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平成 8 年 8 月

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($SU(2,2)$ の許容表現に付随するウィッタカー関数)

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# INTRODUCTION

A Whittaker model belonging to an admissible representation is a realization of the representation in an induced representation related to a maximal unipotent subgroup. To make it help studying automorphic $L$-functions, the explicit formula has been studied in each places. In the case of a quasi-split algebraic group over a $p$-adic field, Casselman and Shalika gave an explicit formula of class-1 Whittaker functions [3]. At real places, it was Stade that gave an integral expression of Whittaker functions of the general linear group $GL(n, \mathbb{R})$ in a recursive way [22].

A remarkable method of deriving the differential equations is the usage of the Schmid operator. This was first appeared in the paper of Schmid [21], who used it to characterize discrete series of real semisimple Lie groups. Then, using the fact that the discrete series representation is isomorphic to the kernel of the Schmid operator (the differential operator of gradient type in his paper), Yamashita realize the discrete series both in principal series representations and in generalized Gelfand-Graev representations [30, 31]. We remark that some highest weight modules also have such characterization [5].

In [17], Miyazaki and Oda regard the Schmid operator as the sum of shift operators; each of them causes a shift between two $K$-types, irreducible representations of a maximal compact subgroup $K$. Then, they derived the differential equations of principal series Whittaker functions on $Sp(2;\mathbb{R})$. Further, they regard the meaning of the kernel of the Schmid operator as shift operators making the $K$-type cross through the "wall", and succeeded to obtain the differential equations of Whittaker functions of generalized principal series representations on $Sp(2;\mathbb{R})$.

In this paper, we deal admissible representations of $SU(2,2)$, the special unitary group of signature $(2,2)$. Our conclusion is that the analogous results hold on $SU(2,2)$. We obtain the differential equations belonging to admissible representations and explicit integral expressions under some parity condition in some cases.

Here is a precise description of the main result of this paper.

Let $\pi$ be an irreducible admissible representation of $G = SU(2,2)$. Let $\eta$ be a unitary character of a maximal unipotent subgroup $N$ of $G$. The space of intertwining operators

$$\mathrm{Hom}_{(\mathfrak{g},K)}(H_\pi^K, C_\eta^\infty(N\backslash G))$$

is called the space of the (algebraic) Whittaker vectors defined by the representation $\pi$, where $K$ is a maximal compact subgroup of $G$. Let $(\tau, V_\tau)$ be an irreducible representation of $K$ and assume that the contragredient representation $(\tau^*, V_\tau^*)$ of $\tau$ appears in $\pi|_K$ with multiplicity one; in this case $\tau^*$ is called a $K$-type of $\pi$. We fix an injection $\iota_{\tau^*}$ of $V_\tau^*$ to $H_\pi^K$. Then, by irreducibility of $\pi$, the mapping $\iota_{\tau^*}$ gives rise to a natural injection

$$\mathrm{Hom}_{(\mathfrak{g},K)}(H_\pi^K, C_\eta^\infty(N\backslash G)) \to \mathrm{Hom}_K(V_\tau^*, C_\eta^\infty(N\backslash G)).$$

Given an element $\phi$ in $\mathrm{Hom}_K(V_\tau^*, C_\eta^\infty(N\backslash G))$, we define the $V_\tau$-valued function $\phi_\tau \in C_{\eta,\tau}^\infty(N\backslash G/K)$ (§3.1) by

$$\phi(v^*)(g) = < v^*, \phi_\tau(g) >, \qquad (v^* \in V^*,\ g \in G).$$

Here $<\,,\,>$ is the pairing of $V_\tau^*$ and $V_\tau$. Thus, given a Whittaker vector $\Phi_\pi$, we get the function $\Phi_{\pi,\tau}$ called the Whittaker function of $\pi$ with $K$-type $\tau^*$.

We next derive the differential equations for Whittaker functions by utilizing the Schmid operator $\nabla$, which is a mapping from $C^\infty_{\eta,\tau}(N\backslash G/K)$ to $C^\infty_{\eta,\tau\otimes \mathrm{Ad}}(N\backslash G/K)$ (§3.3). Let $(\tau', V_{\tau'})$ be an irreducible representation of $K$ that appears in $\tau \otimes \mathrm{Ad}$ as an irreducible constituent and $P'$ the projector to $V_{\tau'}$. By composition, we get a shift operator from $C^\infty_{\eta,\tau}(N\backslash G/K)$ to $C^\infty_{\eta,\tau'}(N\backslash G/K)$. For any $v'^* \in V^*_{\tau'}$, we have

$$< v'^*, P' \circ \nabla \Phi_{\pi,\tau}(g) > = c(\tau, \iota_{\tau^*}, \tau', \iota_{(\tau')^*}) \Phi_\pi(\iota_{(\tau')^*}(v'^*))(g), \quad (g \in G)$$

where $\iota_{(\tau')^*}$ is an injection of $V^*_{\tau'}$ and $c = c(\tau, \iota_{\tau^*}, \tau', \iota_{(\tau')^*})$ is a constant. If $(\tau')^*$ is not a $K$-type of $\pi$, $c$ is understood to be zero.

Using the shift operator repeatedly, if necessary, we can construct a differential operator on $C^\infty_{\eta,\tau}(N\backslash G/K)$. Its radial part, the restriction to the maximal $\mathbb{R}$-split torus, defines a differential equation for $\Phi_{\pi,\tau}$. The Casimir operator acting on $\Phi_{\pi,\tau}$ as a constant multiplication, another differential equation can be obtained from considering the radial part. Appropriate choice of these operators makes a holonomic system. If the rank of the system agrees with the dimension formula, we obtain the theorem. This is done case by case for the representation of $G$ (Theorems 6.1, 6.2 and 5.4).

Further in discrete series and principal $P_J$-series representations, the rapidly decreasing solution of the derived differential equations can be expressed using an Laplace transformation (Theorems 4.6 and 5.5).

This paper consists of two parts. In Part I, basic notations are prepared here with three sections to prove main theorems described in Part II. In Section 1, we review the fundamental facts about the Lie group $G = SU(2,2)$ and the Lie algebra. We fix the notation of both restricted and absolute root systems. In Section 2 we discuss the representations of $G$ and of the maximal compact subgroup $K$. According to Langlands' classification, all the admissible representations of the real semi-simple Lie groups appear in the standard representations as a quotient. It is known that the cuspidal parabolic subgroups of $G$ are $P_m$, $P_J$ and $G$ itself up to conjugacy. Thus we precisely denote the parabolic induction from them and calculate the multiplicity of $K$-types.

In Section 3, we introduce the Schmid operator and state the relation between the operator and the Whittaker model of a given admissible representation. We also calculate the radial part of the Schmid operator, shift operators coming from the Schmid operator and the Casimir operator. The necessary and sufficient condition for existence of a Whittaker model is that the representation is large in the sense of Vogan [26]. Also according to [13], the dimension of the Whittaker vectors, the intertwining operators between the representation and the Whittaker model, is at most the order of the little Weyl group. To be exact, their dimensions are, in general, 8, 4 and 4 in the case of $P_m$-series, $P_J$-series and discrete series, respectively.

Part II contains three sections. In this part, we did the case study; we state our main results according to each series representation. In Section 4, we consider the discrete series. The representation we consider should be large; this is interpreted by the words of Harish-Chandra parameter in this case, i.e., the parameter should be in the cones of type II or of type V, following Yamashita's notation. Thus we take the minimal $K$-type, which is of multiplicity one, and then investigate the radial part of the Schmid operator; they become the differential equations of rank 4. Furthermore in this case, using the Laplace transformation we find the integral expression of rapidly decreasing Whittaker functions under some parity condition as carried out in the case of $Sp(2;\mathbb{R})$ [20].

In Section 5, we consider the $P_J$-series representations. We assume they are irreducible, then they are known to be necessarily large. To derive the differential equations of $P_J$-series Whittaker functions, we need the help of the Casimir operator as well as the Schmid operator. In this case, the notion of the corner $K$-type has an advantage; it is smaller than minimal $K$-types, that is, the Whittaker functions with the corner $K$-type is simpler than those with a minimal $K$-type, which can be seen in the deduced integral expression.

In Section 6, we consider $P_m$-series. It is a most general representation and has no characterization using the kernel of a single operator. Instead, we consider the composition of the Schmid operator. Then using an "eigenvalue" we make an differential equation satisfied by the Whittaker function. With the help of the Casimir operator, the system becomes holonomic of rank 8 when the dimension of $K$-types is one and two, which agrees with the dimension formula.

For further application, we hope that these explicit integral formula could be used to calculate the $\Gamma$-factor of an automorphic $L$-function as is seen in the case of $SU(2,1)$ [12] and in the case of Bessel models of $Sp(2;\mathbb{R})$ [16]. We also remark that these Whittaker functions are typical examples of "generalized spherical functions" determined by the spherical subgroups developed by Brion and Krämer [2, 14]. Particularly in the case of symmetric pair of real semisimple Lie groups such as, $SU(2,2)$, $Sp(2,\mathbb{R})$ and rank 1 groups, many works has been carried out using the Schmid operator [24, 25, 9].

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# Contents

PART I

# General Results on the Representations of $SU(2,2)$

## 1. The structure of $SU(2,2)$ and its Lie algebra

In this section, we fix several notations for the group $SU(2,2)$, its subgroups and the corresponding Lie algebras.

**1.1. $SU(2,2)$ and its Lie algebra.** Let $G$ be the special unitary group of signature $(2,2)$ defined by

$$G = SU(2,2) = \left\{ g \in SL_4(\mathbb{C}) \,\middle|\, {}^t\overline{g} I_{2,2}\, g = I_{2,2} \right\}$$

where $I_{2,2} = \mathrm{diag}(1,1,-1,-1)$. We denote by ${}^tg$, $\overline{g}$, and $1_2$, the transpose of the matrix $g$, the complex conjugation of $g$ and the identity matrix of size two, respectively. This group $G$ is semisimple, of hermitian type and of real rank two.

A maximal compact subgroup $K$ of $G$ is given by the following:

$$K = S(U(2)\times U(2)) = \left\{ \begin{pmatrix} u & \\ & v \end{pmatrix} \,\middle|\, u,\, v \in U(2), \det(u)\det(v) = 1 \right\}.$$

Hereafter we use the convention that blank entries in matrices are zero. Note that $G/K$ is a hermitian upper-half plane.

The Lie algebra $\mathfrak{g}$ of $G$ is expressed as follows:

$$\mathfrak{g} = \left\{ X = \begin{pmatrix} X_1 & X_{12} \\ {}^t\overline{X}_{12} & X_2 \end{pmatrix} \,\middle|\, {}^t\overline{X}_i = -X_i \ (i=1,2), X_{12} \in M_2(\mathbb{C}), \mathrm{tr}\,(X_1 + X_2) = 0 \right\}.$$

It is of real dimension 15. We write its complexification by $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_4(\mathbb{C})$.

The Lie algebra of $K$ is:

$$\mathfrak{k} = \left\{ X = \begin{pmatrix} X_1 & \\ & X_2 \end{pmatrix} \,\middle|\, {}^t\overline{X}_i = -X_i \ (i=1,2), \mathrm{tr}\, X = 0 \right\}.$$

We set

$$\mathfrak{p} = \left\{ X = \begin{pmatrix} & X_{12} \\ {}^t\overline{X}_{12} & \end{pmatrix} \,\middle|\, X_{12} \in M_2(\mathbb{C}) \right\}.$$

Then we have a Cartan decomposition: $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. We also denote the complexification of $\mathfrak{k}$ and $\mathfrak{p}$ by $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ whose dimensions are 7 and 8, respectively.

Next we describe a restricted root system of $\mathfrak{g}$. Put

$$H_1 = \left(\begin{array}{cc|cc} & & 1 & \\ & & & 0 \\ \hline 1 & & & \\ & 0 & & \end{array}\right),\ H_2 = \left(\begin{array}{cc|cc} & & 0 & \\ & & & 1 \\ \hline 0 & & & \\ & 1 & & \end{array}\right).$$

We fix a maximal abelian subalgebra contained in $\mathfrak{p}$, $\mathfrak{a} = \{H_1, H_2\}_{\mathbb{R}}$. The Lie group $A_m = \exp(\mathfrak{a}_m)$ is the identity component of a maximal $\mathbb{R}$-split torus in $G$. We identify $A_m$ with $\mathbb{R}^2_{>0}$ by,

$$A_m = \left\{ a = (a_1, a_2) = (e^s, e^t) = \exp(sH_1 + tH_2) \,\middle|\, s,\, t \in \mathbb{R} \right\}.$$

Choose a basis $\{\lambda_1,\ \lambda_2\}$ of the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ such that $\lambda_i(H_j) = \delta_{ij}$. Then we can describe the restricted root system of $\mathfrak{g}$ associated to $\mathfrak{a}$ as

$$\Delta = \Delta(\mathfrak{g},\ \mathfrak{a}) = \{\pm\lambda_1 \pm \lambda_2,\ \pm 2\lambda_1,\ \pm 2\lambda_2\}.$$

It is of type $C_2$. We take a positive root system $\Delta_+ = \{\lambda_1 \pm \lambda_2,\ 2\lambda_1,\ 2\lambda_2\}$, and a fundamental root system $\Delta_{\text{fund}} = \{\lambda_1 - \lambda_2,\ 2\lambda_2\}$. We set,

$$H_{12} = \operatorname{diag}(\sqrt{-1}, -\sqrt{-1}, 0, 0), \qquad H_{34} = \operatorname{diag}(0, 0, \sqrt{-1}, -\sqrt{-1}),$$

$$E_1 = \sqrt{-1}\left(\begin{array}{cc|cc} 1 & & -1 & \\ & 0 & & 0 \\ \hline 1 & & -1 & \\ & 0 & & 0 \end{array}\right), \qquad E_2 = \sqrt{-1}\left(\begin{array}{cc|cc} 0 & & 0 & \\ & 1 & & -1 \\ \hline 0 & & 0 & \\ & 1 & & -1 \end{array}\right),$$

$$E_3 = \frac{1}{2}\left(\begin{array}{cc|cc} & 1 & & -1 \\ -1 & & 1 & \\ \hline & 1 & & -1 \\ -1 & & 1 & \end{array}\right), \qquad E_4 = \frac{\sqrt{-1}}{2}\left(\begin{array}{cc|cc} & 1 & & -1 \\ 1 & & -1 & \\ \hline & 1 & & -1 \\ 1 & & -1 & \end{array}\right),$$

$$E_5 = \frac{1}{2}\left(\begin{array}{cc|cc} & 1 & & 1 \\ -1 & & 1 & \\ \hline & 1 & & 1 \\ 1 & & -1 & \end{array}\right), \qquad E_6 = \frac{\sqrt{-1}}{2}\left(\begin{array}{cc|cc} & 1 & & 1 \\ 1 & & -1 & \\ \hline & 1 & & 1 \\ -1 & & 1 & \end{array}\right).$$

Also fix a basis of $\mathfrak{k}_{\mathbb{C}}$:

$$(1)\ h^1 = \left(\begin{array}{c|c} h & \\ \hline & \end{array}\right),\ h^2 = \left(\begin{array}{c|c} & \\ \hline & h \end{array}\right),\ e^1_{\pm} = \left(\begin{array}{c|c} e_{\pm} & \\ \hline & \end{array}\right),\ e^2_{\pm} = \left(\begin{array}{c|c} & \\ \hline & e_{\pm} \end{array}\right),\ I_{2,2},$$

where $h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $e_+ = \begin{pmatrix} & 1 \\ 0 & \end{pmatrix}$, $e_- = \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}$ are $2 \times 2$ matrices. Using this notation, one has $\mathfrak{g} = \mathfrak{c}(\mathfrak{a}) + \sum_{\lambda \in \Delta} \mathfrak{g}_\lambda$ where,

$$\begin{aligned} \mathfrak{c}(\mathfrak{a}) &= \text{the centralizer of } \mathfrak{a} = \{H_1,\ H_2,\ H_{12} + H_{34}\}_{\mathbb{R}}, \\ \mathfrak{g}_{2\lambda_1} &= \{E_1\}_{\mathbb{R}},\ \mathfrak{g}_{2\lambda_2} = \{E_2\}_{\mathbb{R}},\ \mathfrak{g}_{\lambda_1+\lambda_2} = \{E_3,\ E_4\}_{\mathbb{R}}, \\ \mathfrak{g}_{\lambda_1-\lambda_2} &= \{E_5,\ E_6\}_{\mathbb{R}}, \quad \mathfrak{g}_{-\mu} = {}^t\mathfrak{g}_\mu = \{{}^tX \mid X \in \mathfrak{g}_\mu\}. \end{aligned}$$

The Weyl group $W$ with respect to $(\mathfrak{g}, \mathfrak{a})$ is the semi-direct product of the symmetric group of degree 2 and two copies of $\mathbb{Z}/2\mathbb{Z}$, thus its order is 8.

Take a compact Cartan subalgebra $\mathfrak{t}$ defined by

$$\mathfrak{t} = \mathbb{R}\sqrt{-1}h^1 + \mathbb{R}\sqrt{-1}h^2 + \mathbb{R}\sqrt{-1}I_{2,2},$$

and let $\mathfrak{t}_{\mathbb{C}}$ be its complexification. Then the absolute root system, of type $A_3$, is expressed as,

$$\tilde{\Delta} = \tilde{\Delta}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\ [\pm 2, 0; 0], [0, \pm 2; 0], [\pm 1, \pm 1; \pm 2]\ \}.$$

where $\beta = [r, s; u]$ means $r = \beta(h^1)$, $s = \beta(h^2)$ and $u = \beta(I_{2,2})$. Let

$$\widetilde{\Delta}^+ = \{ [2,0;0], [0,2;0], [\pm 1, \pm 1; 2] \}.$$

We write the set of compact positive roots by $\widetilde{\Delta}_c^+ = \{ [2,0;0], [0,2;0] \}$ and we fix it hereafter. The Weyl group $\widetilde{W} = \widetilde{W}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is generated by $s_1, s_2, s_3$ where,

$$(2) \qquad \begin{aligned} s_1[r,s;u] &= [-r, s; u], \\ s_2[r,s;u] &= [(r-s+u)/2, (-r+s+u)/2; r+s], \\ s_3[r,s;u] &= [r, -s; u]. \end{aligned}$$

We identify $\widetilde{W}$ and the symmetric group $\mathfrak{S}_4$ of degree 4 by the map: $s_j \mapsto (j, j+1)$. The compact Weyl group is given by $\widetilde{W}_c = < s_1, s_3 >$, also identified canonically with the subgroup $\mathfrak{S}_2 \times \mathfrak{S}_2$.

**1.2. Cuspidal parabolic subgroups of $SU(2,2)$.** It is known that $SU(2,2)$ has three non-isomorphic cuspidal parabolic subgroups; one is maximal, one is minimal and the other is $G$ itself. So, in this subsection, we consider a minimal parabolic subgroup $P_m$, and a Jacobi parabolic subgroup $P_J$ of $G$ with Langlands decomposition $P_m = M_m A_m N_m$, and $P_J = M_J A_J N_J$, respectively.

Let $A_* = \exp \mathfrak{a}_*$ is a split component of $P_*$ ($*$ means either "$m$" or "$J$") with

$$(3) \qquad \mathfrak{a} \;\; = \mathfrak{a}_{\emptyset} = \{ H_1, \; H_2 \}_{\mathbb{R}},$$

$$(4) \qquad \mathfrak{a}_J = \mathfrak{a}_{\{2\lambda_2\}} = \{H_1\}_{\mathbb{R}}.$$

Their unipotent radicals $N_* = \exp(\mathfrak{n}_*)$ can be described as follows:

$$(5) \qquad \mathfrak{n} \;\; = \mathfrak{g}_{2\lambda_1} + \mathfrak{g}_{2\lambda_2} + \mathfrak{g}_{\lambda_1+\lambda_2} + \mathfrak{g}_{\lambda_1-\lambda_2} = \{ E_j \mid j = 1, \ldots, 6 \}_{\mathbb{R}}$$

$$(6) \qquad \mathfrak{n}_J = \mathfrak{g}_{2\lambda_1} \qquad\qquad + \mathfrak{g}_{\lambda_1+\lambda_2} + \mathfrak{g}_{\lambda_1-\lambda_2} = \{ E_j \mid j = 1, 3, \ldots, 6 \}_{\mathbb{R}}.$$

The maximal unipotent subgroup $N_m = \exp \mathfrak{n}_m$ is expressed as,

$$N_m = \exp \mathfrak{n}_m = \left\{ \kappa^{-1} \left( \begin{array}{cc|cc} 1 & \alpha & & \\ & 1 & & \\ \hline & & 1 & \\ & & -\overline{\alpha} & 1 \end{array} \right) \left( \begin{array}{cc|cc} 1 & & \multicolumn{2}{c}{S} \\ & 1 & & \\ \hline & & 1 & \\ & & & 1 \end{array} \right) \kappa \;\middle|\; \alpha \in \mathbb{C}, \; {}^t\overline{S} = S \right\}$$

where,

$$\kappa = \frac{1}{\sqrt{2}} \left( \begin{array}{cc|cc} 1 & & 1 & \\ & 1 & & 1 \\ \hline -\sqrt{-1} & & \sqrt{-1} & \\ & -\sqrt{-1} & & \sqrt{-1} \end{array} \right).$$

By definition, the Levi parts are $M_* = Z_K(\mathfrak{a}_*) \exp \mathfrak{m}_*$ with Lie algebras:

$$(7) \qquad \mathfrak{m} \;\; = \mathbb{R}\sqrt{-1} I_0, \qquad I_0 = \mathrm{diag}(1, -1, 1, -1),$$

$$(8) \qquad \mathfrak{m}_J = \{ H_2, E_2, \sqrt{-1} I_0, H_{24} = \tfrac{\sqrt{-1}}{2}(I_{2,2} - h^1 + h^2) \}_{\mathbb{R}},$$

and therefore,

$$(9) \qquad M = \left\{\exp(\theta I_0)\cdot\gamma^j \mid \theta\in\mathbb{R},\ j=0,1\right\},\quad \gamma=\operatorname{diag}(1,-1,1,-1),$$

$$(10) \qquad M_J = \left\{\exp(\theta I_0)\cdot\left(\begin{array}{cc|cc} 1 & & & \\ & \alpha & & \beta \\ \hline & & 1 & \\ & \bar\beta & & \bar\alpha \end{array}\right) \;\middle|\; \begin{array}{c}\theta\in\mathbb{R},\ \alpha,\beta\in\mathbb{C},\\ |\alpha|^2-|\beta|^2=1\end{array}\right\}$$

$$(=:\mathbb{T}\cdot G_0)\simeq \mathbb{C}^{(1)}\times SU(1,1).$$

### 1.3. Iwasawa decomposition of absolute root vectors.

Let $X_{ij}=\{\delta_{ip}\delta_{jq}\}_{pq}$ and $H_{ij}=\sqrt{-1}(X_{ii}-X_{jj})$. Put,

$$(11) \qquad \mathfrak{p}_+=\{X_{ij}\mid i=1,2,\ j=3,4\}_{\mathbb{C}},\quad \mathfrak{p}_-=\{X_{ij}\mid i=3,4,\ j=1,2\}_{\mathbb{C}}.$$

Then the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is decomposed as $\mathfrak{g}_{\mathbb{C}}=\mathfrak{p}_+ + \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_- = \mathfrak{n}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}} + \mathfrak{k}_{\mathbb{C}}$. Thus we obtain the following formulae by direct calculation.

$$\begin{aligned}
X_{13} &= \tfrac12\left(\sqrt{-1}E_1 + H_1 + \tfrac12(I_{2,2}+h^1-h^2)\right)\\
&= \tfrac12\left(\sqrt{-1}E_1 + H_1 - \sqrt{-1}H_{13}\right),\\
X_{14} &= \tfrac12\left(-E_3+E_5+\sqrt{-1}(E_4-E_6)-2e^2_+\right),\\
X_{23} &= \tfrac12\left(E_3+E_5+\sqrt{-1}(E_4+E_6)+2e^1_-\right),\\
X_{24} &= \tfrac12\left(\sqrt{-1}E_2 + H_2 + \tfrac12(I_{2,2}-h^1+h^2)\right)\\
&= \tfrac12\left(\sqrt{-1}E_2 + H_2 - \sqrt{-1}H_{24}\right),\\
X_{31} &= \tfrac12\left(-\sqrt{-1}E_1 + H_1 - \tfrac12(I_{2,2}+h^1-h^2)\right)\\
&= \tfrac12\left(-\sqrt{-1}E_1 + H_1 + \sqrt{-1}H_{13}\right),\\
X_{32} &= \tfrac12\left(E_3+E_5-\sqrt{-1}(E_4+E_6)-2e^1_+\right),\\
X_{41} &= \tfrac12\left(-E_3+E_5-\sqrt{-1}(E_4-E_6)+2e^2_-\right),\\
X_{42} &= \tfrac12\left(-\sqrt{-1}E_2 + H_2 - \tfrac12(I_{2,2}-h^1+h^2)\right)\\
&= \tfrac12\left(-\sqrt{-1}E_2 + H_2 + \sqrt{-1}H_{24}\right).
\end{aligned}$$

## 2. Admissible representations of $G$ and their $K$-types

In this section, We recall the parametrization of the irreducible representations of $K$. We also treat the induced representations from the cuspidal parabolic subgroups of $G$. According to the Langlands' classification, the irreducible admissible representations of real semisimple Lie groups appears in such representations as a quotient. In addition, we determine the multiplicity of a $K$-type in a given principal series representation and realize $K$-types in the representation space by using their matrix coefficients. The tensor products between $K$-types and the adjoint representation are also given here, to define the Schmid operator mainly. Furthermore, we construct certain operators in $U(\mathfrak{g}_{\mathbb{C}})$ and $M_2(U(\mathfrak{g}_{\mathbb{C}}))$ closely related to the shift operators considered in §3.3.

### 2.1. Parametrization of irreducible representations of $K$.

Firstly, we review the parametrization of the finite-dimensional irreducible representations of $SL_2(\mathbb{C})$. Let $\{f_1, f_2\}$ be a standard basis of the 2-dimensional vector space $V = V_1$. Then $SL_2(\mathbb{C})$ acts on $V$ by matrix multiplication. We denote the symmetric tensor space by $V_d = S^d(V)$ regarded as a $(d+1)$-dimensional vector space with the basis $\{f_p^{(d)} \mid 0 \le p \le d\}$, where $f_p^{(d)} = f_1^{\otimes p} \otimes f_2^{\otimes (d-p)}$ (symmetric tensor). Here $V_0 = \mathbb{C}$. We consider $V_d$ as an $SL_2(\mathbb{C})$-module by :

$$\mathrm{sym}^d(g)(v_1 \otimes v_2 \otimes \ldots \otimes v_d) = gv_1 \otimes gv_2 \otimes \ldots \otimes gv_d.$$

It is well-known that all the finite-dimensional irreducible (polynomial) representations of $SL_2(\mathbb{C})$ can be obtained in this way. By Weyl's unitary trick, all irreducible unitary representations of $SU(2)$ are obtained by restriction. The irreducible representations of $SU(2) \times SU(2) \times \mathbb{C}^{(1)}$ are obtained by tensoring irreducible representations of the simple components.

Let $V_{rs} = V_r \otimes V_s$ and choose the standard basis of $V_{rs}$ as

$$\{f_{pq}^{(rs)} = f_p^{(r)} \otimes f_q^{(s)} \mid 0 \le p \le r,\ 0 \le q \le s\}.$$

Note that $\dim_{\mathbb{C}} V_{rs} = (r+1)(s+1)$. Define

$$\tau_{[r,s;u]}(g_1, g_2; e^{\sqrt{-1}\theta}) = \mathrm{sym}^r(g_1) \otimes \mathrm{sym}^s(g_2) \otimes e^{\sqrt{-1}u\theta}. \tag{12}$$

Then $\{(\tau_{[r,s;u]}, V_{rs}) \mid r, s \in \mathbb{Z}_{\ge 0},\ u \in \mathbb{Z}\}$ gives a complete system of representatives of the unitary dual of $SU(2) \times SU(2) \times \mathbb{C}^{(1)}$. We remark that the notation $[r, s; u]$ will turn out to agree with an element of $\mathfrak{t}_{\mathbb{C}}^*$.

Let us consider the representations of $K$. The group $SU(2) \times SU(2) \times \mathbb{C}^{(1)}$ can be regarded as a covering of $K$ by the map $pr$ :

$$pr(g_1, g_2; e^{\sqrt{-1}\theta}) = \begin{pmatrix} e^{\sqrt{-1}\theta} g_1 & \\ & e^{-\sqrt{-1}\theta} g_2 \end{pmatrix}.$$

Since the kernel of $pr$ is $\{\pm(1_2, 1_2; 1)\}$, this is actually a two-fold cover. The representation $\tau_{[r,s;u]}$ induces a representation of $K$ if and only if the restriction of $\tau_{[r,s;u]}$ to $\ker(pr)$ is trivial, hence we can state the following proposition.

**Proposition 2.1.** $\widehat{K} = \{\tau_{[r,s;u]} \mid r, s \in \mathbb{Z}_{\geq 0}, u \in \mathbb{Z}, r+s+u \in 2\mathbb{Z}\}$.

### 2.2. Infinitesimal representations of $K$.

In this subsection, we collect explicit formulae for the action of standard generators of $\mathfrak{k}_{\mathbb{C}}$. These will be frequently used later, especially in the calculation of shift operators.

Let $(\tau_{[r,s;u]}, V_{rs})$ be a representation of $K$. By definition (cf. (12)),

$$\tau_{[r,s;u]}(\begin{pmatrix} Y_1 & \\ & Y_2 \end{pmatrix}) = d\,\mathrm{sym}^r(Y_1) \otimes id_{V_s} + id_{V_r} \otimes d\,\mathrm{sym}^s(Y_2) \qquad (Y_1, Y_2 \in \mathfrak{sl}_2(\mathbb{C})),$$
$$\tau_{[r,s;u]}(I_{2,2}) = u\, id_{V_{rs}}.$$

Using the basis of $\mathfrak{k}_{\mathbb{C}}$ defined by (1) in §1.1, we have,

**Lemma 2.2.** *Let* $\tau = \tau_{[r,s;u]}$ *and* $f_{pq} = f_{pq}^{(rs)}$. *Then,*

$$(13)\qquad \begin{aligned} &\tau(h^1)\, f_{kl} = (2k-r) f_{kl}, && \tau(h^2) f_{kl} = (2l-s) f_{kl},\\ &\tau(e^1_+)\, f_{kl} = (r-k) f_{k+1,l}, && \tau(e^2_+) f_{kl} = (s-l) f_{k,l+1},\\ &\tau(e^1_-)\, f_{kl} = k f_{k-1,l}, && \tau(e^2_-) f_{kl} = l f_{k,l-1},\\ &\tau(I_{2,2}) f_{kl} = u f_{kl}. \end{aligned}$$

Thus, hereafter, we think of $[r, s; u]$ as the element in $\mathfrak{t}^*_{\mathbb{C}}$.

