On the Existence and Multiplicity of Solutions to Quasilinear p-Laplace Equations on R[n]

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博士論文

On the Existence and Multiplicity of Solutions to Quasilinear p-Laplace Equations on $\mathbf{R}^n$

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博士論文

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（$\mathbf{R}^n$ 上の準線形 p-Laplace 方程式の解の存在と多重性について）

PREFACE

In this dissertation I investigate some quasilinear p−Lapalace equations on $\mathbb{R}^n$. My main tool is "variational methods" developed by Palais, Smale, Ambrosetti, Rabinowitz et al. in 1970's. These methods are widely used in the study of nonlinear elliptic equations, and many authors have contributed to this region since 1970's. The basic idea is to consider a functional corresponding to agiven nonlinear elliptic equation and its critical points in a (probably infinite dimensional) function space. Critical points of a functional are characterized as "min-max" forms, which were established by Ambrosetti and Rabinowitz.

Especially, if the so-called "Palais-Smale" condition holds, the existence of a nontrivial solution to the nonlinear elliptic problems is ensured. The condition ensures the compactness of a bounded sequence in a Banach space. This result is much related to the Rellich-Kondrachev theorem.

Hence, for example, existence theorems to the following problem are well-known:

$$\begin{cases} -\Delta u = |u|^{a-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $2 < a < 2n/(n-2)$ and $n \geq 3$. In this case, the Palais-Smale condition holds because the embedding $H_0^1(\Omega) \hookrightarrow L^a(\Omega)$ is compact.

On the other hand, in case of $\Omega = \mathbb{R}^n$ the situation is quite different. The Palais-Smale condition fails. Though the "concentration compactness argument"(1985) due to P.-L. Lions can overcome this difficulty, there are many unsolved problems in this case. He gave a sufficient condition (but not a necessary one) for a weakly convergent sequence to be a strongly convergent one. There would be weaker conditions than Lions' ones.

In case of the "critical nonlinearity", the Palais-Smale condition also fails. Here I use the method developed by Brezis and Nirenberg in 1983. The method is to use the extremal function for the Sobolev best constant. This case is much delicate. The extremal function for the embedding $H_0^1(\mathbb{R}^n) \hookrightarrow L^{2n/(n-2)}(\mathbb{R}^n)$ is a constant multiple of

$$u(x) = (1 + |x|^2)^{-(n-2)/2}.$$

Taking into account the results as stated above, I investigate here the unsolved problem

$$-\text{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = q|u|^{\sigma}u \quad \text{on } \mathbb{R}^n$$

with $1 < p < n$ and $p - 2 < \sigma \leq np/(n-p) - 2$.

In this problem, I seek a solution in the Banach space $W^{1,p}(\mathbb{R}^n)$. Since the space is not a Hilbert space, the space does not have an inner product. Unlike the case $p = 2$, I must notice this difference.

I show the basic existence theorems (Chapter 1), existence theorems in the class of radial symmetry with the critical Soblev exponent (Chapter 2) and multiplicity of solutions (Chapter 3). Higher order cases are also treated in Chapter 1. The asymptotic behavior is shown in Chapters 2 and 3.

Contents

Chapter 1. Existence Theorems for Quasilinear Elliptic Problems on $\mathbf{R}^n$.

§1-1. Introduction

In this chapter we consider the following quasilinear elliptic problem:

$$-\text{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = q(x)|u|^\sigma u \quad \text{on } \mathbf{R}^n \tag{1}$$

where p and σ are constants which satisfy certain conditions stated later, and λ is a positive constant. We seek a nontrivial solution of (1) as a critical point of the functional

$$\Phi_\lambda(u) = \frac{1}{p}\int_{\mathbf{R}^n}(|\nabla u|^p + \lambda|u|^p)dx - \frac{1}{\sigma+2}\int_{\mathbf{R}^n} q(x)|u|^{\sigma+2}dx. \tag{2}$$

in the Banach space $W^{1,p}(\mathbf{R}^n)$.

From the homogeneity of the first term of $\Phi_\lambda(u)$, under appropriate assumptions on the potential $q(x)$, we can get a nontrivial solution of (1) by solving the constrained minimization problem:

$$\inf_{u\in W^{1,p}(\mathbf{R}^n),\|u\|_\lambda=1}\left(-\int_{\mathbf{R}^n} q(x)|u|^{\sigma+2}dx\right)$$

where $\|u\|_\lambda = \left\{\int_{\mathbf{R}^n}(|\nabla u|^p + \lambda|u|^p)dx\right\}^{1/p}$. Unlike the case $p = 2$, it seems that few papers have treated the case of p-Laplace equations with a potential $q(x)$ which may change its sign.

For the case $p = 2$, many authors including Ding and Ni [1] and Rother [4], [5] considered equations of this type. The former authors studied the case of positive potentials and the latter potentials which may change its sign. In both papers, they used " *à la* uniform integrability" so that the treatment of the problem on $\mathbf{R}^n$ could be similar to that in a bounded domain. Following the idea of them, we consider a more general case, i.e. the case of p-Laplace equations (1).

In Section 1-2 we consider a general case (the non-radial case; but the result in this case is still valid for the radial case). We assume here that $1 < p \leq n$, and $p - 2 < \sigma < pn/(n-p) - 2$. (If $p = n$, we regard the latter inequality as $p - 2 < \sigma$. Note that for $u \in W^{1,n}(\mathbf{R}^n)$, we have $u \in L^\alpha(\mathbf{R}^n)$ for all α such that $1 \leq \alpha < \infty$ by the Sobolev embedding theorem). In Section 1-3 we will treat the radial case. In this case only assuming $1 < p < n$ and $p - 2 < \sigma$, we can prove the existence of a nontrival solution of (1) even if $q(x)$ tends to infinity as $|x| \to \infty$ with some growth order. The precise expression of the order will be given in this section. In Section 1-4 we will treat higher order cases of p-Laplacian forms. But we will investigate only the non-radial case.

The regularity of our weak solution will be taken up in a later paper.

We conclude the introduction with some notations. We denote $W^{1,p}(\mathbf{R}^n)$ by $W^{1,p}$ and $L^t(\mathbf{R}^n)$ by L^t. For a function f, we define $f_+ = \max(f,0)$ and $f_- = \max(-f,0)$ which are the positive part and negative part of f respectively. We write p^* as $p^* = np/(n-p)$ (p^* is the critical Sobolev exponent corresponding to the embedding of $W^{1,p}$ into L^{p^*}). We define the $W^{1,p}$-norm by

$$\|u\|_\lambda = \left\{ \int_{\mathbf{R}^n} (|\nabla u|^p + \lambda |u|^p) dx \right\}^{\frac{1}{p}},$$

which is equivalent to the usual $W^{1,p}$-norm if $\lambda > 0$. We denote a ball in $\mathbf{R}^n$ with center x and radius r by $B_r(x)$.

§1-2. The non-radial case.

Our main result in this paper is the following existence theorem.

Theorem 1. *We assume $1 < p \leq n$, $p - 2 < \sigma < \frac{np}{n-p} - 2$ and $q : \mathbf{R}^n \to \mathbf{R}$ is measurable and satisfies the following assumptions.*

(A 1)
$$q = q_+ - q_-, \quad q_- \in L^1_{\text{loc}}.$$

(A 2)
$$q_+ = q_1 + q_2,$$

where

$$q_1 (\geq 0) \in L^t \quad \text{for} \quad t \in \left(\frac{pn}{p(n+\sigma+2) - n(\sigma+2)}, +\infty\right)$$

and

$$q_2 (\geq 0) \in L^\infty, \ \lim_{x \to \infty} q_2(x) = 0.$$

(A 3) *There exists a function $u_0 \in W^{1,p}$ such that* $\int_{\mathbf{R}^n} q|u_0|^{\sigma+2} dx > 0.$

Then, for all positive constant λ, there exists a nontrivial weak solution u of (1) *in $W^{1,p}$ such that*

$$\int_{\mathbf{R}^n} \left\{ |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi + \lambda |u|^{p-2} u \varphi \right\} dx = \int_{\mathbf{R}^n} q|u|^\sigma u \varphi dx \tag{3}$$

for every $\varphi \in C_0^\infty(\mathbf{R}^n)$.

Remark. If we assume $q(x) \leq 0$ on $\mathbf{R}^n$, we find easily that the equation (1) has onlya trivial solution in $W^{1,p}$. This is also true for higher order cases treated in Section 1-4. Thus assumptions (A 7), (A 10) which will be stated later are indispensable to our existence theorems (so is (A 3)).

If $q(x) \geq a > 0$ on some ball $B_c(x_0)$ where a and c are constants, and $x_0 \in \mathbf{R}^n$, then the assumption (A 3) is fulfilled.

The above remark will be appleid to Theorem 3 and Theorem 4. But, in Theorem 3, a sufficient condition for (A 7) will be changed to "$q(x)$ is uniformly positive on some annulus in $\mathbf{R}^n$".

Proof. As stated in section 1-1, a critical value of functional (2) is expressed

$$\inf_{u\in W^{1,p}, \|u\|_\lambda=1} \Big(-\int_{\mathbf{R}^n} q(x)|u|^{\sigma+2} dx \Big)$$

on the unit sphere of $W^{1,p}$.

Below we will show that this infimum is attained by a function v in $W^{1,p}$ and that a constant multiple of v becomes a nontrivial weak solution of (1).

For this purpose, we set

$$I(u) = -\int_{\mathbf{R}^n} q(x)|u|^{\sigma+2} dx,$$

and minimize $I(u)$ on the set

$$D = \big\{u \in W^{1,p} \mid \int_{\mathbf{R}^n} q_-|u|^{\sigma+2} dx < +\infty,\ \|u\|_\lambda = 1\big\}.$$

We denote the infimum of $I(u)$ by S_λ, i.e.

$$S_\lambda = \inf_{u\in D} I(u).$$

First we estimate $\int_{\mathbf{R}^n} q_+|u|^{\sigma+2}dx$. Denoting the Hölder conjugate of t by t', i.e. $t' = \frac{t}{t-1}$, from (A 2), we have

$$\begin{aligned}(\sigma+2)t' &= (\sigma+2)\frac{t}{t-1} = (\sigma+2)(1+\frac{1}{t-1})\\ &< (\sigma+2)\Big(1+(\frac{pn}{p(\sigma+n+2)-n(\sigma+2)}-1)^{-1}\Big)\\ &= (\sigma+2)\Big(1+\frac{p\sigma+pn+2p-n\sigma-2n}{(n-p)(\sigma+2)}\Big)\\ &= (\sigma+2)\frac{np}{(n-p)(\sigma+2)}\\ &= \frac{np}{n-p} = p^*.\end{aligned}$$

If $p = n$ then, from the Sobolev embedding theorem, we know $u \in W^{1,n}$ belongs to $L^{(\sigma+2)t'}$. Hence the Hölder inequality makes sense and we have

$$\int_{\mathbf{R}^n} q_1|u|^{\sigma+2}dx \leq (\int_{\mathbf{R}^n} q_1^t dx)^{\frac{1}{t}}(\int_{\mathbf{R}^n} |u|^{(\sigma+2)t'}dx)^{\frac{1}{t'}}.$$

From the Sobolev inequality and the fact that $||u||_\lambda = 1$, there exists a positive constant C such that

$$\int_{\mathbf{R}^n} |u|^{(\sigma+2)t'} dx \leq C||\nabla u||_{L^p}^{(\sigma+2)t'} \leq C||u||_\lambda^{(\sigma+2)t'} = C$$

holds independent of $u \in D$. On the other hand, by an easy calculation in view of the Sobolev inequality,

$$\int_{\mathbf{R}^n} q_2|u|^{\sigma+2} dx \leq ||q_2||_{L^\infty} C',$$

where C' is a constant independent of $u \in D$. Therefore, as $q_+ = q_1 + q_2$ with $q_1 \in L^t$ and $q_2 \in L^\infty$, we conclude that there is a constant C'' such that

$$\int_{\mathbf{R}^n} q_+|u|^{\sigma+2} dx \leq C''$$

independent of $u \in D$.

Next, we will show $S_\lambda < 0$. But this fact follows immediately from the assumption (A 3) and the last inequality. We have

$$-\infty < S_\lambda \leq I(u_0) < 0.$$

Now we prove any minimizing sequence has a minimizer in D. Let $\{u_j\}$ be a minimizing sequence for S_λ in D. As $W^{1,p}$ is a reflexive Banach space and $||u_j||_\lambda = 1$, the sequence $\{u_j\}$ is sequentially weakly compact in $W^{1,p}$. We may assume without loss of generality, that $\{u_j\}$ converges weakly to v in $W^{1,p}$. Moreover, for all bounded smooth domains Ω in $\mathbf{R}^n$, we set

$$\tilde{u}_j = u_j|_\Omega.$$

Obviously, $\{\tilde{u}_j\}$ is bounded in $W^{1,p}(\Omega)$. Since the embedding

$$W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$$

in a bounded domain Ω is compact, we can choose a subsequence of $\{u_j\}$ if necessary (still denoted by $\{u_j\}$), such that

$$\tilde{u}_j \to \tilde{v} \text{ a.e. in } \Omega$$

where $\tilde{v} = v|_\Omega$. Since Ω can be chosen arbitrarily, we use a diagonal argument to conclude

$$u_j \to v \text{ a.e. in } \mathbf{R}^n.$$

As the norm $||\cdot||_\lambda$ is lower semi-continuous under the weak convergence in $W^{1,p}$, we have

$$||v||_\lambda \leq \liminf_{j\to\infty} ||u_j||_\lambda = 1.$$

Since $\{u_j\}$ is a minimizing sequence for $S_\lambda(<0)$, we may further assume $I(u_j)<0$. From the fact that

$$\int_{\mathbf{R}^n} q_+|u_j|^{\sigma+2}dx \le C''$$

and

$$S_\lambda \le -\int_{\mathbf{R}^n} q_+|u_j|^{\sigma+2}dx + \int_{\mathbf{R}^n} q_-|u_j|^{\sigma+2}dx < 0,$$

we get

$$\int_{\mathbf{R}^n} q_-|u_j|^{\sigma+2}dx \le C''.$$

Then, from the Fatou lemma, we obtain

$$\int_{\mathbf{R}^n} q_-|v|^{\sigma+2}dx \le \liminf_{j\to\infty}\int_{\mathbf{R}^n} q_-|u_j|^{\sigma+2}dx \le C''.$$

Similarly, in view of the inequality

$$\int_{\mathbf{R}^n} q_+|u|^{\sigma+2}dx \le C' \qquad \text{for all } u\in D,$$

we have

$$\int_{\mathbf{R}^n} q_+|v|^{\sigma+2}dx \le \liminf_{j\to\infty}\int_{\mathbf{R}^n} q_+|u_j|^{\sigma+2}dx \le C''.$$

From (A 2) we can easily choose $R_\varepsilon>0$ for arbitrary $\varepsilon>0$ such that

$$\int_{|x|\ge R_\varepsilon} q_+|u_j|^{\sigma+2}dx \le \varepsilon \qquad \text{for all } j. \tag{4}$$

Furthermore, taking into account the Rellich-Kondrachev theorem, we find

$$u_j \to v \text{ strongly in } L^{(\sigma+2)t'}(B_{R_\varepsilon}(0)) \quad \text{as } j\to\infty.$$

Hence we have

$$\int_{|x|\le R_\varepsilon} q_+|u_j|^{\sigma+2}dx \longrightarrow \int_{|x|\le R_\varepsilon} q_+|v|^{\sigma+2}dx \quad \text{as } j\to\infty.$$

Then we get in view of (4)

$$\begin{aligned}
I(v) &= -\int_{\mathbf{R}^n}(q_+-q_-)|v|^{\sigma+2}dx\\
&= \int_{\mathbf{R}^n} q_-|v|^{\sigma+2}dx - \int_{\mathbf{R}^n} q_+|v|^{\sigma+2}dx\\
&\le \int_{\mathbf{R}^n} q_-|v|^{\sigma+2}dx - \int_{|x|\le R_\varepsilon} q_+|v|^{\sigma+2}dx\\
&\le \liminf_{j\to\infty}\Big(\int_{\mathbf{R}^n} q_-|u_j|^{\sigma+2}dx - \int_{|x|\le R_\varepsilon} q_+|u_j|^{\sigma+2}dx\Big)\\
&\le \liminf_{j\to\infty}\Big(\int_{\mathbf{R}^n} q_-|u_j|^{\sigma+2}dx - \int_{\mathbf{R}^n} q_+|u_j|^{\sigma+2}dx + \varepsilon\Big)\\
&= S_\lambda + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, we obtain $I(v) \le S_\lambda$ and $v \not\equiv 0$.

Finally we must show $v \in D$. We set $\alpha = \|v\|_\lambda$, then $\alpha \in (0,1]$ and $\frac{1}{\alpha}v \in D$. Thus

$$S_\lambda \le I(\frac{1}{\alpha}v) = \alpha^{-(\sigma+2)}I(v) \le \alpha^{-(\sigma+2)}S_\lambda < 0.$$

Since $S_\lambda < 0$, we get $\alpha = 1$. Hence $v \in D$ and $I(v) = S_\lambda$. We note that $|q||v|^{\sigma+1}$ is locally integrable. This is because

$$\begin{aligned}\int_B |q||v|^{\sigma+1}dx &= \int_B |q|^{1/(\sigma+2)}|q|^{(\sigma+1)/(\sigma+2)}|v|^{\sigma+1}dx \\ &\le \Big(\int_B |q|dx\Big)^{1/(\sigma+2)} \cdot \Big(\int_B |q||v|^{\sigma+2}\Big)^{(\sigma+1)/(\sigma+2)} < +\infty\end{aligned}$$

holds for all bounded domains $B \subset \mathbf{R}^n$ in view of the Hölder inequality. By the Gateaux derivative at v in D, we have

$$\int_{\mathbf{R}^n} \Big\{|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + \lambda|u|^{p-2}u\varphi\Big\}dx = |S_\lambda|^{-1}\int_{\mathbf{R}^n} q|u|^\sigma u\varphi dx$$

for every $\varphi \in C_0^\infty(\mathbf{R}^n)$.

Thus in view of the Lagrange multiplier rule(see Struwe [6]), we find that $u = |S_\lambda|^{-1/(\sigma-p+2)}v$ is a nontrivial weak solution of (1).

The proof is complete.

Remark. For the regularity of a weak solution of (1), some additional assumptions are needed. However, we cannot hope the existence of classical solutions of (1) because of the result of DiBenedetto[1] or Uhlenbeck[7](the solution has at most $C^{1,\alpha}$ regularity).

§1-3. The radial case

In this section we will study the radial case,i.e., the cae when the potential $q(x)$ in (1) is a function of the variable $r = |x|$.

We define

$$C_{0,r}^\infty = \{u \in C_0^\infty(\mathbf{R}^n) \mid u \text{ is radial}\}.$$

and denote by $W_r^{1,p}$ the completion of $C_{0,r}^\infty$ with respect to the norm $W^{1,p}$. We also denote the area of $\partial B_1(0)$ by ω_n. We use the same letter C for expressing various constants in this section.

We can now prove the following radial lemma which helps us to weaken the assumptions on q.

Lemma 2 (the radial lemma). *For $u \in W_r^{1,p}$ and $1 < p < n$, if $x \neq 0$, then for almost all $x \in \mathbf{R}^n$*

$$|u(x)|^p \leq C|x|^{p-n}\|u\|_\lambda^p$$

holds.

