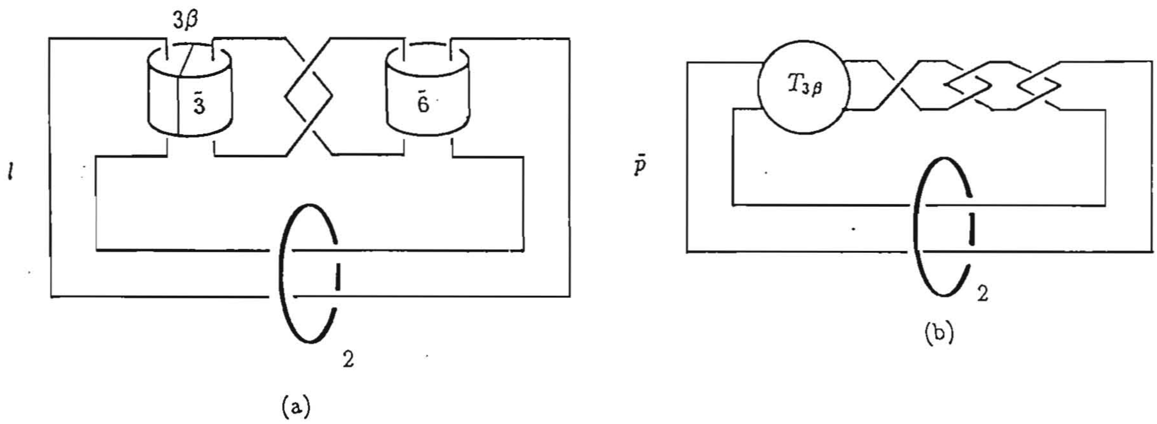


Figure 2.32

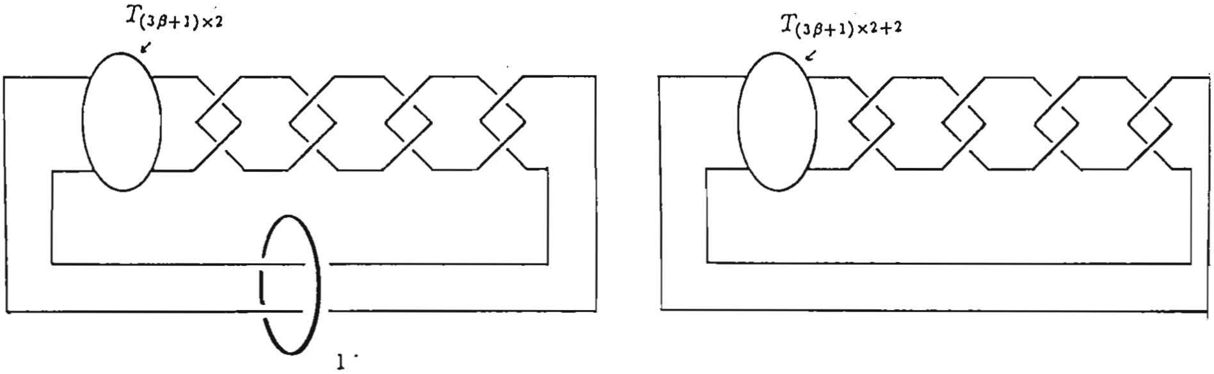


Figure 2.33

Since $| a | \geq 3$ and $p_1 \equiv 0 \pmod 2$, we can have a 4-component pretzel link, which is not $p(0; 2, 2, -2, -2)$, or a pretzel link with five or more components link, by the $| a |$-fold dihedral covering space branched over $l_1 = T(2, a)$. So this pretzel link is universal.