**2.3. Contragredient representations of $K$.** Let $(\tau_d^*, V_d^*)$ be the contragredient representation of $(\tau_d, V_d)$. Then, using the action of $h^1$, $h^2$ and $I_{2,2}$, we know that $f^*_{00}$ is a highest weight vector with highest weight $[r, s; -u]$. Therefore $\tau^*$ is isomorphic to $\tau_{[r,s;-u]}$; an isomorphism is induced by $f^*_{00} \mapsto f_{rs}^{([r,s;-u])}$. Summing up,

**Proposition 2.3.** *The contragredient representation* $\tau^*$ *of* $\tau_{[r,s;u]}$ *is isomorphic to* $\tau_{[r,s;-u]}$. *More precisely, let* $f^*_{kl}$ *be the dual basis of* $f_{kl}^{([r,s;u])}$, *i.e.,* $< f^*_{kl}, f_{ij}^{([r,s;u])} > = \delta_{ki}\delta_{lj}$. *Then the correspondence between the bases,*

$$f^*_{kl} \longmapsto (-1)^{k+l} \binom{r}{k} \binom{s}{l} f_{r-k,s-l}^{([r,s;-u])}$$

*determines the unique isomorphism up to a constant multiple.*

**2.4. Adjoint representation of $K$.** In this subsection, we consider the adjoint representation of $K$ restricted to $\mathfrak{p}_{\mathbb{C}}$. Write simply $\mathrm{Ad} = \mathrm{Ad}_{\mathfrak{p}_{\mathbb{C}}}$. Since both $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are $K$-invariant subspaces, $\mathrm{Ad}_{\mathfrak{p}_{\mathbb{C}}}$ decomposes into two representations $\mathrm{Ad}_+ = \mathrm{Ad}_{\mathfrak{p}_+}$ and $\mathrm{Ad}_- = \mathrm{Ad}_{\mathfrak{p}_-}$.

**Proposition 2.4** ([30, §4.2]). *The representation* $\mathrm{Ad}_+$ (*resp.* $\mathrm{Ad}_-$) *is irreducible and equivalent to* $\tau_{[1,1;2]}$ (*resp.* $\tau_{[1,1;-2]}$) *and the $K$-isomorphism is given by*

$$(14)\qquad \begin{aligned} &\iota_+ : (X_{23}, X_{13}, X_{24}, X_{14}) \mapsto (f_{00}^{(11)}, f_{10}^{(11)}, -f_{01}^{(11)}, -f_{11}^{(11)}),\\ &(\textit{resp. } \iota_- : (X_{41}, X_{31}, X_{42}, X_{32}) \mapsto (f_{00}^{(11)}, f_{01}^{(11)}, -f_{10}^{(11)}, -f_{11}^{(11)})). \end{aligned}$$

We again begin with the case of $SL_2(\mathbb{C})$. If we take an $SL_2(\mathbb{C})$-module $V_r$, it has a decomposition $V_r \otimes V_1 \simeq V_{r+1} \oplus V_{r-1}$ as $SL_2(\mathbb{C})$-module (cf. [30, Lemma 4.1]). Let $P_r^{\pm}$ be the projectors from $V_r \otimes V_1$ to $V_{r\pm 1}$ defined by:

$$(15)\quad \begin{aligned} &P_r^+(f_r^{(r)} \otimes f_1^{(1)}) = f_{r+1}^{(r+1)}, \quad P_r^+(f_{r-1}^{(r)} \otimes f_1^{(1)} - f_r^{(r)} \otimes f_0^{(1)}) = 0, \\ &P_r^-(f_r^{(r)} \otimes f_1^{(1)}) = 0, \qquad P_r^-(f_{r-1}^{(r)} \otimes f_1^{(1)} - f_r^{(r)} \otimes f_0^{(1)}) = (r+1) f_{r-1}^{(r-1)}. \end{aligned}$$

These conditions determine $P_r^{\pm}$ uniquely because they give the action on the highest weight vectors. We have,

**Lemma 2.5.** *Let* $0 \le j \le r$ *and* $e = 0, 1$. *Then,*

$$(16)\quad \begin{aligned} P_r^+(f_j^{(r)} \otimes f_e^{(1)}) &= f_{j+e}^{(r+1)}, \\ P_r^-(f_j^{(r)} \otimes f_e^{(1)}) &= (r\delta_{e1} - j) f_{j-1+e}^{(r-1)}. \end{aligned}$$

PROOF. The $K$-equivariance of the projectors gives correspondences of other weight vectors. First, considering the weights of each vectors, we can put, for $e = 0, 1$,

$$(17)\quad P_r^+(f_j^{(r)} \otimes f_e^{(1)}) = \alpha_j^{(e)} f_{j+e}^{(r+1)}, \qquad P_r^-(f_j^{(r)} \otimes f_e^{(1)}) = \beta_j^{(e)} f_{j-1+e}^{(r-1)}$$

where $\alpha_j^{(e)}$ and $\beta_j^{(e)}$ are constants. Equations (15) imply that $\alpha_r^{(1)} = 1$, $\alpha_{r-1}^{(1)} = \alpha_r^{(0)}$, $\beta_r^{(1)} = 0$ and $\beta_{r-1}^{(1)} - \beta_r^{(0)} = r+1$. By applying $e_+$ or $e_-$ on both sides of (17), we have,

$$\begin{aligned} &(r-j)\alpha_{j+1}^{(1)} = (r+1-(j+1))\alpha_j^{(1)}, \\ &(r-j)\alpha_{j+1}^{(0)} + \alpha_j^{(1)} = (r+1-j)\alpha_j^{(0)}, \\ &j\beta_j^{(1)} = j\beta_{j-1}^{(1)} + \beta_j^{(0)}, \qquad j\beta_{j-1}^{(0)} = (j-1)\beta_j^{(0)}. \end{aligned}$$

These equations imply that $\alpha_j^{(1)} = \alpha_j^{(0)} = 1$, $\beta_j^{(1)} = r - j$ and $\beta_j^{(0)} = -j$ for all $j$. $\square$

We see that $V_{rs} \otimes \mathfrak{p}_{\pm}$ decomposes into four irreducible $[\mathfrak{k}, \mathfrak{k}]$-submodules $V_{r\pm 1, s\pm 1}$ by the argument above. We define the projectors

$$(18)\quad \begin{aligned} P_{rs}^{(\epsilon_1,\epsilon_2)} &= P_r^{\epsilon_1} \otimes P_s^{\epsilon_2} : V_{rs} \otimes \mathfrak{p}_+ \to V_{r+\epsilon_1, s+\epsilon_2}, \\ \overline{P}_{rs}^{(\epsilon_1,\epsilon_2)} &= P_r^{\epsilon_1} \otimes P_s^{\epsilon_2} : V_{rs} \otimes \mathfrak{p}_- \to V_{r+\epsilon_1, s+\epsilon_2}, \end{aligned}$$

where $\epsilon_1, \epsilon_2$ are in $\{+, -\}$. Here $x + \epsilon_j$ means $x + 1$ if $\epsilon_j$ is $+$ and $x - 1$ if $\epsilon_j$ is $-$.

**Lemma 2.6.** *Put* $P^{(\epsilon_1,\epsilon_2)} = P^{(\epsilon_1,\epsilon_2)}_{rs}$ *and* $\overline{P}^{(\epsilon_1,\epsilon_2)} = \overline{P}^{(\epsilon_1,\epsilon_2)}_{rs}$. *Then we have,*

$$
\begin{aligned}
P^{(-,-)}(f_{kl}\otimes X_{13}) &= -\overline{P}^{(-,-)}(f_{kl}\otimes X_{42}) = (k-r)l f_{k,l-1},\\
P^{(-,-)}(f_{kl}\otimes X_{24}) &= -\overline{P}^{(-,-)}(f_{kl}\otimes X_{31}) = k(s-l) f_{k-1,l},\\
P^{(-,-)}(f_{kl}\otimes X_{23}) &= \phantom{-}\overline{P}^{(-,-)}(f_{kl}\otimes X_{41}) = kl f_{k-1,l-1},\\
P^{(-,-)}(f_{kl}\otimes X_{14}) &= \phantom{-}\overline{P}^{(-,-)}(f_{kl}\otimes X_{32}) = (k-r)(s-l) f_{kl},\\
P^{(+,-)}(f_{kl}\otimes X_{13}) &= -\overline{P}^{(+,-)}(f_{kl}\otimes X_{42}) = (-l) f_{k+1,l-1},\\
P^{(+,-)}(f_{kl}\otimes X_{24}) &= -\overline{P}^{(+,-)}(f_{kl}\otimes X_{31}) = (l-s) f_{kl},\\
P^{(+,-)}(f_{kl}\otimes X_{23}) &= \phantom{-}\overline{P}^{(+,-)}(f_{kl}\otimes X_{41}) = (-l) f_{k,l-1},\\
P^{(+,-)}(f_{kl}\otimes X_{14}) &= \phantom{-}\overline{P}^{(+,-)}(f_{kl}\otimes X_{32}) = (l-s) f_{k+1,l},\\
P^{(-,+)}(f_{kl}\otimes X_{13}) &= -\overline{P}^{(-,+)}(f_{kl}\otimes X_{42}) = (r-k) f_{kl},\\
P^{(-,+)}(f_{kl}\otimes X_{24}) &= -\overline{P}^{(-,+)}(f_{kl}\otimes X_{31}) = k f_{k-1,l+1},\\
P^{(-,+)}(f_{kl}\otimes X_{23}) &= \phantom{-}\overline{P}^{(-,+)}(f_{kl}\otimes X_{41}) = (-k) f_{k-1,l},\\
P^{(-,+)}(f_{kl}\otimes X_{14}) &= \phantom{-}\overline{P}^{(-,+)}(f_{kl}\otimes X_{32}) = (k-r) f_{k,l+1}.
\end{aligned}
\tag{19}
$$

PROOF. By Proposition 2.4 and Lemma 2.5, we readily get these equations. □

### 2.5. Principal $P_m$-series representations.

To see the definition of the principal $P_m$-series representations of $G$, we start with constructing characters of the minimal parabolic subgroup $P_m = M_m A_m N_m$. Let $n$ be an integer and let $\epsilon$ be a character of the group $\{\pm 1\}$ , which we also identify with an element of $\{\pm1\}$. Define the unitary character of $M_m$ as follows:

$$\sigma_{n,\epsilon}(\exp(\sqrt{-1}\theta I_0)\gamma^j) = \epsilon(-1)^j e^{\sqrt{-1}n\theta}.$$

Denoting by $\rho = 3\lambda_1 + \lambda_2$, we define a (not necessarily unitary) character $e^{\mu+\rho}$ of $A_m$ by

$$e^{\mu+\rho}(a) = e^{(\mu+\rho)(\log a)}$$

for $\mu \in \mathfrak{a}^*_{\mathbb{C}}$. We extend it to a character of $A_m N_m$ so that the restriction to $N_m$ is trivial. We get a character of $P_m$ by tensoring these characters.

Now we define an induced representation of $G$. Let $H^0_\pi$ be the space of continuous functions on $G$ satisfying the equation,

$$f(pg) = \sigma_{n,\epsilon}\otimes e^{\mu+\rho}(p) f(g)$$

for $p \in P_m$ and $g \in G$. This space is a pre-Hilbert space endowed with the inner product $(\phi_1, \phi_2) = \int_K \phi_1(k)\overline{\phi_2(k)}\,dk$. Then $\pi = \mathrm{ind}^G_{P_m}(\sigma_{n,\epsilon}\otimes e^{\mu+\rho}\otimes 1)$ acts on $H^0_\pi$ by right translation:

$$\pi(g)\phi(x) = \phi(xg) \quad (g,\, x\in G,\ \phi\in H^0_\pi).$$

We remark that $\pi$ is unitary if and only if $\mu$ is purely imaginary. We also denote the $K$-finite vectors by $H_\pi^K$.

**2.6. Principal $P_J$-series representations.** Let $G_0 = SU(1,1)$ and $K_0 \simeq \mathbb{C}^{(1)}$ be its maximal compact subgroup. We regard these groups as subgroups of $M_J$. Let $\chi_m(e^{\sqrt{-1}\theta}) = e^{m\sqrt{-1}\theta}$ be a character of $\mathbb{C}^{(1)}$. The weight lattice of $\mathfrak{g}_o = \mathrm{Lie}(G_0)$ can be identified with $\mathbb{Z}$ with property:

$$\chi_m(\mathrm{diag}(1, e^{\sqrt{-1}\theta}, 1, e^{-\sqrt{-1}\theta})) = \chi_m(e^{\sqrt{-1}\theta}) = e^{m\sqrt{-1}\theta} \quad \text{for } m \in \mathbb{Z}.$$

Let $D_k^\pm$ be the discrete series representation with Blattner parameter $\pm k$. Namely, the minimal $K_0$-type of $D_k^+$ (resp. $D_k^-$) is $\chi_k$, $(k \geq 2)$, (resp. $\chi_{-k}$, $(k \leq -2)$) and the other $K_0$-types are in the form $\chi_{k+2j}$, (resp. $\chi_{-k-2j}$), with non-negative integers $j$. We say that the suffix $\pm$ is the signature of $D_k^\pm$ and denote it by $\mathrm{sgn}(D_k^\pm)$. We note that the contragredient representation of $D_k^+$ is isomorphic to $D_k^-$.

Let $\sigma = (\chi_m, D_k^\pm)$ be a discrete series representation of $M_J$. Choose $\nu \in \mathfrak{a}_{J,\mathbb{C}}^*$. By the symbol $e^\nu$, we denote the character defined by $e^\nu(a_1) = e^{\nu(\log a_1)}$. Then we can define $\pi_J = \mathrm{ind}_{P_J}^G(\sigma \otimes e^{\nu+\rho_J} \otimes 1)$ acting by right translation on $H_{\pi_J}$ similarly, with $\rho_J = 3\lambda_1$. We say $\pi_J$ the principal $P_J$-series representation of $SU(2,2)$.

**2.7. Discrete series representations.** The discrete series representations of $G$ is parametrized efficiently by Harish-Chandra, which we describe.

There are exactly six positive systems of the absolute roots, $\widetilde{\Delta}_{\mathrm{I}}^+$, $\widetilde{\Delta}_{\mathrm{II}}^+, \ldots, \widetilde{\Delta}_{\mathrm{VI}}^+$ containing $\widetilde{\Delta}_c^+$, defined by $\widetilde{\Delta}_J^+ = w_J\widetilde{\Delta}^+$, where the elements $w_J \in \widetilde{W}$ are given as,

$$w_{\mathrm{I}} = 1,\ w_{\mathrm{II}} = s_2,\ w_{\mathrm{III}} = s_2 s_3,\ w_{\mathrm{IV}} = s_2 s_1,\ w_{\mathrm{V}} = s_2 s_3 s_1,\ w_{\mathrm{VI}} = s_2 s_1 s_3 s_2.$$

We denote by $\widetilde{\Delta}_{n,J}^+$ the noncompact positive roots in $\widetilde{\Delta}_J^+$.

By definition, the space of the Harish-Chandra parameters $\Xi_c$ is given by,

$$\Xi_c = \left\{\Lambda \in \mathfrak{t}_{\mathbb{C}}^* \mid \Lambda \text{ is } \widetilde{\Delta}\text{-regular, } K\text{-analytically integral and } \widetilde{\Delta}_c^+\text{-dominant}\right\}.$$

Put

$$\Xi_J = \left\{\Lambda \in \Xi_c \mid \widetilde{\Delta}_J^+\text{-dominant}\right\}.$$

We also put $\rho_{G,J} = \frac{1}{2}\sum_{\beta\in\widetilde{\Delta}_J^+}\beta$ (resp. $\rho_K = \frac{1}{2}\sum_{\beta\in\widetilde{\Delta}_c^+}\beta$), the half sum of positive roots (resp. the half sum of compact positive roots.) Then the space $\Xi_c \subset \mathfrak{t}_{\mathbb{C}}^*$ are divided into six parts: $\Xi_c = \bigcup_{\mathrm{I}\leq J\leq \mathrm{V}} \Xi_J$. We note that $\Xi_{\mathrm{I}}$ (resp. $\Xi_{\mathrm{VI}}$) corresponds to the holomorphic (resp. anti-holomorphic) discrete series. For $\Lambda \in \bigcup_{\mathrm{I}\leq J\leq \mathrm{VI}} \Xi_J$, we denote the corresponding discrete series by $\pi_\Lambda$ and call this $\Lambda$ a Harish-Chandra parameter of $\pi_\Lambda$. As determined in [30, §10.4], the Gelfand-Kirillov dimensions of the discrete series representations $\pi_\Lambda$ are given as follows:

$$\text{(20)} \qquad \mathrm{GK\text{-}dim}(\pi_\Lambda) = \begin{cases} 4 & (\Lambda \in \Xi_{\mathrm{I}} \cup \Xi_{\mathrm{VI}}), \\ 6 & (\Lambda \in \Xi_{\mathrm{II}} \cup \Xi_{\mathrm{V}}), \\ 5 & (\Lambda \in \Xi_{\mathrm{III}} \cup \Xi_{\mathrm{IV}}). \end{cases}$$

Therefore the representations belonging to $\Xi_{\mathrm{II}} \cup \Xi_{\mathrm{V}}$ is a large representation in the sense of Vogan [26, Th. 6.2, f)], hence has an algebraic Whittaker model. (Harish-Chandra parameters of discrete series of $SU(2,2)$ are described as in Figure I.1.)

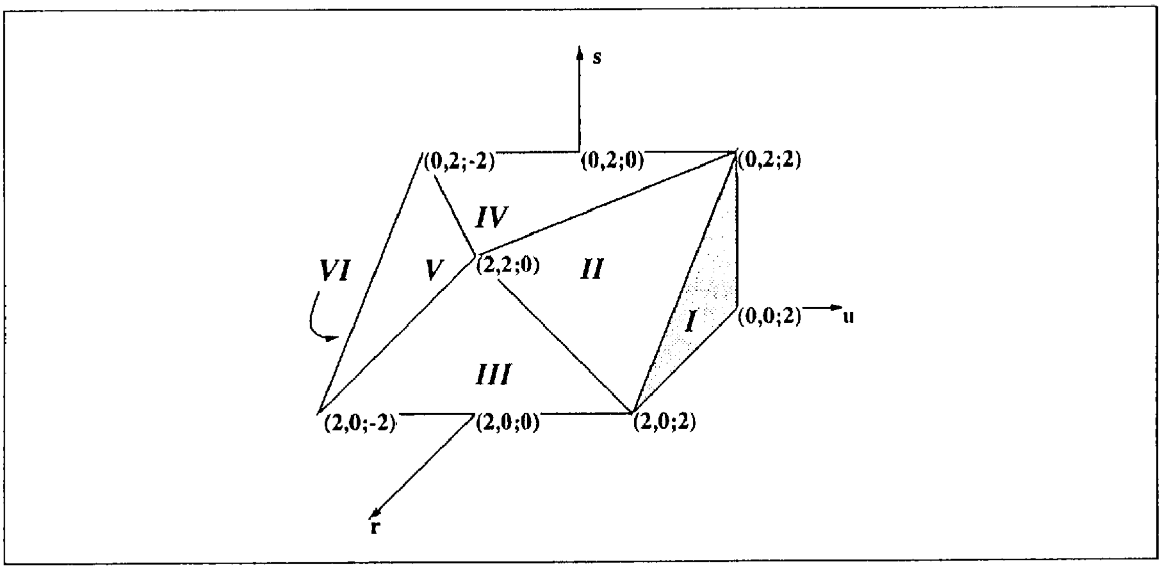


FIGURE I.1. Harish-Chandra parameters in $\mathfrak{t}_{\mathbb{C}}^*$

## 2.8. The multiplicity of $K$-types.

For an admissible representation $\pi$ of $G$, the restriction of $\pi$ to $K$ decomposes into the Hilbert-space sum of irreducible representations of $K$ with finite multiplicities:

$$\pi|_K = \widehat{\bigoplus_{\tau \in \widehat{K}}} [\pi|_K : \tau]\tau.$$

Here $[\pi|_K : \tau]$ is the multiplicity of $\tau$ in $\pi|_K$. If $[\pi|_K : \tau] \neq 0$, we call $\tau$ a $K$-type of $\pi$.

To find an multiplicity-one $K$-type of a given admissible representation, we calculate the multiplicity explicitly in this subsection in the case of principal series. The minimal $K$-type of a discrete series representation is always multiplicity one.

The multiplicity is given by Frobenius reciprocity: (see [10, Theorem 1.14].)

**Lemma 2.7.** *Let* $\pi_* = \mathrm{ind}_{M_* A_* N_*}^G (\sigma_{M_*} \otimes e^{\mu+\rho_*} \otimes 1)$ *with* $\sigma_{M_*}$ *a discrete series representation of* $M_*$, *and* $\tau \in \widehat{K}$. *Then*

$$[\pi_*|_K : \tau] = \sum_{\omega \in (K \cap M_*)^\wedge} [\sigma_{M_*}|_{K \cap M_*} : \omega][\tau|_{K \cap M_*} : \omega]. \tag{21}$$

Particularly in the case of $P_m$-series representations $\pi_m$, we have

$$[\pi_m|_K : \tau] = [\tau|_M : \sigma_{M_m}]$$

since $K \cap P_m = M_m$.

We proceed to calculate the multiplicity in the case of $P_m$-series. Let $\tau_d = \tau_{[r,s;u]}$ with $d = [r, s; u]$ and let the representation space be $V_d = V_{rs}$, given as,

$$V_d = \bigoplus_{0\le p\le r,\ 0\le q\le s} \mathbb{C} f^{(d)}_{pq}.$$

This can be regarded as a direct sum of 1-dimensional $M$-submodules $\mathbb{C}f^{(d)}_{pq}$.
**Lemma 2.8.** *We have,*

$$\tau_d(\gamma) f^{(d)}_{pq} = \sqrt{-1}^{(r-2p+2q-s+u)} f^{(d)}_{pq}, \tag{22}$$

$$\tau_d(\exp(\sqrt{-1}\theta I_0)) f^{(d)}_{pq} = e^{\sqrt{-1}(2p-r+2q-s)\theta} f^{(d)}_{pq}. \tag{23}$$

PROOF. We see that, by definition,

$$pr(\mathrm{diag}(-\sqrt{-1}, \sqrt{-1}), \mathrm{diag}(\sqrt{-1}, -\sqrt{-1}); \sqrt{-1}) = \gamma,$$
$$pr(\mathrm{diag}(e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}), \mathrm{diag}(e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}); 1) = \exp(\sqrt{-1}\theta I_0).$$

Thus, from the values of $\tau_d$ at $\gamma$ and $\exp(\sqrt{-1}\theta I_0)$, we obtain the lemma. □

**Proposition 2.9.** *Let* $\pi = \mathrm{ind}^G_{P_m}(\sigma_{n,\epsilon} \otimes e^{\mu+\rho} \otimes 1)$ *and let* $\tau_d$ *be an irreducible representation of* $K$ *with parameter* $d = [r, s; u]$. *Assume that,*

$$r + s \ge |n| \text{ and } -2s + u \equiv n + 1 - \epsilon(-1) \pmod 4. \tag{24}$$

*Then,*

$$[\pi|_K : \tau_d] = \tfrac{1}{4}\left\{2(r+s) - |r-s| - |n| - \big||r-s| - |n|\big|\right\} + 1. \tag{25}$$

PROOF. By Lemma 2.7, it is enough to calculate the multiplicity $[\tau_d|_M : \sigma_{n,\epsilon}]$. First, $[\tau_d|_M : \sigma_{n,\epsilon}]$ is equal to the number of indices $(p, q)$ which satisfy

$$\tau_d(m) f^{(d)}_{pq} = \sigma(m) f^{(d)}_{pq} \qquad (m \in M).$$

By comparing Equation (22) in Lemma 2.8 with the definition of $\sigma(\gamma)$, we get,

$$\sqrt{-1}^{(r-2p+2q-s+u)} = \epsilon(-1), \tag{26}$$

or equivalently,

$$r - 2p + 2q - s + u \equiv 1 - \epsilon(-1) \pmod 4. \tag{27}$$

Similarly, from Equation (23) and the value of $\sigma(\exp(\sqrt{-1}\theta I_0))$, we have,

$$2p - r + 2q - s = n. \tag{28}$$

Adding equations (27) and (28), we get the necessary condition for the multiplicity to be positive. This is exactly the assumption (24) in the statement. Now, to determine the multiplicity, we count the indices $(p, q)$ satisfying (28).

Assume $r \ge s$. Then (See Figure I.2),

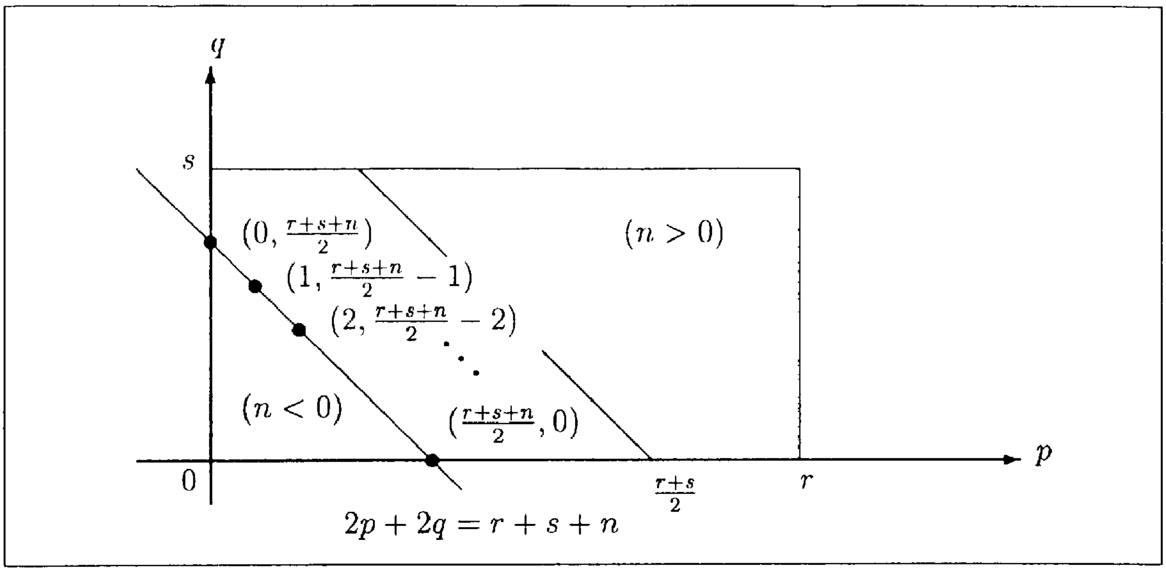


FIGURE I.2

(i). If $n \geq 0$ and $2r \leq r + s + n$, then,

$$(p, q) = (\tfrac{r-s+n}{2}, s), (\tfrac{r-s+n}{2} + 1, s - 1), \ldots, (r, \tfrac{-r+s+n}{2})$$

satisfy (28), thus the number of $(p, q)$'s is $(r + s - n)/2 + 1$.

(ii). If $2s \leq r + s + n < 2r$, then,

$$(p, q) = (\tfrac{r-s+n}{2}, s), (\tfrac{r-s+n}{2} + 1, s - 1), \ldots, (\tfrac{r+s+n}{2}, 0)$$

satisfy (28), thus the number of $(p, q)$'s is $s + 1$.

(iii). If $n < 0$ and $r + s + n < 2s$, then,

$$(p, q) = (0, \tfrac{r+s+n}{2}), (1, \tfrac{r+s+n}{2} - 1), \ldots, (\tfrac{r+s+n}{2}, 0)$$

satisfy (28), thus the number of $(p, q)$'s is $(r + s + n)/2 + 1$.

In conclusion, the number of $(p, q)$'s satisfying (28) is $\min\{s+1, \frac{r+s-|n|}{2} + 1\}$. Similarly, in the case $r < s$, we get the formula: $\min\{r + 1, \frac{r+s-|n|}{2} + 1\}$. Thus, the multiplicity of $\tau_d$ in $\pi|_K$ equals $\min\{s + 1, r + 1, \frac{r+s-|n|}{2} + 1\}$, which coincides with the right-hand side of (25) in the proposition. □

Next, we handle the case of $P_J$-series. To find the multiplicity of $\tau$ in $\pi_J|_K$, we prepare several lemmas. First, we see that

$$K \cap M_J = \{\, (e^{\sqrt{-1}\theta}, e^{\sqrt{-1}\zeta}) = \exp(\theta I_0)\exp(\zeta H_{24}) \mid \theta, \zeta \in \mathbb{R} \,\}.$$

Thus the characters of $K \cap M_J$ can by parametrized along the following:

$$\omega_{(l_1,l_2)}(e^{\sqrt{-1}\theta}, e^{\sqrt{-1}\zeta}) = e^{\sqrt{-1}(l_1\theta + l_2\zeta)}.$$

Clearly, we have,

**Lemma 2.10.** *Let* $\sigma = (\chi_m, D_k^{\pm})$. *Then,*

$$[\sigma|_{K\cap M_J} : \omega_{(l_1,l_2)}] = \begin{cases} 1 & \text{if } m = l_1,\ \operatorname{sgn}(D_k^{\pm})l_2 \geq k \text{ and } l_2 \equiv k \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

Since $V_\tau$ decomposes into 1-dimensional $K \cap M_J$-modules, we have the following:

**Lemma 2.11.** *Let* $\tau = \tau_{[r,s;u]} \in \widehat{K}$. *Then,*

$$[\tau|_{K\cap M_J} : \omega_{(l_1,l_2)}] = \begin{cases} 1, & \begin{array}{c} -u - 2r \leq l_1 - 2l_2 \leq 2r - u, \\ u - 2s \leq l_1 + 2l_2 \leq 2s + u, \\ l_1 - 2l_2 + u + 2r \equiv l_1 + 2l_2 - u + 2s \equiv 0 \pmod 4, \end{array} \\ 0, & \text{otherwise.} \end{cases}$$

Summing up, the multiplicity is given by the following.