Proof. It suffices to show the lemma for $u \in C_{0,r}^\infty$. For such u, we have

$$|u(x)|^p = -\int_{|x|}^\infty \frac{d}{dr}\{|u(r)|^p\}dr.$$

The right-hand side is estimated as follows:

$$\Big|\int_{|x|}^\infty \frac{d}{dr}\{|u(r)|^p\}dr\Big| \leq p\int_{|x|}^\infty |u(r)|^{p-1}\Big|\frac{d}{dr}u(r)\Big|dr.$$

Now we decompose the last integrand in such a way that identity

$$|u(r)|^{p-1}\Big|\frac{d}{dr}u(r)\Big| = r^{-(n-1)(n+1-p)/n}\{|u(r)|r^{(n-1)/p^*}\}^{p-1}\Big|\frac{d}{dr}u(r)\Big|r^{(n-1)/p}$$

holds. The total sum of the exponents of r is equal to 0. In fact,

$$\begin{aligned}
&-\frac{(n-1)(n+1-p)}{n}+\frac{(n-p)(n-1)(p-1)}{pn}+\frac{(n-1)}{p}\\
=&\frac{n-1}{pn}\{-p(n+1-p)+(n-p)(p-1)\}+\frac{n(n-1)}{pn}\\
=&\frac{-n(n-1)+n(n-1)}{pn}\\
=&0.
\end{aligned}$$

We will estimate the integral using the Hölder inequality. First we observe that the Hölder inequality can be applied, because we can raise the power of the decomposed parts to α, β, γ, respectively, where $\alpha = n/(p-1)$, $\beta = p^*/(p-1)$, $\gamma = p$, since

$$\begin{aligned}
\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} &= \frac{p-1}{n}+\frac{(n-p)(p-1)}{pn}+\frac{1}{p}\\
&=\frac{p(p-1)+(n-p)(p-1)+n}{pn}\\
&=1.
\end{aligned}$$

Hence

$$\begin{aligned}
&\Big|\int_{|x|}^\infty \frac{d}{dr}\{|u(r)|^p\}dr\Big| \leq p\int_{|x|}^\infty |u(r)|^{p-1}\Big|\frac{d}{dr}u(r)\Big|dr\\
\leq\, & p\Big(\int_{|x|}^\infty r^{-(n-1)(n+1-p)/(p-1)}dr\Big)^{(p-1)/n}\\
&\times\Big(\int_{|x|}^\infty |u(r)|^{pn/(n-p)}r^{n-1}dr\Big)^{(n-p)(p-1)/pn}\Big(\int_{|x|}^\infty \Big|\frac{d}{dr}u(r)\Big|^p r^{n-1}dr\Big)^{1/p}.
\end{aligned}$$

If we observe

$$\begin{aligned}
&\int_{|x|}^{\infty} r^{-(n-1)(n+1-p)/(p-1)}dr \\
&= \left\{1 - \frac{(n-1)(n+1-p)}{p-1}\right\}^{-1}\left[r^{1-(n-1)(n+1-p)/(p-1)}\right]_{|x|}^{\infty} \\
&= \frac{p-1}{n(n-p)}|x|^{-n(n-p)/(p-1)},
\end{aligned}$$

we get

$$\begin{aligned}
|u(x)|^p &\leq p\{\frac{p-1}{n(n-p)}\}^{(p-1)/n}\omega_n^{-(n-p)(p-1)/pn}\omega_n^{-1/p}|x|^{p-n}\|u\|_{L^{p^*}}^{p-1}\|\nabla u\|_{L^p} \\
&\leq p\{\frac{p-1}{n(n-p)}\}^{(p-1)/n}\omega_n^{-(n-p)(p-1)/pn}\omega_n^{-1/p}|x|^{p-n}\left(\|u\|_{L^{p^*}} + \|\nabla u\|_{L^p}\right)^p \\
&\leq C|x|^{p-n}\|\nabla u\|_{L^p}^p \quad \text{(by the Sobolev embedding theorem)} \\
&\leq C|x|^{p-n}\|u\|_{\lambda}^p,
\end{aligned}$$

where C is a constant independent of $u \in C_{0,r}^{\infty}$, but depending on p and n.

The proof is complete.

We are now in a position to state our main theorem in this section. We assume that $q(x)$ is a radially symmetric function which is allowed to satisfy some growth condition at infinity.

Theorem 3. *Let $1 < p < n$, and $p - 2 < \sigma$. We assume $q : \mathbf{R}^n \to \mathbf{R}$ is measurable, radially symmetric, and satisfies the following assumptions:*

(*A* 4)
$$q = q_+ - q_-, \; q_- \in L^1_{loc}.$$

(*A* 5)
$$0 \leq q_+(|x|) \leq f(|x|)|x|^{k(\sigma)}$$

where $f \in L^\infty$ and $k(\sigma) = \frac{n-p}{p}\{(\sigma+2) - p^*\} - \delta$, where δ is a positive constant. Furthermore

(*A* 6)
$$0 \leq f(|x|) \leq C|x|^{2\delta} \text{ on } B_\eta(0)$$

where $\eta > 0$ is a small constant.

(*A* 7) *There exists $u_0 \in W_r^{1,p}$ such that* $\int_{\mathbf{R}^n} q|u_0|^{\sigma+2}dx > 0$

Then for all positive λ, there exists a nontrivial weak solution u of rm (1) in $W_r^{1,p}$.

Remark. Theorem 3 is valid for all $\delta > 0$, not only for a suitable δ. But, according to δ in (A 5), $f(|x|)$ must vanish at the origin as stated in (A 6). In the case $p < \sigma+2 < p^*$, we can add q_1, q_2 in Theorem 1 to q_+ here (the validity of this statement is ensured by the proof of Theorem 1.).

Proof. We prove the theorem by following the proof of Theorem 1. Instead of D in it, we define $D_r = \{u \in W_r^{1,p} \mid \int_{\mathbf{R}^n} q_-|u|^{\sigma+2}dx < \infty,\ \|u\|_\lambda = 1\ \}$.

Then, by the radial lemma, we have

$$\begin{aligned}\int_{\mathbf{R}^n} q_+|u|^{\sigma+2}dx =&\omega_n \int_0^\infty q_+|u|^{\sigma+2}r^{n-1}dr \\ \leq& C\omega_n\Big(\int_0^\eta q_+\|u\|_\lambda^{\sigma+2}r^{(p-n)(\sigma+2)/p+n-1}dr \\ &+\int_\eta^\infty q_+\|u\|_\lambda^{\sigma+2}r^{(p-n)(\sigma+2)/p+n-1}dr\Big).\end{aligned}$$

We take u in $W_r^{1,p}$ and, from Assumptions (A 5) and (A 6), we get

$$\int_{\mathbf{R}^n} q_+|u|^{\sigma+2}dx \leq C\omega_n\Big\{\int_0^\eta r^\mu dr + \|f\|_\infty \int_\eta^\infty r^\nu dr\Big\},$$

where

$$\begin{aligned}\mu =&2\delta + \frac{n-p}{p}\{(\sigma+2) - \frac{np}{n-p}\} - \delta - \frac{n-p}{p}(\sigma+2) + n - 1,\\ \nu =&\frac{n-p}{p}\{(\sigma+2) - \frac{np}{n-p}\} - \delta - \frac{n-p}{p}(\sigma+2) + n - 1.\end{aligned}$$

Then these values yield $\mu = \delta - 1,\ \nu = -\delta - 1$. Hence, finally, we have

$$\begin{aligned}\int_{\mathbf{R}^n} q_+|u|^{\sigma+2}dx \leq& C\omega_n\Big\{\big[\frac{1}{\delta}r^\delta\big]_0^\eta + \|f\|_{L^\infty}\big[-\frac{1}{\delta}r^{-\delta}\big]_\eta^\infty\Big\} \\ =& C\omega_n\frac{1}{\delta}\Big\{\eta^\delta + \|f\|_{L^\infty}\eta^{-\delta}\Big\}.\end{aligned} \tag{5}$$

This value is independent of $u \in D_r$. Let $\{u_j\}$ be a minimizing sequence for S_λ in D_r. From the same reason as in the proof of Theorem 1 (by the Assumption (A 7)), we have $S_\lambda < 0$, and $\int_{\mathbf{R}^n} q_-|u_j|^{\sigma+2}dx \leq C$ for all j. Moreover, we may assume

$$u_j \rightharpoonup v \text{ weakly in } W^{1,p}, \text{ and } u_j \to v \text{ a.e. in } \mathbf{R}^n.$$

Then we have

$$\|v\|_\lambda \leq \liminf_{j\to\infty} \|u_j\|_\lambda \leq 1$$

and

$$\int_{\mathbf{R}^n} q_-|v|^{\sigma+2}dx \leq \liminf_{j\to\infty}\int_{\mathbf{R}^n} q_-|u_j|^{\sigma+2}dx \leq C.$$

By the fact that $\|u_j\|_\lambda = 1$ and the above estimate (5), for every $\varepsilon > 0$ there exist positive R_ε and r_ε such that

$$\int_{|x|\geq R_\varepsilon} q_+|u|^{\sigma+2}dx \leq \varepsilon, \quad \int_{|x|\leq r_\varepsilon} q_+|u|^{\sigma+2}dx \leq \varepsilon$$

for $u \in D_r$.

We now set $T_\varepsilon = \{x \in \mathbf{R}^n \mid r_\varepsilon \leq |x| \leq R_\varepsilon\}$ and apply the Lebesgue dominant convergence theorem (from Lemma 2 and (A 5) , we can take a summable dominant function; see the above estimate on $q_+|u|^{\sigma+2}$) to obtain

$$\int_{T_\varepsilon} q_+|u_j|^{\sigma+2}dx \longrightarrow \int_{T_\varepsilon} q_+|v|^{\sigma+2}dx \quad \text{as } j \to \infty.$$

Since

$$I(v) \leq \int_{\mathbf{R}^n} q_-|v|^{\sigma+2}dx - \int_{T_\varepsilon} q_+|v|^{\sigma+2}dx,$$

using the same argument as in Section 1-2, we have

$$I(v) \leq \liminf_{j\to\infty}(I(u_j) + 2\varepsilon) = S_\lambda + 2\varepsilon.$$

Hence, we obtain $I(v) \leq S_\lambda$. The remaining proof that $v \in D_r$ and $u = |S|^{-1/(\sigma-p+2)}v$ is the same as in the proof of Theorem 1.

The proof is complete.

§1-4. The Higher order case.

In this section we will extend our existence theorem in the non-radial case to a higher order problem of the form

$$(6) \qquad \sum_{l=1}^{k}(-\nabla)^l\left\{|\nabla^l u|^{p-2}\nabla^l u\right\} + \lambda|u|^{p-2}u = q(x)|u|^\sigma u \quad \text{on } \mathbf{R}^n,$$

where

$$\nabla^l u = \begin{cases} \Delta^m u & \text{if } l = 2m, \\ \nabla(\Delta^{m-1}u) & \text{if } l = 2m-1, \end{cases}$$

and $(-\nabla)^l$ which operates on $\{\cdots\}$ means, if $l = 2m-1$,

$$(-\nabla)^l u = -\mathrm{div}(\Delta^{m-1}u).$$

As in the previous section, we seek a nontrivial solution of (6) as a critical point of the functional

$$\Psi(u) = \frac{1}{p}\sum_{l=1}^{k}\int_{\mathbf{R}^n}|\nabla^l u|^p dx + \frac{\lambda}{p}\int_{\mathbf{R}^n}|u|^p dx - \frac{1}{\sigma+2}\int_{\mathbf{R}^n} q|u|^{\sigma+2}dx$$

in the Banach space $D^{k,p}$. Here $D^{k,p}$ is the completion of C_0^∞ with respect to the norm $(\sum_{l=0}^{k}\|\nabla^l \cdot\|_{L^p}^p)^{1/p}$. If $p \neq 2$, $D^{k,p}$may be different from $W^{k,p}$ (see P.-L.Lions[3]). Note that the $D^{k,p}$-norm is equivalent to $(\sum_{l=1}^{k}\|\cdot\|_{L^p}^p + \lambda\|\cdot\|_{L^p}^p)^{1/p}$.

We have the following existence theorem.

Theorem 4. *For $1 < p \le n/k$ $(k < n;\ n, k \in \mathbf{N})$, we assume $p - 2 < \sigma < \frac{np}{n-kp} - 2$ and $q : \mathbf{R}^n \to \mathbf{R}$ is measurable and satisfies the following assumptions.*

$$(A\ 8) \qquad q = q_+ - q_-,\ q_- \in L^1_{\text{loc}}.$$

$$(A\ 9) \qquad q_+ = q_1 + q_2,\ q_1(\ge 0) \in L^t \quad \text{for} \quad t \in \left(\frac{pn}{np - (n-kp)(\sigma+2)}, +\infty\right)$$

and

$$q_2(\ge 0) \in L^\infty,\ \lim_{|x|\to\infty} q_2 = 0.$$

$$(A\ 10) \qquad \textit{There exists } u_0(\not\equiv 0) \in D^{k,p} \textit{ such that } \int_{\mathbf{R}^n} q|u_0|^{\sigma+2}dx > 0$$

Then there exists a nontrivial weak solution u of (6) in $D^{k,p}$ such that, for all positive λ,

$$\int_{\mathbf{R}^n} \Big\{ \sum_{l=1}^{n} |\nabla^l u|^{p-2}\nabla^l u \cdot \nabla^l \varphi + \lambda |u|^{p-2}u\varphi \Big\} dx = \int_{\mathbf{R}^n} q|u|^\sigma u\varphi dx$$

for every $\varphi \in C_0^\infty(\mathbf{R}^n)$.

Proof. First we note by the Sobolev embedding theorem, that the inclusion map $D^{k,p} \hookrightarrow L^{np/(n-kp)}$ is continuous. We need only to check the exponent of the last term $q_+|u|^{\sigma+2}$. From (A 9) we have

$$\begin{aligned}(\sigma+2)t' &= (\sigma+2)(1 + \frac{1}{t-1}) \\ &< (\sigma+2)(1 + \frac{np - (n-kp)(\sigma+2)}{(n-kp)(\sigma+2)}) \\ &= (\sigma+2)\frac{np}{(\sigma+2)(n-kp)} \\ &= \frac{np}{n-kp},\end{aligned}$$

and hence $\int_{\mathbf{R}^n} q_+|u|^{\sigma+2}dx \le C$ for all $u \in D_k$ where D_k is the set of minimization

$$D_k = \Big\{ u \in D^{k,p} \mid \int_{\mathbf{R}^n} q_-|u|^{\sigma+2}dx < \infty, \|u\|_{D^{k,p}} = 1 \Big\}.$$

The rest of the proof is the same as in Section 1-2.

The proof is complete.

Remark. In Section 1-2 and Section 1-3, we may change λ into $a(x)$ such that $a(x) \in L^\infty$ and $a(x) \geq c > 0$. In Section 1-4, we may change

$$\sum_{l=1}^{k}(-\nabla)^l\left\{|\nabla^l u|^{p-2}\nabla^l u\right\} + \lambda|u|^{p-2}u$$

into

$$\sum_{l=0}^{k}(-\nabla)^l\left\{a_l(x)|\nabla^l u|^{p-2}\nabla^l u\right\}$$

where $a_l(x) \in L^\infty$ such that $a_l(x) \geq c > 0$.

For a radial higher order case, a radial lemma like lemma 2 would be much more complicated and several assumptions would be needed.

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Chapter 2. The Critical Sobolev Exponent Case.

§2-1. Introduction.

We consider in this chapter

$$(1) \qquad -\mathrm{div}(|\nabla u|^{p-2}\nabla u) - Q(x)|u|^{p-2}u - K(x)|u|^{p^*-2}u = 0$$

in $W^{1,p}_{loc,rad}(\mathbf{R}^n)$, and seek a radially symmetric solution of (1). Our aims are to extend the results of C.-S. Lin and S.-S. Lin [18] to $p-$Laplacian and to investigate the asymptotic behavior of the solution. We also consider the bounded case with zero-Dirichlet boundary condition. In this case, we can prove the existence of a nontrivial radially symmetric solution. Our results do not include those of Guedda and Veron [13] or Egnell [9], [10]. But the difference of assumptions enables us to show the existence of nontrivial solutions under assumptions on $Q(x)$ which are weaker than those of them.

The study of elliptic equations involving critical Sobolev exponents is mainly owed to Brezis and Nirenberg [4]. In their cerebrated paper, they have succeeded in showing the existence of a nontrivial solution of

$$-\Delta u - \lambda u = |u|^{2^*-2}u \qquad \text{in } \mathrm{H}^1_0(\Omega)$$

where Ω is a bounded domain in $\mathbf{R}^n$ and $2^* = 2n/(n-2)$.

Since the break-through paper of them, many authors has studied various equations involving critical Sobolev exponents. They are for example, Ambrosetti and Struwe [1], Cappozi, Fortunato, and Palmieri [5], Cerami, Fortunato, and Struwe [6], Cerami, Solimini, and Struwe [7], Egnell [9], Fortunato and Jannelli [12], and Zhang [24].

Later, unbounded case (i.e. $\Omega = \mathbf{R}^n$) was treated by Benci and Cerami [3], C.-S. Lin and S.-S. Lin [17] and others. For the approach by the ordinary differential equations, there are also many results, for exsample, due to Kawano, Yanagida, Yotsutani [15]

We should notice that as well as the Laplacian, a quasilinear version of this type ($p-$Laplacian) has also studied by Azorero and Alonso [2], Egnell[10], Guedda and Veron [8], Kawano, Yanagida and Yotsutani [16] and others.

Though it is well-known that the Palais-Smale condition fails for these problems even in the bounded case, the deficiency of the compactness of embedding to the space with the critical Sobolev exponent has been overcome by the method used by Brezis and Nirenberg. The method has become a standard type for these problems.

The key point of these problems is that the function

$$U(x) = \left\{n\left(\frac{n-p}{p-1}\right)^{p-1}\right\}^{(n-1)/p^2}(1+|x|^{p/(p-1)})^{(p-n)/p}$$

is an entire solution of

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = u^{p^*-1} \quad \text{on } \mathbf{R}^n$$

and it is the extremal function of the functional

$$I(u) = \frac{\int_{\mathbf{R}^n} |\nabla u|^p dx}{(\int_{\mathbf{R}^n} |u|^{p^*} dx)^{p/p^*}}$$

where $p^* = np/(n-p)$.

Using the fact, we compare the best Sobolev constant with the value of the functional for the problem at the point of an "approximate " function. If the best constant is greater than the above value, we can prove the existence of a nontrival solution. For details see Brezis and Nirenberg [4], P.-L. Lions [19], Talenti [20].

The bounded case is considered in Section 2-2. We consider there in the radially symmetric class, that is, in $W^{1,p}_{0,r}(B_R)$

$$(2) \qquad \begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) - Q(x)|u|^{p-2}u - K(x)|u|^{p^*-2}u = 0 & \text{in } B_R \\ u = 0 & \text{on } \partial B_R \end{cases}$$

where B_R is a closed ball with radius $R(>0)$ whose center is the origin. Here we can prove the $C^{1,\alpha}$-regularity of a solution of (2). To show it, we invoke the results of Guedda and Veron and DiBenedetto [8]. But if $Q(x)$ and $K(x)$ are not radially symmetric, readers should refer to Egnell [10].

In Section 2-3, we seek a radially symmetric solution of (1) on $\mathbf{R}^n$ as a limit of a sequence of radially symmetric solutions of (2). In Kabeya [14], if K tends to 0 with some order near the origin, the existence of nontrivial solution has been proved. But in this case, the method in [14] never applies. We use the method by Lin and Lin. In this case Q has to be nonnegative.

Now we state some assumptions often used in Section 2-2 and Section 2-3.

$$(K-1) \qquad K(|x|) = 1 + \beta|x|^{\gamma p} \quad \text{near the origin and } K(|x|) \geq 0,$$

where $\beta > 0$ and $\gamma < 1$.

$$(Q-1) \qquad Q(|x|) \in C^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n).$$

But in Section 2-3, Q must be nonpositive.

According to the mountain pass theorem, one of the critical values of the functional

$$\Phi(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p - Q(x)|u|^p)dx - \frac{1}{p^*}\int_\Omega K(x)|u|^{p^*} dx$$

is expressed as

$$S_{p^*,Q}(\Omega) = \inf_{u \in W_0^{1,p}(\Omega), u \not\equiv 0} \frac{\int_\Omega (|\nabla u|^p - Q(x)|u|^p)dx}{\left(\int_\Omega K(x)|u|^{p^*} dx\right)^{p/p^*}},$$

but if $K \geq 0$ it is equivalent to the follwing expression stated in "notations".

We will check whether $S_{p^*,Q}(\Omega)$ is attained in a suitable function space.

We conclude the introduction introducing some notations:

$$B_R : \text{a closed ball with radius } R \text{ whose center is the origin,}$$

$$\omega_n = |\partial B_1|,$$

$$M_q(B_R) = \left\{ u \in W_{0,r}^{1,p}(B_R) | \int_{B_R} K(x)|u|^q dx = 1 \right\},$$

$$S_{q,Q}(B_R) = \inf \left\{ \int_{B_R} (|\nabla u|^p - Q(x)|u|^p) dx \mid u \in M_q(B_R) \right\},$$

$$S = \inf \left\{ \int_{B_R} |\nabla u|^p dx \mid u \in W_{0,r}^{1,p}(B_R), \int_{B_R} |u|^{p^*} dx = 1 \right\},$$

where $2 \leq q \leq p^*$. It is well known that S is never attained in $W_{0,r}^{1,p}(B_R)$ but is attained if $W^{1,p}(\mathbf{R}^n)$ (see Guedda and Veron [13] or Talenti [20]). It is also known that S is independent of R, and its value is equal to the case $\mathbf{R}^n$.

If we can show $S_{p^*,Q}(B_R) < S$, we are able to prove the exintence of a nontrivial solution to (2).

§2-2. The Dirichlet problem in a bounded domain.

In this section we consider

$$(3) \qquad \begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) - Q(x)|u|^{p-2}u - K(x)|u|^{p^*-2}u = 0 \text{ in } B_R \\ \qquad\qquad\qquad\qquad\qquad\qquad u = 0 \text{ on } \partial B_R. \end{cases}$$

As stated in the introduction, we first show $S_{p^*,Q}(B_R) < S$ under suitable conditions. Hereafter we denote the best Poincaré constant by $\lambda(R)$ i.e.

$$\lambda(R) = \inf_{u \in W_0^{1,p}(B_R),\ u \not\equiv 0} \frac{\int_{B_R} |\nabla u|^p dx}{\int_{B_R} |u|^p dx}.$$

We find out that the behavior of K near the origin plays a crucial role for the existence theorems. The next lemma will be used to ensure the existence of a nontrivial solution.

Lemma 2.1. *Assume $(K-1)$, $(Q-1)$. For any positive R, if $Q < \lambda_1(R)$ in B_R, and $1 < p < \sqrt{n}$, $1 > \gamma > 1/p$, then we have $0 < S_{p^*,Q}(B_R) < S$.*

Remark. This lemma is still valid for non-radial K and Q.