(ii) $p_1 \equiv 0,\ p_2 \equiv 1 \pmod 2$

If $| a | \geq 5$, then we perform the same operation of (i). Thus it is universal. In the case of $| a |= 3$. If $a = 3$, we denote this pretzel link by $p = p(-\beta; 4m, 3, 2n)$ $(m, n \in \mathbf{Z}\backslash\{0\})$. And we consider the 3-fold dihedral covering space branched over $l_1 = T(2, 3)$. (See Figure 2.32.) Then we have a new pretzel link $p(-(3\beta + 1); 2n, 4m, n, 2m)$ — (1). This link has three components. The 3-component pretzel link, which is not universal, is $p(2, -2, s)$ (s:even) and $p(4, 4, -2)$. From (1), if $(m, n, \beta) \neq (1, 1, -1)$, then p is universal. Hence p is universal, but except $p(0; 4, 3, -2)$ ($= p(0; 3, 4, -2)$).
If $a = -3$, then we perform same operation. Thus p is universal, but except $p(0; -4, -3, 2)$ $(= p(0; -3, -4, 2))$, completing the proof.

Proof of Case (III). This pretzel link is a 2-bridge link. From [6], it is universal, but except $p(0; 2s, 2t)$ $(s, t \in \mathbf{Z}\backslash\{0\})$. This completes the proof of Proposition 2.25.

3. Detecting inequivalence of some unknotting tunnels for two-bridge knots

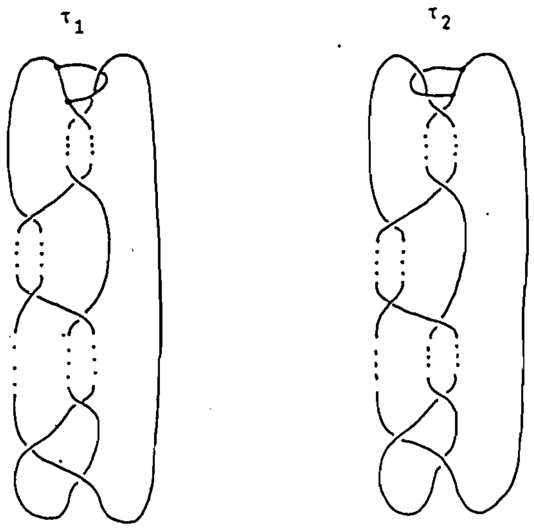


Figure 3.1

A. Introduction

3.1. Let k be a knot in the 3-sphere S^3. An *exterior* of k is the closure of the complement of a regular neighborhood of k, and is denoted by $E(k)$. An *unknotting tunnel* for k is an embedded arc τ in S^3 such that $\tau \cap k = \partial\tau$ and $S^3 - IntN(k \cup \tau, S^3)$ is a genus two handlebody. $N(H, K)$ denotes a regular neighborhood of H in K. Then we denote $\tau \cap E(k)$ by $\hat{\tau}$. Let τ_1, τ_2 be unknotting tunnels for k. We say that τ_1 and τ_2 are *homeomorphic* if there is a self-homeomorphism f of $E(k)$ such that $f(\hat{\tau}_1) = \hat{\tau}_2$, and τ_1 and τ_2 are *isotoic* if $\hat{\tau}_1$ is ambient isotopic to $\hat{\tau}_2$ in $E(k)$.

Suppose that k is a two-bridge knot. Then by [10. Prop. 3.1], τ_1 and τ_2 shown in Figure 3.1 are unknoting tunnels.

For two-bridge knots, we refer to [6]. To each rational number a/b with a odd, there is associated the two-bridge knot $K(a/b)$ shown in Figure 3.2. In Figure 3.2, the central tangle consists of lines of slope