**Proposition 2.12.** $[\pi_J|_K : \tau]$ *equals the number of integers* $l_2$ *satisfying the following*:

(i) $l_2 \equiv k \pmod 2$,
(ii) $\operatorname{sgn}(D_k^{\pm})l_2 \geq k$,
(iii) $2l_2 - u \equiv m + 2r \equiv -m + 2s \pmod 4$,
(iv) $\max(m - 2r, -m - 2s) \leq 2l_2 - u \leq \min(m + 2r, -m + 2s)$.

In particular the necessary and sufficient condition for multiplicity one can be described as follows: (see Figure I.3 as an example.)

**Theorem 2.13.** *Let* $\pi_J = \operatorname{ind}_{P_J}^G((\chi_m, D_k^{\pm}) \otimes e^{\nu+\rho_J} \otimes 1)$. *Put* $\delta_D = \operatorname{sgn}(D_k^{\pm})$, $\delta_m = \operatorname{sgn}(m)$, $(\delta_0 = 0)$. *Then the multiplicity* $[\pi_J|_K; \tau_{[r,s;u]}] = 1$ *if and only if the parameter* $[r, s; u]$ *satisfies* $2k - u \equiv m + 2r \pmod 4$ *and one of the following*:

(i). $r = 0$, $s \geq |m|$ *and* $\delta_D u \geq 2k - \delta_D m$.
(ii). $0 \leq r \leq s - |m|$ *and* $u = 2\delta_D(-r + k) - m$.
(iii). $r + s = |m|$ *and* $\delta_D u \geq -2\delta_D\delta_m s + 2k + \delta_D m$.
(iv). $|r - s| \leq |m|$, $r + s \geq |m|$ *and*

$$u = \begin{cases} -2\delta_D s + 2\delta_m k + m & (\text{if } \delta_D\delta_m \geq 0), \\ -2\delta_D r - 2\delta_m k - m & (\text{if } \delta_D\delta_m < 0). \end{cases}$$

(v). $r - |m| \geq s \geq 0$ *and* $u = 2\delta_D(-s + k) + m$.
(vi). $r \geq |m|$, $s = 0$ *and* $\delta_D u \geq 2k + \delta_D m$.

PROOF. If $\tau_{[r,s;u]}$ satisfies one of these conditions, we can easily check that its multiplicity in $\pi|_K$ is one.

Let $\tau_{[r,s;u]}$ be a multiplicity-one $K$-type. We assume that $\operatorname{sgn}(D_k^{\pm}) > 0$ and $m > 0$ for clarity. If $r \leq s - m$, then Proposition 2.12 says that there is a unique $l_2$ which satisfies

$$l_2 \equiv k \pmod 2 \quad \text{and} \quad \max(m - 2r, 2k - u) \leq 2l_2 - u \leq m + 2r.$$

By the congruence property, $l_2 s - u$ attains $m + 2r$. In order that the only one $l_2$ satisfies the above conditions, it should be $m-2r \leq 2k-u = m+2r$ or $ks-u \leq m-sr \leq m+2r$, equivalent to

$$\begin{cases} r \geq 0, \\ 2r + u = 2k - m, \end{cases} \text{or} \begin{cases} r = 0, \\ u \geq 2k - m, \end{cases}$$

so we have (i) and (ii). Next, if $|s - r| \leq m$, we see that $m - 2r \leq 2k - u = -m + 2s$ or $2k - u \leq m - 2r = -m + 2s$, which is equivalent to

$$\begin{cases} 2s + u = 2k + m, \\ r + s \geq m, \end{cases} \text{or} \begin{cases} r + s = m, \\ u \geq 2k - m + 2r. \end{cases}$$

So we have (iii) and (iv). If $r \geq s+m$, then we have $-m-2s \leq 2k-u = -m+2s$, $s \geq 0$ or $2k - u \leq -m - 2s = -m + 2s$. This is equivalent to

$$\begin{cases} 2s + u = 2k + m, \\ s \geq 0, \end{cases} \text{or} \begin{cases} s = 0, \\ u \geq 2k + m. \end{cases}$$

Hence (v) and (vi) follow. □

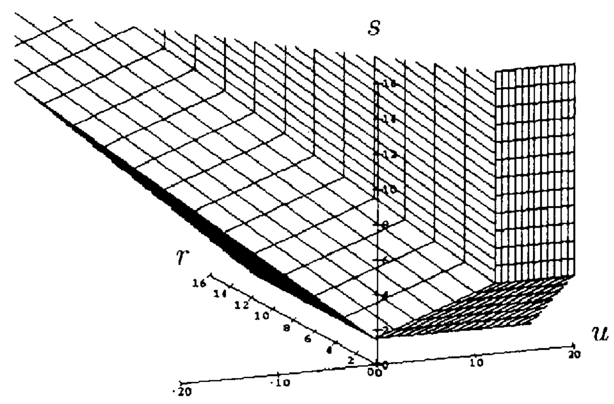


FIGURE I.3. Multiplicity-one $K$-types of $\mathrm{ind}_{P_J}^G((\chi_{-4}, D_4^+) \otimes e^{\nu+\rho_J} \otimes 1)$.

**2.9. $K$-isotypic components of the principal $P_m$-series representations.** We prepare the series of subsections for the calculation about $P_m$-series.

Let $\pi = \mathrm{ind}_{P_m}^G(\sigma_{n,\epsilon} \otimes e^{\mu+\rho} \otimes 1)$ and let $(\tau_d, V_d)$ $(d = [r, s; u])$ be an irreducible representation of $K$ which occurs in $\pi|_K = \mathrm{ind}_{P_m}^G(\sigma_{n,\epsilon} \otimes e^{\mu+\rho} \otimes 1)|_K$. By definition, the $\tau_d$-isotypic component $H_\pi(\tau_d)$ is isomorphic to a $[\pi|_K : \tau_d]$-tuple direct sum of $\tau_d$. In this subsection we describe its basis by using matrix coefficients of $\tau_d$.

To begin with, define the functions $a_{pq,lm}^{(d)}$ on $K$ by the following rule:

$$\tau_d(k) f_{pq}^{(d)} = \sum_{0 \leq l \leq r,\ 0 \leq m \leq s} a_{pq,lm}^{(d)}(k) f_{lm}^{(d)}, \qquad (k \in K).$$

If we rewrite the equation:

$$\tau_d(xy)f_{pq}^{(d)} = \tau_d(x)\tau_d(y)f_{pq}^{(d)}$$

in terms of matrix coefficients, we get,

$$a_{pq,lm}^{(d)}(xy) = \sum_{0\le\mu\le r,\ 0\le\nu\le s} a_{\mu\nu,lm}^{(d)}(x)a_{pq,\mu\nu}^{(d)}(y) \tag{29}$$

for all $x, y \in K$. We also remark that

$$a_{pq,lm}^{(d)}(1_4) = \delta_{pq,lm}. \tag{30}$$

Define the action of $K$ on these matrix coefficients by

$$R_k\, a_{pq,lm}^{(d)}(x) = a_{pq,lm}^{(d)}(xk) \qquad (k \in K).$$

We also define the $K$-module generated by $a_{rs,lm}^{(d)}$ as $W_{lm}^{(d)} = \{R_k a_{rs,lm}^{(d)} \mid k \in K\}_{\mathbb{C}}$. By Equation (29) with $k \in K$ in place of $x$, we see that

$$W_{lm}^{(d)} \subset \bigoplus_{\mu,\nu} \mathbb{C}a_{\mu\nu,lm} \ \big(\subset L^2(K)\big).$$

We can easily check that the vector $a_{rs,lm}^{(d)}$ has the same properties as of the highest weight vector $f_{rs}^{(d)}$. So we have,

**Lemma 2.14.** *For any* $0 \le l \le r$, $0 \le m \le s$, $W_{lm}^{(d)}$ *is* $K$*-isomorphic to* $V_d$.

Let $\sigma = \sigma_{n,\epsilon} \in \widehat{M}_m$, and put

$$L_\sigma^2(K) = \left\{\phi \in L^2(K) \mid \phi(mk) = \sigma(m)\phi(k) \text{ for all } m \in M_m, \text{ a.e. } k \in K\right\}. \tag{31}$$

Through the restriction, the induced representation $\pi = \mathrm{ind}_{P_m}^G(\sigma_{n,\epsilon} \otimes e^{\mu+\rho} \otimes 1)$ can be realized on $L_\sigma^2(K)$ by right translation.

**Lemma 2.15.** *Assume* $r + s \ge |n|$ *and* $u \equiv 2s + n + 1 - \epsilon(-1) \pmod 4$. *Then,*

$$H_\pi(\tau_d) = \bigoplus_l W_{l,\frac{r+s+n}{2}-l}^{(d)},$$

*where* $l$ *runs over integers satisfying,*

$$\begin{cases} (r-s+n)/2 \le l \le r, & \text{if } n \ge 0,\ 2\max(r,s) \le r+s+n, \\ (r-s+n)/2 \le l \le (r+s+n)/2, & \text{if } 2s \le r+s+n < 2r, \\ 0 \le l \le r, & \text{if } 2r \le r+s+n < 2s, \\ 0 \le l \le (r+s+n)/2, & \text{if } n < 0,\ 2\min(r,s) > r+s+n. \end{cases} \tag{32}$$

PROOF. The element $a^{(d)}_{pq,lm}$ belongs to $L^2_\sigma(K)$ if and only if,

$$a^{(d)}_{pq,lm}(kx) = \sigma(k) a^{(d)}_{pq,lm}(x)$$

for any $k \in M_m$. By Lemma 2.8 and Equation (29), we get

$$\begin{aligned} a^{(d)}_{pq,lm}(\gamma x) &= \sqrt{-1}^{(r-2l+2m-s+u)} a^{(d)}_{pq,lm}(x), \\ a^{(d)}_{pq,lm}(\exp(\sqrt{-1}\theta I_0)x) &= e^{\sqrt{-1}\theta(2l-r+2m-s)}\, a^{(d)}_{pq,lm}(x). \end{aligned}$$

Therefore as shown in Proposition 2.9, we see that the sum of the $W^{(d)}_{l,\frac{r+s+n}{2}-l}$'s satisfying (32) is contained in the $\tau_d$-isotypic component of $\pi$. They exhaust $H_\pi(\tau_d)$ since its dimension is $[\pi|_K : \tau_d] \dim V_d$. □

**Corollary 2.16.** *Let $\sigma = \sigma_{n,\epsilon} \in \widehat{M}_m$ and assume that $r + s = |n|$ and $-2s + u \equiv n + 1 - \epsilon(-1) \pmod 4$. Then the map defined by*

$$\begin{cases} f_{pq} \mapsto a^{(d)}_{pq,rs} & (0 \le p \le r,\ 0 \le q \le s), \quad \text{if } n \ge 0, \\ f_{pq} \mapsto a^{(d)}_{pq,00} & (0 \le p \le r,\ 0 \le q \le s), \quad \text{if } n < 0 \end{cases}$$

*is up to a multiple the unique $K$-injection into $L^2_\sigma(K) \simeq H_\pi$.*

PROOF. By Proposition 2.9, we have $[\pi|_K : \tau_d] = 1$ in this case, therefore there is a unique $l$ such that $W^{(d)}_{l,\frac{|n|+n}{2}-l} = H_\pi(\tau_d)$. Lemma 2.15 tells for which $l$ this is valid. □

**2.10. An operator between one-dimensional $K$-types.** From now on, we consider the principal series representation $\pi = \mathrm{ind}^G_{P_m}(\sigma_{n,\epsilon} \otimes e^{\mu+\rho} \otimes 1)$ as a $(U(\mathfrak{g}_\mathbb{C}), K)$-module and identify $H^K_\pi$ with $L^2_\sigma(K)^K$ through restriction (cf. (31)). We will construct certain elements $Y$ of degree 2 in $U(\mathfrak{g}_\mathbb{C})$ such that $\pi(Y)$ maps each 1-dimensional $K$-types to another in $\pi|_K$.

Put $d_0 = [0,0;u]$, $d_{\pm 1} = [1,1;u \pm 2]$ and $d_{\pm 2} = [0,0;u \pm 4]$. Write $\tau_j = \tau_{d_j}$ for simplicity. We assume $n = 0$ so that $\pi = \mathrm{ind}^G_{P_m}(\sigma_{0,\epsilon} \otimes e^{\mu+\rho} \otimes 1)$ contains 1-dimensional $K$-types. Put $\sigma = \sigma_{0,\epsilon}$ and assume that $u \equiv 1 - \epsilon(-1)$, then $[\pi|_K : \tau_0] = 1$, $[\pi|_K : \tau_{\pm 1}] = 2$ and $[\pi|_K : \tau_{\pm 2}] = 1$ (cf. Proposition 2.9). We also write $a^{(j)}_{pq} := a^{(j)}_{pq,00}$ (cf. Corollary 2.16). Define

$$\begin{aligned} g^{(1)}_{00} &= X_{23} \cdot a^{(0)}_{00}, & g^{(-1)}_{00} &= X_{41} \cdot a^{(0)}_{00}, \\ g^{(1)}_{10} &= X_{13} \cdot a^{(0)}_{00}, & g^{(-1)}_{10} &= -X_{42} \cdot a^{(0)}_{00}, \\ g^{(1)}_{01} &= -X_{24} \cdot a^{(0)}_{00}, & g^{(-1)}_{01} &= X_{31} \cdot a^{(0)}_{00}, \\ g^{(1)}_{11} &= -X_{14} \cdot a^{(0)}_{00}, & g^{(-1)}_{11} &= -X_{32} \cdot a^{(0)}_{00}. \end{aligned}$$

**Proposition 2.17.** *The space generated by $g^{(1)}_{ij}$ (resp. $g^{(-1)}_{ij}$) $(0 \le i \le 1, 0 \le j \le 1)$, as a $K$-module, is isomorphic to $\tau_1$ (resp. $\tau_{-1}$).*

PROOF. In fact, this space turns out to be isomorphic to $(\mathrm{Ad}_+, \mathfrak{p}_+)$ (resp. $(\mathrm{Ad}_-, \mathfrak{p}_-)$) if we restrict to the commutator $[\mathfrak{k}, \mathfrak{k}]$. We thus get the result by Proposition 2.4. □

**Lemma 2.18.** *We have,*

$$\begin{aligned} &g_{00}^{(1)}(1_4) = g_{11}^{(1)}(1_4) = 0, && g_{00}^{(-1)}(1_4) = g_{11}^{(-1)}(1_4) = 0,\\ &g_{10}^{(1)}(1_4) = \quad (2\mu_1 + u + 6)/4, && g_{01}^{(-1)}(1_4) = \quad (2\mu_1 - u + 6)/4,\\ &g_{01}^{(1)}(1_4) = -(2\mu_2 + u + 2)/4, && g_{10}^{(-1)}(1_4) = -(2\mu_2 - u + 2)/4. \end{aligned} \tag{33}$$

PROOF. Using §1.3,

$$g_{00}^{(1)}(1_4) = \tfrac{1}{2}\left(E_3 + E_5 + \sqrt{-1}(E_4 + E_6) + 2e_-^1\right) \cdot a_{00}^{(0)}(k)|_{k=1_4}.$$

Note that $X \cdot a_{00}^{(0)}(k) \equiv 0$ for any $X \in \mathfrak{n}$. Thus we conclude that

$$g_{00}^{(1)}(1_4) = 0.$$

Similarly, we get $g_{11}^{(1)}(1_4) = 0$.

Next, considering the action of the center of $\mathfrak{k}$, we have,

$$\begin{aligned} g_{10}^{(1)}(1_4) &= \tfrac{1}{2}\left(\sqrt{-1}E_1 + H_1 + \tfrac{1}{2}(I_{2,2} + h^1 - h^2)\right) \cdot a_{00}^{(0)}(k)|_{k=1_4}\\ &= \tfrac{1}{2}(H_1 + \tfrac{1}{2}I_{2,2}) \cdot a_{00}^{(0)}(k)|_{k=1_4} = (2\mu_1 + u + 6)/4. \end{aligned}$$

Correspondingly,

$$g_{01}^{(1)}(1_4) = -\tfrac{1}{2}(H_2 + \tfrac{1}{2}I_{2,2}) \cdot a_{00}^{(0)}(k)|_{k=1_4} = -(2\mu_2 + u + 2)/4.$$

By using exactly the same method, we find the relations for the $g_{kl}^{(-1)}(1_4)$. □

Define,

$$\begin{aligned} g^{(2)} &= X_{14} \cdot g_{00}^{(1)} + X_{13} \cdot g_{01}^{(1)} - X_{24} \cdot g_{10}^{(1)} - X_{23} \cdot g_{11}^{(1)},\\ g^{(-2)} &= X_{32} \cdot g_{00}^{(-1)} + X_{31} \cdot g_{10}^{(-1)} - X_{42} \cdot g_{01}^{(-1)} - X_{41} \cdot g_{11}^{(-1)}. \end{aligned} \tag{34}$$

We can check that the weight of $g^{(\pm 2)}$ is $d_{\pm 2}$. Hence $g^{(\pm 2)}$ generates a 1-dimensional $K$-module isomorphic to $\tau_{\pm 2}$, respectively.

**Proposition 2.19.** *We have,*

$$g^{(2)} = {}^t\begin{pmatrix} X_{14}\\ X_{13}\\ -X_{24}\\ -X_{23} \end{pmatrix} \begin{pmatrix} X_{23}\\ -X_{24}\\ X_{13}\\ -X_{14} \end{pmatrix} a_{00}^{(0)} = -\left(\mu_1 + \frac{u}{2} + 1\right)\left(\mu_2 + \frac{u}{2} + 1\right) a_{00}^{(2)},$$

$$g^{(-2)} = {}^t\begin{pmatrix} X_{32}\\ X_{31}\\ -X_{42}\\ -X_{41} \end{pmatrix} \begin{pmatrix} X_{41}\\ -X_{42}\\ X_{31}\\ -X_{32} \end{pmatrix} a_{00}^{(0)} = -\left(\mu_1 - \frac{u}{2} + 1\right)\left(\mu_2 - \frac{u}{2} + 1\right) a_{00}^{(-2)}.$$

PROOF. We calculate the value at $1_4$:

$$\begin{aligned} g^{(2)}(1_4) &= -e_+^2 \cdot g_{00}^{(1)}(1_4) + \tfrac{1}{2}\left(H_1 + \tfrac{1}{2}(I_{2,2} + h^1 - h^2)\right) \cdot g_{01}^{(1)}(1_4) \\ &\quad - \tfrac{1}{2}\left(H_2 + \tfrac{1}{2}(I_{2,2} - h^1 + h^2)\right) \cdot g_{10}^{(1)}(1_4) - e_-^1 \cdot g_{11}^{(1)}(1_4) \\ &= -g_{01}^{(1)}(1_4) + \tfrac{1}{2}\left(\mu_1 + 3 + \tfrac{1}{2}(u + 2 - 1 - 1)\right) g_{01}^{(1)}(1_4) \\ &\quad - \tfrac{1}{2}\left(\mu_2 + \tfrac{1}{2}(u + 2 - 1 - 1)\right) g_{10}^{(1)}(1_4) - g_{01}^{(1)}(1_4) \\ &= -(\mu_1 + u/2 + 1)(\mu_2 + u/2 + 1). \end{aligned}$$

Similarly, we get,

$$g^{(-2)}(1_4) = -(\mu_1 - u/2 + 1)(\mu_2 - u/2 + 1).$$

Equation $a_{00}^{(\pm 2)}(1_4) = 1$ implies the proposition. □

**Remark 2.20.** We know that $[\pi|_K : \tau_{\pm 1}] = 2$, that is, there are two independent $K$-injections of $\tau_{\pm 1}$ into the induced representation $L^2_\sigma(K)$ (cf. Lemma 2.15). Set,

$$\iota_{\pm 1,1} : f_{pq}^{(\pm 1)} \mapsto a_{pq,01}^{(\pm 1)}, \qquad \iota_{\pm 1,2} : f_{pq}^{(\pm 1)} \mapsto a_{pq,10}^{(\pm 1)}, \qquad \jmath_{\pm 1} : f_{pq}^{(\pm 1)} \mapsto g_{pq}^{(\pm 1)}.$$

By taking the value at $1_4$ of the functions above, we get

$$\begin{aligned} \jmath_1 &= -\tfrac{1}{4}(2\mu_2 + u + 2)\iota_{1,1} + \tfrac{1}{4}(2\mu_1 + u + 6)\,\iota_{1,2}, \\ \jmath_{-1} &= \tfrac{1}{4}(2\mu_1 + u + 6)\iota_{-1,1} - \tfrac{1}{4}(2\mu_2 + u + 2)\iota_{-1,2}. \end{aligned}$$

### 2.11. An operator between two-dimensional $K$-types.

If there are two $K$-types of dimension 2 in $\pi$ and if these are "close", we can construct an element in $M_2(U(\mathfrak{g}_\mathbb{C}))$ which maps one to the other.

Put $d = d_0 = [r, s; u]$, $d_{\pm 1} = [r + 1, s - 1; u \pm 2]$ and $d_{\pm 2} = [r - 1, s + 1; u \pm 2]$. For $\pi = \mathrm{ind}_{P_m}^G(\sigma_{n,\epsilon} \otimes e^{\mu+\rho} \otimes 1)$, assume that $r + s = |n|$, $-2s + u \equiv n + 1 - \epsilon(-1) \pmod 4$, then $\tau_0$ is in $\pi|_K$ with multiplicity one. Throughout this subsection we assume that all $\tau_j$ appear in $\pi|_K$ (cf. Proposition 2.9). Then, the injection of $\tau_j$ into $\pi|_K$ is unique up to a scalar. Corollary 2.16 says that $a_{pq,rs}^{(j)}$ (resp. $a_{pq,00}^{(j)}$) ($j = 0, \pm 1, \pm 2$) are in $L^2_\sigma(K)$ if $n \geq 0$ (resp. $n < 0$).

Define

$$\begin{aligned} g^{(1)} &= g_{r+1,s-1}^{(1)} = X_{13} \cdot a_{rs}^{(0)} + X_{14} \cdot a_{r,s-1}^{(0)}, \\ g^{(2)} &= g_{r-1,s+1}^{(2)} = X_{24} \cdot a_{rs}^{(0)} - X_{14} \cdot a_{r-1,s}^{(0)}, \\ g^{(-1)} &= g_{r+1,s-1}^{(-1)} = X_{42} \cdot a_{rs}^{(0)} - X_{32} \cdot a_{r,s-1}^{(0)}, \\ g^{(-2)} &= g_{r-1,s+1}^{(-2)} = X_{31} \cdot a_{rs}^{(0)} + X_{32} \cdot a_{r-1,s}^{(0)}. \end{aligned}$$

**Proposition 2.21.** *The $K$-module generated by the vector $g^{(j)}$ is isomorphic to $\tau_j$ ($j = \pm 1, \pm 2$).*

PROOF. Each vector $g^{(j)}$ has weight $d_j$ and is annihilated by $e_+^1$ and $e_+^2$. □

Define the other weight vectors as follows:

$$\begin{aligned}
g_{kl}^{(1)} &= \tfrac{r-k+1}{r+1}(X_{23}\cdot a_{k,l+1}^{(0)} + X_{24}\cdot a_{kl}^{(0)}) + \tfrac{k}{r+1}(X_{13}\cdot a_{k-1,l+1}^{(0)} + X_{14}\cdot a_{k-1,l}^{(0)}),\\
g_{kl}^{(2)} &= \tfrac{s-l+1}{s+1}(X_{13}\cdot a_{kl}^{(0)} - X_{23}\cdot a_{k+1,l}^{(0)}) + \tfrac{l}{s+1}(X_{24}\cdot a_{k+1,l-1}^{(0)} - X_{14}\cdot a_{k,l-1}^{(0)}),\\
g_{kl}^{(-1)} &= \tfrac{r-k+1}{r+1}(X_{31}\cdot a_{kl}^{(0)} - X_{41}\cdot a_{k,l+1}^{(0)}) + \tfrac{k}{r+1}(X_{42}\cdot a_{k-1,l+1}^{(0)} - X_{32}\cdot a_{k-1,l}^{(0)}),\\
g_{kl}^{(-2)} &= \tfrac{s-l+1}{s+1}(X_{42}\cdot a_{kl}^{(0)} + X_{41}\cdot a_{k+1,l}^{(0)}) + \tfrac{l}{s+1}(X_{31}\cdot a_{k+1,l-1}^{(0)} + X_{32}\cdot a_{k,l-1}^{(0)}).
\end{aligned}$$

In other words, the $g_{kl}^{(j)}$'s are defined by the following recurrent relation:

$$g_{kl}^{(j)} = \tfrac{1}{k+1}e_-^1\cdot g_{k+1,l}^{(j)} = \tfrac{1}{l+1}e_-^2\cdot g_{k,l+1}^{(j)}. \tag{35}$$

These imply that an isomorphism between $\{g_{k,l}^{(j)}\}$ and $V_{\tau_j}$ is given by $g_{kl}^{(j)} \mapsto f_{kl}^{(j)}$.

Our main result in this subsection is:

**Proposition 2.22.** *Suppose that* $n \geq 0$ (*resp.* $n < 0$). *Assume* $u \equiv 2s + n + 1 - \epsilon(-1)$ $(\mathrm{mod}\ 4)$. *Write* $a_{kl}^{(j)} := a_{kl,rs}^{(d_j)}$ (*resp.* $a_{kl}^{(j)} := a_{kl,00}^{(d_j)}$). *Then,*

$$g^{(\pm1)} = \frac{2\mu_{\delta(\pm1)} + 2 + r - s \pm u}{4} a_{r+1,s-1}^{(\pm1)}, \tag{36}$$

$$g^{(\pm2)} = \frac{2\mu_{\delta(\mp1)} + 2 - r + s \pm u}{4} a_{r-1,s+1}^{(\pm2)}. \tag{37}$$

*where* $(\delta(1),\delta(-1)) = (1,2)$ (*resp.* $(\delta(1),\delta(-1)) = (2,1)$).

PROOF. It suffices to check the value of each $g^{(j)}$ (resp. $g_{00}^{(j)}$) at $1_4$ for $j = \pm1, \pm2$ in the case $n \geq 0$ (resp. $n < 0$). We assume that $n \geq 0$ for clarity. Using §1.3, we have,

$$\begin{aligned}
g_{r+1,s-1}^{(1)}(1_4) &= X_{13}\cdot a_{rs}^{(0)}(1_4) + X_{14}\cdot a_{r,s-1}^{(0)}(1_4)\\
&= (\tfrac{1}{2}H_1 + \tfrac{1}{4}(I_{2,2} + h^1 - h^2))\cdot a_{rs}^{(0)}(1_4) + (-e_+^2)\cdot a_{r,s-1}^{(0)}(1_4)\\
&= (\tfrac{1}{2}(\mu_1 + 3) + \tfrac{1}{4}(u + r - s))a_{rs}^{(0)}(1_4) - a_{rs}^{(0)}(1_4)\\
&= (2\mu_1 + 2 + r - s + u)/4.
\end{aligned}$$

Here we also use the fact that $a_{rs}^{(0)}, a_{r,s-1}^{(0)}$ are in the image of the $K$-type $\tau_{[r,s;u]}$ and that $a_{rs}(1_4) = 1$ by (30).

Accordingly, we can check the values: $g_{r-1,s+1}^{(2)}(1_4)$, $g_{r+1,s-1}^{(-1)}(1_4)$ and $g_{r-1,s+1}^{(-2)}(1_4)$, which give the proposition. □

We now discuss the case when $\tau_0$ is 2-dimensional; thus consider the special case, $d_0 = [0,1;u]$ and $d_{\pm1} = [1,0;u\pm2]$.

**Corollary 2.23.** *Suppose that* $n = 1$ (*resp.* $n = -1$). *Assume* $u \equiv -n\epsilon(-1)$ $(\mathrm{mod}\ 4)$. *Write* $a_{pq}^{(j)} := a_{pq,rs}^{(d_j)}$ (*resp.* $a_{pq,00}^{(d_j)}$). *Then, we have,*

$$\begin{pmatrix} X_{24} & X_{23} \\ X_{14} & X_{13} \end{pmatrix} \begin{pmatrix} a_{00}^{(0)} \\ a_{01}^{(0)} \end{pmatrix} = \frac{(2\mu_{\delta(1)} + u + 1)}{4} \begin{pmatrix} a_{00}^{(1)} \\ a_{10}^{(1)} \end{pmatrix}, \tag{38}$$

$$\begin{pmatrix} X_{13} & -X_{23} \\ -X_{14} & X_{24} \end{pmatrix} \begin{pmatrix} a_{00}^{(-1)} \\ a_{10}^{(-1)} \end{pmatrix} = \frac{(2\mu_{\delta(2)} + u - 1)}{4} \begin{pmatrix} a_{00}^{(0)} \\ a_{01}^{(0)} \end{pmatrix}, \tag{39}$$

$$\begin{pmatrix} X_{42} & X_{41} \\ X_{32} & X_{31} \end{pmatrix} \begin{pmatrix} a_{00}^{(1)} \\ a_{10}^{(1)} \end{pmatrix} = \frac{(2\mu_{\delta(1)} - u - 1)}{4} \begin{pmatrix} a_{00}^{(0)} \\ a_{01}^{(0)} \end{pmatrix}, \tag{40}$$

$$\begin{pmatrix} X_{31} & -X_{41} \\ -X_{32} & X_{42} \end{pmatrix} \begin{pmatrix} a_{00}^{(0)} \\ a_{01}^{(0)} \end{pmatrix} = \frac{(2\mu_{\delta(2)} - u + 1)}{4} \begin{pmatrix} a_{00}^{(-1)} \\ a_{10}^{(-1)} \end{pmatrix}. \tag{41}$$

*where* $(\delta(1), \delta(2)) = (1, 2)$ (*resp.* $(\delta(1), \delta(2)) = (2, 1)$).

## 3. Whittaker functions, shift operators and the Casimir operator

### 3.1. Whittaker functions.

First of all, we define Whittaker functions. throughout this subsection, let $G$ be a real semisimple Lie group, $K$ its maximal compact subgroup and $P$ a parabolic subgroup of $G$. Let $(\pi, H_\pi)$ be an irreducible admissible representation. For the maximal unipotent subgroup $N$ of $G$ and its unitary character $\eta$, let $C_\eta^\infty(N\backslash G)$ be a space of the $C^\infty$-functions satisfying the relation:

$$\phi(ng) = \eta(n)\phi(g) \qquad (n \in N, g \in G).$$

Let $H_\pi^K$ be the $(\mathfrak{g}, K)$-module that consists of the $K$-finite vectors in $H_\pi$. We call the elements of $\mathrm{Hom}_{(\mathfrak{g},K)}(H_\pi^K, C_\eta^\infty(N\backslash G))$ the algebraic Whittaker vectors of $\pi$. We also call the image of a $K$-finite vector a Whittaker function of $\pi$.

