Twist-untwist intertwining currents and supercurrents on asymmetric orbifolds

Imamura, Yasumasa

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博 士 論 文

TWIST-UNTWIST INTERTWINING CURRENTS AND SUPERCURRENTS ON ASYMMETRIC ORBIFOLDS

(非対称オービフォールド上のツイストーアンツイスト
相互入れ換えカレントと超カレント)

Yasumasa IMAMURA

Department of Fundamental Science of Materials
Division of Science of Materials
Graduate School of Science and Technology, Kobe University
Rokko-dai, Nada, Kobe 657, Japan

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Abstract

We study two symmetries of asymmetric orbifold models. One is a symmetry between untwisted and twisted string states and the other is space-time supersymmetry. In both the cases, careful analysis of massless states is required for a special class of asymmetric orbifold models. Twist-untwist intertwining currents which convert untwisted states to twisted ones and vice versa may appear in twisted sectors for the models and "enhance" the symmetries. Gauge symmetries of the models are analyzed and lists for them are given. We also investigate supersymmetry of $E_8 \times E_8$ heterotic string compactified on asymmetric orbifolds. It is pointed out that unexpected supercurrents may emerge in a special type of asymmetric orbifolds and enlarge the space-time supersymmetry. Finally we give an example of N=1 space-time supersymmetric models.

Contents

1 Introduction

String theory was originally introduced to explain the enormous proliferation of hadrons in strong interaction physics. The theory includes some attractive ideas, and duality is the most remarkable one. This requires that the whole sum of scattering amplitudes in the s-channel is given by the same analytic function as that in the t-channel. In 1968, Veneziano[1] found that such an amplitude is written by Beta functions. In those days, this theory did not yet have string pictures and developed as the dual resonance model describing strong interactions. Nambu and Susskind[2][3] noticed that the dual resonance model is equivalent to the string theory which treats one-dimensional objects instead of point particles. A string propagating in space-time sweeps out a two-dimensional surface known as a world sheet, which is a generalization of a world line in the case of the point particle. The world sheet is parametrized by two parameters, τ and σ, and so string coordinates $X^{\mu}(\tau,\sigma)$ give a map from the world sheet to space-time. The action is given by the area of the world sheet with an appropriate coefficient[2][4].

The development of the string theory as a candidate for a unified theory was made after 1972. In this year, Neveu and Scherk [5] discovered that the dual resonance model has spin 1 massless gauge bosons in infinite limit of the string tension. Soon the existence of spin 2 graviton was shown by Yoneya, Scherk and Schwarz[6]. Since then, the string theory has been regarded as a hopeful candidate for a unified theory of all interactions including gravity. Of course, the string theory has some difficulties. Goddard, Goldstone, Rebbi and Thorn[7] pointed out that the critical dimension of the bosonic string theory is 26 in view of quantum anomaly. In order to realize our four-dimensional space-time, the extra 22-dimensional space has to be curled up to such a scale as we can never observe ($\sim 10^{-33}cm$).

In our world, matter is constructed with fermionic particles. We should make the string theory to possess fermionic degrees of freedom. Such a string theory is called a superstring theory. Supersymmetry is a symmetry between fermions and bosons, by which the action is invariant under the exchange of these two types of particles. The supersymmetry has good properties in several respects. It cancels a lot of divergent Feynman graphs. Moreover, it might solve the gauge hierarchy and cosmological constant problems. Historically, fermionic strings were constructed by Ramond [8], and soon the bosonic partners were given by Neveu and Schwarz[9]. However, this theory has no supersymmetry and has an undesirable particle, called tachyon. To make this model consistent, GSO projection[10] had to be introduced.

The discovery of the anomaly cancellation by Green and Schwarz was done in 1984. They showed that the type I superstring theory is anomaly-free and finite when it has $SO(32)$ un compactified internal symmetry. They also pointed out that the same situation will appear when the superstring theory has $E_8 \times E_8$ internal symmetry. Using this fact, Gross, Harvey, Martinec and Rohm proposed a "heterotic string"[11] possessing left-right asymmetric degrees of freedom. Since the critical dimension of superstring is 10, the extra 6-dimensional space has to be compactified. Various methods have been proposed; toroidal compactification, orbifold compactification[12], Calabi-Yau compactification[13], Gepner [14] and Kazama-Suzuki construction[15] and so on. These methods have some advantages and are related with each other. For example, orbifolds are thought to be a singular limit of some Calabi-Yau manifold.

Heterotic string theory may be one of the most promising theories to describe our real world. Various methods have been proposed for the compactification to realize our 4-dimensional space-time. Orbifold compactification is one of them. Toroidal orbifold is a generalization of a torus and is given as a quotient space of the torus with its isometries. A lot of classifications have been proposed by many groups[16] and some "realistic" models which preserve N=1 space-time supersymmetry and possess "standard-like" or "GUTs-like" gauge group and three generations have been obtained. They offered many consistent candidates and information about the structure of classical string vacua. However, we have not yet found a clue to the solution of the problem which the true vacuum is and why the six extra dimensions are compactified. For the present, all we can do is to construct 4d string theories consistently and to analyze the vacuum structures thoroughly.

Asymmetric orbifold models[17] seem to be an intriguing theory. In fact, we do not see the inevitability for treating the left- and right-moving string coordinates symmetrically in spite of the left-right asymmetric nature of the heterotic string theory. However, classification of asymmetric orbifold models have not been achieved sufficiently, probably because of a great number of their possibilities. We need some rules of model building.

In this paper, we will investigate gauge symmetry and space-time supersymmetry of asymmetric orbifold models. The gauge symmetry is analyzed for bosonic string theories and the supersymmetry is for $E_8 \times E_8$ heterotic string theories.

Throughout this paper, we will impose the following two conditions for $\mathbf{Z}_N$-orbifold models.

(i) Momenta in the internal space are on a Lorentzian even self-dual lattice associated with semi-simple simply-laced Lie algebra G.

(ii) $\mathbf{Z}_N$-twists θ act as an automorphism of the lattice and the action of θ^l ($l = 1, ..., N-1$) does not have any fixed directions.

The condition (i) ensures the invariance of the one-loop partition function.

As we stated before, lists of the orbifold models which have "standard-like" or "GUTs-like" gauge symmetry and preserve N=1 space-time supersymmetry already exist[18]. However, for a special class of asymmetric orbifold models, we found unexpected phenomena to occur and the classification of this class of orbifold models have not yet been completed. In previous papers [19], we investigated the classical vacuum configurations of such asymmetric orbifold models. The models consist of left-moving string coordinates on tori and right-moving ones on orbifolds. We will call this class of orbifold models "chiral" asymmetric orbifold models. In the case of bosonic string compactification, modular invariance severely restricts the allowed models. We analyzed their gauge symmetry and found that extra conserved currents which concern the gauge symmetry appear from the twisted sectors and hence the symmetry of the total Hilbert space becomes larger than we expected. It is noteworthy that the "symmetry enhancement" occurs accidentally for specific dimensions of the orbifolds. The conserved currents are called "twist-untwist intertwining currents"[20], which convert untwisted string states to twisted ones and vice versa. Moreover we analyzed the supersymmetry of $E_8 \times E_8$ heterotic string theory compactified on asymmetric orbifolds, whose left-moving bosonic string coordinates are on tori and right-moving superstring coordinates are on orbifolds. Orbifolds are defined by dividing tori with discrete rotations $\{\theta\}$. In the complex basis, the rotations are generated by

$$\theta = \exp[2\pi i(v^1 J^{12} + v^2 J^{34} + v^3 J^{56})], \tag{1.1}$$

where J^{ij} are the $SO(6)$ Cartan generators and v^i satisfy $Nv^i \in \mathbf{Z}$. In symmetric orbifold models, the condition for the preservation of N=1 space-time supersymmetry is given by

$$\pm v^1 \pm v^2 \pm v^3 = 0, \tag{1.2}$$

for some choice of signs[12]. Then the supersymmetry will be broken and the $\mathbf{Z}_N$-invariant part will survive if they are symmetric orbifold models. However, we have to emphasize that this condition holds for symmetric orbifold models but not for asymmetric ones. In the asymmetric case, unexpected supercurrents may appear in the twisted sectors and play a role of twist-untwist intertwining currents to "enhance" the supersymmetry of the total Hilbert space. We found that these models preserve more than N=1 supersymmetry in the chiral asymmetric case. This implies that the condition for the conservation of N=1 supersymmetry (1.2) is too restrictive for the asymmetric orbifold models.

The classification of the "enhanced" models for the chiral asymmetric orbifolds was completed under the restrictions (i) and (ii)[19]. However, classification for general asymmetric orbifolds has not yet been completed. We attempt to construct unexpected, consistent asymmetric orbifold models with N=1 space-time supersymmetry by using the "enhancement" mechanism, because such models have hitherto been ruled out by the reason that they do not satisfy the condition (1.2). We will consider left-right asymmetric orbifold models[17] with different left-right twists θ_L, θ_R and show an example which preserves N=1 supersymmetry without satisfying the condition (1.2)[21].

The outline of this paper is as follows. In sect.2 we provide a review of symmetric orbifold models. The notations of this section are used throughout this paper. The classifications of lattices and automorphisms satisfying the conditions (i) and (ii) are given near the end of this section. Sect.3 is devoted to explaining the results of ref.[19]. We describe chiral asymmetric orbifold models and point out that twist-untwist intertwining currents may appear in those models and "enhance" the symmetry. The list of such models are given in table 2–6. In sect.4 we consider $E_8 \times E_8$ heterotic string compactified on asymmetric orbifolds. As was pointed out in ref.[19], the symmetry "enhancement" may occur in the case of supersymmetry. We show four chiral asymmetric orbifold models. Finally, we give a "realistic" superstring theory with N=1 space-time supersymmetry by using the fact that E_6-lattice allows two different twists.

2 Orbifold compactification of bosonic strings

In this section, we review closed bosonic strings on a torus and a symmetric orbifold. The bosonic string theory is formulated in 26-dimensions. For simplicity of the quantization, we will take the light-cone gauge. Then the string coordinates are given by $X^i(\tau,\sigma),(i=1,...,24)$. We will consider the models whose D-dimensional space ($D \leq 24$) is compactified on a torus and a orbifold. In order to realize our 4-dimensional world, we should set D =22 so that the transverse two dimensional directions remain uncompactified. Here, for the sake of applications, we weakly restrict D to be even.

2.1 TOROIDAL COMPACTIFICATION

We will consider a string theory compactified on a D-dimensional torus $\mathbf{T}^D$. The torus is defined by identifying points in the D-dimensional space, $X^i \sim X^i + \pi w^i$, where $\{w^i\}$ are vectors on a D-dimensional lattice Λ and represent the winding number of the strings around the torus. Thus, the torus is obtained by dividing the Euclidean D-dimensional space by $\pi\Lambda$:

$$\mathbf{T}^D = \mathbf{R}^D/\pi\Lambda. \tag{2.1}$$

We will start with the action of the internal part,

$$S[X] = \int d\tau \int_0^{\pi} d\sigma \frac{1}{2\pi}\{\eta^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^i + \epsilon^{\alpha\beta}B^{ij}\partial_\alpha X^i \partial_\beta X^j\}, \tag{2.2}$$

where τ and σ are the world sheet coordinates and $\eta^{\alpha\beta}$ is a flat metric on the world sheet and $\epsilon^{01} = -\epsilon^{10} = 1$. $B^{ij},(i,j=1,...,D)$ is a constant antisymmetric background field introduced by Narain, Sarmadi and Witten[22] to explain Narain torus compactification[23]. Note that the second term is a total derivative which does not affect the Hamiltonian H or equations of motion.

On tori, the string coordinates $X^i(\tau,\sigma)$ obey the following boundary conditions,

$$X^i(\tau,\sigma+\pi) = X^i(\tau,\sigma) + \pi w^i. \tag{2.3}$$

Thus, solutions to the equations of motion are given by the form,

$$X^i(\tau,\sigma) = x^i + (p^i - B^{ij}w^j)\tau + w^i\sigma + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\left\{\alpha^i_{Ln}e^{-2in(\tau+\sigma)} + \alpha^i_{Rn}e^{-2in(\tau-\sigma)}\right\}, \tag{2.4}$$

where p^i are the canonical momenta conjugate to the center-of-mass coordinates x^i. The existence of the anti-symmetric background field B^{ij} affects the canonical momentum conjugate to $X^i(\tau,\sigma)$ as follows:

$$\Pi^i = \frac{1}{\pi}\left(\partial_\tau X^i(\tau,\sigma) + B^{ij}\partial_\sigma X^j(\tau,\sigma)\right). \tag{2.5}$$

Note that the canonical momenta p^i take discrete values according to the periodicity of x^i. We assume the following commutation relations introduced by Sakamoto[24],

$$\begin{aligned}
\left[x^i, p^j\right] &= i\delta^{ij}, \\
\left[Q^i, w^j\right] &= i\delta^{ij}, \\
\left[x^i, Q^j\right] &= -i\frac{\pi}{2}\delta^{ij}, \\
\left[\alpha^i_{Lm}, \alpha^j_{Ln}\right] &= m\delta^{ij}\delta_{m+n,0} = \left[\alpha^i_{Rm}, \alpha^j_{Rn}\right],
\end{aligned} \tag{2.6}$$

where Q^i are the canonical "coordinates" conjugate to w^i and all other commutation relations of operators vanish. Under these assumptions, we can get correct commutation relations of vertex operators without putting cocycle factors by hand.

We can separate the string coordinates into left- and right-moving ones:

$$\begin{aligned}
X^i(\tau,\sigma) &= \frac{1}{2}(X^i_L(\tau+\sigma) + X^i_R(\tau-\sigma)), \\
X^i_L(\tau+\sigma) &= x^i_L + 2p^i_L(\tau+\sigma) + i\sum_{n\neq 0}\frac{1}{n}\alpha^i_{Ln}e^{-2in(\tau+\sigma)}, \\
X^i_R(\tau-\sigma) &= x^i_R + 2p^i_R(\tau-\sigma) + i\sum_{n\neq 0}\frac{1}{n}\alpha^i_{Rn}e^{-2in(\tau-\sigma)},
\end{aligned} \tag{2.7}$$

where we have put the relations between x^i, p^i, Q^i, w^i and $x^i_L, x^i_R, p^i_L, p^i_R$ as follows:

$$\begin{aligned}
x^i_L &= (1-B)^{ij}x^j + Q^i, \\
x^i_R &= (1+B)^{ij}x^j - Q^i, \\
p^i_L &= \frac{1}{2}p^i + \frac{1}{2}(1-B)^{ij}w^j, \\
p^i_R &= \frac{1}{2}p^i - \frac{1}{2}(1+B)^{ij}w^j,
\end{aligned} \tag{2.8}$$

or equivalently,

$$x^i = \frac{1}{2}(x^i_L + x^i_R),$$

$$
\begin{aligned}
Q^i &= \frac{1}{2}(1+B)^{ij}x_L^j - \frac{1}{2}(1-B)^{ij}x_R^j, \\
p^i &= (1+B)^{ij}p_L^j + (1-B)^{ij}p_R^j, \\
w^i &= p_L^i - p_R^i.
\end{aligned}
\tag{2.9}
$$

Then the commutation relations are rewritten into the forms

$$
\begin{aligned}
\left[x_L^i, p_L^j\right] &= i\delta^{ij} = \left[x_R^i, p_R^j\right], \\
\left[x_L^i, x_L^j\right] &= i\pi B^{ij} = \left[x_R^i, x_R^j\right], \\
\left[x_L^i, x_R^j\right] &= i\pi(1-B)^{ij}, \\
\left[\alpha_{Lm}^i, \alpha_{Ln}^j\right] &= m\delta^{ij}\delta_{m+n,0} = \left[\alpha_{Rm}^i, \alpha_{Rn}^j\right], \\
\left[\alpha_{Lm}^i, \alpha_{Rn}^j\right] &= 0.
\end{aligned}
\tag{2.10}
$$

From the eq.(2.9), the left- and right-moving momenta (p_L^i, p_R^i) have to lie on a $(D+D)$-dimensional Lorenzian even self-dual lattice $\Gamma^{D,D}$[25], which we will explain at the last of this section. Note that this choice of lattices guarantees modular invariance of the one-loop partition function.

