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中谷, 実伸

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博士論文

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平成8年1月

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（量子グラスマン多様体に付随した$q$-超幾何系について）

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## Preface

The important purpose of the study for the hypergeometric functions is to find the relation with Lie algebra. It is known that the contiguity relation for some hypergeometric functions associated with Grassmann manifold gives the representation of Lie algebra. For example, the contiguity relation for the Gauss hypergeometric functions, associated with Grassmannian $G_{2,4}$, gives a representation of $gl(4)$.

Hopf algebra $U_q(gl(n))$ is the quantum analogue of the universal enveloping algebra of $gl(n)$. E. Horikawa [H] showed that the contiguity relation for the $q$-hypergeometric function ${}_2\varphi_1$, that is $q$-analogue of Gauss hypergeometric function, gives the representation of $U_q(gl(4))$.

From the different viewpoint, M. Noumi [N] showed the relation between $U_q(gl(n))$ and a $q$-hypergeometric function $\varphi_D$, which is a $q$-analogue of the hypergeometric function (Lauricella's) $F_D$.

In the classical case, the hypergeometric function associated with Grassmannian $G_{k,n}$ is the solution of the system of differential equations defined by the representation of Lie algebra $gl(n)$. And the hypergeometric function $F_D$ associates with Grassmannian $G_{2,n}$.

So Noumi defined the system of $q$-difference equations by the representation of $U_q(gl(n))$ on the quantum Grassmannian which is a quantum analogue of the space $G_{2,n}$. He realized $\varphi_D$ as a solution of this system and showed that the contiguity relation for $\varphi_D$ gives the representation of $U_q(gl(n))$.

Then, in this paper, I tried to give the definition of several $q$-hypergeometric functions as a solution of the $q$-difference system defined by the representation of $U_q(gl(n))$, relying on Noumi's work.

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# Chapter 0. Introduction

## 0.0. Contents of the Dissertation

In the classical case, the hypergeometric function $F_{3,6}$, associated with the $G_{3,6}$, is investigated from several viewpoints and various properties have been known (See [MSY]). So, at first, we started to consider a quantum Grassmannian that is a $q$-analogue of the space $G_{3,6}$ in chapter 1. Then, by the representation of $U_q(gl(6))$ on it, we defined the $q$-hypergeometric sytem of type $(3,6)$, and got two types of $q$-hypergeometric functions as the solutions of this system. Furthermore, we showed that the contiguity relation for these functions gives the representation of $U_q(gl(6))$.

We next tried to give the generalized $q$-hypergeometric systems of type $(k,n)$, in chapter 2. By the slightly different way from the case of chapter 1, we defined $q$-hypergeometric systems and realized the $q$-hypergeometric function $\varphi_{k,n}$ as a solution of this system. And we showed that the contiguity relation for $\varphi_{k,n}$ gives a representation of $U_q(gl(n))$.

Finally, we considered the $q$-confluent hypergeometric functions including $q$-Kummer and $q$-Bessel functions in chapter 3.

In what follows, we fix a complex number $q$ with $0<|q|<1$.

## 0.1. Notation of $\mathbf{U_q(gl(n))}$

We recall the definition of $U_q(gl(n))$ briefly. Refer to [N, NUW]. The quantum analogue of $U(gl(n))$, denoted by $U_q(gl(n))$, is the algebra generated by $e_j$, $f_j$ $(1\le j\le n-1)$ and $q^{\pm\epsilon_j}$ $(1\le j\le n)$. We use the notation $q^h=q^{a_1\epsilon_1}\cdots q^{a_n\epsilon_n}$ for any linear combination $h=a_1\epsilon_1+\cdots+a_n\epsilon_n$ with integral coefficients. Then the relations for the generators are given by

$$
\begin{aligned}
&q^0=1, \quad q^hq^{h'}=q^{h+h'},\\
&q^he_jq^{-h}=q^{\langle h,\epsilon_j-\epsilon_{j+1}\rangle}e_j, \quad q^hf_jq^{-h}=q^{\langle h,-\epsilon_j+\epsilon_{j+1}\rangle}f_j \quad (1\le j\le n-1),\\
&e_if_j-f_je_i=\delta_{i,j}\frac{q^{\epsilon_i-\epsilon_{i+1}}-q^{-\epsilon_i+\epsilon_{i+1}}}{q-q^{-1}} \quad (1\le i,j\le n-1)\\
&e_ie_j=e_je_i, \quad f_if_j=f_jf_i \quad (|i-j|\ge 2),\\
&{e_i}^2e_j-(q+q^{-1})e_ie_je_i+e_j{e_i}^2=0 \quad (|i-j|=1),\\
&{f_i}^2f_j-(q+q^{-1})f_if_jf_i+f_j{f_i}^2=0 \quad (|i-j|=1),
\end{aligned}
\tag{0.1.1}
$$

where $\langle\ ,\ \rangle$ stands for the canonical symmetric bilinear form such that $\langle\epsilon_i,\epsilon_j\rangle=\delta_{i,j}$. This algebra has the structure of Hopf algebra with the following convention of the *coproduct* $\Delta$, the *counit* $\varepsilon$, and the *antipode* $S$:

$$\begin{aligned}
\Delta(q^h) &= q^h \otimes q^h, \\
\Delta(e_j) &= e_j \otimes 1 + q^{\epsilon_j - \epsilon_{j+1}} \otimes e_j, \\
\Delta(f_j) &= f_j \otimes q^{-\epsilon_j + \epsilon_{j+1}} + 1 \otimes f_j, \\
&\varepsilon(q^h) = 1, \quad \varepsilon(e_j) = \varepsilon(f_j) = 0, \\
&S(q^h) = q^{-h}, \quad S(e_j) = -q^{-\epsilon_j+\epsilon_{j+1}} e_j, \quad S(f_j) = -f_j q^{\epsilon_j - \epsilon_{j+1}},
\end{aligned} \tag{0.1.2}$$

for $1 \leq j \leq n-1$.

In the classical case, the root vectors $E_{ij}$ of Lie algebra $gl(n)$ are given inductively as follows:

$$E_{ij} = [\,E_{ik}, E_{kj}\,] \quad (i \lessgtr^{\forall} k \lessgtr j).$$

Here $[\,X, Y\,] = XY - YX$.

Then we define the elements $\hat{E}_{ij} \in U_q(gl(n))$, which can be seen as the analogues of the root vectors $E_{ij}$ of $gl(n)$, by the following formulas:

$$\begin{aligned}
&\hat{E}_{jj+1} = e_j, \quad \hat{E}_{j+1j} = f_j \\
&\hat{E}_{ij} = \hat{E}_{ik}\hat{E}_{kj} - q^{\pm 1}\hat{E}_{kj}\hat{E}_{ik} \qquad (i \lessgtr^{\forall} k \lessgtr j).
\end{aligned} \tag{0.1.3}$$

## Chapter 1. A quantum analogue of the hypergeometric system $\mathbf{E}_{3,6}$

### 1.0. Introduction of this chapter

In this chapter, we will discuss a quantum analogue of $G_{3,6}$ and the associated hypergeometric functions, relying on Noumi's work.

In the first section, we recall the definition of the hypergeometric system of differential equations of type $(k, n)$ and the hypergeometric function $F_{3,6}$. Furthermore we give a short preview of this chapter.

In the second section, we define a noncommutative algebra generated by quantum analogues of the Plücker coordinates of the $G_{3,6}$ and compute the representation of $U_q(gl(6))$ on it. Furthermore we define a localization of this noncommutative algebra and two types of decomposition of it.

In the third section, we translate the action of $U_q(gl(6))$ on the above localization to that on the commutative algebra. By defining two isomorphisms of vector spaces, associated with the decompositions, between the localization and commutative algebras, we exhibit the action of $U_q(gl(6))$ in terms of usual $q$-difference operators.

In the fourth section, we define the Casimir operators and, in the fifth section, consider the $q$-hypergeometric functions of type $(3,6)$, denoted by $\varphi_\lambda^{\mathrm{I}}$ and $\varphi_\lambda^{\mathrm{II}}$. One of the functions is defined by the series

$$\begin{aligned}\varphi_\lambda^{\mathrm{I}} &= \varphi_\lambda^{\mathrm{I}}(x_1, x_2, x_3, x_4)\\ &= \sum_{a,b,c,d\geq 0} \frac{(q^{-2\lambda_5}; q^2)_{a+c}(q^{-2\lambda_6}; q^2)_{b+d}(q^{2\lambda_2+2}; q^2)_{a+b}(q^{2\lambda_3+2}; q^2)_{c+d}}{(q^{2\lambda_2+2\lambda_3+2\lambda_4+6}; q^2)_{a+b+c+d}(q^2; q^2)_a(q^2; q^2)_b(q^2; q^2)_c(q^2; q^2)_d}\\ &\quad\times q^{2bc} x_1^a x_2^b x_3^c x_4^d.\end{aligned} \tag{1.5.3}$$

In the last section, we compute the contiguity relations and, in view of these relations, propose to define the *$q$-hypergeometric system of type* $(3, 6; \mathrm{I})$ and $(3, 6; \mathrm{II})$. Furthermore we define another system of $q$-difference equations, which has $q$-hypergeometric functions as solutions. Refer to Proposition 1.6.6.

### 1.1. Preliminaries of this chapter

In this section, we give a short review of the hypergeometric system of differential equations associated with the Grassmannian $G_{k,n}$ $(k \leq n)$ and a preview of this chapter.

Let us denote by $M(k,n)$ the space of all $k \times n$ matrices of the form

$$T = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ t_{21} & t_{22} & \dots & t_{2n} \\ & \dots & \dots & \\ t_{k1} & t_{k2} & \dots & t_{kn} \end{pmatrix}.$$

The *hypergeometric system of differential equations of type* $(k,n)$, denoted by $E_{k,n}$, is the following system of differential equations defined on the space $M(k,n)$:

$$\Phi(gT) = \det(g)^{-1}\Phi(T) \quad (g \in GL(k)) \tag{1.1.1}$$

$$\Phi(T\mathrm{diag}(c_1,\dots,c_n)) = \Phi(T){c_1}^{\lambda_1}\cdots {c_n}^{\lambda_n} \tag{1.1.2}$$

$$\frac{\partial^2}{\partial t_{ri}\partial t_{sj}}\Phi(T) = \frac{\partial^2}{\partial t_{si}\partial t_{rj}}\Phi(T) \quad (1 \le r < s \le k, 1 \le i < j \le n), \tag{1.1.3}$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is a set of complex numbers. For the conditions (1.1.1) and (1.1.2) to be compatible, it is necessary to assume $\lambda_1 + \cdots + \lambda_n = -k$. Because, for $c \in \mathbf{C}^\times$ and identity matrices $I_k (\in GL(k))$ and $I_n (\in GL(n))$, $(cI_k)T = T(cI_n)$.

Multivalued holomorphic solutions $\Phi(T)$ of the system (1.1.1)-(1.1.3), defined on a Zariski open set of the space $M(k,n)$, are called the hypergeometric functions of type $(k,n)$.

Now let $T$ be a general $k \times n$ matrix. We can decompose $T$ as follows:

$$\begin{aligned} T \quad = \quad & \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1k} \\ t_{21} & t_{22} & \dots & t_{2k} \\ & \dots & \dots & \\ t_{k1} & t_{k2} & \dots & t_{kk} \end{pmatrix} \begin{pmatrix} 1 & & & & 1 & 1 & \dots & 1 \\ & 1 & & & x_{2k+1} & x_{2k+2} & \dots & x_{2n} \\ & & \ddots & & & \dots & \dots & \\ & & & 1 & x_{kk+1} & x_{kk+2} & \dots & x_{kn} \end{pmatrix} \\ & \times \mathrm{diag}(1,1,\dots,1,-{\xi_{1\cdots k}}^{-1}\xi_{\hat{1}k+1},\dots,-{\xi_{1\cdots k}}^{-1}\xi_{\hat{1}n}), \end{aligned} \tag{1.1.4}$$

where

$$x_{rj} = (-1)^{r+1}{\xi_{\hat{1}j}}^{-1}\xi_{\hat{r}j} \quad \text{for} \quad r = 2,\cdots,k \quad j = k+1,\cdots,n.$$

$$\xi_{i_1 i_2 \dots i_k} = \sum_{\omega \in S_k} (-1)^{l(\omega)} t_{\omega(1)i_1} t_{\omega(2)i_2} \cdots t_{\omega(k)i_k} \quad \text{for} \quad 1 \le i_1, \dots, i_k \le n,$$

$$\xi_{\hat{r}j} = \xi_{1\dots\hat{r}\dots kj}.$$

Here $S_k$ is the permutation group of $k$ letters and, for each $\omega \in S_k$, $l(\omega)$ denotes the number of inversions in $\omega$. Note that $\{\xi_{i_1 i_2 \dots i_k}\}$ are the Plücker coordinates of the $G_{k,n}$, and satisfy the Plücker relations as follows:

$$\sum_{i=1}^{k+1} (-1)^{i-1} \xi_{\alpha_1 \cdots \alpha_{k-1}\beta_i} \xi_{\hat{\beta}_i} = 0 \tag{1.1.5}$$

for $1 \leq \alpha_1, \ldots, \alpha_{k-1} \leq n$, $\quad 1 \leq \beta_1 < \beta_2 < \cdots < \beta_{k+1} \leq n$ and where $\xi_{\hat{\beta}_i} = \xi_{\beta_1 \cdots \hat{\beta}_i \cdots \beta_{k+1}}$. The relative invariances (1.1.1), (1.1.2) imply that $\Phi(T)$ can be written in the form

$$\Phi(T) = G(x_{2k+1}, x_{2k+2}, \ldots, x_{kn}) \xi_{1\cdots k}{}^{\lambda_1+\lambda_2+\cdots+\lambda_k+k-1} \xi_{\hat{1}k+1}{}^{\lambda_{k+1}} \cdots \xi_{\hat{1}n}{}^{\lambda_n},$$

for some function $G(x_{2k+1}, \ldots, x_{kn})$.

Furthermore, because of invariances (1.1.1), (1.1.2) again, $G(x_{2k+1}, \ldots, x_{kn})$ takes the form

$$G(x_{2k+1}, \ldots, x_{kn}) = x_{2k+1}{}^{-\lambda_2-1} \cdots x_{kk+1}{}^{-\lambda_k-1} F(z_{2k+2}, z_{2k+3}, \ldots, z_{kn}), \tag{1.1.6}$$

where

$$z_{rj} = x_{rk+1}{}^{-1} x_{rj}$$

for $2 \leq r \leq k$, $k+2 \leq j \leq n$. In the coordinates $z = (z_{2k+2}, z_{2k+3}, \ldots, z_{kn})$, it is known that a solution of the equation (1.1.3), holomorphic around the origin $z = 0$, is given as follows:

$$F_{k,n}(z; \lambda) = \sum_{a_j^r \geq 0} \frac{\prod_{j=k+2}^{n} (-\lambda_j)_{|A_j|} \prod_{r=2}^{k} (\lambda_r + 1)_{|A^r|}}{(\gamma)_{|A|} \prod_{r,j} (1)_{a_j^r}} z^A, \tag{1.1.7}$$

where $\gamma = \lambda_2 + \cdots + \lambda_{k+1} + k$, and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$. Here, for the matrix $A = (a_j^r)_{2 \leq r \leq k, k+2 \leq j \leq n}$, we use the notations

$$|A_j| = a_j^2 + a_j^3 + \cdots + a_j^k, \quad |A^r| = a_{k+2}^r + a_{k+3}^r + \cdots + a_n^r, \quad |A| = \sum_{r,j} a_j^r \tag{1.1.8}$$

and

$$z^A = \prod_{r,j} z_j^{r a_j^r}. \tag{1.1.9}$$

The hypergeometric function $F_D$ associated with $G_{2,n}$ has been generalized to the hypergeometric function $F_{k,n}$ associated with the $G_{k,n}$, and it is known that the contiguity relation for $F_{k,n}$ is described in terms of the Lie algebra $gl(n)$. Refer to [G, S]. Therefore the contiguity relation for the function $F_{3,6}$, defined by the series

$$F_{3,6}(z, \lambda) = \sum_{a_{rj} \geq 0} \frac{(-\lambda_5)_{a_{25}+a_{35}} (-\lambda_6)_{a_{26}+a_{36}} (\lambda_2+1)_{a_{25}+a_{26}} (\lambda_3+1)_{a_{35}+a_{36}}}{(\lambda_2+\lambda_3+\lambda_4+3)_{a_{25}+a_{26}+a_{35}+a_{36}} (1)_{a_{25}} (1)_{a_{26}} (1)_{a_{35}} (1)_{a_{36}}} \times z_{25}{}^{a_{25}} z_{26}{}^{a_{26}} z_{35}{}^{a_{35}} z_{36}{}^{a_{36}}, \tag{1.1.10}$$

is descrived in terms of the Lie algebra $gl(6)$.

Now let us set

$$E_{ij} = \sum_{r=1}^{k} t_{ri} \frac{\partial}{\partial t_{rj}}.$$

Then $\{E_{ij}\}_{1 \le i,j \le n}$ generate a Lie algebra isomorphic to $gl(n)$ and define, as differential operators, a $U(gl(n))$-module structure on the polynomial algebra $\mathbf{C}[t_{11}, t_{12}, \cdots, t_{kn}]$ of variables $(t_{rj})_{1 \le r \le k, 1 \le j \le n}$. It holds a remarkable identity

$$(E_{ii}+1)E_{jj} - E_{ji}E_{ij} = \sum_{1 \le r < s \le k} (t_{ri}t_{sj} - t_{si}t_{rj}) \left( \frac{\partial^2}{\partial t_{ri} \partial t_{sj}} - \frac{\partial^2}{\partial t_{si} \partial t_{rj}} \right), \tag{1.1.11}$$

for each $i \neq j$. Then the function

$$G(x_{2k+1}, \dots, x_{kn}) = x_{2k+1}{}^{-\lambda_2 - 1} \cdots x_{kk+1}{}^{-\lambda_k - 1} F_{k,n}(z, \lambda)$$

is a solution of the system of equation (1.1.6) and

$$\{(E_{ii}+1)E_{jj} - E_{ji}E_{ij}\}(G(x_{2k+1}, \dots, x_{kn})) = 0. \tag{1.1.12}$$

We define in the next section a noncommutative algebra as a quantum analogue of the space $G_{3,6}$, which has a structure of left $U_q(gl(6))$-module. Furthermore we give two kinds of left $U_q(gl(6))$-module structure on the commutative algebra $\mathbf{C}[x_{24}, x_{25}, x_{26}, x_{34}, x_{35}, x_{36}]$ of polynomials in 6 variables.

In view of the identity (1.1.12), we look for the analogous elements in $U_q(gl(n))$; they are the central elements of the subalgebra of $U_q(gl(6))$, isomorphic to $U_q(gl(2))$, defined by

$$(q - q^{-1})^2 C_{ij} = (q^{1+\varepsilon_i} - q^{-1-\varepsilon_i})(q^{\varepsilon_j} - q^{-\varepsilon_j}) - (q - q^{-1})^2 \hat{E}_{ji} \hat{E}_{ij} \tag{1.1.13}$$

for $1 \le i \neq j \le 6$. In this chapter, we call such elements the *Casimir* elements. When we need to regard them as the operators, we also call them the *Casimir* operators.

## 1.2. The algebra $\mathbf{U_q(gl(6))}$, Quantum Grassmannians, and localization

We first in this section define the noncommutative algebra, which is the quantum analogue of the coordinate ring $A(M(k,n))$.

Let $A_q(M(k,n))$ denote the algebra defined as follows:

generators: $\quad t_{rj} \quad (1 \le r \le k,\ 1 \le j \le n)$

relations: $$ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \qquad (1.2.1)$$
$$bc = cb, \quad ad - da = (q - q^{-1})bc,$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} t_{ri} & t_{rj} \\ t_{si} & t_{sj} \end{pmatrix} \quad \begin{pmatrix} 1 \le r < s \le 3 \\ 1 \le i < j \le 6 \end{pmatrix}.$$

The commutation relations as (1.2.1) are called by the $Mat_q(2)$-relations.

The algebra $A_q(M(k,n))$ has a structure of a left $U_q(gl(n))$-module; it is determined by assuming the $U_q(gl(n))$-symmetry in the sense that

(a) $a.1 = \varepsilon(a)1$ for all $a \in U_q(gl(n))$;

(b) if $a \in U_q(gl(n))$ and $\Delta(a) = \sum_i a'_i \otimes a''_i$, then

$$a.(\varphi\psi) = \sum_i a'_i.\varphi \; a''_i.\psi \text{ for all } \varphi, \psi \in A_q(M(k,n))$$

and by the action of the generators of $U_q(gl(n))$ on $t_{rj}$ of $A_q(M(k,n))$ given by the formula

$$q^h.t_{rj} = q^{\langle h,\epsilon_j\rangle} t_{rj}, \quad e_k.t_{rj} = \delta_{k+1,j} t_{rk}, \quad f_k.t_{rj} = \delta_{k,j} t_{rk+1}. \qquad (1.2.2)$$

In this chapter, we consider the algebra $A_q(M(3,6))$ which has a left $U_q(gl(6))$-module structure.

Since we can regard the ordinary Grassmann manifold as the space whose coordinate ring is the algebra generated by the Plücker coordinates, we define the quantum Grassmannian by giving the quantum analogue of the Plücker coordinates.

We denote the subalgebra in $A_q(M(3,6))$ generated by the *quantum minor determinants*

$$\xi_{ijk} = \sum_{\omega \in S_3} (-q)^{l(\omega)} t_{\omega(1)i} t_{\omega(2)j} t_{\omega(3)k} \qquad (1 \le i,j,k \le 6)$$

by $\mathcal{A}$. In the classical case, the Plücker coordinates satisfy the Plücker relations (1.1.5). Simiraly, these minors satisfy the Plücker relations as follows:

$$\sum_{i=1}^{4} (-q)^{i-1} \xi_{\alpha_1 \alpha_2 \beta_i} \xi_{\hat{\beta}_i} = 0 \qquad (1.2.3)$$

for $1 \le \alpha_1, \alpha_2 \le 6$ and $1 \le \beta_1 < \beta_2 < \beta_3 < \beta_4 \le 6$, where $\xi_{\hat{\beta}_i} = \xi_{\beta_1 \cdots \hat{\beta}_i \cdots \beta_4}$.

The subalgebra $\mathcal{A}$ is then a left $U_q(gl(6))$-submodule of $A_q(M(3,6))$; the action of $U_q(gl(6))$ on the quantum minors is determined by

$$\begin{aligned} q^h.\xi_{ijk} &= q^{\langle h, \epsilon_i + \epsilon_j + \epsilon_k \rangle} \xi_{ijk}, \\ e_l.\xi_{ijk} &= \delta_{l+1,i} \xi_{ljk} + \delta_{l+1,j} q^{\langle \epsilon_l - \epsilon_{l+1}, \epsilon_i \rangle} \xi_{ilk} + \delta_{l+1,k} q^{\langle \epsilon_l - \epsilon_{l+1}, \epsilon_i + \epsilon_j \rangle} \xi_{ijl}, \\ f_l.\xi_{ijk} &= \delta_{l,i} q^{-\langle \epsilon_l - \epsilon_{l+1}, \epsilon_j + \epsilon_k \rangle} \xi_{l+1jk} + \delta_{l,j} q^{-\langle \epsilon_l - \epsilon_{l+1}, \epsilon_j \rangle} \xi_{il+1k} + \delta_{l,k} \xi_{ijl+1}. \end{aligned} \qquad (1.2.4)$$

In order to give the decomposition of the matrix $T = (t_{rj})_{1\le r\le 3,\,1\le j\le 6}$ of generators for $A_q(M(3,6))$, we introduce some localization of $\mathcal{A}$. In the decomposition (1.1.4), the ${\xi_{123}}^{-1}$ in the diagonal matrix has played a special role. A first step is to invert the principal minor $\xi_{123}$. By the Plücker relations (1.2.3), the localization $\mathcal{A}[{\xi_{123}}^{-1}]$ has the monomial basis

$$\xi_{\hat{1}6}^{a_{16}}\xi_{\hat{1}5}^{a_{15}}\xi_{\hat{1}4}^{a_{14}}\xi_{\hat{2}6}^{a_{26}}\xi_{\hat{2}5}^{a_{25}}\xi_{\hat{2}4}^{a_{24}}\xi_{\hat{3}6}^{a_{36}}\xi_{\hat{3}5}^{a_{35}}\xi_{\hat{3}4}^{a_{34}}\xi_{123}^{\mu} \tag{1.2.5}$$

where $a_{ij} \in \mathbf{N}$, $\mu \in \mathbf{Z}$ and $\xi_{\hat{i}j} = \xi_{1\cdots\hat{i}\cdots 3j}$. Here note that we have fixed an ordering of the variables $\xi_{\hat{i}j}$, called a normal ordering. By the Plücker relations (1.2.3) again, the commutation relations among $\xi_{ijk}$ and $\xi_{\hat{i}\hat{j}}$ are generated by the following relations:

$$\xi_{123}\xi_{\hat{i}j} = q\xi_{\hat{i}j}\xi_{123} \quad \text{for} \quad 1 \le i \le 3, \quad 4 \le j \le 6, \tag{1.2.6}$$

and

$$\begin{gathered}
\xi_{\hat{i}_1 j}\xi_{\hat{i}_2 j} = q^{-1}\xi_{\hat{i}_2 j}\xi_{\hat{i}_1 j}, \qquad \xi_{\hat{i}j_1}\xi_{\hat{i}j_2} = q\xi_{\hat{i}j_2}\xi_{\hat{i}j_1}, \\
\xi_{\hat{i}_1 j_1}\xi_{\hat{i}_2 j_2} = \xi_{\hat{i}_2 j_2}\xi_{\hat{i}_1 j_1}, \qquad \xi_{\hat{i}_2 j_1}\xi_{\hat{i}_1 j_2} - \xi_{\hat{i}_1 j_2}\xi_{\hat{i}_2 j_1} = (q-q^{-1})\xi_{\hat{i}_1 j_1}\xi_{\hat{i}_2 j_2}, \\
\text{for} \quad 1 \le i_1 < i_2 \le 3, \quad 4 \le j_1 < j_2 \le 6.
\end{gathered} \tag{1.2.7}$$

In order to get variables analogous to $x_{2j}$ and $x_{3j}$ in (1.1.4), we next adjoin the inverses of the quantum minors $\xi_{\hat{1}4}$, $\xi_{\hat{1}5}$, $\xi_{\hat{1}6}$. The localized algebra is $\mathcal{A}[{\xi_{123}}^{-1}, {\xi_{\hat{1}4}}^{-1}, {\xi_{\hat{1}5}}^{-1}, {\xi_{\hat{1}6}}^{-1}]$ and, by the commutation relations, it has the monomial base of type (1.2.7) with $a_{2j}, a_{3j} \in \mathbf{N}$ and $a_{1j}, \mu \in \mathbf{Z}$.