Let $(\tau, V_\tau)$ be an irreducible representation of $K$ and $(\tau^*, V_\tau^*)$ be its contragredient representation. Assume that $[\pi|_K : \tau^*] = 1$. Fix a $K$-injection $\iota_{\tau^*} : V_\tau^* \to H_\pi$. By composition, there is a canonical $K$-homomorphism

$$\tilde{\iota}_{\tau^*} : \mathrm{Hom}_{(\mathfrak{g},K)}(H_\pi^K, C_\eta^\infty(N\backslash G)) \to \mathrm{Hom}_K(V_\tau^*, C_\eta^\infty(N\backslash G)),$$

defined by $\tilde{\iota}_{\tau^*}(\Phi_\pi) = \Phi_\pi \circ \iota_{\tau^*}$ for a Whittaker vector $\Phi_\pi$. This is injective by virtue of the irreducibility of $\pi$. Let $C_{\eta,\tau}^\infty(N\backslash G/K)$ be the space of $V_\tau$-valued $C^\infty$-functions satisfying the relation

$$\phi(ngk) = \eta(n)\tau^{-1}(k)\phi(g) \qquad (n \in N, g \in G \text{ and } k \in K).$$

The space $\mathrm{Hom}_K(V_\tau^*, C_\eta^\infty(N\backslash G))$ is identified with $C_{\eta,\tau}^\infty(N\backslash G/K)$ through the identification map $\phi \mapsto \phi_\tau$ by the rule:

$$\phi(v^*)(g) = \langle v^*, \phi_\tau(g) \rangle \qquad (v^* \in V^*, g \in G) \tag{42}$$

where $\langle\, , \rangle$ is the pairing of $V_\tau^*$ and $V_\tau$. In particular, we write $\Phi_{\pi,\tau} = (\tilde{\iota}_{\tau^*}(\Phi_\pi))_\tau \in C_{\eta,\tau}^\infty(N\backslash G/K)$. We also call $\Phi_{\pi,\tau}$ a Whittaker function or, more exactly, a Whittaker function of $\pi$ with $K$-type $\tau^*$.

Now any element in $C_{\eta,\tau}^\infty(N\backslash G/K)$ is uniquely determined by its restriction to a maximal $\mathbb{R}$-split torus $A$. Therefore, for $f \in C_{\eta,\tau}^\infty(N\backslash G/K)$, we denote by $\mathrm{Rad}(f) = \mathrm{Rad}_\tau(f) = f|_A \in C^\infty(A)$ the radial part of $f$ and, given an operator $D$ on $C_{\eta,\tau}^\infty(N\backslash G/K)$, we define the radial part of $D$, $\mathrm{Rad}(D) = \mathrm{Rad}_\tau(D)$ by

$$\mathrm{Rad}(D)\,\mathrm{Rad}(f) = \mathrm{Rad}(D(f)).$$

Actually the differential equations for a Whittaker function shall mean those for its radial part.

**3.2. Dimension of the space of the Whittaker vectors.** The dimension of the space of Whittaker vectors can be found as follows: Let

$$F^{\#} = \exp(\mathfrak{a}_{\mathbb{C}}) = <\gamma,\ \alpha = \sqrt{-1}\left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline 1 & & & \\ & 1 & & \end{array}\right)>, \tag{43}$$

namely, $F^{\#}$ is the group of order 8 generated by $\gamma$ and $\alpha$. It is known that $F^{\#}G = \{g \in G_{\mathbb{C}} \mid (\operatorname{Ad} g)\mathfrak{g} = \mathfrak{g}\}$ where $G_{\mathbb{C}}$ is a complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Let $(\pi, H_\pi)$ be an admissible $G$-module. For $a \in F^{\#}$, define another $G$-module $(\pi^{(a)}, H_\pi^{(a)})$ by,

$$\pi^{(a)}(g)v = \pi(a^{-1}ga)v, \quad H_\pi^{(a)} = H_\pi \qquad (g \in G,\ v \in H_\pi).$$

If $\pi$ has a $(\mathfrak{g}, K)$-module structure, so does $\pi^{(a)}$. Choose $\{a_1, \dots, a_p\}$ so that $\{\pi^{(a_i)}\}$ is a complete system of mutually infinitesimally non-isomorphic classes of $\{\pi^{(a)} \mid a \in F^{\#}\}$.

In the case of $G = SU(2,2)$, if $\pi_J = \operatorname{ind}_{P_J}^G((\chi_m, D_k^{\pm}) \otimes e^{\nu+\rho_J} \otimes 1)$ in principal $P_J$-series, then we find that $p = 2$, i.e., $\{a_1, a_2\} = \{1, \alpha\}$ with $\pi_J^{(\alpha)} \simeq \operatorname{ind}_{P_J}^G((\chi_m, D_k^{\mp}) \otimes e^{\nu+\rho_J} \otimes 1)$. We have the following:

**Theorem 3.1.** *Assume that $\pi_J$ is irreducible. Then,*

$$\dim \operatorname{Hom}_{(\mathfrak{g},K)}(H_{\pi_J}^K, C_\eta^\infty(N_m \backslash G)) = 4.$$

PROOF. If $\pi_J$ is irreducible, then it is large in the sense of [26, Th. 6.2, f)]. Thus we have, from [13, Th. 6.8.1],

$$\dim \operatorname{Hom}_{(\mathfrak{g},K)}(\pi_J, C_\eta^\infty(N_m \backslash G)) + \dim \operatorname{Hom}_{(\mathfrak{g},K)}(\pi_J^{(\alpha)}, C_\eta^\infty(N_m \backslash G)) = 8.$$

On the other hand, the Whittaker models with $\eta$ of $\pi^{(\alpha)}$ is isomorphic to those with $\eta^{(\alpha)}$ of $\pi$. But the dimension of the space of algebraic Whittaker vectors is determined independently of the choice of $\eta$, whence the dimension is 4. □

**Remark 3.2.** One has $\eta^{(\alpha)}(E_2) = -\eta(E_2)$, which will also explain the relation (93).

Besides, if $\pi_m = \operatorname{ind}_{P_m}^G(\sigma_{n,\epsilon} \otimes e^{\mu+\rho} \otimes 1)$, then, we have $\pi_m \simeq \pi_m^{(\alpha)}$, therefore,

**Theorem 3.3** ([13, Theorem 5.5]). *Let $\pi = \operatorname{ind}_{P_m}^G(\sigma_{n,\epsilon} \otimes e^{\mu+\rho} \otimes 1)$. Assume that the unitary character $\eta$ of $N$ is nondegenerate. Then,*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{(\mathfrak{g},K)}(H_\pi^K, C_\eta^\infty(N_m \backslash G)) = 8.$$

In the case of discrete series $\pi_\Lambda$, the same method can be applied.

**Theorem 3.4.**

$$\dim_{\mathbb{C}}(\operatorname{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, C_\eta^\infty(N_m \backslash G))) = \begin{cases} 4 & (J = \mathrm{II}, \mathrm{V}), \\ 0 & \textit{otherwise}. \end{cases}$$

PROOF. We see that

$$\tau_{[r,s;u]}(\alpha^{-1}X\alpha) = \tau_{[s;r;-u]}(X) = \tau_{w_{\mathrm{V}}w_{\mathrm{II}}^{-1}[r,s;u]}(X)$$

and that the Harish-Chandra parameter of $\pi_\Lambda^{(\alpha)}$ belongs to $\Xi_{\mathrm{V}}$. Using the same argument as above and about the largeness (20), we obtain the theorem. □

We remark that the rapid decreasing solution is unique if exists.

### 3.3. The Schmid operator.

Let $(\tau, V_\tau)$ be a finite-dimensional irreducible representation of $K$, $\eta$ a unitary character of $N$. and $\{X_j\}_{j=1}^8$ an orthonormal basis of $\mathfrak{p}$. We define the Schmid operator $\nabla$ by

$$(44)\ \nabla = \nabla_{\eta,\tau}\colon C^\infty_{\eta,\tau}(N\backslash G/K) \ni F \longmapsto \sum_{j=1}^{8} X_j{\cdot}F(\cdot)\otimes X_j \in C^\infty_{\eta,\tau\otimes\mathrm{Ad}}(N\backslash G/K).$$

It is independent of the choice of an orthonormal basis of $\mathfrak{p}$. Besides, $\nabla$ is $K$-equivariant. Let us choose as the orthogonal basis of $\mathfrak{p}$,

$$\{X_{ij}+X_{ji}, \sqrt{-1}(X_{ij}-X_{ji})\}_{i=1,2,\,j=3,4}$$

We define the $\pm$-part of the Schmid operator,

$$\nabla^{\pm}\colon C^\infty_{\eta,\tau}(N\backslash G/K) \longrightarrow C^\infty_{\eta,\tau\otimes\mathrm{Ad}_\pm}(N\backslash G/K)$$

by

$$(45)\quad \begin{aligned} \nabla^{+}F &= \sum_{i=1,2,\ j=3,4} X_{ji}{\cdot}F(\cdot)\otimes X_{ij} \\ \nabla^{-}F &= \sum_{i=1,2,\ j=3,4} X_{ij}{\cdot}F(\cdot)\otimes X_{ji}. \end{aligned}$$

for $F \in C^\infty_{\eta,\tau}(N\backslash G/K)$. Then we can check that (const.)$\nabla = \nabla^+ + \nabla^-$.

Let $\tau'$ be an irreducible component of $\tau\otimes\mathrm{Ad}$ and $P_{\tau'}$ be its projection to $\tau'$. Then, $P_{\tau'}\circ\nabla$ or their compositions are called shift operators. Let $P^*_{\tau'}$ be the canonical $K$-injection defined by

$$< P^*_{\tau'}(w^*),\ v\otimes X > = < w^*,\ P_{\tau'}(v\otimes X) >$$

for $w^* \in V^*_{\tau'}$, $v\in V_\tau$, $X\in\mathfrak{p}_{\mathbb{C}}$. Note that Ad is self-dual. Considering

$$\mathrm{mul}: V_{\tau^*}\otimes\mathfrak{p}_{\mathbb{C}} \ni v^*\otimes X \mapsto \pi(X)\iota_{\tau^*}(v^*) \in H_\pi$$

and a composition $\mathrm{mul}\circ P^*_{\tau'}$, there is an constant $c = c(\tau, \iota_{\tau^*}; \tau', \iota_{(\tau')^*})$ such that

$$\mathrm{mul}\circ P^*_{\tau'} = c\cdot\iota_{(\tau')^*}$$

by virtue of irreducibility of $\tau'$. Here we use the convention that if $\iota_{(\tau')^*}$ is meaningless (*i.e.* $(\tau')^*$ is not a $K$-type of $\pi$), then the constant $c$ is equal to 0.

From this equation, we have, by using $(\mathfrak{g}, K)$-homomorphism $\Phi_\pi$,

$$(46)\qquad \sum_k \Phi_\pi(\mathrm{mul}\circ P^*_{\tau'}((v'_k)^*))v'_k = c\sum_k \Phi_\pi(\iota_{(\tau')^*}(v'_k))v'_k = c\,\Phi_{\pi,\tau'},$$

where $\{v'_k\}$ is a basis of $V_{\tau'}$ and $\{(v'_k)^*\}$, its dual basis. The left-hand side of (46) turns out to be equal to $P_{\tau'} \circ \nabla\Phi_{\pi,\tau}$. This indicates that the shift operators have Whittaker functions as "eigenfunctions".

**Proposition 3.5.** *Let $(\pi, H_\pi)$ be an irreducible admissible representation of a real semisimple Lie group $G$ and $(\tau, V_\tau)$ be an irreducible representation of a maximal compact subgroup $K$ such that $[\pi|_K : \tau^*] \neq 0$. Let $(\tau', V_{\tau'})$ be an irreducible component of $\tau \otimes \mathrm{Ad}$ and $K_{\tau'}$ be its projector to $V_{\tau'}$. Fix a $K$-injection $\iota_{\tau^*}$, (resp. $\iota_{(\tau')^*}$) of $V_{\tau^*}$ (resp. $V_{(\tau')^*}$) to $H_\pi$. Then there exists a constant $c = c(\tau, \iota_{\tau^*}; \tau', \iota_{(\tau')^*})$ such that*

$$P_{\tau'} \circ \nabla\Phi_{\pi,\tau} = c \cdot \Phi_{\pi,\tau'}.$$

*Here if $\iota_{(\tau')^*}$ does not exist, $c$ is understood to be zero.*

### 3.4. Radial part of the Schmid operator.

In what follows, we use for notation:

$$\begin{aligned}
&\partial_j\phi = (H_j \cdot \phi)|_{A_m} = a_j \frac{\partial \phi}{\partial a_j} \quad (j = 1, 2),\\
&\mathcal{L}_1\,\phi = \tfrac{1}{2}\partial_j\phi,\\
&\mathcal{L}_2^{\pm}\phi = \tfrac{1}{2}(\partial_2 \pm \sqrt{-1}e^{2\lambda_2}\eta(E_2))\phi,\\
&\mathcal{S} \ = \tfrac{1}{2}e^{\lambda_1-\lambda_2}(\eta(E_5) + \sqrt{-1}\eta(E_6)) = \frac{1}{2}\xi\left(\frac{a_1}{a_2}\right),\\
&\mathcal{S}' \ = \tfrac{1}{2}e^{\lambda_1-\lambda_2}(\eta(E_5) - \sqrt{-1}\eta(E_6)) = \frac{1}{2}\xi'\left(\frac{a_1}{a_2}\right),
\end{aligned} \tag{47}$$

where we put $\phi = F|_{A_m}$ for a given $F \in C^\infty_{\eta,\tau}(N\backslash G/K)$.

Since the character $\eta$ satisfies $\eta|_{[\mathfrak{n},\mathfrak{n}]} = 0$, $\eta$ is uniquely determined by the value of $E_\alpha$ $(\alpha \in \Delta_{\mathsf{fund}})$. We write for simplicity,

$$\begin{aligned}
&\eta_2 = \sqrt{-1}\eta(E_2),\ \eta_5 = \sqrt{-1}\eta(E_5),\ \eta_6 = \sqrt{-1}\eta(E_6),\\
&\qquad \xi \ = \eta(E_5) + \sqrt{-1}\eta(E_6) = -\sqrt{-1}\eta_5 + \eta_6,\\
&\qquad \xi' = \eta(E_5) - \sqrt{-1}\eta(E_6) = -\sqrt{-1}\eta_5 - \eta_6,\\
&\qquad \eta_0 = \xi\xi' = -(\eta_5^2 + \eta_6^2),
\end{aligned} \tag{48}$$

where $\eta_j$, $(j = 2, 5, 6)$ are real numbers since $\eta$ is unitary. We say $\eta$ is nondegenerate if the $\eta_j$'s $(j = 2, 5, 6)$ are all non-zero.

The radial parts of the $\pm$-part of the Schmid operator $\nabla^\pm$ are as follows.

**Proposition 3.6.** *Let $F \in C^\infty_{\eta,\tau}(N\backslash G/K)$ and let $\phi = F|_{A_m}$. Then,*

$$\begin{aligned}
\mathrm{Rad}(\nabla^+)\phi = {} & (\mathcal{L}_1 - \tfrac{\sqrt{-1}}{2}(\tau \otimes \mathrm{Ad}_+)(H_{13}) - 3)(\phi \otimes X_{13})\\
& + (\mathcal{L}_2^- - \tfrac{\sqrt{-1}}{2}(\tau \otimes \mathrm{Ad}_+)(H_{24}) - 1)(\phi \otimes X_{24})\\
& + (\mathcal{S}' + (\tau \otimes \mathrm{Ad}_+)(e^1_+))(\phi \otimes X_{23})\\
& + (\mathcal{S} - (\tau \otimes \mathrm{Ad}_+)(e^2_-))(\phi \otimes X_{14}),
\end{aligned}$$

$$\begin{aligned}\operatorname{Rad}(\nabla^-)\phi = {} & (\mathcal{L}_1 + \tfrac{\sqrt{-1}}{2}(\tau\otimes \operatorname{Ad}_-)(H_{13}) - 3)(\phi\otimes X_{31}) \\ & + (\mathcal{L}_2^+ + \tfrac{\sqrt{-1}}{2}(\tau\otimes \operatorname{Ad}_-)(H_{24}) - 1)(\phi\otimes X_{42}) \\ & + (\mathcal{S} - (\tau\otimes \operatorname{Ad}_-)(e_-^1))(\phi\otimes X_{32}) \\ & + (\mathcal{S}' + (\tau\otimes \operatorname{Ad}_-)(e_+^2))(\phi\otimes X_{41}).\end{aligned}$$

PROOF. By definition, we can calculate the actions of $E_j$'s given by

$$(E_j\cdot F)|_A(a) = \begin{cases} e^{\lambda_2(\log a)}\eta(E_2)\phi(a) & (j=2), \\ e^{(\lambda_1-\lambda_2)(\log a)}\eta(E_j)\phi(a) & (j=5,6), \\ 0 & (\text{otherwise}).\end{cases}$$

So, if we rewrite the Schmid operator (44) by using §1.3, we have,

$$\begin{aligned}\nabla^+F = {} & \tfrac12\Big\{(H_1+\sqrt{-1}H_{13})\cdot F\otimes X_{13} \\ & + (E_5 - \sqrt{-1}E_6 - 2e_+^1)\cdot F\otimes X_{23} \\ & + (E_5+\sqrt{-1}E_6 + 2e_-^2)\cdot F\otimes X_{14} \\ & + (-\sqrt{-1}E_2 + H_2 + \sqrt{-1}H_{24})\cdot F\otimes X_{24}\Big\} \\ = {} & (\mathcal{L}_1 + \tfrac{\sqrt{-1}}{2}H_{13})\cdot F\otimes X_{13} + (\mathcal{S}' - e_+^1)\cdot F\otimes X_{23} \\ & + (\mathcal{S} + e_-^2)\cdot F\otimes X_{14} + (\mathcal{L}_2^- + \tfrac{\sqrt{-1}}{2}H_{24})\cdot F\otimes X_{24}.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\nabla^-F = {} & (\mathcal{L}_1 - \tfrac{\sqrt{-1}}{2}H_{13})\cdot F\otimes X_{31} + (\mathcal{S}' - e_+^2)\cdot F\otimes X_{41} \\ & + (\mathcal{S} + e_-^1)\cdot F\otimes X_{32} + (\mathcal{L}_2^+ - \tfrac{\sqrt{-1}}{2}H_{24})\cdot F\otimes X_{42}.\end{aligned}$$

Noting that $X\cdot F = -\tau(X)F$ for $X\in\mathfrak{k}$, we get

$$\begin{aligned}(\tau\otimes \operatorname{Ad}_+)(H_{13})(F\otimes X_{13}) &= (-H_{13}+2\sqrt{-1})\,F\otimes X_{13}, \\ (\tau\otimes \operatorname{Ad}_+)(e_+^1)(F\otimes X_{23}) &= -e_+^1\cdot F\otimes X_{23} + F\otimes X_{13}, \\ (\tau\otimes \operatorname{Ad}_+)(e_-^2)(F\otimes X_{14}) &= -e_-^2\cdot F\otimes X_{14} - F\otimes X_{13}, \\ (\tau\otimes \operatorname{Ad}_+)(H_{24})(F\otimes X_{24}) &= (-H_{24}+2\sqrt{-1})\,F\otimes X_{24}, \\ (\tau\otimes \operatorname{Ad}_-)(H_{13})(F\otimes X_{31}) &= (-H_{13}-2\sqrt{-1})\,F\otimes X_{31}, \\ (\tau\otimes \operatorname{Ad}_-)(e_+^2)(F\otimes X_{41}) &= -e_+^2\cdot F\otimes X_{41} + F\otimes X_{31}, \\ (\tau\otimes \operatorname{Ad}_-)(e_-^1)(F\otimes X_{32}) &= -e_-^1\cdot F\otimes X_{32} - F\otimes X_{31}, \\ (\tau\otimes \operatorname{Ad}_-)(H_{24})(F\otimes X_{42}) &= (-H_{24}-2\sqrt{-1})\,F\otimes X_{42}.\end{aligned}$$

Our proposition follows using these relations. □

**3.5. Radial part of shift operators.** Let $P_\tau^{(\epsilon_1,\epsilon_2)}$ (resp. $\overline{P}_\tau^{(\epsilon_1,\epsilon_2)}$) be the projectors to $\tau_{[r\pm\epsilon_1,s\pm\epsilon_2;u+2]}$ (resp. $\tau_{[r\pm\epsilon_1,s\pm\epsilon_2;u-2]}$) (cf. (18)). We define the following shift operators:

$$\begin{aligned} &\mathcal{D}^{\mathrm{up}} = P^{(-,-)} \circ \nabla^+ \circ P^{(+,+)} \circ \nabla^+, \\ (49) \qquad &\mathcal{D}^{\mathrm{down}} = \overline{P}^{(-,-)} \circ \nabla^- \circ \overline{P}^{(+,+)} \circ \nabla^-, \\ &\mathcal{E}^{(\epsilon_1,\epsilon_2)} = P^{(\epsilon_1,\epsilon_2)} \circ \nabla^+, \quad \overline{\mathcal{E}}^{(\epsilon_1,\epsilon_2)} = \overline{P}^{(\epsilon_1,\epsilon_2)} \circ \nabla^-, \quad (\epsilon_j \in \{\pm\}). \end{aligned}$$

Put

$$(50) \qquad \begin{aligned} d_0 &= [r,s;u], & d_{\pm 1} &= [r+1,s-1;u\pm 2], \\ d_{\pm 2} &= [r-1,s+1;u\pm 2], & d_{\pm 3} &= [r-1,s-1;u\pm 2]. \end{aligned}$$

We use the convention $\tau_j = \tau_{d_j}$, $f_{pq}^{(j)} = f_{pq}^{(d_j)}$ for given $d_j$. For $\phi_0 \in C_{\eta,\tau_0}^\infty(N\backslash G/K)$, we write

$$(51) \qquad \phi_0(a) = \sum_{0\le k\le r,\ 0\le l\le s} c_{kl}^{(0)}(a) f_{kl}^{(0)}.$$

Put

$$(52) \qquad \begin{aligned} \phi_1 &= \mathcal{E}^{(+,-)}\phi_0, & \phi_{-1} &= \overline{\mathcal{E}}^{(+,-)}\phi_0, \\ \phi_2 &= \mathcal{E}^{(-,+)}\phi_0, & \phi_{-2} &= \overline{\mathcal{E}}^{(-,+)}\phi_0, \\ \phi_3 &= \mathcal{E}^{(-,-)}\phi_0, & \phi_{-3} &= \overline{\mathcal{E}}^{(-,-)}\phi_0. \end{aligned}$$

Put also,

$$(53) \qquad \phi_j(a) = \sum_{k,l} c_{kl}^{(j)}(a) f_{kl}^{(j)} \qquad (j = \pm 1, \pm 2, \pm 3).$$

Throughout this subsection we use the convention that undefined coefficients $c_{kl}^{(j)}(a)$ are zero. Then the coefficients $c_{kl}^{(j)} = c_{kl}^{(j)}(a)$ are described as follows:

**Lemma 3.7.** *Put* $\nu_1 = (-r+s+u)/4$, $\nu_2 = (r-s+u)/4$. *Then,*

$$(54) \qquad c_{kl}^{(1)} = {}^t\begin{pmatrix} (l-s)\mathcal{S} \\ -(l+1)(\mathcal{L}_1 - 3\nu_1 + u - (k-l)/2) \\ (l-s)(\mathcal{L}_2^- + \nu_2 - (k-l)/2) \\ -(l+1)\mathcal{S}' \end{pmatrix} \begin{pmatrix} c_{k-1,l}^{(0)} \\ c_{k-1,l+1}^{(0)} \\ c_{kl}^{(0)} \\ c_{k,l+1}^{(0)} \end{pmatrix},$$

$$(55) \qquad c_{kl}^{(2)} = {}^t\begin{pmatrix} (k-r)\mathcal{S} \\ (r-k)(\mathcal{L}_1 + \nu_1 - (k-l)/2 - 1) \\ (k+1)(\mathcal{L}_2^- + \nu_2 - (k-l)/2 - 1) \\ -(k+1)\mathcal{S}' \end{pmatrix} \begin{pmatrix} c_{k,l-1}^{(0)} \\ c_{kl}^{(0)} \\ c_{k+1,l-1}^{(0)} \\ c_{k+1,l}^{(0)} \end{pmatrix},$$

$$(56)\qquad c_{kl}^{(-1)} = {}^t\begin{pmatrix} (l-s)\mathcal{S} \\ (l+1)(\mathcal{L}_2^+ - \nu_2 + (k-l)/2 - 1) \\ (s-l)(\mathcal{L}_1 - \nu_1 + (k-l)/2 - 1) \\ -(l+1)\mathcal{S}' \end{pmatrix} \begin{pmatrix} c_{k-1,l}^{(0)} \\ c_{k-1,l+1}^{(0)} \\ c_{kl}^{(0)} \\ c_{k,l+1}^{(0)} \end{pmatrix},$$

$$(57)\qquad c_{kl}^{(-2)} = {}^t\begin{pmatrix} (k-r)\mathcal{S} \\ (k-r)(\mathcal{L}_2^+ - \nu_2 + (k-l)/2) \\ -(k+1)(\mathcal{L}_1 + 3\nu_1 - u + (k-l)/2) \\ -(k+1)\mathcal{S}' \end{pmatrix} \begin{pmatrix} c_{k,l-1}^{(0)} \\ c_{kl}^{(0)} \\ c_{k+1,l-1}^{(0)} \\ c_{k+1,l}^{(0)} \end{pmatrix},$$

$$(58)\qquad c_{kl}^{(3)} = {}^t\begin{pmatrix} (k-r)(s-l)\mathcal{S} \\ (k-r)(l+1)(\mathcal{L}_1 - (k-l+3)/2 + \nu_1 - s) \\ (k+1)(s-l)(\mathcal{L}_2^- - (k-l+1)/2 + \nu_2) \\ (k+1)(l+1)\mathcal{S}' \end{pmatrix} \begin{pmatrix} c_{k,l}^{(0)} \\ c_{k,l+1}^{(0)} \\ c_{k+1,l}^{(0)} \\ c_{k+1,l+1}^{(0)} \end{pmatrix},$$

$$(59)\qquad c_{kl}^{(-3)} = {}^t\begin{pmatrix} (k-r)(s-l)\mathcal{S} \\ (r-k)(l+1)(\mathcal{L}_2^+ + (k-l-1)/2 - \nu_2) \\ (k+1)(l-s)(\mathcal{L}_1 + (k-l-3)/2 - \nu_1 - r) \\ (k+1)(l+1)\mathcal{S}' \end{pmatrix} \begin{pmatrix} c_{k,l}^{(0)} \\ c_{k,l+1}^{(0)} \\ c_{k+1,l}^{(0)} \\ c_{k+1,l+1}^{(0)} \end{pmatrix}.$$

PROOF. We write $c_{kl}(a) := c_{kl}^{(0)}(a)$. Using $K$-equivariance of $P^{(\epsilon_1,\epsilon_2)}$, $\overline{P}^{(\epsilon_1,\epsilon_2)}$ and the fact that $[\mathfrak{n},\mathfrak{n}]$ acts trivially on $\phi_j$'s, Proposition 3.6 says that

$$(60)\qquad \begin{aligned} \phi_j(a) &= \mathcal{E}^{(\epsilon_1,\epsilon_2)}\phi_0(a) = \sum_{k,l} P^{(\epsilon_1,\epsilon_2)} \circ \nabla^+(c_{kl}(a)f_{kl}^{(0)}) \\ &= \sum_{k,l}\Big\{\Big(\mathcal{L}_1 - \tfrac{\sqrt{-1}}{2}\tau_j(H_{13}) - 3\Big)c_{kl}(a)P^{(\epsilon_1,\epsilon_2)}(f_{kl}^{(0)}\otimes X_{13}) \\ &\qquad + \Big(\mathcal{L}_2^- - \tfrac{\sqrt{-1}}{2}\tau_j(H_{24}) - 1\Big)c_{kl}(a)P^{(\epsilon_1,\epsilon_2)}(f_{kl}^{(0)}\otimes X_{24}) \\ &\qquad + (\mathcal{S}' + \tau_j(e_+^1))c_{kl}(a)P^{(\epsilon_1,\epsilon_2)}(f_{kl}^{(0)}\otimes X_{23}) \\ &\qquad + (\mathcal{S} - \tau_j(e_-^2))c_{kl}(a)P^{(\epsilon_1,\epsilon_2)}(f_{kl}^{(0)}\otimes X_{14})\Big\}, \end{aligned}$$

$$(61)\qquad \begin{aligned} \phi_{-j}(a) &= \overline{\mathcal{E}}^{(\epsilon_1,\epsilon_2)}\phi_0(a) = \sum_{k,l} \overline{P}^{(\epsilon_1,\epsilon_2)} \circ \nabla^-(c_{kl}(a)f_{kl}^{(0)}) \\ &= \sum_{k,l}\Big\{\Big(\mathcal{L}_1 + \tfrac{\sqrt{-1}}{2}\tau_{-j}(H_{13}) - 3\Big)c_{kl}(a)\overline{P}^{(\epsilon_1,\epsilon_2)}(f_{kl}^{(0)}\otimes X_{31}) \\ &\qquad + \Big(\mathcal{L}_2^- + \tfrac{\sqrt{-1}}{2}\tau_{-j}(H_{24}) - 1\Big)c_{kl}(a)\overline{P}^{(\epsilon_1,\epsilon_2)}(f_{kl}^{(0)}\otimes X_{42}) \\ &\qquad + (\mathcal{S} - \tau_{-j}(e_-^1))c_{kl}(a)\overline{P}^{(\epsilon_1,\epsilon_2)}(f_{kl}^{(0)}\otimes X_{32}) \\ &\qquad + (\mathcal{S}' + \tau_{-j}(e_+^2))c_{kl}(a)\overline{P}^{(\epsilon_1,\epsilon_2)}(f_{kl}^{(0)}\otimes X_{41})\Big\}, \end{aligned}$$

for $j = 1, 2, 3$. We transform these equations case by case, by using Lemma 2.6.