The mass formulae are then given by

$$
\begin{aligned}
\frac{1}{8}m_L^2 &= \frac{1}{2}\sum_{i=1}^{D}(p_L^i)^2 + \frac{1}{2}\sum_{a=1}^{2}(p_L^a)^2 + N_L - 1, \\
\frac{1}{8}m_R^2 &= \frac{1}{2}\sum_{i=1}^{D}(p_R^i)^2 + \frac{1}{2}\sum_{a=1}^{2}(p_R^a)^2 + N_R - 1,
\end{aligned}
\tag{2.11}
$$

where p^a $(a = 1, 2)$ are uncompactified external space-time momenta in the light-cone gauge and -1 is due to the contribution from the oscillators to the zero-point energy and N_L and N_R are the number operators defined by

$$
N_L = \sum_{i=1}^{D}\sum_{n=1}^{\infty}\alpha_{L-n}^i\alpha_{Ln}^i + \sum_{a=1}^{2}\sum_{n=1}^{\infty}\alpha_{L-n}^a\alpha_{Ln}^a, \tag{2.12}
$$

$$
N_R = \sum_{i=1}^{D}\sum_{n=1}^{\infty}\alpha_{R-n}^i\alpha_{Rn}^i + \sum_{a=1}^{2}\sum_{n=1}^{\infty}\alpha_{R-n}^a\alpha_{Rn}^a. \tag{2.13}
$$

Let us calculate the one-loop partition function defined by

$$
Z(\tau) = \mathrm{Tr} q^{H_L}\bar{q}^{H_R}, \quad q = e^{2\pi i\tau}, \tag{2.14}
$$

where τ is the modular parameter and $H_L(H_R)$ is the left-(right-) Hamiltonian. A one-loop diagram is a 2-dimensional torus. The modular parameter τ characterizes this torus. Using the zero mode of Virasoro operators, we can rewrite the partition function into the form

$$Z(\tau) = \mathrm{Tr} q^{L_0 - \frac{D}{24}} \bar{q}^{\bar{L}_0 - \frac{D}{24}}, \quad q = e^{2\pi i \tau}, \tag{2.15}$$

where D is a dimension of the torus in which the strings are embedded and the trace is taken over the Hilbert space. In the torus models, Virasoro operators are given by

$$\begin{aligned} L_0 &= \sum_{i=1}^{D} \{ \frac{1}{2}(p_L^i)^2 + \sum_{n=1}^{\infty} \alpha_{L-n}^i \alpha_{Ln}^i \}, \\ \bar{L}_0 &= \sum_{i=1}^{D} \{ \frac{1}{2}(p_R^i)^2 + \sum_{n=1}^{\infty} \alpha_{R-n}^i \alpha_{Rn}^i \}. \end{aligned} \tag{2.16}$$

Substituting them into eq.(2.15), we get

$$Z(\tau) = \frac{1}{|\eta(\tau)|^D} \sum_{(p_L^i, p_R^i) \in \Gamma^{D,D}} e^{i\pi\tau (p_L^i)^2} e^{-i\pi\bar{\tau}(p_R^i)^2}, \tag{2.17}$$

where $\eta(\tau)$ is the Dedekind η-function defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{2.18}$$

String theory has a global symmetry called modular invariance. This symmetry reflects the invariance of the complex structure of the world sheet torus under the following transformation,

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbf{Z}, \quad ad - bc = 1. \tag{2.19}$$

This transformation is called modular transformation and makes a group isomorphic to $SL(2, \mathbf{Z})/\mathbf{Z}_2$. The modular transformation is generated by the following two transformations,

$$\begin{aligned} T &: \quad \tau \rightarrow \tau + 1, \\ S &: \quad \tau \rightarrow -1/\tau. \end{aligned} \tag{2.20}$$

It is easy to check the invariance of the partition function under these transformations for each model. We may use the following formulae:

$$\begin{aligned} \eta(\tau + 1) &= e^{i\pi/12} \eta(\tau), \\ \eta(-1/\tau) &= \sqrt{-i\tau}\, \eta(\tau), \end{aligned} \tag{2.21}$$

and the fact that $\Gamma^{D,D}$ is an even and self-dual lattice. Invariance under the T-transformation is trivial. Note that the momentum sum is transformed under the S-transformation as follows:

$$\sum_{(p^i_L,p^i_R)\in\Gamma^{D,D}} e^{\frac{-i\pi}{\tau}(p^i_L)^2} e^{\frac{i\pi}{\bar{\tau}}(p^i_R)^2} = \frac{(-i\tau)^{D/2}(i\bar{\tau})^{D/2}}{\mathrm{vol}\left(\Gamma^{D,D}\right)} \sum_{(p^i_L,p^i_R)\in\Gamma^{D,D*}} e^{i\pi\tau(p^i_L)^2} e^{-i\pi\bar{\tau}(p^i_R)^2}, \quad (2.22)$$

where we used the Poisson resummation formula to derive the right-hand side. Since the lattice $\Gamma^{D,D}$ is even, $(p^i_{L,R})^2$ =even, and self-dual, $\mathrm{vol}\left(\Gamma^{D,D}\right) = 1$, the partition function is, together with the oscillator part and zero-point energy, invariant under the S-transformation.

2.2 SYMMETRIC ORBIFOLD MODELS

An orbifold $\mathbf{O}^D$ is defined as a quotient space of the Euclidean D-dimensional space $\mathbf{R}^D$ with a space group $\mathbf{S}$ specified by discrete rotations $\{\theta\}$ and discrete translations $\{w\}$; $\mathbf{S} = \{\theta, w\}$. This implies that an orbifold is defined by dividing a torus $\mathbf{T}^D$ with its isometries called point group $\mathbf{P}$,

$$\mathbf{O}^D = \frac{\mathbf{T}^D}{\mathbf{P}} \quad (2.23)$$

In this paper, we will consider $\mathbf{Z}_N$-orbifold models, $\theta^N = \mathbf{1}$, with $[\theta, B] = 0$ and $\det(\mathbf{1} - \theta) \neq 0$ for simplicity. If it is not the case, the $\mathbf{Z}_N$-transformation properties of the zero mode of the string coordinate are nontrivial and some topological effects may appear. There are two types of strings on orbifolds. One is in the untwisted sector and follows the boundary condition of the form,

$$X^i(\tau, \sigma + \pi) = X^i(\tau, \sigma) + \pi w^i, \quad i = 1, ..., D. \quad (2.24)$$

The others are in twisted sectors and the boundary conditions take the form,

$$X^i(\tau, \sigma + \pi) = \theta^{ij} X^j(\tau, \sigma) + \pi w^i, \quad i = 1, ..., D. \quad (2.25)$$

Then the string coordinates are expanded in the untwisted sector as

$$X^i(\tau, \sigma) = x^i + (p^i - B^{ij} w^j)\tau + w^i\sigma + X^i_{osc}(\tau, \sigma), \quad (2.26)$$

and in the twisted sectors as

$$X^i(\tau, \sigma) = x^i_{fix} + X^i_{osc}(\tau, \sigma), \quad (2.27)$$

where x^i_{fix} satisfy the fixed point equation,

$$x^i_{fix} = \theta^{ij} x^j_{fix} + \pi w^i, \tag{2.28}$$

and $X^i_{osc}(\tau, \sigma)$ denotes the oscillator part of the string coordinates satisfying

$$X^i_{osc}(\tau, \sigma + \pi) = \theta^{ij} X^j_{osc}(\tau, \sigma). \tag{2.29}$$

It is convenient to introduce a complex basis which makes θ diagonal. The orthogonal matrix θ can always be diagonalized by a unitary matrix M,

$$M^\dagger \theta M = \theta_{\rm diag}, \quad M^\dagger M = \mathbf{1}, \tag{2.30}$$

and

$$\theta_{\rm diag} = \begin{pmatrix} \omega^{r_1} & & & 0 \\ & \omega^{r_2} & & \\ & & \ddots & \\ 0 & & & \omega^{r_D} \end{pmatrix}, \tag{2.31}$$

where $0 < r_i \leq N-1$ ($r_i \in \mathbf{Z}$) and $\omega = e^{2\pi i/N}$. The set of eigenvalues $\{\omega^{r_i}\}$ is identical to the set $\{\omega^{-r_i}\}$ because θ is an orthogonal matrix. Thus, we may write the eigenvalues of θ as

$$\{\omega^{r_i} \quad \text{and} \quad \omega^{-r_i}, \quad i = 1, ..., D/2\}. \tag{2.32}$$

Then the string coordinates transform as $M^{\dagger ij} X^j$. We can rewrite the coordinates into $X^i_{new}(\tau, \sigma)$ and $\bar{X}^i_{new}(\tau, \sigma)$, ($i = 1, ..., D/2$) such as they follow the boundary conditions,

$$\begin{aligned} X^i_{new}(\tau, \sigma + \pi) &= \omega^{r_i} X^i_{new}(\tau, \sigma) + (\text{torus} \quad \text{shift}), \\ \bar{X}^i_{new}(\tau, \sigma + \pi) &= \omega^{-r_i} \bar{X}^i_{new}(\tau, \sigma) + (\text{torus} \quad \text{shift}), \end{aligned} \tag{2.33}$$

Hereafter, we use these new coordinates and omit the index *new*. We will introduce a description developed in 2d conformal field theory (CFT)[26]. In the new basis, the string coordinates on orbifolds are expanded of the form,

$$\begin{aligned} X^i_L(z) &= x^i_L - i p^i_L \ln z + i \sum_{n_j \in \mathbf{Z} - r_j/N > 0} \frac{1}{n_j} \left\{ M^{ij} \gamma^j_{n_j} z^{-n_j} - M^{*ij} \gamma^{j\dagger}_{n_j} z^{n_j} \right\}, \\ X^i_R(z) &= x^i_R - i p^i_R \ln \bar{z} + i \sum_{n_j \in \mathbf{Z} + r_j/N > 0} \frac{1}{n_j} \left\{ M^{ij} \bar{\gamma}^j_{n_j} z^{-n_j} - M^{*ij} \bar{\gamma}^{j\dagger}_{n_j} \bar{z}^{n_j} \right\}, \end{aligned} \tag{2.34}$$

with $p_L^i = p_R^i = 0$ for the twisted sectors. Then the quantization condition for the oscillators are given by

$$\begin{aligned}
\left[\gamma^i_{m_i}, \gamma^j_{n_j}\right] &= m_i \delta^{ij} \delta_{m_i,n_j} \quad \text{for } m_i \in \mathbf{Z} - \frac{r_i}{N} > 0 \text{ and } n_j \in \mathbf{Z} - \frac{r_j}{N} > 0, \\
\left[\bar{\gamma}^i_{m_i}, \bar{\gamma}^j_{n_j}\right] &= m_i \delta^{ij} \delta_{m_i,n_j} \quad \text{for } m_i \in \mathbf{Z} + \frac{r_i}{N} > 0 \text{ and } n_j \in \mathbf{Z} + \frac{r_j}{N} > 0, \\
\left[\gamma^i_{m_i}, \bar{\gamma}^j_{n_j}\right] &= 0.
\end{aligned} \tag{2.35}$$

The mass formulae for θ^l-twisted sector (l=0, for untwisted sector) are given by

$$\begin{aligned}
\frac{1}{8}(m_L^{(l)})^2 &= \frac{1}{2}\sum_{i=1}^{D}(p_L^i)^2 \delta_{l,0} + \frac{1}{2}\sum_{a=1}^{2}(p_L^a)^2 + N_L^{(l)} + E_0^{(l)} - 1, \\
\frac{1}{8}(m_R^{(l)})^2 &= \frac{1}{2}\sum_{i=1}^{D}(p_R^i)^2 \delta_{l,0} + \frac{1}{2}\sum_{a=1}^{2}(p_R^a)^2 + N_R^{(l)} + E_0^{(l)} - 1,
\end{aligned} \tag{2.36}$$

where $N_L^{(l)}$ and $N_R^{(l)}$ are the number operators and $E_0^{(l)}$ is the contribution from twisted oscillators to the zero-point energy, and they are given by

$$N_L^{(l)} = \sum_{i=1}^{D} \sum_{n_i \in \mathbf{Z} - r_i/N > 0} \gamma^{i\dagger}_{n_i} \gamma^i_{n_i} + \sum_{a=1}^{2}\sum_{n=1}^{\infty} \alpha^a_{L-n} \alpha^a_{Ln}, \tag{2.37}$$

$$N_R^{(l)} = \sum_{i=1}^{D} \sum_{n_i \in \mathbf{Z} + r_i/N > 0} \bar{\gamma}^{i\dagger}_{n_i} \bar{\gamma}^i_{n_i} + \sum_{a=1}^{2}\sum_{n=1}^{\infty} \alpha^a_{R-n} \alpha^a_{Rn}, \tag{2.38}$$

$$E_0^{(l)} = \frac{1}{2}\sum_{i=1}^{D/2} \zeta_i^{(l)}(1 - \zeta_i^{(l)}), \quad \zeta_i^{(l)} = l\zeta_i^{(1)} \mod 1, \quad 0 < \zeta_i^{(l)} < 1, \tag{2.39}$$

where $\zeta_i^{(1)} = r_i/N$. In the symmetric orbifold models, the eigenvalues of the zero mode of the Virasoro operators in θ^l-twisted sector are given by

$$\begin{aligned}
L_0^{(l)} &= \frac{1}{2}\sum_{i=1}^{D}(p_L^i)^2 \delta_{l,0} + N_L^{(l)} + E_0^{(l)}, \\
\bar{L}_0^{(l)} &= \frac{1}{2}\sum_{i=1}^{D}(p_R^i)^2 \delta_{l,0} + N_R^{(l)} + E_0^{(l)}.
\end{aligned} \tag{2.40}$$

Note that the energy of the ground states is zero in the untwisted sector. However, in the twisted sectors the energy is non-zero.

To construct the models explicitly we have to select the momentum lattice. We choose the Lorentzian even self-dual lattice $\Gamma^{D,D}$ introduced in the torus models. Let us consider the one-loop partition function here. Because of the nontrivial boundary conditions of the orbifold models, modular invariance of the partition function is not manifest. Modular invariance may give constraints on the models.

In $\mathbf{Z}_N$-orbifold models, the one-loop partition function in the operator formalism will be of the form,

$$
\begin{aligned}
Z(\tau) &= \frac{1}{N}\sum_{l,m=0}^{N-1} Z(g^l, g^m; \tau), \\
Z(g^l, g^m; \tau) &= \mathrm{Tr}\left[g_{(l)}^m q^{L_0 - \frac{D}{24}} \bar{q}^{\bar{L}_0 - \frac{D}{24}}\right]_{g^l\text{-sector}},
\end{aligned} \tag{2.41}
$$

where g is a generator of $\mathbf{Z}_N$ and $g_{(l)}$ is a twist operator in the g^l-sector. Then the partition function of the g^l-sector is written in terms of the projection operator $\mathcal{P}$ as

$$
\begin{aligned}
Z(\tau)_{g^l\text{-sector}} &= \mathrm{Tr}\left[\mathcal{P} q^{L_0 - \frac{D}{24}} \bar{q}^{\bar{L}_0 - \frac{D}{24}}\right]_{g^l\text{-sector}}, \\
\mathcal{P} &= \frac{1}{N}\sum_{m=0}^{N-1} g_{(l)}^m,
\end{aligned} \tag{2.42}
$$

where the trace is taken over the Hilbert space of the g^l-sector. The projection operator in the untwisted sector is well known. However, in the twisted sectors, it is not obvious. We can determine the projection operators in the twisted sectors by means of the requirement of modular invariance. In general, a modular invariant one-loop partition function satisfies the relations:

$$
\begin{aligned}
Z(g^l, g^m; \tau + 1) &= Z(g^l, g^{l+m}; \tau), \\
Z(g^l, g^m; -1/\tau) &= Z(g^{-m}, g^l; \tau).
\end{aligned} \tag{2.43}
$$

Using these relations as a guiding principle, we can construct a modular invariant partition function.