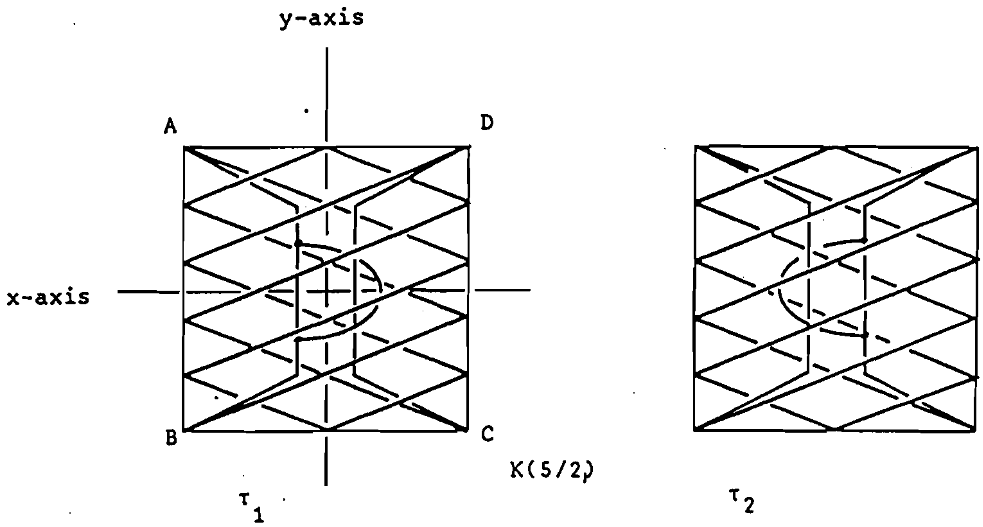


Figure 3.2

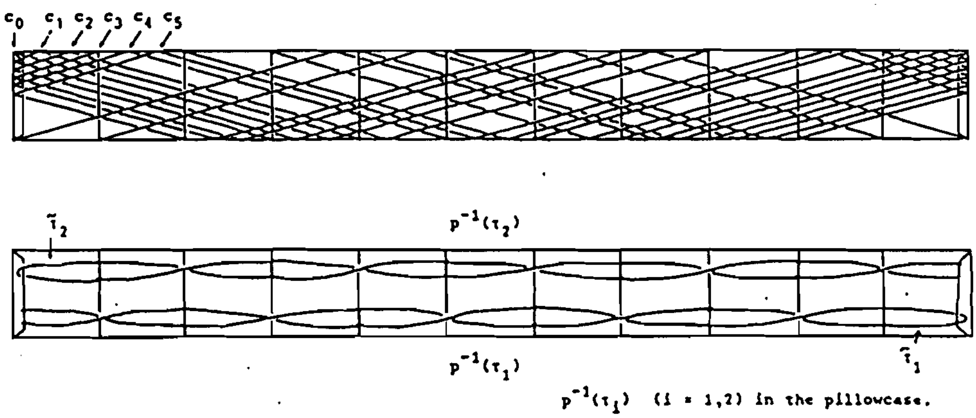


Figure 3.3

$\pm b/a$ which are drawn on a square "pillowcase". In Figure 3.2 we

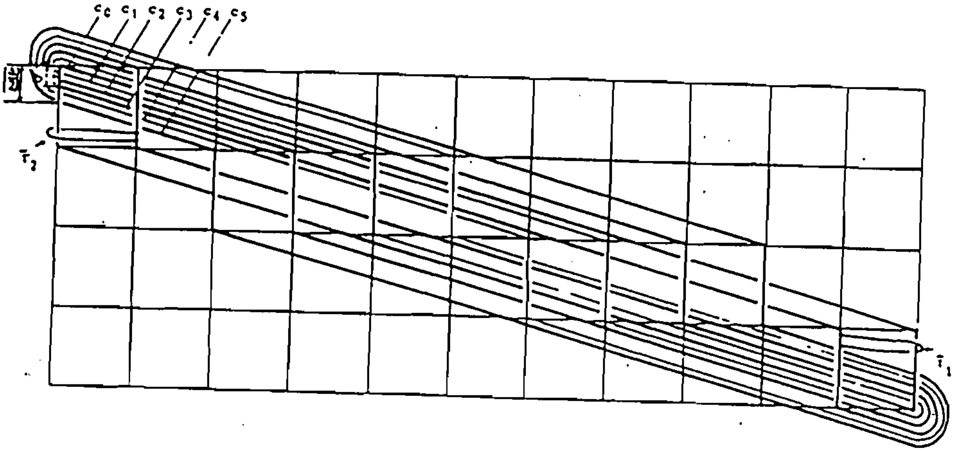


Figure 3.4

also describe τ_1 and τ_2. By π-twist of $\overline{CD}$ about x-axis, τ_i $(i = 1, 2)$ for $K(a/b)$ is isotopic to τ_i for $K(a/b \pm a)$ and $K(a/-b)$ is the mirror image of $K(a/b)$. Then we can assume that $0 < b \leq a/2$.

3.2. Theorem. *For a two-bridge knot $K(a/b)$ with a odd and $0 < b \leq a/2$, τ_1 and τ_2 are isotopic if and only if $(a, b) = (2n+1, n)$ or $(2n+1, 1)$ $(n \in N)$.*

3.3. Remark. π-rotation about y-axis, shows that τ_1 and τ_2 are homeomorphic. Morimoto and Sakuma[11] has proved this theorem by the argument of algebraic argument.

B. Proof of Theorem