With this localization and the commutation relations the matrix $T = (t_{rj})_{1\le r\le 3,\,1\le j\le 6}$ of generators for $A_q(M(3,6))$ can be decomposed as follows:

$$\begin{aligned}
T \quad &= \quad \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & u_{24} & u_{25} & u_{26} \\ 0 & 0 & 1 & u_{34} & u_{35} & u_{36} \end{pmatrix} \\
&\quad \times \operatorname{diag}(1,1,1,-q^{-2}{\xi_{123}}^{-1}\xi_{\hat{1}4}, -q^{-2}{\xi_{123}}^{-1}\xi_{\hat{1}5}, -q^{-2}{\xi_{123}}^{-1}\xi_{\hat{1}6}),
\end{aligned} \tag{1.2.8}$$

where

$$u_{2j} = -\xi_{\hat{1}j}^{-1}\xi_{\hat{2}j} \quad \text{and} \quad u_{3j} = q\xi_{\hat{1}j}^{-1}\xi_{\hat{3}j}. \tag{1.2.9}$$

Since $A_q(M(3,6))$ is the algebra with $U_q(gl(6))$-symmetry, we can extend the action of $U_q(gl(6))$ on $\mathcal{A}$ to the action on $\mathcal{A}[{\xi_{123}}^{-1}, {\xi_{\hat{1}4}}^{-1}, {\xi_{\hat{1}5}}^{-1}, {\xi_{\hat{1}6}}^{-1}]$. For an element $a$ of $U_q(gl(6))$, we denote its action on $\mathcal{A}[{\xi_{123}}^{-1}, {\xi_{\hat{1}4}}^{-1}, {\xi_{\hat{1}5}}^{-1}, {\xi_{\hat{1}6}}^{-1}]$ by

$$\hat{\rho}(a) \quad : \quad \mathcal{A}[{\xi_{123}}^{-1}, {\xi_{\hat{1}4}}^{-1}, {\xi_{\hat{1}5}}^{-1}, {\xi_{\hat{1}6}}^{-1}] \to \mathcal{A}[{\xi_{123}}^{-1}, {\xi_{\hat{1}4}}^{-1}, {\xi_{\hat{1}5}}^{-1}, {\xi_{\hat{1}6}}^{-1}]. \tag{1.2.10}$$

The algebra $\mathcal{A}[\xi_{123}{}^{-1}, \xi_{\hat{1}4}{}^{-1}, \xi_{\hat{1}5}{}^{-1}, \xi_{\hat{1}6}{}^{-1}]$, as a vector space, has several kinds of decomposition. But the commutation relations among $u_{rj}$ are not so simple. So, in order to check the action of $U_q(gl(6))$ on $\mathcal{A}[\xi_{123}{}^{-1}, \xi_{\hat{1}4}{}^{-1}, \xi_{\hat{1}5}{}^{-1}, \xi_{\hat{1}6}{}^{-1}]$ explicitly, we here treat the following two decompositions.

$$\text{(I)}\quad \mathcal{A}[\xi_{123}{}^{-1}, \xi_{\hat{1}4}{}^{-1}, \xi_{\hat{1}5}{}^{-1}, \xi_{\hat{1}6}{}^{-1}] = \bigoplus_{\mu,\lambda_j \in \mathbf{Z}} R_2 \xi_{\hat{1}6}{}^{\lambda_6} \xi_{\hat{1}5}{}^{\lambda_5} \xi_{\hat{1}4}{}^{\lambda_4} \xi_{123}{}^{\mu} R_3 \tag{1.2.11}$$

$R_r$ is the *noncommutative* algebra generated by $u_{rj}$ $(r = 2, 3\ j = 4, 5, 6)$

$$\text{(II)}\quad \mathcal{A}[\xi_{123}{}^{-1}, \xi_{\hat{1}4}{}^{-1}, \xi_{\hat{1}5}{}^{-1}, \xi_{\hat{1}6}{}^{-1}] = \bigoplus_{\mu,\lambda_j \in \mathbf{Z}} \xi_{\hat{1}6}{}^{\lambda_6} \xi_{\hat{1}5}{}^{\lambda_5} \xi_{\hat{1}4}{}^{\lambda_4} \xi_{123}{}^{\mu} R \tag{1.2.12}$$

$R$ is the *noncommutative* algebra generated by $u_{rj}$ $(j = 4, 5, 6)$

## 1.3. Translation of the action of $\mathbf{U_q(gl(6))}$

The action of $U_q(gl(6))$ on the *noncommutative* algebra $\mathcal{A}[\xi_{123}{}^{-1}, \xi_{\hat{1}4}{}^{-1}, \xi_{\hat{1}5}{}^{-1}, \xi_{\hat{1}6}{}^{-1}]$ is translated to that on the *commutative* algebra $\mathbf{C}[x_{24}, x_{25}, x_{26}, x_{34}, x_{35}, x_{36}]$ of polynomials in terms of usual $q$-difference operators in this section.

To do this we define a vector space

$$\mathcal{G} \quad = \quad \bigoplus_{\mu,\lambda_4,\lambda_5,\lambda_6 \in \mathbf{Z}} \mathcal{G}_{\mu,\lambda_4,\lambda_5,\lambda_6}, \tag{1.3.1}$$

where $\mathcal{G}_{\mu,\lambda_4,\lambda_5,\lambda_6} = \mathbf{C}[x_{24}, \cdots, x_{36}]$ for all $\mu, \lambda_4, \lambda_5, \lambda_6 \in \mathbf{Z}$ and the two isomorphisms of vector spaces

$$\omega^{\mathrm{I}} : \mathcal{G} \to \mathcal{A}[\xi_{123}{}^{-1}, \xi_{\hat{1}4}{}^{-1}, \xi_{\hat{1}5}{}^{-1}, \xi_{\hat{1}6}{}^{-1}], \tag{1.3.2}$$

$$\omega^{\mathrm{II}} : \mathcal{G} \to \mathcal{A}[\xi_{123}{}^{-1}, \xi_{\hat{1}4}{}^{-1}, \xi_{\hat{1}5}{}^{-1}, \xi_{\hat{1}6}{}^{-1}], \tag{1.3.3}$$

as follows. Let us fix the monomial bases of $\mathcal{A}[\xi_{123}{}^{-1}, \xi_{\hat{1}4}{}^{-1}, \xi_{\hat{1}5}{}^{-1}, \xi_{\hat{1}6}{}^{-1}]$ in two ways, referring to the decompositions (1.2.11) and (1.2.12):

$$\begin{array}{lll} \mathrm{I} & u_{24}{}^{\nu_{24}} u_{25}{}^{\nu_{25}} u_{26}{}^{\nu_{26}} \xi_{\hat{1}6}{}^{\lambda_6} \xi_{\hat{1}5}{}^{\lambda_5} \xi_{\hat{1}4}{}^{\lambda_4} \xi_{123}{}^{\mu} u_{36}{}^{\nu_{36}} u_{35}{}^{\nu_{35}} u_{34}{}^{\nu_{34}} & (\nu_{rj} \in \mathbf{N}) \\ \mathrm{II} & \xi_{\hat{1}6}{}^{\lambda_6} \xi_{\hat{1}5}{}^{\lambda_5} \xi_{\hat{1}4}{}^{\lambda_4} \xi_{123}{}^{\mu} u_{26}{}^{\nu_{26}} u_{36}{}^{\nu_{36}} u_{25}{}^{\nu_{25}} u_{35}{}^{\nu_{35}} u_{24}{}^{\nu_{24}} u_{34}{}^{\nu_{34}} & (\nu_{rj} \in \mathbf{N}), \end{array} \tag{1.3.4}$$

where $u_{2j} = -\xi_{\hat{1}j}{}^{-1} \xi_{\hat{2}j}$ and $u_{3j} = q\xi_{\hat{1}j}{}^{-1} \xi_{\hat{3}j}$. Then the isomorphism $\omega^{\mathrm{I}}$ is determined as the direct sum of linear mappings

$$\omega^{\mathrm{I}}_{\mu,\lambda_4,\lambda_5,\lambda_6} : \mathbf{C}[x_{24}, \cdots, x_{36}] \to R_2 \xi_{\hat{1}6}{}^{\lambda_6} \xi_{\hat{1}5}{}^{\lambda_5} \xi_{\hat{1}4}{}^{\lambda_4} \xi_{123}{}^{\mu} R_3$$

such that

$$\omega^{\mathrm{I}}_{\mu,\lambda_4,\lambda_5,\lambda_6}({x_{24}}^{\nu_{24}}{x_{25}}^{\nu_{25}}{x_{26}}^{\nu_{26}}{x_{34}}^{\nu_{34}}{x_{35}}^{\nu_{35}}{x_{36}}^{\nu_{36}})$$
$$= {u_{24}}^{\nu_{24}}{u_{25}}^{\nu_{25}}{u_{26}}^{\nu_{26}}{\xi_{\hat{1}6}}^{\lambda_6}{\xi_{\hat{1}5}}^{\lambda_5}{\xi_{\hat{1}4}}^{\lambda_4}{\xi_{123}}^{\mu}{u_{36}}^{\nu_{36}}{u_{35}}^{\nu_{35}}{u_{34}}^{\nu_{34}},$$

and the isomorphism $\omega^{\mathrm{II}}$ as the direct sum of linear mappings

$$\omega^{\mathrm{II}}_{\mu,\lambda_4,\lambda_5,\lambda_6} : \mathbf{C}[x_{24},\cdots,x_{36}] \to {\xi_{\hat{1}6}}^{\lambda_6}{\xi_{\hat{1}5}}^{\lambda_5}{\xi_{\hat{1}4}}^{\lambda_4}{\xi_{123}}^{\mu}R$$

such that

$$\omega^{\mathrm{II}}_{\mu,\lambda_4,\lambda_5,\lambda_6}({x_{24}}^{\nu_{24}}{x_{25}}^{\nu_{25}}{x_{26}}^{\nu_{26}}{x_{34}}^{\nu_{34}}{x_{35}}^{\nu_{35}}{x_{36}}^{\nu_{36}})$$
$$= {\xi_{\hat{1}6}}^{\lambda_6}{\xi_{\hat{1}5}}^{\lambda_5}{\xi_{\hat{1}4}}^{\lambda_4}{\xi_{123}}^{\mu}{u_{26}}^{\nu_{26}}{u_{36}}^{\nu_{36}}{u_{25}}^{\nu_{25}}{u_{35}}^{\nu_{35}}{u_{24}}^{\nu_{24}}{u_{34}}^{\nu_{34}}.$$

Through the isomorphisms $\omega^{\mathrm{I}}$ and $\omega^{\mathrm{II}}$ thus defined, we obtain two kinds of left $U_q(gl(6))$-module structure on the vector space $\mathcal{G}$. More precisely, for an element $a$ of $U_q(gl(6))$ of weight $\kappa$, i.e., $q^h a q^{-h} = q^{\langle h,\kappa\rangle}a$, its action on $\mathcal{G}$ induces a family of operators

$$\tilde{\rho}^*_{\mu,\lambda_4,\lambda_5,\lambda_6}(a) = {\omega^*_{\mu',\lambda'_4,\lambda'_5,\lambda'_6}}^{-1} \circ \hat{\rho}(a) \circ \omega^*_{\mu,\lambda_4,\lambda_5,\lambda_6} : \mathcal{G}_{\mu,\lambda_4,\lambda_5,\lambda_6} \to \mathcal{G}_{\mu',\lambda'_4,\lambda'_5,\lambda'_6}, \qquad (1.3.5)$$

where $\mu' = \mu+\kappa_1+\kappa_2+\kappa_3$ and $\lambda'_j = \lambda_j+\kappa_j$ $(j=4,5,6)$ for $\kappa = \kappa_1\epsilon_1+\cdots+\kappa_6\epsilon_6$ $(\kappa_j \in \mathbf{Z})$, and $* = \mathrm{I}$ or $\mathrm{II}$. The explicit formulas of the operators will now be given. They are rationally written by $q$-difference operators with coefficients in $\mathbf{C}[x_{24},\ldots,x_{36}]$.

Here we denote the $q$-shift operator for $x_{rj}$ by $T_{q,x_{rj}}$, that is,

$$T_{q,x_{rj}}f(x_{24},\ldots,x_{rj},\ldots,x_{36}) \quad = \quad f(x_{24},\ldots,qx_{rj},\ldots,x_{36}). \qquad (1.3.6)$$

For example, $T_{q,x_{25}}\prod_{r,j}{x_{rj}}^{\nu_{rj}} = q^{\nu_{25}}\prod_{r,j}{x_{rj}}^{\nu_{rj}}$.

**Remark 1.3.1.** Generally, the $q$-shift operator for a variable $x$ is denoted by $T_{q,x}$ or $q^{\theta_x}$. In this chapter, we use the first notation $T_{q,x}$. But, in the chapter 2 and the chapter 3, we use the second notation $q^{\theta_x}$.

For each 6-tuple $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_6)$ of complex numbers with $\sum\lambda_j = -3$, let $\mathcal{F}_\lambda$ be the vector space of polynomials $G = G(x_{24},\ldots,x_{36})$ such that

$$T_{q,x_{24}}T_{q,x_{25}}T_{q,x_{26}}G = q^{-\lambda_2-1}G, \quad T_{q,x_{34}}T_{q,x_{35}}T_{q,x_{36}}G = q^{-\lambda_3-1}G. \qquad (1.3.7)$$

Then, for each element $a \in U_q(gl(6))$ of weight $\kappa$, the operators

$$\rho^{\mathrm{I}}_\lambda(a) = \tilde{\rho}^{\mathrm{I}}_{\lambda_1+\lambda_2+\lambda_3+2,\lambda_4,\lambda_5,\lambda_6}(a), \quad \rho^{\mathrm{II}}_\lambda(a) = \tilde{\rho}^{\mathrm{II}}_{\lambda_1+\lambda_2+\lambda_3+2,\lambda_4,\lambda_5,\lambda_6}(a) \qquad (1.3.8)$$

define mappings

$$\rho_\lambda^{\mathrm{I}}(a),\ \rho_\lambda^{\mathrm{II}}(a) : \mathcal{F}_\lambda \to \mathcal{F}_{\lambda+\kappa}. \tag{1.3.9}$$

Here we have the explicit formulas of the operators

$$\begin{aligned} \rho_\lambda^*(q^{\epsilon_j}) &: \mathcal{F}_\lambda \to \mathcal{F}_\lambda \quad (1 \le j \le 6), \\ \rho_\lambda^*(e_j) &: \mathcal{F}_\lambda \to \mathcal{F}_{\lambda+\alpha_j} \quad (1 \le j \le 5), \\ \rho_\lambda^*(f_j) &: \mathcal{F}_\lambda \to \mathcal{F}_{\lambda-\alpha_j} \quad (1 \le j \le 5), \end{aligned} \tag{1.3.10}$$

where $* = \mathrm{I}$ or II.

We set for simplicity $\alpha_j = \epsilon_j - \epsilon_{j+1}$. For instance, for the case (I) put

$$T_{rj} = T_{q^2, x_{rj}};$$

then we see

$$\begin{aligned}
&\rho_\lambda^{\mathrm{I}}(q^{\varepsilon_j}) = q^{\lambda_j} \qquad (1 \le j \le 6), \\
&(q-q^{-1})\rho_\lambda^{\mathrm{I}}(e_1) = x_{24}(q^{-2\lambda_4}T_{24}T_{34} - 1) + x_{25}q^{-2\lambda_4}T_{24}T_{34}(q^{-2\lambda_5}T_{25}T_{35} - 1) \\
&\qquad + x_{26}q^{-2\lambda_4-2\lambda_5}T_{24}T_{34}T_{25}T_{35}(q^{-2\lambda_6}T_{26}T_{36} - 1), \\
&(q-q^{-1})\rho_\lambda^{\mathrm{I}}(e_2) = q^{-\lambda_1-\lambda_3-3}\left[\frac{1}{x_{24}}\{(x_{36}-x_{35})(1-q^{-2\lambda_6}T_{26}T_{36})\right. \\
&\qquad + (x_{35}-x_{34})(1-q^{-2\lambda_6-2\lambda_5}T_{26}T_{36}T_{25}T_{35}) + x_{34}\}(1-T_{24}) \\
&\qquad + \frac{1}{x_{25}}\{(x_{36}-x_{35})(1-q^{-2\lambda_6}T_{26}T_{36}) + x_{35}\}(1-T_{25})T_{24} \\
&\qquad \left. + \frac{1}{x_{26}}x_{36}T_{24}T_{25}(1-T_{26})\right], \\
&(q-q^{-1})\rho_\lambda^{\mathrm{I}}(e_3) = q^{-\lambda_1-\lambda_2+\lambda_3-\lambda_4+1}\frac{1}{x_{34}}(T_{34}-1), \\
&(q-q^{-1})\rho_\lambda^{\mathrm{I}}(e_j)_{(j=4,5)} = q^{\lambda_{j+1}}{T_{2j+1}}^{-1} - q^{-\lambda_{j+1}}T_{3j+1} \\
&\qquad + \frac{x_{2j}}{x_{2j+1}}q^{\lambda_{j+1}}(1-{T_{2j+1}}^{-1}) + \frac{x_{3j}}{x_{3j+1}}q^{-\lambda_{j+1}}(T_{3j+1}-1), \\
&(q-q^{-1})\rho_\lambda^{\mathrm{I}}(f_1) = -q^{-1-\lambda_1-\lambda_2}\left\{\frac{1}{x_{24}}{T_{25}}^{-1}{T_{26}}^{-1}(1-{T_{24}}^{-1})\right. \\
&\qquad \left. + \frac{1}{x_{25}}{T_{26}}^{-1}(1-{T_{25}}^{-1}) + \frac{1}{x_{26}}(1-{T_{26}}^{-1})\right\}, \\
&(q-q^{-1})\rho_\lambda^{\mathrm{I}}(f_2) = q^{2+\lambda_3-\lambda_4-\lambda_5-\lambda_6} \times \left[\frac{x_{26}}{x_{36}}(1-T_{36})\right. \\
&\qquad + \frac{1}{x_{35}}\{x_{25}q^{2\lambda_6}{T_{26}}^{-1}{T_{36}}^{-1} + x_{26}(1-q^{2\lambda_6}{T_{26}}^{-1}{T_{36}}^{-1})\}T_{36}(1-T_{35})
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{x_{34}}\{x_{24}q^{2\lambda_6+2\lambda_5}T_{26}{}^{-1}T_{36}{}^{-1}T_{25}{}^{-1}T_{35}{}^{-1})\\
&\quad+x_{25}q^{2\lambda_6}T_{26}{}^{-1}T_{36}{}^{-1}(1-q^{2\lambda_5}T_{25}{}^{-1}T_{35}{}^{-1})\\
&\quad+x_{26}(1-q^{2\lambda_6}T_{26}{}^{-1}T_{36}{}^{-1})\}T_{36}T_{35}(1-T_{34})\big],
\end{aligned}
$$

$$
\begin{aligned}
(q-q^{-1})\rho^{\mathrm{I}}_{\lambda}(f_3)&=q^{-2-\lambda_3-\mu}\times\\
&\big[x_{36}(q^{-2\lambda_6}T_{36}-T_{26}{}^{-1})T_{25}{}^{-1}\\
&+x_{35}(q^{-2\lambda_5}T_{35}-T_{25}{}^{-1})q^{-2\lambda_6}T_{36}+x_{34}(q^{2\mu}-q^{-2\lambda_6-2\lambda_5}T_{36}T_{35})\\
&+\frac{x_{24}}{x_{25}}\{x_{36}(q^{-2\lambda_6}T_{36}-T_{26}{}^{-1})-x_{35}q^{-2\lambda_6}T_{36}\}(1-T_{25}{}^{-1})\\
&-\frac{x_{24}}{x_{26}x_{36}}(1-T_{26}{}^{-1})\Big],
\end{aligned}
$$

$$
\begin{aligned}
(q-q^{-1})\rho^{\mathrm{I}}_{\lambda}(f_j)_{(j=4,5)}&=(q^{\lambda_j}T_{3j}{}^{-1}-q^{-\lambda_j}T_{2j})\\
&+\frac{x_{3j+1}}{x_{3j}}q^{\lambda_j}(1-T_{3j}{}^{-1})+\frac{x_{2j+1}}{x_{2j}}q^{-\lambda_j}(T_{2j}-1).
\end{aligned}
$$

We should remark here that the coefficients belong to the polynomial ring, because $x_{rj}$ in the denominators cancells with the operator $(1-T_{rj})$ or $(1-T_{rj}{}^{-1})$ as difference operators, as often as it appears. We have already seen the formula for $q^{\epsilon_1}$. The actions of $e_1\in U_q(gl(6))$, for instance, on $u_{2j}$ and $u_{3j}$ is determined by

$$
\hat\rho(e_1)u_{2j}{}^{\nu_{2j}}=(q^{2\nu_{2j}}-1)u_{2j}^{\nu_{2j}},\qquad \hat\rho(e_1)u_{3j}{}^{\nu_{3j}}=(q^{2\nu_{3j}}-1)u_{2j}u_{3j}^{\nu_{3j}}.
$$

So, the second formula is seen by the computation

$$
\begin{aligned}
&(q-q^{-1})\hat\rho(e_1)R_2(\nu_{24},\nu_{25},\nu_{26})\xi_{\lambda,\mu}R_3(\nu_{36},\nu_{35},\nu_{34})\\
&\quad=\sum_{j=4}^{6}q^{2\nu_{24}+\cdots+2\nu_{2j-1}}(q^{2\mu_{2j}}-1)R_2(\ldots,\nu_{2j}+1,\ldots)\xi_{\lambda,\mu}R_2(\nu_{36},\nu_{35},\nu_{34})\\
&\quad-q^{2\nu_{24}+2\nu_{25}+2\nu_{26}}R_2(\nu_{24},\nu_{25},\nu_{26})\\
&\qquad\times\sum_{j=4}^{6}q^{-2\lambda_4-\cdots-2\lambda_{j-1}}(1-q^{-2\lambda_j})u_{2j}\xi_{\lambda,\mu}R_3(\nu_{36},\nu_{35},\nu_{34})\\
&\quad+q^{2\nu_{24}+2\nu_{25}+2\nu_{26}-\lambda_4-\lambda_5-\lambda_6}R_2(\nu_{24},\nu_{25},\nu_{26})\xi_{\lambda,\mu}\\
&\qquad\times\sum_{j=4}^{6}q^{2\nu_{34}+\cdots+2\nu_{3j-1}}(q^{2\nu_{3j}}-1)u_{2j}R_3(\nu_{36},\nu_{35},\nu_{34})\\
&\quad+q^{2\nu_{24}+2\nu_{25}+2\nu_{26}-\lambda_4-\lambda_5-\lambda_6}R_2(\nu_{24},\nu_{25},\nu_{26})\xi_{\lambda,\mu}\\
&\qquad\times\sum_{j=4}^{6}q^{2\nu_{34}+\cdots+2\nu_{3j-1}}(q^{2\nu_{3j}}-1)u_{2j}R_3(\nu_{36},\nu_{35},\nu_{34})
\end{aligned}
$$

$$
\begin{aligned}
&= (q^{2\nu_{24}+2\nu_{34}-2\lambda_4} - 1)R_2(\nu_{24}+1,\nu_{25},\nu_{26})\xi_{\lambda,\mu}R_3(\nu_{36},\nu_{35},\nu_{34})\\
&+ q^{2\nu_{24}+2\nu_{34}-2\lambda_4}(q^{2\nu_{25}+2\nu_{35}-2\lambda_5} - 1)R_2(\nu_{24},\nu_{25}+1,\nu_{26})\xi_{\lambda,\mu}R_3(\nu_{36},\nu_{35},\nu_{34})\\
&+ q^{2\nu_{24}+2\nu_{34}+2\nu_{25}+2\nu_{35}-2\lambda_4-2\lambda_5}(q^{2\nu_{26}+2\nu_{36}-2\lambda_6} - 1)\\
&\qquad R_2(\nu_{24},\nu_{25},\nu_{26}+1)\xi_{\lambda,\mu}R_3(\nu_{36},\nu_{35},\nu_{34}).
\end{aligned}
$$

Namely, we have

$$
\begin{aligned}
&(q-q^{-1})\rho^{\mathrm{I}}_{\lambda}(e_1)({x_{24}}^{\nu_{24}}\dots {x_{36}}^{\nu_{36}})\\
&= (q-q^{-1}){\omega_{\lambda_1+\lambda_2+\lambda_3+2,\lambda_4,\lambda_5,\lambda_6}}^{-1}\circ\hat{\rho}(e_1)\circ\omega_{\lambda_1+\lambda_2+\lambda_3+2,\lambda_4,\lambda_5,\lambda_6}({x_{24}}^{\nu_{24}}\dots {x_{36}}^{\nu_{36}})\\
&= (q-q^{-1}){\omega_{\lambda_1+\lambda_2+\lambda_3+2,\lambda_4,\lambda_5,\lambda_6}}^{-1}\circ\hat{\rho}(e_1)R_2(\nu_{24},\nu_{25},\nu_{26})\xi_{\lambda,\lambda_1+\lambda_2+\lambda_3+2}R_3(\nu_{36},\nu_{35},\nu_{34})\\
&= (q^{2\nu_{24}+2\nu_{34}-2\lambda_4} - 1){x_{24}}^{\nu_{24}+1}\dots {x_{36}}^{\nu_{36}}\\
&\quad + q^{2\nu_{24}+2\nu_{34}-2\lambda_4}(q^{2\nu_{25}+2\nu_{35}-2\lambda_5} - 1){x_{24}}^{\nu_{24}}{x_{25}}^{\nu_{25}+1}\dots {x_{36}}^{\nu_{36}}\\
&\quad + q^{2\nu_{24}+2\nu_{34}-2\lambda_4}(q^{2\nu_{25}+2\nu_{35}-2\lambda_5} - 1){x_{24}}^{\nu_{24}}{x_{25}}^{\nu_{25}+1}\dots {x_{36}}^{\nu_{36}}\\
&\quad + q^{2\nu_{24}+2\nu_{34}+2\nu_{25}+2\nu_{35}-2\lambda_4-2\lambda_5}(q^{2\nu_{26}+2\nu_{36}-2\lambda_6} - 1){x_{24}}^{\nu_{24}}\dots {x_{26}}^{\nu_{26}+1}\dots {x_{36}}^{\nu_{36}},
\end{aligned}
$$

which shows the second formula. The remaining formulas can be shown similarly.