2.3 LATTICES AND AUTOMORPHISMS

We will consider the models whose left- and right-moving momenta (p_L^i, p_R^i) lie on a $(D+D)$-dimensional Lorenzian even self-dual lattice $\Gamma_G^{D,D}$ associated with a semi-simple simply-laced Lie algebra G

$$
\Gamma_G^{D,D} = \{(p_L^i, p_R^i) |\, p_L^i, p_R^i \in \Lambda_W(G) \quad \text{and} \quad p_L^i - p_R^i \in \Lambda_R(G)\}, \tag{2.44}
$$

where $\Lambda_W(G)$ ($\Lambda_R(G)$) is the weight (root) lattice of G and the root vectors are normalized so that their squared length is two. On this lattice, the inner product of the $P = (p_L^i, p_R^i)$ and $P' = (p_L'^i, p_R'^i)$ are calculated with Lorentzian signature $\left[(+1)^D, (-1)^D\right]$; i.e. $P \cdot P' = \sum_{i=1}^{D}(p_L^i p_L'^i - p_R^i p_R'^i)$ and $P \cdot P' \in \mathbf{Z}$ for all $P,\ P' \in \Gamma_G^{D,D}$. All lattice vectors have even $(\text{length})^2$. This implies that this lattice is even integral. Furthermore this lattice is self-dual, i.e. $\Gamma_G^{D,D*} = \Gamma_G^{D,D}$, in other words, $\text{vol}\left(\Gamma_G^{D,D}\right) = 1$. Then these choices of the lattices limit the twists $\{\theta\}$ because they have to be automorphisms of the lattices.

The automorphisms of Lie algebras are well known[27]. Let Φ be a root system of the semi-simple simply-laced Lie algebra G. The automorphisms of Φ make a group AutΦ. It is known that AutΦ is a semi-direct product of two groups [27][28],

$$\begin{aligned} \text{Aut}\Phi &= W \rtimes \text{Aut}(\Phi,\Delta), \\ W &\cap \text{Aut}(\Phi,\Delta) = \{1\}, \end{aligned} \tag{2.45}$$

where W is the Weyl group of Φ and Δ is a fixed basis of Φ. Aut(Φ,Δ) is defined as

$$\text{Aut}(\Phi,\Delta) = \{\varphi \in \text{Aut}\Phi \mid \varphi(\Delta) = \Delta\}, \tag{2.46}$$

and corresponds to the symmetries of the Dynkin diagram of G. This symmetry is called outer automorphisms of Φ.

In this paper, we restrict our consideration to the models whose automorphisms $\theta^l, (l = 1, ..., N-1)$ are inner ones and do not have any fixed directions. For the semi-simple simply-laced Lie algebras, i.e. $SU(n+1), SO(2n), E_6, E_7, E_8$ and direct products of them, the allowed automorphisms are classified by Myhill[29].

$$\begin{aligned}
SU(n+1) \ &: \ [2\widehat{(n+1)}] \to [n+1] \to [\hat{2}] && \text{for} \ \ n+1 = \text{prime} \\
& \ \ [\hat{2}] && \text{for} \ \ n+1 \neq \text{prime} \\
SO(2n) \ &: \ [\hat{2}^{N+1}] \to [2^N] \to \cdots \to [2] && \text{for} \ \ n = 2^N p,\ n > 4 \\
& \ \ [\hat{2}] && \text{for} \ \ n = p, (p : \text{odd}) \\
SO(8) \ &: \ [\hat{12}] \to [\hat{6}] \to [4] \to [\hat{3}] \to [2] \\
&: \ [\hat{8}] \to [4] \to [2] \\
E_6 \ &: \ [\hat{18}] \to [9] \to [\hat{6}] \to [3] \to [\hat{2}]
\end{aligned}$$

$$
\begin{aligned}
E_7 \;&:\; [2] \\
E_8 \;&:\; [30] \to [15] \to [10] \to [6] \to [5] \to [3] \to [2] \\
&\quad\; [24] \to [12] \to [8] \to [6] \to [4] \to [3] \to [2] \\
&\quad\; [20] \to [10] \to [5] \to [4] \to [2],
\end{aligned} \tag{2.47}
$$

where the numbers denote the order of the automorphisms and $\wedge$ denotes an outer automorphism. Suppose that A is a matrix which generates the highest order automorphisms in the above rows of numbers, lower ones are induced from its powers. For example, order 18 outer automorphism of E_6 is generated by a matrix,

$$
A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 & 1 & -2 \\ -2 & 1 & 1 & -1 & 2 & -2 \\ -2 & 1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 & -1 \end{pmatrix} \tag{2.48}
$$

where we adopted a basis of simple root. Then the automorphisms of E_6 are given by the following matrices,

$$
\begin{array}{cccccccccc}
 & A & & A^2 & & A^3 & & A^6 & & A^9 \\
E_6 : & [\hat{18}] & \to & [9] & \to & [\hat{6}] & \to & [3] & \to & [\hat{2}] \\
 & \text{OUTER} & & \text{INNER} & & \text{OUTER} & & \text{INNER} & & \text{OUTER.}
\end{array} \tag{2.49}
$$

3 Asymmetric orbifold models

In this section, we will describe left-right asymmetric orbifold models with different left-right twists θ_L, θ_R. In this case, the twists act on the string coordinates as follows:

$$g: \quad (X_L, X_R) \to (\theta_L X_L, \theta_R X_R), \tag{3.1}$$

where g is an element of $\mathbf{Z}_N$. In the complex basis, these twists are specified by eigenvalues of $\theta_{L,R}$, v_L^i and $v_R^i, (i = 1, ..., D/2)$. We will consider the momentum lattice. We consider the following the Lorentzian even self-dual lattice $\Gamma_G^{D,D}$[25] associated with semi-simple simply-laced Lie algebras G with rank D,

$$\Gamma_G^{D,D} = \{(p_L^i, p_R^i) |\ p_L^i, p_R^i \in \Lambda_W(G) \quad \text{and} \quad p_L^i - p_R^i \in \Lambda_R(G)\}, \tag{3.2}$$

Then the action of $\mathbf{Z}_N$ on the momenta is

$$g: \quad (p_L^i, p_R^i) \to (\theta_L^{ij} p_L^j, \theta_R^{ij} p_R^j). \tag{3.3}$$

Since $\mathbf{Z}_N$-transformation has to act as an automorphism of the momentum lattice;

$$(\theta^{ij} p_L^j, \theta^{ij} p_R^j) \in \Gamma_G^{D,D} \quad \text{for all} \quad (p_L^i, p_R^i) \in \Gamma_G^{D,D}. \tag{3.4}$$

Variety of models can be considered. Here we restrict ourselves to the models whose twists θ^l $(l = 1, ..., N-1)$ have the origin as the only fixed point.

3.1 CHIRAL ASYMMETRIC ORBIFOLD MODELS

There are a lot of candidates for asymmetric twists. The following $\mathbf{Z}_N$-twists will be the simplest ones.

$$g: \quad (X_L, X_R) \to (X_L, \theta X_R), \tag{3.5}$$

We will call this class of orbifold models "chiral" type. We found that these models have particular properties[19]. As we discussed in the previous section, the conformal weight of the ground state in the twisted sectors is positive, while that in the untwisted sector is zero. Thus, in these chiral asymmetric models, there is a possibility that (0, 1) states may appear in the twisted sectors. In general, the states with conformal weight (1, 0) and (0, 1) correspond to conserved currents and they ensure the existence of some symmetries. In this case, the appearance of the (0, 1) states will imply a symmetry between untwisted and twisted sectors. These currents convert untwisted string states into twisted ones and vice versa. Therefore we will call these currents "twist-untwist intertwining currents". Because of the existence of these extra currents, the symmetry of the total Hilbert space will become greater than that of each (untwisted and twisted) sectors. It should be emphasized that these (1, 0) or (0, 1) currents cannot appear in the symmetric orbifold models because the left- and right-conformal weights of the ground states in the twisted sectors are both positive and equal.

Let us first consider the untwisted sector. Since the boundary condition of the string coordinates is the same as in the torus case, the left- and right-moving string coordinates will be expanded as

$$\begin{aligned} X_L^i(z) &= x_L^i - ip_L^i \ln z + i \sum_{n\in\mathbf{Z}} \frac{1}{n}\alpha_{Ln}^i z^{-n}, \\ X_R^i(\bar{z}) &= x_R^i - ip_R^i \ln \bar{z} + i \sum_{n\in\mathbf{Z}} \frac{1}{n}\alpha_{Rn}^i \bar{z}^{-n}. \end{aligned} \tag{3.6}$$

We will introduce a twist operator $g_{(0)}$ which induce a $\mathbf{Z}_N$- transformation, i.e.

$$g: \quad g_{(0)}\left(X_L^i(z), X_R^i(\bar{z})\right) g_{(0)}^{-1} = \left(X_L^i(z), \theta^{ij} X_R^j(\bar{z})\right). \tag{3.7}$$

Using the same notations with the symmetric orbifold models, the operator $g_{(0)}$ is given by

$$\begin{aligned} g_{(0)} &= \omega^{K_R^{(0)}} \sum_{(p_L^i, p_R^i)\in\Gamma^{D,D}} |p_L^i, p_R^i> < p_L^i, \theta^{ij} p_R^j|, \\ K_R^{(0)} &= \sum_{i=1}^{D}\sum_{n=1}^{\infty} \frac{r_i}{n} \bar{\gamma}_{Rn}^{i\dagger} \bar{\gamma}_{Rn}^i, \end{aligned} \tag{3.8}$$

where $\bar{\gamma}_{Rn}^i \equiv M^{\dagger ij}\alpha_{Rn}^j$ $(n \in \mathbf{Z}, n > 0)$ and $\omega = e^{2\pi i/N}$. Then the one-loop partition function in the untwisted sector is given by .

$$Z^{(0)}(\tau) = \frac{1}{N}\sum_{m=0}^{N-1} Z(1, g^m; \tau), \tag{3.9}$$

where

$$Z(1, g^m; \tau) = \mathrm{Tr}\left[g_{(0)}^m q^{L_0 - \frac{D}{24}} \bar{q}^{\bar{L}_0 - \frac{D}{24}}\right], \quad q = e^{2\pi i \tau}, \tag{3.10}$$

$$
\begin{aligned}
L_0 &= \sum_{i=1}^{D}\{\frac{1}{2}(p_L^i)^2 + \sum_{n=1}^{\infty} \alpha_{L-n}^i \alpha_{Ln}^i\}, \\
\bar{L}_0 &= \sum_{i=1}^{D}\{\frac{1}{2}(p_R^i)^2 + \sum_{n=1}^{\infty} \alpha_{R-n}^i \alpha_{Rn}^i\}.
\end{aligned} \tag{3.11}
$$

For the model building, we must be careful in that the $\mathbf{Z}_N$-transformation (3.5) should act as an automorphism of the momentum lattice $\Gamma_G^{D,D}$,i.e.

$$
(p_L^i, \theta^{ij} p_R^j) \in \Gamma_G^{D,D} \quad \text{for all} \quad (p_L^i, p_R^i). \tag{3.12}
$$

This implies that the $\mathbf{Z}_N$-transformation must not change the conjugacy class and we have to exclude the outer automorphisms, which change the conjugacy class.

Let us consider the g^l-twisted sector. In this case, the string coordinates will obey the following boundary conditions,

$$
\begin{aligned}
X_L^i(e^{2\pi i} z) &= X_L^i(z) + (\text{torus shift}), \\
X_R^i(e^{-2\pi i} \bar{z}) &= \theta^{ij} X_R^j(\bar{z}) + (\text{torus shift}).
\end{aligned} \tag{3.13}
$$

Thus the string coordinates will be expanded as follows:

$$
X_L^i(z) = x_L^i - i p_L^i ln z + i \sum_{n \in \mathbf{Z}} \frac{1}{n} \alpha_{Ln}^i z^{-n}, \tag{3.14}
$$

$$
X_R^i(z) = x_R^i + i \sum_{n_j \in \mathbf{Z} + r_j/N > 0} \frac{1}{n_j} \left\{ M^{ij} \tilde{\gamma}_{n_j}^j z^{-n_j} - M^{*ij} \tilde{\gamma}_{n_j}^{j\dagger} \bar{z}^{n_j} \right\}. \tag{3.15}
$$

In these models, the mass formulae for g^l-sector (l=0 for the untwisted sector) are given by the form,

$$
\frac{1}{8}(m_L^{(l)})^2 = \frac{1}{2}\sum_{i=1}^{6}(p_L^i)^2 + \frac{1}{2}\sum_{a=1}^{2}(p_L^a)^2 + N_L - 1, \tag{3.16}
$$

$$
\frac{1}{8}(m_R^{(l)})^2 = \frac{1}{2}\sum_{i=1}^{6}(p_R^i)^2 \delta_{l,0} + \frac{1}{2}\sum_{a=1}^{2}(p_R^a)^2 + N_R^{(l)} + E_{R0}^{(l)} - 1, \tag{3.17}
$$

where N_L and $N_R^{(l)}$ are the number operators defined in eqs.(2.12) and (2.38), and $E_{R0}^{(l)}$ is the contribution from twisted oscillators to the zero-point energy and is given by

$$
E_{R0}^{(l)} = \frac{1}{2}\sum_{i=1}^{D/2} \zeta_i^{(l)}(1 - \zeta_i^{(l)}), \quad \zeta_i^{(l)} = l\zeta_i^{(1)} \mod 1, \quad 0 < \zeta_i^{(l)} < 1. \tag{3.18}
$$

Then the partition function in the g^l-twisted sector will be given by

$$Z^{(l)}(\tau) = \frac{1}{N}\sum_{m=0}^{N-1} Z(g^l, g^m; \tau), \quad l = 1, 2, ..., N-1, \tag{3.19}$$

where

$$Z(g^l, g^m; \tau) = \mathrm{Tr}\left[g_{(l)}^m q^{L_0 - \frac{D}{24}} \bar{q}^{\bar{L}_0 - \frac{D}{24}}\right]_{g^l-\mathrm{sector}}. \tag{3.20}$$

As in the symmetric orbifold case, we can construct a modular invariant partition function by making use of the relation eq.(2.43).

