Iterated Integral Representations of Hypergeometric Functions and Intersection Theory

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博士論文

Iterated integral representations of hypergeometric functions and intersection theory

超幾何関数の逐次積分表示と交点理論

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July 8, 1998

Contents

Chapter 0

Introduction

In this paper, we study twisted (co)homology groups associated with hypergeometric functions using iterations of integrations. Hypergeometric functions have multiple integral representations and can be regarded as pairings of twisted cycles and twisted cocycles.

We can derive various properties and formulae of hypergeometric functions by applying the theory of twisted (co)homology groups. For instance, we can interpret quadratic relations of hypergeometric functions as quadratic forms of intersection matrices of twisted (co)cycles as shown in [3].

The origin of the general theory of twisted cohomology group is Deligne's works in 1970's. However, there was not a systematic method of explicit computation of intersection matrices until quite recently. As to explicit computation of intersection numbers of (co)cycles, we have a long wait for recent results motivated by a study of period mappings. In [3], K. Cho and K. Matsumoto gave a method of evaluating intersection numbers for twisted cohomology groups associated with configuration spaces on the 1-dimensional projective space. In [15], K. Matsumoto developed a method to evaluate intersection numbers for twisted cohomology groups associated with hyperplane arrangements in general position. In [12], M. Kita and M. Yoshida derived a formula of intersection numbers for twisted homology groups associated with hyperplane arrangements in general position. For these methods and formulae, we refer to the book [?] of K. Aomoto and M. Kita on the modern theory of hypergeometric functions. However, these methods and formulae are hard to apply for degenerate arrangements unless we give a procedure of blowing up a given devisor to a normally crossing devisor. Moreover, in order to perform a computation of intersection numbers, we need some geometric insights.

In this paper, we study twisted (co)homology groups with locally constant sheaves of which ranks are more than one and apply them to study hypergeometric functions associated with degenerate arrangements; we study monodromy groups and intersection numbers for hypergeometric functions associated with degenerate arrangements such as ${}_pF_{p-1}$ and the Selberg type integrals. We regard a multiple integral representation of a hypergeometric function as an iteration of integration, that is, a single integral representation. We regard the single integral representation as a pairing of cycles and cocycles with a locally constant sheaves of which rank is more than one. For example, consider the following two integral representations.

$$\int_{0\le s\le t\le x} t^{\lambda_5}(x-t)^{\lambda_4}s^{\lambda_3}(t-s)^{\lambda_2}(1-s)^{\lambda_1}dsdt$$
$$= \int_0^x t^{\lambda_5}(x-t)^{\lambda_4}f(s)dt, \qquad f(s)=\int_0^t s^{\lambda_3}(t-s)^{\lambda_2}(1-s)^{\lambda_1}ds.$$

We identify the integral of the left hand side with a pairing of cycles and cocycles with a locally constant sheaf of rank 1 generated by $t^{\lambda_5}(x-t)^{\lambda_4}s^{\lambda_3}(t-s)^{\lambda_2}(1-s)^{\lambda_1}$ and the integral of the right hand side with a pairing of cycles and cocycles with a locally constant sheaf of rank 2 generated by $t^{\lambda_5}(x-t)^{\lambda_4}f(s)$.

We remark that an integral whose kernel is written by the Gauss hypergeometric function was treated in Takano [24] to compute the monodromy group of Appell's F_4. We also note that n-folds integrals of Selberg type have single integral representations whose kernels are written by $(n-1)$-folds integrals of Selberg type. K. Aomoto utilized this fact to study Gauss-Manin connections of Selberg type integrals ([1]). These are successful cases of an application of iterated integral representations and we are going to give more general and systematic discussions to study monodromy groups and intersection matrices.