We second consider the case (II). Set

$$T_{rj} = T_{q,x_{rj}};$$

then we see

$$
\begin{aligned}
&\rho^{\mathrm{II}}_{\lambda}(q^{e_j}) = q^{\lambda_j} \qquad (1\le j\le 6),\\
&(q-q^{-1})\rho^{\mathrm{II}}_{\lambda}(e_1) = q^{\lambda_4+\lambda_5}x_{26}(q^{-\lambda_6}{T_{26}}^2{T_{36}}^2 - q^{\lambda_6})\\
&\qquad\qquad + q^{\lambda_4-\lambda_6}x_{25}{T_{26}}^2T_{36}(q^{-\lambda_5}{T_{25}}^2{T_{35}}^2 - q^{\lambda_5})\\
&\qquad\qquad + q^{-\lambda_5-\lambda_6}x_{24}{T_{26}}^2T_{36}{T_{25}}^2T_{35}(q^{-\lambda_4}{T_{24}}^2{T_{34}}^2 - q^{\lambda_4}),\\
&(q-q^{-1})\rho^{\mathrm{II}}_{\lambda}(e_2) = q^{-1}\left\{\frac{x_{36}}{x_{26}}{T_{26}}^{-1}(1-{T_{26}}^2)\frac{x_{35}}{x_{25}}{T_{26}}^{-1}{T_{25}}^{-1}T_{36}(1-{T_{25}}^2)\right.\\
&\qquad\qquad \left. + \frac{x_{34}}{x_{24}}{T_{26}}^{-1}{T_{25}}^{-1}{T_{24}}^{-1}T_{36}T_{35}(1-{T_{24}}^2)\right\},\\
&(q-q^{-1})\rho^{\mathrm{II}}_{\lambda}(e_3) = q^{-\lambda_1+\lambda_3-\lambda_4+2}\frac{1}{x_{34}}({T_{34}}^2 - 1),\\
&(q-q^{-1})\rho^{\mathrm{II}}_{\lambda}(e_j)_{(j=4,5)} = q^{\lambda j+1} - q^{-\lambda j+1}{T_{3j+1}}^2{T_{2j+1}}^2\\
&\qquad\qquad - q^{-\lambda_{j+1}}\left\{\frac{x_{2j}}{x_{2j+1}}T_{3j+1}(1-{T_{2j+1}}^2) + \frac{x_{3j}}{x_{3j+1}}T_{2j}(1-{T_{3j+1}}^2)\right\},
\end{aligned}
$$

$$(q-q^{-1})\rho_\lambda^{\mathrm{II}}(f_1) = -q\left\{\frac{1}{x_{26}}T_{25}{}^{-2}T_{24}{}^{-2}T_{34}{}^{-1}T_{35}{}^{-1}T_{36}{}^{-1}(1-T_{26}{}^{-2})\right.$$
$$+\frac{1}{x_{25}}T_{24}{}^{-2}T_{35}{}^{-1}T_{34}{}^{-1}(1-T_{25}{}^{-2})$$
$$\left.+\frac{1}{x_{24}}T_{34}{}^{-1}(1-T_{24}{}^{-2})\right\},$$
$$(q-q^{-1})\rho_\lambda^{\mathrm{II}}(f_2) = q\left\{\frac{x_{26}}{x_{36}}T_{24}T_{25}T_{34}{}^{-1}T_{35}{}^{-1}T_{36}{}^{-1}(1-T_{36}{}^2)\right.$$
$$\left.+\frac{x_{25}}{x_{35}}T_{24}T_{34}{}^{-1}T_{35}{}^{-1}(1-T_{35}{}^2)+\frac{x_{24}}{x_{34}}T_{34}{}^{-1}(1-T_{34}{}^2)\right\},$$
$$(q-q^{-1})\rho_\lambda^{\mathrm{II}}(f_3) = q^{-2-\mu-\lambda_3}\times$$
$$\left\{x_{36}T_{26}(1-q^{-2\lambda_6}T_{26}{}^2T_{36}{}^2)\right.$$
$$+q^{-2\lambda_6}x_{35}T_{26}T_{25}T_{36}{}^2(1-q^{-2\lambda_5}T_{25}{}^2T_{35}{}^2)$$
$$-q^{2\mu}x_{34}T_{26}T_{25}T_{24}(1-q^{-2\mu-2\lambda_6-2\lambda_5}T_{36}{}^2T_{35}{}^2)$$
$$-q^{-1-2\lambda_6-2\lambda_5}\frac{x_{24}x_{35}}{x_{25}}T_{26}T_{25}T_{36}{}^2T_{35}(1-T_{25}{}^2)$$
$$-q^{-1-2\lambda_6-2\lambda_5}\frac{x_{24}x_{36}}{x_{26}}T_{26}T_{25}{}^2T_{36}T_{35}(1-T_{26}{}^2)$$
$$\left.-q^{-1-2\lambda_6}\frac{x_{25}x_{36}}{x_{26}}T_{26}T_{36}(1-T_{26}{}^2)(1-q^{-2\lambda_5}T_{25}{}^2T_{35}{}^2)\right\},$$
$$(q-q^{-1})\rho_\lambda^{\mathrm{II}}(f_j)_{(j=4,5)} = q^{\lambda_j}T_{3j}{}^{-2}T_{2j}{}^{-2}-q^{-\lambda_j}$$
$$+q^{\lambda_j}\left\{\frac{x_{2j+1}}{x_{2j}}T_{3j+1}T_{3j}{}^{-2}(1-T_{2j}{}^{-2})+\frac{x_{3j+1}}{x_{3j}}T_{2j}{}^{-1}(1-T_{3j}{}^{-2})\right\}.$$

Thus we have translated the action of $U_q(gl(6))$ on the commutative algebra $\bigoplus_\lambda \mathcal{F}_\lambda$.

### 1.4. Casimir operators

In this section we introduce the Casimir operators, instead of Laplacian in the classical case, and compute their action on the algebra $\bigoplus_\lambda \mathcal{F}_\lambda$. The Casimir elements are defined by

$$(q-q^{-1})^2C_j = (q^{1+\varepsilon_j}-q^{-1-\varepsilon_j})(q^{\varepsilon_{j+1}}-q^{-\varepsilon_{j+1}})-(q-q^{-1})^2f_je_j, \tag{1.4.1}$$

for $1\le j\le 5$. They are central elements of the subalgebra $\mathbf{C}[q^{\pm\varepsilon_j},q^{\pm\varepsilon_{j+1}},e_j,f_j]$ of $U_q(gl(6))$, isomorphic to $U_q(gl(2))$.

Here recall the homogeneity condition (1.3.7); to take care of this, we introduce new coordinates by

$$z_1=\frac{x_{25}}{x_{24}},\quad z_2=\frac{x_{26}}{x_{24}},\quad z_3=\frac{x_{35}}{x_{34}},\quad z_4=\frac{x_{36}}{x_{34}},$$

and the new unknown $\varphi$ by

$$G = x_{24}^{-\lambda_2-1} x_{34}^{-\lambda_3-1} \varphi(z_1, z_2, z_3, z_4). \tag{1.4.2}$$

Then, relative to these coordinates, we can compute the actions of the Casimir operators by using the concrete forms of the actions of $q^{\pm\epsilon_j}$, $e_j$, and $f_j$, which have been given in the previous section.

**Proposition 1.4.1.** For the case (I), define the $q$-difference operators by

$$\begin{gathered} T_{25} = T_{q^2,z_1} = T_1, \quad T_{26} = T_{q^2,z_2} = T_2, \quad T_{35} = T_{q^2,z_3} = T_3, \\ T_{36} = T_{q^2,z_4} = T_4, \quad T_{24} = q^{-2\lambda_2-2} T_1{}^{-1} T_2{}^{-1}, \quad T_{34} = q^{-2\lambda_3-2} T_3{}^{-1} T_4{}^{-1}, \\ \mathbf{T} = T_1 T_2 T_3 T_4. \end{gathered} \tag{1.4.3}$$

Then we have

$$\begin{aligned} (q-q^{-1})^2 \rho_\lambda^{\mathrm{I}}(C_4) = {} & q^{-2\lambda_2-\lambda_4+\lambda_5-1}(1-z_1{}^{-1}) T_1{}^{-2} T_2{}^{-1} \times \\ & \{(1-q^{2\lambda_2+2\lambda_3+2\lambda_4+4}\mathbf{T})(1-T_1) - q^2 z_1 (1-q^{2\lambda_2+2} T_1 T_2)(1-q^{-2\lambda_5} T_1 T_3)\} \\ & + q^{-2\lambda_2-\lambda_4-\lambda_5-3}(1-z_3{}^{-1}) T_1{}^{-1} \times \\ & \{T_2{}^{-1}(1-q^{2\lambda_2+2\lambda_3+2\lambda_4+4}\mathbf{T})(1-T_3) \\ & \quad - q^{2\lambda_2+2\lambda_4+2\lambda_5+4} z_3 (1-q^{2\lambda_3+2} T_3 T_4)(1-q^{-2\lambda_5} T_1 T_3)\} \\ & + q^{-2\lambda_2-\lambda_4-\lambda_5-1} \left(\frac{1}{z_3} - \frac{1}{z_1}\right) T_1{}^{-1} \times \\ & \{z_1 T_2{}^{-1}(1-q^{2\lambda_2+2} T_1 T_2)(1-T_3) \\ & \quad - q^{2\lambda_2+2\lambda_4+2\lambda_5+2} z_3 (1-q^{2\lambda_3+2} T_3 T_4)(1-T_1)\}, \end{aligned}$$

$$\begin{aligned} (q-q^{-1})^2 \rho_\lambda^{\mathrm{I}}(C_5) = {} & q^{\lambda_5+\lambda_6+1} \left(\frac{1}{z_2} - \frac{1}{z_1}\right) T_2{}^{-1} \times \\ & \{z_1 T_3{}^{-1}(1-q^{-2\lambda_5} T_1 T_3)(1-T_2) - q^{-2\lambda_5} z_2 (1-q^{-2\lambda_6} T_2 T_4)(1-T_1)\} \\ & + q^{\lambda_5-\lambda_6+1} \left(\frac{1}{z_4} - \frac{1}{z_3}\right) T_2{}^{-1} \times \\ & \{z_3 (1-q^{-2\lambda_5} T_1 T_3)(1-T_4) - q^{2\lambda_6} z_4 T_2{}^{-1}(1-q^{-2\lambda_6} T_2 T_4)(1-T_3)\} \\ & + q^{\lambda_5+\lambda_6+1} \left(\frac{1}{z_2 z_3} - \frac{1}{z_1 z_4}\right) \times \\ & \{q^{-2\lambda_5-2\lambda_6-2} z_2 z_3 (1-T_1)(1-T_4) - z_1 z_4 T_2{}^{-1} T_3{}^{-1}(1-T_2)(1-T_3)\}. \end{aligned}$$

For the case (II), define the difference operators by

$$\begin{gathered} T_{25} = T_{q,z_1} = T_1, \quad T_{26} = T_{q,z_2} = T_2, \quad T_{35} = T_{q,z_3} = T_3, \\ T_{36} = T_{q,z_4} = T_4, \quad T_{24} = q^{-\lambda_2-1} T_1{}^{-1} T_2{}^{-1}, \quad T_{34} = q^{-\lambda_3-1} T_3{}^{-1} T_4{}^{-1}, \end{gathered} \tag{1.4.4}$$

Then we have

$$
\begin{aligned}
(q-q^{-1})^2\rho^{\mathrm{II}}_{\lambda}(C_4) = q^{-\lambda_4-\lambda 5-1}&\left(T_3-\frac{1}{z_1}\right)T_3\times \\
&\{(1-q^{2\lambda_2+2\lambda_3+2\lambda_4+4}\mathbf{T}^2)(1-{T_1}^2) \\
&\quad -q^{2\lambda_3+2\lambda_4+2\lambda_5+4}T_3{T_4}^2z_1(1-q^{2\lambda_2+2}{T_1}^2{T_2}^2)(1-q^{-2\lambda_5}{T_1}^2{T_3}^2)\} \\
&+q^{-\lambda_2-\lambda_4-\lambda 5-2}\left(T_1T_2-\frac{1}{z_3}\right)\times \\
&\{{T_1}^{-1}{T_2}^{-1}(1-q^{2\lambda_2+2\lambda_3+2\lambda_4+4}\mathbf{T}^2)(1-{T_3}^2) \\
&\quad -q^{\lambda_2+2\lambda_4+2\lambda_5+3}z_3(1-q^{2\lambda_3+2}{T_3}^2{T_4}^2)(1-q^{-2\lambda_5}{T_1}^2{T_3}^2)\} \\
&+q^{-\lambda_2+2\lambda_3+\lambda_4-\lambda_5-1}\left(\frac{1}{z_3}T_3-\frac{1}{z_1}T_1T_2\right)T_3\times \\
&\{z_1{T_1}^{-1}{T_2}^{-1}T_3{T_4}^2(1-q^{2\lambda_2+2}{T_1}^2{T_2}^2)(1-{T_3}^2) \\
&\quad -q^{\lambda_2-2\lambda_3}z_3(1-q^{2\lambda_3+2}{T_3}^2{T_4}^2)(1-{T_1}^2)\},
\end{aligned}
$$

$$
\begin{aligned}
(q-q^{-1})^2\rho^{\mathrm{II}}_{\lambda}(C_5) = q^{\lambda_5-\lambda_6+1}&\left(\frac{1}{z_2}-\frac{1}{z_1}{T_4}^{-1}\right){T_1}^{-2}{T_3}^{-2}\times \\
&\{z_1T_4(1-q^{-2\lambda_5}{T_1}^2{T_3}^2)(1-T_2^2)-q^{2\lambda_6}z_2(1-q^{-2\lambda_6}{T_2}^2{T_4}^2)(1-{T_1}^2)\} \\
&+q^{\lambda_5-\lambda_6+1}\left(\frac{1}{z_4}-\frac{1}{z_3}{T_1}^{-1}\right)T_3^{-2}\times \\
&\{z_3{T_1}^{-1}(1-q^{-2\lambda_5}{T_1}^2{T_3}^2)(1-{T_4}^2)-q^{2\lambda_6}z_4(1-q^{-2\lambda_6}{T_2}^2{T_4}^2)(1-{T_3}^2)\} \\
&+q^{\lambda_5-\lambda_6+2}\left(\frac{1}{z_2z_3}-\frac{1}{z_1z_4}{T_1}^{-1}{T_4}^{-1}\right){T_1}^{-2}{T_3}^{-2}\times \\
&\{q^{-1}z_2z_3(1-{T_1}^2)(1-{T_4}^2)-z_1z_4T_1T_4(1-{T_1}^2)(1-{T_3}^2)\}.
\end{aligned}
$$

Note that we translated all actions of the elements of $U_q(gl(6))$ in terms of $q$-difference operators. Therefore, we can extend the action of $U_q(gl(6))$ to the space of formal power series.

### 1.5. $q$-hypergeometric functions of type (3,6)

In this section we define two kinds of functions, which are both $q$-analogue of $F_{3,6}$. Let us consider the following system of defference equations:

$$
T_{q,x_{24}}T_{q,x_{25}}T_{q,x_{26}}G=q^{-\lambda_2-1}G,\qquad T_{q,x_{34}}T_{q,x_{35}}T_{q,x_{36}}G=q^{-\lambda_3-1}G, \tag{1.5.1*}
$$

$$
(q-q^{-1})^2\rho^{*}_{\lambda}(C_j)G=0\qquad(1\le j\le 5), \tag{1.5.2*}
$$

for $* =$I or II. We define functions that are solutions of this system, assuming that $\lambda_2 + \lambda_3 + \lambda_4 + 3 \neq 0, -1, -2, \dots$. Put

$$\begin{aligned}\varphi_\lambda^{\mathrm{I}} &= \varphi_\lambda^{\mathrm{I}}(x_1, x_2, x_3, x_4)\\ &= \sum_{a,b,c,d=0}^{\infty} \frac{(q^{-2\lambda_5}; q^2)_{a+c}(q^{-2\lambda_6}; q^2)_{b+d}(q^{2\lambda_2+2}; q^2)_{a+b}(q^{2\lambda_3+2}; q^2)_{c+d}}{(q^{2\lambda_2+2\lambda_3+2\lambda_4+6}; q^2)_{a+b+c+d}(q^2; q^2)_a(q^2; q^2)_b(q^2; q^2)_c(q^2; q^2)_d}, \\ &\quad \times q^{2bc} x_1^a x_2^b x_3^c x_4^d,\end{aligned} \tag{1.5.3}$$

where

$$\begin{aligned} x_1 &= q^2 z_1, & x_2 &= q^{2-2\lambda_5} z_2, \\ x_3 &= q^{2\lambda_2+2\lambda_4+2\lambda_5+4} z_3, & x_4 &= q^{2\lambda_2+2\lambda_4+2\lambda_5+2\lambda_6+4} z_4, \end{aligned} \tag{1.5.4}$$

and

$$\begin{aligned}\varphi_\lambda^{\mathrm{II}} &= \varphi_\lambda^{\mathrm{II}}(x_1, x_2, x_3, x_4)\\ &= \sum_{a,b,c,d=0}^{\infty} \frac{(q^{-2\lambda_5}; q^2)_{a+c}(q^{-2\lambda_6}; q^2)_{b+d}(q^{2\lambda_2+2}; q^2)_{a+b}(q^{2\lambda_3+2}; q^2)_{c+d}}{(q^{2\lambda_2+2\lambda_3+2\lambda_4+6}; q^2)_{a+b+c+d}(q^2; q^2)_a(q^2; q^2)_b(q^2; q^2)_c(q^2; q^2)_d} \\ &\quad \times q^{c(a+b)+d(2a+b)} x_1^a x_2^b x_3^c x_4^d,\end{aligned} \tag{1.5.5}$$

where

$$\begin{aligned} x_1 &= q^{2\lambda_3+2\lambda_4+2\lambda_5+4} z_1, & x_2 &= q^{2\lambda_3+2\lambda_4+2\lambda_5+2\lambda_6+4} z_2, \\ x_3 &= q^{\lambda_2+2\lambda_4+2\lambda_5+3} z_3, & x_4 &= q^{\lambda_2+2\lambda_4+2\lambda_5+2\lambda_6+3} z_4. \end{aligned} \tag{1.5.6}$$

Here $(\alpha, q)_n = (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1})$.

Then we can see the next proposition by a simple but a little longer direct computation.

**Proposition 1.5.1.** The functions

$$G_\lambda^{\mathrm{I}} = x_{24}{}^{-\lambda_2-1} x_{34}{}^{-\lambda_3-1} \varphi_\lambda^{\mathrm{I}}, \qquad G_\lambda^{\mathrm{II}} = x_{24}{}^{-\lambda_2-1} x_{34}{}^{-\lambda_3-1} \varphi_\lambda^{\mathrm{II}}$$

satisfy the $q$-difference equations (1.5.1$*$) and (1.5.2$*$) for $* =$I and II, respectively.

**Definition 1.5.2.** We call the functions $\varphi_\lambda^{\mathrm{I}}$ and $\varphi_\lambda^{\mathrm{II}}$ defined above the $q$-hypergeometric series of type (3,6;I) and type (3,6;II), respectively.

**Remark 1.5.3.** When $q$ tends to 1, $(1-q^\alpha)/(1-q)$ converges to $\alpha$. Hence, both of $\varphi_\lambda^{\mathrm{I}}$ and $\varphi_\lambda^{\mathrm{II}}$ converge to the function $F_{3,6}$ in (1.1.10), when $q$ tends to 1.

### 1.6. Contiguity relations and $q$-hypergeometric systems of type (3,6)

Define the operators $\pi_\lambda^{\rm I}(a)$ and $\pi_\lambda^{\rm II}(a)$ as follows:

$$\begin{aligned} \rho_\lambda^{\rm I}(a)G_\lambda^{\rm I} &= x_{24}{}^{-\lambda_2'-1}x_{34}{}^{-\lambda_3'-1}\pi_\lambda^{\rm I}(a)\varphi_\lambda^{\rm I}, \\ \rho_\lambda^{\rm II}(a)G_\lambda^{\rm II} &= x_{24}{}^{-\lambda_2'-1}x_{34}{}^{-\lambda_3'-1}\pi_\lambda^{\rm II}(a)\varphi_\lambda^{\rm II}, \end{aligned} \tag{1.6.1}$$

where $a$ is an element of $U_q(gl(6))$ of weight $\kappa$ and $\lambda' = (\lambda_j') = \lambda + \kappa$. Then we have

**Proposition 1.6.1.** The following is the list of contiguity relations for the $q$-hypergeometric series $\varphi_\lambda^{\rm I}$:

$$\begin{aligned}
\pi_\lambda^{\rm I}(q^{\epsilon_j})\varphi_\lambda^{\rm I} &= q^{\lambda_j}\varphi_\lambda^{\rm I} \quad (1 \le j \le 6), \\
\pi_\lambda^{\rm I}(e_1)\varphi_\lambda^{\rm I} &= -q^{-\mu}[\mu]\varphi_{\lambda+\alpha_1}^{\rm I}, \\
\pi_\lambda^{\rm I}(e_2)\varphi_\lambda^{\rm I} &= q^{\lambda_4-\lambda_5-\lambda_6-1}[\lambda_2+1]\varphi_{\lambda+\alpha_2}^{\rm I}, \\
\pi_\lambda^{\rm I}(e_3)\varphi_\lambda^{\rm I} &= -q^{\lambda_3+\lambda_5+\lambda_6+3}[\lambda_3+1]\varphi_{\lambda+\alpha_3}^{\rm I}, \\
\pi_\lambda^{\rm I}(e_4)\varphi_\lambda^{\rm I} &= q^{2\lambda_2+\lambda_4-\mu}\frac{[\lambda_4+1][\lambda_5]}{[\mu+1]}\varphi_{\lambda+\alpha_4}^{\rm I}, \\
\pi_\lambda^{\rm I}(e_5)\varphi_\lambda^{\rm I} &= [\lambda_6]\varphi_{\lambda+\alpha_5}^{\rm I}, \\
\pi_\lambda^{\rm I}(f_1)\varphi_\lambda^{\rm I} &= -q^{\mu+1}\frac{[\lambda_1][\lambda_2+1]}{[\mu+1]}\varphi_{\lambda-\alpha_1}^{\rm I}, \\
\pi_\lambda^{\rm I}(f_2)\varphi_\lambda^{\rm I} &= q^{-\lambda_4+\lambda_5+\lambda_6+1}[\lambda_3+1]\varphi_{\lambda-\alpha_2}^{\rm I}, \\
\pi_\lambda^{\rm I}(f_3)\varphi_\lambda^{\rm I} &= -q^{-\lambda_3-\lambda_5-\lambda_6-2}[\lambda_4+1]\varphi_{\lambda-\alpha_3}^{\rm I}, \\
\pi_\lambda^{\rm I}(f_4)\varphi_\lambda^{\rm I} &= q^{-2\lambda_2-\lambda_4+\mu-2}[\mu]\varphi_{\lambda-\alpha_4}^{\rm I}, \\
\pi_\lambda^{\rm I}(f_5)\varphi_\lambda^{\rm I} &= [\lambda_5]\varphi_{\lambda-\alpha_5}^{\rm I}.
\end{aligned}$$

The contiguity relations for the $q$-hypergeometric series $\varphi_\lambda^{\rm II}$ are given as follows:

$$\begin{aligned}
\pi_\lambda^{\rm II}(q^{\epsilon_j})\varphi_\lambda^{\rm II} &= q^{\lambda_j}\varphi_\lambda^{\rm II} \quad (1 \le j \le 6), \\
\pi_\lambda^{\rm II}(e_1)\varphi_\lambda^{\rm II} &= -q^{\lambda_1+\lambda_4+1}[\mu]\varphi_{\lambda+\alpha_1}^{\rm II}, \\
\pi_\lambda^{\rm II}(e_2)\varphi_\lambda^{\rm II} &= q^{-1}[\lambda_2+1]\varphi_{\lambda+\alpha_2}^{\rm II}, \\
\pi_\lambda^{\rm II}(e_3)\varphi_\lambda^{\rm II} &= q^{-\lambda_1-\lambda_4+1}[\lambda_3+1]\varphi_{\lambda+\alpha_3}^{\rm II}, \\
\pi_\lambda^{\rm II}(e_4)\varphi_\lambda^{\rm II} &= q^{-\lambda_2-\lambda_3-2}\frac{[\lambda_4+1][\lambda_5]}{[\mu+1]}\varphi_{\lambda+\alpha_4}^{\rm II}, \\
\pi_\lambda^{\rm II}(e_5)\varphi_\lambda^{\rm II} &= [\lambda_6]\varphi_{\lambda+\alpha_5}^{\rm II}, \\
\pi_\lambda^{\rm II}(f_1)\varphi_\lambda^{\rm II} &= -q^{-\lambda_1-\lambda_4}\frac{[\lambda_1][\lambda_2+1]}{[\mu+1]}\varphi_{\lambda-\alpha_1}^{\rm II}, \\
\pi_\lambda^{\rm II}(f_2)\varphi_\lambda^{\rm II} &= q[\lambda_3+1]\varphi_{\lambda-\alpha_2}^{\rm II}, \\
\pi_\lambda^{\rm II}(f_3)\varphi_\lambda^{\rm II} &= q^{\lambda_1+\lambda_4}[\lambda_4+1]\varphi_{\lambda-\alpha_3}^{\rm II},
\end{aligned}$$

$$\begin{aligned}
\pi^{\mathrm{II}}_\lambda(f_4)\varphi^{\mathrm{II}}_\lambda &= q^{\lambda_2+\lambda_3+2}[\mu]\varphi^{\mathrm{II}}_{\lambda-\alpha_4},\\
\pi^{\mathrm{II}}_\lambda(f_5)\varphi^{\mathrm{II}}_\lambda &= [\lambda_5]\varphi^{\mathrm{II}}_{\lambda-\alpha_5}.
\end{aligned}$$

Here $\mu = \lambda_2 + \lambda_3 + \lambda_4 + 2$, and $[a] = (q^a - q^{-a})/(q - q^{-1})$.