Case I: $c_{kl}^{(1)}$. We write $f_{kl} := f_{kl}^{(1)}$. By Equation (60),

$$\begin{aligned}
\phi_1 = \sum_{k,l} \Big\{ & -l\left(\mathcal{L}_1 + \tfrac{1}{4}\tau_1(I_{2,2} + h^1 - h^2) - 3\right) c_{kl}\, f_{k+1,l-1} \\
& + (l-s)\left(\mathcal{L}_2^- + \tfrac{1}{4}\tau_1(I_{2,2} - h^1 + h^2) - 1\right) c_{kl}\, f_{kl} \\
& \quad - l(\mathcal{S}' + \tau_1(e_+^1))c_{kl}\, f_{k,l-1} + (l-s)(\mathcal{S} - \tau_1(e_-^2))c_{kl}\, f_{k+1,l} \Big\} \\
= \sum_{k,l} \Big\{ & -l\left(\mathcal{L}_1 + (u+2+(2k-r+1)-(2l-s-1))/4 - 3\right) c_{kl}\, f_{k+1,l-1} \\
& + (l-s)\left(\mathcal{L}_2^- + (u+2-(2k-r-1)+(2l-s+1))/4 - 1\right) c_{kl}\, f_{kl} \\
& \quad - l\mathcal{S}' c_{kl}\, f_{k,l-1} - l(r+1-k)c_{kl}\, f_{k+1,l-1} \\
& \qquad + (l-s)\mathcal{S} c_{kl}\, f_{k+1,l} - l(l-s)c_{kl}\, f_{k+1,l-1} \Big\} \\
= \sum_{k,l} \Big\{ & -l\left(\mathcal{L}_1 + \nu_1 - (k-l)/2 + r - s - 1\right) c_{kl}\, f_{k+1,l-1} - l\mathcal{S}' c_{kl}\, f_{k,l-1} \\
& + (l-s)\left(\mathcal{L}_2^- + \nu_2 - (k-l)/2\right) c_{kl}\, f_{kl} + (l-s)\mathcal{S} c_{kl}\, f_{k+1,l} \Big\} \\
= \sum_{k,l} \Big\{ & -(l+1)\left(\mathcal{L}_1 - 3\nu_1 + u - (k-l)/2\right) c_{k-1,l+1} - (l+1)\mathcal{S}' c_{k,l+1} \\
& + (l-s)\left(\mathcal{L}_2^- + \nu_2 - (k-l)/2\right) c_{kl} + (l-s)\mathcal{S} c_{k-1,l} \Big\} f_{kl}.
\end{aligned}$$

This implies Equation (54).

Case II: $c_{kl}^{(2)}$. We write $f_{kl} := f_{kl}^{(2)}$ here. By Equation (60),

$$\begin{aligned}
\phi_2 = \sum_{k,l} \Big\{ & (r-k)\left(\mathcal{L}_1 + \tfrac{1}{4}\tau_2(I_{2,2} + h^1 - h^2) - 3\right) c_{kl}\, f_{kl} \\
& + k\left(\mathcal{L}_2^- + \tfrac{1}{4}\tau_2(I_{2,2} - h^1 + h^2) - 1\right) c_{kl}\, f_{k-1,l+1} \\
& \quad - k(\mathcal{S}' + \tau_2(e_+^1))c_{kl}\, f_{k-1,l} + (k-r)(\mathcal{S} - \tau_2(e_-^2))c_{kl}\, f_{k,l+1} \Big\} \\
= \sum_{k,l} \Big\{ & (r-k)\left(\mathcal{L}_1 + \nu_1 - (k-l)/2 - 1\right) c_{kl}\, f_{kl} - k\mathcal{S}' c_{kl}\, f_{k-1,l} \\
& + k\left(\mathcal{L}_2^- + \nu_2 - (k-l)/2\right) c_{kl}\, f_{k-1,l+1} + (k-r)\mathcal{S} c_{kl}\, f_{k,l+1} \Big\} \\
= \sum_{k,l} \Big\{ & (r-k)\left(\mathcal{L}_1 + \nu_1 - (k-l)/2 - 1\right) c_{kl} - (k+1)\mathcal{S}' c_{k+1,l} \\
& + (k+1)\left(\mathcal{L}_2^- + \nu_2 - (k-l)/2 - 1\right) c_{k+1,l-1} + (k-r)\mathcal{S} c_{k,l-1} \Big\} f_{kl}.
\end{aligned}$$

This shows Equation (55).

Case III: $c_{kl}^{(3)}$. We write $f_{kl} := f_{kl}^{(3)}$ here. By Equation (60),

$$\begin{aligned}
\phi_3 = \sum_{k,l}\Big\{&(k-r)l\left(\mathcal{L}_1 + \tfrac{1}{4}\tau_3(I_{2,2}+h^1-h^2)-3\right)c_{kl}\,f_{k,l-1}\\
&+k(s-l)\left(\mathcal{L}_2^- + \tfrac{1}{4}\tau_3(I_{2,2}-h^1+h^2)-1\right)c_{kl}\,f_{k-1,l}\\
&\quad+kl(\mathcal{S}'+\tau_3(e_+^1))c_{kl}\,f_{k-1,l-1}+(k-r)(s-l)(\mathcal{S}-\tau_3(e_-^2))c_{kl}\,f_{k,l}\Big\}\\
=\sum_{k,l}\Big\{&(k-r)(l+1)\left(\mathcal{L}_1-2+\nu_1-s+(l-k+1)/2\right)c_{k,l+1}\\
&+(k+1)(s-l)\left(\mathcal{L}_2^-+\nu_2+(l-k-1)/2\right)c_{k+1,l+1}\\
&\quad+(k+1)(l+1)\mathcal{S}'c_{k+1,l+1}+(k-r)(s-l)\mathcal{S}c_{k,l}\Big\}f_{kl}.
\end{aligned}$$

Case IV: $c_{kl}^{(-1)}$. In this case, we set $f_{kl} := f_{kl}^{(-1)}$. Then Equation (61) implies,

$$\begin{aligned}
\phi_{-1} = \sum_{k,l}\Big\{&(s-l)\left(\mathcal{L}_1 - \tfrac{1}{4}\tau_{-1}(I_{2,2}+h^1-h^2)-3\right)c_{kl}\,f_{kl}\\
&+l\left(\mathcal{L}_2^+ - \tfrac{1}{4}\tau_{-1}(I_{2,2}-h^1+h^2)-1\right)c_{kl}\,f_{k+1,l-1}\\
&\quad+(l-s)(\mathcal{S}-\tau_{-1}(e_-^1))c_{kl}\,f_{k+1,l}-l(\mathcal{S}'+\tau_{-1}(e_+^2))c_{kl}\,f_{k,l-1}\Big\}\\
=\sum_{k,l}\Big\{&(s-l)\left(\mathcal{L}_1-\nu_1+(k-l)/2-1\right)c_{kl}\,f_{kl}+(l-s)\mathcal{S}c_{kl}\,f_{k+1,l}\\
&+l\left(\mathcal{L}_2^+-\nu_2+(k-l)/2\right)c_{kl}\,f_{k+1,l-1}-l\mathcal{S}'c_{kl}\,f_{k,l-1}\Big\}\\
=\sum_{k,l}\Big\{&(s-l)\left(\mathcal{L}_1-\nu_1+(k-l)/2-1\right)c_{kl}+(l-s)\mathcal{S}c_{k-1,l}\\
&+(l+1)\left(\mathcal{L}_2^+-\nu_2+(k-l)/2-1\right)c_{k-1,l+1}-(l+1)\mathcal{S}'c_{k,l+1}\Big\}f_{kl}.
\end{aligned}$$

Therefore Equation (56) follows.

Case V: $c_{kl}^{(-2)}$. Set $f_{kl} := f_{kl}^{(-2)}$, then Equation (61) shows,

$$\begin{aligned}
\phi_{-2} = \sum_{k,l}\Big\{&(-k)\left(\mathcal{L}_1 - \tfrac{1}{4}\tau_{-2}(I_{2,2}+h^1-h^2)-3\right)c_{kl}\,f_{k-1,l+1}\\
&+(k-r)\left(\mathcal{L}_2^+ - \tfrac{1}{4}\tau_{-2}(I_{2,2}-h^1+h^2)-1\right)c_{kl}\,f_{kl}\\
&\quad+(k-r)(\mathcal{S}-\tau_{-2}(e_-^1))c_{kl}\,f_{k,l+1}-k(\mathcal{S}'+\tau_{-2}(e_+^2))c_{kl}\,f_{k-1,l}\Big\}\\
=\sum_{k,l}\Big\{&(-k)\left(\mathcal{L}_1-\nu_1+(k-l)/2-r+s-1\right)c_{kl}\,f_{k-1,l+1}-k\mathcal{S}'c_{kl}\,f_{k-1,l}\\
&+(k-r)\left(\mathcal{L}_2^+-\nu_2+(k-l)/2\right)c_{kl}\,f_{kl}+(k-r)\mathcal{S}c_{kl}\,f_{k,l+1}\Big\}\\
=\sum_{k,l}\Big\{&-(k+1)\left(\mathcal{L}_1+3\nu_1-u+(k-l)/2\right)c_{k+1,l-1}+(k-r)\mathcal{S}c_{k,l-1}\\
&+(k-r)\left(\mathcal{L}_2^+-\nu_2+(k-l)/2\right)c_{kl}-(k+1)\mathcal{S}'c_{k+1,l}\Big\}f_{kl}.
\end{aligned}$$

This implies Equation (57).

Case VI: $c_{kl}^{(-3)}$. Set $f_{kl} := f_{kl}^{(-3)}$, then Equation (61) shows,

$$\begin{aligned}\phi_{-3} = \sum_{k,l} \Big\{ & k(l-s)\left(\mathcal{L}_1 - \tfrac{1}{4}\tau_{-3}(I_{2,2}+h^1-h^2)-3\right) c_{kl}\, f_{k-1,l} \\ & + (r-k)l\left(\mathcal{L}_2^+ - \tfrac{1}{4}\tau_{-3}(I_{2,2}-h^1+h^2)-1\right) c_{kl}\, f_{k,l-1} \\ & + (k-r)(s-l)(\mathcal{S}-\tau_{-3}(e_-^1))c_{kl}\, f_{k,l} + kl(\mathcal{S}'+\tau_{-3}(e_+^2))c_{kl}\, f_{k-1,l-1}\Big\} \\ = \sum_{k,l}\Big\{ & (k+1)(l-s)\,(\mathcal{L}_1 - 2-\nu_1-r+(k-l+1)/2)\, c_{k+1,l} \\ & + (r-k)(l+1)\left(\mathcal{L}_2^+ -\nu_2+(k-l-1)/2\right) c_{k,l+1} \\ & + (k+1)(l+1)\mathcal{S}'c_{k+1,l} + (k-r)(s-l)\mathcal{S}c_{k,l}\Big\} f_{kl}.\end{aligned}$$

and the lemma is proved. □

**3.6. Radial part of the Casimir operator.** Let $Z(\mathfrak{g}_\mathbb{C})$ be the center of the universal enveloping algebra of the complexification of the Lie algebra $\mathfrak{g}$. We mainly have an interest in the Casimir operator, a generator of $Z(\mathfrak{g}_\mathbb{C})$; it will give one of the differential equations in Theorem 5.4, Theorems 6.1 and 6.2. Thus we calculate the radial part of the Casimir operator $\Omega$ and compute the infinitesimal character of the principal series representations.

By definition, the Casimir operator $\Omega$ is of the form $\sum_j X_j^* X_j$, with a basis $\{X_j\}$ of $\mathfrak{g}$ and its dual basis $\{X_j^*\}$. Precisely, it is given by,

$$\Omega = H_1^2 + H_2^2 + \frac{1}{2}I_0^2 - \frac{1}{2}\sum_{j=1,2}(E_j{}^tE_j + {}^tE_jE_j) + \sum_{j=3}^{6}(E_j{}^t\overline{E}_j + {}^t\overline{E}_jE_j). \tag{62}$$

First, we rewrite $\Omega$ in $Z(\mathfrak{g}_\mathbb{C})$ using the Poincaré-Birkoff-Witt basis. Paying attention to $[E_j, {}^t\overline{E_j}] = 4H_j$, $(j=1,2)$, $[E_j, {}^t\overline{E_j}] = H_1+H_2$, $(j=3,4)$ and $[E_j, {}^t\overline{E_j}] = H_1-H_2$, $(j=5,6)$, we get

$$\begin{aligned}\Omega &= H_1^2 + H_2^2 - 6H_1 - 2H_2 + \frac{1}{2}I_0^2 + \sum_{j=1,2} E_j{}^t\overline{E}_j + 2\sum_{j=3}^{6} E_j{}^t\overline{E}_j \\ &= H_1^2 + H_2^2 - 6H_1 - 2H_2 + \frac{1}{2}I_0^2 + \sum_{j=1,2} E_j^2 + 2\sum_{j=3}^{6} E_j^2 \\ &\quad - \sum_{j=1,2} E_j(E_j - {}^t\overline{E}_j) - 2\sum_{j=3}^{6} E_j(E_j - {}^t\overline{E}_j).\end{aligned} \tag{63}$$

Now we determine the radial part of the Casimir operator. We remember the following notation:

$$(64)\qquad \begin{aligned} \partial_j\phi &= (H_j\cdot\phi)|_{A_m} \quad (j=1,2),\\ \mathcal{S} &= \tfrac{1}{2}e^{\lambda_1-\lambda_2}(\eta(E_5)+\sqrt{-1}\eta(E_6)) = \tfrac{1}{2}\xi\, e^{\lambda_1-\lambda_2},\\ \mathcal{S}' &= \tfrac{1}{2}e^{\lambda_1-\lambda_2}(\eta(E_5)-\sqrt{-1}\eta(E_6)) = \tfrac{1}{2}\xi' e^{\lambda_1-\lambda_2}. \end{aligned}$$

Refer to Equation (47).

**Lemma 3.8.** *Let $\tau=\tau_{[r,s;u]}$ be an irreducible representation of $K$. Put*

$$(65)\qquad \begin{aligned} \phi\,(a) &= F|_{A_m}(a) = \sum_{0\le k\le r,\ 0\le l\le s} c_{kl}(a)f_{kl},\\ \phi^{(\Omega)}(a) &= \mathrm{Rad}_\tau(\Omega)\phi(a) = \sum_{0\le k\le r,\ 0\le l\le s} c^{(\Omega)}_{kl}(a)f_{kl}, \end{aligned}$$

*for $F\in C^\infty_{\eta,\tau}(N\backslash G/K)$. Then,*

$$(66)\qquad c^{(\Omega)}_{kl} = {}^t\begin{pmatrix} 4(r-k+1)\mathcal{S}\\ 4(s-l+1)\mathcal{S}\\ L_0+(u-2k+2l+r-s)\eta_2e^{2\lambda_2}+\alpha_{k,l}\\ -4(l+1)\mathcal{S}'\\ -4(k+1)\mathcal{S}' \end{pmatrix}\begin{pmatrix} c_{k-1,l}\\ c_{k,l-1}\\ c_{kl}\\ c_{k,l+1}\\ c_{k+1,l}\end{pmatrix}$$

*with,*

$$\begin{aligned} L_0 &= \partial_1^2+\partial_2^2-6\partial_1-2\partial_2-\eta_2^2a_2^4+8\mathcal{S}\mathcal{S}',\\ \alpha_{k,l} &= (2k-r+2l-s)^2/2. \end{aligned}$$

PROOF. Using the fact that $E_1\cdot F=E_3\cdot F=E_4\cdot F=0$ we see that

$$\left(H_1^2+H_2^2-6H_1-2H_2+\frac{1}{2}I_0^2+\sum_{j=1,2}E_j^2+2\sum_{j=3}^{6}E_j^2\right)F(a) = (L_0+\alpha_{k,l})\phi(a).$$

Furthermore, we know that

$$\begin{aligned} E_2+{}^tE_2 &= \sqrt{-1}(I_{2,2}-h^1+h^2),\\ E_5-{}^tE_5 &= e_+^1-e_-^1+e_+^2-e_-^2,\\ E_6+{}^tE_6 &= \sqrt{-1}(e_+^1+e_-^1+e_+^2+e_-^2). \end{aligned}$$

Therefore we get

$$\begin{aligned} &\Big(-\sum_{j=1,2}E_j(E_j-{}^t\overline{E}_j)-2\sum_{j=3}^{6}E_j(E_j-{}^t\overline{E}_j)\Big)F(a)\\ &=\sum_{k,l}\Big\{E_2\cdot c_{kl}(a)\sqrt{-1}\tau(I_{2,2}-h^1+h^2)f_{kl}+2E_5\cdot c_{kl}(a)\tau(e_+^1-e_-^1+e_+^2-e_-^2)f_{kl}\\ &\quad+2E_6\cdot c_{kl}(a)\tau(\sqrt{-1}(e_+^1+e_-^1+e_+^2+e_-^2))f_{kl}\Big\} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k,l}\Big\{\sqrt{-1}\eta(E_2)a_2^2c_{kl}(a)(u-2k+r+2l-s)f_{kl} \\
&\quad + 2\eta(E_5)(a_1/a_2)c_{kl}(a)((r-k)f_{k+1,l}-kf_{k-1,l}+(s-l)f_{k,l+1}-lf_{k,l-1}) \\
&\quad + 2\sqrt{-1}\eta(E_6)(a_1/a_2)c_{kl}(a)((r-k)f_{k+1,l}+kf_{k-1,l}+(s-l)f_{k,l+1}+lf_{k,l-1})\Big\} \\
&= \sum_{k,l}\Big\{(u-2k+r+2l-s)\eta_2a_2^2c_{kl}(a)+(a_1/a_2)\big(2(r-k+1)\xi c_{k-1,l}(a) \\
&\quad -2(k+1)\xi' c_{k+1,l}(a)+2(s-l+1)\xi c_{k,l-1}(a)-2(l+1)\xi' c_{k,l+1}(a)\big)\Big\}f_{kl}.
\end{aligned}
$$

This proves the lemma. □

**3.7. Infinitesimal character of principal series representations.** We again rewrite the Casimir operator $\Omega$ in the following form:

$$\Omega = H_1^2 + H_2^2 + 6H_1 + 2H_2 + \frac{1}{2}I_0^2 + \sum_{j=1,2} {}^t\overline{E}_jE_j + 2\sum_{j=3}^{6} {}^t\overline{E}_jE_j.$$

$$= H_1^2 + 6H_1 + I_0^2/2 + {}^t\overline{E}_1E_1 + 2\sum_{j=3}^{6} {}^t\overline{E}_jE_j + \Omega_D, \tag{67}$$

where,

$$
\begin{aligned}
\Omega_D &= H_2^2 + 2H_2 + {}^t\overline{E}_2E_2 \\
&= -H_{24}^2 - 2\sqrt{-1}H_{24} + 4X_{42}X_{24} - H_{24}^2 + 2\sqrt{-1}H_{24} + 4X_{24}X_{42}
\end{aligned}
$$

According to [10, Proposition 8.22], the infinitesimal character of a principal series representation is, so to speak, the sum of the infinitesimal character of the discrete series representation of the Levi part and the exponent of the split component. Namely, if $\pi = \mathrm{ind}_{P_m}^G(\sigma_{n,\epsilon} \otimes e^{\mu+\rho} \otimes 1)$ is a principal $P_m$-series, $\chi_\pi$ is

$$\sigma_n + \mu, \qquad \text{(here we set } \sigma_n(I_0) = n\text{)},$$

considered as an element of $(\mathfrak{m} + \mathfrak{a})_{\mathbb{C}}^*$ via the Harish-Chandra homomorphism and $\chi_{\pi_J}$ of a principal $P_J$-series $\pi_J = \mathrm{ind}_{P_J}^G((\chi_m, D_k^{\mp}) \otimes e^{\nu+\rho_J} \otimes 1)$ is

$$m + \mathrm{sgn}(D_k^{\pm})(k-1) + \nu$$

considered as the element of $(\mathfrak{t} + \mathfrak{k}_0 + \mathfrak{a}_J)_{\mathbb{C}}^*$. Concludingly, we have,

**Lemma 3.9.** *Let* $\pi = \mathrm{ind}_{P_m}^G(\sigma_{n,\epsilon} \otimes e^{\mu+\rho} \otimes 1)$ *and* $\pi_J = \mathrm{ind}_{P_J}^G((\chi_m, D_k^{\pm}) \otimes e^{\nu+\rho_J} \otimes 1)$. *Then,*

$$\chi_\pi(\Omega) = \mu_1^2 + \mu_2^2 + \frac{n^2}{2} - 10, \tag{68}$$

$$\chi_{\pi_J}(\Omega) = \nu^2 + (k-1)^2 - 10 + \frac{m^2}{2}. \tag{69}$$

PROOF. Tracing the Harish-Chandra homomorphism carefully, we have,

$$\begin{aligned}\chi_\pi(\Omega) &= \chi_\pi(H_1^2 + H_2^2 + 6H_1 + 2H_2 + \frac{1}{2}I_0^2)\\ &= (\sigma_n + \mu)((H_1 - 3)^2 + (H_2 - 1)^2 + 6(H_1 - 3) + 2(H_2 - 1) + \frac{1}{2}I_0^2)\\ &= \mu(H_1^2 + H_2^2) - 10 + \frac{n^2}{2}.\end{aligned}$$

On the other hand, the value $\chi_{\pi_J}(\Omega)$ is,

$$\begin{aligned}&\chi_{\pi_J}(\Omega)\\ &\quad = \chi_{\pi_J}((H_1 - 3)^2 + 6(H_1 - 3) + I_0^2/2 - (H_{24} \mp \sqrt{-1})^2 \mp 2\sqrt{-1}(H_{24} \mp \sqrt{-1}))\\ &\quad = (m + \operatorname{sgn}(D_k^\pm)(k - 1) + \nu)(H_1^2 - 9 - H_{24}^2 - 1 + I_0^2/2)\\ &\quad = \nu^2 + (k - 1)^2 - 10 + m^2/2.\end{aligned}$$

Thus we obtain the lemma. □

PART II

# Whittaker Functions of Admissible Representations

## 4. Discrete series Whittaker functions

**4.1. Result of Yamashita.** In the case of a large discrete series representation, the Whittaker model is explicitly determined by means of the kernel of the Schmid operator, also known as a differential operator of gradient type, which we will introduce in this subsection.

Let $\eta$ be a unitary character of $N = \exp \mathfrak{n}$ and $F(g) \in C^{\infty}_{\eta,\tau_d}(N\backslash G/K)$. Using the Schmid operator $\nabla_{\eta,\tau_d}$, and the following projectors

$$P_d^{(J)} : V_d \otimes \mathfrak{p}_{\mathbb{C}} \to V_d^- = \bigoplus_{\beta\in\tilde{\Delta}^+_{n,J}} V_{d-\beta},$$

we define $\mathcal{D}^{(J)}_{\eta,d} = P_d^{(J)} \circ \nabla_{\eta,\tau_d}$.

Let $\pi_\Lambda$ be the discrete series representation of $G$ where $\Lambda \in \Xi_J$ is a Harish-Chandra parameter. Then $d = [r, s; u] = \Lambda + \rho_{G,J} - 2\rho_K$, called the Blatter parameter of $\pi_\Lambda$, is the highest weight of the minimal $K$-type of $\pi_\Lambda$. For a Whittaker vector $\Phi \in \mathrm{Hom}_{\mathfrak{g},K}(\pi^*_\Lambda, C^\infty_\eta(N\backslash G))$, define $\Phi_{\Lambda,d} = \Phi_{\pi^*_\Lambda,\tau_d} \in C^{\infty}_{\eta,\tau_d}(N\backslash G/K)$ by the following:

$$< v^*, \Phi_{\Lambda,d}(g) >= \Phi(v^*)(\iota_{\tau^*}(g)) \qquad (v^* \in V_{\tau_d^*},\ g \in G).$$

The algebraic Whittaker function $\Phi_{\Lambda,d}$ for the representation of discrete series is uniquely determined at the value $a \in A_m = \exp(\mathfrak{a}_m)$ by virtue of the Iwasawa decomposition.

**Theorem 4.1** ([30]). *Let $\Lambda \in \Xi_J$. Then,*

$$\mathrm{Hom}_{(\mathfrak{g},K)}(\pi^*_\Lambda, C^\infty_\eta(N\backslash G)) \simeq \ker(\mathcal{D}^{(J)}_{\eta,\tau_d}),$$

*where $d$ is the Blattner parameter of $\pi_\Lambda$.*

**4.2. Radial $A_m$-part of the differential operator $\mathcal{D}_{\eta,d}$.** In the following $\eta$ is assumed to be nondegenerate. According to Theorem 3.4, we treat the case of $J = \mathrm{II}$ and $V$. Moreover, since $\eta^{(\alpha)}(E_2) = -\eta(E_2)$, the Whittaker functions of $\pi_\Lambda$ ($\Lambda \in \Xi_{\mathrm{V}}$) can be obtained from $\pi_{\Lambda'}$ ($\Lambda' = w_{\mathrm{II}} w_{\mathrm{V}}^{-1}\Lambda \in \Xi_{\mathrm{II}}$), by taking the parameters $(s, r, -u, -\eta_2)$ in the place of $(r, s, u, \eta_2)$. Thus we only treat the case $\Lambda \in \Xi_{\mathrm{II}}$. In this case, the Blattner parameter of $\pi_\Lambda$ is $d = \Lambda + [0, 0; 2]$ and the projector $P_d^{(J)}$ decomposes into four projectors as follows:

**Lemma 4.2.**

$$P_d^{(\mathrm{II})} = P^{(-,-)} \oplus \overline{P}^{(-,-)} \oplus \overline{P}^{(-,+)} \oplus \overline{P}^{(+,-)}. \tag{70}$$

Proof. We find that

$$\tilde{\Delta}^+_{\mathrm{II}} = \{[1, -1; 2],\ [1, 1; \pm 2],\ [-1, 1; 2],\ [0, 2; 0],\ [2, 0; 0]\}$$

and that

$$\tilde{\Delta}^+_{n,\mathrm{II}} = \{[1, -1; 2],\ [1, 1; \pm 2],\ [-1, 1; 2]\}.$$

Thus the lemma follows. □

According to Theorem 4.1, Whittaker functions are characterized by the differential equations derived by the composition of the Schmid operator and projectors which appears in the decomposition of $P_d^{(\mathrm{II})}$. Let $\Phi = \Phi_{\Lambda,d} = \sum_{kl} c_{k,l}(a) f_{k,l}$. For notation, we write $(\Phi)_{k,l} = c_{k,l}$. Then, $c_{k,l}$'s satisfy the following system which is equivalent to $\mathcal{D}_{\eta,d}^{(\mathrm{II})}\Phi = 0$:

$(\widetilde{C}_1^+ : 1)_{kl}$

$$(P^{(-,-)} \circ \nabla\Phi)_{k,l} = 0,$$

$(\widetilde{C}_2^- : 1)_{kl}$

$$(\overline{P}^{(-,-)} \circ \nabla\Phi)_{k,l} - (s-l)(\overline{P}^{(-,+)} \circ \nabla\Phi)_{k,l+1} = 0,$$

$(\widetilde{C}_2^- : 2)_{kl}$

$$(\overline{P}^{(-,-)} \circ \nabla\Phi)_{k,l} + (l+1)(\overline{P}^{(-,+)} \circ \nabla\Phi)_{k,l+1} = 0,$$

$(\widetilde{C}_3^- : 1)_{kl}$

$$(\overline{P}^{(-,-)} \circ \nabla\Phi)_{k,l} + (k+1)(\overline{P}^{(+,-)} \circ \nabla\Phi)_{k+1,l} = 0,$$

$(\widetilde{C}_3^- : 2)_{kl}$

$$(\overline{P}^{(-,-)} \circ \nabla\Phi)_{k,l} - (r-k)(\overline{P}^{(+,-)} \circ \nabla\Phi)_{k+1,l} = 0,$$

where we simply write $\nabla = \nabla_{\eta,d}$. We put

$$b_0 = b_0(d) = (r+s+u)/2, \qquad b_1 = b_1(d) = (-r+s+u)/2, \tag{71}$$
$$b_2 = b_2(d) = (r-s+u)/2, \qquad b_3 = b_3(d) = (-r-s+u)/2$$

for given $d = [r, s; u]$. Then, using Lemma 3.7 (we note that $\nu_j = b_j/2 \quad (j = 1, 2)$), the coefficients $\{c_{k,l}\}$ satisfy the following concrete equations:

$$(C_1^+ : 1)_{kl} \quad \begin{aligned} &(k-r)(l+1)(\mathcal{L}_1 + \tfrac{1}{2}(b_3 - s - k + l - 3))c_{k,l+1} \\ &\quad + (k+1)(s-l)(\mathcal{L}_2^- + \tfrac{1}{2}(b_2 - k + l - 1))c_{k+1,l} \\ &\quad\quad + (k+1)(l+1)\mathcal{S}'\, c_{k+1,l+1} + (k-r)(s-l)\mathcal{S}\, c_{k,l} = 0, \end{aligned}$$

$$(C_2^- : 1)_{kl} \quad \begin{aligned} &(r-k)(\mathcal{L}_2^+ - \tfrac{1}{2}(b_2 - k + l + 1))c_{k,l+1} \\ &\quad\quad + (k+1)\mathcal{S}'\, c_{k+1,l+1} + (k+1)(s-l)c_{k+1,l} = 0, \end{aligned}$$

$$(C_2^- : 2)_{kl} \quad (k+1)(\mathcal{L}_1 - \tfrac{1}{2}(b_0 + r - k - l + 1))c_{k+1,l} - (k-r)\mathcal{S}c_{k,l} = 0,$$

$$(C_3^- : 1)_{kl} \quad \begin{aligned} &(l+1)(\mathcal{L}_2^+ - \tfrac{1}{2}(b_2 - k + l + 1))c_{k,l+1} \\ &\quad\quad - (s-l)\mathcal{S}c_{k,l} + (k+1)(s-l)c_{k+1,l} = 0, \end{aligned}$$

$$(C_3^- : 2)_{kl} \quad (s-l)(\mathcal{L}_1 - \tfrac{1}{2}(b_1 + k + l + 3))c_{k+1,l} - (l+1)\mathcal{S}'c_{k+1,l+1} = 0.$$

Each equation $(C_p^\pm : q)$ was derived from $(\widetilde{C}_p^\pm : q)$.