Proof. As in Guedda and Veron [13] or Egnell [9] or Brezis and Nirenberg [4], we prove the lemma by using the function

$$u_\varepsilon(x) = \phi(x)(\varepsilon + |x|^{p/(p-1)})^{1-n/p}$$

where ε is an arbitrary positive number and $\phi \in C_0^\infty(\mathbf{R}^n)$ is a cut off function defined as

$$\phi \equiv 1 \text{ near } 0 \text{ and } 0 \le \phi \le 1$$

with support in B_δ $(\delta > 0)$. Now we calculate

$$\int_{B_R} |\nabla u_\varepsilon|^p \, dx, \quad \int_{B_R} Q(x)|u_\varepsilon|^p \, dx, \text{ and } \int_{B_R} K(x)|u_\varepsilon|^{p^*} \, dx.$$

First, we have as $\varepsilon \to 0$,

$$\begin{aligned}
&\int_{B_R} |\nabla u_\varepsilon|^p \, dx \\
&= (\frac{n-p}{p-1})^p \int_{B_R} (\varepsilon + |x|^{p/(p-1)})^{-n} |x|^{p/(p-1)} dx + O(1) \\
&= (\frac{n-p}{p-1})^p \int_{\mathbf{R}^n} (\varepsilon + |x|^{p/(p-1)})^{-n} |x|^{p/(p-1)} dx + O(1).
\end{aligned}$$

Now we set $|x| = \varepsilon^{(p-1)/p}\rho$ to get

$$\int_{B_R} |\nabla u_\varepsilon|^p \, dx = \varepsilon^{1-n/p} (\frac{n-p}{p-1})^p \omega_n \int_0^\infty (1 + \rho^{p/(p-1)})^{-n} \rho^{n-1+p/(p-1)} d\rho + O(1).$$

We set

$$K_1 = (\frac{n-p}{p-1})^p \omega_n \int_0^\infty (1 + \rho^{p/(p-1)})^{-n} \rho^{n-1+p/(p-1)} d\rho$$

to have

$$\int_{B_R} |\nabla u_\varepsilon|^p \, dx = K_1 \varepsilon^{1-n/p} + O(1). \tag{4}$$

Similarly, we have

$$\begin{aligned}
\int_{B_R} Q(x)|u_\varepsilon|^p \, dx &\ge d \int_{B_R} \Big\{(\phi(x))^p - 1 + 1\Big\} (\varepsilon + |x|^{p/(p-1)})^{p-n} dx \\
&= \varepsilon^{p-n/p} \omega_n d \int_0^\infty (1 + \rho^{p/(p-1)})^{p-n} \rho^{n-1} d\rho + O(1),
\end{aligned}$$

where $d = \min_{B_R} Q$. Setting

$$K_3 = \omega_n \int_0^\infty (1 + \rho^{p/(p-1)})^{p-n} \rho^{n-1} d\rho,$$

we have

$$\int_{B_R} |u_\varepsilon|^p dx \geq K_3 d\varepsilon^{p-n/p} + O(1). \tag{5}$$

Finally, using the assumptions on $K(x)$ and $\phi(x)$, we have

$$\begin{aligned}
&\int_{B_R} K(x)|u_\varepsilon|^{p^*} dx \\
&\geq \int_{B_\delta} (1 + \beta|x|^{\gamma p})(\varepsilon + |x|^{p/(p-1)})^{-n} dx \\
&= \int_{\mathbf{R}^n} (\varepsilon + |x|^{p/(p-1)})^{-n} dx + \beta \int_{B_\delta} |x|^{\gamma p}(\varepsilon + |x|^{p/(p-1)})^{-n} dx + O(1) \\
&= \varepsilon^{-n/p}\omega_n \int_0^\infty (1 + \rho^{p/(p-1)})^{-n} \rho^{n-1} d\rho \\
&+ \beta\varepsilon^{(p-1)\gamma - n/p}\omega_n \int_0^{\delta\varepsilon^{-(p-1)/p}} \rho^{\gamma p}(1 + \rho^{p/(p-1)})^{-n} \rho^{n-1} d\rho + O(1).
\end{aligned}$$

We set

$$\omega_n \int_0^\infty (1 + \rho^{p/(p-1)})^{-n} \rho^{n-1} d\rho = \tilde{K}_2$$

and

$$\omega_n \int_0^\infty \rho^{\gamma p}(1 + \rho^{p/(p-1)})^{-n} \rho^{n-1} d\rho = K_\gamma$$

for convenience.

Now we check whether K_γ is finite. We only show the behavior of the integrand at infinity is $o(\rho^{-1})$ as $\rho \to \infty$. Let

$$F(\gamma) = -(\gamma p - pn/(p-1) + n - 1) + 1 = -\gamma p + n/(p-1),$$

we have $F(1) > 0$ by the assumption on p. According to the linearity of $F(\gamma)$, we have $F(\gamma) > 0$ for all $\gamma < 1$. Hence we find K_γ is finite. Consequently, we obtain

$$\begin{aligned}
\int_{B_R} K|u_\varepsilon|^{p^*} dx &\geq \tilde{K}_2\varepsilon^{-n/p} + \beta K_\gamma \varepsilon^{-n/p + (p-1)\gamma} + O(1) \\
&= \tilde{K}_2\varepsilon^{-n/p}\Big(1 + \frac{\beta K_\gamma}{\tilde{K}_2}\varepsilon^{(p-1)\gamma} + O(\varepsilon^{n/p})\Big).
\end{aligned}$$

Hence we get

$$\Big(\int_{B_R} K|u_\varepsilon|^{p^*} dx\Big)^{p/p^*} \geq K_2\varepsilon^{-(n-p)/p}\Big(1 + \frac{\beta K_\gamma}{\tilde{K}_2}\varepsilon^{(p-1)\gamma} + O(\varepsilon^{n/p})\Big)^{(n-p)/n},$$

$$(6) \qquad \Big(\int_{B_R} K|u_\varepsilon|^{p^*}dx\Big)^{p/p^*} \ge K_2\varepsilon^{-(n-p)/p}\big(1+\frac{\beta K_\gamma}{\tilde K_2}\varepsilon^{(p-1)\gamma}\big)^{(n-p)/n} + O(1)$$

with $K_2 = (\tilde K_2)^{p/p^*}$. Let

$$R_Q(u) = \frac{\int_{B_R}\Big(|\nabla u|^p - Q|u|^p\Big)dx}{\Big(\int_{B_R}K(x)|u|^{p^*}dx\Big)^{p/p^*}},$$

then we have by (4), (5), and (6),

$$\begin{aligned} R_Q(u_\varepsilon) &\le \frac{K_1\varepsilon^{1-n/p} - K_3 d\varepsilon^{p-n/p} + O(1)}{K_2\varepsilon^{\frac{-(n-p)}{p}}\big(1+\frac{\beta K_\gamma}{\tilde K_2}\varepsilon^{(p-1)\gamma}\big)^{(n-p)/n} + O(1)} \\ &= \frac{K_1 - K_3 d\varepsilon^{p-1} + O(\varepsilon^{(n-p)/p})}{K_2\big(1+\frac{\beta K_\gamma}{\tilde K_2}\varepsilon^{(p-1)\gamma}\big)^{(n-p)/n} + O(\varepsilon^{(n-p)/p})} \\ &= \frac{K_1 - K_3 d\varepsilon^{p-1} + O(\varepsilon^{(n-p)/p})}{K_2\big(1+\frac{(n-p)\beta K_\gamma}{n\tilde K_2}\varepsilon^{(p-1)\gamma} + O(\varepsilon^{2(p-1)\gamma})\big) + O(\varepsilon^{(n-p)/p})}. \end{aligned}$$

Setting $l = \min(2(p-1)\gamma,\ (n-p)/p)$, we get

$$\begin{aligned} R_Q(u_\varepsilon) &\le \frac{K_1 - K_3 d\varepsilon^{p-1} + O(\varepsilon^{(n-p)/p})}{K_2\big(1+\frac{(n-p)\beta K_\gamma}{n\tilde K_2}\varepsilon^{(p-1)\gamma}\big) + O(\varepsilon^l)} \\ &= \frac{1}{K_2}\Big(K_1 - K_3 d\varepsilon^{p-1} + O(\varepsilon^{(n-p)/p})\Big)\Big(1 - \frac{(n-p)\beta K_\gamma}{n\tilde K_2}\varepsilon^{(p-1)\gamma} + O(\varepsilon^l)\Big) \\ &= \frac{K_1}{K_2} - \frac{K_3}{K_2}d\varepsilon^{p-1} - \frac{(n-p)\beta K_1 K_\gamma}{n\tilde K_2 K_2}\varepsilon^{(p-1)\gamma} + o(\varepsilon^{(p-1)\gamma}). \end{aligned}$$

Noting that $p-1 > (p-1)\gamma$ and $S = K_1/K_2$, we obtain for every small $\varepsilon > 0$,

$$S_{p^*,Q}(B_R) \le R_Q(u_\varepsilon) < S.$$

For $0 < S_{p^*,Q}(B_R)$, we only notice that $0 < Q(x) < \lambda_1(B_R)$. It follows from that

$$\begin{aligned} \int_{B_R}(|\nabla u|^p - Q(x)|u|^p)dx &\ge \int_{B_R}(|\nabla u|^p - \frac{Q}{\lambda_1(B_R)})dx \\ &\ge \frac{\lambda_1(B_R) - \max Q}{S\lambda_1(B_R)}\Big(\int_{B_R}|u|^{p^*}dx\Big)^{p/p^*} \\ &\ge \frac{\lambda_1(B_R) - \max Q}{S\lambda_1(B_R)}(\max K)^{-p/p^*} \\ &> 0 \end{aligned}$$

under the condition $\int_{B_R} K|u|^{p^*}dx = 1$.

The proof is complete.

Remark. If $p = \sqrt{n}$, we have for some positive K

$$\int_{B_\mu} |u_\varepsilon|^p dx \geq K|\log \varepsilon| + O(1)$$

and $R_Q(u_\varepsilon)$ yields

$$R_Q(u_\varepsilon) = \frac{K_1}{K_2} - \frac{K}{K_2} d\varepsilon^{p-1}|\log \varepsilon| - \frac{(n-p)\beta K_1 K_\gamma}{n\tilde{K}_2 K_2}\varepsilon^{(p-1)\gamma} + o(\varepsilon^{(p-1)\gamma}).$$

Thus even if $p = \sqrt{n}$, the result of Lemma 2.1 still holds.

In the proof, we find that the assumption $(K-1)$ may be changed into

$$(K-1)' \qquad 1 + \tilde{\beta}|x|^{p-\varepsilon} \leq K(|x|) \leq 1 + \hat{\beta}|x|^{1+\varepsilon}$$

where $0 < \tilde{\beta} < \hat{\beta}$ and ε is arbitrary positive.

For the radial functions, the radial lemma is a key for the existence theorems (see for instance Egnell [11] or the author [14]).

Lemma 2.2.(the radial Lemma). *Let u be a radially symmetric function such that*

$$\int_{\mathbf{R}^n} |\nabla u|^p dx < \infty,$$

then

$$|u(x)| \leq C(n,p)|x|^{(p-n)/p}\|\nabla u\|_{L^p(\mathbf{R}^n)}$$

holds for all $x \neq 0$, where $1 < p < n$.

By the ssumption $(K-1)$, $Q(|x|)$ may change its sign or may be nonpositive. The minimizer obtained here plays an important role in Section 2-3.

Theorem 2.3. *If $(C^1(B_R) \ni) Q(|x|) < \lambda_1(B_R)$ on $\bar{B}_R$ and $K \geq 0$ satisfies $(K-1)$, then there exists a minimizer for $S_{p^*,Q}(B_R)$ in $W^{1,p}_{0,r}(B_R)$. Moreover the minimizer belongs to $C^{1,\alpha}_0(B_R)$ where $\alpha \in (0,1)$.*

Proof. We devide the proof into four parts.
1st step. To show the boundedness of a minimizing sequence in $W^{1,p}_{0,r}(B_R)$.

Let us consider the problem

$$(7) \qquad \begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) - Q(x)|u|^{p-2}u - S_{q,Q}(B_R)K(x)|u|^{q-2}u = 0 & \text{in } B_R \\ u = 0 & \text{on } \partial B_R, \end{cases}$$

and let u_q be a solution of this problem, where $p \le q < p^*$.

Since the domain B_R is bounded, we can easily know that the solution $u_q$belongs to $M_q(B_R)$ using the compactness of the embedding $W^{1,p}_{0,r}(B_R) \hookrightarrow L^q(B_R)$ for $p \le q < p^*$. Taking

$$\int_{B_R} (|\nabla u_q|^p - Q(x)|u_q|^p)dx = S_{q,Q}(B_R)$$

into account, we find out

$$\int_{B_R} |\nabla u_q|^p dx \le \frac{\lambda_1(B_R)}{\lambda_1(B_R) - \max Q} S_{q,Q}(B_R),$$

by using

$$\int_{B_R} |\nabla u_q|^p dx = \int_{B_R} Q(x)|u_q|^p dx + S_{q,Q}(B_R) \le \frac{\max Q}{\lambda_1(R)} \int_{B_R} |\nabla u_q|^p dx + S_{q,Q}(R).$$

Invoking the result of Guedda and Veron, we get

$$\lim_{q \uparrow p^*} S_{q,Q}(B_R) = S_{p^*,Q}(B_R)$$

to obtain the boundedness of $\{u_q\}$ in $W^{1,p}_{0,r}(B_R)$. Thus we can choose a subsequence $\{q_j\}$ as $q_j \uparrow p^*$ such that the sequence $\{u_{q_j}\}$satisfies

$$(8) \qquad \begin{cases} u_{q_j} \rightharpoonup u_\infty & \text{weakly in } W^{1,p}_{0,r}(B_R) \\ u_{q_j} \to u_\infty & \text{stongly in } L^p(B_R) \\ u_{q_j} \to u_\infty & \text{a.e. in } B_R. \end{cases}$$

Hereafter we denote u_{q_j} by u_j.

2nd step. We show u_∞ is nontrivial. Suppose $u_\infty \equiv 0$. Recalling that

$$S_{p^*,Q}(B_R) = \lim_{j\to\infty} \frac{\int_{B_R}(|\nabla u_j|^p - Q(|x|)|u_j|^p)dx}{(\int_{B_R} K(|x|)|u_j|^{q_j} dx)^{p/q_j}},$$

and the strong convergence of $\{u_j\}$ in $L^p(B_R)$, we have

$$S_{p^*,Q}(B_R) = \lim_{j\to\infty} \frac{\int_{B_R} |\nabla u_j|^p dx + o(1)}{(\int_{B_R} K(|x|)|u_j|^{q_j} dx)^{p/q_j} + o(1)}.$$

Moreover by the radial lemma and the Lebesgue dominant convergence theorem, we get

$$\lim j \to \infty \int_{B_R \setminus B_\delta} |\nabla u_j|^p dx = 0$$

and

$$\lim j \to \infty \int_{B_R \setminus B_\delta} K|u_j|^{p^*}\,dx = 0.$$

Hence for $\delta > 0$ we have

$$S_{p^*,Q}(B_R) \geq \lim_{j\to\infty} \frac{\int_{B_\delta} |\nabla u_j|^p\,dx + o(1)}{(\int_{B_\delta} K(|x|)|u_j|^{q_j}\,dx)^{p/q_j} + o(1)}.$$

In addition, from $(K-1)$ for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$1 \leq K(|x|) \leq (1+\varepsilon)^{p^*/p} \quad \text{in } B_\delta$$

holds. Then using the Sobolev embedding theorem, we have

$$S_{p^*,Q}(B_R) \geq \lim_{j\to\infty} \frac{S}{1+\varepsilon} \frac{\left(\int_{B_\delta} |u_j|^{p^*}\,dx\right)^{p/p^*} + o(1)}{(\int_{B_\delta} |u_j|^{q_j}\,dx)^{p/q_j} + o(1)}.$$

Now by the Hölder inequality, we have

$$\left(\int_{B_\delta} |u_j|^{q_j}\,dx\right)^{p/q_j} \leq \left(\int_{B_\delta} dx\right)^{(p^*-q_j)p/p^*q_j} \left(\int_{B_\delta} |u_j|^{p^*}\,dx\right)^{p/p^*}.$$

Using this inequality, we have

$$S_{p^*,Q}(B_R) \geq \frac{S}{1+\varepsilon}$$

Since ε is arbitrary, we have

$$S_{p^*,Q}(B_R) \geq S.$$

But this fact contradicts to Lemma 2.1. We get $u_\infty \not\equiv 0$.

3rd step. Now we prove that $\{u_j\}$ is bounded in $C_0^{1,\alpha}(B_R)$.

To show the boundedness, we utilize some propositions due to Guedda and Veron and DiBenedetto [7] without proof. For a proof, see their papers.

Proposition 2.4.(Guedda and Veron). *Suppose u is a solution of*

$$(9) \qquad \begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) - g(x)|u|^{p-2}u = f & \text{in } G \\ u = 0 & \text{on } \partial G, \end{cases}$$

where G is a bounded domain in $\mathbb{R}^n$, $1 < p < n$, $f \in L^{n/p}(G)$, and $g \in L^{n/p}(G)$. Then

$$\|u\|_{L^t(G)} \leq C(t, G, \|f\|_{L^{n/p}}, \|g\|_{L^{n/p}}) \quad \text{for } t \in [1,\infty).$$

Proposition 2.5.(Guedda and Veron). *Assume* $1 < p \le n$, $f \in L^s(G)$ *for some* $s > n/p$ *and* $u \in W_0^{1,p}(G)$ *is a solution of*

$$(10) \qquad \begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f & \text{in } G \\ u = 0 & \text{on } \partial G. \end{cases}$$

Then

$$\|u\|_{L^\infty(G)} \le C(n,p,G)\|f\|_{L^s(G)}^{1/(p-1)}$$

where G *is a bounded domain in* $\mathbf{R}^n$.

Proposition 2.6.(Guedda and Veron). *Let* u_ε *is a solution of*

$$(11) \qquad \begin{cases} -\operatorname{div}\{(\varepsilon + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u\} = f_\varepsilon & \text{in } G \\ u = 0 & \text{on } \partial G, \end{cases}$$

where $\varepsilon > 0$, G *is a bounded domain,* $f_\varepsilon \in C^1(\bar{G})$*and the solution* u_ε *is in* $C^{2,\beta}(\bar{G})$. *Then there exist* $\alpha \in (0,1)$ *and* $C = C(p,n,G,\|f_\varepsilon\|_{L^\infty(G)}) > 0$ *such that*

$$\|u_\varepsilon\|_{C^{1,\alpha}(G)} \le C \quad \text{for any} \varepsilon \in (0,1).$$

Moreover if $\|f_\varepsilon\|_{L^\infty(G)}$ *does not depend on* ε, *we can pass* ε *to 0.*

Now we continue the proof. First, we show the sequence $\{u_j\}$ belongs to $L^t(B_R)$, invoking the Proposition 2.4.

We regard (2) as

$$(12) \quad \begin{cases} -\operatorname{div}(|\nabla u_j|^{p-2}\nabla u_j) - (Q + S_{q_j,Q}(B_R)K|u_j|^{q_j-p})|u_j|^{p-2}u_j = 0 & \text{in } B_R \\ u_j = 0 & \text{on } \partial B_R. \end{cases}$$

We only check $Q(x) + S_{q_j,\lambda}(B_R)K(x)|u_j|^{q_j-p} \in L^{n/p}(B_R)$. (We regard $f_. \equiv 0$). From the assumptions $(K-1)$ ($K \in L^\infty(\mathbf{R}^n)$), and $(Q-1)$ we show $|u_j|^{q_j-p} \in L^{n/p}(B_R)$. In view of

$$p^* - (q_j - p)\frac{n}{p} = \frac{n\{pn - (n-p)q_j\}}{p(n-p)},$$

and using $q_j < p^*$, we find $p^* - (q_j - p)n/p > 0$ to get $|u_j|^{q_j-p} \in L^{n/p}(B_R)$. Using the boundedness of $\{u_j\}$ in $W_0^{1,p}(B_R)$, we get $\{|u_j|^{p^*-p}\}$ is uniformly bounded in $L^{n/p}(B_R)$, hence we have $\{u_j\}$ is uniformly bounded in $L^t(B_R)$ for every $t \in [1,\infty)$.

Next we show $\{u_j\}$ is in $L^\infty(B_R)$.

In Proposition 2.5 we think of f as $f_j = Q|u_j|^{p-2}u_j + S_{q_j,Q}(B_R)|u_j|^{q_j-2}u_j$. Using the result obtained above, we have $f \in L^s(B_R)$ $(s > n/p)$ and

$$\|u_j\|_{L^\infty(B_R)} \le C\|f_j\|_{L^s(B_R)}^{1/(p-1)}.$$

The boundedness of $\{u_j\}$ in $L^\infty(B_R)$ is obtained from the boundedness of $\{f_j\}$ in $L^s(B_R)$ (this comes from the boundedness of $\{u_j\}$ in $L^t(B_R)$).