Now we consider the action of the elements $\hat{E}_{ij}$. $\hat{E}_{ij} \in U_q(gl(6))$ $(i \neq j)$

The element $\hat{E}_{ij}$ is of weight $\alpha_{ij} = \epsilon_i - \epsilon_j$. When being considered as $q$-difference operators, they define endomorphisms

$$\rho^*_\lambda(\hat{E}_{ij}) \quad : \quad \mathcal{F}_\lambda \to \mathcal{F}_{\lambda+\alpha_{ij}} \tag{1.6.2}$$

for $*$ =I and II.

Their actions on $q$-hypergeometric series are symmarized in the following list.

**Proposition 1.6.2.** The following is the list of contiguity relations appending those in Proposition 1.6.1 for the $q$-hypergeometric series $\varphi^{\mathrm{I}}_\lambda$:

$$\begin{aligned}
\pi^{\mathrm{I}}_\lambda(\hat{E}_{13})\varphi^{\mathrm{I}}_\lambda &= -q^{-\lambda_2+\lambda_4-\lambda_5-\lambda_6-\mu-1}[\mu]\varphi^{\mathrm{I}}_{\lambda+\alpha_{13}},\\
\pi^{\mathrm{I}}_\lambda(\hat{E}_{31})\varphi^{\mathrm{I}}_\lambda &= -q^{\lambda_2-\lambda_4+\lambda_5+\lambda_6+\mu+2}\frac{[\lambda_1][\lambda_3+1]}{[\mu+1]}\varphi^{\mathrm{I}}_{\lambda+\alpha_{31}},\\
\pi^{\mathrm{I}}_\lambda(\hat{E}_{14})\varphi^{\mathrm{I}}_\lambda &= q^{-2\lambda_2-\lambda_3-1}[\mu]\varphi^{\mathrm{I}}_{\lambda+\alpha_{14}},\\
\pi^{\mathrm{I}}_\lambda(\hat{E}_{41})\varphi^{\mathrm{I}}_\lambda &= q^{2\lambda_2+\lambda_3+1}\frac{[\lambda_1][\lambda_4+1]}{[\mu+1]}\varphi^{\mathrm{I}}_{\lambda+\alpha_{41}},\\
\pi^{\mathrm{I}}_\lambda(\hat{E}_{15})\varphi^{\mathrm{I}}_\lambda &= q^{-\mu-\lambda_3+1}[\lambda_5]\varphi^{\mathrm{I}}_{\lambda+\alpha_{15}},\\
\pi^{\mathrm{I}}_\lambda(\hat{E}_{51})\varphi^{\mathrm{I}}_\lambda &= q^{\mu+\lambda_3-1}[\lambda_1]\varphi^{\mathrm{I}}_{\lambda+\alpha_{51}},\\
\pi^{\mathrm{I}}_\lambda(\hat{E}_{16})\varphi^{\mathrm{I}}_\lambda &= q^{-\mu-\lambda_3-\lambda_5+1}[\lambda_6]\varphi^{\mathrm{I}}_{\lambda+\alpha_{16}},\\
\pi^{\mathrm{I}}_\lambda(\hat{E}_{61})\varphi^{\mathrm{I}}_\lambda &= q^{\mu+\lambda_3+\lambda_5-1}[\lambda_1]\varphi^{\mathrm{I}}_{\lambda+\alpha_{61}},\\
\pi^{\mathrm{I}}_\lambda(\hat{E}_{24})\varphi^{\mathrm{I}}_\lambda &= -q^{\lambda_4+1}[\lambda_2+1]\varphi^{\mathrm{I}}_{\lambda+\alpha_{24}},\\
\pi^{\mathrm{I}}_\lambda(\hat{E}_{42})\varphi^{\mathrm{I}}_\lambda &= -q^{-\lambda_4-2}[\lambda_4+1]\varphi^{\mathrm{I}}_{\lambda+\alpha_{42}},\\
\pi^{\mathrm{I}}_\lambda(\hat{E}_{25})\varphi^{\mathrm{I}}_\lambda &= -q^{\lambda_2-\lambda_3+2}\frac{[\lambda_2+1][\lambda_5]}{[\mu+1]}\varphi^{\mathrm{I}}_{\lambda+\alpha_{25}},\\
\pi^{\mathrm{I}}_\lambda(\hat{E}_{52})\varphi^{\mathrm{I}}_\lambda &= -q^{-\lambda_2+\lambda_4-1}[\mu]\varphi^{\mathrm{I}}_{\lambda+\alpha_{52}},\\
\pi^{\mathrm{I}}_\lambda(\hat{E}_{26})\varphi^{\mathrm{I}}_\lambda &= -q^{\lambda_2-\lambda_3-\lambda_5+2}\frac{[\lambda_2+1][\lambda_6]}{[\mu+1]}\varphi^{\mathrm{I}}_{\lambda+\alpha_{26}},\\
\pi^{\mathrm{I}}_\lambda(\hat{E}_{62})\varphi^{\mathrm{I}}_\lambda &= -q^{-\lambda_2+\lambda_3+\lambda_5-1}[\mu]\varphi^{\mathrm{I}}_{\lambda+\alpha_{62}},\\
\pi^{\mathrm{I}}_\lambda(\hat{E}_{35})\varphi^{\mathrm{I}}_\lambda &= -q^{\lambda_2-\lambda_4+\lambda_5+\lambda_6+2}\frac{[\lambda_3+1][\lambda_5]}{[\mu+1]}\varphi^{\mathrm{I}}_{\lambda+\alpha_{35}},\\
\pi^{\mathrm{I}}_\lambda(\hat{E}_{53})\varphi^{\mathrm{I}}_\lambda &= -q^{-\lambda_2+\lambda_4-\lambda_5-\lambda_6-3}[\mu]\varphi^{\mathrm{I}}_{\lambda+\alpha_{53}},
\end{aligned}$$

$$
\begin{aligned}
\pi^{\mathrm{I}}_\lambda(\hat{E}_{36})\varphi^{\mathrm{I}}_\lambda &= -q^{\lambda_2-\lambda_4+\lambda_6+2}\frac{[\lambda_3+1][\lambda_6]}{[\mu+1]}\varphi^{\mathrm{I}}_{\lambda+\alpha_{36}},\\
\pi^{\mathrm{I}}_\lambda(\hat{E}_{63})\varphi^{\mathrm{I}}_\lambda &= -q^{-\lambda_2-\lambda_4-\lambda_6-3}[\mu]\varphi^{\mathrm{I}}_{\lambda+\alpha_{63}},\\
\pi^{\mathrm{I}}_\lambda(\hat{E}_{46})\varphi^{\mathrm{I}}_\lambda &= q^{\lambda_2-\lambda_3-\lambda_5}\frac{[\lambda_4+1][\lambda_6]}{[\mu+1]}\varphi^{\mathrm{I}}_{\lambda+\alpha_{46}},\\
\pi^{\mathrm{I}}_\lambda(\hat{E}_{64})\varphi^{\mathrm{I}}_\lambda &= q^{-\lambda_2+\lambda_3+\lambda_5}[\mu]\varphi^{\mathrm{I}}_{\lambda+\alpha_{64}}.
\end{aligned}
$$

For the $q$-hypergeometric series $\varphi^{\mathrm{II}}_\lambda$, we have

$$
\begin{aligned}
\pi^{\mathrm{II}}_\lambda(\hat{E}_{13})\varphi^{\mathrm{II}}_\lambda &= -q^{\lambda_1-\lambda_2+\lambda_4}[\mu]\varphi^{\mathrm{II}}_{\lambda+\alpha_{13}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{31})\varphi^{\mathrm{II}}_\lambda &= -q^{-\lambda_1+\lambda_2-\lambda_4+1}\frac{[\lambda_1][\lambda_3+1]}{[\mu+1]}\varphi^{\mathrm{II}}_{\lambda+\alpha_{31}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{14})\varphi^{\mathrm{II}}_\lambda &= -q^{-\lambda_2-\lambda_3}[\mu]\varphi^{\mathrm{II}}_{\lambda+\alpha_{14}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{41})\varphi^{\mathrm{II}}_\lambda &= -q^{\lambda_2+\lambda_3}\frac{[\lambda_1][\lambda_4+1]}{[\mu+1]}\varphi^{\mathrm{II}}_{\lambda+\alpha_{41}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{15})\varphi^{\mathrm{II}}_\lambda &= -q^{-2\mu+\lambda_4+2}[\lambda_5]\varphi^{\mathrm{II}}_{\lambda+\alpha_{15}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{51})\varphi^{\mathrm{II}}_\lambda &= -q^{2\mu-\lambda_4-2}[\lambda_1]\varphi^{\mathrm{II}}_{\lambda+\alpha_{51}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{16})\varphi^{\mathrm{II}}_\lambda &= -q^{-2\mu+\lambda_4-\lambda_5+2}[\lambda_6]\varphi^{\mathrm{II}}_{\lambda+\alpha_{16}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{61})\varphi^{\mathrm{II}}_\lambda &= -q^{2\mu-\lambda_4+\lambda_5-2}[\lambda_1]\varphi^{\mathrm{II}}_{\lambda+\alpha_{61}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{24})\varphi^{\mathrm{II}}_\lambda &= q^{-\lambda_1-\lambda_3-\lambda_4}[\lambda_2+1]\varphi^{\mathrm{II}}_{\lambda+\alpha_{24}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{42})\varphi^{\mathrm{II}}_\lambda &= q^{\lambda_1+\lambda_3+\lambda_4+1}[\lambda_4+1]\varphi^{\mathrm{II}}_{\lambda+\alpha_{42}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{25})\varphi^{\mathrm{II}}_\lambda &= q^{-\lambda_1-\lambda_3-\lambda_4-\mu-1}\frac{[\lambda_2+1][\lambda_5]}{[\mu+1]}\varphi^{\mathrm{II}}_{\lambda+\alpha_{25}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{52})\varphi^{\mathrm{II}}_\lambda &= q^{\lambda_1+\lambda_3+\lambda_4+\mu}[\mu]\varphi^{\mathrm{II}}_{\lambda+\alpha_{52}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{26})\varphi^{\mathrm{II}}_\lambda &= q^{-\lambda_1-\lambda_3-\lambda_4-\lambda_5-\mu-1}\frac{[\lambda_2+1][\lambda_6]}{[\mu+1]}\varphi^{\mathrm{II}}_{\lambda+\alpha_{26}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{62})\varphi^{\mathrm{II}}_\lambda &= -q^{\lambda_1+\lambda_3+\lambda_4+\lambda_5+\mu}[\mu]\varphi^{\mathrm{II}}_{\lambda+\alpha_{62}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{35})\varphi^{\mathrm{II}}_\lambda &= q^{-\lambda_1-\lambda_4-\mu}\frac{[\lambda_3+1][\lambda_5]}{[\mu+1]}\varphi^{\mathrm{II}}_{\lambda+\alpha_{35}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{53})\varphi^{\mathrm{II}}_\lambda &= q^{\lambda_1+\lambda_4+\mu-1}[\mu]\varphi^{\mathrm{II}}_{\lambda+\alpha_{53}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{36})\varphi^{\mathrm{II}}_\lambda &= q^{-\lambda_1-\lambda_4-\lambda_5-\mu}\frac{[\lambda_3+1][\lambda_6]}{[\mu+1]}\varphi^{\mathrm{II}}_{\lambda+\alpha_{36}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{63})\varphi^{\mathrm{II}}_\lambda &= q^{\lambda_1+\lambda_4+\lambda_5+\mu-1}[\mu]\varphi^{\mathrm{II}}_{\lambda+\alpha_{63}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{46})\varphi^{\mathrm{II}}_\lambda &= q^{\lambda_4-\lambda_5-\mu}\frac{[\lambda_4+1][\lambda_6]}{[\mu+1]}\varphi^{\mathrm{II}}_{\lambda+\alpha_{46}},\\
\pi^{\mathrm{II}}_\lambda(\hat{E}_{64})\varphi^{\mathrm{II}}_\lambda &= q^{-\lambda_4+\lambda_5+\mu}[\mu]\varphi^{\mathrm{II}}_{\lambda+\alpha_{64}}.
\end{aligned}
$$

We can now rewrite these relations in a much simpler form; namely, referring to these relations, define the elements of $U_q(gl(6))$ by

$$(q-q^{-1})^2 C_{ij} = (q^{1+\varepsilon_i} - q^{-1-\varepsilon_i})(q^{\varepsilon_j} - q^{-\varepsilon_j}) - (q-q^{-1})^2 \hat{E}_{ji}\hat{E}_{ij}, \tag{1.1.13}$$

for $1 \leq i \neq j \leq 6$. Then we can define a quantum analogue of the classical hypergeometric system $E_{3,6}$ as follows.

**Definition 1.6.3.** We call the following system the *$q$-hypergeometric system of type* $(3,6;*)$:

$$T_{q,x_{24}}T_{q,x_{25}}T_{q,x_{26}}u = q^{-\lambda_2-1}u, \qquad T_{q,x_{34}}T_{q,x_{35}}T_{q,x_{36}}u = q^{-\lambda_3-1}u, \tag{1.6.3}$$

$$\rho_\lambda^*(C_{ij})u = 0 \qquad (1 \leq i \neq j \leq 6), \tag{1.6.4*}$$

where $*$ =I or II. A direct computation by use of the contiguity relations above shows

**Theorem 1.6.4.** The functions $G_\lambda^*$ is a solution of the $q$-hypergeometric system of type $(3,6;*)$ for $*$ =I or II.

Finally we define the $q$-difference equation as follows.

**Proposition 1.6.5.** For $J = (j_1, j_2, j_3, j_4)$ $(j_2 < j_4,\ j_k \neq j_l$ if $k \neq l)$, we have

$$\pi_\lambda^*(\hat{E}_{j_1j_2}\hat{E}_{j_3j_4} - q^{a(J,\lambda)}\hat{E}_{j_1j_4}\hat{E}_{j_3j_2})\varphi_\lambda^* = 0$$

for each $*$ =I or II, where

$$a(J,\lambda) = \begin{cases} -1 & (j_1, j_3 < j_2 < j_4 \text{ or } j_2 < j_4 < j_1, j_3) \\ \lambda_{j_2} + \lambda_{j_3} & (j_1 < j_2 < j_3 < j_4) \\ -\lambda_{j_1} - \lambda_{j_2} & (j_3 < j_2 < j_1 < j_4) \\ -\lambda_{j_1} - \lambda_{j_4} & (j_2 < j_1 < j_4 < j_3) \\ \lambda_{j_3} + \lambda_{j_4} & (j_2 < j_3 < j_4 < j_1) \\ -\lambda_{j_1} + \lambda_{j_3} + 1 & (j_2 < j_1, j_3 < j_4) \\ \lambda_{j_2} - \lambda_{j_4} - 1 & (j_1 < j_2 < j_4 < j_3 \text{ or } j_3 < j_2 < j_4 < j_1). \end{cases}$$

We can compute these equations by the contiguity relations. Let us notice that $a(J,\lambda)$ does not depend on the type of the monomial basis I and II.

## Chapter 2. q-Hypergeometric functions associated with the quantum Grassmannians

### 2.0. Introduction of this chapter

In the previous chapter, we discussed about $q$-hypergeometric system of type $(3,6)$. In this chapter, we will discuss a quantum analogue of $G_{k,n}$, and associated generalized $q$-hypergeometric system and $q$-hypergemetric functions which is $q$-analogue of hypergeometric function associated with $G_{k,n}$.

In the first section, we consider the quantum analogue of the space $G_{k,n}$ and quantum minor of degree $k$.

In the second section, we give the short review of the classical case.

In the third section, we define the localization, which has a $U_q(gl(n))$-submodule structure of the noncommutative algebra .

In the fourth section, we define the action of $U_q(gl(n))$ on the commutative polynomial ring. Furthermore, we exibit the action in terms of $q$-difference operators.

In the fifth section, we define the Casimir operator and, as the solution of the system of $q$-difference equations, realize the $q$-hypergeometric function defined by the series

$$\varphi_{k,n}(\lambda;q,z)=\sum_{a_j^r\geq 0}\frac{\prod_{j=k+2}^{n}(q^{-\lambda_j};q)_{|A_j|}\prod_{r=2}^{k}(q^{\lambda_r+1};q)_{|A^r|}}{(q^{\gamma};q)_{|A|}\prod_{r,j}(q;q)_{a_j^r}}z^A. \tag{2.5.7}$$

In the last section, we compute the contiguity relation for $q$-hypergeometric function.

The contents of this chapter is a joint research of M. Noumi and the author.

### 2.1. Quantum Grassmannians

In this section, we introduce the quantum Grassmannians which is a quantum analogue of the space $G_{k,n}$.

In the previous chapter, we defined the noncommutative algebra $A_q(M(k,n))$, which is the quantum analogue of the coordinate ring $A(M(k,n))$.

Now we define the *quantum minor determinants* $\xi_{j_1\cdots j_k}$ by

$$\xi_{j_1\cdots j_k}=\sum_{\omega\in S_k}(-q)^{l(\omega)}t_{\omega(1)j_1}t_{\omega(2)j_2}\cdots t_{\omega(k)j_k}\qquad(1\leq j_1,\ldots,j_k\leq n). \tag{2.1.1}$$

Here $S_k$ is the permutation group of $k$ letters and, for each $\omega \in S_k$, $l(\omega)$ denotes the number of inversions in $\omega$.

We denote the subalgebra in $A_q(M(k,n))$ generated by the quantum minors $\{\xi_{j_1 \dots j_k}\}$ by $\mathcal{A}$. Then, for the definition, we can calculate the properties of the quantum minors as follows:

**Proposition 2.1.1.** For $j_1 \leq j_2 \leq \dots \leq j_k$ and $\omega \in S_k$,

$$\xi_{j_1 \dots j_k} = (-q)^{-l(\omega)} \xi_{j_{\omega(1)} \dots j_{\omega(k)}}. \tag{2.1.2}$$

Especially, if $j_\alpha = j_\beta$ for $1 \leq \alpha \neq \beta \leq k$,

$$\xi_{j_1 \dots j_\alpha \dots j_\beta \dots j_k} = 0. \tag{2.1.3}$$

**Proof.** For $j_\alpha > j_{\alpha+1}$, we can easily compute that

$$\xi_{j_1 \dots j_\alpha j_{\alpha+1} \dots j_k} = (-q) \xi_{j_1 \dots j_{\alpha+1} j_\alpha \dots j_k},$$

and, for $j_\alpha = j_{\alpha+1}$,

$$\xi_{j_1 \dots j_\alpha j_{\alpha+1} \dots j_k} = 0.$$

Furthermore, these minors satisfy the Plücker relations as follows:

**Proposition 2.1.2(Plücker relations).** For $1 \leq i_1, \dots, i_{k-1} \leq n$ and $1 \leq j_0 \leq \dots \leq j_k \leq n$,

$$\sum_{m=0}^{k} (-q)^m \xi_{i_1 \dots i_{k-1} j_m} \xi_{j_0 \dots \hat{j_m} \dots j_k} = 0 \tag{2.1.4}$$

where $l(J_1; J_2) = \#\{(j_\alpha, j_\beta);\, j_\alpha \in J_1, j_\beta \in J_2, j_\alpha > j_\beta\}$.

**Proof.** See [NYM].

Here, similarly as the previous chapter, we denote the noncommutative algebra, which is generated by the quantum minors of degree $k$, by $\mathcal{A}$.

## 2.2. Classical case (hypergeometric functions associated with the Grassmannians)

In this section, we discuss again the hypergeometric system of differential equations associated with the Grassmannian $G_{k,n}$ $(k \leq n)$ in order to refer on the making of the $q$-hypergeometric function associated with the quantum Grassmannian.

Let us consider the $k \times n$ matrices

$$T = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ t_{21} & t_{22} & \dots & t_{2n} \\ & \dots & \dots & \\ t_{k1} & t_{k2} & \dots & t_{kn} \end{pmatrix} \in M(k,n). \tag{2.2.1}$$

The hypergeometric system of differential equations of type $G_{k,n}$ is the following system on the space $M(k,n)$:

$$\Phi(gT) = \det(g)^{-1}\Phi(T) \quad (g \in GL(k)) \tag{2.2.2}$$

$$\Phi(T\mathrm{diag}(c_1, \dots, c_n)) = \Phi(T){c_1}^{\lambda_1}\cdots {c_n}^{\lambda_n} \tag{2.2.3}$$

$$\frac{\partial^2}{\partial t_{ri}\partial t_{sj}}\Phi(T) = \frac{\partial^2}{\partial t_{si}\partial t_{rj}}\Phi(T) \quad (1 \leq r < s \leq k, 1 \leq i < j \leq n), \tag{2.2.4}$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is a set of complex numbers. For the conditions (2.2.2) and (2.2.3) to be compatible, it is necessary to assume $\lambda_1 + \cdots + \lambda_n = -k$.

Malutivalued holomorphic solutions $\Phi(T)$ of the system (2.2.2) – (2.2.4) are called the hypergeometric functions of type $(k,n)$.

Now we can decompose a general $k \times n$ matrix $T$ as follows:

$$T = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1k} \\ t_{21} & t_{22} & \dots & t_{2k} \\ & \dots & \dots & \\ t_{k1} & t_{k2} & \dots & t_{kk} \end{pmatrix} \times T', \tag{2.2.5}$$

where

$$T' = \begin{pmatrix} 1 & & & (-1)^1{\xi_{1\cdots k}}^{-1}\xi_{\hat{1}k+1} & \cdots & (-1)^1{\xi_{1\cdots k}}^{-1}\xi_{\hat{1}n} \\ & 1 & & (-1)^2{\xi_{1\cdots k}}^{-1}\xi_{\hat{2}k+1} & \cdots & (-1)^2{\xi_{1\cdots k}}^{-1}\xi_{\hat{2}n} \\ & & \ddots & & \cdots & \\ & & 1 & (-1)^k{\xi_{1\cdots k}}^{-1}\xi_{\hat{k}k+1} & \cdots & (-1)^k{\xi_{1\cdots k}}^{-1}\xi_{\hat{k}n} \end{pmatrix}, \tag{2.2.6}$$

$\xi_{\hat{r}j} = \xi_{1\cdots\hat{r}\cdots kj}$ for $r = 1, \dots, k$ $\;j = k+1, \dots, n$, and $\{\xi_{j_1 j_2 \dots j_k}\}$ are defined as (2.1.5) in Remark 2.1.3. Furthermore, $T'$ can be decomposed as follows:

$$\begin{aligned} T' = {} & \mathrm{diag}(1, (-1)^1\xi_{\hat{1}k+1}{\xi_{\hat{2}k+1}}^{-1}, \dots, (-1)^{k-1}\xi_{\hat{1}k+1}{\xi_{\hat{k}k+1}}^{-1}) \\ & \times \begin{pmatrix} 1 & & & & 1 & 1 & 1 & \dots & 1 \\ & 1 & & & 1 & z_{k+2}^2 & z_{k+3}^2 & \dots & z_n^2 \\ & & \ddots & & \vdots & & \dots & \dots & \\ & & & 1 & 1 & z_{k+2}^k & z_{k+3}^k & \dots & z_n^k \end{pmatrix} \\ & \times \mathrm{diag}(1, \overbrace{(-1)^1{\xi_{\hat{1}k+1}}^{-1}\xi_{\hat{2}k+1}, \dots, (-1)^{k-1}{\xi_{\hat{1}k+1}}^{-1}\xi_{\hat{k}k+1}}^{k-1}, \\ & \qquad \overbrace{-{\xi_{1\cdots k}}^{-1}\xi_{\hat{1}k+1}, \dots, -{\xi_{1\cdots k}}^{-1}\xi_{\hat{1}n}}^{n-k}), \end{aligned} \tag{2.2.7}$$

where $z_j^r = \dfrac{\xi_{\hat{r}j}\xi_{\hat{1}k+1}}{\xi_{\hat{r}k+1}\xi_{\hat{1}j}}$ for $r = 2, \dots, k$ and $j = k+2, \dots, n$. Therefore $\Phi(T)$ can be written in the form

$$\begin{aligned}\Phi(T) =& F(z_{k+2}^2, z_{k+3}^2, \dots, z_n^k) \\ & \times \xi_{1\cdots k}{}^{\lambda_1+\lambda_2+\cdots+\lambda_k+k-1} \xi_{\hat{1}k+1}{}^{\lambda_2+\lambda_3+\cdots+\lambda_{k+1}+k-1} \prod_{r=2}^{k} \xi_{\hat{r}k+1}{}^{-\lambda_r-1} \prod_{j=k+2}^{n} \xi_{\hat{1}j}{}^{\lambda_j},\end{aligned} \tag{2.2.8}$$

for some function $F(z_{k+2}^2, \dots, z_n^k)$ in the $(k-1)\times(n-k-1)$ variables $(z_{k+2}^2, z_{k+3}^2, \dots, z_n^k)$.