Next, define

$$h_{k,l} := k!\, l!\, (r-k)!\, (s-l)!\, e^{a_2^2\eta_2/2} a_1^{-b_1-k+l-2} a_2^{-b_2+k-l}\, c_{k,l}. \tag{72}$$

Then the system satisfied by $h_{k,l}$'s is given as follows:

$$(H_1^+ : 1)_{kl} \quad \begin{aligned}&(\partial_1 + 2b_3 - 2)h_{k,l+1} - \xi'(a_1/a_2)^2 h_{k+1,l+1} + \xi(a_1/a_2)^2 h_{k,l}\\ &- (\partial_2 - 2\eta_2 a_2^2 + 2b_2 - 2k + 2l)(a_1/a_2)^2 h_{k+1,l} = 0\\ &\qquad (0 \le k \le r-1,\ 0 \le l \le s-1),\end{aligned}$$

$$(H_2^- : 1)_{kl} \quad \begin{aligned}&\partial_2 h_{k,l+1} + \xi'(a_1/a_2)^2 h_{k+1,l+1} + 2(l+1)(a_1/a_2)^2 h_{k+1,l} = 0\\ &\qquad (0 \le k \le r-1,\ 0 \le l \le s-1),\end{aligned}$$

$$(H_2^- : 1)_{k\,-1} \quad \partial_2 h_{k,0} + \xi'(a_1/a_2)^2 h_{k+1,0} = 0 \qquad (0 \le k \le r-1),$$

$$(H_2^- : 2)_{kl} \quad (\partial_1 - 2r + 2k + 2)h_{k+1,l} + \xi h_{k,l} = 0 \qquad (0 \le k \le r-1),$$

$$(H_3^- : 1)_{kl} \quad \begin{aligned}&\partial_2 h_{k,l+1} - \xi(a_1/a_2)^2 h_{k,l} + 2(r-k)(a_1/a_2)^2 h_{k+1,l} = 0\\ &\qquad (0 \le k \le r-1,\ 0 \le l \le s-1),\end{aligned}$$

$$(H_3^- : 1)_{rl} \quad \partial_2 h_{r,l+1} - \xi(a_1/a_2)^2 h_{r,l} = 0 \qquad (0 \le l \le s-1),$$

$$(H_3^- : 2)_{kl} \quad (\partial_1 - 2l)h_{k+1,l} - \xi' h_{k+1,l+1} = 0 \qquad (0 \le l \le s-1).$$

Each $(H_p^\pm : q)_{kl}$ was a direct consequence of corresponding $(C_p^\pm : q)_{kl}$. Concludingly, we obtain the differential equations satisfied by the Whittaker functions.

**Proposition 4.3.** *Let* $\Phi_{\Lambda,d} = \sum_{k,l} c_{k,l} f_{k,l}^{(d)}$ *and let* $h_{k,l}$ *be as in* (72). *Then* $h_{k,l}$ *satisfies the following*:

$$\begin{aligned}
&\text{(i)} \quad (\partial_1 + 2b_3 - 2)h_{k,l+1}\\
&\qquad - (2\partial_1 + \partial_2 - 2\eta_2 a_2^2 + 2b_3 - 2)(a_1/a_2)^2 h_{k+1,l} = 0\\
&\qquad\qquad (0 \le k \le r-1,\ 0 \le l \le s-1),\\
&\text{(ii)} \quad \partial_2 h_{k,l+1} + (a_1/a_2)^2(\partial_1 + 2)h_{k+1,l} = 0\\
&\qquad\qquad (0 \le k \le r-1,\ 0 \le l \le s-1),\\
&\text{(iii)} \quad (\partial_1 - 2r + 2k + 2)h_{k+1,l} + \xi h_{k,l} = 0 \qquad (0 \le k \le r-1),\\
&\text{(iv)} \quad (\partial_1 - 2l)h_{k+1,l} - \xi' h_{k+1,l+1} = 0 \qquad (0 \le l \le s-1),\\
&\text{(v)} \quad \partial_2 h_{k,0} + \xi'(a_1/a_2)^2 h_{k+1,0} = 0 \qquad (0 \le k \le r-1),\\
&\text{(vi)} \quad \partial_2 h_{r,l+1} - \xi(a_1/a_2)^2 h_{r,l} = 0 \qquad (0 \le l \le s-1).
\end{aligned}$$

PROOF. Equation (i) can be obtained from Equations $(H_1^+ : 1)_{kl}$, $(H_2^+ : 1)_{kl}$ and $(H_3^- : 1)_{kl}$. Equation (ii) is from Equations $(H_2^- : 2)_{kl}$ and $(H_3^- : 1)_{kl}$. Equations from (iii) to (vi) had no change. □

Equations (iii) and (iv) in Proposition 4.3 show that $h_{r,0}$ determines the system $\{h_{k,l}\}$. From the proposition above, we see that from (iii) and (V),

$$(\partial_1\partial_2 - \eta_0(a_1/a_2)^2)h_{r,0} = 0, \tag{73}$$

and from (i), (iii) and (iv),

$$(\partial_1 + 2b_3 - 2)\partial_1^2 h_{r,0} + (2\partial_1 + \partial_2 - 2\eta_2 a_2^2 + 2b_3 - 2)(a_1/a_2)^2 \eta_0 h_{r,0} = 0. \tag{74}$$

Operating $\partial_2$ to (74), we obtain the following with (73):

**Corollary 4.4.**

$$(\partial_1\partial_2 - \eta_0(a_1/a_2)^2)h_{r,0} = 0, \tag{75}$$

$$\left((\partial_1 + \partial_2)^2 + (2b_3 - 2)(\partial_1 + \partial_2) - 2\eta_2 a_2^2 \partial_2\right) h_{r,0} = 0. \tag{76}$$

Equation (74) can be recovered from (76) by operating $\partial_1$. Also we can check that this system becomes holonomic (See Appendix).

**4.3. Integral expression of discrete series Whittaker functions.** To get an integral representation, first we consider,

$$\mathcal{W}(a_1, a_2) = \int_0^\infty \phi(u) \exp\left(\frac{\eta_0 a_1^2}{u} - \frac{u}{4a_2^2}\right) \frac{du}{u} \tag{77}$$

for $\phi \in C^\infty(\mathbb{R}_{>0})$. Then $\mathcal{W}$ formally satisfies the differential equation (75). Suppose $\mathcal{W}$ satisfy (76), then $\phi$ should be the solution of the following differential equation,

$$4u^2 \frac{d^2\phi}{du^2} + 4b_3 \frac{d\phi}{du} - \eta_2 u \phi = 0. \tag{78}$$

Putting $v = \sqrt{u}$ and $\phi(u) = v^{-b_3+1/2}\psi(v)$, the equation (78) becomes,

$$\frac{d^2\psi}{dv^2} + \left(-\frac{1}{4}\cdot(4\eta_2) + \frac{1/4 - (b_3 - 1)^2}{v^2}\right)\psi = 0.$$

If $\eta_2 > 0$ (i.e. $\mathrm{Im}(\eta(E_2)) < 0$), then we can find the unique rapidly-decreasing solution,

$$\psi(v) = W_{0,b_3-1}(2\sqrt{\eta_2}v)$$

where $W_{k,l}(z)$ is the (classical) Whittaker function. Returning to the equation (77), we can confirm the absolute convergence of the integral in this case. Concluding,

**Theorem 4.5 (Oda).** *Let $\Lambda = [r, s; u - 2] \in \Xi_{\mathrm{II}}$. Assume that*

$$-\eta_2 = \mathrm{Im}(\eta(E_2)) < 0$$

*for a nondegenerate character $\eta$ on $N$. Then there exists a rapidly decreasing Whittaker function $\Phi_{\Lambda,d}$ characterized by the following $h_{r,0}$:*

$$h_{r,0}(a_1, a_2) = C \int_0^\infty t^{-b_3+1/2} W_{0,b_3-1}(t) \exp\left(\frac{4\eta_2\eta_0 a_1^2}{t^2} - \frac{t^2}{16\eta_2 a_2^2}\right) \frac{dt}{t}, \tag{79}$$

*where $\eta_2 = -\mathrm{Im}(\eta(E_2))$, $C$ is a constant multiple.*

PROOF. The situation is completely similar to [20, Theorem (9.1)]. So we omit it. □

From this theorem, $\pi^*_\Lambda$, $(\Lambda \in \Xi_V)$ has no rapidly decreasing solution in its 4-dimensional space of Whittaker vectors, which agree with the Shalika's multiplicity-one result.

Now draw out the $h_{k,l}$'s formula from that of $h_{r,0}$. By using Proposition 4.3, we have,

**Theorem 4.6.** *Let* $\Lambda = [r, s; u-2] \in \Xi_{\mathrm{II}}$. *Assume that* $\mathrm{Im}(\eta(E_2)) < 0$ *for a nondegenerate character* $\eta$. *Consider the rapidly decreasing Whittaker function* $\Phi_{\Lambda,d}(g) = \sum_{k,l} c_{k,l}(g) f_{k,l}$ *and define* $\{h_{k,l}\}$ *by means of* (72). *Then they can be expressed as follows:*

$$(80)\quad h_{k,l} = C\sum_{i=0}^{l}(-1)^l(k-r;l-i)\binom{l}{i}(\xi')^{r-k-l+i}\xi^i(-8\eta_2 a_1^2)^{r-k+i} \times \int_0^\infty t^{-b_3+2(k-i-r)+1/2}W_{0,b_3-1}(t)\exp\left(\frac{4\eta_0\eta_2 a_1^2}{t^2} - \frac{t^2}{16\eta_2 a_2^2}\right)\frac{dt}{t}.$$

*Here* $C$ *is a constant multiple independent of* $j$ *and* $k$, *the binomial* $\binom{0}{0} = 1$ *and*

$$(p;q) = \begin{cases} p(p+1)\cdots(p+q-1) & (q>0), \\ 1 & (q=0), \\ 0 & (-q+1 \le p \le 0,\ q \neq 0), \end{cases}$$

*for a non-positive integer* $p$ *and a non-negative integer* $q$.

PROOF. One can directly check that this formula satisfies the recursive condition in Proposition 4.3. Its initial condition is exactly (79) in Theorem 4.5. □

Generally speaking, there is no rule characterizing the $P_J$-series Whittaker functions; the Whittaker model is just a subspace of the kernel of appropriately selected shift operators. In the case of $SU(2,2)$, however, we will eventually see that the Whittaker model with the corner $K$-type is characterized by such a kernel with the help of the Casimir operator.

We begin with introducing the notion of the corner $K$-type; analogous to the minimal $K$-type of discrete series representations.

**5.1. Corner $K$-types of $P_J$-series representations.** As in §2.6, let $\pi_J = \mathrm{ind}_{P_J}^G((\chi_m, D_k^\pm)\otimes e^{\nu+\rho_J}\otimes 1)$ be a principal $P_J$-series representation. The corner $K$-type $\tau_d^*$ is characterized by the following property:

(i). $\dim \tau_d^*$ is minimum in $\pi|_K$.

(ii). $\tau_d^*|_{K_0}$ has the minimal $K_0$-type of $D_k^\pm$.

(iii). If $m \neq 0$, there exists a non-compact root $\delta$ with respect to $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ such that $\tau_{d+\delta} \in \widehat{K}$ and $[\pi|_K : \tau_{d+\delta}^*] = 0$.

We remark that the corner $K$-type is not necessarily minimal. Choose $\tau$ so as that its contragredient representation $\tau^*$ becomes the corner $K$-type of $\pi_J$, determined uniquely as in Table II.1.

In the following we write $\pi = \pi_J$, $\tau_j^{(\pm)} = \tau_{d_j^{(\pm)}}$ for simplicity. Consider the Whittaker function $\Phi_{\pi,\tau}$. Then,

**Proposition 5.1.** (i). *If* $\mathrm{sgn}(D_k^\pm) > 0$, *we have, for* $j = 1, 2$,

$$\mathcal{E}^{(\mathrm{sgn}\,(m),\,-\,\mathrm{sgn}\,(m))}\Phi_{\pi,\tau_j} = 0 \qquad (m \neq 0), \tag{81}$$

$$\mathcal{D}^{\mathrm{up}}\Phi_{\pi,\tau_0^+} = 0 \qquad (m = 0). \tag{82}$$

(ii). *If* $\mathrm{sgn}(D_k^\pm) < 0$, *we have, for* $j = -1, -2$,

$$\overline{\mathcal{E}}^{(-\,\mathrm{sgn}\,(m),\,\mathrm{sgn}\,(m))}\Phi_{\pi,\tau_j} = 0 \qquad (m \neq 0), \tag{83}$$

$$\mathcal{D}^{\mathrm{down}}\Phi_{\pi,\tau_0^-} = 0 \qquad (m = 0). \tag{84}$$

PROOF. We can check these equations from Proposition 2.12 and Table II.1. □

**5.2. Differential equations for Whittaker functions.** We can take the radial part of the equations in § 5.1 using Lemma 3.7.

**Proposition 5.2.** *Let* $\Phi_{\pi,\tau_j}(a) = \sum_{kl} c_{kl}^{(j)}(a) f_{kl}^{(j)}$ *for* $j = \pm 1, \pm 2$ *and let* $\Phi_{\pi,\tau_0^\pm}(a) = c_{00}^{(0,\pm)}(a) f_{00}^{(0,\pm)}$. *Then we have,*

(i). $\mathrm{sgn}(D_k^\pm) > 0$, $m = 0$ *case*:

$$\left((\partial_1 - k - 2)(\partial_2 - a_2^2\eta_2 - k) - \eta_0\left(\frac{a_1}{a_2}\right)^2\right) c_{00}^{(0,+)} = 0. \tag{85}$$

| $\mathrm{sgn}(D_k^{\pm})$ | $m$ | the parameter of $\tau$ |
|---|---|---|
| $+$ | $+$ | $d_1 = [0, m; -2k+m]$ |
| | $-$ | $d_2 = [-m, 0; -2k-m]$ |
| | $0$ | $d_0^+ = [0, 0; -2k]$ |
| $-$ | $-$ | $d_{-1} = [0, -m; 2k+m]$ |
| | $+$ | $d_{-2} = [m, 0; 2k-m]$ |
| | $0$ | $d_0^- = [0, 0; 2k]$ |

TABLE II.1. Corner $K$-types $\tau^*$ of $\mathrm{ind}_{P_J}^G((\chi_m, D_k^{\pm}) \otimes e^{\nu+\rho_J} \otimes 1)$

(ii). $\mathrm{sgn}(D_k^{\pm}) > 0$, $m > 0$ *case*:

$$(m-j)(\partial_2 - a_2^2\eta_2 - k + j)c_{0j}^{(1)} + (j+1)\left(\frac{a_1}{a_2}\right)\xi' c_{0,j+1}^{(1)} = 0, \tag{86}$$

$$(m-j)\left(\frac{a_1}{a_2}\right)\xi c_{0j}^{(1)} + (j+1)(\partial_1 - m - k - 1 + j)c_{0,j+1}^{(1)} = 0. \tag{87}$$

(iii). $\mathrm{sgn}(D_k^{\pm}) > 0$, $m < 0$ *case*:

$$(m+j)(\partial_1 - k - j - 2)c_{j0}^{(2)} + (j+1)\left(\frac{a_1}{a_2}\right)\xi' c_{j+1,0}^{(2)} = 0,$$

$$(m+j)\left(\frac{a_1}{a_2}\right)\xi c_{j0}^{(2)} + (j+1)(\partial_2 - a_2^2\eta_2 - m - k - 1 - j)c_{j+1,0}^{(2)} = 0.$$

(iv). $\mathrm{sgn}(D_k^{\pm}) < 0$, $m = 0$ *case*:

$$\left((\partial_1 - k - 2)(\partial_2 + a_2^2\eta_2 - k) - \eta_0\left(\frac{a_1}{a_2}\right)^2\right)c_{00}^{(0,-)} = 0.$$

(v). $\mathrm{sgn}(D_k^{\pm}) < 0$, $m > 0$ *case*:

$$(m-j)(\partial_2 + a_2^2\eta_2 - k + j)c_{j0}^{(-2)} + (j+1)\left(\frac{a_1}{a_2}\right)\xi' c_{j+1,0}^{(-2)} = 0,$$

$$(m-j)\left(\frac{a_1}{a_2}\right)\xi c_{j0}^{(-2)} + (j+1)(\partial_1 - m - k - 1 + j)c_{j+1,0}^{(-2)} = 0.$$

(vi). $\mathrm{sgn}(D_k^{\pm}) < 0$, $m < 0$ *case*:

$$(m+j)(\partial_1 - k - j - 2)c_{0j}^{(-1)} + (j+1)\left(\frac{a_1}{a_2}\right)\xi' c_{0,j+1}^{(-1)} = 0,$$

$$(m+j)\left(\frac{a_1}{a_2}\right)\xi c_{0j}^{(-1)} + (j+1)(\partial_2 + a_2^2\eta_2 - m - k - 1 - j)c_{0,j+1}^{(-1)} = 0.$$

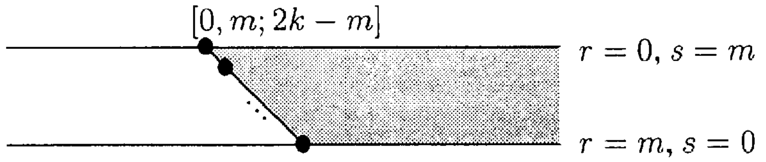


FIGURE II.4. The corner $K$-type in the plane: $r+s=m>0$ consisting of $K$-types of $\mathrm{ind}_{P_J}^{G}((\chi_m, D_k^{+})\otimes e^{\nu+\rho_J}\otimes 1)$

From the Casimir operator, we get another proposition by Lemma 3.8.

**Proposition 5.3.** *Let* $\Phi_{\pi,\tau_j}(a) = \sum_{kl} c_{kl}^{(j)}(a) f_{kl}^{(j)}$, *and so* $\Phi_{\pi,\tau_0^{\pm}}(a) = c_{00}^{(0,\pm)}(a) f_{00}^{(0,\pm)}$. *Then,*

(i). $\mathrm{sgn}(D_k^{\pm})>0$, $m>0$ *case:*

$$\begin{aligned}(88)\quad &\Big(\partial_1^2+\partial_2^2-6\partial_1-2\partial_2-a_2^4\eta_2^2+2\eta_0(a_1/a_2)^2+2(j-k)\eta_2 a_2^2\\ &+(2j-m)^2/2\Big)c_{0j}^{(1)}+2(m-j+1)\xi(a_1/a_2)c_{0,j-1}^{(1)}\\ &-2(j+1)\xi'(a_1/a_2)c_{0,j+1}^{(1)}=(\nu^2+(k-1)^2-10+m^2/2)c_{j0}^{(1)}.\end{aligned}$$

*When* $m=0$, $c_{00}^{(0,+)}$ *also satisfies* (88).

(ii). $\mathrm{sgn}(D_k^{\pm})>0$, $m<0$ *case:*

$$\begin{aligned}(89)\quad &\Big(\partial_1^2+\partial_2^2-6\partial_1-2\partial_2-a_2^4\eta_2^2+2\eta_0(a_1/a_2)^2-2(m+k+j)\eta_2 a_2^2\\ &+(2j+m)^2/2\Big)c_{j0}^{(2)}+2(1-m-j)\xi(a_1/a_2)c_{j-1,0}^{(2)}\\ &-2(j+1)\xi'(a_1/a_2)c_{j+1,0}^{(2)}=(\nu^2+(k-1)^2-10+m^2/2)c_{j0}^{(2)}.\end{aligned}$$

(iii). $\mathrm{sgn}(D_k^{\pm})<0$, $m>0$ *case:*

$$\begin{aligned}(90)\quad &\Big(\partial_1^2+\partial_2^2-6\partial_1-2\partial_2+a_2^4\eta(E_2)^2+2\eta_0(a_1/a_2)^2+2(k-j)\eta_2 a_2^2\\ &+(2j-m)^2/2\Big)c_{j0}^{(-2)}+2(m-j+1)\xi(a_1/a_2)c_{j-1,0}^{(-2)}\\ &-2(j+1)\xi'(a_1/a_2)c_{j+1,0}^{(-2)}=(\nu^2+(k-1)^2-10+m^2/2)c_{j0}^{(-2)}.\end{aligned}$$

*When* $m=0$, $c_{00}^{(0,-)}$ *also satisfies* (90).

(iv). $\mathrm{sgn}(D_k^{\pm})<0$, $m<0$ *case:*

$$\begin{aligned}(91)\quad &\Big(\partial_1^2+\partial_2^2-6\partial_1-2\partial_2+a_2^4\eta(E_2)^2+2\eta_0(a_1/a_2)^2+2(m+k+j)\eta_2 a_2^2\\ &+(m+2j)^2/2\Big)c_{0j}^{(-1)}+2(1-m-j)\xi(a_1/a_2)c_{0,j-1}^{(-1)}\\ &-2(j+1)\xi'(a_1/a_2)c_{0,j+1}^{(-1)}=(\nu^2+(k-1)^2-10+m^2/2)c_{0j}^{(-1)}.\end{aligned}$$

From these propositions, we find that

$$(c_{j0}^{(2)}, m) = (\overline{c_{0,|m|-j}^{(1)}}, -m), \qquad (c_{j0}^{(-2)}, m) = (\overline{c_{0,|m|-j}^{(-1)}}, -m), \tag{92}$$

$$(c_{0j}^{(1)}, \eta_2) = (c_{j0}^{(-2)}, -\eta_2), \qquad (c_{0j}^{(-1)}, \eta_2) = (c_{j0}^{(2)}, -\eta_2). \tag{93}$$

Therefore it is sufficient to treat the typical case: $\mathrm{sgn}(D_k^{\pm}) > 0$, $m \geq 0$.

**Theorem 5.4.** *Let* $\pi_J = \mathrm{ind}_{P_J}^{G}((\chi_m, D_k^{\pm}) \otimes e^{\nu+\rho_J} \otimes 1)$. *Assume that*

$$\mathrm{sgn}(D_k^{\pm}) > 0,\ m \geq 0.$$

*For* $\tau = \tau_{[0,m;m-2k]}$, *consider the Whittaker function* $\Phi_{\pi_J,\tau} = \sum_j c_{0j}^{(1)} f_{0j}^{(1)}$. *Put*

$$c_{0j}^{(1)}(a) = \exp(a_2^2\eta_2/2) a_1^{m+k+2-j} a_2^{k-j} h_j^{(1)}(a).$$

*Then,* $h_j^{(1)}$ $(j = 0, \ldots, m)$ *satisfy the following:*

$$\left(\partial_1\partial_2 - (a_1/a_2)^2\eta_0\right) h_j^{(1)} = 0, \tag{94}$$

$$\begin{aligned}\Big(\partial_1^2 + \partial_2^2 + 2(m+k-2j-1)(\partial_1+\partial_2) + 2\eta_2 a_2^2\partial_2 \\ + (m+k-2j-1)^2 - \nu^2\Big) h_j^{(1)} = 0.\end{aligned} \tag{95}$$

*These two differential equations become a holonomic system of rank* 4.

PROOF. When $m = 0$, they are a direct consequence of Equations (85) and (88). When $m > 0$, Equation (95) is a consequence of (88), while Equation (94) is obtained from Equations (86) and (87). See Appendix A for a proof of holonomicity. □

**5.3. Integral expression of Whittaker functions.** In this subsection, we obtain an integral expression of the rapidly decreasing solution of differential equations in Theorem 5.4.

As in §4.3, using a general solution of Equation (94), one has,

$$\mathcal{W}(a) = \int_0^\infty \phi(t) \exp\left(\frac{\eta_0 a_1^2}{t} - \frac{t}{4a_2^2}\right) \frac{dt}{t}.$$

for $\phi \in C^\infty(\mathbb{R}_{>0})$. We find the formal relation

$$(\partial_1 + \partial_2)\mathcal{W} = \int_0^\infty 2\partial_t\phi(t) \exp\left(\frac{\eta_0 a_1^2}{t} - \frac{t}{4a_2^2}\right) \frac{dt}{t},$$

which tells us, with (95), that

$$\left(4\partial_t^2 + 4(m+k-2j-1)\partial_t + \eta_2 t + (m+k-2j-1)^2 - \nu^2\right)\phi = 0,$$

with $\partial_t = t(d/dt)$. Put $v = \sqrt{t}$ and $\phi(t) = v^{-(m+k-2j-1)-1/2}\psi(v)$ to get

$$\frac{d^2\psi}{dv^2} + \left(-\frac{1}{4}(-4\eta_2) + \frac{1/4 - \nu^2}{v^2}\right)\psi = 0. \tag{96}$$

This turns out to be the Whittaker's differential equation if $\eta_2 < 0$. We denote by $W_{0,\nu}(2\sqrt{-\eta_2 v})$ the rapidly decreasing solution of (96).

**Theorem 5.5.** *Let $\pi_J = \mathrm{ind}_{P_J}^G((\chi_m, D_k^{\pm}) \otimes e^{\nu+\rho_J} \otimes 1)$ be an irreducible generalized principal series representation of $G$. Let $\Phi_{\pi_J,\tau_d} = \sum c_{ij}^{(d)} f_{ij}^{(d)}$ be a rapidly decreasing Whittaker function of $\pi_J$ with the corner $K$-type $\tau_d^*$ given by Table* II.1. *Then,*

(i). $\mathrm{sgn}(D_k^{\pm}) > 0$, $m \geq 0$ *case: if* $\eta_2 < 0$,

$$c_{0j}^{(1)}(a) = C_1(8\eta_2)^{m-j}\xi'^j e^{\eta_2 a_2^2/2} a_1^{m+k+2-j} a_2^{k-j} \times \int_0^\infty t^{2j-m-k+1/2} W_{0,\nu}(t) \exp\left(-\frac{4\eta_2\eta_0 a_1^2}{t^2} + \frac{t^2}{16\eta_2 a_2^2}\right)\frac{dt}{t}. \tag{97}$$

(ii). $\mathrm{sgn}(D_k^{\pm}) > 0$, $m < 0$ *case: if* $\eta_2 < 0$,

$$c_{j0}^{(2)}(a) = C_2(8\eta_2)^{m-j}(-\xi)^j e^{\eta_2 a_2^2/2} a_1^{k+2+j} a_2^{k+m+j} \times \int_0^\infty t^{-2j-m-k+1/2} W_{0,\bar\nu}(t) \exp\left(-\frac{4\eta_2\eta_0 a_1^2}{t^2} + \frac{t^2}{16\eta_2 a_2^2}\right)\frac{dt}{t}. \tag{98}$$

(iii). $\mathrm{sgn}(D_k^{\pm}) < 0$, $m \geq 0$ *case: if* $\eta_2 > 0$,

$$c_{j0}^{(-2)}(a) = C_{-2}(-8\eta_2)^{m-j}\xi'^j e^{-\eta_2 a_2^2/2} a_1^{m+k+2-j} a_2^{k-j} \times \int_0^\infty t^{2j-m-k+1/2} W_{0,\nu}(t) \exp\left(\frac{4\eta_2\eta_0 a_1^2}{t^2} - \frac{t^2}{16\eta_2 a_2^2}\right)\frac{dt}{t}. \tag{99}$$

(iv). $\mathrm{sgn}(D_k^{\pm}) < 0$, $m < 0$ *case: if* $\eta_2 > 0$,

$$c_{0j}^{(-1)}(a) = C_{-1}(8\eta_2)^{m-j}\xi^j e^{-\eta_2 a_2^2/2} a_1^{k+2+j} a_2^{k+m+j} \times \int_0^\infty t^{-2j-m-k+1/2} W_{0,\bar\nu}(t) \exp\left(\frac{4\eta_2\eta_0 a_1^2}{t^2} - \frac{t^2}{16\eta_2 a_2^2}\right)\frac{dt}{t}. \tag{100}$$

*Here, $C_{\pm1}$, $C_{\pm2}$ are constant multiples determined independently of the choice of $j$.*

PROOF. Nondegeneracy of $\eta$ says that $\eta_0$ is negative. So the parity condition of $\eta_2$ tells the convergence of integrals in the right-hand side. Once it converges, it clearly satisfies equations (94) and (95). Keeping in mind the convention for other cases, we can readily deduce the other equations. □

**Remark 5.6.** These expressions are very similar to those in the case of $Sp(2;\mathbb{R})$ ([18, Theorems (9.1), (9.2)]).

## 6. $P_m$-SERIES WHITTAKER FUNCTIONS

In contrast to the preceding two sections, principal $P_m$-series Whittaker functions have no single differential operators which make them vanish. That has been an obstacle to derive the differential equations of Whittaker functions with higher dimensional $K$-types as well as an integral expression of them.

Nevertheless, once we restrict ourselves to the case of smaller $K$-types – their dimensions are one and two –, we can obtain the differential equations, which we will explain.

**6.1. Differential equations for Whittaker functions.** The next theorem describes the Whittaker functions of principal $P_m$-series representations with the one-dimensional $K$-types.

**Theorem 6.1.** *Suppose* $\pi = \mathrm{ind}_{P_m}^G(\sigma_{0,\epsilon} \otimes e^{\mu+\rho} \otimes 1)$ *be irreducible. Let* $\tau = \tau_{[0,0;u]}$ *be an irreducible representation of* $K$ *satisfying* $u \equiv 1 - \epsilon(-1) \pmod 4$. *Choose a nondegenerate character* $\eta$ *of* $N$. *Let* $\Phi_{\pi,\tau} \in C^\infty_{\eta,\tau}(N\backslash G/K)$ *be a Whittaker function of* $\pi$ *with* $K$*-type* $\tau^*$. *Then, for* $a = (a_1, a_2) \in A_m$, $I(a) = a_1^{-3}a_2^{-1}\Phi_{\pi,\tau}(a)$ *satisfies the differential equations*:

(101)
$$\left\{\partial_1^2 + \partial_2^2 - \eta_2^2 a_2^4 + u\eta_2 a_2^2 + 2\eta_0\left(\frac{a_1}{a_2}\right)^2\right\} I = (\mu_1^2 + \mu_2^2) I,$$

$$\begin{aligned}\left\{\left(\partial_1^2 - \left(\frac{u}{2}+1\right)^2\right)\left(\partial_2^2 - \left(\frac{u}{2}+1\right)^2 - \eta_2^2 a_2^4 + u\eta_2 a_2^2\right)\right. \\ - 2\eta_0\left(\frac{a_1}{a_2}\right)^2(\partial_1+1)(\partial_2-1) - u\left(\frac{u}{2}+2\right)\eta_0\left(\frac{a_1}{a_2}\right)^2 + u\eta_0\eta_2 a_1^2 \\ \left. + \eta_0^2\left(\frac{a_1}{a_2}\right)^4\right\} I = \left(\mu_1^2 - \left(\frac{u}{2}+1\right)^2\right)\left(\mu_2^2 - \left(\frac{u}{2}+1\right)^2\right) I,\end{aligned} \tag{102}$$

*where we put* $\partial_j = a_j \frac{\partial}{\partial a_j}$ *for* $j = 1, 2$.

In the case of the two-dimensional $K$-types, we obtain the following.