We can now expect the form of the twist operator in the l-twisted sector. Let us construct the partition function with the trace formula in the operator formalism:

$$\begin{aligned} L_0 &= \sum_{i=1}^{D}\{\frac{1}{2}(p_L^i)^2 + \sum_{n=1}^{\infty}\alpha_{L-n}^i\alpha_{Ln}^i\}, \\ \bar{L}_0 &= \sum_{i=1}^{D}\sum_{n_i\in \mathbf{Z}-r_i/N>0} \bar{\gamma}_{Rn_i}^i \bar{\gamma}_{Rn_i}^i + E_{R0}^{(l)}, \\ g_{(l)}^l &= e^{2\pi i(L_0-\bar{L}_0)}. \end{aligned} \tag{3.21}$$

Note that the trace of the momenta (p_L^i, p_R^i) is taken over the dual lattice of $\Gamma_{\mathrm{inv}}^{D,D}$ defined by

$$\Gamma_{\mathrm{inv}}^{D,D} = \{(p_L^i, p_R^i = 0) \in \Gamma^{D,D}\}, \tag{3.22}$$

and the degeneracy of the ground states in the g^l-twisted sector is given by

$$d = \frac{\sqrt{\det(1-\theta^l)}}{\mathrm{vol}(\Gamma_{\mathrm{inv}}^{D,D})}. \tag{3.23}$$

We can make a modular invariant partition function by using the transformation properties which it should obey. If we denote by N_l the greatest common divisor (N, l) of N and l, a partition function $Z(g^l, g^m; \tau)$ has to be invariant under the modular transformation, $\tau \to \tau + N/N_l$, because $\theta^N = \mathbf{1}$. We get a necessary condition for the modular invariance,

$$\frac{N}{N_l}(L_0 - \bar{L}_0) = 0 \mod 1. \tag{3.24}$$

This is called the left-right level-matching condition[17][30] and is the necessary and sufficient condition for modular invariance. In the chiral asymmetric orbifold models, this condition reduces to

$$\begin{aligned} \frac{N}{N_l} E_0 &= 0 \mod 1, \\ \frac{N}{N_l} (p_L^i)^2 &= 0 \mod 2 \quad \text{for all} \quad p_L^i \in \Gamma_{\text{inv}}^{D,D*}. \end{aligned} \tag{3.25}$$

3.2 TWIST-UNTWIST INTERTWINING CURRENTS

Solutions to the classical field equations of string theory are described by 2d conformal field theories (CFT's)[26]. CFT's are constructed by the vacuum $|0>$ and primary fields $\Phi(w,\bar{w})$, which show the following short distance behavior with the holomorphic and anti-holomorphic stress-energy tensors,

$$\begin{aligned} T(z)\Phi(w,\bar{w}) &= \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\Phi(w,\bar{w}) + (\text{nonsingular}), \\ \bar{T}(\bar{z})\Phi(w,\bar{w}) &= \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\Phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\Phi(w,\bar{w}) + (\text{nonsingular}), \end{aligned} \tag{3.26}$$

where h and $\bar{h}$ are conformal weights of $\Phi(w,\bar{w})$ and we take the description $(h,\bar{h})$. The infinitesimal conformal transformation are generated by the Fourier components of $T(z)$ and $\bar{T}(\bar{z})$,

$$\begin{aligned} L_n &= \oint \frac{dz}{2\pi i} z^{n+1} T(z), \quad T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2}, \\ \bar{L}_n &= \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{n+1} \bar{T}(\bar{z}), \quad \bar{T}(\bar{z}) = \sum_{n=-\infty}^{\infty} \bar{L}_n \bar{z}^{-n-2}, \end{aligned} \tag{3.27}$$

and the L_n and $\bar{L}_n$ are called Virasoro operators. In the string theory, we adopt the radial quantization. Thus we will interpret the dilatation operator $L_0 + \bar{L}_0$ as the Hamiltonian. We will define Fourier components of a general holomorphic primary field $\phi(z)$ with conformal weight h as follows:

$$\phi_n = \oint \frac{dz}{2\pi i} z^{n+h-1} \phi(z), \quad \phi(z) = \sum_{-\infty}^{\infty} \phi_n z^{-n-h}. \tag{3.28}$$

Anti-holomorphic fields may be defined in an analogous way. Let us compute the commutator of the primary field $\phi(z)$ and L_0,

$$[L_0, \phi(z)] = \oint \frac{dw}{2\pi i} w T(w) \phi(z)$$

$$
\begin{aligned}
&= \oint \frac{dw}{2\pi i} w \left[\frac{h}{(w-z)^2}\phi(z) + \frac{1}{w-z}\partial_z\phi(z) \right] \\
&= h\phi(z) + z\partial_z\phi(z).
\end{aligned} \tag{3.29}
$$

Note that this commutator is a total derivative when the conformal weight of $\phi(z)$ is $h_\phi = 1$. Since the $\bar{L}_0$ always commutes with the holomorphic field $\phi(z)$, the following relation is satisfied for the (1, 0) field,

$$
[L_0, \phi(z)] = \text{ total derivative.} \tag{3.30}
$$

This implies that $\phi(z)$ is a conserved current. The same discussion will be made for the anti-holomorphic field $\bar{\phi}(\bar{z})$ and (0, 1) states will become conserved currents for the right Hamiltonian system.

In the chiral asymmetric orbifold models, we found that such currents appear in twisted sectors. As we discussed in the previous subsection, the ground states of the twisted sectors have non-zero and positive energy. In other words, $\bar{h} = E_{R0}^{(l)} > 0, (l = 1, ..., N-1)$. In CFT's, such g^l-twisted ground states are constructed by the twist fields defined by

$$
\Lambda^{(l)}(0)|0>=|\Lambda^{(l)}>, \tag{3.31}
$$

where $|\, 0>$ is the ground state of untwisted sector. The twist fields convert the untwisted sector to twisted sectors. Then the conformal weight of the twist field is given by

$$
\begin{aligned}
E_{R0}^{(l)} &= \frac{1}{2}\sum_{i=1}^{D/2} \zeta_i^{(l)}(1-\zeta_i^{(l)}), \\
\zeta_i^{(l)} &= l\zeta_i^{(1)} \mod 1, \quad 0 < \zeta_i^{(l)} < 1.
\end{aligned} \tag{3.32}
$$

We are interested in (1, 0) or (0, 1) states in the twisted ones. These states may be physical because they manifestly satisfy the level-matching condition[30].

$$
L_0 - \bar{L}_0 = 0 \mod 1, \tag{3.33}
$$

and correspond to "gauge" symmetric currents in the case of bosonic string theory. In the case that there exist states with the conformal weight (0, 1) in the twisted sectors, the following states will represent massless "gauge" bosons,

$$
\alpha_{L-1}^a|0>_L \otimes|T>_R, \tag{3.34}
$$

where a is an uncompactified space-time index and $| T >_R$ is a twisted state with the conformal weight $\bar{h}_T = 1$ created from the twisted vacuum $|\Lambda^{(l)} >_R$ and oscillators in the twisted sectors. Starting with a torus model with gauge symmetry G, we will expect that the resulting $\mathbf{Z}_N$-orbifold model will possess the $\mathbf{Z}_N$-invariant gauge symmetry $G/\mathbf{Z}_N$. In fact, this is true for the symmetric orbifold models. However, the above extra (0, 1) currents appear in the asymmetric orbifold models and unexpected symmetries may be realized. Here we repeat that these (0, 1) states will never appear in the case of the symmetric orbifold models. These (0, 1) or (1, 0) states are constructed by (0, 1) or (1, 0) currents and these currents are called "twist-untwist intertwining currents". The twist-untwist intertwining currents convert untwisted string states to twisted ones and vice versa and "enhance" the gauge symmetry of the total Hilbert space.

Let us consider the condition for the twist-untwist intertwining currents to exist. In general, the conformal weights of the ground states in the twisted sectors $E_0^{(l)} \equiv h_l$ depend on the order N and the dimension D of the orbifolds. We have to take care in the case that N is not prime. The weight h_l is given by the formula:

$$h_l = \frac{D}{\varphi(N_l)} \frac{1}{4} \sum_{\substack{(m,N_l)=1 \\ m=1,\ldots,N_l-1}} \frac{m}{N_l}\left(1 - \frac{m}{N_l}\right), \tag{3.35}$$

where N_l is the minimum positive integer which satisfies $(g^l)^{N_l} = \mathbf{1}$ and $\varphi(N_l)$ is the number of the primitive N_lth roots of unity among $\omega^m, (m = 1, ..., N-1)$, where $\omega = e^{2\pi i/N}$ and m is mutually prime to N. Since we have assumed that the rotation matrices $\theta^l, (l = 1, ..., N-1)$ have no fixed direction, $\omega^{lr_i}, (l = 1, ..., N-1)$ are not equal to unity. Therefore we find that each primitive Nth root of unity appears $D/\varphi(N)$ times. This implies that the dimension of the orbifolds has to be a multiple of $\varphi(N)$, i.e.

$$D = 0 \mod \varphi(N). \tag{3.36}$$

Together with the condition for modular invariance eq.(3.25), we get a condition for the dimension D, which gives consistent asymmetric orbifold models,

$$\begin{aligned} D &= 0 \mod \frac{\varphi(N_l)}{N_l h'_l} \quad \text{for } l = 1, ..., N-1, \\ h'_l &= \frac{1}{4} \sum_{\substack{(m,N_l)=1 \\ m=1,\ldots,N_l-1}} \frac{m}{N_l}\left(1 - \frac{m}{N_l}\right). \end{aligned} \tag{3.37}$$

Since we are interested in the models which possess (0, 1) states in the twisted sectors, models are severely restricted. We give a list of such models in table 1. In the second column, we listed Euler functions. The third column is devoted to a list of allowed dimensions of orbifolds. In the forth column, a list of the dimensions of orbifold models which might possess physical (0, 1) states in the twisted sectors. This list is shown up to $N = 30$ for our purpose of considering the Lie lattice.

3.3 TORUS-ORBIFOLD EQUIVALENCE

In order to analyze the symmetries of asymmetric orbifold models with twist-untwist intertwining currents, we will use a trick called "torus-orbifold equivalence"[31]. Any closed bosonic string theory compactified on a $\mathbf{Z}_N$-orbifold is equivalent to a closed bosonic string theory on a torus if the dimension of the orbifold is equal to the rank of a gauge symmetry of strings in each of the untwisted and twisted sectors of the orbifold model. We have discussed such models in this paper.

Let us start with a D-dimensional torus model associated with a semi-simple simply-laced Lie algebra G. Using the string coordinates $X_L^i(z)$ and $X_R^i(\bar{z})$ (i=1,..., D), we can construct an affine Kač-Moody algebra $\hat{G} \oplus \hat{G}$[32],

$$\begin{aligned} P_L^i(z) &\equiv i\partial_z X_L^i(z), \\ V_L(\alpha; z) &\equiv \, :\exp\{i\alpha \cdot X_L(z)\}:, \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} P_R^i(\bar{z}) &\equiv i\partial_{\bar{z}} X_R^i(\bar{z}), \\ V_R(\alpha; \bar{z}) &\equiv \, :\exp\{i\alpha \cdot X_R(\bar{z})\}:, \end{aligned} \tag{3.39}$$

where α is a root vector of G and its squared length is normalized to two. An orbifold model is given by modding out the torus with its automorphisms. Since the physical operators have to be invariant under the $\mathbf{Z}_N$-transformation, the remaining symmetry should become the $\mathbf{Z}_N$-invariant subgroup G_0 of G. If the rank of G_0 equals to D, we can always construct $\mathbf{Z}_N$-invariant operators, ${P'}_L^i(z)$ and ${P'}_R^i(\bar{z})$ by taking suitable linear combinations of $P_L^i(z)$, $V_L(\alpha; z)$ and $P_R^i(\bar{z})$, $V_R(\alpha; \bar{z})$, respectively. In this basis, a vertex operator transforms under the $\mathbf{Z}_N$- transformation as follows:

$$gV'(k_L^i, k_R^i; z)g^{-1} = e^{i2\pi(k_L\cdot v_L - k_R\cdot v_R)}V'(k_L^i, k_R^i; z), \tag{3.40}$$

for some constant vector (v_L^i, v_R^i). $k_L(k_R)$ represents eigenvalues of the momentum operator $\hat{p}'_L(\hat{p}'_R)$ in the new basis. Thus, we can expect that g is given by the form,

$$g = \rho_{(l)}\exp\{i2\pi(\hat{p}'_L \cdot v_L - \hat{p}'_R \cdot v_R)\}, \tag{3.41}$$

where $\rho_{(l)}$is a constant phase with $(\rho_{(l)})^N = 1$. Under this transformation, the string coordinates in the new basis transforms as

$$g(X_L'^i(z), X_R'^i(\bar{z}))g^{-1} = (X_L'^i(z) + 2\pi v_L^i, X_R'^i(\bar{z}) - 2\pi v_R^i). \tag{3.42}$$

This implies that the string coordinate $(X_L'^i(z), X_R'^i(\bar{z}))$ in the g^l-sector obeys the following boundary condition:

$$(X_L'^i(e^{2\pi i}z), X_R'^i(e^{-2\pi i}\bar{z})) = (X_L'^i(z) + 2\pi l v_L^i, X_R'^i(\bar{z}) - 2\pi l v_R^i) + (\text{torus shift}), \tag{3.43}$$

and hence the momentum eigenvalues in the g^l-sector will be included in the following lattice,

$$(p_L'^i, p_R'^i) \in \Gamma^{D,D} + l(v_L^i, v_R^i). \tag{3.44}$$

In the new basis, $g_{(l)}^l$ in the g^l-sector will be given by

$$g_{(l)}^l = e^{i2\pi(L_0' - \bar{L}_0')}, \tag{3.45}$$

where

$$\begin{aligned} L_0' &= \sum_{I=1}^{D}\{\frac{1}{2}(p_L'^I)^2 + \sum_{n=1}^{\infty}\alpha_{L-n}'^I\alpha_{Ln}'^I\}, \\ \bar{L}_0' &= \sum_{I=1}^{D}\{\frac{1}{2}(p_R'^I)^2 + \sum_{n=1}^{\infty}\alpha_{R-n}'^I\alpha_{Rn}'^I\}, \end{aligned} \tag{3.46}$$

and

$$\begin{aligned} \left[\alpha'^i_{Lm}, \alpha'^j_{Ln}\right] &= m\delta^{ij}\delta_{m+n,0} = \left[\alpha'^i_{Rm}, \alpha'^j_{Rn}\right], \\ \left[\alpha'^i_{Lm}, \alpha'^j_{Rn}\right] &= 0. \end{aligned} \tag{3.47}$$

Comparing eq.(3.41) with eq.(3.45), we can get the form of the phase factor $\rho_{(l)}$,

$$\rho_{(l)} = \exp\{-i\pi l\left((v_L^i)^2 - (v_R^i)^2\right)\}. \tag{3.48}$$

This phase factor plays an important role in taking out the physical states. Since $g^N = 1$, the shift vector (v_L^i, v_R^i) must satisfy the following conditions,

$$\begin{aligned} N\left((v_L^i)^2 - (v_R^i)^2\right) &= 0 \mod 2, \\ N(v_L^i, v_R^i) &\in \Gamma^{D,D}. \end{aligned} \tag{3.49}$$

We require that every physical state should be invariant under the $\mathbf{Z}_N$-transformation. Then the operator g has to act as an identity for the physical states. Thus the allowed momentum eigenvalues $(p_L'^i, p_R'^i)$ of the physical states in the g^l-sector are restricted to

$$(p_L'^i, p_R'^i) \in \Gamma^{D,D} + l(v_L^i, v_R^i) \quad \text{with } p_L' \cdot v_L - p_R' \cdot v_R - \frac{1}{2}l((v_L^i)^2 - (v_R^i)^2) = 0 \mod 1. \tag{3.50}$$

The total physical Hilbert space $\mathcal{H}$ of the $\mathbf{Z}_N$-orbifold model is given as a direct sum of the physical spaces $\mathcal{H}_{(l)}$ of all the sectors:

$$\mathcal{H} = \mathcal{H}_{(0)} \oplus \mathcal{H}_{(1)} \oplus \cdots \oplus \mathcal{H}_{(N-1)}. \tag{3.51}$$

The above considerations imply that the total physical Hilbert space of the $\mathbf{Z}_N$-orbifold model is equivalent to that of the torus model associated with the lattice $\Gamma'^{D,D}$,

$$\Gamma'^{D,D} = \{(p'^i_L, p'^i_R) \in \bigcup_{l=0}^{N-1} \left[\Gamma^{D,D} + l(v_L^i, v_R^i)\right] \mid p_L' \cdot v_L - p_R' \cdot v_R - \frac{1}{2}l\left((v_L^i)^2 - (v_R^i)^2\right) \in \mathbf{Z}\}. \tag{3.52}$$

Note that $\Gamma'^{D,D}$ is a Lorentzian even self-dual lattice if $\Gamma^{D,D}$ is so.