Since the "if" part is easily seen (cf. [11, Theorem 5.2]), we prove "only if" part. Assume that τ_1 and τ_2 are isotopic for $K(a/b)$ with $(a, b) \neq (2n+1, n)$ and $(2n+1, 1)$. Then, there exists an ambient isotopy $f_t(0 \leq t \leq 1)$ of $E(K(a/b))$ such that $f_1(\hat{\tau}_1) = \hat{\tau}_2$. Let $p : M \longrightarrow E(K(a/b))$ be a covering of $E(K(a/b))$, $\tilde{f}_t : M \longrightarrow M$ a

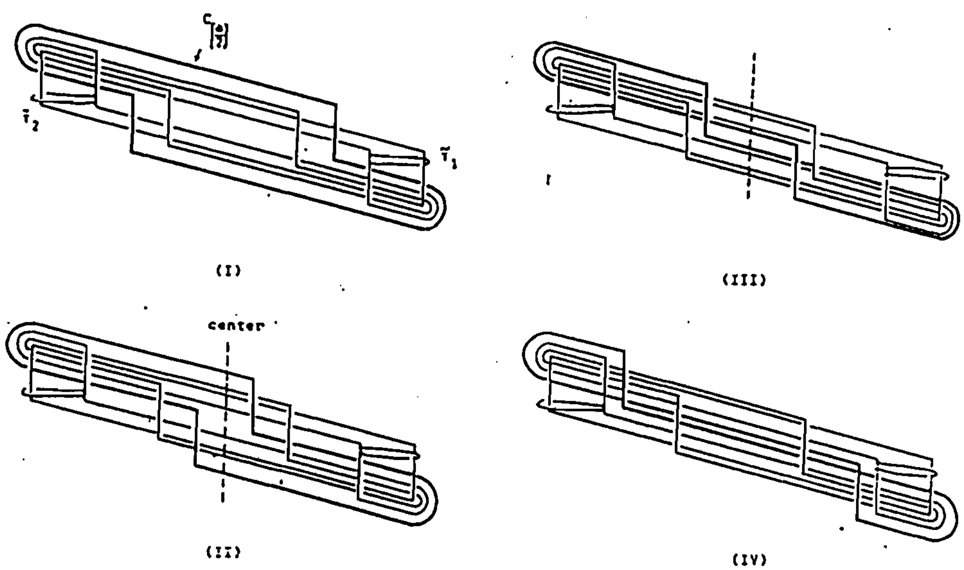


Figure 3.5

lift of f_t, $\tilde{\tau}_i (i = 1, 2)$ a component of $p^{-1}(\hat{\tau}_i)$ such that $\tilde{f}_1(\tilde{\tau}_1) = \tilde{\tau}_2$. By restrincting $\tilde{f}_t : M \times I \longrightarrow M$ to $\tilde{\tau}_1 \times I$ ($\cong I \times I$), we get a continuous map $g : I \times I \longrightarrow M$ with the following properties.