**Theorem 6.2.** *Let* $|n| = 1$, $\pi = \mathrm{ind}_{P_m}^G(\sigma_{n,\epsilon} \otimes e^{\mu+\rho} \otimes 1)$ *be irreducible and* $\eta$ *be nondegenerate. Assume that the contragredient representation* $\tau^*$ *of* $\tau = \tau_{[r,s;u]}$ *appears in* $\pi|_K$ *with multiplicity one. Let* $\Phi_{\pi,\tau} \in C^\infty_{\eta,\tau}(N\backslash G/K)$ *be the Whittaker function of* $\pi$ *with* $K$*-type* $\tau^*$. *For the standard basis* $\{f_{kl}\}$ *of* $V_\tau$ *and* $a = (a_1, a_2) \in A_m$, *put*,

$$I(a) = a_1^{-3}a_2^{-1}\Phi_{\pi,\tau}(a) = \sum_{0 \le k \le r,\ 0 \le l \le s} b_{kl}(a) f_{kl}.$$

(i). *If* $r = 0$, $s = 1$, *then* $b_{00}$, $b_{01}$ *satisfy the equations*:

$$\begin{pmatrix} \mathcal{P}_2 + (u-1)\eta_2 a_2^2 & \xi'(a_1/a_2)(\partial_1 + \partial_2 + \eta_2 a_2^2 - 1) \\ \xi(a_1/a_2)(\partial_1 + \partial_2 + \eta_2 a_2^2 + 1) & \mathcal{P}_1 \end{pmatrix} \begin{pmatrix} b_{00} \\ b_{01} \end{pmatrix} \tag{103}$$

$$= \begin{cases} \mu_2^2 \begin{pmatrix} b_{00} \\ b_{01} \end{pmatrix} & (n = 1,\ u \equiv \epsilon(-1) \pmod 4), \\ \mu_1^2 \begin{pmatrix} b_{00} \\ b_{01} \end{pmatrix} & (n = -1,\ u \equiv -\epsilon(-1) \pmod 4). \end{cases}$$

$$\text{(104)} \quad \begin{pmatrix} \tilde{L}_0 + (u-1)\eta_2 a_2^2 & -2\xi'(a_1/a_2) \\ 2\xi(a_1/a_2) & \tilde{L}_0 + (u+1)\eta_2 a_2^2 \end{pmatrix} \begin{pmatrix} b_{00} \\ b_{01} \end{pmatrix} = \left(\mu_1^2 + \mu_2^2\right) \begin{pmatrix} b_{00} \\ b_{01} \end{pmatrix}.$$

(ii). *If* $r = 1$, $s = 0$, *then* $b_{00}, b_{10}$ *satisfy the equations*:

$$\text{(105)} \quad \begin{pmatrix} \mathcal{P}_1 & -\xi'(a_1/a_2)(\partial_1 + \partial_2 - \eta_2 a_2^2 + 1) \\ -\xi(a_1/a_2)(\partial_1 + \partial_2 - \eta_2 a_2^2 + 1) & \mathcal{P}_2 + (u-1)\eta_2 a_2^2 \end{pmatrix}$$

$$\times \begin{pmatrix} b_{00} \\ b_{10} \end{pmatrix} = \begin{cases} \mu_1^2 \begin{pmatrix} b_{00} \\ b_{10} \end{pmatrix} & (n = 1,\ u \equiv -\epsilon(-1) \pmod 4), \\ \mu_2^2 \begin{pmatrix} b_{00} \\ b_{10} \end{pmatrix} & (n = -1,\ u \equiv \epsilon(-1) \pmod 4). \end{cases}$$

$$\text{(106)} \quad \begin{pmatrix} \tilde{L}_0 + (u+1)\eta_2 a_2^2 & -2\xi'(a_1/a_2) \\ 2\xi(a_1/a_2) & \tilde{L}_0 + (u-1)\eta_2 a_2^2 \end{pmatrix} \begin{pmatrix} b_{00} \\ b_{10} \end{pmatrix} = \left(\mu_1^2 + \mu_2^2\right) \begin{pmatrix} b_{00} \\ b_{10} \end{pmatrix}.$$

*Here we write,*

$$\mathcal{P}_1 = \partial_1^2 + \eta_0 \left(\frac{a_1}{a_2}\right)^2, \quad \mathcal{P}_2 = \partial_2^2 - \eta_2^2 a_2^4 + \eta_0 \left(\frac{a_1}{a_2}\right)^2,$$
$$\tilde{L}_0 = \partial_1^2 + \partial_2^2 - \eta_2^2 a_2^4 + 2\eta_0 \left(\frac{a_1}{a_2}\right)^2.$$

**Remark 6.3.** As a matter of fact, the above differential equations are essentially the same as those in [17, Theorems (10.1) and (11.3)]. We recover those equations by putting $\eta_2, \eta_0, u$ in place of $-4\pi c_3$, $-4\pi^2 c_0^2$, $2l$, respectively.

To see that the system of differential equations in Theorems 6.1 and 6.2 becomes holonomic, we must compute the corresponding characteristic variety. See Appendix A in detail.

**Remark 6.4.** Our calculation is valid without the assumption that $\pi$ is irreducible. However, the injectivity of $\tilde{\iota}_{\tau^*}$ fails. In this case we might consider some subquotient representation of $\pi$ whose $K$-types occur in $\pi|_K$ with multiplicity one.

**6.2. Radial part of shift operators (one-dimensional $K$-type case).** We handle the representation $\pi = \mathrm{ind}_{P_m}^{G}(\sigma_{0,\epsilon} \otimes e^{\mu+\rho} \otimes 1)$ admitting a 1-dimensional $K$-type $\tau_0 = \tau^*_{[0,0,u]}$ with $-u \equiv 1 - \epsilon(-1) \pmod 4$.

In this case we consider $\tau \otimes \mathrm{Ad}_{\pm} \otimes \mathrm{Ad}_{\pm}$. Then, in each contragredient representation of them, the unique 1-dimensional irreducible constituent occurs in $\pi|_K$ with multiplicity one. Define the operators $\mathcal{D}^{up}$ and $\mathcal{D}^{down}$ (cf. (49).) Since $\dim \tau = 1$, they become,

$$\begin{aligned}\mathcal{D}^{up} &= P^{(-,-)}_{1,1} \circ \nabla^{+}_{\tau\otimes\mathrm{Ad}_+} \circ \nabla^{+}_{\tau},\\ \mathcal{D}^{down} &= \overline{P}^{(-,-)}_{1,1} \circ \nabla^{-}_{\tau\otimes\mathrm{Ad}_-} \circ \nabla^{-}_{\tau}.\end{aligned}$$

Then, we find that the composition $\mathcal{D}^{down} \circ \mathcal{D}^{up}$ will give one of the differential equations in Theorem 6.1.

Now we calculate the radial parts of these operators. We write

$$(107) \qquad \begin{aligned} d_0 &= [0,0;u], \\ d_1 &= [1,1;u+2], \\ d_2 &= [0,0;u+4], \end{aligned} \qquad \begin{aligned} d_3 &= [1,1;u-2], \\ d_4 &= [0,0;u-4]. \end{aligned}$$

For given $d_j$, we write simply $\tau_j = \tau_{d_j}$, $f^{(j)}_{pq} = f^{(d_j)}_{pq}$.

**Lemma 6.5.** *Put $\phi_0(a) = F_0|_{A_m}(a) = c^{(0)}(a) f^{(0)}_{00}$ for $F_0 \in C^{\infty}_{\eta,\tau_0}(N\backslash G/K)$ and put also*

$$\begin{aligned}\phi_1(a) &= F_1|_{A_m}(a) = \mathrm{Rad}(\nabla^+)\phi_0(a) = \sum_{0\le k,l\le 1} c^{(1)}_{kl}(a) f^{(1)}_{kl},\\ \phi_2(a) &= F_2|_{A_m}(a) = \mathrm{Rad}(P^{(-,-)}_{1,1} \circ \nabla^+)\phi_1(a) = c^{(2)}(a) f^{(2)}_{00},\\ \phi_3(a) &= F_3|_{A_m}(a) = \mathrm{Rad}(\nabla^-)\phi_0(a) \doteq \sum_{0\le k,l\le 1} c^{(3)}_{kl}(a) f^{(3)}_{kl},\\ \phi_4(a) &= F_4|_{A_m}(a) = \mathrm{Rad}(\overline{P}^{(-,-)}_{1,1} \circ \nabla^-)\phi_3(a) = c^{(4)}(a) f^{(4)}_{00}.\end{aligned}$$

*Then the image of the shift operators can be described as follows*:

$$(108) \qquad \begin{aligned} c^{(1)}_{00} &= \mathcal{S}' c^{(0)}, & c^{(1)}_{01} &= (-\mathcal{L}_2^- - u/4) c^{(0)},\\ c^{(1)}_{10} &= (\mathcal{L}_1 + u/4) c^{(0)}, & c^{(1)}_{11} &= -\mathcal{S} c^{(0)},\end{aligned}$$

$$(109) \qquad \begin{aligned} c^{(3)}_{00} &= \mathcal{S}' c^{(0)}, & c^{(3)}_{01} &= (\mathcal{L}_1 - u/4) c^{(0)},\\ c^{(3)}_{10} &= (-\mathcal{L}_2^+ + u/4) c^{(0)}, & c^{(3)}_{11} &= -\mathcal{S} c^{(0)},\end{aligned}$$

$$(110) \qquad c^{(2)} = (-\mathcal{L}_1 + 2 - u/4) c^{(1)}_{01} + (\mathcal{L}_2^- + u/4) c^{(1)}_{10} + \mathcal{S}' c^{(1)}_{11} - \mathcal{S} c^{(1)}_{00},$$

$$(111) \qquad c^{(4)} = (-\mathcal{L}_1 + 2 + u/4) c^{(3)}_{10} + (\mathcal{L}_2^+ - u/4) c^{(3)}_{01} + \mathcal{S}' c^{(3)}_{11} - \mathcal{S} c^{(3)}_{00}.$$

PROOF. Equations (110) and (111) are no other than specializations of Lemma 3.7.

First we consider $c^{(1)}_{ij}$ and $c^{(3)}_{ij}$. Using Propositions 2.4, 3.6 and Lemma 2.2,

$$\phi_1(a) = \left(\mathcal{L}_1 - \tfrac{\sqrt{-1}}{2}\tau_1(H_{13}) - 3\right) \iota_+(F_0(a) \otimes X_{13})$$

$$
\begin{aligned}
&+ \left(\mathcal{L}_2^- - \tfrac{\sqrt{-1}}{2}\tau_1(H_{24}) - 1\right) \iota_+(F_0(a) \otimes X_{24}) \\
&\quad + \left(\mathcal{S}' + \tau_1(e_+^1)\right) \iota_+(F_0(a) \otimes X_{23}) + \left(\mathcal{S} - \tau_1(e_-^2)\right) \iota_+(F_0(a) \otimes X_{14}) \\
&= \left(\mathcal{L}_1 + \tfrac{1}{4}\tau_1(I_{2,2} + h^1 - h^2) - 3\right) c^{(0)}(a) f_{10}^{(1)} \\
&\quad - \left(\mathcal{L}_2^- + \tfrac{1}{4}\tau_1(I_{2,2} - h^1 + h^2) - 1\right) c^{(0)}(a) f_{01}^{(1)} \\
&\quad + \left(\mathcal{S}' + \tau_1(e_+^1)\right) c^{(0)}(a) f_{00}^{(1)} - \left(\mathcal{S} - \tau_1(e_-^2)\right) c^{(0)}(a) f_{11}^{(1)} \\
&= (\mathcal{L}_1 + u/4)\, c^{(0)}(a) f_{10}^{(1)} - \left(\mathcal{L}_2^- + u/4\right) c^{(0)}(a) f_{01}^{(1)} + \mathcal{S}' c^{(0)}(a) f_{00}^{(1)} - \mathcal{S} c^{(0)}(a) f_{11}^{(1)}.
\end{aligned}
$$

We get (108) by comparing the definition of $c_{ij}^{(1)}$ with the last equation. In a similar manner,

$$
\begin{aligned}
\phi_3(a) &= \left(\mathcal{L}_1 + \tfrac{\sqrt{-1}}{2}\tau_3(H_{13}) - 3\right) \iota_-(F_0(a) \otimes X_{31}) \\
&\quad + \left(\mathcal{L}_2^+ + \tfrac{\sqrt{-1}}{2}\tau_3(H_{24}) - 1\right) \iota_-(F_0(a) \otimes X_{42}) \\
&\quad + \left(\mathcal{S} - \tau_3(e_-^1)\right) \iota_-(F_0(a) \otimes X_{32}) + \left(\mathcal{S}' + \tau_3(e_+^2)\right) \iota_-(F_0(a) \otimes X_{41}) \\
&= \left(\mathcal{L}_1 - \tfrac{1}{4}\tau_3(I_{2,2} + h^1 - h^2) - 3\right) c^{(0)}(a) f_{01}^{(3)} \\
&\quad - \left(\mathcal{L}_2^+ - \tfrac{1}{4}\tau_3(I_{2,2} - h^1 + h^2) - 1\right) c^{(0)}(a) f_{10}^{(3)} \\
&\quad - \left(\mathcal{S} - \tau_3(e_-^1)\right) c^{(0)}(a) f_{11}^{(3)} + \left(\mathcal{S}' + \tau_3(e_+^2)\right) c^{(0)}(a) f_{00}^{(3)} \\
&= (\mathcal{L}_1 - u/4)\, c^{(0)}(a) f_{01}^{(3)} - \left(\mathcal{L}_2^+ - u/4\right) c^{(0)}(a) f_{10}^{(3)} + \mathcal{S}' c^{(0)}(a) f_{00}^{(3)} - \mathcal{S} c^{(0)}(a) f_{11}^{(3)}.
\end{aligned}
$$

The last equation implies (109). □

**Proposition 6.6.** *Let $F \in C^\infty_{\eta,\tau_0}(N\backslash G/K)$ and put $\phi = F|_{A_m} \in C^\infty(A_m)$. Then,*

$$\mathrm{Rad}_{d_0}(\mathcal{D}^{up})\,\phi = 2\left\{(\mathcal{L}_1 + u/4 - 1)(\mathcal{L}_2^- + u/4) - \mathcal{S}\mathcal{S}'\right\}\phi, \tag{112}$$

$$\mathrm{Rad}_{d_0}(\mathcal{D}^{down})\phi = 2\left\{(\mathcal{L}_1 - u/4 - 1)(\mathcal{L}_2^+ - u/4) - \mathcal{S}\mathcal{S}'\right\}\phi. \tag{113}$$

This is the direct consequence of Lemma 6.5. We can also check directly that they commute with the Casimir operator, namely:

**Proposition 6.7.** *Let $F \in C^\infty_{\eta,\tau_0}(N\backslash G/K)$ and $\phi = F|_A$. Then,*

$$\mathrm{Rad}_{d_0}(\mathcal{D}^{up}) \circ \mathrm{Rad}_{d_0}(\Omega)\phi = \mathrm{Rad}_{d_2}(\Omega) \circ \mathrm{Rad}_{d_0}(\mathcal{D}^{up})\phi,$$

$$\mathrm{Rad}_{d_0}(\mathcal{D}^{down}) \circ \mathrm{Rad}_{d_0}(\Omega)\phi = \mathrm{Rad}_{d_4}(\Omega) \circ \mathrm{Rad}_{d_0}(\mathcal{D}^{down})\phi.$$

**6.3. Radial part of shift operators (two-dimensional $K$-type case).** Next, we consider the case: $\dim\tau = 2$ and $[\pi|_K : \tau^*] = 1$, precisely, $|n| = 1$ and $(r,s) \in \{(0,1),(1,0)\}$. In this case, $\tau \otimes \mathrm{Ad}_\pm$ has the unique irreducible constituent, whose contragredient is a multiplicity-one $K$-type of $\pi$. So we use the shift operators, $\mathcal{E}^{(\pm,\mp)}$ and $\overline{\mathcal{E}}^{(\pm,\mp)}$ (cf. (49)).

For suitable choice of $\pm$, the composition $\overline{\mathcal{E}}^{(\pm,\mp)} \circ \mathcal{E}^{(\mp,\pm)}$ will yield the differential equation.

We shall see in §6.4 that these shift operators characterize Whittaker functions of $\pi$ as their eigenfunctions through the calculation essentially done in §§2.10 and 2.11.

We rewrite Lemma 3.7 as follows:

**Corollary 6.8.** (i). *Let $r = 0$ and $s = 1$. Then,*

$$\begin{pmatrix} c_{00}^{[1,0;u+2]} \\ c_{10}^{[1,0;u+2]} \end{pmatrix} = -\begin{pmatrix} \mathcal{L}_2^- + (u-1)/4 & \mathcal{S}' \\ \mathcal{S} & \mathcal{L}_1 + (u-5)/4 \end{pmatrix} \begin{pmatrix} c_{00}^{[0,1;u]} \\ c_{01}^{[0,1;u]} \end{pmatrix}, \tag{114}$$

$$\begin{pmatrix} c_{00}^{[1,0;u-2]} \\ c_{10}^{[1,0;u-2]} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_1 - (u+5)/4 & -\mathcal{S}' \\ -\mathcal{S} & \mathcal{L}_2^+ - (u+1)/4 \end{pmatrix} \begin{pmatrix} c_{00}^{[0,1;u]} \\ c_{01}^{[0,1;u]} \end{pmatrix}. \tag{115}$$

(ii). *Let $r = 1$ and $s = 0$. Then,*

$$\begin{pmatrix} c_{00}^{[0,1;u+2]} \\ c_{01}^{[0,1;u+2]} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_1 + (u-5)/4 & -\mathcal{S}' \\ -\mathcal{S} & \mathcal{L}_2^- + (u-1)/4 \end{pmatrix} \begin{pmatrix} c_{00}^{[1,0;u]} \\ c_{10}^{[1,0;u]} \end{pmatrix}, \tag{116}$$

$$\begin{pmatrix} c_{00}^{[0,1;u-2]} \\ c_{01}^{[0,1;u-2]} \end{pmatrix} = -\begin{pmatrix} \mathcal{L}_2^+ - (u+1)/4 & \mathcal{S}' \\ \mathcal{S} & \mathcal{L}_1 - (u+5)/4 \end{pmatrix} \begin{pmatrix} c_{00}^{[1,0;u]} \\ c_{10}^{[1,0;u]} \end{pmatrix}. \tag{117}$$

We see from this Corollary, that Equation (115) (resp. (117)) can be recovered from (114) (resp. (116)) with the following rule:

$$\begin{aligned} &(c_{00}^{(0)}, c_{01}^{(0)}, c_{00}^{(1)}, c_{10}^{(1)}, u) \leftrightarrow (-\overline{c_{01}^{(0)}}, -\overline{c_{00}^{(0)}}, \overline{c_{10}^{(-1)}}, \overline{c_{00}^{(-1)}}, -u), \\ \text{(resp. } &(c_{00}^{(0)}, c_{10}^{(0)}, c_{00}^{(2)}, c_{01}^{(2)}, u) \leftrightarrow (-\overline{c_{10}^{(0)}}, -\overline{c_{00}^{(0)}}, \overline{c_{01}^{(-2)}}, \overline{c_{00}^{(-2)}}, -u)). \end{aligned} \tag{118}$$

In addition, one can check the following directly:

**Proposition 6.9.** *Let $\mathrm{Rad}_d = \mathrm{Rad}_{\tau_d}$ be the radial part. Then we have,*

$$\begin{aligned} \mathrm{Rad}_{d_0}(\mathcal{E}^{(+,-)}) \circ \mathrm{Rad}_{d_0}(\Omega) &= \mathrm{Rad}_{d_1}(\Omega) \circ \mathrm{Rad}_{d_0}(\mathcal{E}^{(+,-)}), \\ \mathrm{Rad}_{d_0}(\mathcal{E}^{(-,+)}) \circ \mathrm{Rad}_{d_0}(\Omega) &= \mathrm{Rad}_{d_2}(\Omega) \circ \mathrm{Rad}_{d_0}(\mathcal{E}^{(-,+)}), \\ \mathrm{Rad}_{d_0}(\overline{\mathcal{E}}^{(+,-)}) \circ \mathrm{Rad}_{d_0}(\Omega) &= \mathrm{Rad}_{d_{-1}}(\Omega) \circ \mathrm{Rad}_{d_0}(\overline{\mathcal{E}}^{(+,-)}), \\ \mathrm{Rad}_{d_0}(\overline{\mathcal{E}}^{(-,+)}) \circ \mathrm{Rad}_{d_0}(\Omega) &= \mathrm{Rad}_{d_{-2}}(\Omega) \circ \mathrm{Rad}_{d_0}(\overline{\mathcal{E}}^{(-,+)}). \end{aligned}$$

### 6.4. Proof of Theorem 6.1.

Throughout the following subsections, we use for notation $\tau_j$, $f_{kl}^{(j)}$, $c_{kl}^{(j)}$ for a given $d_j$, as in the previous subsection.

For $d_0 = [0,0;u]$, put $\Phi = \Phi_{\pi,\tau_0} = c_{00}^{(0)} f_{00}^{(d_0)}$. We show Equation (101) firstly. Lemma 3.9 says that the Casimir operator acts on Whittaker vectors by a constant multiple. Putting $r = 0$, $s = 0$ in Lemma 3.8, we have

$$(L_0 + u\eta_2 a_2^2)\Phi = (\mu_1^2 + \mu_2^2 - 10)\Phi.$$

The translation from $\Phi(a)$ to $I(a) = a_1^{-3}a_2^{-1}c_{00}^{(0)}(a)$ yields (101).

Next, we verify Equation (102). Put $d_1 = [1,1;u+2]$ and $d_2 = [0,0;u+4]$. Write,

$$\nabla^+\Phi = \sum_{0\le k\le 1,\ 0\le l\le 1} c'_{kl}f^{(1)}_{kl}, \qquad \mathcal{D}^{up}\,\Phi = c''_{00}f^{(2)}_{00},$$
$$\nabla^- \circ \mathcal{D}^{up}\,\Phi = \sum_{0\le k\le 1,\ 0\le l\le 1} c'''_{kl}f^{(1)}_{kl}.$$

Considering (45), (14) and Lemma 2.6, we have the following equations:

$$\begin{aligned}
\nabla^+\Phi &= (X_{32}\cdot c^{(0)}_{00})f^{(1)}_{00} - (X_{42}\cdot c^{(0)}_{00})f^{(1)}_{01} + (X_{31}\cdot c^{(0)}_{00})f^{(1)}_{10} - (X_{41}\cdot c^{(0)}_{00})f^{(1)}_{11},\\
\mathcal{D}^{up}\,\Phi &= (-X_{41}\cdot c'_{00} - X_{31}\cdot c'_{01} + X_{42}\cdot c'_{10} + X_{32}\cdot c'_{11})f^{(2)}_{00},\\
\nabla^-\circ\mathcal{D}^{up}\,\Phi &= (X_{14}\cdot c''_{00})f^{(1)}_{00} + (X_{13}\cdot c''_{00})f^{(1)}_{01} - (X_{24}\cdot c''_{00})f^{(1)}_{10} - (X_{23}\cdot c''_{00})f^{(1)}_{11},\\
\mathcal{D}^{down}\circ\mathcal{D}^{up}\,\Phi &= (-X_{23}\cdot c'''_{00} - X_{24}\cdot c'''_{01} + X_{13}\cdot c'''_{10} + X_{14}\cdot c'''_{11})f^{(0)}_{00}.
\end{aligned}$$

Concluding, we obtain,

$$(119)\qquad {}^t\begin{pmatrix}-X_{23}\\ X_{24}\\ -X_{13}\\ X_{14}\end{pmatrix}\begin{pmatrix}X_{14}\\ X_{13}\\ -X_{24}\\ -X_{23}\end{pmatrix}{}^t\begin{pmatrix}-X_{41}\\ -X_{31}\\ X_{42}\\ X_{32}\end{pmatrix}\begin{pmatrix}X_{32}\\ -X_{42}\\ X_{31}\\ -X_{41}\end{pmatrix}\Phi = \mathcal{D}^{down}\circ\mathcal{D}^{up}\,\Phi.$$

We know from Proposition 2.19 that the left-hand side of (119) has $\Phi$ as an eigenfunction with eigenvalue $(\mu_1 - \frac{u}{2} - 1)(\mu_2 - \frac{u}{2} - 1)(\mu_1 + \frac{u}{2} + 1)(\mu_2 + \frac{u}{2} + 1)$. On the other hand, using Proposition 6.6, we have,

$$\begin{aligned}
I' := a_1^{-3}a_2^{-1}\,\mathrm{Rad}_{d_0}(\mathcal{D}^{up})c^{(0)}_{00} &= 2\left\{(\mathcal{L}_1 + \tfrac{u}{4} + \tfrac12)(\mathcal{L}_2^- + \tfrac{u}{4} + \tfrac12) - \mathcal{S}\mathcal{S}'\right\} I,\\
a_1^{-3}a_2^{-1}\,\mathrm{Rad}_{d_0}(\mathcal{D}^{down}\circ\mathcal{D}^{up})c^{(0)}_{00} &= 2\left\{(\mathcal{L}_1 - \tfrac{u}{4} - \tfrac12)(\mathcal{L}_2^+ - \tfrac{u}{4} - \tfrac12) - \mathcal{S}\mathcal{S}'\right\} I'.
\end{aligned}$$

Therefore the radial part of $\mathcal{D}^{down}\circ\mathcal{D}^{up}$ is,

$$\begin{aligned}
&a_1^{-3}a_2^{-1}\,\mathrm{Rad}_{d_2}(\mathcal{D}^{down})\circ\mathrm{Rad}_{d_0}(\mathcal{D}^{up})c^{(0)}_{00}\\
&\quad= \frac14\left\{\left(\partial_1 - \left(\frac{u}{2}+1\right)\right)\left(\partial_2 - \left(\frac{u}{2}+1\right) + \eta_2 a_2^2\right) - \eta_0\left(\frac{a_1}{a_2}\right)^2\right\}\\
&\qquad\times\left\{\left(\partial_1 + \frac{u}{2}+1\right)\left(\partial_2 + \frac{u}{2} + 1 - \eta_2 a_2^2\right) - \eta_0\left(\frac{a_1}{a_2}\right)^2\right\} I\\
&\quad= \frac14\left\{\left(\partial_1^2 - \left(\frac{u}{2}+1\right)^2\right)\left(\partial_2^2 - \left(\frac{u}{2}+1\right)^2 + \eta_2 a_2^2\right)\right.\\
&\qquad\left. - \eta_0\left(\frac{a_1}{a_2}\right)^2\left(2(\partial_1+1)(\partial_2-1) + \frac{u^2}{2} + 2u - u\eta_2 a_2^2\right) + \eta_0^2\left(\frac{a_1}{a_2}\right)^4\right\} I.
\end{aligned}$$

In conclusion, Equation (119) and the last equation imply Equation (102). □

**Remark 6.10.** We also obtain these equations by the method used in [17, §10]. See also [11]. (cf. [17, Remark (12.1)].)

### 6.5. Proof of Theorem 6.2.

Assume $|n| = 1$. Put

$$d_{\pm 1} = [0, 1; \pm u], d_{\pm 2} = [1, 0; \pm(u+2)], d_{\pm 3} = [1, 0; \pm u], d_{\pm 4} = [0, 1; \pm(u+2)].$$

Then, $[\pi|_K : \tau_1^*] = 1$ if $u \equiv n\epsilon(-1)$ and $[\pi|_K : \tau_3^*] = 1$ if $u \equiv -n\epsilon(-1)$. We identify $\tau_{-j}$ with $\tau_j^*$ through the map in Proposition 2.3.