With this method, we can easily analyze the symmetries of the orbifold models. In the chiral asymmetric case, all we have to do is to set $v_L = 0$. Then the above lattice is reduced to the following one,

$$\Gamma'^{D,D} = \{(p'^i_L, p'^i_R) \in \bigcup_{l=0}^{N-1} \left[\Gamma^{D,D} + l(0, v_R^i)\right] \mid p_R' \cdot v_R - \frac{1}{2}l(v_R^i)^2 \in \mathbf{Z}\}. \tag{3.53}$$

where the shift vector v_R^i must satisfy

$$\begin{aligned} Nv_R^i &\in \Lambda, \\ lv_R^i &\notin \Lambda \quad (l = 1, ..., N-1). \end{aligned} \tag{3.54}$$

The shift vector can always be chosen to satisfy the following condition:

$$\frac{1}{2}(v_L^i)^2 = h_1, \tag{3.55}$$

where h_1 is the conformal weight of the ground state of the right-moving strings twisted by g and is determined by eq.(3.35). We investigate the massless spectrum of these models. The results are summarized in table 2–6. The G_0 denotes the $\mathbf{Z}_N$-invariant subalgebra of G and is the symmetry of each right-moving Hilbert space. Note that the symmetry of the untwisted left-moving Hilbert space remains to be G. The G' denotes the symmetry of the full Hilbert space. The twist-untwist intertwining currents appear in the twisted sectors and build the physical (0, 1) states in these models. We can construct an adjoint representation of G' by the $\mathbf{Z}_N$-invariant operators which belong to G_0, together with the intertwining currents.

4 Heterotic strings on asymmetric orbifolds

Heterotic string consists of a left-moving 26-dimensional bosonic string and a right-moving 10-dimensional superstring. In order to obtain a 4d string theory, 22 left-moving and 6 right-moving string coordinates have to be compactified on a compact space such as torus, orbifold or some manifolds. In this section, we will consider asymmetric orbifold compactification of 10-dimensional $E_8 \times E_8$ heterotic string. It seems natural to study the asymmetric orbifold models because of the asymmetric nature of the heterotic string theory. Then the left-moving sixteen bosonic coordinates out of 26 ones are compactified on a 16-dimensional torus and the momenta of the extra bosons take discrete values because of the periodicity of the center-of-mass coordinates. They span a $E_8 \times E_8$-lattice $\Gamma_L^{E_8 \times E_8}$. Since this theory is formulated in 10 dimensions, the extra 6-dimensional space has to be compactified. We will analyze the supersymmetry of $E_8 \times E_8$ heterotic string theory whose extra 6-dimensional space is compactified on asymmetric orbifolds.

4.1 HETEROTIC STRING AND ASYMMETRIC TWISTS

We start by describing the ingredients of the heterotic string in the light-cone gauge. The heterotic string consists of eight left-moving bosonic fields $X_L^i(\sigma + \tau), (i = 1, ..., 8)$ and extra sixteen fields $X_L^I(\sigma + \tau), (I = 1, ..., 16)$. Right-movers include eight bosonic fields $X_R^i(\sigma - \tau), (i = 1, ..., 8)$ and Neveu-Schwarz-Ramond (NSR) fermions $\lambda^i(\sigma - \tau)$. We will introduce bosonized fields $H^t(t = 1, ..., 4)$ instead of the NSR fermions, whose momenta lie on the weight lattice of $SO(8)$. Then the NS sector corresponds to vectorial weights and the R sector corresponds to spinorial weights of $SO(8)$, respectively.

Orbifolds $\mathbf{O}$ are defined as quotient spaces of tori $\mathbf{T}$ by the action of discrete rotations $\{\theta\}$. Now we will consider the following models,

$$\mathbf{O} = \frac{\mathbf{T}^6_{L+R}}{\mathbf{Z}_N} \times \mathbf{T}_L^{E_8 \times E_8}, \tag{4.1}$$

where $\mathbf{Z}_N$ is a discrete group and acts as isometries of the 6-dimensional lattice with order N; $\mathbf{Z}_N = \{\theta^l, l = 0, ..., N-1\}$ and $\theta^N = \mathbf{1}$. In order to specify an orbifold, we will choose a torus $\mathbf{T}^6_{L+R}$ and rotations $\{\theta\}$. It is convenient to introduce complex coordinates $Y^i = \frac{1}{\sqrt{2}}(X^{2i-1} + iX^{2i})$, $\bar{Y}^i = \frac{1}{\sqrt{2}}(X^{2i-1} - iX^{2i})$, (i=1,...,3). In this basis, the $\mathbf{Z}_N$-rotations are generated by

$$\theta = \exp[2\pi i(v^1 J^{12} + v^2 J^{34} + v^3 J^{56})], \tag{4.2}$$

where J^{ij} are the $SO(6)$ Cartan generators and v^i satisfy $Nv^i \in \mathbf{Z}$ and the eigenvalues can be diagonalized to the form $e^{2\pi i v^i}$. When we divide the left- and right-movers in the same manner, we obtain symmetric orbifold models. If this is not the case, we will obtain asymmetric orbifold models.

We will investigate the following asymmetric $\mathbf{Z}_N$-transformation denoted by g:

$$\begin{aligned} g: \quad (X_L^i, X_R^i) &\rightarrow (\theta_L^{ij} X_L^j, \theta_R^{ij} X_R^j), \quad i,j = 1,...,6, \\ & \qquad \text{with} \quad \theta_L^{M_L} = \mathbf{1},\ \theta_R^{M_R} = \mathbf{1} \\ X_L^I &\rightarrow X_L^I, \qquad\qquad I = 1,...,16, \\ p^t &\rightarrow p^t + v^t, \qquad\quad t = 1,2,3, \end{aligned} \tag{4.3}$$

where p^t are momenta of the bosonized fields H^t and then the $SO(8)$ weight lattice is divided by the shifts v^t. The remaining fields $X_{L,R}^i (i = 7,8)$ and H^4 are uncompactified coordinates in transverse directions. Note that N is the least common multiple of M_L and M_R and v^t is a $\mathbf{Z}_N$-shift.

In order to construct the models explicitly, we will restrict ourselves to the defining tori $\mathbf{T}^6_{L+R}$ such that they are spanned by root or weight vectors associated with semi-simple simply-laced Lie algebras G with rank 6. Let us consider the (6+6)-dimensional momentum lattice $\Gamma_G^{6,6}$. To ensure modular invariance of the one-loop partition function, the lattice has to be a Lorentzian even self-dual lattice[25],

$$\Gamma_G^{6,6} = \{(p_L^i, p_R^i) |\, p_L^i, p_R^i \in \Lambda_W(G) \quad \text{and} \quad p_L^i - p_R^i \in \Lambda_R(G)\}, \tag{4.4}$$

where $\Lambda_W(G)$ ($\Lambda_R(G)$) is the weight (root) lattice of G and the root vectors are normalized such that their squared length is two. G may be given by all the possible combinations of the following groups allowing the multiplicity with total rank 6: $\{SU(2), SU(3), SU(4), SU(5), SU(6), SU(7), SO(8), SO(10), SO(12), E_6\}$. Then these choices of the lattices limit the twists $\{\theta\}$ because they have to be automorphisms of the lattices.We select the models such that the automorphisms are inner ones. Moreover, throughout this paper, we restrict our attention to the $\mathbf{Z}_N$-orbifold models whose

$\mathbf{Z}_N$-twists θ^l ($l = 1, ..., N-1$) leave only the origin fixed. These allowed inner automorphisms are classified in ref.[29]. Under these restrictions, we find that there are four candidates,i.e.,

$$
\begin{aligned}
\frac{\mathbf{T}_{SU(3)^3}}{\mathbf{Z}_3} \quad &, \quad (v^1, v^2, v^3) = (\frac{1}{3}, \frac{1}{3}, \frac{2}{3}), \\
\frac{\mathbf{T}_{E_6}}{\mathbf{Z}_3} \quad &, \quad (v^1, v^2, v^3) = (\frac{1}{3}, \frac{1}{3}, \frac{2}{3}), \\
\frac{\mathbf{T}_{SU(7)}}{\mathbf{Z}_7} \quad &, \quad (v^1, v^2, v^3) = (\frac{1}{7}, \frac{2}{7}, \frac{3}{7}), \\
\frac{\mathbf{T}_{E_6}}{\mathbf{Z}_9} \quad &, \quad (v^1, v^2, v^3) = (\frac{1}{9}, \frac{2}{9}, \frac{5}{9}).
\end{aligned}
\tag{4.5}
$$

Thus we will consider $\mathbf{Z}_3$- $\mathbf{Z}_7$- and $\mathbf{Z}_9$-asymmetric orbifold models. We will analyze the space-time supersymmetry of the asymmetric orbifold models with these four twists.

4.2 HETEROTIC STRING ON CHIRAL ASYMMETRIC ORBIFOLDS

Let us consider the heterotic strings on chiral asymmetric orbifold models whose twists act as follows:

$$
\begin{aligned}
g: \quad (X_L^i, X_R^i) &\rightarrow (X_L^i, \theta^{ij} X_R^j), \\
X_L^I &\rightarrow X_L^I, \\
p^t &\rightarrow p^t + v^t,
\end{aligned}
\tag{4.6}
$$

where θ is a $\mathbf{Z}_N$-rotation matrix and v^t a $\mathbf{Z}_N$- shift, and they have to satisfy the conditions,

$$\theta^N = \mathbf{1}, \quad N v^t \in \mathbf{Z} \quad \text{with} \quad N \sum_{t=1}^{4} v^t = 0 \mod 2. \tag{4.7}$$

This transformation implies that the left-moving bosonic string is on a torus and the right-moving superstring is on an orbifold.

If 10-dimensional heterotic string is compactified on a 6-dimensional torus, the resulting 4-dimensional theory possesses N=4 space-time supersymmetry. In orbifold models, the supersymmetry may be broken by the twists. In the case of symmetric orbifold models, it is known that for the preservation of 4d N=1 space-time supersymmetry, v^i must satisfy the condition,

$$\pm v^1 \pm v^2 \pm v^3 = 0, \tag{4.8}$$

for some choice of signs[12]. Therefore these four models (4.5) will have N=1, 1, 1 and 0 space-time supersymmetry, respectively, if they are symmetric orbifold models. However, for asymmetric orbifold models, the above condition for the preservation of 4d N=1 space-time supersymmetry is too restrictive. In ref.[19], we investigated the supersymmetry of the $E_8 \times E_8$ heterotic string theories on asymmetric orbifolds of "chiral" type ; $\mathbf{Z}_N : (X_L, X_R) \to (X_L, \theta X_R)$ and found that the four models (4.5) possess N=2, 4, 4 and 4 space-time supersymmetry, respectively. We will show the details of the supersymmetry "enhancement" in the case of the asymmetric $Z_9(E_6)$-orbifold model.

We will consider the $Z_9(E_6)$-asymmetric orbifold model as an illustrative example. In this model, the momenta lie on the following 6-dimensional lattice:

$$\Gamma^{6,6}_{E_6} = \{(p^i_L, p^i_R) | \, p^i_L, p^i_R \in \Lambda_W(E_6) \quad \text{and} \quad p^i_L - p^i_R \in \Lambda_R(E_6)\}, \tag{4.9}$$

where the squared length of the root vectors of E_6 is normalized to two. In the complex basis, the rotation matrices can always be diagonalized as

$$\theta = \text{diag}[\exp 2\pi i(\zeta^{(1)}_1, \zeta^{(1)}_2, \zeta^{(1)}_3)]. \tag{4.10}$$

This asymmetric Z_9-orbifold model has components

$$\zeta^{(1)} = (\frac{1}{9}, \frac{2}{9}, \frac{5}{9}). \tag{4.11}$$

Note that the gauge sector does not have any effects by the right-moving twists in this model. Therefore, the "gauge" symmetry is still unbroken,i.e. $E_6 \times E_8 \times E_8$. However, supersymmetry may be broken. In order to see that, we have to analyze the massless fermionic states.

Before analyzing these states, we will consider the one-loop partition function of the θ^l-twisted sector twisted by g^m in order to confirm the consistency of this model. As in the bosonic case, the necessary and sufficient condition for modular invariance are given by the level-matching.

$$\frac{N}{N_l}(E_0 - \bar{E}_0) = 0 \mod 1, \tag{4.12}$$

where E_0 ($\bar{E}_0$) is an eigenvalue of the left-(right-) Hamiltonian,i.e. energy. For our purpose, it is sufficient to check the level-matching for the massless states which concern the symmetries. The mass formulae for g^l-twisted sector (l=0, for untwisted sector) are given by

$$\frac{1}{8}(m_L^{(l)})^2 = \frac{1}{2}\sum_{i=1}^{6}(p_L^i)^2 + \frac{1}{2}\sum_{I=1}^{16}(p_L^I)^2 + N_L - 1, \qquad (4.13)$$

$$\frac{1}{8}(m_R^{(l)})^2 = \frac{1}{2}\sum_{i=1}^{6}(p_R^i)^2\delta_{l,0} + \frac{1}{2}\sum_{t=1}^{4}(p^t + lv^t)^2 + N_R^{(l)} + E_{R0}^{(l)} - \frac{1}{2}, \qquad (4.14)$$

where N_L and $N_R^{(l)}$ are the number operators and $E_{R0}^{(l)}$ is the contribution from the twisted oscillators to the zero-point energy. They are given by

$$N_L = \sum_{i=1}^{6}\sum_{n=1}^{\infty}\alpha_{L-n}^i\alpha_{Ln}^i + \sum_{a=1}^{2}\sum_{n=1}^{\infty}\alpha_{L-n}^a\alpha_{Ln}^a, \qquad (4.15)$$

$$N_R^{(l)} = \sum_{i=1}^{6}\sum_{n_i\in Z+r_i/N>0}\bar{\gamma}_{n_i}^{i\dagger}\bar{\gamma}_{n_i}^i + \sum_{a=1}^{2}\sum_{n=1}^{\infty}\alpha_{R-n}^a\alpha_{Rn}^a, \qquad (4.16)$$

$$E_{R0}^{(l)} = \frac{1}{2}\sum_{i=1}^{3}\zeta_i^{(l)}(1-\zeta_i^{(l)}), \quad \zeta_i^{(l)} = l\zeta_i^{(1)} \mod 1, \quad 0 < \zeta_i^{(l)} < 1. \qquad (4.17)$$