**Remark 2.2.1.** For the condition (2.2.2) and (2.2.3), the decomposition (2.2.5) means that $\Phi(T)$ can be written as the function of the $k\times(n-k)+2$ variables $(\xi_{1\cdots k}{}^{\pm1}, \xi_{\hat{1}_{k+1}}, \dots, \xi_{\hat{k}_n})$ as follows:

$$\Phi(T) = G(\xi_{1\cdots k}{}^{\pm1}, \xi_{\hat{1}k+1}, \xi_{\hat{1}k+2}, \dots, \xi_{\hat{k}n}). \tag{2.2.9}$$

Furthermore, because of invariances (2.2.2) and (2.2.3) again, the decomposition (2.2.6) and (2.2.7) mean that the function $G(\xi_{1\cdots k}{}^{\pm1}, \dots, \xi_{\hat{k}n})$ has the homogeneities:

$$\sum_{r=1}^{k} \xi_{\hat{r}j} \frac{\partial}{\partial \xi_{\hat{r}j}} G(\xi_{1\cdots k}{}^{\pm1}, \dots, \xi_{\hat{k}n}) = \lambda_j G(\xi_{1\cdots k}{}^{\pm1}, \dots, \xi_{\hat{k}n}) \quad \text{for } k \le j \le n, \tag{2.2.10}$$

$$\sum_{j=k+1}^{n} \xi_{\hat{r}j} \frac{\partial}{\partial \xi_{\hat{r}j}} G(\xi_{1\cdots k}{}^{\pm1}, \dots, \xi_{\hat{k}n}) = (-\lambda_r - 1) G(\xi_{1\cdots k}{}^{\pm1}, \dots, \xi_{\hat{k}n}) \quad \text{for } 1 \le r \le k. \tag{2.2.11}$$

In these coordinates $z = (z_{k+2}^2, z_{k+3}^2, \dots, z_n^k)$, it is known that a solution of the equation (2.2.4), holomorphic around the origin $z = 0$, is given as follows:

$$F_{k,n}(z;\lambda) = \sum_{a_j^r \ge 0} \frac{\prod_{j=k+2}^{n}(-\lambda_j)_{|A_j|} \prod_{r=2}^{k}(\lambda_r+1)_{|A^r|}}{(\gamma)_{|A|}\prod_{r,j}(1)_{a_j^r}} z^A, \tag{2.2.12}$$

where $\gamma = \lambda_2 + \cdots + \lambda_{k+1} + k$. Here, for the matrix $A = (a_j^r)_{2\le r\le k, k+2\le j\le n}$, we use the notations

$$|A_j| = a_j^2 + a_j^3 + \cdots + a_j^k, \quad |A^r| = a_{k+2}^r + a_{k+3}^r + \cdots + a_n^r, \quad |A| = \sum_{r,j} a_j^r \tag{2.2.13}$$

and

$$z^A = \prod_{r,j} z_j^{r\, a_j^r}. \tag{2.2.14}$$

**Remark 2.2.2.** The hypergeometric function $F_{2,n}$ associated with $G_{2,n}$ is known as the *Lauricella's* hypergeometric series $F_D$ in $n-3$ variables. Especially, $F_{2,4}$ is known as Gauss hypergeometric series ${}_2F_1$.

Now let us set

$$E_{ij} = \sum_{r=1}^{k} t_{ri}\frac{\partial}{\partial t_{rj}}. \tag{2.2.15}$$

Then $\{E_{ij}\}_{1\leq i,j\leq n}$ generate a Lie algebra isomorphic to $gl(n)$ and, for ${}^{\exists}\nu(\lambda;i,j)\in\mathbf{C}$, we have

$$E_{i,j}F_{k,n}(z;\lambda) = \nu(\lambda;i,j)F_{k,n}(z;\lambda^i_j)$$

where $\lambda^i_j = (\lambda_1,\ldots,\lambda_i+1,\ldots,\lambda_j-1,\ldots,\lambda_n)$ for $1\leq i,j\leq n$. In that sense, the contiguity relations for the hypergeometric function $F_{k,n}$ give a representation of $U(gl(n))$ which is the universal enveloping algebra of Lie algebra $gl(n)$. Furthermore, as differential operators, $\{E_{ij}\}$ define a $U(gl(n))$-module structure on the polynomial algebra $\mathbf{C}[t_{11},t_{12},\ldots,t_{kn}]$ of $k\times n$ variables $(t_{rj})_{1\leq r\leq k,1\leq j\leq n}$. Remember that it holds an identity

$$(E_{ii}+1)E_{jj} - E_{ji}E_{ij} = \sum_{1\leq r<s\leq k}(t_{ri}t_{sj}-t_{si}t_{rj})\left(\frac{\partial^2}{\partial t_{ri}\partial t_{sj}} - \frac{\partial^2}{\partial t_{si}\partial t_{rj}}\right), \tag{1.1.11}$$

for each $i\neq j$.

Then the function

$$\begin{aligned} G_\lambda(&{\xi_{1\cdots k}}^{\pm1},\xi_{\hat{1}k+1},\ldots,\xi_{\hat{k}n}) \\ &= F_{k,n}(z,\lambda){\xi_{1\cdots k}}^{\gamma+\lambda_1-\lambda_{k+1}-1}{\xi_{\hat{1}k+1}}^{\gamma-1}\prod_{r=2}^{k}{\xi_{\hat{r}k+1}}^{-\lambda_r-1}\prod_{j=k+2}^{n}{\xi_{\hat{1}j}}^{\lambda_j}, \end{aligned} \tag{2.2.16}$$

is a solution of the system of differential equations (2.2.10), (2.2.11), and

$$\{(E_{ii}+1)E_{jj}-E_{ji}E_{ij}\}G({\xi_{1\cdots k}}^{\pm1},\xi_{\hat{1}k+1},\ldots,\xi_{\hat{k}n}) = 0. \tag{2.2.17}$$

**Remark 2.2.3.** By the condition (2.2.2) and deconposition (2.2.5), we have

$$\Phi(T) = {\xi_{1\cdots k}}^{-1}\Phi(T'). \tag{2.2.18}$$

Therefore the function $G({\xi_{1\cdots k}}^{\pm1},\ldots,\xi_{\hat{k}n})$ can be written as follows:

$$G({\xi_{1\cdots k}}^{\pm1},\ldots,\xi_{\hat{k}n}) = \sum_{\nu\in(\mathbf{Z}_{\geq0})^{k(n-k)}}\alpha(\nu)\xi_{\hat{1}n}^{\nu_n^1}\xi_{\hat{1}n-1}^{\nu_{n-1}^1}\cdots\xi_{\hat{1}k+1}^{\nu_{k+1}^1}\xi_{\hat{2}n}^{\nu_n^2}\xi_{\hat{2}n-1}^{\nu_{n-1}^2}\cdots\xi_{\hat{k}k+1}^{\nu_{k+1}^k}{\xi_{1\cdots k}}^{-|\nu|-1} \tag{2.2.19}$$

where $|\nu| = \sum_{r,j} \nu_j^r$ for the $k \times (n-k)$ matrix $\nu = (\nu_j^r)_{1 \le r \le k, k+1 \le j \le n}$.

### 2.3. The representation of the algebra $\mathbf{U_q(gl(n))}$ and a localization of $\mathcal{A}$

In this section, we define the action of $U_q(gl(n))$ on the noncommutative algebra $A_q(M(k,n))$. Furthermore, we define a localization of the noncommutative algebra $\mathcal{A}$ generated by quantum minors of degree $k$.

The algebra $A_q(M(k,n))$ has a structure of a left $U_q(gl(n))$-module; it is determined by assuming the $U_q(gl(n))$-symmetry.

Furthermore, the subalgebra $\mathcal{A}$ is a left $U_q(gl(n))$-submodule of $A_q(M(k,n))$. The actions of the generators of $U_q(gl(n))$ are determined by

$$
\begin{aligned}
q^h \cdot \xi_{j_1 \dots j_k} &= q^{\langle h, \epsilon_{j_1} + \dots + \epsilon_{j_k} \rangle} \xi_{j_1 \dots j_k}, \\
e_l \cdot \xi_{j_1 \dots j_k} &= \sum_{m=1}^{k} \delta_{l+1, j_m} q^{\langle \epsilon_l - \epsilon_{l+1}, \epsilon_{j_1} + \dots + \epsilon_{j_{m-1}} \rangle} \xi_{j_1 \dots j_m - 1 \dots j_k}, \\
f_l \cdot \xi_{j_1 \dots j_k} &= \sum_{m=1}^{k} \delta_{l, j_m} q^{\langle -\epsilon_l + \epsilon_{l+1}, \epsilon_{j_{m+1}} + \dots + \epsilon_{j_k} \rangle} \xi_{j_1 \dots j_m + 1 \dots j_k}.
\end{aligned}
\tag{2.3.1}
$$

Here we introduce a localization of $\mathcal{A}$. In the decomposition (2.2.5), the ${\xi_{1\dots k}}^{-1}$ has played a special role. Herefore, we invert the principal minor $\xi_{1\dots k}$.

**Theorem 2.3.1.** The localization $\mathcal{A}[{\xi_{1\dots k}}^{-1}]$ is the algebra generated by ${\xi_{1\dots k}}^{\pm 1}$ and $\xi_{\hat{r}j} (1 \le r \le k, k+1 \le j \le n)$.

**Proof.** Take any index set $\{r_1, \dots, r_{k-s}, j_1, \dots, j_s\}$ such as $1 \le r_1 < \dots < r_{k-s} \le k$ and $k+1 \le j_1 < \dots < j_s \le n$. Then by the Plücker relations (2.1.4), we have a form

$$\xi_{r_1 \dots r_{k-s} j_1 \dots j_s} \xi_{1 \dots k} = -\sum_{i=1}^{k} (-q)^{i-k} \xi_{r_1 \dots r_{k-s} j_1 \dots j_{s-1} i} \xi_{\hat{i} j_s}.$$

In the right hand side, the number of indeces which are larger than $k+1$ in the first factors are $r-1$. By using this method inductively, we can see that $\xi_{r_1 \dots r_{k-s} j_1 \dots j_s} \xi_{1 \dots k}^{r-1}$ lies in the algebra generated by $\xi_{1\dots k}^{\pm 1}$ and $\xi_{\hat{r}j} (1 \le r \le k, k+1 \le j \le n)$.

Furthermore, by the property (2.1.3) and the Plücker relations (2.1.4), the commutation relations among $\xi_{1\dots k}^{\pm 1}$ and $\xi_{\hat{r}j}$ are generated by the following relations:

$$\xi_{1 \dots k} \xi_{\hat{r}j} = q \xi_{\hat{r}j} \xi_{1 \dots k} \quad \text{for} \quad 1 \le r \le k, \quad k+1 \le j \le n, \tag{2.3.2}$$

and

$$\xi_{\hat{r}j}\xi_{\hat{s}j} = q^{-1}\xi_{\hat{s}j}\xi_{\hat{r}j}, \qquad \xi_{\hat{r}i}\xi_{\hat{r}j} = q\xi_{\hat{r}j}\xi_{\hat{r}i},$$
$$\xi_{\hat{r}i}\xi_{\hat{s}j} = \xi_{\hat{s}j}\xi_{\hat{r}i}, \qquad \xi_{\hat{s}i}\xi_{\hat{r}j} - \xi_{\hat{r}j}\xi_{\hat{s}i} = (q-q^{-1})\xi_{\hat{r}i}\xi_{\hat{s}j}, \tag{2.3.3}$$
$$\text{for} \quad 1 \le r < s \le k, \quad k+1 \le i < j \le n.$$

**Remark 2.3.2.** We can see that the $2 \times 2$ matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \xi_{\hat{s}i} & \xi_{\hat{s}j} \\ \xi_{\hat{r}i} & \xi_{\hat{r}j} \end{pmatrix} \qquad \begin{pmatrix} 1 \le r < s \le k \\ 1 \le i < j \le n \end{pmatrix}$$

again satisfies the $Mat_q(2)$-relations (1.2.1).

Now the algebra $\mathcal{A}[\xi_{1\cdots k}{}^{-1}]$ has the monomial basis

$$\xi_{\hat{1}n}^{\nu_n^1}\xi_{\hat{1}n-1}^{\nu_{n-1}^1}\cdots\xi_{\hat{1}k+1}^{\nu_{k+1}^1}\xi_{\hat{2}n}^{\nu_n^2}\xi_{\hat{2}n-1}^{\nu_{n-1}^2}\cdots\xi_{\hat{k}k+1}^{\nu_{k+1}^k}\xi_{1\cdots k}^{\mu} \tag{2.3.4}$$

where $\nu_j^r \in \mathbf{Z}_{\ge 0}$ and $\mu \in \mathbf{Z}$. In the following, we set

$$\xi^{\nu} = \xi_{\hat{1}n}^{\nu_n^1}\xi_{\hat{1}n-1}^{\nu_{n-1}^1}\cdots\xi_{\hat{1}k+1}^{\nu_{k+1}^1}\xi_{\hat{2}n}^{\nu_n^2}\xi_{\hat{2}n-1}^{\nu_{n-1}^2}\cdots\xi_{\hat{k}k+1}^{\nu_{k+1}^k} \tag{2.3.5}$$

for each matrix $\nu = (\nu_j^r)_{1\le r\le k, k+1\le j\le n}$ of nonnegative integers. With this notation, we have

$$\mathcal{A}[\xi_{1\cdots k}{}^{-1}] = \bigoplus_{\nu\in(\mathbf{Z}_{\ge 0})^{k(n-k)};\mu\in\mathbf{Z}} \mathbf{C}\xi^{\nu}\xi_{1\cdots k}{}^{\mu}. \tag{2.3.6}$$

Furthermore, we can see that the algebra $\mathcal{A}[\xi_{1\cdots k}{}^{-1}]$ has a structure of a left $U_q(gl(n))$-submodule.

We now consider the homogeneous component of degree $-1$ of $\mathcal{A}[\xi_{1\cdots k}{}^{-1}]$ with the respect to the quantum minor $\xi_{j_1\ldots j_k}$:

$$\mathcal{M}_{k,n} = \bigoplus_{\nu\in(\mathbf{Z}_{\ge 0})^{k(n-k)}} \mathbf{C}\xi^{\nu}\xi_{1\cdots k}{}^{-|\nu|-1}. \tag{2.3.7}$$

where $|\nu| = \sum_{r,j}\nu_j^r$. It is easy to see that this subspace $\mathcal{M}_{k,n}$ has the $U_q(gl(n))$-submodule structure of $\mathcal{A}[\xi_{1\cdots k}{}^{-1}]$. Here, for all $a \in U_q(gl(n))$, we denote its action on $\mathcal{M}_{k,n}$ by

$$\bar{\rho}(a) : \mathcal{M}_{k,n} \to \mathcal{M}_{k,n}. \tag{2.3.8}$$

## 2.4. Translation of the action of $\mathbf{U_q(gl(n))}$

In this section, we translate the action of the $U_q(gl(n))$ on the *noncommutative* algebra $\mathcal{M}_{k,n}$ to that on the *commutative* polynomial ring. Moreover, we give the explicit formulas of the action of the generators of $U_q(gl(n))$ in the terms of the $q$-difference operators.

In order to translate the action of the algebra $U_q(gl(n))$ on the *noncommutative* algebra $\mathcal{M}_{k,n}$ into the $q$-shift operators, we consider the *commutative* polynomial ring $\mathbf{C}[x_k^1, x_{k+1}^1, \ldots, x_n^k]$ in $k \times (n-k)$ variables $x = (x_j^r)_{1\leq r\leq k, k+1\leq j\leq n}$. Here, for $\nu = (\nu_j^r)$, we use the notation of multi-indices

$$x^\nu = \prod_{1\leq r\leq k, k+1\leq j\leq n} x_j^{r\nu_j^r}. \tag{2.4.1}$$

Here let $\phi : M(k, n-k; \mathbf{Z}_{\geq 0}) \to \mathbf{Z}$ be an arbitrary function of degree at most two with values in $\mathbf{Z}$ in the form

$$\phi(\nu) = \sum_{r,s,i,j} \alpha_{r,s,i,j} \nu_i^r \nu_j^s + \sum_{r,j} \beta_{r,j} \nu_j^r + c, \tag{2.4.2}$$

for the $k \times (n-k)$ matrix $\nu = (\nu_j^r)_{1\leq r\leq k, k+1\leq j\leq n}$. For such a $\phi$, we define an isomorphism of vector space

$$\psi_\phi : \mathbf{C}[x] \to \mathcal{M}_{k,n}$$

by setting

$$\psi_\phi(x^\nu) = \xi^\nu {\xi_{1\cdots k}}^{-|\nu|-1} q^{\phi(\nu)}, \tag{2.4.3}$$

for all $\nu \in (\mathbf{Z}_{\geq 0})^{k(n-k)}$. Then we obtain a left $U_q(gl(n))$-module structure on the vector space $\mathbf{C}[x]$. Namely we have

$$\rho_\phi(a) = {\psi_\phi}^{-1} \circ \bar{\rho}(a) \circ \psi_\phi \tag{2.4.4}$$

for all $a \in U_q(gl(n))$. In the following, we denote by $q^{\theta_{r,j}}$ the $q$-shift operator in the variables $x_j^r$:

$$q^{\theta_{r,j}} x^\nu = q^{\nu_j^r} x^\nu. \tag{2.4.5}$$

**Remark 2.4.1.** Let $\phi$ and $\phi_0$ be two functions of degree $\leq 2$. Then it is easy to see the two representations $\rho_\phi$ and $\rho_{\phi_0}$ are related by the formula

$$\rho_\phi(a) = q^{-(\phi-\phi_0)(\theta)} \circ \rho_{\phi_0}(a) \circ q^{(\phi-\phi_0)(\theta)} \tag{2.4.6}$$

for all $a \in U_q(gl(n))$, where $\theta = (\theta_{r,j})_{1\leq r\leq k, k+1\leq j\leq n}$.

Now we set

$$\phi(\nu)=\sum_{1\le r<s\le k,k+1\le i<j\le n}\nu_i^r\nu_j^s-\sum_{j=k+2}^{n}(\nu_j^1)^2-\sum_{r=2}^{k}\nu_{k+1}^r(\nu_{k+1}^r+2). \tag{2.4.7}$$

In the following, we simply write $\rho=\rho_\phi$. We give the representation of $\rho$ by $q$-difference operators as follows:

$$\begin{aligned}
\rho(q^{\epsilon_r})&=q^{-1-\theta_{r,k+1\cdots n}}\quad(1\le r\le k),\\
\rho(q^{\epsilon_j})&=q^{\theta_{1\cdots k,j}}\quad(k+1\le j\le n),\\
(q-q^{-1})\rho(e_1)&=\sum_{j=k+2}^{n}\frac{x_j^2}{x_j^1}(1-q^{-2\theta_{1,j}})q^{1-\theta_{1,k+1\cdots n}+2\theta_{2,j+1\cdots n}}\\
&\quad+\frac{x_{k+1}^2}{x_{k+1}^1}(q^{2\theta_{1,k+1}}-1)q^{3-\theta_{1,k+1\cdots n}+2\theta_{2,k+1\cdots n}},\\
(q-q^{-1})\rho(e_r)&=\sum_{j=k+2}^{n}\frac{x_j^{r+1}}{x_j^r}(q^{2\theta_{r,j}}-1)q^{-\theta_{r,k+1\cdots n}+2\theta_{r+1,j+1\cdots n}}\\
&\quad+\frac{x_{k+1}^{r+1}}{x_{k+1}^r}(1-q^{-2\theta_{r,k+1}})q^{2-\theta_{r,k+1\cdots n}+2\theta_{r,k+1\cdots n}}\quad(2\le r\le k-1),\\
(q-q^{-1})\rho(e_k)&=\frac{1}{x_{k+1}^k}(1-q^{-2\theta_{k,k+1}})q^{-1+\theta_{1\cdots k-1,k+2\cdots n}},\\
(q-q^{-1})\rho(e_{k+1})&=\sum_{r=2}^{k}\frac{x_{k+1}^r}{x_{k+2}^r}(q^{2\theta_{r,k+2}}-1)q^{3+\theta_{1\cdots r,k+1}-\theta_{1\cdots k,k+2}}\\
&\quad+\frac{x_{k+1}^1}{x_{k+2}^1}(1-q^{-2\theta_{1,k+2}})q^{1+2\theta_{1,k+1}-\theta_{1\cdots k,k+2}},\\
(q-q^{-1})\rho(e_j)&=\sum_{r=2}^{k}\frac{x_j^r}{x_{j+1}^r}(q^{2\theta_{r,j+1}}-1)q^{2\theta_{1\cdots r-1,j}-\theta_{1\cdots k,j+1}}\\
&\quad+\frac{x_j^1}{x_{j+1}^1}(1-q^{-2\theta_{1,j+1}})q^{2+2\theta_{1,j}-\theta_{1\cdots k,j+1}}\quad(k+2\le j\le n-1),\\
(q-q^{-1})\rho(f_1)&=\sum_{j=k+2}^{n}\frac{x_j^1}{x_j^2}(q^{2\theta_{2,j}}-1)q^{1+2\theta_{1,k+1\cdots j}-\theta_{2,k+1\cdots n}}\\
&\quad+\frac{x_{k+1}^1}{x_{k+1}^2}(1-q^{-2\theta_{2,k+1}})q^{-1-\theta_{2,k+1\cdots n}},\\
(q-q^{-1})\rho(f_r)&=\sum_{j=k+2}^{n}\frac{x_j^r}{x_j^{r+1}}(q^{2\theta_{r+1,j}}-1)q^{2\theta_{r,k+1\cdots j-1}-\theta_{r+1,k+1\cdots n}}\\
&\quad+\frac{x_{k+1}^r}{x_{k+1}^{r+1}}(1-q^{-2\theta_{r+1,k+1}})q^{2+2\theta_{r,k+1}-\theta_{r+1,k+1\cdots n}}\quad(2\le r\le k-1),
\end{aligned}$$

$$
\begin{aligned}
(q-q^{-1})\rho(f_k) &= x_{k+1}^k q^{2-\theta_{1\cdots k,k+1\cdots n}+2\theta_{k,k+1}}(1-q^{2+2\theta_{1\cdots k,k+1}+2\theta_{k,k+2\cdots n}}) \\
&\quad + \sum_{p=1}^{k-1}(-1)^{p-1} \sum_{\substack{1\le r_1<\cdots<r_p<r_{p+1}=k \\ k+1=j_0<j_1<\cdots<j_p\le n}} \frac{x_{j_0}^{r_1}x_{j_1}^{r_2}\cdots x_{j_p}^{r_{p+1}}}{x_{j_1}^{r_1}x_{j_2}^{r_2}\cdots x_{j_p}^{r_p}} \\
&\quad \times \prod_{i=1}^{p}(1-q^{2\theta_{r_i,j_i}})q^{2+\eta(r_1)-2\theta_{1\cdots k-1,k+2\cdots n}-2\theta_{k,k+2\cdots j_p}} \\
&\quad \times q^{-2\sum_{i=1}^{p}(\theta_{r_i\cdots k-1,j_{i-1}}+\theta_{r_i+1\cdots k-1,j_{i-1}+1\cdots j_i-1})} \\
(q-q^{-1})\rho(f_{k+1}) &= \sum_{r=2}^{k}\frac{x_{k+2}^r}{x_{k+1}^r}(1-q^{-2\theta_{r,k+1}})q^{-1-\theta_{1\cdots k,k+1}+2\theta_{r+1\cdots k,k+2}} \\
&\quad + \frac{x_{k+2}^1}{x_{k+1}^1}(q^{2\theta_{1,k+1}}-1)q^{1-\theta_{1\cdots k,k+1}+2\theta_{1\cdots k,k+2}}, \\
(q-q^{-1})\rho(f_j) &= \sum_{r=2}^{k}\frac{x_{j+1}^r}{x_j^r}(q^{2\theta_{r,j}}-1)q^{-\theta_{1\cdots k,j}+2\theta_{r+1\cdots k,j+1}} \\
&\quad + \frac{x_{j+1}^1}{x_j^1}(1-q^{-2\theta_{1,j}})q^{2-\theta_{1\cdots k,j}+2\theta_{1\cdots k,j+1}} \quad (k+2\le j\le n-1),
\end{aligned}
$$

where $\theta_{r_1\cdots r_2,j_1\cdots j_2}=\sum_{r_1\le r\le r_2;j_1\le j\le j_2}\theta_{r,j}$ and

$$
\eta(r_1)=\begin{cases}-1-2\theta_{1,j_1} & r_1=1\\ 1+2\theta_{r_1,k+1} & \text{otherwise}\end{cases}. \tag{2.4.8}
$$

Thus we have translated the action of $U_q(gl(n))$ on the commutative algebra $\mathbf{C}[x]$.

### 2.5. Casimir elements and q-hypergeometric function of type (k,n)

In this section, we compute the action of the Casimir elements and define the $q$-hypergeometric systems of $q$-difference equations.