Finally, we prove the $C^{1,\alpha}(B_R)$-boundedness of $\{u_j\}$. The fact that u_j belongs to $C^{1,\alpha}(B_R)$ follows from the reguraliry result due to DiBenedetto.

Proposition 2.7.(DiBenedetto). *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak local solution of*

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \varphi, \quad p > 1,\ \varphi \in L^q_{\mathrm{loc}}(\Omega)$$

where $q > pn/(p-1)$. Then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$.

In our case, we may take $\varphi = f_j = Q|u_j|^{p-2}u_j + S_{q_j,Q}(B_R)|u_j|^{q_j-2}u_j$. On the uniform boundedness, we use Proposition 2.5. Noticing that

$$f_\epsilon = f_j = Q|u_j|^{p-2}u_j + S_{q_j,Q}(B_R)K(x)|u_j|^{q_j-2}u_j \in C^1(B_R),$$

we have a solution of (11) $u_{\epsilon,j} \in C^{2,\beta}(B_R)$ by the usual bootstrap method (because the approximate equation is not degenerate.). Applying Proposition 2.6, we obtain

$$\|u_j\|_{C^{1,\alpha}(B_R)} \le C(p, n, B_R, \|f_j\|_{L^\infty(B_R)}).$$

By the boundedness of $u_j \in L^\infty(B_R)$ (so is f_j), the constant $C(p,n,B_R,\|f_j\|_{L^\infty(B_R)})$ is bounded. Cosequently, we obtain $\{u_j\}$ is bounded in $C^{1,\alpha}(B_R)$.

4th step. Conclusion.

Noticing that the embedding $C^{1,\alpha}(G) \hookrightarrow C^{1,\acute{\alpha}}(G)$ is compact if $0 < \acute{\alpha} < \alpha$ and if G is a bounded domain in $\mathbf{R}^n$, we find

$$u_j \to u_\infty \quad \text{strongly in } C^{1,\acute{\alpha}}(B_R).$$

Letting

$$v = S_{p^*,Q}(B_R)^{1/(p^*-p)}u_\infty,$$

we have v is a solution of (2) and $v \in C^{1,\acute{\alpha}}_0(B_R)$.

The proof is complete.

Remark. Invoking the maximum principle for p-Laplacian due to Vazquez [23] (Stated below), we find the solution u_∞ is positive in B_R.

Proposition 2.8.(Vazquez [23]). *Let $u(\ge 0) \in C^1(\Omega)$ satisfies the following assumptions:*

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) \in L^2_{\mathrm{loc}}(\Omega), \text{ and } \operatorname{div}(|\nabla u|^{p-2}\nabla u) \le \beta(u) \in \Omega,$$

and $\beta : [0,+\infty) \longrightarrow \mathbf{R}^n$ is continuous, decreasing, $\beta(0) = 0$, $\beta(s) = 0$ for some $s > 0$ or $\beta(s) > 0$ for all $s > 0$, but $\int_0^1 (\beta(s)s)^{-1/p}ds = +\infty$. Then if u does not vanish identically on Ω, u is positive everywhere in Ω.

Noticing that our solution u_∞ may be assumed to be nonnegative, we only check the property of

$$\beta(s) = |\max Q|s^{p-1}.$$

Obviously, $\beta(0) = 0$ and $\beta(s) > 0$ for all $s > 0$. If $0 \le s \le 1$,

$$(\beta(s)s)^{-1/p} \ge c's^{-1}$$

holds and we have

$$\int_0^1 (\beta(s)s)^{-1/p} ds \ge c' \int_0^1 s^{-1} ds = \infty.$$

Finally we obtain $u_\infty > 0$ in B_R.

Remark. In general, we can't expect the more regularity of the solutions of this type. A counterexample is introduced by Tolksdolf [22].

As is well-known for the Laplacian, the Pohozaev type identity also holds for the p-Laplacian. It is due to Guedda and Veron (also due to Egnell).

The extended Pohozaev Identity.

$$n \int_\Omega H(x,u)dx + \int_\Omega x \cdot \nabla_x H(x,u)dx + (1 - \frac{n}{p}) \int_\Omega ug(x,u)dx$$
$$= (1 - \frac{1}{p}) \int_{\partial\Omega} x \cdot \nu |u_\nu|^p dS$$

where $g(x,u) = Q(x)|u|^{p-2}u + K(x)|u|^{p^*-2}u$ *and* $H(x,u) = \int_0^u g(x,s)dS$.

Calculating directly the relation, we get

$$\int_\Omega Q|u|^p dx + \int_\Omega \{x \cdot \nabla(Q+K)\}|u|^p(1 + |u|^{p^*-p})dx = (1 - \frac{1}{p}) \int_{\partial\Omega} (x \cdot \nu)|u_\nu|^p ds.$$

Assuming that $Q(x) = Q(|x|) < 0$, $K(x) = K(|x|)$ (Q and K is radial), $Q'(s) + K'(s) \le 0$ for every $s \in \mathbf{R}$ and Ω is star-shaped with respect to the origin, we get any nonnegative solution which belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is only the trivial one. In our framework, Q has to be decreasing near the origin so that $Q' + K' \le 0$ holds.

If $Q(x) \ge \lambda_1(R)$, then there exists no positive solution of (2). Namely, the following theorem holds.

Theorem 2.9. *If $Q(x) \ge \lambda_1(R)$ and $K(x) > 0$, then there exists no positive solution of (2).*

Proof. The proof is almost same as in that of Guedda and Veron (Theorem 3.3., using the Vazquez maximum principle and the extended Hopf boundary point theorem), so we omit it.

§2-3. A Radially symmetric solution on $\mathbf{R}^n$.

In this section we consider

$$(13) \qquad -\mathrm{div}(|\nabla u|^{p-2}\nabla u) - Q(x)|u|^{p-2}u - K(x)|u|^{p^*-2}u = 0 \quad \text{on } \mathbf{R}^n,$$

in the radially symmetric class. We seek a solution of (13) as the limit of the sequence of the minimizers obtained in Theorem 3.3. Since $\lambda_1(B_R) \to 0$ as $R \to \infty$, we can not show the existence of nontrivial solution of (13) if $Q > 0$(for given Q, Lemma 2.1 fails in a large ball.). Noting that Theorem 2.3 holds for nonpositive $Q \in C^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$, we assume here

$$(Q-2) \qquad Q(|x|) \leq 0.$$

Now we set for radially symmetric functions

$$M_{rad}^{\infty} = \left\{ u \in L^p_{loc}(\mathbf{R}^n) \mid \nabla u \in L^p(\mathbf{R}^n) \int_{\mathbf{R}^n} |Q||u|^p dx < +\infty, \int_{\mathbf{R}^n} K|u|^{p^*} dx = 1 \right\}$$

$$S_{p^*,Q}^{\infty} = \inf_{u \in M_{rad}^{\infty}} \int_{\mathbf{R}^n} (|\nabla u|^p - Q(x)|u|^p) dx.$$

Noting that $u \in M_{p^*}(B_r)$ (assuming u is radial) means $u \in M_{rad}^{\infty}$, we find out that $S_{p^*,Q}(B_r)$ is decreasing as $B_r \uparrow \mathbf{R}^n$. Namely, for any sequence of domains $\{B_{R_j}\}$ such that $B_{R_j} \subset B_{R_{j+1}}$ and $B_{R_j} \uparrow \mathbf{R}^n$ as $j \to \infty$, we get

$$\lim_{j \to \infty} S_{p^*,Q}(B_{R_j}) \geq S_{p^*,Q}^{\infty}.$$

On the other hand, let $\{\psi_j\} \in C_0^\infty(\mathbf{R}^n)$ be a minimizing sequence in radially symmetric class for $S_{p^*,Q}^{\infty}$ with $\mathrm{supp}\psi_j \subset B_{R_j}$ for any j, where $B_{R_j} \subset B_{R_{j+1}}$ and $B_{R_j} \uparrow \mathbf{R}^n$ as $j \to \infty$. Then for any $\varepsilon > 0$, there exists $j(\varepsilon) \in \mathbf{N}$ such that for all $j > j(\varepsilon)$

$$\left\{ \int_{\mathbf{R}^n} (|\nabla \psi_j|^p - Q|\psi_j|^p) dx \right\} \left\{ \int_{\mathbf{R}^n} K(x)|\psi_j|^{p^*} dx \right\}^{-p/p^*} < S_{p^*,Q}^{\infty} + \varepsilon.$$

Hence we have

$$S_{p^*,Q}(B_{R_j}) \leq \left\{ \int_{\mathbf{R}^n} (|\nabla \psi_j|^p - Q|\psi_j|^p) dx \right\} \left\{ \int_{\mathbf{R}^n} K(x)|\psi_j|^{p^*} dx \right\}^{-p/p^*} < S_{p^*,Q}^{\infty} + \varepsilon.$$

Since ε is arbitrary, we get

$$(14) \qquad \lim_{j \to \infty} S_{p^*,Q}(B_{R_j}) = S_{p^*,Q}^{\infty}.$$

Noting the relation (14), we obtain the following existence theorem.

Theorem 3.1. *Assume that K, Q are radially symmetric and K and Q satisfy $(K-2)$,*

$$\limsup_{|x|\to\infty} K(|x|) \leq 1$$

and $(Q-2)$, respectively, then there exists a minimizer for $S^{\infty}_{p^*,Q}$ in M^{∞}_{rad} i.e., there exists a solution of

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) - Q(|x|)|u|^{p-2}u - K(|x|)|u|^{p^*-2}u = 0 \quad \text{on } \mathbf{R}^n$$

in M^{∞}_{rad}.

Proof. As a similar way of proof of Theorem 2.3, we will show the theorem in five steps.
1st step. To show the weak convergence of a minimizing sequence.

We take a sequence of balls $\{B_{R_j}\}$ $(B_{R_0} = B)$ such that

$$B \subset B_{R_1} \subset \cdots \subset B_{R_j} \subset B_{R_{j+1}} \subset \cdots$$

and $B_{R_j} \uparrow \mathbf{R}^n$ as $j \to \infty$. Let v_j be the radial minimizer obtained in Theorem 2.3 of

$$(15) \qquad \begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) - Q(|x|)u^{p-1} = S_{p^*,Q}(B_{R_j})K(|x|)u^{p^*-1} & \text{in } B_{R_j} \\ u > 0 \quad \text{in } B_{R_j} \\ u = 0 \quad \text{on } \partial B_{R_j}. \end{cases}$$

By the monotone decreaseness of $S_{p^*,Q}(B_{R_j})$, and the nonpositivity of Q, we have

$$(16) \qquad \int_{B_{R_j}} |\nabla u|^p dx \leq S_{p^*,Q}(B_{R_j}).$$

Since (16) implies the boundedness of $\{\nabla v_j\}$ in $L^p(\mathbf{R}^n)$, we can choose a subsequence of $\{v_j\}$ (still denoted by $\{v_j\}$) such that

$$\nabla v_j \rightharpoonup \nabla v_\infty \quad \text{weakly in } L^p(\mathbf{R}^n)$$

This means $v_\infty \in M^{\infty}_{rad}$.
2nd step. To prove $v_\infty \not\equiv 0$.

Now we invoke the radial lemma 2.2.

By this lemma, we find $\{v_j\} \in L^{\infty}_{\mathrm{loc}}(\mathbf{R}^n \backslash 0)$. Now suppose that $v_\infty \equiv 0$ holds. Using the property of $\{v_j\}$, we have

$$S_{p^*,Q}(B) \geq \lim_{j\to\infty} \frac{\int_{\mathbf{R}^n}(|\nabla v_j|^p - Q|v_j|^p)dx}{\left(\int_{\mathbf{R}^n} K(|x|)|v_j|^{p^*}dx\right)^{p/p^*}}.$$

We note that $\{v_j\}$ is uniformly bounded on any annulus $\hat{A}$, and the Lebesgue dominant convergence theorem holds on $\hat{A}$. Moreover, we should notice that by the assumption on K, for $\varepsilon > 0$ there exist $\delta > 0$ and $R > 0$ such that

$$\begin{aligned} 1 \le K(|x|) &\le (1+\varepsilon)^{p^*/p} \quad \text{in } B_\delta \\ K(|x|) &\le (1+\varepsilon)^{p^*/p} \quad \text{in } \mathbf{R}^n \backslash B_R. \end{aligned}$$

Then we have

$$\begin{aligned} S_{p^*,Q}(B) &\ge \lim_{j\to\infty} \frac{\int_{\mathbf{R}^n}(|\nabla v_j|^p - Q|v_j|^p)dx}{\left(\int_{\mathbf{R}^n} K|v_j|^{p^*}dx\right)^{p/p^*}} \\ &\ge \lim_{j\to\infty} \frac{\int_{B_\delta}|\nabla v_j|^p + \int_{\mathbf{R}^n\backslash B_R}|\nabla v_j|^p dx - \int_{\mathbf{R}^n\backslash B_R} Q|v_j|^p dx + o(1)}{\left(\int_{B_\delta} K|v_j|^{p^*}dx + \int_{\mathbf{R}^n\backslash B_R} K|v_j|^{p^*}dx\right)^{p/p^*} + o(1)}. \end{aligned}$$

Using the Sobolev embedding theorem and the property of K and Q, we get

$$\begin{aligned} S_{p^*,Q}(B) &\ge \frac{S}{1+\varepsilon} \lim_{j\to\infty} \frac{(\int_{B_\delta}|v_j|^{p^*} + \int_{\mathbf{R}^n\backslash B_R}|v_j|^{p^*}dx)^{p/p^*} + o(1)}{\left(\int_{B_\delta}|v_j|^{p^*}dx + \int_{\mathbf{R}^n\backslash B_R}|v_j|^{p^*}dx\right)^{p/p^*} + o(1)} \\ &= \frac{S}{1+\varepsilon} \end{aligned}$$

Since ε is arbitrary, we find this relation is a contradiction.

3rd step. Next we prove $v_\infty \in C^{1,\alpha}_{\rm loc}(\mathbf{R}^n \backslash 0)$.

Since $\{v_j\}$ is bounded in $\mathbf{R}^n \backslash B_\eta$ (for any $\eta > 0$), we get $v_\infty \in L^\infty_{\rm loc}(\mathbf{R}^n\backslash 0)$. Hence invoking Proposition 2.6, we obtain $v_\infty \in C^{1,\alpha}_{\rm loc}(\mathbf{R}^n\backslash 0)$.

4th step. To show $v_\infty \in M^\infty_{rad}$.

The above consideration shows that v_∞ solves locally in the weak sense of $W^{1,p}_{\rm loc}(\mathbf{R}^n)$,

$$-\mathrm{div}(|\nabla v_\infty|^{p-2}\nabla v_\infty) - Q(|x|)|v_\infty|^{p-2}v_\infty - S^\infty_{p^*,Q}K(|x|)|v_\infty|^{p^*-2}v_\infty = 0. \tag{17}$$

Then multiplying the both sides by v_∞ and integrating (17) over B_R, we get

$$\int_{B_R}(|\nabla v_\infty|^p - Q|v_\infty|^p)dx - \omega_n R^{n-1}|v'_\infty|^{p-2}v'_\infty v_\infty = S^\infty_{p^*Q}\int_{B_R} K|v_\infty|^{p^*}dx \tag{18}$$

Noting that by the radial lemma, $v_\infty(R) \to 0$ as $R \to \infty$ and

$$\int_{\mathbf{R}^n}|\nabla v_\infty|^p dx, \ \int_{\mathbf{R}^n} Q|v_\infty|^p dx, \ \text{and} \ \int_{\mathbf{R}^n} K|v_\infty|^{p^*}dx$$

are finite, we can choose an increasing sequnece $\{R_l\}$ such that

$$R_l \to \infty \quad \text{as } l \to \infty, \text{ and } v'_\infty(R_l) \le 0 \quad \text{for any } l \in \mathbf{N}.$$

Then we have for any $l \in \mathbf{N}$

$$\int_{B_{R_l}} (|\nabla v_\infty|^p - Q|v_\infty|^p)\,dx \le S_{p^*,Q}^\infty \int_{B_{R_l}} K|v_\infty|^{p^*}\,dx. \tag{19}$$

Now letting $l \to \infty$ in (19), we get

$$\int_{\mathbf{R}^n} (|\nabla v_\infty|^p - Q|v_\infty|^p)\,dx \le S_{p^*,Q}^\infty \int_{\mathbf{R}^n} K(|x|)|v_\infty|^{p^*}\,dx.$$

Then

$$\begin{aligned} S_{p^*,Q}^\infty &\le \frac{\int_{\mathbf{R}^n} (|\nabla v_\infty|^p - Q|v_\infty|^p)\,dx}{\left(\int_{\mathbf{R}^n} K|v_\infty|^{p^*}\,dx\right)^{p/p^*}} \\ &\le S_{p^*,Q}^\infty \left(\int_{\mathbf{R}^n} K|v_\infty|^{p^*}\,dx\right)^{1-p/p^*}. \end{aligned}$$

Using the Fatou lemma, we have

$$1 \le \int_{\mathbf{R}^n} K|v_\infty|^{p^*}\,dx \le \varliminf_{j\to\infty} \int_{\mathbf{R}^n} K|v_j|^{p^*}\,dx = 1,$$

i.e.,

$$\int_{\mathbf{R}^n} K|v_\infty|^{p^*}\,dx = 1.$$

5th step. To show the strong convergence of ∇v_j.

In the previous argument, we have

$$\int_{\mathbf{R}^n} (|\nabla v_\infty|^p - Q|v_\infty|^p)\,dx = \lim_{j\to\infty} \int_{\mathbf{R}^n} (|\nabla v_j|^p - Q|v_j|^p)\,dx,$$

On the other hand by the Fatou lemma,

$$\int_{\mathbf{R}^n} (|\nabla v_\infty|^p - Q|v_\infty|^p)\,dx \le \varliminf_{j\to\infty} \int_{\mathbf{R}^n} (|\nabla v_j|^p - Q|v_j|^p)\,dx$$

holds and in view of $(Q-2)$, we have

$$\begin{aligned} \int_{\mathbf{R}^n} |\nabla v_\infty|^p\,dx &= \lim_{j\to\infty} \int_{\mathbf{R}^n} |\nabla v_j|^p\,dx \\ \int_{\mathbf{R}^n} Q|v_\infty|^p\,dx &= \lim_{j\to\infty} \int_{\mathbf{R}^n} Q|v_j|^p\,dx. \end{aligned}$$

Finally, we find out

$$\nabla v_j \to \nabla v_\infty \quad \text{strongly in } L^p(\mathbf{R}^n),$$

i.e., $v_\infty \in M_{rad}^\infty$.

As in Theorem 2.2., letting $w_\infty = (S^\infty_{p^*,Q})^{1/(p^*-p)} v_\infty$, we have that w_∞ is a solution of (13), moreover we find w_∞ is positive by Proposition 2.7.

The proof is complete.

Remark. In general, though ∇v_j converges to ∇v_∞ in $L^p(\mathbf{R}^n)$ and $v_\infty \in L^p(\mathbf{R}^n)$, v_j converges to v_∞ only in locally $L^p(\mathbf{R}^n)$. But if $Q < -c < 0$ holds on $\mathbf{R}^n$ (c is a positive constant.), v_j converges to v_∞ in $L^p(\mathbf{R}^n)$.

If Q is uniformly bounded away from 0, then we expect that the solution obtained in Theorem 3.1. decays exponentially at infinity. Following the idea of Tolksdorf [21] or Li and Yan [17], we have the follwing theorem.