The chapter 1 and 3 are based on [20]. In chapter 1, we explain a theory of twisted (co)homology groups with a locally constant sheaf of which rank is more than 1. A noteworthy fact is that a certain quadratic form give a duality of locally constant sheaves. We give a definition of intersection numbers for twisted (co)cycles with a locally constant sheaf by utilizing the quadratic form. By the definition, two intersection numbers defined by multiple integral representations and intersection numbers defined by single integral representations agree.

The chapter 2 is based on [18]. In chapter 2, we construct an explicit basis of twisted homology groups $H_1(T, \operatorname{Ker}\nabla^*)$ on a 1-dimensional space $T=\mathbb{C}\setminus\{x_1,\dots,x_n\}$ and, as an application, we compute monodromy matrices of hypergeometric functions ${}_pF_{p-1}$.

In chapter 3, we explain a method to evaluate intersection numbers of twisted cycles for hypergeometric functions ${}_3F_2$ and for 2-dimensional Selberg-type integrals. The intersection matrix for ${}_2F_1$ or the 1-dimensional Selberg-type integral determines a quadratic form which gives a dual pairing of locally constant sheaves associated with these functions, which yields ${}_3F_2$ or the 2-dimensional Selberg-type integral. Therefore, we can evaluate intersection numbers for ${}_3F_2$ or 2-dimensional Selberg-type integral by utilizing the intersection matrix for ${}_2F_1$ or the 1-dimensional Selberg-type integral.

The chapter 4 is based on [18]. In chapter 4, we study intersection numbers of a twisted cohomology group $H^1(T, \operatorname{Ker}\nabla_\pm)$ for a Selberg-type arrangement and get a recursion formula of intersection numbers on the dimension n, which is derived by a recursive structure of intersection matrices and dual pairings among locally constant sheaves of which rank is more than one. Using a computer algebra system, we can also get explicit intersection numbers for small dimensions.

In appendix, we discuss an algorithm and an implementation for a computation of βnbc basis defined by M. J. Falk and H. Terao ([7]). The βnbc basis is a basis of a twisted cohomology group for a degenerate arrangement. The implementation was useful for studying examples in this paper.

Chapter 1

Theory of twisted (co)homology groups

1.1 The dual of a locally constant sheaf

Let T be a complex manifold of dimension r. For a given locally constant sheaf V of rank m on T, the dual of V is defined by $\mathcal{H}\mathrm{om}_{\mathbb{C}}(V, \mathbb{C})$. The dual is again a locally constant sheaf of rank m. We are given a locally constant sheaf as a kernel sheaf of a connection. In order to compute the intersection number for cycles appearing in the special function theory, we need an explicit presentation of the dual locally constant sheaf as a kernel sheaf of a connection. Let us see examples of the explicit presentation.

Example 1.1.1. Let $\nabla = d + \Omega$ be an integrable connection ($\nabla \circ \nabla = 0$) on the trivial vector bundle $\mathcal{O} \otimes \mathbb{C}^m$ (free $\mathcal{O}$-module of rank m) on T and $\nabla^* = d - {}^t\Omega$ be the adjoint integrable connection on the trivial vector bundle. Here, Ω is a matrix valued holomorphic 1-form on T. Let us denote by $\mathrm{Ker}\,\nabla$ the associated sheaf on T to the presheaf

$$\begin{aligned} \mathrm{Ker}\,\nabla(U) &= \{f \in \mathcal{O}(U) \otimes \mathbb{C}^m \mid \nabla f = 0\}, \\ \rho_{UW} &: \mathrm{Ker}\,\nabla(U) \ni f \mapsto f \in \mathrm{Ker}\,\nabla(W) \end{aligned}$$

which we call the kernel sheaf of ∇. For $f \in \mathrm{Ker}\,\nabla(U)$ and $g \in \mathrm{Ker}\,\nabla^*(U)$ where U is a domain in T, we have

$$\begin{aligned} d(f, g) &= (df, g) + (f, dg) \\ &= -(\Omega f, g) + (f, {}^t\Omega g) \\ &= -(f, {}^t\Omega g) + (f, {}^t\Omega g) = 0 \end{aligned}$$