If we solve the condition

$$E_0 = \frac{1}{8}(m_L^{(l)})^2 = 0, \qquad \bar{E}_0 = \frac{1}{8}(m_R^{(l)})^2 = 0, \qquad (4.18)$$

we get candidates of the physical massless states. In the untwisted sector, all the massless fermionic states cannot survive under the generalized GSO projection introduced for the sake of modular invariance. Since the projection operator has a form of $\mathcal{P} = \exp 2\pi i(\sum_{t=1}^{4}\hat{p}^t v^t)$ with $v^4 = 0$, where $\wedge$ denotes the operators, we find that the physical massless states in the untwisted sector are obtained as follows:

$$|p^t = (0,0,0,\pm1)>_R^{unt} \otimes \alpha_{L-1}^a|0>_L, \quad a = 1,2, \qquad (4.19)$$

where a is a uncompactified space-time index in the light-cone gauge. These states correspond to graviton, antisymmetric background field and a scalar field, and

$$|p^t = (0,0,0,\pm1)>_R^{unt} \otimes \begin{cases} \alpha_{L-1}^i|0>_L \\ |(p_L^i)^2 = 2>_L \\ \alpha_{L-1}^I|0>_L \\ |(p_L^I)^2 = 2>_L \end{cases} \qquad (4.20)$$

correspond to $E_6 \times E_8 \times E_8$ Yang-Mills fields. It should be emphasized that there are no fermionic states in this sector. This fact is coincident with the expectation from the condition (4.8) in the case of symmetric orbifold models. In the symmetric case, this result of non-supersymmetry reflects the symmetry of the total Hilbert space including the twisted and untwisted sectors. Supersymmetric orbifold models should have their supermultiplets in each sector. However, in the chiral asymmetric models, unexpected conserved currents appear in the twisted sectors. We will investigate the massless states in the twisted sectors as in the untwisted sectors. In this case, the contributions from the twisted oscillators to the zero-point energy are given by $E_{R0}^{(l)} = \frac{7}{27}$ ($l \neq 3,6$) and $E_{R0}^{(l)} = \frac{1}{3}$ ($l = 3,6$). From the mass formulae (4.13) and (4.14) there may exist (0, 1) physical states in these twisted sectors and hence symmetry "enhancement" may occur. After analyzing the one-loop partition function, we have found that the following states are physical and massless in the twisted sectors:

g-sector:

$$\left.\begin{array}{l} \gamma^{i\dagger}_{R-\frac{1}{9}}|p^t = (0,0,-1,0) >^g_R \\ |p^t = (-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}) >^g_R \end{array}\right\} \otimes \alpha^a_{L-1}|0>_L, \quad a = 1,2 \tag{4.21}$$

$$\left.\begin{array}{l} \gamma^{i\dagger}_{R-\frac{1}{9}}|p^t = (0,0,-1,0) >^g_R \\ |p^t = (-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}) >^g_R \end{array}\right\} \otimes \left\{\begin{array}{l} \alpha^i_{L-1}|0>_L \\ |(p^i_L)^2 = 2>_L \\ \alpha^I_{L-1}|0>_L \\ |(p^I_L)^2 = 2>_L \end{array}\right. \tag{4.22}$$

g^2-sector:

$$\left.\begin{array}{l} \gamma^{i\dagger}_{R-\frac{1}{9}}|p^t = (0,0,-1,0) >^{g^2}_R \\ |p^t = (-\frac{1}{2},-\frac{1}{2},-\frac{3}{2},\frac{1}{2}) >^{g^2}_R \end{array}\right\} \otimes \alpha^a_{L-1}|0>_L, \quad a = 1,2 \tag{4.23}$$

$$\left.\begin{array}{l} \gamma^{i\dagger}_{R-\frac{1}{9}}|p^t = (0,0,-1,0) >^{g^2}_R \\ |p^t = (-\frac{1}{2},-\frac{1}{2},-\frac{3}{2},\frac{1}{2}) >^{g^2}_R \end{array}\right\} \otimes \left\{\begin{array}{l} \alpha^i_{L-1}|0>_L \\ |(p^i_L)^2 = 2>_L \\ \alpha^I_{L-1}|0>_L \\ |(p^I_L)^2 = 2>_L \end{array}\right. \tag{4.24}$$

g^3-sector:

$$|p^t = (-\frac{1}{2},-\frac{1}{2},-\frac{3}{2},\frac{1}{2}) >^{g^3}_R \otimes \alpha^a_{L-1}|0>_L, \quad a = 1,2 \tag{4.25}$$

$$|p^t = (-\frac{1}{2},-\frac{1}{2},-\frac{3}{2},\frac{1}{2}) >^{g^3}_R \otimes \left\{\begin{array}{l} \alpha^i_{L-1}|0>_L \\ |(p^i_L)^2 = 2>_L \\ \alpha^I_{L-1}|0>_L \\ |(p^I_L)^2 = 2>_L \end{array}\right. \tag{4.26}$$

g^4-sector:

$$\left.\begin{array}{l} \gamma^{i\dagger}_{R-\frac{1}{9}}|p^t=(0,-1,-2,0)>^{g^4}_R \\ |p^t=(-\frac{1}{2},-\frac{1}{2},-\frac{5}{2},-\frac{1}{2})>^{g^4}_R \end{array}\right\} \otimes \alpha^a_{L-1}|0>_L, \quad a=1,2 \tag{4.27}$$

$$\left.\begin{array}{l} \gamma^{i\dagger}_{R-\frac{1}{9}}|p^t=(0,-1,-2,0)>^{g^4}_R \\ |p^t=(-\frac{1}{2},-\frac{1}{2},-\frac{5}{2},-\frac{1}{2})>^{g^4}_R \end{array}\right\} \otimes \left\{\begin{array}{l} \alpha^i_{L-1}|0>_L \\ |(p^i_L)^2=2>_L \\ \alpha^I_{L-1}|0>_L \\ |(p^I_L)^2=2>_L \end{array}\right. \tag{4.28}$$

g^5-sector:

$$\left.\begin{array}{l} \gamma^{i\dagger}_{R-\frac{1}{9}}|p^t=(-1,-1,-3,0)>^{g^5}_R \\ |p^t=(-\frac{1}{2},-\frac{3}{2},-\frac{5}{2},\frac{1}{2})>^{g^5}_R \end{array}\right\} \otimes \alpha^a_{L-1}|0>_L, \quad a=1,2 \tag{4.29}$$

$$\left.\begin{array}{l} \gamma^{i\dagger}_{R-\frac{1}{9}}|p^t=(-1,-1,-3,0)>^{g^5}_R \\ |p^t=(-\frac{1}{2},-\frac{3}{2},-\frac{5}{2},\frac{1}{2})>^{g^5}_R \end{array}\right\} \otimes \left\{\begin{array}{l} \alpha^i_{L-1}|0>_L \\ |(p^i_L)^2=2>_L \\ \alpha^I_{L-1}|0>_L \\ |(p^I_L)^2=2>_L \end{array}\right. \tag{4.30}$$

g^6-sector:

$$|p^t=(-\frac{1}{2},-\frac{3}{2},-\frac{7}{2},-\frac{1}{2})>^{g^6}_R \otimes \alpha^a_{L-1}|0>_L, \quad a=1,2 \tag{4.31}$$

$$|p^t=(-\frac{1}{2},-\frac{3}{2},-\frac{7}{2},-\frac{1}{2})>^{g^6}_R \otimes \left\{\begin{array}{l} \alpha^i_{L-1}|0>_L \\ |(p^i_L)^2=2>_L \\ \alpha^I_{L-1}|0>_L \\ |(p^I_L)^2=2>_L \end{array}\right. \tag{4.32}$$

g^7-sector:

$$\left.\begin{array}{l} \gamma^{i\dagger}_{R-\frac{1}{9}}|p^t=(-1,-2,-4,0)>^{g^7}_R \\ |p^t=(-\frac{1}{2},-\frac{3}{2},-\frac{7}{2},-\frac{1}{2})>^{g^7}_R \end{array}\right\} \otimes \alpha^a_{L-1}|0>_L, \quad a=1,2 \tag{4.33}$$

$$\left.\begin{array}{l} \gamma^{i\dagger}_{R-\frac{1}{9}}|p^t=(-1,-2,-4,0)>^{g^7}_R \\ |p^t=(-\frac{1}{2},-\frac{3}{2},-\frac{7}{2},-\frac{1}{2})>^{g^7}_R \end{array}\right\} \otimes \left\{\begin{array}{l} \alpha^i_{L-1}|0>_L \\ |(p^i_L)^2=2>_L \\ \alpha^I_{L-1}|0>_L \\ |(p^I_L)^2=2>_L \end{array}\right. \tag{4.34}$$

g^8-sector:

$$\left.\begin{array}{l} \gamma^{i\dagger}_{R-\frac{1}{9}}|p^t=(-1,-2,-4,0)>^{g^8}_R \\ |p^t=(-\frac{1}{2},-\frac{3}{2},-\frac{9}{2},\frac{1}{2})>^{g^8}_R \end{array}\right\} \otimes \alpha^a_{L-1}|0>_L, \quad a=1,2 \tag{4.35}$$

$$\left.\begin{array}{l} \gamma^{i\dagger}_{R-\frac{1}{9}}|p^t=(-1,-2,-4,0)>^{g^8}_R \\ |p^t=(-\frac{1}{2},-\frac{3}{2},-\frac{9}{2},\frac{1}{2})>^{g^8}_R \end{array}\right\} \otimes \left\{\begin{array}{l} \alpha^i_{L-1}|0>_L \\ |(p^i_L)^2=2>_L \\ \alpha^I_{L-1}|0>_L \\ |(p^I_L)^2=2>_L \end{array}\right. \tag{4.36}$$

Here we have to mention that the degeneracy of the ground states in the g^l-twisted sectors is given by

$$n = \frac{\sqrt{\det(1-\theta^l)}}{V_{\Lambda_R(E_6)}} = \begin{cases} 1 & l \neq 3,6 \\ 3 & l = 3,6 \end{cases} \tag{4.37}$$

where $V_{\Lambda_R(E_6)}$ is the volume of a unit cell of $\Lambda_R(E_6)$. In order to evaluate the physical states, a detailed analysis of the one-loop partition function is required. For $\mathbf{Z}_N$-orbifolds with non-prime N, a further non-trivial projection may appear.

$$Z(\tau)^{(3)} = \frac{1}{9}\sum_{m=0}^{8} Z(g^3, g^m; \tau), \tag{4.38}$$

$$Z(\tau)^{(6)} = \frac{1}{9}\sum_{m=0}^{8} Z(g^6, g^m; \tau). \tag{4.39}$$

It is easy to expand these partition functions in powers of $q = e^{2\pi i\tau}$ and $\bar{q} = e^{-2\pi i\bar{\tau}}$ and to show that one of the three degeneracy states survive.

In the asymmetric $\mathbf{Z}_9$-orbifold model, the shift vector $(v^1, v^2, v^3) = (\frac{1}{9}, \frac{2}{9}, \frac{5}{9})$ does not satisfy the condition (4.8) and in fact this shift breaks the supersymmetry in the untwisted sector. However, supercurrents appear from the twisted sectors because the ground state of the untwisted left-movers has eigenvalue zero with respect to the zero-mode of the Virasoro operator L_0 and $(0, 1)$ conserved currents exist. These currents play a role of "twist-untwist intertwining currents". The twist-untwist intertwining currents convert untwisted sector to twisted ones and vice versa, and possess a conformal weight $(1, 0)$ or $(0, 1)$ [20]. Owing to the existence of the currents, the fermionic currents become to the supercurrents of the total Hilbert space. Thus, together with the states (4.19)-(4.36), we can conclude that this model has N=4 supergravity and super Yang-Mills multiplet and realizes the N=4 supersymmetry.

It is easy to apply the same analysis to the other three models in (4.5). We get one N=2 supersymmetric model and three N=4 models. These models are interesting in itself but not realistic. In order to obtain more "realistic" models, we have to seek other constructions.

4.3 AN EXAMPLE OF N=1 SUSY MODEL

We attempt to construct 4d N=1 space-time supersymmetric models. Let us consider the E_6-model whose left-moving bosonic string is living on a Z_3-orbifold and the right-moving superstring on a $\mathbf{Z}_9$-orbifold. The E_6-lattice has order 3 and 9 inner automorphisms. Then the twists act as follows.

$$\begin{aligned} g: \quad (X_L^i, X_R) &\rightarrow (\theta_L^{ij} X_L^j, \theta_R^{ij} X_R^j), \quad \text{with} \quad \theta_L^3 = \mathbf{1},\ \theta_R^9 = \mathbf{1}, \\ X_L^I &\rightarrow X_L^I, \\ p^t &\rightarrow p^t + v^t, \quad 9v^t \in \mathbf{Z}. \end{aligned} \tag{4.40}$$

In the complex basis, the rotation matrices can always be diagonalized

$$\theta_L = \operatorname{diag}[\exp 2\pi i(\xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)})], \tag{4.41}$$

$$\theta_R = \operatorname{diag}[\exp 2\pi i(\zeta_1^{(1)}, \zeta_2^{(1)}, \zeta_3^{(1)})]. \tag{4.42}$$

This left-Z_3 and right-Z_9 model has components

$$\xi^{(1)} = (\frac{1}{3}, \frac{1}{3}, \frac{2}{3}), \qquad \zeta^{(1)} = (\frac{1}{9}, \frac{2}{9}, \frac{5}{9}). \tag{4.43}$$

Because of our choice of the twists, we can expect that (0, 1) states appear only from g^3- and g^6-twisted sectors and supercurrents appear from these sectors.

Let us consider the one-loop partition function of this model. It consists of the following parts, uncompactified space-time part, boson parts, NSR fermion part and zero-point energy part,

$$Z(\tau) = Z(\tau)_{\text{space-time}} Z(\tau)_{X_L^i, X_R^i} Z(\tau)_{X_L^I} Z(\tau)_{\text{NSR}} q^{E_{L0}} \bar{q}^{E_{R0}}, \tag{4.44}$$

where $E_{L0} = -1$ and $E_{R0} = -\frac{1}{2}$. The explicit form of orbifoldized parts in the untwisted sector is as follows:

$$Z(\tau)^{(0)} = \frac{1}{N} \sum_{m=1}^{N-1} Z(1, g^m; \tau), \tag{4.45}$$

$$Z(1, g^m; \tau) = Z(1, g^m; \tau)_{X_L^i, X_R^i} Z(1, g^m; \tau)_{X_L^I} Z(1, g^m; \tau)_{\text{NSR}}, \tag{4.46}$$

$m \neq 3, 6$

$$Z(1, g^m; \tau)_{X_L^i, X_R^i} = \prod_{i=1}^{3} \left[\frac{-2\sin(\pi \xi_i^{(m)}) \eta(\tau)}{\vartheta_1(\xi_i^{(m)}|\tau)} \right] \prod_{i=1}^{3} \left[\frac{-2\sin(\pi \zeta_i^{(m)}) \overline{\eta(\tau)}}{\vartheta_1^*(\zeta_i^{(m)}|\tau)} \right], \tag{4.47}$$

$m = 3, 6$

$$Z(1, g^m; \tau)_{X_L^i, X_R^i} = \frac{1}{\left(\overline{\eta(\tau)}\right)^6} \prod_{i=1}^{3} \left[\frac{-2\sin(\pi \zeta_i^{(m)}) \overline{\eta(\tau)}}{\vartheta_1^*(\zeta_i^{(m)}|\tau)} \right] \sum_{(p_L^i, p_R^i) \in \Gamma_{\text{inv}}^{D,D}} q^{\frac{1}{2}(p_L^i)^2}, \tag{4.48}$$

$$Z(1,g^m;\tau)_{X_L^I} = \frac{1}{(\eta(\tau))^8}\left[\vartheta_2^8(0|\tau)+\vartheta_4^8(0|\tau)+\vartheta_3^8(0|\tau)\right]^2. \tag{4.49}$$

$$Z(1,g^m;\tau)_{\rm NSR} = \frac{1}{\left(\overline{\eta(\tau)}\right)^4}\left[\prod_{t=1}^4 \vartheta_3^*(mv^t|\tau) - \prod_{t=1}^4 \vartheta_4^*(mv^t|\tau) - \prod_{t=1}^4 \vartheta_2^*(mv^t|\tau) - \prod_{t=1}^4 \vartheta_1^*(mv^t|\tau)\right]. \tag{4.50}$$