Theorem 3.2. *If there exists R_0 such that $Q \le -q_0 < 0$ for all $|x| \ge R_0$, then for some positive $q < q_0$, $R_1 \ge R_0$ and $C > 0$, we have for the solution obtained in the previous theorem,*

$$w_\infty \le C \exp(-(\frac{q}{p-1})^{1/p}|x|) \quad |x| \ge R_1$$

Proof. Since $w_\infty \to 0$ as $|x| \to \infty$ (by the radial lemma), there exist R_1 and q such that

$$-Qw_\infty^{p-1} - Kw_\infty^{p^*-1} \ge qw_\infty^{p-1} \ge 0 \tag{20}$$

for $R_1 \le |x|$. Now let $\tilde{C} \ge w_\infty(R_1)$ and $V = \tilde{C}\exp(\theta(R_1 - |x|))$. Then we have

$$-\mathrm{div}(|\nabla V|^{p-2}\nabla V) = \tilde{C}\theta^{p-1}\Big(\exp\{\theta(p-1)(R_1 - |x|)\}\Big)\{-(p-1)\theta + \frac{(n-1)}{|x|}\}.$$

Hence we get

$$\begin{aligned}-\mathrm{div}(|\nabla V|^{p-2}\nabla V) + qV^{p-1} \ge \tilde{C}\Big(\exp\{\theta(p-1)(R_1 - |x|)\}\Big) \times \\ \times \{-(p-1)\theta^p + \frac{(n-1)}{|x|}\theta^{p-1} + q\}.\end{aligned}$$

Letting $\theta = (\frac{q}{p-1})^{1/p}$, we obtain

$$-\mathrm{div}(|\nabla V|^{p-2}\nabla V) + qV^{p-1} > 0. \tag{21}$$

Putting $\varphi = (w_\infty - V)^+$, then we have $\varphi \in W^{1,p}_r(\mathbf{R}^n)$ and $\varphi(R_1) = 0$. Using (20), we have for $|x| \ge R_1$,

$$-\mathrm{div}(|\nabla w_\infty|^{p-2}\nabla w_\infty) + qw_\infty^{p-1} < 0. \tag{22}$$

Now multiplying the both sides of (22) with φ and integrating over $\{|x| \ge R_1\}$, we have

$$\int_{|x|\ge R_1} \Big(|\nabla w_\infty|^{p-2}\nabla w_\infty \cdot \nabla\varphi + qw_\infty^{p-1}\varphi\Big)dx \le 0. \tag{23}$$

Similarly for (21), we have

$$\int_{|x|\geq R_1}\left(|\nabla V|^{p-2}\nabla V\cdot\nabla\varphi+qV^{p-1}\varphi\right)dx\geq 0. \tag{24}$$

Calculating (23) − (24), we have

$$0\geq\int_{|x|\geq R_1}(|\nabla w_\infty|^{p-2}\nabla w_\infty-|\nabla V|^{p-2}\nabla V)\cdot\nabla\varphi dx$$
$$+\int_{|x|\geq R_1}q(w_\infty^{p-1}-V^{p-1})\varphi dx.$$

Using

$$\begin{aligned}\sum_{i=1}^{n}(|\xi|^{p-2}\xi_i-|\eta|^{p-2}\eta_i)(\xi_i-\eta_i)&\geq|\xi|^p+|\eta|^p-(|\xi|^{p-2}+|\eta|^{p-2})|\xi||\eta|\\&=(|\xi|^{p-1}-|\eta|^{p-1})(|\xi|-|\eta|)\\&>0\quad\text{if }\xi\neq\eta,\end{aligned}$$

we have

$$0\geq\int_{\{|x|\geq R_1\}\cap\{w_\infty\geq V\}}q(w_\infty^{p-1}-V^{p-1})\varphi dx.$$

Since w_∞ and V are continuous in $|x|\geq R_1$, we obtain $\{|x|\geq R_1\}\cap\{w_\infty\geq V\}=\emptyset$.

Finally we have

$$w_\infty\leq C\exp(-(\frac{q}{p-1})^{1/p}|x|)\quad |x|\geq R_1.$$

The proof is complete.

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Chapter 3. On Multiple Solutions.

§3-1. Introduction.

In this chapter we study the number of nontrivial solutions of

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = q(x)|u|^{\sigma}u \quad \text{on } \mathbf{R}^n, \tag{1}$$

and

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = f(x,u) \quad \text{on } \mathbf{R}^n, \tag{2}$$

where $1 < p < n$, $p-2 < \sigma < np/(n-p) - 2$. The equation (1) is a typical case of (2). In order to make clear the essence of proofs of our existence theorem for (2), we discuss (1) in Section 3-2. In [7], Y.-Y. Li treated (1) and (2) in the case $p = 2$. In [4], the author showed some existence theorems of (1) (though the multiplicity of solutions was not discussed) under some assumptions which were different from those of Y.-Y. Li. Our aim is to unify the results of Y.-Y. Li and the author. Moreover we show that the solutions obtained here decay exponentially at infinity.

In the case $p = 2$, P.-L. Lions [8] discussed the existence of a nontrivial solution of this type of equations using his "concentration compactness argument". Ding and Ni [2], Ni [9] and Rother [11], [12] also considered similar type of equations, that is, semilinear elliptic equations with "subcritical growth conditions" on $\mathbf{R}^n$. For quasilinear equations, Egnell [3] dicussed mainly the existence and non-existence of the radial solutions of (2) with $f(x,u) \geq 0$. He also considered a bounded domain case. G.-B. Li [6] also considered a p-Laplace equation on $\mathbf{R}^n$. His main concern is the regurality of solutions. The multiplicity of the solutions is not discussed in both of them.

In the following, we introduce some notations often used in this paper:

$\langle \cdot,\cdot \rangle$ means the dual coupling between $(W^{1,p}(\mathbf{R}^n))'$ and $W^{1,p}(\mathbf{R}^n)$, and let

$$\|u\|^p = \int (|\nabla u|^p + |u|^p),$$

$$\Phi(u) = \frac{1}{p}\int (|\nabla u|^p + |u|^p) - \frac{1}{\sigma+2}\int q(x)|u|^{\sigma+2},$$

$$\Phi_\infty(u) = \frac{1}{p}\int (|\nabla u|^p + |u|^p) - \frac{1}{\sigma+2}\int q_\infty |u|^{\sigma+2},$$

$$V_\infty = \{u \in W^{1,p}(\mathbf{R}^n)\backslash\{0\} \mid \langle \Phi'_\infty(u), u\rangle = 0\},$$

$$S = \inf_{u \in D^{1,p}(\mathbf{R}^n)\backslash\{0\}} \left(\int |\nabla u|^p\right)^{1/p} \left(\int |u|^{p^*}\right)^{-1/p^*} \quad \text{(the best Sobolev constant)},$$

$$M_\infty = \inf_{u \in V_\infty} \Phi_\infty(u),$$

$$p^* = \frac{np}{n-p},$$

where q_∞ is a positive constant and $D^{1,p}(\mathbf{R}^n) = \{u \in L^p(\mathbf{R}^n) | \nabla u \in L^p(\mathbf{R}^n)\}$. Note that the best Sobolev constant depends only on n and p.

Throughout this paper, if the integration is taken over the whole space, we do not indicate the domain of integration, and the Lebesgue measure "dx" is always omitted (except for Lemma 2.3). We denote various positive constants by C.

Now we state our main theorems. Condtions $(Q_1)-(Q_3)$, $(F_1)-(F_7)$ and (G_1), (G_2) are stated later. For the equation (1), we have:

Theorem 1.1. *Suppose that (Q_1), (Q_2), and (Q_3) hold. Moreover we assume there exist N functions $u_k \in W^{1,p}(\mathbf{R}^n)$ $(k = 1, \cdots, N)$ with disjoint supports such that*

$$\int q|u_k|^{\sigma+2} > 0, \quad k = 1, \cdots, N,$$
$$\sum_{k=1}^{N} \frac{\left(\int |\nabla u_k|^p + |u_k|^p\right)^{(\sigma+2)/(\sigma+2-p)}}{\left(\int q|u_k|^{\sigma+2}\right)^{p/(\sigma+2-p)}} < \left(\frac{1}{p} - \frac{1}{\sigma+2}\right)^{-1} M_\infty.$$

Then (1) has at least N pairs of nontrivial solutions in $W^{1,p}(\mathbf{R}^n)$. Moreover, if

$$q(x) \in C^0(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n),$$

then each weak solution to (1) belongs to $L^\infty(\mathbf{R}^n) \cap C^{1,\mu}(\mathbf{R}^n)$ $(0 < \mu < 1)$, provided $n > p \geq 2$.

For (2), we get:

Theorem 1.2. *Suppose that Conditions $(F_1)-(F_7)$ and (G_1), (G_2) hold. Moreover we assume there exist N functions $u_k \in W^{1,p}(\mathbf{R}^n)$ $(k = 1, \cdots, N)$ with disjoint supports such that*

$$\int f(x, u_k)u_k > 0, \quad k = 1, \cdots, N,$$

and let

$$X_N = \{t_1 u_1 + \cdots + t_N u_N \mid (t_1, \cdots, t_N) \in \mathbf{R}^N\}.$$

If

$$\sup_{X_N} J(u) < M_g$$

holds, then (2) has at least N pairs of nontrivial solutions in $W^{1,p}(\mathbf{R}^n)$.

Since the proofs of these theorems require some lemmas, we discuss the details in Section 3-2 and Section 3-3.

§3-2. The case of separation of variables.

In this section we consider only the equation

$$(1) \qquad -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = q(x)|u|^{\sigma}u \quad \text{on } \mathbf{R}^n,$$

where $1 < p < n$, and $p-2 < \sigma < p^* - 2$. We assume that $q(x)$ satisfies the following conditions:

$$(Q_1) \qquad \begin{cases} q(x) \in L^t(\mathbf{R}^n), & \text{where } t > \dfrac{p^*}{p^* - (\sigma+2)}, \\ q(x) \in L^\infty_{loc}(\mathbf{R}^n) \cap L^t(\mathbf{R}^n) & \text{for } t = \dfrac{p^*}{p^* - (\sigma+2)}. \end{cases}$$

$$(Q_2) \qquad \int q(x)|u_0|^{\sigma+2} > 0 \quad \text{for some } u_0 \in W^{1,p}(\mathbf{R}^n).$$

(Q_3) There exists a number $q_\infty (> 0)$ such that $\limsup_{|x|\to\infty} q(x) \le q_\infty$.

(Q_4) $q(x) \in C^0(\mathbf{R}^n)$ and $q(x) > 0$ for some $x \in \mathbf{R}^n$.

First we state some lemmas, which ensure Proposition 2.5 and Lemma 2.6.

LEMMA 2.1. *If $u \in L^p(\mathbf{R}^n) \cap L^{p^*}(\mathbf{R}^n)$ then $u \in L^s(\mathbf{R}^n)$ for all $s \in [p, p^*]$ with estimate*

$$(3) \qquad \int |u|^s \le \Big(\int |u|^p\Big)^{(p^*-s)/(p^*-p)} \Big(\int |u|^{p^*}\Big)^{(s-p)/(p^*-p)}.$$

Especially, by the Sobolev inequality, we have

$$(4) \qquad \int |u|^s \le C\Big(\int (|\nabla u|^p + |u|^p)\Big)^{s/p}.$$

LEMMA 2.2. *For any $x, y \in \mathbf{R}$, we have*

$$\Big||x|^{a-1}x - |y|^{a-1}y\Big| \le \begin{cases} C|x-y|^a & \text{if } 0 < a < 1, \\ C(|x|^{a-1} + |y|^{a-1})|x-y| & \text{if } 1 \le a. \end{cases}$$

Proof. If $0 < a < 1$, then the inequality follows from the Hölder continuity with exponent a of the function $|x|^{a-1}x$. If $a \ge 1$, then the mean value theorem can be applied. Hence the inequality holds.

The proof is complete.

LEMMA 2.3. *Let $p > 1$. For a bounded sequence of functions $f_j \in L^p(\mathbf{R}^n)$, we assume that $f_j \rightharpoonup f(\in L^p(\mathbf{R}^n))$ weakly in $L^p(\mathbf{R}^n)$ and $f_j \to g(\in L^p(\mathbf{R}^n))$ a.e. in $\mathbf{R}^n$ as $j \to \infty$. Then*

$$f = g \quad \text{a.e. in } \mathbf{R}^n$$

holds.

Proof. Let

$$A_k^{(l)} = \left\{ x \in \mathbf{R}^n \middle| \, |f_k(x)| \le l \right\},$$

$$A^{(l)} = \cup_{m=1}^{\infty} \cap_{k \ge m}^{\infty} A_k^{(l)}, \text{and } A = \cup_{l=1}^{\infty} A^{(l)}.$$

Since $f_j \in L^p(\mathbf{R}^n)$ converges almost everwhere to g, the (n-dimensional) Lebesgue measure of $\mathbf{R}^n - A$ equals to 0, that is, $|\mathbf{R}^n - A| = 0$. Now for any $\psi \in C_0^\infty(\mathbf{R}^n)$, we have by the Lebesgue convergence theorem

$$\int_{A^{(l)}} f_j \psi dx \to \int_{A^{(l)}} g \psi dx \quad \text{as } j \to \infty.$$

On the other hand, we get by the definition of weak convergence

$$\int_{A^{(l)}} f_j \psi dx = \int_{\mathbf{R}^n} f_j \psi \Big|_{A^{(l)}} dx \to \int_{\mathbf{R}^n} f \psi \Big|_{A^{(l)}} dx = \int_{A^{(l)}} f \psi dx \quad \text{as } j \to \infty.$$

Letting $l \to \infty$, we obtain

$$\int_{\mathbf{R}^n} g \psi dx = \int_{\mathbf{R}^n} f \psi dx.$$

This implies $f = g$ a.e. in $\mathbf{R}^n$.

The proof is complete.

Now we show the crucial proposition for the multiplicity results. We recall the definition of the local Palais-Smale condition (denoted by $(PS)_c$).

DEFINITION 2.4 (the $(PS)_c$ condition). *Let E be a real Banach space and $J \in C^1(E, \mathbf{R})$. We say that the functional J satisfies the $(PS)_c$ if any sequence $\{u_j\}$ satisfying*

$$J(u_j) \to c, \; J'(u_j) \to 0 \tag{5}$$

contains a strongly convergent subsequence.

The proposition is :

Proposition 2.5. *Φ satisfies $(PS)_c$ for all $c \in (-\infty, M_\infty)$.*

Proof. Let $\{u_j\} \subset W^{1,p}(\mathbf{R}^n)$ be a sequence satisfying (5) for $c \in (-\infty, M_\infty)$, that is,

$$\text{(6)} \qquad \begin{cases} \dfrac{1}{p}\displaystyle\int (|\nabla u_j|^p + |u_j|^p) - \dfrac{1}{\sigma+2}\int q|u_j|^{\sigma+2} \to c \\ |\langle \Phi'(u_j), v\rangle| \le o(1)\|v\| \quad \text{for any } v \in W^{1,p}(\mathbf{R}^n). \end{cases}$$

Letting $v = u_j$ in the second inequality of (6), we have

$$\text{(7)} \qquad -o(1)\|u_j\| \le \int (|\nabla u_j|^p + |u_j|^p) - \int q|u_j|^{\sigma+2}.$$

By (6) and (7), we get

$$(\frac{\sigma+2}{p} - 1)\|u_j\|^p \le o(1)\|u_j\| + (\sigma+2)c + o(1).$$

Thus we obtain the boundedness of $\{u_j\}$ in $W^{1,p}(\mathbf{R}^n)$. Hence we may suppose that a subsequence of $\{u_j\}$ (still denoted by $\{u_j\}$) satisfies

$$\begin{cases} u_j \rightharpoonup u_\infty & \text{weakly in } W^{1,p}(\mathbf{R}^n), \\ u_j \to u_\infty & \text{locally strongly in } L^\theta(\mathbf{R}^n)\ 1 \le \theta < p^*, \\ u_j \to u_\infty & \text{a.e. in } \mathbf{R}^n. \end{cases}$$

The proof of Proposition 2.5 needs the following lemma.

Lemma 2.6. *$\langle \Phi'(u_\infty), u_\infty\rangle = 0$ holds.*

Proof. Obviously, we have $\langle \Phi'(u_j), u_\infty\rangle = o(1)$, that is,

$$\int (|\nabla u_j|^{p-2}\nabla u_j \nabla u_\infty + |u_j|^{p-2}u_j u_\infty) - \int q|u_j|^\sigma u_j u_\infty = o(1).$$

First we observe

$$\int (|\nabla u_j|^{p-2}\nabla u_j \nabla u_\infty + |u_j|^{p-2}u_j u_\infty) - \int (|\nabla u_\infty|^p + |u_\infty|^p).$$

Since the two sequences $\{|\nabla u_j|^{p-2}\nabla u_j\}$ and $\{|u_j|^{p-2}u_j\}$ are bounded in $L^{p/(p-1)}(\mathbf{R}^n)$, we may assume these are weakly convergent in $L^{p/(p-1)}(\mathbf{R}^n)$ (choosing a subsequence once more if necessary). By Lemma 2.3, the weak limit of $\{|\nabla u_j|^{p-2}\nabla u_j\}$ coincides with $|\nabla u_\infty|^{p-2}\nabla u_\infty$. Hence we have

$$\int (|\nabla u_j|^{p-2}\nabla u_j \nabla u_\infty + |u_j|^{p-2}u_j u_\infty) - \int (|\nabla u_\infty|^p + |u_\infty|^p) = o(1).$$

Next we show
$$\int q|u_j|^\sigma u_j u_\infty - \int q|u_\infty|^{\sigma+2} = o(1).$$
Since $q \in L^t(\mathbf{R}^n)$, for every $\varepsilon > 0$ there exists a bounded domain Ω such that
$$\Big(\int_{\mathbf{R}^n\setminus\Omega} |q|^t\Big)^{1/t} < \varepsilon$$
holds. Noticing that
$$\begin{aligned}&\int q|u_j|^\sigma u_j u_\infty - \int q|u_\infty|^{\sigma+2}\\ &= \int_\Omega (q|u_j|^\sigma u_j u_\infty - q|u_\infty|^{\sigma+2}) + \int_{\mathbf{R}^n\setminus\Omega} (q|u_j|^\sigma u_j u_\infty - q|u_\infty|^{\sigma+2}),\end{aligned}$$
we find that the integration over Ω yields
$$\begin{aligned}&\int_\Omega \Big|q|u_j|^\sigma u_j u_\infty - q|u_\infty|^{\sigma+2}\Big|\\ &\le \Big(\int_\Omega |q|^t\Big)^{1/t}\Big(\int_\Omega \big||u_j|^\sigma u_j - |u_\infty|^\sigma u_\infty\big|^\alpha\Big)^{1/\alpha}\Big(\int_\Omega |u_\infty|^\beta\Big)^{1/\beta}\end{aligned}$$
where $1/t + 1/\alpha + 1/\beta = 1$, α and β satisfy if
$$p^*/(p^* - (\sigma+2)) \le t \le p^*(\sigma+2)/(p^* - (\sigma+2)),$$
then
$$\alpha = \frac{p^* t}{(p^*-1)t - p^*},\ \beta = p^* \tag{8}$$
and if $t > p^*(\sigma+2)/(p^* - (\sigma+2))$,
$$\alpha = \frac{\sigma+2}{\sigma+1},\ \beta = \frac{(\sigma+2)t}{t-(\sigma+2)}. \tag{9}$$
In both cases, $p \le (\sigma+1)\alpha \le p^*$ and $p \le \beta \le p^*$ hold. If $t > p^*/(p^* - (\sigma+2))$, then $p < (\sigma+1)\alpha < p^*$. By the strong convergence of $\{u_j\}$ in $L^\theta(\Omega)$ for each θ with $1 \le \theta < p^*$ and Lemma 2.2, we have
$$\int_\Omega (q|u_j|^\sigma u_j u_\infty - q|u_\infty|^{\sigma+2}) = o(1).$$
If $t = p^*/(p^* - (\sigma+2))$, we use another inequality:
$$\begin{aligned}&\int_\Omega \Big|q|u_j|^\sigma u_j u_\infty - q|u_\infty|^{\sigma+2}\Big|\\ &\le \sup_\Omega |q|\Big(\int_\Omega \big||u_j|^\sigma u_j - |u_\infty|^\sigma u_\infty\big|^{(\sigma+2)/(\sigma+1)}\Big)^{(\sigma+1)/(\sigma+2)}\Big(\int_\Omega |u_\infty|^{\sigma+2}\Big)^{1/(\sigma+2)}\end{aligned}$$

Using the same fact stated above, we have again in this case

$$\int_{\Omega}(q|u_j|^{\sigma}u_j u_{\infty} - q|u_{\infty}|^{\sigma+2}) = o(1).$$

For $\mathbf{R}^n\backslash\Omega$, we have

$$\begin{aligned}
&\int_{\mathbf{R}^n\backslash\Omega}\left|q|u_j|^{\sigma}u_j u_{\infty} - q|u_{\infty}|^{\sigma+2}\right| \\
&\leq \Big(\int_{\mathbf{R}^n\backslash\Omega}|q|^t\Big)^{1/t}\Big(\int_{\mathbf{R}^n\backslash\Omega}\big||u_j|^{\sigma}u_j - |u_{\infty}|^{\sigma}u_{\infty}\big|^{\alpha}\Big)^{1/\alpha}\Big(\int_{\mathbf{R}^n\backslash\Omega}|u_{\infty}|^{\beta}\Big)^{1/\beta} \\
&\leq \varepsilon\Big(\int_{\mathbf{R}^n\backslash\Omega}\big||u_j|^{\sigma}u_j - |u_{\infty}|^{\sigma}u_{\infty}\big|^{\alpha}\Big)^{1/\alpha}\Big(\int_{\mathbf{R}^n\backslash\Omega}|u_{\infty}|^{\beta}\Big)^{1/\beta}
\end{aligned}$$

where α and β are defined by (8) and (9). Then the boundedness of

$$\int_{\mathbf{R}^n\backslash\Omega}\big||u_j|^{\sigma}u_j - |u_{\infty}|^{\sigma}u_{\infty}\big|^{\alpha} \text{ and } \Big(\int_{\mathbf{R}^n\backslash\Omega}|u_{\infty}|^{\beta}\Big)^{1/\beta}$$

follows from the boundedness of $\{u_j\}$ in $W^{1,p}(\mathbf{R}^n)$,(3), and Lemma 2.2. Hence we have

$$\int_{\mathbf{R}^n\backslash\Omega}\left|q|u_j|^{\sigma}u_j u_{\infty} - q|u_{\infty}|^{\sigma+2}\right| \leq C\varepsilon.$$

Since ε is arbitrary, we have

$$\langle\Phi'(u_{\infty}), u_{\infty}\rangle = 0.$$

The proof is complete.

Now we get back to the proof of Proposition 2.5. As in Y.-Y. Li, there is no other possibility of $\{u_j\}$ except for the following two cases.