where $(x, y) = \sum_{i=1}^{m} x_i y_i,\ x, y \in \mathbb{C}^m$. The inner product (f, g) is constant, then $(\cdot, g)$ defines an element of $\mathrm{Hom}_{\mathbb{C}}(\mathrm{Ker}\,\nabla(U), \mathbb{C})$, i.e.,

$$(\cdot, \cdot)\ :\ \mathrm{Ker}\,\nabla \times \mathrm{Ker}\,\nabla^* \longrightarrow \mathbb{C}$$

is the sheaf homomorphism. We can prove that the quadratic form defines the sheaf isomorphism between the dual $\mathcal{H}\mathrm{om}_{\mathbb{C}}(\mathrm{Ker}\,\nabla, \mathbb{C})$ and the locally constant sheaf $\mathrm{Ker}\,\nabla^*$.

Example 1.1.2 (twisted Riemann's period relation [3, Sections 3 and 4]). In our point of view, the twisted Riemann's period relation ([3]) is nothing but an explicit expression of the pairing of two sheaves which are dual each other. Let us figure out the pairing of two sheaves by an example.

Set

$$
\begin{aligned}
u &= t^{\alpha}(1-t)^{\gamma-\alpha}(1-xt)^{-\beta}, \quad \alpha, \gamma-\alpha, \beta, \gamma-\beta \notin \mathbb{Z} \\
\omega_1 &= \frac{dt}{t} - \frac{dt}{t-1} = \frac{dt}{t(1-t)}, \\
\omega_2 &= -\left(\frac{dt}{t-1} - \frac{dt}{t-\frac{1}{x}}\right) = \frac{1-x}{(1-t)(1-xt)}dt.
\end{aligned}
$$

Note $\frac{1}{\beta}(1-x)\frac{d}{dx}(u\omega_1) = u\omega_2$, $\frac{1}{-\beta}(1-x)\frac{d}{dx}(u^{-1}\omega_1) = u^{-1}\omega_2$ and $\frac{\partial u}{\partial t} = \left(\frac{\alpha}{t} + \frac{\gamma-\alpha}{1-t} + \frac{\beta}{\frac{1}{x}-t}\right) u$. Assume $1 < 1/x$ and define cycles $\gamma_1 = [0,1]$, $\gamma_2 = [1, 1/x]$, $\gamma_3 = [1/x, +\infty]$ and period matrices

$$
P_+ = \begin{pmatrix} \int_{\gamma_2} u\omega_1 & \int_{\gamma_3} u\omega_1 \\ \int_{\gamma_2} u\omega_2 & \int_{\gamma_3} u\omega_2 \end{pmatrix},
$$

$$
P_- = \begin{pmatrix} \int_{\gamma_2} u^{-1}\omega_1 & \int_{\gamma_3} u^{-1}\omega_1 \\ \int_{\gamma_2} u^{-1}\omega_2 & \int_{\gamma_3} u^{-1}\omega_2 \end{pmatrix}.
$$

Since the integral $\int_{\gamma_i} u^{\pm}\omega_1$ satisfies the Gauss hypergeometric equation $[\theta_x(\theta_x \pm \gamma - 1) - x(\theta_x \pm \alpha)(\theta_x \pm \beta)]f = 0$, $\theta_x = x\frac{d}{dx}$, we have $\nabla_+ P_+ = 0$ and $\nabla_- P_- = 0$ where $\nabla_+ = d - \Omega$, $\nabla_- = d + \Omega$ and

$$
\Omega = \begin{pmatrix} 0 & 0 \\ \alpha & -\gamma \end{pmatrix} \frac{dx}{x} + \begin{pmatrix} 0 & \beta \\ 0 & \alpha+\beta-\gamma \end{pmatrix} \frac{d(x-1)}{x-1}.
$$