The twisted parts can be constructed by using modular transformation properties. As in the model described in the previous subsection, the level-matching condition eq.(4.12) is the necessary and sufficient condition for modular invariance. We analyze the level-matching for the massless states which concern the symmetries. The mass formulae for the g^l-twisted sector (l=0, for untwisted sector) are given by

$$\frac{1}{8}(m_L^{(l)})^2 = \begin{cases} \frac{1}{2}\sum_{i=1}^6 (p_L^i)^2 + \frac{1}{2}\sum_{I=1}^{16}(p_L^I)^2 + N_L^{(l)} - 1, & \text{for } l=0,3,6 \\ \frac{1}{2}\sum_{I=1}^{16}(p_L^I)^2 + N_L^{(l)} + E_{L0}^{(l)} - 1, & \text{for } l \neq 0,3,6 \end{cases} \tag{4.51}$$

$$\frac{1}{8}(m_R^{(l)})^2 = \frac{1}{2}\sum_{i=1}^6 (p_R^i)^2\delta_{l,0} + \frac{1}{2}\sum_{t=1}^4 (p^t+lv^t)^2 + N_R^{(l)} + E_{R0}^{(l)} - \frac{1}{2}, \tag{4.52}$$

where $N_L^{(l)}$ and $N_R^{(l)}$ are the number operators. $E_{L0}^{(l)}$ and $E_{R0}^{(l)}$ are the contributions from twisted oscillators to the the zero-point energy,

$$E_{L0}^{(l)} = \frac{1}{2}\sum_{i=1}^3 \xi_i^{(l)}(1-\xi_i^{(l)}), \quad \xi_i^{(l)} = l\xi_i^{(1)} \mod 1, \quad 0<\xi_i^{(l)}<1 \tag{4.53}$$

$$E_{R0}^{(l)} = \frac{1}{2}\sum_{i=1}^6 \zeta_i^{(l)}(1-\zeta_i^{(l)}), \quad \zeta_i^{(l)} = l\zeta_i^{(1)} \mod 1, \quad 0<\zeta_i^{(l)}<1. \tag{4.54}$$

Since the right-movers are on the $\mathbf{Z}_9$-orbifold, all massless fermionic fields will not survive under the generalized GSO projection. The remaining physical massless states are as follows:

$$|p^t=(0,0,0,\pm1)>_R^{unt} \otimes \alpha_{L-1}^a|0>_L, \quad a=1,2. \tag{4.55}$$

These states correspond to graviton, antisymmetric background field and a scalar field, and

$$|p^t=(0,0,0,\pm1)>_R^{unt} \otimes \begin{cases} \alpha_{L-1}^i|0>_L \\ |(p_L^i)^2=2>_L \\ \alpha_{L-1}^I|0>_L \\ |(p_L^I)^2=2>_L \end{cases} \tag{4.56}$$

correspond to $SU(3)^3 \times E_8 \times E_8$ Yang-Mills fields. Here, $SU(3)^3$ is a $\mathbf{Z}_3$-invariant subgroup of E_6 under the $\mathbf{Z}_3$-twist (4.42). Since there are no superpartners in the untwisted sector, we will search into the twisted sectors for them. Because of our choice of the model, the contributions from the twisted oscillators to the zero-point energy are given by $(E_{L0}^{(l)}, E_{R0}^{(l)}) = (\frac{1}{3}, \frac{7}{27})\,(l \neq 3,6)$, and we find that there are no massless fermionic states in these sectors since massless states cannot have space-time indices. However, for the g^3- and g^6-sectors, the orbifold becomes chiral type with $(E_{L0}^{(l)}, E_{R0}^{(l)}) = (0, \frac{1}{3}), (l = 3,6)$, and the massless states are given as follows:

g^3-sector:

$$|p^t = (-\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \frac{1}{2}) >_R^{g^3} \otimes \alpha_{L-1}^a |0>_L, \quad a = 1,2 \tag{4.57}$$

$$|p^t = (-\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \frac{1}{2}) >_R^{g^3} \otimes \begin{cases} \alpha_{L-1}^i |0>_L \\ |(p_L^i)^2 = 2>_L \\ \alpha_{L-1}^I |0>_L \\ |(p_L^I)^2 = 2>_L \end{cases} \tag{4.58}$$

g^6-sector:

$$|p^t = (-\frac{1}{2}, -\frac{3}{2}, -\frac{7}{2}, -\frac{1}{2}) >_R^{g^6} \otimes \alpha_L^a |0>_L, \quad a = 1,2 \tag{4.59}$$

$$|p^t = (-\frac{1}{2}, -\frac{3}{2}, -\frac{7}{2}, -\frac{1}{2}) >_R^{g^6} \otimes \begin{cases} \alpha_{L-1}^i |0>_L \\ |(p_L^i)^2 = 2>_L \\ \alpha_{L-1}^I |0>_L \\ |(p_L^I)^2 = 2>_L \end{cases} \tag{4.60}$$

The states (4.57) (4.59) are superpartners of graviton, antisymmetric background field and a scalar field (4.55). (4.58) (4.60) correspond to Yang-Mills fields (4.56). Usually, the supersymmetry is realized in each sector. In this model, note that each sector does not have N=1 super Yang-Mills and supergravity multiplets but has some parts. The existence of the twist-untwist intertwining currents which convert untwisted string states to twisted ones and vice versa make it possible to possess N=1 space-time supersymmetry in this model. The twist-untwist intertwining currents are constructed from fields with total conformal weight (1, 0) or (0, 1). We will soon discuss such currents that have a conformal weight (0, 1). By analyzing the massless spectrum explicitly, we can conclude that this model has N=1 supergravity and super Yang-Mills multiplet and realizes the N=1 space-time supersymmetry.

In this model, we have to mention that the degeneracy of the ground states in the g^3- and g^6-twisted sectors as in the chiral model discussed in the previous subsection. After the detailed analysis, we find that one of the three degeneracy states survive as in the chiral asymmetric $\mathbf{Z}_9$-orbifold model.

4.4 SUPERCURRENTS ON ORBIFOLDS

The supercurrents are constructed from fermion vertex operators[33]. In the covariant formalism, they are constructed by introducing a spin field $e^{-\phi/2}$ and five free scalar fields H^σ (σ=1,...,5) representing the NSR fermions through bosonization. The fermion vertex operators for the untwisted sector are given by

$$V_{-\frac{1}{2}} = e^{-\phi/2} e^{i\alpha_s H} e^{ikX}, \tag{4.61}$$

and for the g^l-twisted sector

$$V_{-\frac{1}{2}} = e^{-\phi/2} e^{i(\alpha_s + v^{(l)})H} e^{ikX} \Lambda, \tag{4.62}$$

where $e^{-\phi/2}$ is a spin field with conformal dimension $\frac{3}{8}$ for the (β, γ) superconformal ghost system. α_s are spinorial vectors of $SO(10)$, $\alpha_s = (\pm\frac{1}{2}, ..., \pm\frac{1}{2})$ with even number of + signs modulo root vectors and $v^{(l)} = (v_1^{(l)}, v_2^{(l)}, v_3^{(l)}, 0, 0)$, $v_i^{(l)} = l v_i$. The forth and fifth components correspond to transverse and longitudinal directions in the four dimensional space-time and k^μ is a four dimensional momentum. We have replaced $SO(8)$ by $SO(10)$ in order to follow the covariant formalism. Λ is a twist field which creates a twisted vacuum out of the untwisted one and is defined by

$$\Lambda(0)|0> = |\Lambda> . \tag{4.63}$$

The conformal dimension of the twist field is easily calculated by

$$h_\Lambda = \frac{1}{2} \sum_{i=1}^{3} \zeta_i^{(l)} (1 - \zeta_i^{(l)}). \tag{4.64}$$

The conformal weight of the currents is contributed additively. For the supercurrents in the untwisted sector, the spin field contributes by $\frac{5}{8}$ and the vertex operator with spinorial weights contributes by $\frac{3}{8}$ if they exist. As a result, we can get (1, 0) or (0, 1) conserved fermionic currents. However, in our model, there is no such spinorial fields because of the physical state condition. As in the untwisted sector, the supercurrents in the twisted sectors have a conformal weight $\frac{3}{8} + (\alpha_s + v^{(l)})^2/2 + h_\Lambda$ for θ^l-twisted sector. It is easy to confirm that the total conformal weight of the currents is 1 when we substitute the values of p^t, which we gave in eqs.(4.57), (4.58), (4.59) and (4.60), into $\alpha_s = (p^1, p^2, p^3, p^4, \pm\frac{1}{2})$ with appropriate sign and $v^{(l)}$ and h_Λ. Then the states (4.55) are transformed to the states (4.57) (4.59) and the states (4.56) are transformed to the states (4.58) (4.60) by the supercurrents respectively. If the conformal weight of the supercurrents is (0, 1) and these currents will play a role of twist-untwist intertwining currents and the total Hilbert space becomes to possess N=1 space-time supersymmetry. We will emphasize again that the supercurrents appear from the twisted sectors but not the untwisted sector and hence the massless fermionic matters are all in the twisted sectors.

It is convenient to rewrite the fermion vertex operators into the form that 4d space-time supersymmetry is manifest,

$$\begin{aligned} V^{\alpha}_{-\frac{1}{2}}(\bar{z}) &= e^{-\phi/2} S^{\alpha} \Sigma(\bar{z}), \\ V^{\dot{\alpha}}_{-\frac{1}{2}}(\bar{z}) &= e^{-\phi/2} S^{\dot{\alpha}} \Sigma^{\dagger}(\bar{z}), \end{aligned} \tag{4.65}$$

where S^α are the spin fields given as the exponentials of two free bosons $H^{1,2}$ with conformal dimension $\frac{1}{4}$ and $\Sigma(\Sigma^\dagger)$ are dimension $\frac{3}{8}$ fields constructed from exponential of three free bosons $H^{3,4,5}$ and Λ. The 4d super charges are given by $Q_\alpha = \oint d\bar{z} V^{\alpha}_{-1/2}(\bar{z})$, $Q_{\dot{\alpha}} = \oint d\bar{z} V^{\dot{\alpha}}_{-1/2}(\bar{z})$, where $\alpha = (\pm\frac{1}{2}, \pm\frac{1}{2}), \dot{\alpha} = (\pm\frac{1}{2}, \mp\frac{1}{2})$.

4.5 STANDARD EMBEDDING

Finally we will discuss gauge sector of this left-$\mathbf{Z}_3$ and right-$\mathbf{Z}_9$ asymmetric orbifold model. We consider standard embedding that the gauge twisting group acts on the gauge sector with the same action as on the six dimensional compactified dimensions. The gauge twisting group can be realized by a shift $V = (V^1, ..., V^8) \times (V'^1, ..., V'^8)$

and then standard embedding is given by setting $V = (v^1, v^2, v^3, 0, ..., 0) \times (0, ..., 0)$. In this model, if we assume that the left $\mathbf{Z}_3$-twist is embedded in the gauge sector, the gauge group will reduce to $E_6 \times SU(3) \times E_8$. Then the N=1 space-time supersymmetry is preserved. However, the gauge twists should be the same twists that act to the NSR fermions. In this case, the gauge group is broken to $SO(10) \times U(1)^3 \times E_8$ by the $\mathbf{Z}_9$-twist. Since the gauge twist contributes to the left mass spectrum, $p_L^I \to p_L^I + lV^I$ in eq.(4.51), the level matching condition tells us that the gauge breaking effect also breaks the space-time supersymmetry of this model above the GUT energy scale.

5 Conclusions

In the first half of this paper, we explained asymmetric $\mathbf{Z}_N$-orbifold models with physical (0, 1) states in the twisted sectors. These states correspond to twist-untwist intertwining currents which convert untwisted states to twisted ones and vice versa. As we mentioned in sect.3, the existence of such (0, 1) states restricts the models to asymmetric ones. Starting with the torus models with gauge symmetry G, we constructed $\mathbf{Z}_N$-asymmetric orbifold models. We investigated the gauge symmetry of the asymmetric orbifold models and found that the symmetry has become larger than we expected. In the latter half, we discussed $E_8 \times E_8$ heterotic string compactified on asymmetric orbifolds. Modular invariance and the conditions for the lattice and automorphisms severely restrict the models. We found four models, i.e. $\mathbf{T}_{SU(3)^3}/\mathbf{Z}_3$, $\mathbf{T}_{E_6}/\mathbf{Z}_3$, $\mathbf{T}_{SU(7)}/\mathbf{Z}_7$, $\mathbf{T}_{E_6}/\mathbf{Z}_9$, to be allowed. We considered the space-time supersymmetry of these models. Detailed analysis of massless states tells us that these four models (4.5) have N=1, 1, 1 and 0 space-time supersymmetry, respectively, if they are symmetric orbifold models. However, for chiral asymmetric orbifold models, $\mathbf{Z}_N : (X_L, X_R) \longrightarrow (X_L, \theta X_R)$, we found that the four models possess N=2, 4, 4 and 4 space-time supersymmetry, respectively. They imply that the condition for the preservation of N=1 space-time supersymmetry $\pm v^1 \pm v^2 \pm v^3 = 0$ for any choice of signs is too restrictive for the asymmetric orbifold models. Lastly, we gave a new 4d N=1 space-time supersymmetric model compactified on an asymmetric orbifold associated with the Lie algebra E_6. Here, we also restricted the models whose $\mathbf{Z}_N$-twists θ^l $(l = 1, ..., N-1)$ act as inner automorphisms and do not have fixed direction. Under these conditions, $\mathbf{Z}_3$- and $\mathbf{Z}_9$-twists are allowed for the E_6-model. We divided the bosonic left-moving string coordinates with the $\mathbf{Z}_3$-twist and the right-moving superstring coordinates with the $\mathbf{Z}_9$-twist. The $\mathbf{Z}_9$-twist does not satisfy the condition for the preservation of N=1 space-time supersymmetry. In fact this model has no fermionic massless state in the untwisted sector. However, supercurrents appear from g^3- and g^6-twisted sectors and N=1 space-time supersymmetry is realized in the total Hilbert space. Then the supercurrent is given by the form of eqs.(4.65) and plays a role of intertwiner between the untwisted and twisted sectors. It is remarkable that all massless matter fields appear from the twisted sectors.

There may be some other examples which possess 4d N=1 supersymmetry in the same way. It is possible to make the condition (ii) loose because it may be too restrictive. In this paper, we restricted our consideration to non-embedding and standard embedding models. Other approaches, such as non-standard embedding or Wilson-line mechanism, may give more realistic models.

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6 Appendix

In this appendix we give the shift vectors v_R^I introduced in rewriting the asymmetric $\mathbf{Z}_N$-orbifold models into the equivalent torus models. In the orthonormal basis, the root lattices Λ_R of the simple Lie algebras are spanned by

$$\begin{aligned}
SU(n+1) &: \quad e_i - e_j \quad (i \neq j,\ 1 \leq i \leq n+1,\ 1 \leq j \leq n+1), \\
SO(2n) &: \quad \pm e_i \pm e_j \quad (1 \leq i < j \leq n), \\
E_6 &: \quad \pm e_i \pm e_j \quad (1 \leq i < j \leq 5), \\
&\quad\ \pm\frac{1}{2}\left(e_8 - e_7 - e_6 + \sum_{i=1}^{5}(-1)^{\nu(i)}e_i\right) \quad \text{with} \quad \sum_{i=1}^{5}\nu(i) = \text{even}, \\
E_8 &: \quad \pm e_i \pm e_j \quad (1 \leq i < j \leq 8), \\
&\quad\ \frac{1}{2}\sum_{i=1}^{8}(-1)^{\nu(i)}e_i \quad \text{with} \quad \sum_{i=1}^{8}\nu(i) = \text{even},
\end{aligned}$$

where we have normalized the squared length of the root vectors to two.