(A) For all $\delta > 0$, there exists an $R > 0$ such that

$$\int_{|x|\geq R}(|\nabla u_j|^p + |u_j|^p) < \delta$$

holds for each integer $j(\geq n_0(R))$.

(B) The negative proposition of (A): There exists a $\delta_0 > 0$ such that

$$\int_{|x|\geq R'}(|\nabla u_j|^p + |u_j|^p) \geq \delta_0$$

holds for each $R' > 0$ and for a suitable integer $j(\geq R')$.

First we consider the case (A). Recalling the lower semi-continuity of the norm, we have

$$\int_{|x|\geq R}(|\nabla u_{\infty}|^p + |u_{\infty}|^p) \leq \liminf_{j\to\infty}\int_{|x|\geq R}(|\nabla u_j|^p + |u_j|^p) < \delta.$$

As in (7), we get

$$\Big|\int(|\nabla u_j|^{p-2}\nabla u_j\nabla v+|u_j|^{p-2}u_jv)-\int q|u_j|^\sigma u_jv\Big|\le o(1)\|v\|\text{ for }v\in W^{1,p}(\mathbf{R}^n).$$

In the unbounded domain $\{|x|\ge R\}$, we have as in the proof of Lemma 2.6,

$$|\int_{|x|\ge R}q|u_j|^\sigma u_jv|\le\Big(\int_{|x|\ge R}|q|^t\Big)^{1/t}\Big(\int_{|x|\ge R}|u_j|^{(\sigma+1)\alpha}\Big)^{1/\alpha}\Big(\int_{|x|\ge R}|v|^\beta\Big)^{1/\beta}$$

with $1/t+1/\alpha+1/\beta=1$. α and β are determined by (8) and (9). In addition, by (3) we get

$$\int_{|x|\ge R}|u_j|^{(\sigma+1)\alpha}\le\Big(\int_{|x|\ge R}|u_j|^p\Big)^{\kappa/p}\Big(\int_{|x|\ge R}|u_j|^{p^*}\Big)^{\lambda/p^*}$$

where $\kappa=p(p^*-(\sigma+1)\alpha)/(p^*-p)$ and $\lambda=p^*((\sigma+1)\alpha-p)/(p^*-p)$. Now the Sobolev inequality, we have

$$\int_{|x|\ge R}|u_j|^{(\sigma+1)\alpha}\le C\Big(\int_{|x|\ge R}|u_j|^p\Big)^{\kappa/p}\Big(\int_{|x|\ge R}|\nabla u_j|^p+|u_j|^p\Big)^{\lambda/p}.$$

Hence in the case (A), we obtain

$$\Big(\int_{|x|\ge R}|u_j|^{(\sigma+1)\alpha}\Big)^{1/\alpha}\le C\Big(\delta^{(\kappa+\lambda)/p}\Big)^{1/\alpha}=C\delta^{(\sigma+1)/p}.$$

Hence we have

$$\Big|\int_{|x|\ge R}q|u_j|^\sigma u_jv\Big|\le C\delta^{(\sigma+1)/p}\|v\|,$$

using (3) for v. For the domain $B_R=\{|x|\le R\}$, we also have in a similar way,

$$\begin{aligned}&\Big|\int_{B_R}q|u_j|^\sigma u_jv-\int_{B_R}q|u_\infty|^\sigma u_\infty v\Big|\\&\le\Big(\int_{B_R}|q|^t\Big)^{1/t}\Big(\int_{B_R}\big||u_j|^\sigma u_j-|u_\infty|^\sigma u_\infty\big|^\alpha\Big)^{1/\alpha}\Big(\int_{B_R}|v|^\beta\Big)^{1/\beta}\\&\le o(1)C\Big(\int_{B_R}|q|^t\Big)^{1/t}\|v\|\text{ (by Lemma 2.1)}\end{aligned}$$

if $t>p^*/(p^*-(\sigma+2))$. Even if $t=p^*/(p^*-(\sigma+2))$, we have

$$\begin{aligned}&\Big|\int_{B_R}q|u_j|^\sigma u_jv-\int_{B_R}q|u_\infty|^\sigma u_\infty v\Big|\\&\le C\sup_{B_R}|q|\Big(\int_{B_R}\big||u_j|^\sigma u_j-|u_\infty|^\sigma u_\infty\big|^{(\sigma+2)/(\sigma+1)}\Big)^{(\sigma+1)/(\sigma+2)}\|v\|\\&\le o(1)\|v\|,\end{aligned}$$

Thus we have by letting $v = u_j$,

$$\Big|\int(|\nabla u_j|^p + |u_j|^p) - \int q|u_\infty|^\sigma u_\infty u_j\Big| \le o(1)\|u_j\| + \delta^{(\sigma+1)/p}\|u_j\|.$$

Similarly, we calculate the behavior of the term

$$\int_{B_R} |q||u_\infty|^{\sigma+1}|u_j - u_\infty|,$$

as $j \to \infty$. We get if $t = p^*/(p^* - (\sigma+2))$,

$$\begin{aligned}&\int_{B_R} |q||u_\infty|^{\sigma+1}|u_j - u_\infty| \\ &\quad\le \sup_{B_R}|q|\Big(\int_{B_R}|u_\infty|^{p^*}\Big)^{(\sigma+1)/p^*}\Big(\int_{B_R}|u_j - u_\infty|^{p^*/(p^*-(\sigma+1))}\Big)^{(p^*-(\sigma+1))/p^*} = o(1).\end{aligned}$$

The last estimate follows from the fact that u_j converges strongly to u_∞ in $L^{\frac{p^*}{p^*-(\sigma+1)}}(B_R)$ (this exponent is subcritical). If $t > p^*/(p^* - (\sigma+2))$, we get

$$\begin{aligned}&\int_{B_R} |q||u_\infty|^{\sigma+1}|u_j - u_\infty| \\ &\quad\le \Big(\int_{B_R}|q|^t dx\Big)^{1/t}\Big(\int_{B_R}|u_\infty|^{p^*}\Big)^{(\sigma+1)/p^*}\Big(\int_{B_R}|u_j - u_\infty|^{\hat{p}}\Big)^{1/\hat{p}} = o(1).\end{aligned}$$

where $1/t + (\sigma+1)/p^* + 1/\hat{p} = 1$. Since $t > p^*/(p^* - (\sigma+2))$, $\hat{p} < p^*$ and the last estimate follows.

Hence we get

$$\begin{aligned}&\Big|\int(|\nabla u_j|^p + |u_j|^p) - \int(|\nabla u_\infty|^p + |u_\infty|^p)\Big| \\ &= \Big|\int(|\nabla u_j|^p + |u_j|^p) - \int q|u_\infty|^{\sigma+2}\Big| \\ &\le \Big|\int(|\nabla u_j|^p + |u_j|^p) - \int q|u_\infty|^\sigma u_\infty u_j\Big| + \Big|\int q|u_\infty|^\sigma u_\infty u_j - \int q|u_\infty|^{\sigma+2}\Big| \\ &\le C(\delta^{(\sigma+1)/p} + o(1))\end{aligned}$$

Since δ is arbitrary, we have

$$\|u_j\| \to \|u_\infty\| \quad \text{as } j \to \infty,$$

which implies, together with the weak convergence $u_j \rightharpoonup u_\infty$ in a uniformly convex Banach space $W^{1,p}(\mathbf{R}^n)$, the strong convergence

$$u_j \to u_\infty \text{ in } W^{1,p}(\mathbf{R}^n).$$

In the case (A), we obtain the strong limit u_∞ of $\{u_j\}$.

Next we consider the case (B). Here we may assume, for some $\delta_0 > 0$, that

$$\int_{|x|\geq l}(|\nabla u_{j_l}|^p + |u_{j_l}|^p) \geq \delta_0$$

holds for all $l \in \mathbf{N}$ and an appropriate $j_l(\geq l)$. For $\varepsilon > 0$, there exists an $R_0 > 0$ such that

$$\int_{|x|\geq R_0}(|\nabla u_\infty|^p + |u_\infty|^p + |q||u_\infty|^{\sigma+2}) < \varepsilon \tag{10}$$

and

$$q(x) < q_\infty + \varepsilon \quad \text{for } |x| \geq R_0 \tag{11}$$

hold. From the boundedness of $\{u_{j_l}\}$, we can choose an integer $m(\varepsilon) \in \mathbf{N}$ such that

$$m(\varepsilon)\varepsilon > \|u_{j_l}\|^p$$

holds for all l. Now let

$$I_k = \{x \in \mathbf{R}^n \mid R_0 + k - 1 \leq |x| \leq R_0 + k\} \quad \text{for } k = 1, 2, \cdots, m(\varepsilon).$$

Then we have

$$\|u_{j_l}\|^p \geq \int_{\cup_{k=1}^{m(\varepsilon)} I_k}(|\nabla u_{j_l}|^p + |u_{j_l}|^p) \geq m(\varepsilon)\int_{I_i}(|\nabla u_{j_l}|^p + |u_{j_l}|^p)$$

for a suitable $i \in \{1, \cdots, m(\varepsilon)\}$. Hence we get, choosing a subsequence of $\{l\}$ if necessary (still denoted by $\{l\}$), independently of l,

$$\int_{I_i}(|\nabla u_{j_l}|^p + |u_{j_l}|^p) < \varepsilon.$$

Hereafter we set $I_i = \{x \in \mathbf{R}^n \mid \hat{R} \leq |x| \leq \hat{R} + 1\}$ $(R \leq \hat{R})$.

Now we define a cut off function $\rho_1(x) \in C_0^\infty(\mathbf{R}^n)$ such that

$$0 \leq \rho_1 \leq 1, \ |\nabla \rho_1| \leq 2,$$

$$\rho_1 = \begin{cases} 1 & |x| \leq \hat{R} \\ 0 & |x| \geq \hat{R} + 1 \end{cases}$$

and $\rho_2 = 1 - \rho_1$. Let $v_l = \rho_1 u_{j_l}$, and $w_l = \rho_2 u_{j_l}$. Then we have supp $v_l \subset B_{\hat{R}+1}$ and supp $w_l \subset \mathbf{R}^n \setminus B_{\hat{R}}$. Now we estimate

$$\left|\langle \Phi'(u_{j_l}), v_l\rangle - \langle \Phi'(v_l), v_l\rangle\right|.$$

The calculation shows

$$\begin{aligned}
&\left|\langle \Phi'(u_{j_l}), v_l\rangle - \langle \Phi'(v_l), v_l\rangle\right| \\
&\le \left|\int (|\nabla u_{j_l}|^{p-2}\nabla u_{j_l}\nabla v_l + |u_{j_l}|^{p-2}u_{j_l}v_l) - \int (|\nabla v_l|^p + |v_l|^p)\right| \\
&\quad + \left|\int q|u_{j_l}|^\sigma u_{j_l}v_l - \int q|v_l|^{\sigma+2}\right| \\
&\le \int_{I_i}\left||\nabla u_{j_l}|^{p-2}\nabla u_{j_l}\nabla v_l + |u_{j_l}|^{p-2}u_{j_l}v_l\right| + \int_{I_i}\Big(|\nabla v_l|^p + |v_l|^p\Big) \\
&\quad + \left|\int_{I_i} q|u_{j_l}|^\sigma u_{j_l}v_l - \int_{I_i} q|v_l|^{\sigma+2}\right|,
\end{aligned}$$

where the last inequality follows from $u_{j_l} = v_l$ on $B_{\hat{R}}$. For the integral

$$\int_{I_i}\left||\nabla u_{j_l}|^{p-2}\nabla u_{j_l}\nabla v_l + |u_{j_l}|^{p-2}u_{j_l}v_l\right| \text{ and } \int_{I_i}\Big(|\nabla v_l|^p + |v_l|^p\Big),$$

we have the following estimate by using the Young inequality:

$$\begin{aligned}
&\int_{I_i}\left||\nabla u_{j_l}|^{p-2}\nabla u_{j_l}\nabla v_l + |u_{j_l}|^{p-2}u_{j_l}v_l\right| \\
&\le \Big(\int_{I_i}|\nabla u_{j_l}|^p\Big)^{(p-1)/p}\Big(\int_{I_i}|\nabla v_l|^p\Big)^{1/p} + \Big(\int_{I_i}|u_{j_l}|^p\Big)^{(p-1)/p}\Big(\int_{I_i}|v_l|^p\Big)^{1/p} \\
&\le \frac{p-1}{p}\Big(\int_{I_i}|\nabla u_{j_l}|^p + \int_{I_i}|u_{j_l}|^p\Big) + \frac{1}{p}\Big(\int_{I_i}|\nabla v_l|^p + |v_l|^p\Big) \\
&\le C\varepsilon.
\end{aligned}$$

The second term is estimated as follows:

$$\int_{I_i}(|\nabla v_l|^p + |v_l|^p) \le \int_{I_i}(|\nabla v_l|^p + |u_{j_l}|^p) \le C\int_{I_i}(|\nabla u_{j_l}|^p + |u_{j_l}|^p) \le C\varepsilon.$$

The last term yields, by using the Hölder inequality and (4) with $s = (\sigma+2)t/(t-1)$,

$$\begin{aligned}
&\left|\int_{I_i} q|u_{j_l}|^\sigma u_{j_l}v_l - \int_{I_i} q|v_l|^{\sigma+2}\right| \le \int_{I_i}|q|\rho_1(1-\rho_1^{\sigma+1})|u_{j_l}|^{\sigma+2} \\
&\le \int_{I_i}|q||u_{j_l}|^{\sigma+2} \le \Big(\int_{I_i}|q|^t\Big)^{1/t}\Big(\int_{I_i}|u_{j_l}|^{(\sigma+2)t/(t-1)}\Big)^{(t-1)/t} \\
&\le C\Big(\int_{I_i}|\nabla u_{j_l}|^p + |u_{j_l}|^p\Big)^{s(t-1)/pt} \\
&\le C\varepsilon^{(\sigma+2)/p} = o(\varepsilon).
\end{aligned}$$

Note that $(\sigma+2) < s \le p^*$ if $t \ge p^*/(p^*-(\sigma+2))$. Finally, we get

$$\left|\langle \Phi'(u_{j_l}), v_l\rangle - \langle \Phi'(v_l), v_l\rangle\right| \le C\varepsilon.$$

Similarly, we have

$$\left|\langle \Phi'(u_{j_l}), w_l\rangle - \langle \Phi'(w_l), w_l\rangle\right| \le C\varepsilon.$$

Since $\langle \Phi'(u_{j_l}), v_l\rangle = o(1)$ and $\langle \Phi'(u_{j_l}), w_l\rangle = o(1)$, we have

$$\left\{\begin{aligned} &\int(|\nabla v_l|^p + |v_l|^p) = \int q|v_l|^{\sigma+2} + O(\varepsilon) + o(1), \\ &\int(|\nabla w_l|^p + |w_l|^p) = \int q|w_l|^{\sigma+2} + O(\varepsilon) + o(1). \end{aligned}\right. \tag{12}$$

Because $\{u_{j_l}\}$ is a sequence for c satisfying (6),

$$c + o(1) = \Phi(u_{j_l}) = \Phi(v_l) + \Phi(w_l) + O(\varepsilon)$$

holds. In addition, from (12) we have

$$\Phi(v_l) = (\frac{1}{p} - \frac{1}{\sigma+2})\|v_l\|^p + O(\varepsilon) + o(1) \ge O(\varepsilon) + o(1).$$

Then we have

$$\begin{aligned} c + o(1) &\ge \Phi(w_l) + O(\varepsilon) \\ &= \frac{1}{p}\int(|\nabla w_l|^p + |w_l|^p) - \frac{1}{\sigma+2}\int q|w_l|^{\sigma+2} + O(\varepsilon). \\ &= (\frac{1}{p} - \frac{1}{\sigma+2})\int(|\nabla w_l|^p + |w_l|^p) + O(\varepsilon) \end{aligned} \tag{13}$$

Moreover, we get in this case, for large $l(\ge \hat{R}+1)$,

$$\int(|\nabla w_l|^p + |w_l|^p) \ge \delta_0.$$

By (11) and (12), we also get

$$\int q_\infty |w_l|^{\sigma+2} \ge \frac{\delta_0}{2}.$$

Now let

$$\xi = \left\{\frac{\int(|\nabla w_l|^p + |w_l|^p)}{\int q_\infty |w_l|^{\sigma+2}}\right\}^{1/(\sigma+2-p)}$$

and $\hat{w}_l = \xi w_l$. Then $\hat{w}_l \in V_\infty$. We should note $\xi \ne +\infty$. For $|x| \ge l$, by assumption (Q_3), $q \le q_\infty + \varepsilon$ holds. Then we have

$$\int(|\nabla w_l|^p + |w_l|^p) \le \int q_\infty |w_l|^{\sigma+2} + \varepsilon\int |w_l|^{\sigma+2} + O(\varepsilon) + o(1),$$

and $\xi = \{1 + O(\varepsilon)\}^{1/(\sigma+2-p)}$. Hence we get

$$\begin{aligned} M_\infty &\leq \frac{\xi^p}{p}\int(|\nabla w_l|^p + |w_l|^p) - \frac{\xi^{\sigma+2}}{\sigma+2}\int q_\infty |w_l|^{\sigma+2} \\ &= \xi^p(\frac{1}{p} - \frac{1}{\sigma+2})\int(|\nabla w_l|^p + |w_l|^p) \\ &= (\frac{1}{p} - \frac{1}{\sigma+2})(1+O(\varepsilon))^{p/(\sigma+2-p)}\int(|\nabla w_l|^p + |w_l|^p) \\ &\leq (\frac{1}{p} - \frac{1}{\sigma+2})\int(|\nabla w_l|^p + |w_l|^p) + O(\varepsilon^\vartheta), \end{aligned}$$

where $\vartheta = \min(1, p/(\sigma+2-p))$. From (13) and the above, we obtain

$$M_\infty - c \leq O(\varepsilon^\vartheta).$$

Since ε is arbitrary, we have

$$M_\infty \leq c,$$

which is a contradiction.

The proof is complete.

Now we are in a position to prove the following theorem.

Theorem 1.1. *Suppose that (Q_1), (Q_2), and (Q_3) hold. Moreover we assume there exist N functions $u_k \in W^{1,p}(\mathbf{R}^n)$ $(k = 1, \cdots, N)$ with disjoint supports such that*

$$\int q|u_k|^{\sigma+2} > 0, \quad k = 1, \cdots, N,$$

$$\sum_{k=1}^N \frac{\left(\int |\nabla u_k|^p + |u_k|^p\right)^{(\sigma+2)/(\sigma+2-p)}}{\left(\int q|u_k|^{\sigma+2}\right)^{p/(\sigma+2-p)}} < (\frac{1}{p} - \frac{1}{\sigma+2})^{-1} M_\infty.$$

Then (1) has at least N pairs of nontrivial solutions in $W^{1,p}(\mathbf{R}^n)$. Moreover, if

$$q(x) \in C^0(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n),$$

then our weak solution to (1) belongs to $L^\infty(\mathbf{R}^n) \cap C^{1,\mu}(\mathbf{R}^n)$ $(0 < \mu < 1)$, provided $n > p \geq 2$.

Remark. From (Q_2), there exists at least one function u such that $\int q|u|^{\sigma+2} > 0$.

Proof. Let

$$X_N = \{t_1 u_1 + \cdots + t_N u_N \mid (t_1, \cdots, t_N) \in \mathbf{R}^N\}$$

where $u_1, \cdots, u_N$ satisfy the assumptions of the theorem. Then

$$X_N \cap \{u \in W^{1,p}(\mathbf{R}^n) \mid \Phi(u) \geq 0\}$$

is a bounded set. By the Theorem of Rabinowitz in the appendix, it is sufficient to prove $M_\infty > 0$ and

$$\sup_{X_N} \Phi(u) < M_\infty.$$

First we show $M_\infty > 0$. By (4) with $s = \sigma + 2$, we have

$$\int |u|^{\sigma+2} \leq S^{-p^*(\sigma+2-p)/(p^*-p)} \Big(\int |\nabla u|^p + |u|^p \Big)^{(\sigma+2)/p}.$$

Hence we obtain for $u \in V_\infty$

$$S^{p^*(\sigma+2-p)/(p^*-p)} q_\infty^{-1} \leq \Big(\int |\nabla u|^p + |u|^p \Big)^{(\sigma+2-p)/p}.$$

Then we have

$$\Phi_\infty(u) \geq \Big(\frac{1}{p} - \frac{1}{\sigma+2}\Big) S^{p^* p/(p^*-p)} q_\infty^{-p/(\sigma+2-p)}, \tag{14}$$

hence $M_\infty > 0$.

Next we show $\sup_{X_N} \Phi(u) < M_\infty$.