The sheaf $\mathrm{Ker}\,\nabla_-$ is isomorphic to $\mathcal{H}\mathrm{om}_{\mathbb{C}}(\mathrm{Ker}\,\nabla_+, \mathbb{C})$; the isomorphism is given by the quadratic form

$$
S^*(f,g) = \frac{1}{2\pi i}{}^t f\, {}^t I_{ch}^{-1} g, \quad f, g \in \mathbb{C}^2 \tag{1.1}
$$

where I_{ch} is the twisted intersection matrix of cocycles ω_1 and ω_2 ([3, Theorem 1]);

$$
I_{ch} = \begin{pmatrix} \langle \omega_1, \omega_1 \rangle & \langle \omega_1, \omega_2 \rangle \\ \langle \omega_2, \omega_1 \rangle & \langle \omega_2, \omega_2 \rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha} + \frac{1}{\gamma-\alpha} & \frac{1}{\gamma-\alpha} \\ \frac{1}{\gamma-\alpha} & -\frac{1}{\beta} + \frac{1}{\gamma-\alpha} \end{pmatrix}.
$$

The matrix I_{ch} does not depend on x because of our choice of bases of the rational cohomology group. For given $g \in \mathrm{Ker}\,\nabla_-(U)$, $S^*(\cdot, g)$ is an element of $\mathrm{Hom}_{\mathbb{C}}(\mathrm{Ker}\,\nabla_+(U), \mathbb{C})$;

$$
S^* : \mathrm{Ker}\,\nabla_+ \times \mathrm{Ker}\,\nabla_- \longrightarrow \mathbb{C}
$$

is the sheaf homomorphism and non-degenerate. The correspondence gives the isomorphism between $\mathcal{H}\mathrm{om}_{\mathbb{C}}(\mathrm{Ker}\,\nabla_+, \mathbb{C})$ and $\mathrm{Ker}\,\nabla_-$. Surprisingly, the value of S^* can be expressed by the twisted intersection number defined by [12] of the two cycles by virtue of the twisted Riemann's period relation ([3, Theorem 2]):

$$
S^*(p_i^+, p_j^-) = \mathrm{reg}(\gamma_j \otimes u) \cdot (\gamma_i \otimes u^{-1})
$$

where $\mathrm{reg}(\gamma_j \otimes u)$ is the regularization of the twisted cycle $\gamma_j \otimes u$ ([12]) and

$$p_i^+ = \begin{pmatrix} \int_{\gamma_i} u\omega_1 \\ \int_{\gamma_i} u\omega_2 \end{pmatrix} \quad \text{and} \quad p_j^- = \begin{pmatrix} \int_{\gamma_j} u^{-1}\omega_1 \\ \int_{\gamma_j} u^{-1}\omega_2 \end{pmatrix}. \tag{1.2}$$

Note that we have $S^*(\Omega f, g) = S^*(f, \Omega g), \quad f, g \in \mathbb{C}^2$, and ${}^t\Omega I_{ch}^{-1} = I_{ch}^{-1}\Omega,\ I_{ch}{}^t\Omega = \Omega I_{ch}$.

1.2 Definitions of the intersection number of twisted (co)cycles

Consider two integrable connections $\nabla_+ = d + \Omega_+$ and $\nabla_- = d - \Omega_-$ on the trivial vector bundle $\mathcal{O} \otimes \mathbb{C}^m$ on T. We assume that $\mathrm{Ker}\,\nabla_-$ is isomorphic to the dual $\mathcal{H}\mathrm{om}_{\mathbb{C}}(\mathrm{Ker}\,\nabla_+, \mathbb{C})$ of $\mathrm{Ker}\,\nabla_+$ and the isomorphism is given by the $\mathcal{O}$-bilinear extension of a non-degenerate bilinear form

$$S\ :\ \mathbb{C}^m \times \mathbb{C}^m \longrightarrow \mathbb{C}.$$

The extension is also denoted by S and the $m \times m$ matrix that defines S is also denoted by S.