Furthermore we define the $q$-hypergeometric function of type $(k,n)$ as a solution of this system.

Note that, in the classical case, the hypergeometric function associated with $G_{k,n}$ is a solution of the system of differential equations (2.2.10), (2.2.11) and (2.2.17).

In view of the equation (2.2.17), we define the analogous elements of $U_q(gl(n))$

$$
(q-q^{-1})^2C_j=(q^{1+\epsilon_j}-q^{-1-\epsilon_j})(q^{\epsilon_{j+1}}-q^{-\epsilon_{j+1}})-(q-q^{-1})^2f_je_j \tag{2.5.1}
$$

for $1\le j\le n-1$. They are central elements of the subalgebra of $U_q(gl(n))$, isomorphic to $U_q(gl(2))$, and called the *Casimir elements.*

In the representations, the actions of the Casimir elements are computed as follows:

$$
\begin{aligned}
&(q-q^{-1})^2\rho(C_1)\\
&=\sum_{k+2\le i<j\le n}(q^{-1+\theta_{1,k+1\cdots i}-\theta_{1,i+1\cdots n}-\theta_{2,k+1\cdots j}+\theta_{2,j+1\cdots n}}x_i^1x_j^2\\
&\qquad\qquad -q^{1+\theta_{1,k+1\cdots j}-\theta_{1,j+1\cdots n}-\theta_{2,k+1\cdots i}+\theta_{2,i+1\cdots n}}x_j^1x_i^2)\\
&\qquad\times\{\frac{1}{x_j^1x_i^2}(1-q^{-2\theta_{1,j}})(q^{2\theta_{2,i}}-1)\\
&\qquad\qquad -\frac{1}{x_i^1x_j^2}(1-q^{-2\theta_{1,i}})(q^{2\theta_{2,j}}-1)\}\\
&\quad+\sum_{j=k+2}^{n}(q^{-1-\theta_{1,k+1\cdots n}-\theta_{2,k+1\cdots j}+\theta_{2,j+1\cdots n}}x_{k+1}^1x_j^2\\
&\qquad\qquad -q^{1+\theta_{1,k+1\cdots j}-\theta_{1,j+1\cdots n}+\theta_{2,k+1\cdots n}}x_j^1x_{k+1}^2)\\
&\qquad\times\{\frac{1}{x_j^1x_{k+1}^2}(1-q^{-2\theta_{2,k+1}})(1-q^{-2\theta_{1,j}})\\
&\qquad\qquad -\frac{1}{x_{k+1}^1x_j^2}(q^{2\theta_{1,k+1}}-1)(q^{2\theta_{2,j}}-1)\},\\
&(q-q^{-1})^2\rho(C_r)_{(2\le r\le k)}\\
&=\sum_{k+2\le i<j\le n}(q^{-1+\theta_{r,k+1\cdots i-1}-\theta_{r,i\cdots n}-\theta_{r+1,k+1\cdots j}+\theta_{r+1,j+1\cdots n}}x_i^rx_j^{r+1}\\
&\qquad\qquad -q^{1+\theta_{r,k+1\cdots j-1}-\theta_{r,j\cdots n}-\theta_{r+1,k+1\cdots i}+\theta_{r+1,i+1\cdots n}}x_j^rx_i^{r+1})\\
&\qquad\times\{\frac{1}{x_j^rx_i^{r+1}}(q^{2\theta_{r,j}}-1)(q^{2\theta_{r+1,i}}-1)\\
&\qquad\qquad -\frac{1}{x_i^rx_j^{r+1}}(q^{2\theta_{r,i}}-1)(q^{2\theta_{r+1,j}}-1)\}\\
&\quad+\sum_{j=k+2}^{n}(q^{-1+\theta_{r,k+1}-\theta_{r,k+2\cdots n}-\theta_{r+1,k+1\cdots j}+\theta_{r+1,j+1\cdots n}}x_{k+1}^rx_j^{r+1}\\
&\qquad\qquad -q^{1+\theta_{r,k+1\cdots j-1}-\theta_{r,j\cdots n}+\theta_{r+1,k+1\cdots n}}x_j^rx_{k+1}^{r+1})\\
&\qquad\times\{\frac{1}{x_j^rx_{k+1}^{r+1}}(1-q^{-2\theta_{r+1,k+1}})(q^{2\theta_{r,j}}-1)\\
&\qquad\qquad -\frac{1}{x_{k+1}^rx_j^{r+1}}(1-q^{-2\theta_{r,k+1}})(q^{2\theta_{r+1,j}}-1)\},\\
&(q-q^{-1})^2\rho(C_k)\\
&=(q^{\theta_{1\cdots k,k+1}+\theta_{k,k+2\cdots n}}-q^{-\theta_{1\cdots k,k+1}-\theta_{k,k+2\cdots n}})(q^{\theta_{k,k+1}}-q^{-\theta_{k,k+1}})\\
&\qquad -(q^{\theta_{1\cdots k,k+1}}-q^{-\theta_{1\cdots k,k+1}})(q^{\theta_{k,k+1\cdots n}}-q^{-\theta_{k,k+1\cdots n}})
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{p=1}^{k-1}(-1)^{p-1} \sum_{\substack{1 \leq r_1 < \cdots < r_p < r_{p+1} = k \\ k+1 = j_0 < j_1 < \cdots < j_p \leq n}} \frac{x_{j_0}^{r_1} x_{j_1}^{r_2} \cdots x_{j_p}^{r_{p+1}}}{x_{j_1}^{r_1} x_{j_2}^{r_2} \cdots x_{j_p}^{r_p} x_{k+1}^{k}} \\
&\quad \times (1 - q^- 2\theta_{k,k+1}) \prod_{i=1}^{p} (1 - q^{2\theta_{r_i, j_i}}) q^{1+\eta(r_1) - \theta_{1\cdots k-1, k+2 \cdots n} - 2\theta_{k, k+2 \cdots j_p}} \\
&\quad \times q^{-2\sum_{i=1}^{p} (\theta_{r_i \cdots k-1, j_{i-1}} + \theta_{r_i+1 \cdots k-1, j_{i-1}+1 \cdots j_i - 1})}
\end{aligned}
$$

$$
\begin{aligned}
&(q - q^{-1})^2 \rho(C_{k+1}) \\
&= \sum_{2 \leq r < s \leq k} (q^{-1+\theta_{1\cdots r,k+1} - \theta_{r+1\cdots k,k+1} - \theta_{1\cdots s,k+2} + \theta_{s+1\cdots k,k+2}} x_{k+1}^{r} x_{k+2}^{s} \\
&\qquad - q^{1+\theta_{1\cdots s,k+1} - \theta_{s+1\cdots k,k+1} - \theta_{1\cdots r,k+2} + \theta_{r+1\cdots k,k+2}} x_{k+2}^{r} x_{k+1}^{s}) \\
&\quad \times \{\frac{1}{x_{k+2}^{r} x_{k+1}^{s}} (1 - q^{-2\theta_{s,k+1}})(q^{2\theta_{r,k+2}} - 1) \\
&\qquad - \frac{1}{x_{k+1}^{r} x_{k+2}^{s}} (1 - q^{-2\theta_{r,k+1}})(q^{2\theta_{s,k+2}} - 1)\} \\
&\quad + \sum_{s=2}^{k} (q^{-1+\theta_{1,k+1} - \theta_{2\cdots k,k+1} - \theta_{1\cdots s,k+2} + \theta_{s+1\cdots k,k+2}} x_{k+1}^{1} x_{k+2}^{s} \\
&\qquad - q^{1+\theta_{1\cdots s,k+1} - \theta_{s+1\cdots k,k+1} + \theta_{1\cdots k,k+2}} x_{k+2}^{1} x_{k+1}^{s}) \\
&\quad \times \{\frac{1}{x_{k+2}^{1} x_{k+1}^{s}} (1 - q^{-2\theta_{s,k+1}})(1 - q^{-2\theta_{1,k+2}}) \\
&\qquad - \frac{1}{x_{k+1}^{1} x_{k+2}^{s}} (q^{2\theta_{1,k+1}} - 1)(q^{2\theta_{s,k+2}} - 1)\},
\end{aligned}
$$

$$
\begin{aligned}
&(q - q^{-1})^2 \rho(C_j)_{(k+2 \leq j \leq n-1)} \\
&= \sum_{2 \leq r < s \leq k} (q^{-1+\theta_{1\cdots r-1,j} - \theta_{r\cdots k,j} - \theta_{1\cdots s,j+1} + \theta_{s+1\cdots k,j+1}} x_{j}^{r} x_{j+1}^{s} \\
&\qquad - q^{1+\theta_{1\cdots s-1,j} - \theta_{s\cdots k,j} - \theta_{1\cdots r,j+1} + \theta_{r+1\cdots k,j+1}} x_{j+1}^{r} x_{j}^{s}) \\
&\quad \times \{\frac{1}{x_{j+1}^{r} x_{j}^{s}} (q^{2\theta_{s,j}} - 1)(q^{2\theta_{r,j+1}} - 1) - \frac{1}{x_{j}^{r} x_{j+1}^{s}} (q^{2\theta_{r,j}} - 1)(q^{2\theta_{s,j+1}} - 1)\} \\
&\quad + \sum_{s=2}^{k} (q^{-1+\theta_{1,j} - \theta_{2\cdots k,j} - \theta_{1\cdots s,j+1} + \theta_{s+1\cdots k,j+1}} x_{j}^{1} x_{j+1}^{s} \\
&\qquad - q^{1+\theta_{1\cdots s-1,j} - \theta_{s\cdots k,j} + \theta_{1\cdots k,j+1}} x_{j+1}^{1} x_{j}^{s}) \\
&\quad \times \{\frac{1}{x_{j+1}^{1} x_{j}^{s}} (q^{2\theta_{s,j}} - 1)(1 - q^{-2\theta_{1,j+1}}) - \frac{1}{x_{j}^{1} x_{j+1}^{s}} (1 - q^{-2\theta_{1,j}})(q^{2\theta_{s,j+1}} - 1)\}.
\end{aligned}
$$

Now, by fixing the complex parameters $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 + \cdots + \lambda_n = -k$, we consider the following system of $q$-difference equations for $G(x)$.

$$q^{\theta_{1\cdots k,j}}G(x) = q^{\lambda_j}G(x) \tag{2.5.2}$$

$$q^{\theta_{r,k+1\cdots n}}G(x) = q^{-\lambda_r-1}G(x) \tag{2.5.3}$$

$$\rho(C_j)G(x) = 0 \qquad (1 \leq j \leq n-1) \tag{2.5.4}$$

**Remark 2.5.1.** We can regard these equations as the analogous of the equation (2.10), (2.11) and (2.18), respectively.

Then, in view of the homogeneities (2.5.2) and (2.5.3), we introduce new $(k-1) \times (n-k-1)$ variables $z = (z_j^r)$:

$$z_j^r = \frac{x_j^r x_{k+1}^1}{x_{k+1}^r x_j^1} \tag{2.5.5}$$

for $2 \leq r \leq k$, $k+2 \leq j \leq n$, and one can show that the system above has a solution in the form

$$G(x) = {x_{k+1}^1}^{\gamma-1} \prod_{r=2}^{k} {x_{k+1}^r}^{-\lambda_r-1} \prod_{j=k+2}^{n} {x_j^1}^{\lambda_j} \varphi(\lambda; z), \tag{2.5.6}$$

for $\gamma = \lambda_2 + \lambda_3 + \cdots + \lambda_{k+1} + k$.

We next define the $q$-hypergeometric function of type $(k,n)$ as follows:

$$\varphi_{k,n}(\lambda; q, z) = \sum_{a_j^r \geq 0} \frac{\prod_{j=k+2}^{n}(q^{-\lambda_j}; q)_{|A_j|} \prod_{r=2}^{k}(q^{\lambda_r+1}; q)_{|A^r|}}{(q^{\gamma}; q)_{|A|} \prod_{r,j}(q;q)_{a_j^r}} z^A. \tag{2.5.7}$$

For the matrices $A = (a_j^r)_{2\leq r\leq k, k+2\leq j\leq n}$, the notations $|A^r|$, $|A_j|$, $|A|$ and $z^A$ are defined in (2.2.12).

Then we have

**Proposition 2.5.2.** The function

$$G_\lambda = {x_{k+1}^1}^{\gamma-1} \prod_{r=2}^{k} {x_{k+1}^r}^{-\lambda_r-1} \prod_{j=k+2}^{n} {x_j^1}^{\lambda_j} \varphi_{k,n}(\lambda; q^2, z) \tag{2.5.8}$$

is a solution of the system of the $q$-difference equations (2.5.2) – (2.5.4).

In the following, we simply write $\varphi_\lambda = \varphi_{k,n}(\lambda; q^2, z)$.

**Remark 2.5.3.** $(1-q^{\alpha})/(1-q)$ converges to $\alpha$ when $q$ tends to 1. Therefore, both of $\varphi_\lambda$ and $\varphi_{k,n}(\lambda; q, z)$ converge to the function $F_{k,n}$, when $q$ tends to 1.

## 2.6. Contiguity relations and q-hypergeometric systems

If $a$ is an element of $U_q(gl(n))$ of weight $\kappa$, the $q$-difference operator $\rho(a)$ transform $G_\lambda$ to a constant multiple of $G_{\lambda+\kappa}$. In this section, we compute the contiguity relations in terms of $\varphi_\lambda$. Furthermore, we define the $q$-hypergeometric systems.

We define the operators $\pi_\lambda(a)$ as follows:

$$\rho_\lambda(a)G_\lambda = x^1_{k+1}{}^{\gamma'-1} \prod_{r=2}^{k} x^r_{k+1}{}^{-\lambda'_r-1} \prod_{j=k+2}^{n} x^1_j{}^{\lambda'_j} \pi_\lambda(a)\varphi_\lambda, \tag{2.6.1}$$

where $a$ is an element of $U_q(gl(n))$ of weight $\kappa = \kappa_1\epsilon_1 + \cdots + \kappa_n\epsilon_n$, $\lambda' = (\lambda'_j) = (\lambda_j + \kappa_j)$ and $\gamma' = \gamma + \kappa_2 + \cdots + \kappa_{k+1}$. Furthermore, we set for simplicity $\alpha_j = \epsilon_j - \epsilon_{j+1}$ and $[a] = (q^a - q^{-a})/(q - q^{-1})$. Then we have

**Proposition 2.6.1.** For the generators of $U_q(gl(n))$, the contiguity relations for the function $\varphi_\lambda$ are given as follows:

$$\begin{aligned}
\pi_\lambda(q^{\epsilon_j})\varphi_\lambda &= q^{\lambda_j}\varphi_\lambda \qquad (1 \le j \le n),\\
\pi_\lambda(e_1)\varphi_\lambda &= q^{\lambda_1-2\lambda_2+\gamma+1}[\gamma-1]\phi_{\lambda+\alpha_1},\\
\pi_\lambda(e_r)\varphi_\lambda &= -q^{2\lambda_r-2\lambda_{r+1}+2}[\lambda_r+1]\varphi_{\lambda+\alpha_r} \qquad (2 \le r \le k-1),\\
\pi_\lambda(e_k)\varphi_\lambda &= -q^{-\lambda_1-\gamma+2\lambda_k+1}[\lambda_k+1]\varphi_{\lambda+\alpha_k},\\
\pi_\lambda(e_{k+1})\varphi_\lambda &= q^{-\gamma+\lambda_{k+1}-2\lambda_{k+2}+2}\frac{[\lambda_{k+2}][\lambda_{k+1}+1]}{[\gamma]}\varphi_{\lambda+\alpha_{k+1}},\\
\pi_\lambda(e_j)\varphi_\lambda &= q^{2\lambda_j-2\lambda_{j+1}+2}[\lambda_{j+1}]\varphi_{\lambda+\alpha_j} \qquad (k+2 \le j \le n-1),\\
\pi_\lambda(f_1)\varphi_\lambda &= q^{-\lambda_1+2\lambda_2-\gamma+1}\frac{[\lambda_1][\lambda_2+1]}{[\gamma]}\varphi_{\lambda-\alpha_1},\\
\pi_\lambda(f_r)\varphi_\lambda &= -q^{-2\lambda_r+2\lambda_{r+1}+2}[\lambda_{r+1}+1]\varphi_{\lambda-\alpha_r} \qquad (2 \le r \le k-1),\\
\pi_\lambda(f_k)\varphi_\lambda &= -q^{\lambda_1+\gamma-2\lambda_k+1}[\lambda_{k+1}+1]\varphi_{\lambda-\alpha_k},\\
\pi_\lambda(f_{k+1})\varphi_\lambda &= q^{\gamma-\lambda_{k+1}+2\lambda_{k+2}}[\gamma-1]\varphi_{\lambda-\alpha_{k+1}},\\
\pi_\lambda(f_j)\varphi_\lambda &= q^{-2\lambda_j+2\lambda_{j+1}+2}[\lambda_j]\varphi_{\lambda-\alpha_j} \qquad (k+2 \le j \le n-1).
\end{aligned}$$

The elements $\hat{E}_{i,j}$ are of weight $\alpha_{i,j} = \epsilon_i - \epsilon_j$, respectively, and their actions on the function $\varphi_\lambda$ are given as follows.

**Proposition 2.6.2.** The following is the list of contiguity relations for the $q$-hypergeometric function $\varphi_\lambda$:

$$\begin{aligned}
\pi_\lambda(\hat{E}_{1,r})\varphi_\lambda &= (-1)^r q^{\lambda_1-2\lambda_r+\gamma+\tau_{1,r}+1}[\gamma-1]\varphi_{\lambda+\alpha_{1,r}} \quad (2\le r\le k),\\
\pi_\lambda(\hat{E}_{r,1})\varphi_\lambda &= (-1)^r q^{-\lambda_1+2\lambda_r-\gamma-\tau_{1,r}+1}\frac{[\lambda_1][\lambda_r+1]}{[\gamma]}\varphi_{\lambda+\alpha_{r,1}} \quad (2\le r\le k),\\
\pi_\lambda(\hat{E}_{1,k+1})\varphi_\lambda &= (-1)^{k-1} q^{\tau_{1,k+1}}[\gamma-1]\varphi_{\lambda+\alpha_{1,k+1}},\\
\pi_\lambda(\hat{E}_{k+1,1})\varphi_\lambda &= (-1)^{k-1} q^{-\tau_{1,k+1}}\frac{[\lambda_1][\lambda_{k+1}+1]}{[\gamma]}\varphi_{\lambda+\alpha_{1,k+1}},\\
\pi_\lambda(\hat{E}_{1,j})\varphi_\lambda &= (-1)^{k-1} q^{\lambda_{k+1}-\gamma-2\lambda_j+\tau_{1,j}+2}[\lambda_j]\varphi_{\lambda+\alpha_{1,j}} \quad (k+2\le j\le n),\\
\pi_\lambda(\hat{E}_{j,1})\varphi_\lambda &= (-1)^{k-1} q^{-\lambda_{k+1}+\gamma+2\lambda_j-\tau_{1,j}}[\lambda_1]\varphi_{\lambda+\alpha_{j,1}} \quad (k+2\le j\le n),\\
\pi_\lambda(\hat{E}_{r,s})\varphi_\lambda &= (-1)^{r-s} q^{2\lambda_r-2\lambda_s+\tau_{r,s}+2}[\lambda_r+1]\varphi_{\lambda+\alpha_{r,s}} \quad (2\le r<s\le k),\\
\pi_\lambda(\hat{E}_{s,r})\varphi_\lambda &= (-1)^{r-s} q^{-2\lambda_r+2\lambda_s-\tau_{r,s}+2}[\lambda_s+1]\varphi_{\lambda+\alpha_{s,r}} \quad (2\le r<s\le k),\\
\pi_\lambda(\hat{E}_{r,k+1})\varphi_\lambda &= (-1)^{k+1-r} q^{-\lambda_1-\gamma+2\lambda_r+\tau_{r,k+1}+1}[\lambda_r+1]\varphi_{\lambda+\alpha_{r,k+1}},\\
\pi_\lambda(\hat{E}_{k+1,r})\varphi_\lambda &= (-1)^{k+1-r} q^{\lambda_1+\gamma-2\lambda_r-\tau_{r,k+1}+1}[\lambda_{k+1}+1]\varphi_{\lambda+\alpha_{k+1,r}},\\
\pi_\lambda(\hat{E}_{r,j})\varphi_\lambda &= (-1)^{k+1-r} q^{-\lambda_1+2\lambda_r+\lambda_{k+1}-2\lambda_j-2\gamma+\tau_{r,j}+2}\frac{[\lambda_r+1][\lambda_j]}{[\gamma]}\varphi_{\lambda+\alpha_{r,j}}\\
&\qquad (2\le r\le k; k+2\le j\le n),\\
\pi_\lambda(\hat{E}_{j,r})\varphi_\lambda &= (-1)^{k+1-r} q^{\lambda_1-2\lambda_r-\lambda_{k+1}+2\lambda_j+2\gamma-\tau_{r,j}}[\gamma-1]\varphi_{\lambda+\alpha_{j,r}}\\
&\qquad (2\le r\le k; k+2\le j\le n),\\
\pi_\lambda(\hat{E}_{k+1,j})\varphi_\lambda &= q^{\lambda_{k+1}-\gamma-2\lambda_j+\tau_{k+1,j}+2}\frac{[\lambda_{k+1}+1][\lambda_j]}{[\gamma]}\varphi_{\lambda+\alpha_{k+1,j}} \quad (k+2\le j\le n),\\
\pi_\lambda(\hat{E}_{j,k+1})\varphi_\lambda &= q^{-\lambda_{k+1}+\gamma+2\lambda_j-\tau_{k+1,j}}[\gamma-1]\varphi_{\lambda+\alpha_{j,k+1}} \quad (k+2\le j\le n),\\
\pi_\lambda(\hat{E}_{i,j})\varphi_\lambda &= q^{2\lambda_i-2\lambda_j+\tau_{i,j}+2}[\lambda_j]\varphi_{\lambda+\alpha_{i,j}} \quad (k+2\le i<j\le n),\\
\pi_\lambda(\hat{E}_{j,i})\varphi_\lambda &= q^{-2\lambda_i+2\lambda_j-\tau_{i,j}+2}[\lambda_i]\varphi_{\lambda+\alpha_{j,i}} \quad (k+2\le i<j\le n),
\end{aligned}$$

where

$$\tau_{i,j} = \begin{cases} -\lambda_{i+1}-\cdots-\lambda_{j-1} & (j-i>1)\\ 0 & (j-i=1)\end{cases}$$

for $1\le i<j\le n$.

Now, refering to these relations, we define the elements of $U_q(gl(n))$ by

$$(q-q^{-1})^2 C_{ij} = (q^{1+\epsilon_i}-q^{-1-\epsilon_i})(q^{\epsilon_j}-q^{-\epsilon_j}) - (q-q^{-1})^2\hat{E}_{j,i}\hat{E}_{i,j}, \tag{2.6.2}$$

for $1\le i<j\le n-1$. Now we call the following system *the $q$-hypergeometric system of type $(k,n)$*:

$$q^{\theta_{1\cdots k,j}}G(x) = q^{\lambda_j}G(x) \tag{2.5.2}$$

$$q^{\theta_{r,k+1\cdots n}}G(x) = q^{-\lambda_r - 1}G(x) \tag{2.5.3}$$

$$\rho_\lambda(C_{ij})G(x) = 0 \tag{2.6.3}$$

By the contiguity relations, we have

**Theorem 2.6.3.** The function $G_\lambda$ is a solution of the $q$-hypergeometric system of type $(k, n)$.

**Remark 2.6.4.** The definition of the elements $C_{ij}$ in (2.6.2) is including the original definition of the Casimir elements in (2.5.1).

Finally we can check that $\varphi_\lambda$ satisfies Proposition 1.6.5. again, namely,

**Proposition 2.6.5.** For $J = (j_1, j_2, j_3, j_4)$ $(j_2 < j_4, j_k \neq j_l$ if $k \neq l)$, we have

$$\pi_\lambda(\hat{E}_{j_1,j_2}\hat{E}_{j_3,j_4} - q^{a(J,\lambda)}\hat{E}_{j_1,j_4}\hat{E}_{j_3,j_2})\varphi_\lambda = 0, \tag{2.6.4}$$

where

$$a(J,\lambda) = \begin{cases} -1 & (j_1, j_3 < j_2 < j_4 \text{ or } j_2 < j_4 < j_1, j_3) \\ \lambda_{j_2} + \lambda_{j_3} & (j_1 < j_2 < j_3 < j_4) \\ -\lambda_{j_1} - \lambda_{j_2} & (j_3 < j_2 < j_1 < j_4) \\ -\lambda_{j_1} - \lambda_{j_4} & (j_2 < j_1 < j_4 < j_3) \\ \lambda_{j_3} + \lambda_{j_4} & (j_2 < j_3 < j_4 < j_1) \\ -\lambda_{j_1} + \lambda_{j_3} + 1 & (j_2 < j_1, j_3 < j_4) \\ \lambda_{j_2} - \lambda_{j_4} - 1 & (j_1 < j_2 < j_4 < j_3 \text{ or } j_3 < j_2 < j_4 < j_1). \end{cases}$$

**Remark 2.6.6.** Note that $a(J,\lambda)$ does not depend on the type $(k, n)$. The $q$-difference equation (2.6.4) will be proved in the paper written by M. Noumi and the author.

## Chapter 3. q-Kummer and q-Bessel functions associated with the quantum Grassmannians

### 3.0. Introduction of this chapter

The contiguity relations for $q$-confluent functions, including a $q$-analogue of the Kummer functions, has been already discussed by E. Date and E. Horiuchi. Refer to [DH].

In this chapter, we will discuss the $q$-analogues of the confluent hypergeometric functions including Kummer and Bessel functions.

In the first section, we give the short reviews of the general confluent hypergeometric systems.

In the second section, we define the localization which has a structure of $U_q(gl(n))$-module.