Let $\Phi_\pi$ be a Whittaker vector and let $\Phi_{\pi,\tau_j}$ be a Whittaker function of $\pi$ with $K$-type $\tau_j^*$, $(j = 1, \dots, 4)$. Namely, for any $v^* \in V_{\tau_j}^*$,

$$< v^*, \Phi_{\pi,\tau_j}(g) > = \Phi_\pi(\iota_{\tau_j^*}(v^*))(g). \tag{120}$$

We write $\Phi_{\pi,\tau_j}(g) = \sum_{k,l} c_{kl}^{(j)}(g) f_{kl}^{(j)}$. By Corollary 2.16, we can take $a_{kl}^{(-j)}$ in place of $v^*$ in (120); then we have,

$$c_{00}^{(0)}(g) = < f_{00}^{(1)*}, \Phi_{\pi,\tau_j}(g) > = \Phi_\pi(\iota_{\tau_{-j}}(f_{01}^{(-1)}))(g) = \Phi_\pi(a_{01}^{(-1)})(g).$$

Similarly,

$$\begin{aligned} c_{00}^{(4)} &= \Phi_\pi(a_{01}^{(-4)}), & c_{01}^{(j)} &= -\Phi_\pi(a_{00}^{(-j)}) & &\text{for } j = 1, 4, \\ c_{00}^{(j)} &= \Phi_\pi(a_{10}^{(-j)}), & c_{10}^{(j)} &= -\Phi_\pi(a_{00}^{(-j)}) & &\text{for } j = 2, 3. \end{aligned}$$

Since $\Phi_\pi$ is in particular a $\mathfrak{g}$-homomorphism, one obtain the following equations from Corollary 2.23:

$$\begin{pmatrix} X_{42} & X_{32} \\ X_{41} & X_{31} \end{pmatrix} \begin{pmatrix} c_{00}^{(1)} \\ c_{01}^{(1)} \end{pmatrix} = \frac{(2\mu_{\delta(2)} + u + 1)}{4} \begin{pmatrix} c_{00}^{(2)} \\ c_{10}^{(2)} \end{pmatrix}, \tag{121}$$

$$\begin{pmatrix} X_{24} & X_{14} \\ X_{23} & X_{13} \end{pmatrix} \begin{pmatrix} c_{00}^{(2)} \\ c_{10}^{(2)} \end{pmatrix} = \frac{(2\mu_{\delta(2)} - u - 1)}{4} \begin{pmatrix} c_{00}^{(1)} \\ c_{01}^{(1)} \end{pmatrix}, \tag{122}$$

$$\begin{pmatrix} X_{31} & -X_{32} \\ -X_{41} & X_{42} \end{pmatrix} \begin{pmatrix} c_{00}^{(3)} \\ c_{10}^{(3)} \end{pmatrix} = \frac{(2\mu_{\delta(1)} + u + 1)}{4} \begin{pmatrix} c_{00}^{(4)} \\ c_{01}^{(4)} \end{pmatrix}, \tag{123}$$

$$\begin{pmatrix} X_{13} & -X_{14} \\ -X_{23} & X_{24} \end{pmatrix} \begin{pmatrix} c_{00}^{(4)} \\ c_{01}^{(4)} \end{pmatrix} = \frac{(2\mu_{\delta(1)} - u - 1)}{4} \begin{pmatrix} c_{00}^{(3)} \\ c_{10}^{(3)} \end{pmatrix}. \tag{124}$$

where $(\delta(1), \delta(2)) = (1, 2)$, (resp. $(\delta(1), \delta(2)) = (2, 1)$) if $n = 1$, (resp. $n = -1$). We actually have obtained shift operators, i.e., by (45) and Lemma 2.6, Equations (121) and (122) (resp. (123) and (124)) turn out to be,

$$\overline{\mathcal{E}}^{(-,+)} \circ \mathcal{E}^{(+,-)} \Phi_{\pi,\tau_1} = \left( \left( \frac{\mu_{\delta(2)}}{2} \right)^2 - \left( \frac{u+1}{4} \right)^2 \right) \Phi_{\pi,\tau_1}, \tag{125}$$

$$\text{(resp. } \overline{\mathcal{E}}^{(+,-)} \circ \mathcal{E}^{(-,+)} \Phi_{\pi,\tau_3} = \left( \left( \frac{\mu_{\delta(1)}}{2} \right)^2 - \left( \frac{u+1}{4} \right)^2 \right) \Phi_{\pi,\tau_3}). \tag{126}$$

Now, we take the radial part of the both sides of the above equations, which will finish this proof. Corollary 6.8 says that, from (125),

(127)
$$\begin{pmatrix} \mathcal{L}_2^+ - (u+3)/4 & \mathcal{S}' \\ \mathcal{S} & \mathcal{L}_1 - (u+7)/4 \end{pmatrix} \begin{pmatrix} \mathcal{L}_2^- + (u-1)/4 & \mathcal{S}' \\ \mathcal{S} & \mathcal{L}_1 + (u-5)/4 \end{pmatrix} \begin{pmatrix} c_{00}^{(1)} \\ c_{01}^{(1)} \end{pmatrix} = \frac{1}{4}\left\{ \mu_{\delta(2)}^2 - \left(\frac{u+1}{2}\right)^2 \right\} \begin{pmatrix} c_{00}^{(1)} \\ c_{01}^{(1)} \end{pmatrix}.$$

By multiplying matrices in the left-hand side, we have, by Equation (127),

$$\begin{aligned} &\left(\partial_2^2 - 2\partial_2 - \eta_2^2 a_2^4 + (u-1)\eta_2 a_2^2 - \frac{(u+3)(u-1)}{4} + \eta_0 \left(\frac{a_1}{a_2}\right)^2\right) c_{00}^{(1)} \\ &\quad + \xi'\left(\frac{a_1}{a_2}\right)(\partial_1 + \partial_2 + \eta_2 a_2^2 - 5) c_{01}^{(1)} \\ &\quad = \left(\mu_{\delta(2)}^2 - \left(\frac{u+1}{2}\right)^2\right) c_{00}^{(1)}, \end{aligned} \tag{128}$$

and,

$$\begin{aligned} &\xi\left(\frac{a_1}{a_2}\right)(\partial_1 + \partial_2 - \eta_2 a_2^2 - 3) c_{00}^{(1)} \\ &\quad + \left(\partial_1^2 - 6\partial_1 - \frac{(u+7)(u-5)}{4} + \eta_0\left(\frac{a_1}{a_2}\right)^2\right) c_{01}^{(0)} \\ &\quad = \left(\mu_{\delta(2)}^2 - \left(\frac{u+1}{2}\right)^2\right) c_{01}^{(1)}. \end{aligned} \tag{129}$$

Keeping in sight of (118), we obtain the differential equations of $\Phi_{\pi,\tau_3}$ immediately from (128). Therefore, translating Equations (128) and (129) to those of $b_{kl}^{(j)} := a_1^{-3}a_2^{-1}c_{kl}^{(j)}$, we obtain equations (103) and (105).

Equations (104) and (106) is a direct consequence derived from the Casimir operator, Lemmas 3.8 and 3.9. □

**Remark 6.11.** Put $\tau = \tau_{[r,s;u]}$ and $\tau' = \tau_{[r,s;u+4]}$.

If $r = 0$ and $s = 0$, $\mathcal{D}^{up}\,\Phi_{\pi,\tau}$ becomes a Whittaker function of $\pi$ with $K$-type $\tau'$. This can be shown by seeing that $\mathcal{D}^{up}\,\Phi_{\pi,\tau}$ satisfies the differential equations (101) and (102) with $u+4$ in place of $u$.

Similarly if $r = 0$ and $s = 1$ (resp. $r = 1$ and $s = 0$), $\mathcal{E}^{(+,-)}\Phi_{\pi,\tau}$ (resp. $\mathcal{E}^{(-,+)}\Phi_{\pi,\tau}$) satisfies the differential equations (105) and (106) with a suitable change of parameter.

**Remark 6.12.** We also obtain another differential equation using the radial part of the operator $\mathcal{E}^{(-,+)} \circ \overline{\mathcal{E}}^{(+,-)}$ instead. Apparently this equation is dependent; in fact the operator $\mathcal{E}^{(-,+)} \circ \overline{\mathcal{E}}^{(+,-)} + \overline{\mathcal{E}}^{(-,+)} \circ \mathcal{E}^{(+,-)}$ differs from the Casimir operator only on constant terms.

## Appendix: A. HOLONOMICITY OF THE SYSTEM OF DIFFERENTIAL EQUATIONS

In this appendix, we collect the basic results about Gröbner basis, characteristic varieties and holonomicities. It is the theory of Gröbner bases that enable us to show the holonomicity of the differential equations obtained in Part II. The general reference is [19].

In the following, $\mathbb{N}$ is non-negative integers. For multi-index $x = (x_1, \dots, x_n)$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, we use the convention $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$, $|\beta| = \beta_1 + \cdots + \beta_n$ and $D^{(\beta)} = d^{|\beta|}/dx_1^{\beta_1} \cdots dx_n^{\beta_n}$.

**A.1. Term order and Gröbner basis.** Let $R_n = \mathbb{C}[x][D]$ be a Weyl algebra, i.e.,

$$R_n = \left\{ \sum_{|\beta|<\infty} c_\beta(x) D^{(\beta)} \,\middle|\, c_\beta(x) \in \mathbb{C}[x] \right\}.$$

Let $m(f)$ be the total degree of $f \in R_n$ and

$$\sigma(f) = \sum_{|\beta|=m(f)} c_\beta(x)\zeta^\beta,$$

called the principal symbol of $f$.

A total order $\prec$ of $\mathbb{N}^n$ is said to be a term order if it satisfies the following:

(T-1). If $\alpha \prec \beta$, then $\alpha + \gamma \prec \beta + \gamma$ for $\gamma \in \mathbb{N}^n$,
(T-2). $0 = (0, \dots, 0) \prec \alpha$ holds for any $\alpha \in \mathbb{N}^n \backslash \{0\}$.

Define $\prec_r$ be a total order of $\mathbb{N}^n \times \{1, \dots, r\}$ with the following property:

(T-3). $(\alpha, i) \prec_r (\beta, i)$ holds if and only if $\alpha \prec \beta$,
(T-4). If $(\alpha, i) \prec_r (\beta, j)$ then $(\alpha + \gamma, i) \prec_r (\beta + \gamma, j)$ for any $\gamma \in \mathbb{N}^n$.

Given an $f = (\sum_{\alpha_1} c_{\alpha_1} D^{(\alpha_1)}, \dots \sum_{\alpha_r} c_{\alpha_r} D^{(\alpha_r)}) \in (R_n)^r$, let

$$\operatorname{lexp}(f) = \max_{\prec_r} \{(\alpha_i, i) \mid c_{\alpha_i} \neq 0\}$$

be the leading exponent of $f$. If, say, $\operatorname{lexp}(f) = (\alpha, i)$, we denote by $\operatorname{lp}(f) = i$, the leading point of $f$.

For a subset $N$ of $(R_n)^r$, we put

$$E(N) = \{\operatorname{lexp}(f) \mid f \in N\} \subset \mathbb{N}^n \times \{1, \dots, r\}.$$

If $N$ is a left $R_n$-submodule of $(R_n)^r$, $E(N)$ becomes a monoideal, i.e., $(\alpha + \mathbb{N}^n, i) \subset E(N)$ holds for any $(\alpha, i) \in E(N)$.

**Definition A.1.** Let $N$ be a left $R_n$-submodule of $(R_n)^r$. A subset $\mathcal{G}$ of $N$ is said to be a Gröbner basis of $N$ if it satisfies the following:

(i). $\mathcal{G}$ generates $N$ over $R_n$,
(ii). the monoideal generated by $E(\mathcal{G})$ coincides $E(N)$.

**A.2. Characteristic varieties.** To describe characteristic varieties, we specify the term order of $\mathbb{N}^{2n} \times \{1, \ldots, r\}$ as follows:

$$\begin{aligned}(\alpha, \beta, i) \prec_r (\alpha', \beta', j) \text{ holds if and only if } & |\beta| < |\beta'| \\ & \text{or } (|\beta| = |\beta'|,\, i < j) \\ & \text{or } (|\beta| = |\beta'|,\, i = j,\, \beta \prec_{\text{lex}} \beta') \\ & \text{or } (\beta = \beta',\, i = j,\, |\alpha| < |\alpha'|) \\ & \text{or } (\beta = \beta',\, i = j,\, |\alpha| = |\alpha'|,\, \alpha \prec_{\text{lex}'} \alpha')\end{aligned}$$

Here $\prec_{\text{lex}}$ (resp. $\prec_{\text{lex}'}$) is a lexicographical order of homogeneous parts of $\mathbb{C}[x]$ (resp. $\mathbb{C}[x][D]$). Then $\prec_r$ satisfies the conditions (T-3) and (T-4).

Let $\mathcal{M}$ be a system of differential equations defined by:

$$\sum_{j=1}^{r} P_{ij} v_j = 0, \qquad (i = 1, \ldots, s)$$

for $P_i = (P_{i1}, \ldots, P_{ir}) \in (R_n)^r$. Let $N = \sum_{i=1}^{r} R_n P_i$ and choose a Gröbner basis $\mathcal{G}$ of $N$. The characteristic variety $\text{char}(\mathcal{M})$ of $\mathcal{M}$ is an algebraic variety in $\mathbb{C}^{2n}$ defined by

$$\text{char}(\mathcal{M}) = \bigcup_{\nu=1}^{r} \{(x, \zeta) \in \mathbb{C}^{2n} \mid \sigma(P)_\nu(x, \zeta) = 0 \text{ for any } P \in \mathcal{G}_\nu\}. \tag{130}$$

where $\mathcal{G}_\nu = \{P \in \mathcal{G} \mid \text{lp}(P) = \nu\}$ and $\sigma(P) = (\sigma(P)_1, \ldots, \sigma(P)_r)$.

Let $\text{proj}\colon \mathbb{C}^{2n} \ni (x, \zeta) \mapsto x \in \mathbb{C}^n$, then

$$\text{Sing}(\mathcal{M}) = \text{proj}(\{(x, \zeta) \in \text{char}(\mathcal{M}) \mid \zeta \neq 0\})$$

called the singular locus of $\text{char}(\mathcal{M})$. $\mathcal{M}$ is said to be holonomic if $\dim_{\mathbb{C}} \text{char}(\mathcal{M}) = n$.

The following is well-known.

**Theorem A.2** ([19, Th. 5.3.4]). *Given a system of differential equations $\mathcal{M}$, let $\mathcal{G}$ be a Gröbner basis defined by $\mathcal{M}$. Put*

$$m = \#(\{\mathbb{N}^n \times \{1, \ldots, r\}\} \backslash (\textit{monoideal generated by } \varpi(E(\mathcal{G}))))$$

*where*

$$\begin{aligned}\varpi\colon \mathbb{N}^{2n} \times \{1, \cdots, r\} &\to \mathbb{N}^n \times \{1, \ldots, r\} \\ (\alpha, \beta, \nu) &\mapsto (\beta, \nu).\end{aligned}$$

*If $m < \infty$, then $\mathcal{M}$ is holonomic on $\mathbb{C}^n \backslash \text{Sing}(\mathcal{M})$ and the rank of the system $\mathcal{M}$ is equal to $m$.*

**A.3. Proof of holonomicity.** Here, we consider the system of differential equations satisfied by the discrete series, $P_J$-series and $P_m$-series Whittaker functions and show their holonomicities using Theorem A.2. To compute Gröbner bases, we use the compute software "Kan" programmed by N. Takayama ([23]).

Let $\mathcal{M}_*$ be their systems of differential equations defined by

$$\mathcal{M}_d\colon \text{(73) and (74)},$$
$$\mathcal{M}_J\colon \text{(94) and (95)},$$
$$\mathcal{M}_m\colon \text{(101) and (102)}.$$

Let $\mathscr{G}_*$ be a Gröbner basis of $\mathcal{M}_*$. Considering Equation 130, the characteristic variety of both $\mathcal{M}_J$ and $\mathcal{M}_d$ is defined by the following:

$$\operatorname{char}(\mathcal{M}_J) = \operatorname{char}(\mathcal{M}_d) = \{(a_1, a_2, \zeta_1, \zeta_2) \in \mathbb{C}^4 \mid q_{d,j}(a, \zeta) = 0, \quad (j = 1, \ldots, 3)\}$$

where

$$\begin{aligned} q_{d,1} &= a_1^2\zeta_1^2 + 2a_2a_1\zeta_2\zeta_1 + a_2^2\zeta_2^2, \\ q_{d,2} &= a_2^3a_1\zeta_2\zeta_1, \\ q_{d,3} &= -a_2^5\zeta_2^3. \end{aligned}$$

$q_{d,j}$'s are exactly the principal symbols of the elements of $\mathscr{G}_d$, $\mathscr{G}_J$.

On the other hand, the characteristic variety of both $\mathcal{M}_m$ is defined by the following:

$$\operatorname{char}(\mathcal{M}_m) = \{(a_1, a_2, \zeta_1, \zeta_2) \in \mathbb{C}^4 \mid q_{m,j}(a, \zeta) = 0, \quad (j = 1, \ldots, 28)\}$$

where

$$\begin{aligned} q_{m,1} &= a_2^6a_1^2\zeta_2^2\zeta_1^2, \\ q_{m,2} &= a_2^2a_1^2\zeta_1^2 + a_2^4\zeta_2^2, \\ q_{m,3} &= a_2^8\zeta_2^4, \\ q_{m,4} &= -\eta_2^2a_2^8a_1^4\zeta_1^4 + 2\eta_0a_2^6a_1^2\zeta_2^4, \\ q_{m,5} &= -2\eta_0a_2^6a_1^2\zeta_2^4, \\ q_{m,6} &= 2\eta_0a_2^6a_1^2\zeta_2^6, \\ q_{m,7} &= 6\eta_0a_2^6a_1\zeta_2^4\zeta_1 + 14\eta_0a_2^7\zeta_2^5, \\ q_{m,8} &= \eta_2^4a_2^{12}a_1^4\zeta_1^4, \\ q_{m,9} &= 6\eta_0a_2^6a_1\zeta_2^6\zeta_1 + 14\eta_0a_2^7\zeta_2^7, \\ q_{m,10} &= 14\eta_0a_2^7a_1\zeta_2^5, \\ q_{m,11} &= 14\eta_0a_2^7a_1\zeta_2^7, \\ q_{m,12} &= -2\eta_0\mu_1^2a_2^7a_1\zeta_2^4 - 2\eta_0\mu_2^2a_2^7a_1\zeta_2^4 - \eta_0^2a_2a_1^7\zeta_1^4 + 5\eta_0^2a_2^5a_1^3\zeta_2^4 \\ &\quad + 40\eta_0a_2^7a_1\zeta_2^4, \end{aligned}$$

$$
\begin{aligned}
q_{m,13} &= -\eta_0^2 a_1^8 \zeta_1^4 - 4\eta_0^2 a_2 a_1^7 \zeta_2 \zeta_1^3 - 4\eta_0^2 a_2^3 a_1^5 \zeta_2^3 \zeta_1 + 5\eta_0^2 a_2^4 a_1^4 \zeta_2^4, \\
q_{m,14} &= 28\eta_0 \mu_1^2 a_2^7 \zeta_2^5 + 28\eta_0 \mu_2^2 a_2^7 \zeta_2^5 - 3\eta_0^2 a_1^7 \zeta_1^5 - 5\eta_0^2 a_2 a_1^6 \zeta_2 \zeta_1^4 \\
&\quad + 15\eta_0^2 a_2^4 a_1^3 \zeta_2^4 \zeta_1 - 23\eta_0^2 a_2^5 a_1^2 \zeta_2^5 - 420\eta_0 a_2^7 \zeta_2^5, \\
q_{m,15} &= \eta_2^6 a_2^{14} a_1^4 \zeta_1^4 + 2\eta_2^4 \eta_0 a_2^9 a_1^5 \zeta_2 \zeta_1^3 + 2\eta_2^4 \eta_0 a_2^{11} a_1^3 \zeta_2^3 \zeta_1, \\
q_{m,16} &= 7\eta_2^4 a_2^{11} a_1^5 \zeta_1^6 - 14\eta_0 \mu_1^2 a_2^7 a_1 \zeta_2^6 - 14\eta_0 \mu_2^2 a_2^7 a_1 \zeta_2^6 - 7\eta_0^2 a_2 a_1^7 \zeta_2^2 \zeta_1^4 \\
&\quad + 35\eta_0^2 a_2^5 a_1^3 \zeta_2^6, \\
q_{m,17} &= -\eta_2^4 a_2^{10} a_1^6 \zeta_1^6 + \eta_0^2 a_1^8 \zeta_2^2 \zeta_1^4 + 4\eta_0^2 a_2 a_1^7 \zeta_2^3 \zeta_1^3 + 4\eta_0^2 a_2^3 a_1^5 \zeta_2^5 \zeta_1 - 5\eta_0^2 a_2^4 a_1^4 \zeta_2^6, \\
q_{m,18} &= -3\eta_2^4 a_2^{10} a_1^5 \zeta_1^7 + 7\eta_2^4 a_2^{11} a_1^4 \zeta_2 \zeta_1^6 - 28\eta_0 \mu_1^2 a_2^7 \zeta_2^7 - 28\eta_0 \mu_2^2 a_2^7 \zeta_2^7 + 3\eta_0^2 a_1^7 \zeta_2^2 \zeta_1^5 \\
&\quad + 5\eta_0^2 a_2 a_1^6 \zeta_2^3 \zeta_1^4 - 15\eta_0^2 a_2^4 a_1^3 \zeta_2^6 \zeta_1 + 23\eta_0^2 a_2^5 a_1^2 \zeta_2^7 + 420\eta_0 a_2^7 \zeta_2^7, \\
q_{m,19} &= -3\eta_0^2 a_2 a_1^7 \zeta_1^5 + 15\eta_0^2 a_2^5 a_1^3 \zeta_2^4 \zeta_1, \\
q_{m,20} &= 7\eta_0^2 a_2 a_1^7 \zeta_2 \zeta_1^4 - 35\eta_0^2 a_2^5 a_1^3 \zeta_2^5, \\
q_{m,21} &= 2\eta_2^4 \eta_0 a_2^{11} a_1^3 \zeta_2^5 \zeta_1, \\
q_{m,22} &= 7\eta_0^2 a_2 a_1^7 \zeta_2^4 \zeta_1^4 - 35\eta_0^2 a_2^5 a_1^3 \zeta_2^8, \\
q_{m,23} &= \eta_0^2 a_1^8 \zeta_2^4 \zeta_1^4 + 4\eta_0^2 a_2 a_1^7 \zeta_2^5 \zeta_1^3 + 4\eta_0^2 a_2^3 a_1^5 \zeta_2^7 \zeta_1 - 5\eta_0^2 a_2^4 a_1^4 \zeta_2^8, \\
q_{m,24} &= -56\eta_0 \mu_1^2 a_2^7 \zeta_2^9 - 56\eta_0 \mu_2^2 a_2^7 \zeta_2^9 + 6\eta_0^2 a_1^7 \zeta_2^4 \zeta_1^5 + 10\eta_0^2 a_2 a_1^6 \zeta_2^5 \zeta_1^4 \\
&\quad - 30\eta_0^2 a_2^4 a_1^3 \zeta_2^8 \zeta_1 + 46\eta_0^2 a_2^5 a_1^2 \zeta_2^9 + 840\eta_0 a_2^7 \zeta_2^9, \\
q_{m,25} &= -\eta_2^8 a_2^{16} a_1^4 \zeta_1^4 - 2\eta_2^6 \eta_0 a_2^{11} a_1^5 \zeta_2 \zeta_1^3 - 2\eta_2^6 \eta_0 a_2^{13} a_1^3 \zeta_2^3 \zeta_1, \\
q_{m,26} &= \eta_2^8 a_2^{16} a_1^4 \zeta_1^5 + 4\eta_2^6 \eta_0 a_2^{11} a_1^5 \zeta_2 \zeta_1^4 - u\eta_2^5 \eta_0 a_2^{12} a_1^2 \zeta_2^4 \zeta_1, \\
q_{m,27} &= -2\eta_2^6 \eta_0 a_2^{13} a_1^3 \zeta_2^5 \zeta_1, \\
q_{m,28} &= -\eta_2^{10} a_2^{18} a_1^4 \zeta_1^5 + u\eta_2^7 \eta_0 a_2^{14} a_1^2 \zeta_2^4 \zeta_1 - \eta_2^6 (u/2)^4 a_2^{14} \zeta_2^4 \zeta_1 \\
&\quad + \eta_2^6 \mu_1^2 (u/2)^2 a_2^{14} \zeta_2^4 \zeta_1 + \eta_2^6 \mu_2^2 (u/2)^2 a_2^{14} \zeta_2^4 \zeta_1 - \eta_2^6 \mu_2^2 \mu_1^2 a_2^{14} \zeta_2^4 \zeta_1
\end{aligned}
$$

$\{q_{m,j}\}$'s are the principal symbols of the elements of $\mathscr{G}_m$.

As a complex manifold, these characteristic varieties eventually coincide

$$
\begin{aligned}
\mathrm{char}(\mathcal{M}_d) &= \mathrm{char}(\mathcal{M}_J) = \mathrm{char}(\mathcal{M}_m) \\
&= \{(a_1 = a_2 = 0\} \cup \{(\zeta_1 = a_2 = 0\} \cup \{(a_1 = \zeta_2 = 0\} \cup \{(\zeta_1 = \zeta_2 = 0\}.
\end{aligned}
$$

In particular, they are holonomic and their singular locus is $\{a_1 = 0\} \cup \{a_2 = 0\}$.

Considering the leading exponents of the polynomials above, we have,

$$
\begin{aligned}
E(\mathscr{G}_d) &= \{(2,0,2,0),\ (3,1,1,1),\ (0,5,0,3)\} \\
E(\mathscr{G}_m) &= \{(2,6,2,2),\ (2,2,2,0),\ (0,8,0,4),\ (4,8,4,0),\ (2,6,0,4), \\
&\quad (2,6,0,6),\ (1,6,1,4),\ (4,12,4,0),\ (1,6,1,6),\ (1,7,0,5), \\
&\quad (1,7,0,7),\ (7,1,4,0),\ (8,0,4,0),\ (7,0,5,0),\ (4,0,4,0), \\
&\quad (5,11,6,0),\ (6,10,6,0),\ (5,10,7,0),\ (7,1,5,0),\ (7,1,4,1),
\end{aligned}
$$

$$(3,11,1,5),\ (7,1,4,4),\ (8,0,4,4),\ (7,0,5,4),\ (4,16,4,0),$$
$$(4,16,5,0),\ (3,13,1,5),\ (4,18,5,0)\}$$

Thus we know from above that the rank of $\mathrm{char}(\mathcal{M}_d)$, $\mathrm{char}(\mathcal{M}_d)$ and $\mathrm{char}(\mathcal{M}_m)$ is $4$, $4$ and $8$, respectively.

## Appendix: B. The structure of $Z(\mathfrak{g}_{\mathbb{C}})$

Here we recall the structure of $Z(\mathfrak{g}_{\mathbb{C}})$, the center of the universal enveloping algebra of the complexification of the Lie algebra $\mathfrak{g}$ in detail.

Many techniques have been developed to find the generators of $Z(\mathfrak{g}_{\mathbb{C}})$. We review one of them. The general reference of this method is Bourbaki's book [1].

This algebra $Z(\mathfrak{g}_{\mathbb{C}})$ is isomorphic to $S(\mathfrak{g}_{\mathbb{C}})^{\mathrm{Ad}\,\mathfrak{g}}$, the $\mathrm{Ad}\,\mathfrak{g}$-invariant elements of the symmetric algebra of $\mathfrak{g}_{\mathbb{C}}$ as a vector space. First, we consider the generators in a commutative situation, and then, through symmetrization, construct the true generators of the center. In fact, it is well-known that, via Harish-Chandra homomorphism, the center is indeed isomorphic to $S(\mathfrak{g}_{\mathbb{C}})^{\mathrm{Ad}\,\mathfrak{g}}$ as an algebra. These two maps are different, but have a property that they keep the symbols, which warrants this procedure. We should mention that there are many efficient ways of directly finding the generators of the center, namely, usage of Capelli identity, taking the trace of the finite dimensional representations, and so on.

Let us begin with the symmetric algebra. By the theorem of Todd-Shephard, $S(\mathfrak{g}_{\mathbb{C}})^{\mathrm{Ad}\,\mathfrak{g}}$ has three generators of degree 2, 3 and 4, respectively. (cf. [28, Proposition 2.1.3.2]).

Consider the next matrix of degree 4,

$$T = \left(\begin{array}{c|c} T^{(11)} & T^{(12)} \\ \hline T^{(21)} & T^{(22)} \end{array}\right) \tag{131}$$

where

$$T^{(11)} = \left(\begin{array}{c|c} I_0 - \sqrt{-1}{}^tE_1 - \sqrt{-1}E_1 & \begin{array}{c} {}^tE_3 - E_3 - \sqrt{-1}({}^tE_4 + E_4) \\ + {}^tE_5 - E_5 - \sqrt{-1}({}^tE_6 + E_6) \end{array} \\ \hline \begin{array}{c} -{}^tE_3 + E_3 - \sqrt{-1}({}^tE_4 + E_4) \\ - {}^tE_5 + E_5 - \sqrt{-1}({}^tE_6 + E_6) \end{array} & -I_0 - \sqrt{-1}{}^tE_2 - \sqrt{-1}E_2 \end{array}\right),$$

$$T^{(12)} = \left(\begin{array}{c|c} 2H_1 + \sqrt{-1}{}^tE_1 - \sqrt{-1}E_1 & \begin{array}{c} -{}^tE_3 - E_3 + \sqrt{-1}({}^tE_4 - E_4) \\ + {}^tE_5 + E_5 - \sqrt{-1}({}^tE_6 - E_6) \end{array} \\ \hline \begin{array}{c} {}^tE_3 + E_3 + \sqrt{-1}({}^tE_4 - E_4) \\ + {}^tE_5 + E_5 + \sqrt{-1}({}^tE_6 - E_6) \end{array} & 2H_2 + \sqrt{-1}{}^tE_2 - \sqrt{-1}E_2 \end{array}\right),$$

$$T^{(21)} = \left(\begin{array}{c|c} 2H_1 - \sqrt{-1}{}^tE_1 + \sqrt{-1}E_1 & \begin{array}{c} {}^tE_3 + E_3 - \sqrt{-1}({}^tE_4 - E_4) \\ + {}^tE_5 + E_5 - \sqrt{-1}({}^tE_6 - E_6) \end{array} \\ \hline \begin{array}{c} -{}^tE_3 - E_3 - \sqrt{-1}({}^tE_4 - E_4) \\ + {}^tE_5 + E_5 + \sqrt{-1}({}^tE_6 - E_6) \end{array} & 2H_2 - \sqrt{-1}{}^tE_2 + \sqrt{-1}E_2 \end{array}\right),$$

$$T^{(22)} = \left(\begin{array}{c|c} I_0 + \sqrt{-1}{}^tE_1 + \sqrt{-1}E_1 & \begin{array}{c} -{}^tE_3 + E_3 + \sqrt{-1}({}^tE_4 + E_4) \\ + {}^tE_5 - E_5 - \sqrt{-1}({}^tE_6 + E_6) \end{array} \\ \hline \begin{array}{c} {}^tE_3 - E_3 + \sqrt{-1}({}^tE_4 + E_4) \\ - {}^tE_5 + E_5 - \sqrt{-1}({}^tE_6 + E_6) \end{array} & -I_0 + \sqrt{-1}{}^tE_2 + \sqrt{-1}E_2 \end{array}\right).$$

This matrix $T = (T_{ij})$ should be considered as the $M_4(\mathbb{C})$-valued function on $\mathfrak{g}$, as follows:

$$T(X) = (1/4\,\mathrm{tr}(T_{ij}X)).$$

We can easily check that this matrix has the next property:

$$T(X) = X \tag{132}$$

for $X \in \mathfrak{g}$. This implies

$$T(\mathrm{Ad}(g)X) = gT(X)g^{-1}. \tag{133}$$

Consider the characteristic polynomial $\Phi_T(t)$ of $T$:

$$\Phi_T(t) = \det(t1_4 - T) = t^4 + \Phi_T^{(2)}t^2 - \Phi_T^{(3)}t + \Phi_T^{(4)}.$$

Then $\Phi_T^{(j)}$'s are all in $S(\mathfrak{g}_{\mathbb{C}})^{\mathrm{Ad}\,\mathfrak{g}}$ because $\Phi_{gTg^{-1}} = \Phi_T$ by (133). For example, we obtain the Casimir element $\Phi_T^{(2)}$. Its explicit formula is given by the following lemma.

**Lemma B.1.**

$$\Phi_T^{(2)} = H_1^2 + H_2^2 + \frac{1}{2}I_0^2 - {}^tE_2E_2 - {}^tE_1E_1 + 2({}^tE_3E_3 - {}^tE_4E_4 + {}^tE_5E_5 - {}^tE_6E_6).$$

By symmetrizing these $\Phi_T^{(j)}$'s, we get the generators of $Z(\mathfrak{g}_{\mathbb{C}})$. In particular the symmetrized form $L$ of the Casimir element $\Phi_T^{(2)}$ is given by

**Lemma B.2.**

$$L = H_1^2 + H_2^2 + \frac{1}{2}I_0^2 - \frac{1}{2}\sum_{j=1,2}(E_j{}^tE_j + {}^tE_jE_j) + \sum_{j=3}^{6}(E_j{}^t\overline{E}_j + {}^t\overline{E}_jE_j). \tag{134}$$

This coincides Equation (62).

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