For $f \in \mathrm{Ker}\,\nabla_+(U)$ and $g \in \mathrm{Ker}\,\nabla_-(U)$, we have

$$\begin{aligned} dS(f,g) &= S(df,g) + S(f,dg) \\ &= -S(\Omega_+ f, g) + S(f, \Omega_- g) = 0. \end{aligned}$$

Then, for any $f, g \in \mathcal{O} \otimes \mathbb{C}^m$, we have

$$S(\Omega_+ f, g) = S(f, \Omega_- g), \tag{1.3}$$

which means

$${}^t\Omega_+ S - S\Omega_- = 0. \tag{1.4}$$

Let $\mathcal{E}^p$ be the sheaf of smooth p-forms on the real differentiable manifold T of $2r$ dimension. The bilinear form S can be extended to the sheaf homomorphism

$$S\ :\ (\mathcal{E}^p \otimes \mathbb{C}^m) \times (\mathcal{E}^q \otimes \mathbb{C}^m) \longrightarrow \mathcal{E}^{p+q}$$

and satisfies the relation

$$S(\Omega_+ f, g) = (-1)^p S(f, \Omega_- g)$$

for $f \in \mathcal{E}^p \otimes \mathbb{C}^m$ and $g \in \mathcal{E}^q \otimes \mathbb{C}^m$.

The next lemma is the key lemma to define intersection numbers properly.

Lemma 1.2.1. *If $\mathrm{Ker}\,\nabla_+$ and $\mathrm{Ker}\,\nabla_-$ are dual by the bilinear form S, then $\mathrm{Ker}\,\nabla_+^*$ and $\mathrm{Ker}\,\nabla_-^*$ are dual by the bilinear form S^* that is defined by the matrix ${}^tS^{-1}$;*

$$\begin{array}{ccc} \mathrm{Ker}\,\nabla_+ & \overset{S}{\longleftrightarrow} & \mathrm{Ker}\,\nabla_- \\ E \updownarrow & & E \updownarrow \\ \mathrm{Ker}\,\nabla_+^* & \overset{{}^tS^{-1}}{\longleftrightarrow} & \mathrm{Ker}\,\nabla_-^* \end{array}$$

where E is the identity matrix and $M \overset{B}{\longleftrightarrow} N$ means that the matrix B gives a non-degenerate bilinear form $M \times N \overset{B}{\longrightarrow} \mathbb{C}$ and $\nabla_\pm^*$ is the adjoint connection of $\nabla_\pm$.

Proof. Take

$$f \in (\operatorname{Ker} \nabla_+^*)(U) = \{f \in \mathcal{O}(U)^m \,|\, (d - {}^t\Omega_+)f = 0\}$$

and

$$g \in (\operatorname{Ker} \nabla_-^*)(U) = \{g \in \mathcal{O}(U)^m \,|\, (d + {}^t\Omega_-)g = 0\}.$$

We have

$$\begin{aligned} & d({}^t f {}^t S^{-1} g) \\ =& {}^t(df){}^t S^{-1} g + {}^t f {}^t S^{-1} dg \\ =& {}^t f(\Omega_+ {}^t S^{-1} - {}^t S^{-1t}\Omega_-)g. \end{aligned}$$

Since $S^t(\Omega_+ {}^t S^{-1} - {}^t S^{-1t}\Omega_-)S = {}^t\Omega_+ S - S\Omega_- = 0$, we have $d({}^t f {}^t S^{-1} g) = 0$. □

Let us quickly review the cohomology theory for locally constant sheaves (see e.g. [12]).