The shift vectors v_R^I for the asymmetric $\mathbf{Z}_N$-orbifolds satisfying the conditions (3.53), (3.54) and (3.55) are given in the usual orthonormal basis:

(a) Asymmetric $\mathbf{Z}_2$-orbifolds

$v_R^I(E_8) = \frac{1}{2}(2,0,0,0,0,0,0,0)$,

$v_R^I(SO(32)) = \frac{1}{2}(1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0)$,

$v_R^I(SO(24)) = \frac{1}{2}(1,1,1,1,1,1,0,0,0,0,0,0)$,

$v_R^I(SO(16)) = \frac{1}{2}(1,1,1,1,0,0,0,0)$,

$v_R^I(SO(8)) = \frac{1}{2}(1,1,0,0)$,

(b) Asymmetric $\mathbf{Z}_3$-orbifolds

$v_R^I(E_8) = \frac{1}{3}(0,2,1,0,-1,1,0,1)$,

$v_R^I(E_6) = \frac{1}{3}(0,1,2,0,1,0,0,0)$,

$v_R^I(SU(3)) = \frac{1}{3}(1,0,-1)$,

(c) Asymmetric $\mathbf{Z}_4$-orbifold

$v_R^I(E_8) = \frac{1}{4}(0,-1,2,1,0,-1,2,1)$,

$v_R^I(SO(32)) = \frac{1}{4}(0,-1,2,1,0,-1,2,1,0,-1,2,1,0,-1,2,1)$,

$v_R^I(SO(24)) = \frac{1}{4}(0,-1,2,1,0,-1,2,1,0,-1,2,1)$,

$v_R^I(SO(16)) = \frac{1}{4}(0,-1,2,1,0,-1,2,1)$,

$v_R^I(SO(8)) = \frac{1}{4}(0,-1,2,1),$

(d) Asymmetric $\mathbf{Z}_5$-orbifold

$v_R^I(E_8) = \frac{1}{5}(0,-1,3,2,1,0,-1,2),$

$v_R^I(SU(5)) = \frac{1}{5}(2,1,0,-1,-2),$

(e) Asymmetric $\mathbf{Z}_6$-orbifold

$v_R^I(E_8) = \frac{1}{6}(0,-1,-2,3,2,1,0,1),$

(f) Asymmetric $\mathbf{Z}_7$-orbifolds

$v_R^I(SU(7)) = \frac{1}{7}(3,2,1,0,-1,-2,-3),$

(g) Asymmetric $\mathbf{Z}_9$-orbifold

$v_R^I(E_6) = \frac{1}{9}(0,-1,-2,-3,-4,-2,-2,2),$

(h) Asymmetric $\mathbf{Z}_{10}$-orbifolds

$v_R^I(E_8) = \frac{1}{10}(0,-3,4,1,-2,5,2,1),$

(i) Asymmetric $\mathbf{Z}_{11}$-orbifolds

$v_R^I(SU(11)) = \frac{1}{11}(5,4,3,2,1,0,-1,-2,-3,-4,-5),$

(j) Asymmetric $\mathbf{Z}_{13}$-orbifolds

$v_R^I(SU(13)) = \frac{1}{13}(6,5,4,3,2,1,0,-1,-2,-3,-4,-5,-6),$

(k) Asymmetric $\mathbf{Z}_{15}$-orbifolds

$v_R^I(E_8) = \frac{1}{15}(0,-1,-2,-3,-4,-5,-6,7),$

(l) Asymmetric $\mathbf{Z}_{17}$-orbifold

$v_R^I(SU(17)) = \frac{1}{17}(8,7,6,5,4,3,2,1,0,-1,-2,-3,-4,-5,-6,-7,-8),$

(m) Asymmetric $\mathbf{Z}_{19}$-orbifold

$v_R^I(SU(19))$

$= \frac{1}{19}(9,8,7,6,5,4,3,2,1,0,-1,-2,-3,-4,-5,-6,-7,-8,-9),$

(n) Asymmetric $\mathbf{Z}_{20}$-orbifold

$v_R^I(E_8) = \frac{1}{20}(0,1,1,2,2,3,4,15),$

(o) Asymmetric $\mathbf{Z}_{23}$-orbifold

$v_R^I(SU(23))$

$= \frac{1}{23}(11,10,9,8,7,6,5,4,3,2,1,0,-1,-2,-3,-4,-5,-6,-7,-8,-9,-10,-11),$

(p) Asymmetric $\mathbf{Z}_{30}$-orbifold

$v_R^I(E_8) = \frac{1}{30}(0,-1,-2,-3,-4,-5,-6,-23).$

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Table Captions

Table 1. φ denotes the Euler function. In the third column allowed values of D, which are consistent with the constraints (3.37), are given. In the fourth column the dimensions d in which (1, 0) states might appear in twisted sectors are given.

Table 2. G_0 denotes the $\mathbf{Z}_2$-invariant subgroup of G, which is the symmetry in each sector and G' denotes the full symmetry of the total Hilbert space.

Table 3. G_0 denotes the $\mathbf{Z}_3$-invariant subgroup of G, which is the symmetry in each sector and G' denotes the full symmetry of the total Hilbert space.

Table 4. G_0 denotes the $\mathbf{Z}_4$-invariant subgroup of G, which is the symmetry in each sector and G' denotes the full symmetry of the total Hilbert space.

Table 5. G_0 denotes the $\mathbf{Z}_5$-invariant subgroup of G, which is the symmetry in each sector and G' denotes the full symmetry of the total Hilbert space.

Table 6. G_0 denotes the $\mathbf{Z}_N$-invariant subgroup of G, which is the symmetry in each sector and G' denotes the full symmetry of the total Hilbert space.

Table 1.

$\mathbf{Z}_N$	$\varphi(N)$	D	d
$\mathbf{Z}_2$	1	$8\mathbf{Z}$	8, 16
$\mathbf{Z}_3$	2	$6\mathbf{Z}$	6, 12, 18
$\mathbf{Z}_4$	2	$16\mathbf{Z}$	16
$\mathbf{Z}_5$	4	$4\mathbf{Z}$	4, 8, 12, 16, 20
$\mathbf{Z}_6$	2	$24\mathbf{Z}$	24
$\mathbf{Z}_7$	6	$6\mathbf{Z}$	6, 12, 18
$\mathbf{Z}_8$	4	$32\mathbf{Z}$	non
$\mathbf{Z}_9$	6	$18\mathbf{Z}$	18
$\mathbf{Z}_{10}$	4	$8\mathbf{Z}$	8, 16, 24
$\mathbf{Z}_{11}$	10	$10\mathbf{Z}$	10, 20
$\mathbf{Z}_{12}$	4	$48\mathbf{Z}$	non
$\mathbf{Z}_{13}$	12	$12\mathbf{Z}$	12
$\mathbf{Z}_{14}$	6	$24\mathbf{Z}$	24
$\mathbf{Z}_{15}$	8	$24\mathbf{Z}$	24
$\mathbf{Z}_{16}$	8	$64\mathbf{Z}$	non
$\mathbf{Z}_{17}$	16	$16\mathbf{Z}$	16
$\mathbf{Z}_{18}$	6	$72\mathbf{Z}$	non
$\mathbf{Z}_{19}$	18	$18\mathbf{Z}$	18
$\mathbf{Z}_{20}$	8	$16\mathbf{Z}$	16
$\mathbf{Z}_{21}$	12	$12\mathbf{Z}$	12, 24
$\mathbf{Z}_{22}$	10	$40\mathbf{Z}$	non
$\mathbf{Z}_{23}$	22	$22\mathbf{Z}$	22
$\mathbf{Z}_{24}$	8	$96\mathbf{Z}$	non
$\mathbf{Z}_{25}$	20	$20\mathbf{Z}$	20
$\mathbf{Z}_{26}$	12	$24\mathbf{Z}$	24
$\mathbf{Z}_{27}$	18	$54\mathbf{Z}$	non
$\mathbf{Z}_{28}$	12	$48\mathbf{Z}$	non
$\mathbf{Z}_{29}$	28	$28\mathbf{Z}$	non
$\mathbf{Z}_{30}$	8	$24\mathbf{Z}$	24

Table 2.

Asymmetric $\mathbf{Z}_2$-orbifold models with (1,0) twisted states.

G	G_0	G'
$D=8$		
E_8	$SO(16)$	E_8
$SO(16)$	$(SO(8))^2$	$SO(16)$
$(SO(8))^2$	$(SU(2))^8$	$(SO(8))^2$
$D=16$		
$(E_8)^2$	$(SO(16))^2$	$SO(32)$
$SO(32)$	$(SO(16))^2$	$E_8 \times SO(16)$
$SO(24) \times SO(8)$	$(SO(12))^2 \times (SU(2))^4$	$E_7 \times SO(12) \times (SU(2))^3$
$E_8 \times SO(16)$	$SO(16) \times (SO(8))^2$	$SO(24) \times SO(8)$
$(SO(16))^2$	$(SO(8))^4$	$SO(16) \times (SO(8))^2$
$E_8 \times (SO(8))^2$	$SO(16) \times (SU(2))^8$	$SO(20) \times (SU(2))^6$
$SO(16) \times (SO(8))^2$	$(SO(8))^2 \times (SU(2))^8$	$SO(12) \times SO(8) \times (SU(2))^6$
$(SO(8))^4$	$(SU(2))^{16}$	$SO(8) \times (SU(2))^{12}$

Table 3.

Asymmetric $\mathbf{Z}_3$-orbifold models with (1,0) twisted states.

G	G_0	G'
$D=6$		
E_6	$(SU(3))^3$	E_6
$(SU(3))^3$	$(U(1))^6$	$(SU(3))^3$
$D=12$		
$(E_6)^2$	$(SU(3))^6$	$(E_6)^2$
$E_6 \times (SU(3))^3$	$(SU(3))^3 \times (U(1))^6$	$SU(6) \times SO(8) \times (U(1))^3$
$(SU(3))^6$	$(U(1))^{12}$	$(SU(2))^6 \times (U(1))^6$
$E_8 \times (SU(3))^2$	$SU(9) \times (U(1))^4$	$SO(20) \times (U(1))^2$
$D=18$		
$(E_6)^3$	$(SU(3))^9$	$E_6 \times (SU(3))^6$
$(E_6)^2 \times (SU(3))^3$	$(SU(3))^6 \times (U(1))^6$	$SU(6) \times (SU(3))^4 \times (U(1))^5$
$E_6 \times (SU(3))^6$	$(SU(3))^3 \times (U(1))^{12}$	$SU(4) \times (SU(3))^2 \times (U(1))^{11}$
$(SU(3))^9$	$(U(1))^{18}$	$SU(2) \times (U(1))^{17}$
$E_8 \times E_6 \times (SU(3))^2$	$SU(9) \times (SU(3))^3 \times (U(1))^4$	$SU(12) \times (SU(3))^2 \times (U(1))^3$
$E_8 \times (SU(3))^5$	$SU(9) \times (U(1))^{10}$	$SU(10) \times (U(1))^9$
$(E_8)^2 \times SU(3)$	$(SU(9))^2 \times (U(1))^2$	$SU(18) \times U(1)$

Table 4.

Asymmetric $\mathbf{Z}_4$-orbifold models with (1,0) twisted states.

G	G_0	G'
$D = 16$		
$(E_8)^2$	$(SO(10))^2 \times (SU(4))^2$	$(E_8)^2$
$SO(32)$	$SU(8) \times (SO(8))^2 \times U(1)$	$SO(24) \times SO(8)$
$SO(24) \times SO(8)$	$SU(6) \times (SU(4))^2$ $\times SU(2) \times (U(1))^4$	$SU(12) \times SU(4) \times (U(1))^2$
$E_8 \times SO(16)$	$SO(10) \times (SU(4))^2$ $\times (SU(2))^4 \times U(1)$	$(E_7)^2 \times (SU(2))^2$
$(SO(16))^2$	$(SU(4))^2 \times (SU(2))^8 \times (U(1))^2$	$(SO(12))^2 \times (SU(2))^4$
$E_8 \times (SO(8))^2$	$SO(10) \times SU(4)$ $\times (SU(2))^2 \times (U(1))^6$	$(E_6)^2 \times (U(1))^4$
$SO(16) \times (SO(8))^2$	$SU(4) \times (SU(2))^6 \times (U(1))^7$	$(SU(6))^2 \times (SU(2))^2 \times (U(1))^4$
$(SO(8))^4$	$(SU(2))^4 \times (U(1))^{12}$	$(SU(3))^4 \times (U(1))^8$

Table 5.

Asymmetric $\mathbf{Z}_5$-orbifold models with (1,0) twisted states.

G	G_0	G'
$D=4$		
$SU(5)$	$(U(1))^4$	$SU(5)$
$D=8$		
E_8	$(SU(5))^2$	E_8
$(SU(5))^2$	$(U(1))^8$	$(SU(5))^2$
$D=12$		
$E_8 \times SU(5)$	$(SU(5))^2 \times (U(1))^4$	$SO(22) \times U(1)$
$(SU(5))^3$	$(U(1))^{12}$	$(SU(4))^3 \times (U(1))^3$
$D=16$		
$(E_8)^2$	$(SU(5))^4$	$(E_8)^2$
$E_8 \times (SU(5))^2$	$(SU(5))^2 \times (U(1))^8$	$(SO(12))^2 \times (U(1))^4$
$(SU(5))^4$	$(U(1))^{16}$	$(SU(2))^8 \times (U(1))^8$
$D=20$		
$(E_8)^2 \times SU(5)$	$(SU(5))^4 \times (U(1))^4$	$(SU(10))^2 \times (U(1))^2$
$E_8 \times (SU(5))^3$	$(SU(5))^2 \times (U(1))^{12}$	$(SU(6))^2 \times (U(1))^{10}$
$(SU(5))^5$	$(U(1))^{20}$	$(SU(2))^2 \times (U(1))^{18}$

Table 6.

Asymmetric $\mathbf{Z}_N$-orbifold models with (1,0) twisted states.

$\mathbf{Z}_N$	G	G_0	G'
$\mathbf{Z}_6$	$D = 24$		
	$(E_8)^3$	$(SU(5))^3 \times (SU(4))^3 \times (U(1))^3$	$(SU(5))^6$
$\mathbf{Z}_7$	$D = 6$		
	$SU(7)$	$(U(1))^6$	$SU(7)$
	$D = 12$		
	$(SU(7))^2$	$(U(1))^{12}$	$(SU(6))^2 \times (U(1))^2$
	$D = 18$		
	$(SU(7))^3$	$(U(1))^{18}$	$(SU(2))^9 \times (U(1))^9$
$\mathbf{Z}_9$	$D = 18$		
	$(E_6)^3$	$(SU(2))^3 \times (U(1))^{15}$	$(SO(8))^3 \times (U(1))^6$
$\mathbf{Z}_{10}$	$D = 8$		
	E_8	$(SU(3))^2 \times (SU(2))^2 \times (U(1))^2$	E_8
	$D = 16$		
	$(E_8)^2$	$(SU(3))^4 \times (SU(2))^4 \times (U(1))^4$	$SO(32)$
	$D = 24$		
	$(E_8)^3$	$(SU(3))^6 \times (SU(2))^6 \times (U(1))^6$	$(SU(3))^{12}$
$\mathbf{Z}_{11}$	$D = 10$		
	$SU(11)$	$(U(1))^{10}$	$SU(11)$
	$D = 20$		
	$(SU(11))^2$	$(U(1))^{20}$	$(SU(2))^{10} \times (U(1))^{10}$
$\mathbf{Z}_{13}$	$D = 12$		
	$SU(13)$	$(U(1))^{12}$	$SU(12) \times U(1)$
$\mathbf{Z}_{15}$	$D = 24$		
	$(E_8)^3$	$(SU(2))^{12} \times (U(1))^{12}$	$(SU(2))^{24}$
$\mathbf{Z}_{17}$	$D = 16$		
	$SU(17)$	$(U(1))^{16}$	$(SU(8))^2 \times (U(1))^2$
$\mathbf{Z}_{19}$	$D = 18$		
	$SU(19)$	$(U(1))^{18}$	$(SU(6))^3 \times (U(1))^3$
$\mathbf{Z}_{20}$	$D = 16$		
	$(E_8)^2$	$(SU(2))^4 \times (U(1))^{12}$	$(E_8)^2$
$\mathbf{Z}_{23}$	$D = 22$		
	$SU(23)$	$(U(1))^{22}$	$(SU(2))^{11} \times (U(1))^{11}$
$\mathbf{Z}_{30}$	$D = 24$		
	$(E_8)^3$	$(U(1))^{24}$	$(U(1))^{24}$