(1) $g \mid_{I \times \{0\}}$ is an embedding whose image is $\tilde{\tau}_1$,
(2) $g \mid_{I \times \{1\}}$ is an embedding whose image is $\tilde{\tau}_2$, and
(3) $g(\{0, 1\} \times I) \subset$ a component of ∂M.

Then, $\tilde{\tau}_1$ is homotopic (rel.∂) to $\tilde{\tau}_2$ in M.

From [6], the a-fold dihedral irregular branched covering of S^3 ranched over $K(a/b)$ is S^3. And the preimage of $K(a/b)$ (= $K(11/3)$), τ_1 and τ_2 is described as in Figure 3.3. The link $p^{-1}(K(a/b))$ has $1 + [a/2]$ components $C_0, C_1, ..., C_{[a/2]}$. In Figure 3.3, $\partial\tilde{\tau}_1$ is containd in $\partial N(C_b)$ and only $\tilde{\tau}_2$ in $p^{-1}(\tau_2)$ whose boundary is contained in $\partial N(C_b)$. Then $\tilde{f}_1(\tilde{\tau}_1) = \tilde{\tau}_2$. Since $\tilde{\tau}_1$ is homotopic (rel.∂) to $\tilde{\tau}_2$ in $S^3 - Int(\bigcup_{i=2}^{1+[\frac{a}{2}]} N(C_i))$, $\tilde{\tau}_1$ is homotopic (rel. ∂) to