In the third section, we translate the action of $U_q(gl(n))$ on the noncommutative algebra to that on the commutative polynomial ring.

In the last section, we define the $q$-confluent hypergeometric system and $q$-confluent hypergeometric functions including $q$-Kummer function:

$$ {}_1\varphi_1\left(\begin{matrix} a \\ c \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n(-1)^n q^{n(n-1)/2}}{(c;q)_n(q;q)_n} z^n, \tag{3.4.18} $$

and $q$-Bessel function:

$$ J_v(z;q) = z^v \frac{(q^{v+1};q)_\infty}{(q;q)_\infty} {}_1\varphi_1\left(\begin{matrix} 0 \\ q^{v+1} \end{matrix}; q, qz^2\right), \tag{3.4.19} $$

as the solution of the system of the $q$-difference equations. Moreover we compute the contiguity relations for these functions.

### 3.1. Classical case

In this section, we introduce the confluent hypergeometric functions roughly. See [KHT1], [KHT2], and [KHT3].

For a fixed integer $n$, we set $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)_{\alpha_i \in \mathbf{N}}$ as the partition of $n$, that is, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_l$ and $|\alpha| = \alpha_1 + \cdots + \alpha_l = n$. For example, there are 5 types of partition of 4 as follows:

$$ (1,1,1,1), \quad (2,1,1), \quad (2,2), \quad (3,1), \quad (4). $$

Now we define three types of the maximal commutative subgroup of $G = GL(n;\mathbf{C})$ for some patitions of $n$.

$$H_{(1,\dots,1)} = \left\{ h = \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_n \end{pmatrix} ; h_i \in \mathbf{C}^\times \right\},$$

$$H_{(2,1,\dots,1)} = \left\{ h = \begin{pmatrix} h_1 & h_2 & & & \\ & h_1 & & & \\ & & h_3 & & \\ & & & \ddots & \\ & & & & h_n \end{pmatrix} ; \begin{array}{c} h_1, h_3, \dots, h_n \in \mathbf{C}^\times \\ h_2 \in \mathbf{C} \end{array} \right\},$$

$$H_{(2,2,1,\dots,1)} = \left\{ h = \begin{pmatrix} h_1 & h_2 & & & & & \\ & h_1 & & & & & \\ & & h_3 & h_4 & & & \\ & & & h_3 & & & \\ & & & & h_5 & & \\ & & & & & \ddots & \\ & & & & & & h_n \end{pmatrix} ; \begin{array}{c} h_1, h_3, h_5, \dots, h_n \in \mathbf{C}^\times \\ h_2, h_4 \in \mathbf{C} \end{array} \right\}.$$

Moreover we set $\tilde{H}_\alpha$ as the universal covering group of $H_\alpha$.

Then we give the system of differential equations. Let us set $T = (t_{rj})_{1\le r\le 2, 1\le j\le n} \in M(2,n)$. By fixing the partition $\alpha$ of $n$ and for some $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{C}^n$, the confluent hypergeometric system $E(\alpha;\lambda)$ of differential equations on the coordinate ring $A(M(2,n))$ is the following system :

$$\Phi(gT) = \det(g)^{-1}\Phi(T) \quad (g \in GL(2)), \tag{3.1.1}$$

$$\Phi(Th) = \Phi(T)\chi(h,\lambda) \quad (h \in \tilde{H}_\alpha), \tag{3.1.2}$$

$$\frac{\partial^2}{\partial t_{1i}\partial t_{2j}}\Phi(T) = \frac{\partial^2}{\partial t_{2i}\partial t_{1j}}\Phi(T) \quad (1 \le i < j \le n). \tag{3.1.3}$$

$\chi : \tilde{H}_\alpha \to \mathbf{C}$ is the character of $H_\alpha$. For instance, for $\alpha^{(0)} := (1, \dots, 1)$,

$$\chi(h;\lambda) = h_1{}^{\lambda_1} h_2{}^{\lambda_2} \cdots h_n{}^{\lambda_n}. \tag{3.1.4}$$

For $\alpha^{(1)} := (2, 1, \dots, 1)$,

$$\chi(h;\lambda) = h_1{}^{\lambda_1} exp\left(\lambda_2 \frac{h_2}{h_1}\right) h_3{}^{\lambda_3} \cdots h_n{}^{\lambda_n}. \tag{3.1.5}$$

And for $\alpha^{(2)} := (2, 2, 1, \ldots, 1)$,

$$\chi(h; \lambda) = {h_1}^{\lambda_1} exp\left(\lambda_2 \frac{h_2}{h_1}\right) {h_3}^{\lambda_3} exp\left(\lambda_4 \frac{h_4}{h_3}\right) {h_5}^{\lambda_5} \cdots {h_n}^{\lambda_n}. \tag{3.1.6}$$

For the conditions (3.1.1) and (3.1.2) to be compartible, we can easily check that $\lambda_1 + \cdots + \lambda_n = -2$ for $\alpha = \alpha^{(0)}$, $\lambda_1 + \lambda_3 + \cdots + \lambda_n = -2$ for $\alpha = \alpha^{(1)}$, and $\lambda_1 + \lambda_3 + \lambda_5 + \cdots + \lambda_n = -2$ for $\alpha = \alpha^{(2)}$.

**Remark 3.1.1.** It has been shown that we can realize the Lauricella's hypergeometric function $F_D$ as a solution of the system $E(\alpha^{(0)}; \lambda)$.

Now we recall that we can decompose a general $2 \times n$ matrix $T$ as follows:

$$T = \begin{pmatrix} t_{11} & t_{13} \\ t_{21} & t_{23} \end{pmatrix} \times T', \tag{3.1.7}$$

where

$$T' = \begin{pmatrix} 1 & x_2^1 & 0 & x_4^1 & \cdots & x_n^1 \\ 0 & x_2^2 & 1 & x_4^2 & \cdots & x_n^2 \end{pmatrix}. \tag{3.1.8}$$

Here $x_j^1 = -{\xi_{13}}^{-1}\xi_{3j}$ and $x_j^2 = {\xi_{13}}^{-1}\xi_{1j}$. And $\{\xi_{rj}\}$ are the Plücker coordinates of $G_{2,n}$.

Furthermore, we have two types of decomposition of $T'$:

$$\begin{aligned} T' &= \begin{pmatrix} 1 & 0 \\ 0 & -\frac{x_4^2}{x_4^1} \end{pmatrix} T_{(1)} \begin{pmatrix} 1 & x_2^1 & & & & \\ & 1 & & & & \\ & & -\frac{x_4^1}{x_4^2} & & & \\ & & & x_4^1 & & \\ & & & & \ddots & \\ & & & & & x_n^1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & x_2^2 \end{pmatrix} T_{(2)} \begin{pmatrix} 1 & x_2^1 & & & & \\ & 1 & & & & \\ & & \frac{1}{x_2^2} & \frac{x_4^2}{x_2^2} & & \\ & & & \frac{1}{x_2^2} & & \\ & & & & \frac{x_5^2}{x_2^2} & & \\ & & & & & \ddots & \\ & & & & & & \frac{x_n^2}{x_2^2} \end{pmatrix}, \end{aligned} \tag{3.1.9}$$

where

$$T_{(1)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & y_2 & 1 & -1 & y_5 & \cdots & y_n \end{pmatrix}, \tag{3.1.10}$$

and

$$T_{(2)} = \begin{pmatrix} 1 & 0 & 0 & z_4 & z_5 & \cdots & z_n \\ 0 & 1 & 1 & 0 & 1 & \cdots & 1 \end{pmatrix}. \tag{3.1.11}$$

Here $y_2 = -\frac{x_4^1 x_2^2}{x_4^2}$, $y_j = -\frac{x_4^1 x_j^2}{x_4^2 x_j^1}$ $(j \geq 5)$ and $z_4 = x_4^1 x_2^2$, $z_j = -\frac{x_j^1 x_2^2}{x_j^2}$ $(j \geq 5)$.

For the decomposition (3.1.7), $\Phi(T)$ can be written in the form

$$\Phi(T) = {\xi_{13}}^{-1} G(x), \tag{3.1.12}$$

for some function $G(x)$ in the coordinates $x = (x_2^1, x_2^2, x_4^1, x_4^2, \ldots, x_n^1, x_n^2)$ . Furthermore, with $\lambda_2 = 1$, in the case of $E(\alpha^{(1)}; \lambda)$, $G(x)$ can be written in the form

$$G(x) = (x_4^1)^{\lambda_3+\lambda_4+1} (x_4^2)^{-\lambda_3-1} exp(x_2^1) \prod_{j \geq 5} (x_j^1)^{\lambda_j} F_1(y), \tag{3.1.13}$$

for some function $F_1(y)$ of $(n-3)$ variables $y = (y_2, y_5, \ldots, y_n)$.

Now we define the function $F^{(1)}$ as follows:

$$F^{(1)} \left( {a; b_5, \ldots, b_n \atop c} ; y \right) = \sum_{\nu_j \geq 0} \frac{(a)_{|\nu|} (b_5)_{\nu_5} \cdots (b_n)_{\nu_n}}{(c)_{|\nu|} (1)_{\nu_2} (1)_{\nu_5} \cdots (1)_{\nu_n}} {y_2}^{\nu_2} {y_5}^{\nu_5} \cdots {y_n}^{\nu_n} \tag{3.1.14}$$

Then it is known that $\Phi(T) = {\xi_{13}}^{-1} G^{(1)}(x)$;

$$G^{(1)}(x) = (x_4^1)^{\lambda_3+\lambda_4+1} (x_4^2)^{-\lambda_3-1} exp(x_2^1) \prod_{j \geq 5} (x_j^1)^{\lambda_j} F^{(1)} \left( {\lambda_3 + 1; -\lambda_5, \ldots, -\lambda_n \atop \lambda_3 + \lambda_4 + 2} ; y \right), \tag{3.1.15}$$

is a solution of the system $E(\alpha^{(1)}; \lambda)$.

Similarly, in the case of $E(\alpha^{(2)}; \lambda)$, $G(x)$ can be written in the form

$$G(x) = (x_2^2)^{\lambda_1+1} exp(x_2^1) exp(x_4^2) \prod_{j \geq 5} (x_j^2)^{\lambda_j} F_2(z), \tag{3.1.16}$$

for some function $F_2$ of $(n-3)$ variables $z = (z_4, z_5, \ldots z_n)$ with $\lambda_2 = \lambda_4 = 1$.

We define the function $F^{(2)}$ as follows:

$$F^{(2)} \left( {b_5, \ldots, b_n \atop c} ; z \right) = \sum_{\nu_j \geq 0} \frac{(b_5)_{\nu_5} \cdots (b_n)_{\nu_n}}{(c)_{|\nu|} (1)_{\nu_4} (1)_{\nu_5} \cdots (1)_{\nu_n}} {z_4}^{\nu_4} {z_5}^{\nu_5} \cdots {z_n}^{\nu_n}. \tag{3.1.17}$$

Then it is known that $\Phi(T) = {\xi_{13}}^{-1} G^{(2)}(x)$;

$$G^{(2)}(x) = (x_2^2)^{\lambda_1+1} exp(x_2^1) exp(x_4^2) \prod_{j \geq 5} (x_j^2)^{\lambda_j} F^{(2)} \left( {-\lambda_5, \ldots, -\lambda_n \atop \lambda_1 + 2} ; z \right), \tag{3.1.18}$$

is a solution of the system $E(\alpha^{(2)}; \lambda)$.

**Remark 3.1.2.** Especially, when $n=4$, $F^{(1)}\left({a \atop c};y_2\right) = {}_1F_1\left({a \atop c};y_2\right)$ is called as Kummer function. Moreover the Bessel function $J_v$ is defined as follows

$$J_v(t) = \sum_{m\geq 0} \frac{(-1)^m (t/2)^{v+2m}}{m!\Gamma(v+m+1)} = \frac{(t/2)^v}{\Gamma(v+1)} {}_0F_1\left({- \atop v+1};-\frac{t^2}{4}\right). \tag{3.1.19}$$

So

$$F^{(2)}\left({- \atop c};y_4\right) = (-y_4)^{\frac{1-c}{2}}\Gamma(c)J_{c-1}(2\sqrt{-y_4}). \tag{3.1.20}$$

In these senses, Kummer and Bessel functions are realized as a solution of the confluent hypergeometric system.

Here we set

$$E_{ij} = \sum_{r=1}^{2} t_{ri}\frac{\partial}{\partial t_{rj}}. \tag{3.1.21}$$

Then $\{E_{ij}\}_{1\leq i,j\leq n}$ generate a Lie algebra isomorphic to $gl(n)$. And we can verify that

$$(E_{ii}+1)E_{jj} - E_{ji}E_{ij} = (t_{1i}t_{2j} - t_{2i}t_{1j})\left(\frac{\partial^2}{\partial t_{1i}\partial t_{2j}} - \frac{\partial^2}{\partial t_{2i}\partial t_{1j}}\right), \tag{3.1.22}$$

for each $i \neq j$. Here, for $E(\alpha^{(1)};\lambda)$, it is known that the differential equation (3.1.2) is written as follows.

$$\begin{aligned} (E_{11}+E_{22})\Phi(T) &= \lambda_1\Phi(T) \\ E_{12}\Phi(T) &= \Phi(T) \\ E_{jj}\Phi(T) &= \lambda_j\Phi(T) \quad (j\geq 3). \end{aligned} \tag{3.1.23.(1)}$$

We can see that $F^{(1)}$ is a solution of the system of the differential equations (3.1.23.(1)), and

$$\{(E_{ii}+1)E_{jj} - E_{ji}E_{ij}\}\Phi(T) = 0. \tag{3.1.24}$$

Moreover, for $E(\alpha^{(2)};\lambda)$, it is known that (3.1.2) is written as follows.

$$\begin{aligned} (E_{11}+E_{22})\Phi(T) &= \lambda_1\Phi(T) \\ E_{12}\Phi(T) &= \Phi(T) \\ (E_{33}+E_{44})\Phi(T) &= \lambda_3\Phi(T) \\ E_{34}\Phi(T) &= \Phi(T) \\ E_{jj}\Phi(T) &= \lambda_j\Phi(T) \quad (j\geq 5), \end{aligned} \tag{3.1.23.(2)}$$

Then we can see that $F^{(2)}$ is a solution of the system of the differential equations (3.1.23.(2)), and

$$\{(E_{ii}+1)E_{jj} - E_{ji}E_{ij}\}\Phi(T) = 0. \tag{3.1.24}$$

Furthermore, for several $E_{ij}$, we can compute the contiguity relation for $F^{(1)}$ and $F^{(2)}$.

We have already discussed the $q$-analogue of the hypergeometric function of type (k,n). In this chapter, we will discuss the $q$-analogues of confluent hypergeometric functions $F^{(1)}$ and $F^{(2)}$.

## 3.2. Noncommutative localization $\mathbf{A}[{\xi_{13}}^{-1}]$

In this section, we consider the localization $\mathcal{A}[{\xi_{13}}^{-1}]$ which has a $U_q(gl(n))$-module structure.

We in this chapter denote the noncommutative algebra, which is generated by the quantum minors of degree 2, by $\mathcal{A}$. We here consider the locallization $\mathcal{A}[{\xi_{13}}^{-1}]$.

**Theorem 3.2.1.** The localization $\mathcal{A}[{\xi_{13}}^{-1}]$ is the algebra generated by the quantum minors $\{{\xi_{13}}^{\pm 1}, \xi_{12}, \xi_{14}, \ldots, \xi_{1n}, \xi_{32}, \xi_{34}, \ldots \xi_{3n}\}$.

**Proof.** It is proved similarly as Theorem 2.3.1.

The commutation relation among these genarators are as follows:

$$\xi_{13}\xi_{12} = q^{-1}\xi_{12}\xi_{13}, \quad \xi_{13}\xi_{rj} = q\xi_{rj}\xi_{13} \quad (r,j) \neq (1,2),(1,3),$$

$$\xi_{ri}\xi_{rj} = q\xi_{rj}\xi_{ri} \quad (r = 1,3; i < j), \quad \xi_{1j}\xi_{3j} = q\xi_{3j}\xi_{1j} \quad (j = 2,4,\ldots,n),$$

$$\xi_{12}\xi_{3j} = q^2\xi_{3j}\xi_{12}, \quad (j \geq 4), \quad \xi_{1j}\xi_{3i} = \xi_{3i}\xi_{1j} \quad (i > j),$$

$$\xi_{1i}\xi_{3j} - \xi_{3j}\xi_{1i} = (q - q^{-1})\xi_{1j}\xi_{3i} \quad (4 \leq i < j \leq n).$$

Then, we can give the monomial basis of the localization $\mathcal{A}[{\xi_{13}}^{-1}]$,

$$ {\xi_{3n}}^{\nu_{3n}} \cdots {\xi_{34}}^{\nu_{34}} {\xi_{32}}^{\nu_{32}} {\xi_{1n}}^{\nu_{1n}} \cdots {\xi_{14}}^{\nu_{14}} {\xi_{12}}^{\nu_{12}} {\xi_{13}}^{\mu} \tag{3.2.1}$$

where $\nu_{rj} \in \mathbf{Z}_{\geq 0}$ and $\mu \in \mathbf{Z}$.

In the following, we set

$$\xi^{\nu} = {\xi_{3n}}^{\nu_{3n}} \cdots {\xi_{34}}^{\nu_{34}} {\xi_{32}}^{\nu_{32}} {\xi_{1n}}^{\nu_{1n}} \cdots {\xi_{14}}^{\nu_{14}} {\xi_{12}}^{\nu_{12}} \tag{3.2.2}$$

for each matrix $\nu = (\nu_{rj})_{r=1,3;j=2,4\ldots,n}$ of nonnegative integers. With this notation, we have

$$\mathcal{A}[{\xi_{13}}^{-1}] = \bigoplus_{\nu,\mu} \mathbf{C}\xi^{\nu}{\xi_{13}}^{\mu}. \tag{3.2.3}$$

We have already defined the action of $U_q(gl(n))$ on the noncommutative algebra $\mathcal{A}$, and we can see that the algebra $\mathcal{A}[{\xi_{13}}^{-1}]$ has a left $U_q(gl(n))$-module structure.

We next define the subspace of $\mathcal{A}[{\xi_{13}}^{-1}]$ as follows:

$$\mathcal{S}_{2,n} = \bigoplus_{\nu \in (\mathbf{Z}_{\geq 0})^{2\times(n-2)}} \mathbf{C}\xi^{\nu}{\xi_{13}}^{-|\nu|-1}, \tag{3.2.4}$$

where $|\nu| = \sum \nu_{rj}$. We can see that this subspace $\mathcal{S}_{2,n}$ has the $U_q(gl(n))$-submodule structure of $\mathcal{A}[{\xi_{13}}^{-1}]$. Here, for all $a \in U_q(gl(n))$, we denote its action on $\mathcal{S}_{2,n}$ by

$$\bar{\rho}(a) : \mathcal{S}_{2,n} \to \mathcal{S}_{2,n}. \tag{3.2.5}$$

Then we can translate the action of $U_q(gl(n))$ on the noncommutative algebra $\mathcal{S}_{2,n}$ to that on the commutative polynomial ring $\mathbf{C}[x_2^1, x_2^2, x_4^1, x_4^2, \dots, x_n^1, x_n^2]$ in the next section.

## 3.3. The action of $\mathbf{U_q(gl(n))}$ on the polynomial ring

In this section, we define the action of $U_q(gl(n))$ on the commutative polynomial ring, and give the explicit formulas of the action in the term of the $q$-difference operators.

At first, we consider the commutative polynomial ring $\mathbf{C}[x]$ in $2 \times (n-2)$ variables $x = (x_2^1, x_2^2, x_4^1, x_4^2, \dots x_n^1, x_n^2)$. Here, for $\nu = (\nu_{rj})_{r=1,3;j=2,4,\dots n}$, we set the notation

$$x^{\nu} = {x_2^1}^{\nu_{32}}{x_4^1}^{\nu_{34}} \cdots {x_n^1}^{\nu_{3n}}{x_2^2}^{\nu_{12}}{x_4^2}^{\nu_{14}} \cdots {x_n^2}^{\nu_{1n}}. \tag{3.3.1}$$

Let $\phi : M(2, n-2; \mathbf{Z}_{\geq 0}) \to \mathbf{Z}$ be an arbitrary function of degree at most two in the form

$$\phi(\nu) = \sum_{r,s=1,3;i,j=2,4,\dots,n} \alpha_{r,s,i,j}\nu_{r,i}\nu_{s,j} + \beta_{r,j}\nu_{r,j} + c, \tag{3.3.2}$$

for the $2\times(n-2)$ matrix $\nu = (\nu_{r,j})_{r=1,3;j=2,4,\dots,n}$. For such a $\phi$, we define the isomorphism of vector space

$$\eta_{\phi} : \mathbf{C}[x] \to \mathcal{S}_{2,n}$$

by setting

$$\eta_{\phi}(x^{\nu}) = \xi^{\nu}{\xi_{13}}^{-|\nu|-1}q^{\phi(\nu)}, \tag{3.3.3}$$

for all matrix $\nu \in M(2,(n-2); \mathbf{Z}_{\geq 0})$. Then we obtain a left $U_q(gl(n))$-module structure on the vector space $\mathbf{C}[x]$, namely

$$\rho_{\phi}(a) = {\eta_{\phi}}^{-1} \circ \bar{\rho}(a) \circ \eta_{\phi}, \tag{3.3.4}$$

for all $a \in U_q(gl(n))$.

In the following, we denote by $q^{\theta_{r,i}}$ the $q$-shift operator in the variables $x_i^r$, that is,

$$q^{\theta_{r,i}} f(x_2^1, \ldots, x_n^2) = f(x_2^1, \ldots, q x_i^r, \ldots, x_n^2). \tag{3.3.5}$$

With this notation, for example, when $\phi(\nu) \equiv 0$,

$$\begin{aligned}
\rho_0(q^{\epsilon_1}) &= q^{-1-\theta_{1,2}-\theta_{1,4\cdots n}}, \\
\rho_0(q^{\epsilon_3}) &= q^{-1-\theta_{2,2}-\theta_{2,2\cdots n}}, \\
\rho_0(q^{\epsilon_j}) &= q^{\theta_{1,j}+\theta_{2,j}} \quad (j = 2, 4, \ldots n), \\
(q - q^{-1})\rho_0(e_1) &= \frac{1}{x_1}(1 - q^{2\theta_{1,2}}) q^{-\theta_{1,2}-\theta_{2,2}+\theta_{2,4\cdots n}+1}, \\
(q - q^{-1})\rho_0(f_1) &= \sum_{j=4}^{n} \frac{x_2^2 x_j^1}{x_j^2}(q^{2\theta_{2,j}} - 1) q^{2(\theta_{1,2}+\theta_{1,4\cdots j-1})+\theta_{1,j\cdots n}+2(\theta_{2,2}+\theta_{2,j+1\cdots n})+2} \\
&\quad - x_2^1(1 - q^{2(\theta_{1,2}+\theta_{1,4\cdots n}+\theta_{2,2})+2}) q^{-\theta_{1,2}-\theta_{1,4\cdots n}-\theta_{2,4\cdots n}-2}, \\
(q - q^{-1})\rho_0(e_2) &= x_2^2(1 - q^{2(\theta_{1,2}+\theta_{1.4\cdots n}+\theta_{2,4\cdots n})+2}) q^{\theta_{2,2}+\theta_{1,4\cdots n}+1} \\
&\quad - \sum_{j=4}^{n} \frac{x_2^1 x_j^2}{x_j^1}(q^{2\theta_{1,j}} - 1) q^{-\theta_{1,2}-\theta_{1,j\cdots n}+\theta_{2,2}-\theta_{2,4\cdots j}-2}, \\
(q - q^{-1})\rho_0(f_2) &= \frac{1}{x_2^2}(q^{2\theta_{2,2}} - 1) q^{-\theta_{1,2}-\theta_{1,4\cdots n}-3\theta_{2,2}-\theta_{2,4\cdots n}}, \\
(q - q^{-1})\rho_0(e_3) &= -\frac{1}{x_4^2}(1 - q^{2\theta_{2,4}}) q^{\theta_{1,2}+\theta_{1,5\cdots n}-\theta_{2,2}-\theta_{2,4}},
\end{aligned}$$

$$\begin{aligned}
(q - q^{-1})\rho_0(f_3) &= x_4^2(1 - q^{2(\theta_{1,4}+\theta_{2,2}+\theta_{2,4\cdots n})+2}) q^{-\theta_{1,2}-\theta_{1,4\cdots n}-\theta_{2,4\cdots n}-1} \\
&\quad + \frac{x_4^1 x_2^2}{x_2^1}(1 - q^{2\theta_{1,2}}) q^{\theta_{1,4\cdots n}+2(\theta_{2,2}+\theta_{2,4})+\theta_{2,5\cdots n}+3} \\
&\quad + \sum_{j \geq 5} \frac{x_4^1 x_j^2}{x_j^1}(1 - q^{2\theta_{1j}}) q^{-(\theta_{1,2}+\theta_{1,j\cdots n})+2\theta_{2,2}+\theta_{2,4}+\theta_{2,j+1\cdots n}}, \\
(q - q^{-1})\rho_0(e_j)_{j \geq 4} &= \frac{x_j^1}{x_{j+1}^1}(q^{\theta_{1,j+1}} - q^{-\theta_{1,j+1}}) + \frac{x_j^2}{x_{j+1}^2}(q^{\theta_{2,j+1}} - q^{-\theta_{2,j+1}}) q^{\theta_{1,j}-\theta_{1,j+1}}, \\
(q - q^{-1})\rho_0(f_j)_{j \geq 4} &= \frac{x_{j+1}^1}{x_j^1}(q^{\theta_{1,j}} - q^{-\theta_{1,j}}) q^{\theta_{2,j+1}-\theta_{2,j}} + \frac{x_{j+1}^2}{x_j^2}(q^{\theta_{2,j}} - q^{-\theta_{2,j}}).
\end{aligned}$$

Thus we can translate the action of $U_q(gl(n))$ on the noncommutative algebra to that on the polynomial ring.