Since $\mathrm{supp} u_i \cap \mathrm{supp} u_j = \emptyset$, we have for $u = t_1 u_1 + \cdots + t_N u_N$,

$$\int (|\nabla u|^p + |u|^p) = \sum_{j=1}^{N} |t_j|^p \int (|\nabla u_j|^p + |u_j|^p),$$

$$\int q|u|^{\sigma+2} = \sum_{j=1}^{N} |t_j|^{\sigma+2} \int q|u_j|^{\sigma+2}.$$

Now Let $A_j = \int (|\nabla u_j|^p + |u_j|^p)$ and $B_j = \int q|u_j|^{\sigma+2}$. Then, by the Hölder inequality and the assumption on $\{u_j\}$, we have

$$\begin{aligned}
\sum_{j=1}^{N} |t_j|^p A_j &= \sum_{j=1}^{N} |t_j|^p A_j B_j^{p/(\sigma+2)} B_j^{-p/(\sigma+2)} \\
&\leq \Big\{ \sum_{j=1}^{N} (A_j B_j^{-p/(\sigma+2)})^{(\sigma+2)/(\sigma+2-p)} \Big\}^{(\sigma+2-p)/(\sigma+2)} \Big\{ \sum_{j=1}^{N} |t_j|^{\sigma+2} B_j \Big\}^{p/(\sigma+2)} \\
&< \Big(\frac{1}{p} - \frac{1}{\sigma+2}\Big)^{-(\sigma+2-p)/(\sigma+2)} \bar{M}_\infty^{(\sigma+2-p)/(\sigma+2)} \Big\{ \sum_{j=1}^{N} |t_j|^{\sigma+2} B_j \Big\}^{p/(\sigma+2)}
\end{aligned}$$

Hence we get

$$\Big(\int (|\nabla u|^p + |u|^p) \Big) \Big(\int q|u|^{\sigma+2} \Big)^{-p/(\sigma+2)} < \Big(\frac{1}{p} - \frac{1}{\sigma+2}\Big)^{-(\sigma+2-p)/(\sigma+2)} \bar{M}_\infty^{(\sigma+2-p)/(\sigma+2)},$$

where $\bar{M}_\infty (< M_\infty)$ is a constant for which the second inequality in the assumption holds. Since

$$\max_{s\geq 0} \Phi(su) = (\frac{1}{p} - \frac{1}{\sigma+2})\Big\{\frac{\int(|\nabla u|^p + |u|^p)}{(\int q|u|^{\sigma+2})^{p/(\sigma+2)}}\Big\}^{(\sigma+2)/(\sigma+2-p)} < \bar{M}_\infty,$$

we find that

$$\sup_{u\in X_N} \Phi(u) < M_\infty.$$

Thus we have checked all the hypotheses of the Rabinowitz Theorem and the conlusion of our theorem comes immediately from it. The regularity of solutions to our problem comes from the regularity theorem due to G.-B. Li [6].

The proof is complete.

COLLORARY 2.7. *In addition to the assumptions of Theorem 1.1, if (Q_4) holds and*

$$\limsup_{|x|\to\infty} q(x) \leq 0,$$

then there exist infinitely many solutions of (1).

Proof. We can take $q_\infty > 0$ so small that M_∞ is large enough. In fact, this is possible by the estimate (14). Then, according to Theorem 1.1, we can take N functions assumed in it with their supports in the open set $\{x \in \mathbf{R}^n | q(x) > 0\}$ assumed in (Q_4). Since q_∞ can be taken arbitrarily small, N may be arbitrarily large.

The proof is complete.

THEOREM 2.8 **(Exponential decay).** *Under the assumptions of Theorem 1.1, if*

$$q(x) \in C^0(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n),$$

then for given η $(0 < \eta < 1)$, there exists an $R(\eta) > 0$ such that our solution u to (1) is estimated as

$$|u(x)| \leq C(\eta) \exp\big\{ -(\frac{\eta}{p-1})^{1/p}|x|\big\}$$

for $|x| \geq R(\eta)$, where $C(\eta)$ is a positive constant.

Proof. We may assume that the solution u to (1) is nonnegative, continuous and satisfies $\lim_{|x|\to\infty} u(x) = 0$ (due to G.-B. Li [6]). Then for given $\eta > 0$, there exists an $R(\eta) > 0$ such that

$$|u|^{p-2}u - q(x)|u|^\sigma u \geq (1-\eta)|u|^{p-2}u \tag{15}$$

holds for $|x| \geq R(\eta)$.

Now let

$$V(x) = C(\eta)\exp\{-(\frac{1-\eta}{p-1})^{1/p}|x|\}$$

with $C(\eta) = u(R(\eta))\exp\{(\frac{1-\eta}{p-1})^{1/p}R(\eta)\}$, where $u(R(\eta)) = \max_{|x|=R(\eta)} u(x)$. Then putting $r = |x|$, we have

$$\begin{aligned}-\mathrm{div}(|\nabla V|^{p-2}\nabla V) &= -r^{1-n}\frac{d}{dr}(r^{n-1}|\frac{dV}{dr}|^{p-2}\frac{dV}{dr}) \\ &= (\frac{1-\eta}{p-1})^{(p-1)/p} r^{1-n}\frac{d}{dr}(r^{n-1}|V|^{p-2}V) \\ &= \Big\{\frac{(n-1)}{r}(\frac{1-\eta}{p-1})^{(p-1)/p} - (1-\eta)\Big\}|V|^{p-2}V.\end{aligned}$$

Hence we get

$$-\mathrm{div}(|\nabla V|^{p-2}\nabla V) + (1-\eta)|V|^{p-2}V = \frac{n-1}{r}\Big(\frac{1-\eta}{p-1}\Big)^{(p-1)/p}|V|^{p-2}V \geq 0. \tag{16}$$

On the other hand, by (15), we have for solution u to (1)

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + (1-\eta)|u|^{p-2}u \leq 0. \tag{17}$$

Now let $\phi = (u-V)^+$. Then by G.-B. Li's regularity theorem, $\phi \in C^0(\mathrm{R}^n)\cap L^\infty(\mathrm{R}^n)$. In addition, integrating (16) and (17) over $\{|x| \geq R(\eta)\}$ with a test function ϕ ($\phi(R(\eta)) = 0$), we get

$$\int_{|x|\geq R(\eta)} \big(|\nabla u|^{p-2}\nabla u\nabla\phi + (1-\eta)|u|^{p-2}u\phi\big) \leq 0,$$

and

$$\int_{|x|\geq R(\eta)} \big(|\nabla V|^{p-2}\nabla V\nabla\phi + (1-\eta)|V|^{p-2}V\phi\big) \geq 0.$$

Substracting the above two inequalities, we obtain

$$(1-\eta)\int_{\{|x|\geq R(\eta)\}\cap\{u\geq V\}} (|u|^{p-2}u - |V|^{p-2}V)\phi \leq 0.$$

Here we have used for $\xi, \zeta \in \mathrm{R}^n$ with $\xi \neq \zeta$

$$\begin{aligned}\sum_{i=1}^{n}(|\xi|^{p-2}\xi_i - |\zeta|^{p-2}\zeta_i)(\xi_i - \zeta_i) &\geq |\xi|^p + |\zeta|^p - (|\xi|^{p-2} + |\zeta|^{p-2})|\xi||\zeta| \\ &= (|\xi|^{p-1} - |\zeta|^{p-1})(|\xi| - |\zeta|) > 0.\end{aligned}$$

Since u and V are continuous, we have

$$\{|x| \geq R(\eta)\} \cap \{x \in \mathrm{R}^n \; |u > V\} = \emptyset.$$

This implies $u \le V$ for $|x| \le R(\eta)$.

The proof is complete.

§3-3. The general case.

In this section we consider more general equations of the form

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = f(x,u) \quad \text{on } \mathbf{R}^n. \tag{2}$$

We assume that $f(x,u)$ satisfies the following assumptions:

(F_1) $\quad f(x,u)$ is continuous in u for almost all $x \in \mathbf{R}^n$.

(F_2) $\quad f(x,u)$ is odd in $u \in \mathbf{R}$.

(F_3) $$|f(x,u)| \le b_1(x)|u|^{r+1} + b_2(x)|u|^{\sigma+1}$$

where $p-2 < r < \sigma < p^*-2$, $b_1 \in L^{\varsigma}(\mathbf{R}^n) \cap L^\infty_{loc}(\mathbf{R}^n)$ $(\varsigma \ge p^*/(p^*-(r+2)))$, and $b_2 \in L^s(\mathbf{R}^n) \cap L^\infty_{loc}(\mathbf{R}^n)$ $(s \ge p^*/(p^*-(\sigma+2)))$, but $L^\infty_{loc}(\mathbf{R}^n)$ is needed only for the cases when the equality on ς or s holds

(F_4) $$|f(x,u_1) - f(x,u_2)| \le b_1'(x)\big||u_1|^r u_1 - |u_2|^r u_2\big| + b_2'(x)\big||u_1|^\sigma u_1 - |u_2|^\sigma u_2\big|$$

for all $u_1, u_2 \in \mathbf{R}$, where $b_1' \in L^\varsigma \cap L^\infty_{loc}(\mathbf{R}^n)$, and $b_2' \in L^s \cap L^\infty_{loc}(\mathbf{R}^n)$ with the same ς and s in (F_3), but $L^\infty_{loc}(\mathbf{R}^n)$ is again needed only for the cases when the equality on ς or s hold.

(F_5) $$F(x,u) \le \frac{1}{\tilde p} f(x,u)u \quad \text{for } (x,u) \in \mathbf{R}^n \times \mathbf{R} \text{ and some } \tilde p > p$$

where $F(x,u) = \int_0^u f(x,s)ds$.

(F_6) $$\frac{f(x,u)}{|u|^{p-2}u} \text{ is locally bounded in } \mathbf{R}^n \times \mathbf{R}, \text{ and nondecreasing in } u \text{ if } u > 0.$$

There exists a mapping $g : \mathbf{R}^n \times \mathbf{R} \mapsto \mathbf{R}$ satisfying the following properties:

$$(F_7)\quad \begin{cases} f(x,u)u \le g(x,u)u, \quad \text{for all } |x| > R_1,\ u \in \mathbf{R}, \\ g \in C(\mathbf{R}^n \times \mathbf{R}, \mathbf{R}),\ u \mapsto g(\cdot,u)u \text{ is a continuous map } W^{1,p}(\mathbf{R}^n) \to L^1(\mathbf{R}^n), \\ \tau^{p+\iota} g(x,u)u \le g(x,\tau u)\tau u \text{ for } \tau \ge 1,\ x \in \mathbf{R}^n,\ u \in \mathbf{R}\ (\iota > 0), \\ \limsup\limits_{\tau \to +0} \dfrac{1}{\tau^p} \displaystyle\int g(x,\tau w)\tau w = 0 \quad \text{for all } w \in W^{1,p}(\mathbf{R}^n) \\ \text{with supp } w \subset \mathbf{R}^n \backslash B_{R_1} \text{ for some } R_1 > 0. \end{cases}$$

Now let

$$G(x,u) = \int_0^u g(x,s)ds, \quad (x,u) \in \mathbf{R}^n \times \mathbf{R},$$
$$J(u) = \frac{1}{p}\int (|\nabla u|^p + |u|^p) - \int F(x,u),$$
$$K(u) = \frac{1}{p}\int (|\nabla u|^p + |u|^p) - \int G(x,u),$$
$$V_g = \{u \in W^{1,p}(\mathbf{R}^n)\backslash\{0\} \mid \langle K'(u), u\rangle = 0\},$$
$$M_g = \inf_{u \in V_g} K(u).$$

Moreover we assume for G and g of (F_7),

$$(G_1) \qquad G(x,u) < \frac{1}{q} g(x,u)u$$

for some $q(> p)$ and

$$(G_2) \qquad \int |g(x,u)u| \le \sum_{i=1}^{2} C_i\Big(\int (|\nabla u|^p + |u|^p)\Big)^{\alpha_i}$$

for some $C_i \ge 0$ $(C_1^2 + C_2^2 \ne 0), \alpha_i > 1$ $(i = 1,2)$.

EXAMPLE. Let

$$f(x,u) = q_1(x)|u|^{\sigma_1}u + q_2(x)|u|^{\sigma_2}u$$

with $\limsup_{|x|\to\infty} q_i(x) < \hat{q}$ $(i = 1,2)$ and $f(x,u)u > 0$ for some $(x_0,u_0) \in \mathbf{R}^n \times \mathbf{R}$. Here we assume $q_i(\ge 0) \in C^0(\mathbf{R}^n) \cap L^{t_i}(\mathbf{R}^n)$ with $t_i > p^*/(p^* - (\sigma_i + 2))$ $(i = 1,2)$, $p < \sigma_1 + 2 < \sigma_2 + 2 < p^*$ and $\hat{q} > 0$. Then $f(x,u)$ satisfies $(F_1) - (F_7)$ with $g(x,u) = \hat{q}(|u|^{\sigma_1}u + |u|^{\sigma_2}u)$. Moreover $g(x,u)$ satisfies (G_1) and (G_2).

REMARK. Obviously b_1, b_2, b_1', and b_2' are nonnegative. From (F_2), $f(x,u)/(|u|^{p-2}u)$ is even in u. Then (F_6) implies that $f(x,u)/(|u|^{p-2}u)$ is non-increasing in u if $u < 0$. Note that for $w \in W^{1,p}(\mathbf{R}^n)$ with supp $w \subset \mathbf{R}^n\backslash B_{R_1}$, we have by (F_7)

$$(18) \qquad \int F(x,w) \le \int G(x,w).$$

Then, as in Section 3-2, we have a similar proposition to Proposition 2.5.

PROPOSITION 3.1. *J satisfies $(PS)_c$ for $c \in (-\infty, M_g)$.*

Proof. Let $\{u_j\} \subset W^{1,p}(\mathbf{R}^n)$ be a sequence for $c \in (-\infty, M_g)$ satisfying (5), that is,

$$(19) \qquad \begin{cases} \dfrac{1}{p}\displaystyle\int (|\nabla u_j|^p + |u_j|^p) - \int F(x,u) \to c, \\ |\langle J'(u_j), v\rangle| \le o(1)\|v\| \quad \text{for any } v \in W^{1,p}(\mathbf{R}^n). \end{cases}$$

As in the proof of Proposition 2.5, letting $v = u_j$ in (19) and using (F_5), we have

$$\begin{aligned}-o(1)\|u_j\| &\le \int (|\nabla u_j|^p + |u_j|^p) - \int f(x,u_j)u_j \\ &\le \int (|\nabla u_j|^p + |u_j|^p) - \tilde{p}\int F(x,u_j) \\ &= (1-\frac{\tilde{p}}{p})\int (|\nabla u_j|^p + |u_j|^p) + \tilde{p}c + o(1).\end{aligned}$$

Hence we get

$$\frac{\tilde{p}-p}{p}\|u_j\|^p + \tilde{p}c + o(1) \le o(1)\|u_j\|.$$

This implies the boundedness of $\{u_j\}$ in $W^{1,p}(\mathbf{R}^n)$ by $\tilde{p} > p$. Hence we may suppose that a subsequence of $\{u_j\}$ (still denoted by $\{u_j\}$) satisfies

$$\begin{cases} u_j \rightharpoonup u_\infty & \text{weakly in } W^{1,p}(\mathbf{R}^n), \\ u_j \to u_\infty & \text{locally strongly in } L^\theta(\mathbf{R}^n), \quad 1 \le \theta < p^*, \\ u_j \to u_\infty & \text{a.e. in } \mathbf{R}^n. \end{cases}$$

Similarly, to prove Proposition 3.1, we need the next lemma.

LEMMA 3.2. *$\langle J'(u_\infty), u_\infty\rangle = 0$ holds.*

Proof. As in the proof of Lemma 2.6, we have (choosing a subsequence if necessary)

$$\int (|\nabla u_j|^{p-2}\nabla u_j \nabla u_\infty + |u_j|^{p-2}u_j u_\infty) - \int (|\nabla u_\infty|^p + |u_\infty|^p) = o(1).$$

Next we prove

$$\int f(x,u_j)u_\infty \to \int f(x,u_\infty)u_\infty \quad \text{as } j \to \infty.$$

For any $\varepsilon > 0$, there exists an $R > 0$ such that

$$\int_{|x|\ge R} b_1^\varsigma < \varepsilon, \int_{|x|\ge R} b_2^s < \varepsilon.$$

Now we estimate

$$\left|\int_{|x|\ge R} f(x,u_j)u_\infty\right|, \left|\int_{|x|\ge R} f(x,u_\infty)u_\infty\right|, \text{ and } \left|\int_{B_R}\{f(x,u_j)u_\infty - f(x,u_\infty)u_\infty\}\right|.$$

For the first term, we have, in a similar manner to the proof of Lemma 2.6, using (F_3),

$$\begin{aligned}\left|\int_{|x|\ge R} f(x,u_j)u_\infty\right| \le& \int_{|x|\ge R} |f(x,u_j)u_\infty| \\ \le& \int_{|x|\ge R} b_1|u_j|^{r+1}|u_\infty| + \int_{|x|\ge R} b_2|u_j|^{\sigma+1}|u_\infty| \\ \le& \Big(\int_{|x|\ge R} b_1^\varsigma\Big)^{1/\varsigma}\Big(\int_{|x|\ge R}|u_j|^{(r+1)\nu}\Big)^{1/\nu}\Big(\int_{|x|\ge R}|u_\infty|^\varpi\Big)^{1/\varpi} \\ &+ \Big(\int_{|x|\ge R} b_2^s\Big)^{1/s}\Big(\int_{|x|\ge R}|u_j|^{(\sigma+1)\nu'}\Big)^{1/\nu'}\Big(\int_{|x|\ge R}|u_\infty|^{\varpi'}\Big)^{1/\varpi'} \\ \le& C\varepsilon,\end{aligned}$$

where $1/\varsigma + 1/\nu + 1/\varpi = 1$, and $1/s + 1/\nu' + 1/\varpi' = 1$, and they are determined as follows: if $p^*/(p^* - (r+2)) \le \varsigma \le p^*(r+2)/(p^* - (r+2))$,

$$\nu = \frac{p^*\varsigma}{(p^*-1)\varsigma - p^*}, \quad \varpi = p^*. \tag{20}$$

if $\varsigma \ge p^*(r+2)/(p^* - (r+2))$,

$$\nu = \frac{r+2}{r+1}, \quad \varpi = \frac{(r+2)\varsigma}{\varsigma - (r+2)}. \tag{21}$$

ν', ϖ' are determined as (8) and (9). Also for the second term, we have

$$\int_{|x|\ge R} |f(x,u_\infty)u_\infty| \le C\varepsilon.$$

For the last term, we have by (F_4)

$$\begin{aligned}&\Big|\int_{B_R} \{f(x,u_j)u_\infty - f(x,u_\infty)u_\infty\}\Big| \\ &\le \int_{B_R} \big(b_1'\big||u_j|^r u_j - |u_\infty|^r u_\infty\big| + b_2'\big||u_j|^\sigma u_j - |u_\infty|^\sigma u_\infty\big|\big)|u_\infty|.\end{aligned}$$

By Assumptions (F_3) and (F_4), it suffices to calculate only the terms which have r as the exponent of $|u|$. If $\varsigma > p^*/(p^* - (r+2))$ and $s > p^*/(p^* - (\sigma+2))$ then we have, using Lemma 2.2 with exponents (20) and (21), respectively,

$$\begin{aligned}&\int_{B_R} b_1'\big||u_j|^r u_j - |u_\infty|^r u_\infty\big||u_\infty| \\ &\quad\le \Big(\int_{B_R} b_1'^{\varsigma}\Big)^{1/\varsigma}\Big(\int_{B_R}\big||u_j|^r u_j - |u_\infty|^r u_\infty\big|^\nu\Big)^{1/\nu}\Big(\int_{B_R}|u_\infty|^\varpi\Big)^{1/\varpi} = o(1),\end{aligned}$$

and

$$\int_{B_R} b_2'\big||u_j|^\sigma u_j - |u_\infty|^\sigma u_\infty\big||u_\infty| = o(1).$$

If $\varsigma = p^*/(p^* - (r+2))$ or $s = p^*/(p^* - (\sigma+2))$ then we get

$$\begin{aligned}&\int_{B_R} \big(b_1'\big||u_j|^r u_j - |u_\infty|^r u_\infty\big|\big)|u_\infty| \\ &\quad\le \sup_{B_R} b_1' \Big(\int_{B_R}\big||u_j|^r u_j - |u_\infty|^r u_\infty\big|^{(r+2)/(r+1)}\Big)^{(r+1)/(r+2)}\Big(\int_{B_R}|u_\infty|^{r+2}\Big)^{1/(r+2)} = o(1)\end{aligned}$$

and

$$\begin{aligned}&\int_{B_R} \big(b_2'\big||u_j|^\sigma u_j - |u_\infty|^\sigma u_\infty\big|\big)|u_\infty| \\ &\quad\le \sup_{B_R} b_2'\Big(\int_{B_R}\big||u_j|^\sigma u_j - |u_\infty|^\sigma u_\infty\big|^{(\sigma+2)/(\sigma+1)}\Big)^{(\sigma+1)/(\sigma+2)}\Big(\int_{B_R}|u_\infty|^{\sigma+2}\Big)^{1/(\sigma+2)} \\ &\quad= o(1).\end{aligned}$$

Finally, we have

$$\left|\int f(x,u_j)u_\infty - \int f(x,u_\infty)u_\infty\right| \le C\varepsilon + o(1).$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\langle J'(u_\infty), u_\infty\rangle = 0$.

The proof is complete.

Now we get back to the proof of Proposition 3.1.