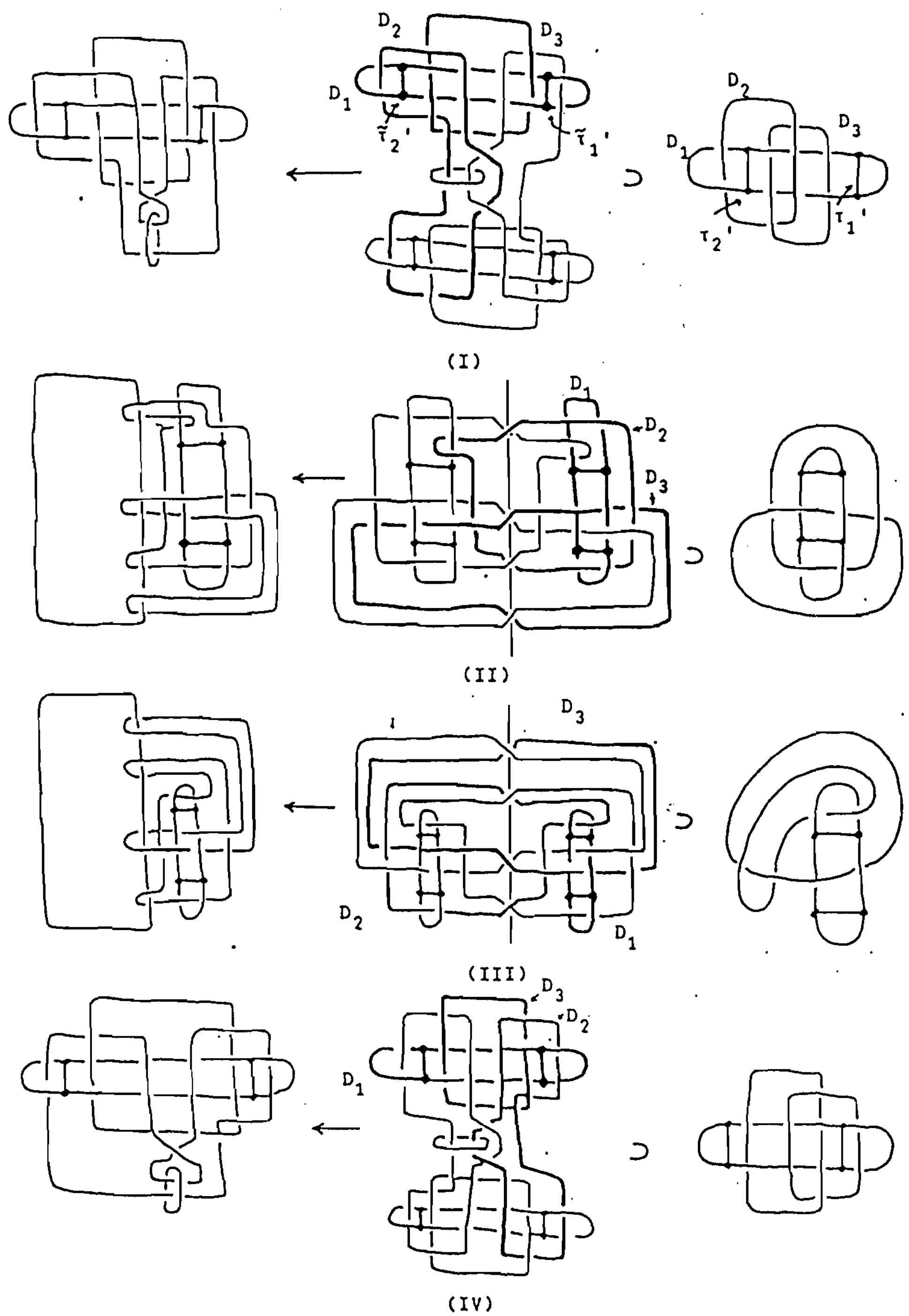


Figure 3.6

$\tilde{\tau}_2$ in $S^3 - Int(N(C_0) \cup N(C_1) \cup N(C_{a-[a/2]b}) \cup N(C_{[a/2]}))$. Figure 3.4 shows the preimage after puncturing the "pillowcase", twisting one of its ends and flatting out onto the plane.

Remark. Since $0 < b < a/2, b \neq 1$, and $(a,b) \neq (2n+1,1)$, then $0 < a - [a/b]b < b < [a/2]$.

From [6. pp.502 - 503], the link $C_0 \cup C_b \cup C_{a-[a/b]b}$ is uniquely determined. And there are only four cases for $C_{[a/2]}$ (Figure 3.5). In the case of Figure 3.5(I), the two-fold branched covering branched over $C_{a-[a/b]b}$ is described in Figure 3.6(I). Then, in Figure 3.6(I), $\tilde{\tau}_1'$ must be homotopic (rel.∂) to $\tilde{\tau}_2'$ in $S^3 - Int(\bigcup_{i=1}^{6} N(D_i))$. So $\tilde{\tau}_1'$ must be homotopic to $\tilde{\tau}_2'$ in $S^3 - Int(N(D_1) \cup N(D_2) \cup N(D_3))$. Now, $H_1(S^3 - Int(N(D_1) \cup N(D_2) \cup N(D_3), \partial N(D_1))) \cong \mathbf{Z} \cong < d_2 \mid >$, where d_2 is a meridian of D_2, and $\tilde{\tau}_2' \simeq d_2$ and $\tilde{\tau}_1' \simeq 0$ in $H_1(S^3 - Int(N(D_1) \cup N(D_2) \cup N(D_3), \partial N(D_1)))$. Then $\tilde{\tau}_1'$ is not homotopic (rel.∂) to $\tilde{\tau}_2'$, so τ_1 and τ_2 are not isotopic.

In the other cases, Figure 3.6 (II) ((III), (IV), resp.) describes the two-fold branched cover branched over $C_{a-[a/b]b}$ in Figure 3.5(II) ((III),(IV), resp.). Then we do the similar operation for the bold lines. Hence, we can show that τ_1 and τ_2 are not isotopic. This completes the proof.

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