### 3.4. The q-hypergeometric system and q-confluent hypergeometric functions

In this section, we define $q$-hypergeometric systems and realize $q$-Kummer and $q$-Bessel function as the solution of this system. Moreover we compute the contiguity relations.

At first we compute the actions of the Casimir elements

$$(q-q^{-1})^2 C_j = (q^{1+\epsilon_j} - q^{-1-\epsilon_j})(q^{\epsilon_{j+1}} - q^{-\epsilon_{j+1}}) - (q-q^{-1})^2 f_j e_j. \tag{2.5.1}$$

Now we set

$$\phi_0(\nu) = \sum_{j=4}^{n-1} \nu_j^3(\nu_{j+1}^1 + \cdots + \nu_n^1) + 2\nu_2^1(\nu_4^3 + \cdots + \nu_n^3) - (\nu_2^3+1)(\nu_4^1 + \cdots + \nu_n^1 + 1). \tag{3.4.1}$$

In the representations, for $\phi = \phi_0$, we have the actions of the Casimir operators as follows:

$$\begin{aligned}
(q-q^{-1})^2 \rho_{\phi_0}(C_1) &= \sum_{j=4}^{n} q^{\theta_{1,4\cdots j-1} - \theta_{1,j\cdots n} - \theta_{2,2}} \\
&\quad \times \left\{ \frac{x_j^1 x_2^2}{x_2^1 x_j^2}(1-q^{2\theta_{1,2}})(1-q^{2\theta_{2,j}}) - (1-q^{2\theta_{2,2}})(1-q^{2\theta_{1,j}}) \right\} \\
(q-q^{-1})^2 \rho_{\phi_0}(C_2) &= \sum_{j=4}^{n} q^{-\theta_{2,4\cdots j} + \theta_{2,j+1\cdots n}} \\
&\quad \times \left\{ \frac{x_2^1 x_j^2}{x_2^2 x_j^1}(1-q^{2\theta_{1,j}})(1-q^{2\theta_{2,2}}) - (1-q^{2\theta_{1,2}})(1-q^{2\theta_{2,j}}) \right\} \\
(q-q^{-1})^2 \rho_{\phi_0}(C_3) &= q^{-\theta_{1,4} - \theta_{2,2} - \theta_{2,5\cdots n}} \\
&\quad \times \left\{ \frac{x_4^1 x_2^2}{x_2^1 x_4^2}(1-q^{2\theta_{1,2}})(1-q^{2\theta_{2,4}}) - (1-q^{2\theta_{1,4}})(1-q^{2\theta_{2,2}}) \right\} \\
&\quad + \sum_{j=5}^{n} q^{-\theta_{1,4} + \theta_{2,2} - \theta_{2,5\cdots j} + \theta_{2,j+1\cdots n}} \\
&\quad \times \left\{ \frac{x_4^1 x_j^2}{x_j^1 x_4^2}(1-q^{2\theta_{1,j}})(1-q^{2\theta_{2,4}}) - (1-q^{2\theta_{1,4}})(1-q^{2\theta_{2,j}}) \right\} \\
(q-q^{-1})^2 \rho_{\phi_0}(C_j)_{j\geq 4} &= q^{1+\theta_{1,j} - \theta_{1,j+1} - \theta_{2,j} + \theta_{2,j+1}} \\
&\quad \times \left\{ -\frac{x_{j+1}^1 x_j^2}{x_{j+1}^2 x_j^1}(1-q^{2\theta_{1,j}})(1-q^{2\theta_{2,j+1}}) + (1-q^{2\theta_{1,j+1}})(1-q^{2\theta_{2,j}}) \right\} \\
&\quad + q^{-1-\theta_{1,j} - \theta_{1,j+1} - \theta_{2,j} - \theta_{2,j+1}} \\
&\quad \times \left\{ -\frac{x_j^1 x_{j+1}^2}{x_j^2 x_{j+1}^1}(1-q^{2\theta_{1,j+1}})(1-q^{2\theta_{2,j}}) + (1-q^{2\theta_{1,j}})(1-q^{2\theta_{2,j+1}}) \right\}
\end{aligned}$$

Here we define the $q$-hypergeometric function $\varphi^{(1)}$ as follows:

$$\varphi^{(1)}\left(\begin{matrix}\alpha;\beta_5,\cdots,\beta_n\\ \gamma\end{matrix},q;y\right)$$
$$=\sum_{\nu_j\geq 0}\frac{(\alpha;q)_{|\nu|}(\beta_5;q)_{\nu_5}\cdots(\beta_n;q)_{\nu_n}(-1)^{\nu_2}}{(\gamma;q)_{|\nu|}(q;q)_{\nu_2}(q;q)_{\nu_5}\cdots(q;q)_{\nu_n}}q^{\nu_2(\nu_2-1)/2}{y_2}^{\nu_2}{y_5}^{\nu_5}\dots{y_n}^{\nu_n}, \tag{3.4.2}$$

in the coordinates $y=(y_2,y_5,\dots,y_n)$.

Furthermore we define the $q$-hypergeometric function $\varphi^{(2)}$ as

$$\varphi^{(2)}\left(\begin{matrix}\beta_5,\dots,\beta_n\\ \gamma\end{matrix},q;z_4,z_5,\dots,z_n\right)=\varphi^{(1)}\left(\begin{matrix}0;\beta_5,\dots,\beta_n\\ \gamma\end{matrix},q;z_4,z_5,\dots,z_n\right) \tag{3.4.3}$$

**Remark 3.4.1.** We can easily check that we have the formal limit

$$\lim_{q\uparrow 1}\varphi^{(1)}\left(\begin{matrix}q^a;q^{b_5},\dots,q^{b_n}\\ q^c\end{matrix},q;(q-1)y_2,y_5,\dots,y_n\right)=F^{(1)}\left(\begin{matrix}a;b_5,\dots,b_n\\ c\end{matrix};y\right) \tag{3.4.4}$$

and

$$\lim_{q\uparrow 1}\varphi^{(2)}\left(\begin{matrix}q^{b_5},\dots,q^{b_n}\\ q^c\end{matrix},q;(1-q)^2z_4,(1-q)z_5,\dots,(1-q)z_n\right)=F^{(2)}\left(\begin{matrix}b_5,\dots,b_n\\ c\end{matrix};z\right) \tag{3.4.5}$$

We now introduce the $L$-operators defined by the following formulas (Refer to [J, NUW]):

$$L^+_{ij} = \begin{cases} q^{\epsilon_i} & (i=j)\\ (q-q^{-1})\hat{E}_{ji}q^{\epsilon_i} & (i<j)\\ 0 & (i>j),\end{cases} \tag{3.4.6}$$

and

$$L^-_{ij} = \begin{cases} q^{-\epsilon_i} & (i=j)\\ -(q-q^{-1})q^{-\epsilon_i}\hat{E}_{ji} & (i>j)\\ 0 & (i<j).\end{cases} \tag{3.4.7}$$

For some $\lambda=(\lambda_1,\dots\lambda_n)$, we give two types of the system of $q$-difference equations $E((i),\lambda,\phi)_{i=1,2}$ on the commutative polynomial ring $\mathbf{C}[x]$.

$E((1),\lambda,\phi)$ :

$$\begin{aligned}\rho_\phi(q^{\epsilon_1+\epsilon_2})G(x)&=q^{\lambda_1}G(x),\\ \rho_\phi(q^{\epsilon_j})G(x)&=q^{\lambda_j}G(x)\quad(j=3,4),\end{aligned} \tag{3.4.8}$$

$$\rho_\phi(L^-_{21})G(x)=(q-q^{-1})q^{-\lambda_1}G(x), \tag{3.4.9}$$

$$\rho_\phi(C_j)G(x) = 0, \tag{3.4.10}$$

$E((2), \lambda, \phi)$ :

$$\begin{aligned} \rho_\phi(q^{\epsilon_1+\epsilon_2})G(x) &= q^{\lambda_1}G(x), \\ \rho_\phi(q^{\epsilon_3+\epsilon_4})G(x) &= q^{\lambda_3}G(x), \\ \rho_\phi(q^{\epsilon_j})G(x) &= q^{\lambda_j}G(x), \end{aligned} \tag{3.4.11}$$

$$\begin{aligned} \rho_\phi(L_{21}^-)G(x) &= (q - q^{-1})q^{-\lambda_1}G(x) \\ \rho_\phi(L_{43}^-)G(x) &= (q - q^{-1})q^{\lambda_5+\cdots+\lambda_n}G(x) \end{aligned} \tag{3.4.12}$$

$$\rho_\phi(C_j)G(x) = 0. \tag{3.4.10}$$

Here we define the $q$-analogue of exponential map as follows:

$$e_q(x) = \frac{1}{((1-q)x;q)_\infty}, \tag{3.4.13}$$

where $(\alpha;q)_\infty = \lim_{n\to\infty}(\alpha;q)_n$.

**Remark 3.4.2.** In the classical case, $exp(x)$ is a solution of the differential equation $\frac{d}{dx}u = u$. Now we can see the $q$-exponential map $e_q(x)$ as a solution of the $q$-difference equation $\frac{1}{x}\frac{(1-q^{\theta_x})}{(1-q)}u = u$. We use the $L$-operators above because their actions on the polynomial ring $\mathbf{C}[x]$ have $q$-exponential map as the eigen function.

Now we set

$$\begin{aligned} G^{(1)}(x;\lambda) =& {x_4^1}^{\lambda_3+\lambda_4+1}{x_4^2}^{-\lambda_3-1}e_{q^2}(x_1)\prod_{j\geq 5}(x_j^1)^{\lambda_j} \\ &\times \varphi^{(1)}\left(\begin{matrix} q^{2\lambda_3+2}; q^{-2\lambda_5}, \ldots, q^{-2\lambda_n} \\ q^{2\lambda_3+2\lambda_4+4} \end{matrix}; q^2, (1-q^2)y_2, y_5, \ldots, y_n\right), \end{aligned} \tag{3.4.14}$$

where $y_2 = -\frac{x_4^1x_3}{x_2^2}$, $y_j = -\frac{x_j^2x_4^1}{x_j^1x_4^2}$ $(j \geq 5)$, and

$$\begin{aligned} G^{(2)}(x;\lambda) =& {x_2^2}^{\lambda_1+1}e_{q^2}(x_2^1)e_{q^2}(x_4^2)\prod_{j\geq 5}(x_j^2)^{\lambda_j} \\ &\times \varphi^{(2)}\left(\begin{matrix} q^{-2\lambda_5}, \ldots, q^{-2\lambda_n} \\ q^{2\lambda_3+4} \end{matrix}; q^2, (1-q^2)^2z_4, (1-q^2)z_5, \ldots, (1-q^2)z_n\right), \end{aligned} \tag{3.4.15}$$

where $z_4 = x_2^2 x_4^1, z_j = \dfrac{x_j^1 x_2^2}{x_j^2}$. Furthermore,

$$\phi_1(\nu) = \phi_0(\nu) - \nu_4^1(\nu_4^1+1) - \sum_{j\geq 5} \nu_j^3(\nu_j^3+1) - \nu_2^1(\nu_2^1-1) \tag{3.4.16}$$

and

$$\phi_2(\nu) = \phi_0(\nu) - \sum_{j\geq 5} \nu_j^1(\nu_j^1+1) - \nu_4^3(\nu_4^3-1). \tag{3.4.17}$$

Then we have the next proposition.

**Proposition 3.4.3.** In the case of $(2,n)$, the functions $G^{(1)}$ and $G^{(2)}$ satisfy $E((1),\lambda,\phi_1)$ and $E((2),\lambda,\phi_2)$ respectively.

Thus we have $q$-confluent hypergeometric functions as a solution of $q$-hypergeometric system of $q$-difference equations.

Generally, $q$-hypergeometric function ${}_1\varphi_1$ and the Hahn-Exton $q$-Bessel function $J_v$ are defined as follows:

$${}_1\varphi_1\left(\begin{matrix} a \\ c \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n(-1)^n q^{n(n-1)/2}}{(c;q)_n(q;q)_n} z^n, \tag{3.4.18}$$

$$J_v(z;q) = z^v \frac{(q^{v+1};q)_\infty}{(q;q)_\infty} {}_1\varphi_1\left(\begin{matrix} 0 \\ q^{v+1} \end{matrix}; q, qz^2\right), \tag{3.4.19}$$

where $(a;q)_\infty = \lim_{n\to\infty}(a;q)_n$. See [GR, K].

Then, in the case of $(2,4)$, we have

$$\varphi^{(1)}\left(\begin{matrix} a \\ c \end{matrix}; q, y_2\right) = {}_1\varphi_1\left(\begin{matrix} a \\ c \end{matrix}; q, y_2\right), \tag{3.4.20}$$

and

$$\varphi^{(2)}\left(\begin{matrix} - \\ c \end{matrix}; q, z_4\right) = C z_4{}^{-(\lambda_1+1)/2} J_{\lambda_1+1}((q^{-1}z_4)^{1/2}; q). \tag{3.4.21}$$

Here $C = \dfrac{(q^2;q^2)_\infty}{(q^{2\lambda_3+2};q^2)_\infty}$.

In these senses, we have realized $q$-Kummer and $q$-Bessel functions as a solution of $q$-confluent hypergeometric system of $q$-difference equations.

Then we next compute the contiguity relations for $q$-hypergeometric functions $\varphi^{(1)}$ and $\varphi^{(2)}$.

We simply write

$$\varphi^{(1)}_\lambda = \varphi^{(1)}\left( {q^{2\lambda_3+2}; q^{-2\lambda_5}, \ldots, q^{-2\lambda_n} \atop q^{2\lambda_3+2\lambda_4+4}} ; q^2, (1-q^2)y_2, y_5, \ldots, y_n \right), \tag{3.4.22}$$

and

$$\varphi^{(2)}_\lambda = \varphi^{(2)}\left( {q^{-2\lambda_5}, \ldots, q^{-2\lambda_n} \atop q^{2\lambda_3+4}} ; q^2, (1-q^2)^2 z_4, (1-q^2)z_5, \ldots, (1-q^2)z_n \right). \tag{3.4.23}$$

For $\varphi^{(1)}_\lambda$, we can give the contiguity relation by the element of $U_q(gl(n))$ which has $e_{q^2}(x^1_2)$ as an eigen function. For such an element $a \in U_q(gl(n))$, we define the operators $\pi_{(1;\lambda)}(a)$ for $a \in U_q(gl(n))$ as follows:

$$\rho_{\phi_1}(a)G^{(1)} = (x^1_4)^{\lambda'_3+\lambda'_4+1}(x^2_4)^{-\lambda'_3-1}e_{q^2}(x^1_2)\prod_{j\geq 5}(x^1_j)^{\lambda'_j}\pi_{(1;\lambda)}(a)\varphi^{(1)}_{\lambda'}. \tag{3.4.24}$$

**Proposition 3.4.4.** The following is the list of contiguity relations for the $q$-hypergeometric function $\varphi^{(1)}_\lambda$:

$$\begin{aligned}
\pi_{(1;\lambda)}(q^{\epsilon_1+\epsilon_2})\varphi^{(1)}_\lambda &= q^{\lambda_1}\varphi^{(1)}_\lambda \\
\pi_{(1;\lambda)}(q^{\epsilon_j})\varphi^{(1)}_\lambda &= q^{\lambda_j}\varphi^{(1)}_\lambda \quad (j \geq 3) \\
(q-q^{-1})^{-1}\pi_{(1;\lambda)}(L^-_{21})\varphi^{(1)}_\lambda &= q^{-\lambda_1-1}\varphi^{(1)}_\lambda \\
(q-q^{-1})^{-1}\pi_{(1;\lambda)}(L^-_{31})\varphi^{(1)}_\lambda &= q^{-\lambda_3+\lambda_4+2}[\lambda_3+\lambda_4+1]\varphi^{(1)}_{\lambda+\alpha_{1,3}} \\
(q-q^{-1})^{-1}\pi_{(1;\lambda)}(L^-_{41})\varphi^{(1)}_\lambda &= -q^{\lambda_4+\cdots+\lambda_n+1}[\lambda_3+\lambda_4+1]\varphi^{(1)}_{\lambda+\alpha_{1,4}} \\
(q-q^{-1})^{-1}\pi_{(1;\lambda)}(L^-_{j1})\varphi^{(1)}_\lambda &= q^{-\lambda_3-\lambda_j+\lambda_{j+1}+\cdots+\lambda_n}[\lambda_j]\varphi^{(1)}_{\lambda+\alpha_{1,j}} \quad (j \geq 5) \\
(q-q^{-1})^{-1}\pi_{(1;\lambda)}(L^+_{23})\varphi^{(1)}_\lambda &= q^{2\lambda_3+\lambda_5+\cdots+\lambda_n-1}\frac{[\lambda_3+1]}{[\lambda_3+\lambda_4+2]}\varphi^{(1)}_{\lambda+\alpha_{3,1}} \\
(q-q^{-1})^{-1}\pi_{(1;\lambda)}(L^+_{24})\varphi^{(1)}_\lambda &= -q^{\lambda_3-2}\frac{[\lambda_4+1]}{[\lambda_3+\lambda_4+2]}\varphi^{(1)}_{\lambda+\alpha_{4,1}} \\
(q-q^{-1})^{-1}\pi_{(1;\lambda)}(L^+_{2j})\varphi^{(1)}_\lambda &= -q^{2\lambda_3-\tau_{3,j}+2\lambda_j+1}\varphi^{(1)}_{\lambda+\alpha_{j,1}} \quad (j \geq 5) \\
\pi_{(1;\lambda)}(\hat{E}_{34})\varphi^{(1)}_\lambda &= q^{2\lambda_3+\lambda_5+\cdots+\lambda_n+1}[\lambda_3+1]\varphi^{(1)}_{\lambda+\alpha_{3,4}} \\
\pi_{(1;\lambda)}(\hat{E}_{3j})\varphi^{(1)}_\lambda &= q^{\lambda_3-\lambda_4-\lambda_j+\lambda_{j+1}+\cdots+\lambda_n-1}\frac{[\lambda_3+1][\lambda_j]}{[\lambda_3+\lambda_4+2]}\varphi^{(1)}_{\lambda+\alpha_{3,j}} \quad (j \geq 5) \\
\pi_{(1;\lambda)}(\hat{E}_{43})\varphi^{(1)}_\lambda &= q^{-2\lambda_3-\lambda_5-\cdots-\lambda_n+1}[\lambda_4+1]\varphi^{(1)}_{\lambda+\alpha_{4,3}} \\
\pi_{(1;\lambda)}(\hat{E}_{j3})\varphi^{(1)}_\lambda{}_{j\geq 5} &= q^{-\lambda_3+\lambda_4+\lambda_j-\lambda_{j+1}-\cdots-\lambda_n}[\lambda_3+\lambda_4+1]\varphi^{(1)}_{\lambda+\alpha_{j,3}} \quad (j \geq 5)
\end{aligned}$$

$$
\begin{aligned}
\pi_{(1;\lambda)}(\hat{E}_{4j})\varphi^{(1)}_{\lambda} &= q^{-\lambda_3+\tau_{4,j}-2\lambda_j-1}\frac{[\lambda_4+1][\lambda_j]}{[\lambda_3+\lambda_4+2]}\varphi^{(1)}_{\lambda+\alpha_{4,j}} \quad (j\geq 5)\\
\pi_{(1;\lambda)}(\hat{E}_{j4})\varphi^{(1)}_{\lambda} &= q^{\lambda_3-\tau_{4,j}+2\lambda_j+3}[\lambda_3+\lambda_4+1]\varphi^{(1)}_{\lambda+\alpha_{j,4}} \quad (j\geq 5)\\
\pi_{(1;\lambda)}(\hat{E}_{ij})\varphi^{(1)}_{\lambda} &= q^{2\lambda_j-2\lambda_{j+1}+\tau_{i,j}+2}[\lambda_{j+1}]\varphi^{(1)}_{\lambda+\alpha_{i,j}} \quad (5\leq i<j)\\
\pi_{(1;\lambda)}(\hat{E}_{ji})\varphi^{(1)}_{\lambda} &= q^{-2\lambda_j+2\lambda_{j+1}-\tau_{i,j}+2}[\lambda_j]\varphi^{(1)}_{\lambda+\alpha_{j,i}} \quad (5\leq i<j).
\end{aligned}
$$

$\alpha_{i,j}$ and $\tau_{i,j}$ were defined in the section 2.6.

Similarly, for $\varphi^{(2)}_{\lambda}$, we can give the contiguity relation by the element of $U_q(gl(n))$ which has $e_{q^2}(x^1_2)$ and $e_{q^2}(x^3_4)$ as the eigen functions. For such an element $a\in U_q(gl(n))$, we define the operators $\pi_{(2;\lambda)}(a)$ for $a\in U_q(gl(n))$ as follows:

$$
\rho_{\phi_2}(a)G^{(2)} = (x^2_2)^{\lambda'_1+1}e_{q^2}(x^1_2)e_{q^2}(x^2_4)\prod_{j\geq 5}(x^2_j)^{\lambda'_j}\pi_{(1;\lambda)}(a)\varphi^{(2)}_{\lambda'}. \tag{3.4.25}
$$

**Proposition 3.4.5.** The following is the list of contiguity relations for the $q$-hypergeometric function $\varphi^{(2)}_{\lambda}$:

$$
\begin{aligned}
\pi_{(2;\lambda)}(q^{\epsilon_1+\epsilon_2})\varphi^{(2)}_{\lambda} &= q^{\lambda_1}\varphi^{(2)}_{\lambda}\\
\pi_{(2;\lambda)}(q^{\epsilon_3+\epsilon_4})\varphi^{(2)}_{\lambda} &= q^{\lambda_3}\varphi^{(2)}_{\lambda}\\
\pi_{(2;\lambda)}(q^{\epsilon_j})\varphi^{(2)}_{\lambda} &= q^{\lambda_j}\varphi^{(2)}_{\lambda} \quad (j\geq 5)\\
(q-q^{-1})^{-1}\pi_{(2;\lambda)}(L^-_{21})\varphi^{(2)}_{\lambda} &= q^{-\lambda_1-1}\varphi^{(2)}_{\lambda}\\
(q-q^{-1})^{-1}\pi_{(2;\lambda)}(L^-_{43})\varphi^{(2)}_{\lambda} &= q^{\lambda_5+\cdots+\lambda_n}\varphi^{(2)}_{\lambda}\\
(q-q^{-1})^{-1}\pi_{(2;\lambda)}(q^{-\epsilon_4}L^-_{41})\varphi^{(2)}_{\lambda} &= -q^{2\lambda_5+\cdots+2\lambda_n}\frac{1}{[\lambda_1+2]}\varphi^{(2)}_{\lambda+\alpha_{1,3}}\\
(q-q^{-1})^{-1}\pi_{(2;\lambda)}(q^{-\epsilon_4}L^-_{j1})\varphi^{(2)}_{\lambda} &= -q^{-\tau_{4,j}+2\lambda_{j+1}+\cdots+2\lambda_n}\frac{[\lambda_j]}{[\lambda_1+2]}\varphi^{(2)}_{\lambda+\alpha_{1,j}} \quad (j\geq 5)\\
(q-q^{-1})^{-1}\pi_{(2;\lambda)}(L^-_{j3})\varphi^{(2)}_{\lambda} &= q^{-\lambda_j+\lambda_{j+1}+\cdots+\lambda_n}[\lambda_j]\varphi^{(2)}_{\lambda+\alpha_{3,j}} \quad (j\geq 5)\\
(q-q^{-1})^{-1}\pi_{(2;\lambda)}(L^+_{23}q^{\epsilon_4})\varphi^{(2)}_{\lambda} &= q^{\lambda_3+1}[\lambda_1+1]\varphi^{(2)}_{\lambda+\alpha_{3,1}}\\
(q-q^{-1})^{-1}\pi_{(2;\lambda)}(L^+_{2j}q^{\epsilon_4})\varphi^{(2)}_{\lambda} &= q^{-\lambda_1+\tau_{4,j}-2\lambda_{j+1}-\cdots-2\lambda_n}[\lambda_1+1]\varphi^{(2)}_{\lambda+\alpha_{3,1}}\\
(q-q^{-1})^{-1}\pi_{(2;\lambda)}(L^-_{4j})\varphi^{(2)}_{\lambda} &= -q^{-\tau_{4,j}+2\lambda_j+1}\varphi^{(2)}_{\lambda+\alpha_{j,3}} \quad (j\geq 5)\\
\pi_{(2;\lambda)}(\hat{E}_{ij})\varphi^{(2)}_{\lambda} &= q^{2\lambda_j-2\lambda_{j+1}+\tau_{i,j}+2}[\lambda_{j+1}]\varphi^{(2)}_{\lambda+\alpha_{i,j}} \quad (5\leq i<j)\\
\pi_{(2;\lambda)}(\hat{E}_{ji})\varphi^{(2)}_{\lambda} &= q^{-2\lambda_j+2\lambda_{j+1}-\tau_{i,j}+2}[\lambda_j]\varphi^{(2)}_{\lambda+\alpha_{j,i}} \quad (5\leq i<j).
\end{aligned}
$$

We can show that the contiguity relation for $\varphi^{(1)}_{\lambda}$ gives a representation of an algebra which has a subalgebra generated by $q^{\epsilon_j}$, $e_j$, $f_j$ $(j\geq 3)$, isomorphic to $U_q(gl(n-2))$.

Furthermore, the contiguity relation for $\varphi_\lambda^{(2)}$ gives a representation of an algebra which has a subalgebra generated by $q^{\epsilon_j}$, $e_j$, $f_j$ $(j \geq 5)$, isomorphic to $U_q(gl(n-4))$.

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