As in the proof of Lemma 2.6, let us consider two cases:

(C) For all $\delta > 0$, there exists an $R > 0$ such that

$$\int_{|x|\ge R} (|\nabla u_j|^p + |u_j|^p) < \delta$$

holds for each integer $j(\ge n_0(R))$.

(D) The negative proposition of (C): There exists a $\delta_0 > 0$ such that

$$\int_{|x|\ge R'} (|\nabla u_j|^p + |u_j|^p) \ge \delta_0$$

holds for each $R' > 0$ and for a suitable integer $j(\ge R')$.

According to the proof of Proposition 2.5, we first consider the case (C). To show the strong convergence of $\{u_j\}$, it remains to estimate

$$\left|\int_{|x|\ge R} f(x,u_j)v\right|,\ \left|\int_{|x|\ge R} f(x,u_\infty)v\right|,\ \text{and}\ \left|\int_{B_R} (f(x,u_j)-f(x,u_\infty))v\right|.$$

The first term yields

$$\begin{aligned}\left|\int_{|x|\ge R} f(x,u_j)v\right| &\le \int_{|x|\ge R} |f(x,u_j)v| \\ &\le \int_{|x|\ge R} b_1|u_j|^{r+1}|v| + \int_{|x|\ge R} b_2|u_j|^{\sigma+1}|v|.\end{aligned}$$

On the other hand we have

$$\begin{aligned}&\int_{|x|\ge R} b_1|u_j|^{r+1}|v| \\ &\le \Big(\int_{|x|\ge R} b_1^{\varsigma}\Big)^{1/\varsigma}\Big(\int_{|x|\ge R} |u_j|^{(r+1)\nu}\Big)^{1/\nu}\Big(\int_{|x|\ge R} |v|^{\varpi}\Big)^{1/\varpi} \\ &\le C\left\{\Big(\int_{|x|\ge R} |u_j|^p\Big)^{\zeta/p}\Big(\int_{|x|\ge R} |u_j|^{p^*}\Big)^{\xi/p^*}\right\}^{1/\nu} \|v\| \le C\delta^{(r+1)/p}\|v\|\end{aligned}$$

and

$$\int_{|x|\geq R} b_2|u_j|^{\sigma+1}|v|$$
$$\leq \Big(\int_{|x|\geq R} b_2^s\Big)^{1/s}\Big(\int_{|x|\geq R}|u_j|^{(\sigma+1)\nu'}\Big)^{1/\nu'}\Big(\int_{|x|\geq R}|v|^{\varpi'}\Big)^{1/\varpi'} \leq C\delta^{(\sigma+1)/p}\|v\|$$

where ν, ϖ, ν', ϖ' are defined by in (20) and (21), and $\zeta/p+\xi/p^* = 1$, $\zeta+\xi = (r+1)\nu$. The last inequality comes from the same reason in Section 3-2.

Hence we get

$$\Big|\int_{|x|\geq R} f(x,u_j)v\Big| \leq C(\delta^{(r+1)/p}+\delta^{(\sigma+1)/p})\|v\|.$$

Similarly, we have

$$\Big|\int_{|x|\geq R} f(x,u_\infty)v\Big| \leq C(\delta^{(r+1)/p}+\delta^{(\sigma+1)/p})\|v\|.$$

For the domain B_R, we have as in the proof of Lemma 3.2,

$$\Big|\int_{B_R}(f(x,u_j)-f(x,u_\infty))v\Big| \leq o(1)\|v\|.$$

Then we get

$$\Big|\int(|\nabla u_j|^{p-2}\nabla u_j\nabla v+|u_j|^{p-2}u_jv-\int f(x,u_\infty)v\Big| \leq \Big\{C(\delta^{(r+1)/p}+\delta^{(\sigma+1)/p})+o(1)\Big\}\|v\|.$$

Especially letting $v = u_j$, we get

$$\Big|\int(|\nabla u_j|^p+|u_j|^p)-\int f(x,u_\infty)u_j\Big| \leq \Big\{C(\delta^{(r+1)/p}+\delta^{(\sigma+1)/p})+o(1)\Big\}\|u_j\|.$$

Now we consider

$$\Big|\int(|\nabla u_j|^p+|u_j|^p)-\int(|\nabla u_\infty|^p+|u_\infty|^p)\Big|.$$

Using

$$\begin{cases} \int(|\nabla u_j|^p+|u_j|^p) = \int f(x,u_j)u_j+o(1), \\ \int(|\nabla u_\infty|^p+|u_\infty|^p) = \int f(x,u_\infty)u_\infty, \end{cases}$$

we have

$$\begin{aligned}
&\Big|\int(|\nabla u_j|^p+|u_j|^p)-\int(|\nabla u_\infty|^p+|u_\infty|^p)\Big| \\
&\leq \Big|\int(f(x,u_j)u_j-f(x,u_\infty)u_\infty)\Big|+o(1) \\
&\leq \int\big|f(x,u_j)u_j-f(x,u_\infty)u_j\big|+\int\big|f(x,u_\infty)u_j-f(x,u_\infty)u_\infty\big|+o(1) \\
&\leq \Big\{C(\delta^{(r+1)/p}+\delta^{(\sigma+1)/p})+o(1)\Big\}+\int\big|f(x,u_\infty)u_j-f(x,u_\infty)u_\infty\big| \\
&\leq \Big\{C(\delta^{(r+1)/p}+\delta^{(\sigma+1)/p})+o(1)\Big\}+\int_{B_R}\big|f(x,u_\infty)u_j-f(x,u_\infty)u_\infty\big|.
\end{aligned}$$

$\int_{|x|\geq R} \left|f(x,u_\infty)u_j - f(x,u_\infty)u_\infty\right|$ has been estimated in the proof of Lemma 3.2. For the term

$$\int_{B_R} \left|f(x,u_\infty)u_j - f(x,u_\infty)u_\infty\right|,$$

we have by using (F_3)

$$\int_{B_R} \left|f(x,u_\infty)u_j - f(x,u_\infty)u_\infty\right| \leq \int_{B_R} (b_1|u_\infty|^{r+1} + b_2|u_\infty|^{\sigma+1})|u_j - u_\infty|.$$

If $\varsigma > p^*/(p^* - (r+2))$, we have

$$\begin{aligned}&\int_{B_R} b_1|u_\infty|^{r+1}|u_j - u_\infty| \\ &\qquad \leq \Big(\int_{B_R} b_1^\varsigma\Big)^{1/\varsigma}\Big(\int_{B_R}|u_\infty|^{p^*}\Big)^{(r+1)/p^*}\Big(\int_{B_R}|u_j - u_\infty|^\rho\Big)^{1/\rho} = o(1),\end{aligned}$$

where $1/\varsigma + (r+1)/p^* + 1/\rho = 1$. Since $\varsigma > p^*/(p^* - (r+2))$, $\rho < p^*$ and the last inequality follows. If $\varsigma = p^*/(p^* - (r+2))$, we have

$$\begin{aligned}&\int_{B_R} b_1|u_\infty|^{r+1}|u_j - u_\infty| \\ &\quad \leq \sup_{B_R} b_1\Big(\int_{B_R}|u_\infty|^{p^*}\Big)^{(r+1)/p^*}\Big(\int_{B_R}|u_j - u_\infty|^{p^*/(p^*-(r+1))}\Big)^{(p^*-(r+1))/p^*} = o(1).\end{aligned}$$

The last inequality of the above follows from the fact that u_j converges strongly to u_∞ in $L^{p^*/(p^*-(r+1))}(B_R)$ (this exponent is subcritical).

Similarly, for the latter term we have

$$\int_{B_R} b_2|u_j|^{\sigma+1}|u_j - u_\infty| = o(1).$$

Hence, finally we have

$$\Big|\int(|\nabla u_j|^p + |u_j|^p) - \int(|\nabla u_\infty|^p + |u_\infty|^p)\Big| \leq \Big\{C(\delta^{\frac{r+1}{p}} + \delta^{\frac{\sigma+1}{p}}) + o(1)\Big\}$$

Since $\delta > 0$ is arbitrary, we have

$$\|u_j\| \to \|u_\infty\|,$$

which implies the strong convergence of $\{u_j\}$.

If (D) occurs, then we may assume that for some $\delta_0 > 0$

$$\int_{|x|\geq j}(|\nabla u_j|^p + |u_j|^p) \geq \delta_0$$

holds for all $j \in \mathbf{N}$. In addition, there exists an $R_2 > 0$ such that for any $\varepsilon > 0$,

$$\int_{|x|\geq R_2} (|\nabla u_\infty|^p + |u_\infty|^p + |f(x, u_\infty)u_\infty|) < \varepsilon$$

holds. As in the proof of Proposition 2.5, for a subsequence of $\{u_j\}$ (denoted by $\{u_{j_l}\}$), there exists an annulus $I = \{x \in \mathbf{R}^n \mid \check{R} \leq |x| \leq \check{R}+1\}$ ($\check{R} \geq R_2$) such that

$$\int_I (|\nabla u_{j_l}|^p + |u_{j_l}|^p) < \varepsilon.$$

Now let $v_l = \rho_1(x)u_{j_l}$, and $w_l = \rho_2(x)u_{j_l}$ where ρ_1 and ρ_2 are stated in Section 3-2. As in Section 3-2, we estimate

$$\Big|\langle J'(u_{j_l}), v_l\rangle - \langle J'(v_l), v_l\rangle\Big|.$$

The estimate is as follows:

$$\begin{aligned}
&\Big|\langle J'(u_{j_l}), v_l\rangle - \langle J'(v_l), v_l\rangle\Big| \\
\leq &\int_I \Big||\nabla u_{j_l}|^{p-2}\nabla u_{j_l}\nabla v_l + |u_{j_l}|^{p-2}u_{j_l}v_l\Big| + \int_I (|\nabla v_l|^p + |v_l|^p) \\
&+ \int_I |f(x, u_{j_l}) - f(x, v_l)||v_l| \\
\leq &C\varepsilon + \int_I |f(x, u_{j_l}) - f(x, v_l)||v_l| \\
\leq &C\varepsilon + C\Big\{\Big(\int_I (|\nabla u_{j_l}|^p + |u_{j_l}|^p)\Big)^{(r+2)/p} + \Big(\int_I (|\nabla u_{j_l}|^p + |u_{j_l}|^p)\Big)^{(\sigma+2)/p}\Big\} \\
\leq &C\varepsilon + o(\varepsilon) = O(\varepsilon) \text{ (by } p-2 < r < \sigma).
\end{aligned}$$

Here we have estimated $\int_I |f(x, u_{j_l}) - f(x, v_l)||v_l|$ by using (F_4) and (4) in the same spirit of the proof of Proposition 2.5. Hence we have

$$\langle J'(v_l), v_l\rangle = O(\varepsilon) + o(1).$$

Similarly, we have

$$\langle J'(w_l), w_l\rangle = O(\varepsilon) + o(1)$$

and

$$J(v_l) + J(w_l) = c + O(\varepsilon) + o(1).$$

Using

$$J(v_l) = \frac{1}{p}\int_{B_{\check{R}+1}} (|\nabla v_l|^p + |v_l|^p) - \int_{B_{\check{R}+1}} F(x, v_l)$$

and

$$\langle J'(v_l), v_l\rangle = \int_{B_{\hat{R}+1}} (|\nabla v_l|^p + |v_l|^p) - \int_{B_{\hat{R}+1}} f(x, v_l)v_l = O(\varepsilon) + o(1),$$

we have

$$J(v_l) = \int_{B_{\hat{R}+1}} \{\frac{1}{p} f(x, v_l)v_l - F(x, v_l)\} + O(\varepsilon) + o(1).$$

From (F_5), we get

$$J(v_l) \geq O(\varepsilon) + o(1).$$

Thus we obtain

$$J(w_l) \leq c + O(\varepsilon) + o(1),$$

that is,

$$\frac{1}{p}\int (|\nabla w_l|^p + |w_l|^p) - \int F(x, w_l) \leq c + O(\varepsilon) + o(1).$$

In addition, by

$$\int (|\nabla w_l|^p + |w_l|^p) = \int f(x, w_l)w_l + O(\varepsilon) + o(1),$$

we have

$$J(w_l) = \frac{1}{p}\int f(x, w_l)w_l - \int F(x, w_l) + O(\varepsilon) + o(1). \tag{22}$$

Now let

$$\Xi(t) = \frac{1}{t^p}\int g(x, tw_l)tw_l.$$

Then, for $t = 1 + C\varepsilon$, we have by (F_7) and the assumption $\int(|\nabla w_l|^p + |w_l|^p) \geq \delta_0$,

$$\begin{aligned}\Xi(1 + C\varepsilon) &\geq (1 + C\varepsilon)^{\iota} \int f(x, w_l)w_l \geq \int f(x, w_l)w_l \\ &= \int (|\nabla w_l|^p + |w_l|^p) + O(\varepsilon) + o(1) \geq \delta_0 + O(\varepsilon) + o(1).\end{aligned}$$

Since $\lim_{t\to+0} \Xi(t) = 0$ holds by (F_7), there exists a $t_0(\leq 1 + C\varepsilon)$, such that

$$\Xi(t_0) = \int (|\nabla w_l|^p + |w_l|^p).$$

Let $\hat{w}_l = t_0 w_l$. Then $\hat{w}_l \in V_g$ and by (18) we have

$$\begin{aligned}M_g \leq K(\hat{w}_l) &= \frac{t_0^p}{p}\int (|\nabla w_l|^p + |w_l|^p) - \int G(x, t_0 w_l) \\ &\leq \frac{t_0^p}{p}\int f(x, w_l)w_l - \int F(x, t_0 w_l) + O(\varepsilon).\end{aligned}$$

Now set

$$\Lambda(t) = \frac{t^p}{p}\int f(x, w_l)w_l - \int F(x, tw_l).$$

Then $\Lambda(t)$ is non-decreasing in $t \in [0, 1]$. This is because

$$\begin{aligned}\Lambda'(t) &= t^{p-1}\int f(x, w_l)w_l - \int f(x, tw_l)w_l \\ &= t^{p-1}\int \Big(\frac{f(x, w_l)}{|w_l|^{p-2}w_l} - \frac{f(x, tw_l)}{t^{p-1}|w_l|^{p-2}w_l}\Big)|w_l|^p \geq 0\end{aligned}$$

by (F_6). Hence if $t_0 \in [0, 1]$ we have

$$M_g \leq J(w_l) + O(\varepsilon) \leq c + O(\varepsilon) + o(1).$$

If $1 \leq t_0 \leq 1 + C\varepsilon$, we have from the mean value theorem, for some $t_1 \in [1, t_0]$,

$$|\Lambda(t_0) - \Lambda(1)| \leq C\varepsilon\Big\{t_1^{p-1}\int |f(x, w_l)w_l| + \int |f(x, t_1w_l)w_l|\Big\}. \tag{23}$$

By (22)

$$\Lambda(1) = J(w_l) + O(\varepsilon) + o(1)$$

holds and the integrals on the right hand side of (23) are bounded. Hence we have

$$M_g \leq J(w_l) + O(\varepsilon) \leq c + O(\varepsilon) + o(1).$$

Since $\varepsilon > 0$ is arbitrary, $M_g \leq c$, but this is a contradiction.

The proof is complete.

Now we prove Theorem 1.2.

Theorem 1.2. *Suppose that conditions $(F_1) - (F_7)$ and (G_1), (G_2) hold. Moreover we assume there exist N functions $u_k \in W^{1,p}(\mathbf{R}^n)$ $(k = 1, \cdots, N)$ with disjoint supports such that*

$$\int f(x, u_k)u_k > 0 \quad k = 1, \cdots, N,$$

and let

$$X_N = \{t_1u_1 + \cdots + t_Nu_N \mid (t_1, \cdots, t_N) \in \mathbf{R}^N\}$$

If

$$\sup_{X_N} J(u) < M_g$$

holds, then (2) has at least N pairs of nontrivial solutions in $W^{1,p}(\mathbf{R}^n)$.

Remark. From (F_2), $u \equiv 0$ is also a solution of (2). But we are concerned with nontrivial solutions. Also in this case, all our solutions decay exponentially at infinity under similar assumptions stated in Theorem 2.8. But we do not prove it.

Proof. First we note

$$X_N \cap \{u \in W^{1,p}(\mathbf{R}^n) \mid J(u) \geq 0\}$$

is a bounded set. By the Theorem of Rabinowitz in the appendix, it is sufficient to prove $M_g > 0$. For $u \in V_g$, we have by (G_1),

$$\begin{aligned} K(u) = \frac{1}{p}\int g(x,u)u - \int G(x,u)u &> (\frac{1}{p} - \frac{1}{q})\int g(x,u)u \\ &= (\frac{1}{p} - \frac{1}{q})\int (|\nabla u|^p + |u|^p). \end{aligned}$$

In addition, by (G_2) we get

$$\int (|\nabla u|^p + |u|^p) \leq \int |g(x,u)u| \leq \sum_{i=1}^{2} C_i \Big(\int (|\nabla u|^p + |u|^p)\Big)^{\alpha_i}.$$

This means

$$\int (|\nabla u|^p + |u|^p) > C \tag{24}$$

for some $C > 0$ and each $u \in V_g$. Hence under the assumption on α_i, we obtain

$$K(u) > (\frac{1}{p} - \frac{1}{q})C$$

for all $u \in V_g$. This implies $M_g > 0$.

Thus we have checked all the hypotheses of the Rabinowitz Theorem and the conlusion of this theorem comes immediately from it.

The proof is complete.

COLLORARY 3.3. *Let*

$$f(x,u) = q_1(x)|u|^{\sigma_1}u + q_2(x)|u|^{\sigma_2}u$$

satisfies $(F_1) - (F_7)$ *with* $q_1, q_2 \in C^0(\mathbf{R}^n)$. *If*

$$\limsup_{|x| \to \infty} q_i(x) \leq 0 \quad (i = 1, 2)$$

and $f(x,u)u > 0$ *for some* $(x_0, u_0) \in \mathbf{R}^n \times \mathbf{R}$ *then (2) has infinitely many solutions.*

REMARK. As stated in Example, if $q_i(\geq 0) \in C^0(\mathbf{R}^n) \cap L^{t_i}(\mathbf{R}^n)$ with $t_i > p^*/(p^* - (\sigma_i + 2))$ $(i = 1, 2)$ and $p < \sigma_1 + 2 < \sigma_2 + 2 < p^*$, then Assumptions $(F_1) - (F_7)$ are satisfied.

Proof. For every $\varepsilon > 0$, we can take g in (F_7) as

$$g(x,u) = \varepsilon(|u|^{\sigma_1+1}u + |u|^{\sigma_2+1}u).$$

Then in (24) the constant C tends to infinity as $\varepsilon \to 0$ (a similar estimate to (14) holds), and M_g can be arbitrarily large. Hence by (F_1) there exist N functions with disjoint supports (in a neighborhood of (x_0, u_0)) which satisfy the assumptions of Theorem 1.2. Moreover we can take N arbitrarily large.

The proof is complete.

§3-4. **Appendix.**

In this section, we state a minimax lemma due to Rabinowitz [9] which ensures the multiplicity of critical points. Suppose that E is a real Banach space, and let $J \in C^1(E, \mathbf{R})$ be an even functional satisfying the following conditions:

For some $C > 0$,

(J_1) $$J \text{ satisfies } (PS)_c \text{ for all } c \in (0, C).$$

There exist $\rho,\ \alpha > 0$ such that

(J_2) $$\begin{cases} J > 0 \text{ in} B_\rho \backslash \{0\} \\ J \geq \alpha \text{ on } \partial B_\rho. \end{cases}$$

There exists an n-dimensional subspace X_n of E such that

(J_3) $$\begin{cases} X_n \cap A_0 \text{ is bounded in } E, \\ \sup\limits_{X_n} J < C, \end{cases}$$

where $A_0 = \{u \in E \mid J(u) \geq 0\}$. Let Γ^* be a set of odd homeomorphisms from E onto E with the following property, that is,

$$\Gamma^* = \{h \in Homeo(E, E) \mid h(B_1) \subset A_0,\ h \text{ is odd}\}$$

and let

$$\hat{\Gamma} = \{K \subset E \mid K \text{ is compact, symmetric with respect to } 0\},$$

and

$$\Gamma_m = \{K \subset \hat{\Gamma} | \gamma(K \cap h(S)) \geq m \text{ for any } h \in \Gamma^*\},$$

where $\gamma(\Sigma)$ indicates the Krasnoselskii genus of a symmetric set Σ and $S = \partial B_1$. For the Krasnoselskii genus, refer to Deimling [1] or Krasnoselskii and Zabreiko [5].

Now we are in a position to state the theorem due to Rabinowitz.

THEOREM 4.1. *(Rabinowitz). Let E be a real Banach space, and let $J \in C^1(E, \mathbf{R})$ satisfy (J_1), (J_2), (J_3). Let*

$$b_m = \inf_{K \in \Gamma_m} \max_{u \in K} J(u) \quad m = 1, \ldots, n.$$

Then we have

$$0 < \alpha \leq b_1 \leq \ldots \leq b_n \leq C$$

and $\{b_j\}$ are critical values of J. Moreover if $b_j = b_{j+1}$ for some $j \in \{1, \ldots, n-1\}$, then J has infinetely many critical points corresponding to b_m.

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