We take fine resolutions of $\operatorname{Ker} \nabla_\pm$:

$$\begin{array}{cccccccccc} 0 & \longrightarrow & \operatorname{Ker} \nabla_\pm & \longrightarrow & \mathcal{E}^0 \otimes \mathbb{C}^m & \xrightarrow{\nabla_\pm} & \cdots & \xrightarrow{\nabla_\pm} & \mathcal{E}^{2r} \otimes \mathbb{C}^m & \longrightarrow & 0 \\ 0 & \longrightarrow & \operatorname{Ker} \nabla_\pm & \longrightarrow & \mathcal{D}^0 \otimes \mathbb{C}^m & \xrightarrow{\nabla_\pm} & \cdots & \xrightarrow{\nabla_\pm} & \mathcal{D}^{2r} \otimes \mathbb{C}^m & \longrightarrow & 0, \end{array}$$

where $\mathcal{E}^p$ is the sheaf of smooth p-forms on the real $2r$-dimensional manifold T and $\mathcal{D}^p$ is the sheaf of currents of degree p on T.

Then we have

$$\begin{aligned} H^p(T, \operatorname{Ker} \nabla_\pm) &\simeq \frac{\operatorname{Ker}(\Gamma(T, \mathcal{E}^p \otimes \mathbb{C}^m) \xrightarrow{\nabla_\pm} \Gamma(T, \mathcal{E}^{p+1} \otimes \mathbb{C}^m))}{\nabla_\pm \Gamma(T, \mathcal{E}^{p-1} \otimes \mathbb{C}^m)} \\ &\simeq \frac{\operatorname{Ker}(\Gamma(T, \mathcal{D}^p \otimes \mathbb{C}^m) \xrightarrow{\nabla_\pm} \Gamma(T, \mathcal{D}^{p+1} \otimes \mathbb{C}^m))}{\nabla_\pm \Gamma(T, \mathcal{D}^{p-1} \otimes \mathbb{C}^m)}. \end{aligned}$$

The cohomology with compact support $H_c^p(T, \operatorname{Ker} \nabla_\pm)$ is isomorphic to

$$\begin{aligned} & \frac{\operatorname{Ker}(\Gamma(T, \mathcal{E}_c^p \otimes \mathbb{C}^m) \xrightarrow{\nabla_\pm} \Gamma(T, \mathcal{E}_c^{p+1} \otimes \mathbb{C}^m))}{\nabla_\pm \Gamma(T, \mathcal{E}_c^{p-1} \otimes \mathbb{C}^m)} \\ \simeq & \frac{\operatorname{Ker}(\Gamma(T, \mathcal{D}_c^p \otimes \mathbb{C}^m) \xrightarrow{\nabla_\pm} \Gamma(T, \mathcal{D}_c^{p+1} \otimes \mathbb{C}^m))}{\nabla_\pm \Gamma(T, \mathcal{D}_c^{p-1} \otimes \mathbb{C}^m)} \end{aligned}$$

where $\mathcal{E}_c^p$ is the sheaf of smooth p-forms with compact supports on T and $\mathcal{D}_c^p$ is the sheaf of currents of degree p with compact supports on T. The natural inclusion $\Gamma(T, \mathcal{E}_c^p \otimes \mathbb{C}^m) \longrightarrow \Gamma(T, \mathcal{E}^p \otimes \mathbb{C}^m)$ induces a morphism

$$H_c^p(T, \operatorname{Ker} \nabla_\pm) \longrightarrow H^p(T, \operatorname{Ker} \nabla_\pm).$$

These two cohomology groups agree under conditions: for example, suppose that the complex manifold T is a submanifold of a complex manifold $\bar{T}$ of dimension r and $D = \bar{T} \setminus T$ is a normal crossing divisor of $\bar{T}$. If the connection $\nabla_{\pm}$ has logarithmic poles along D and the monodromy along each component of D does not have unit eigenvalue, then we have

$$H^p(T, \mathrm{Ker}\,\nabla_{\pm}) \simeq H^p_c(T, \mathrm{Ker}\,\nabla_{\pm})$$

([6, (1.6)]).

It follows from the existence of two fine resolutions of $\mathrm{Ker}\,\nabla_{\pm}$ that the two expressions of r-dimensional cohomology groups

$$H^r(\Gamma(T, \mathcal{E}^{\cdot} \otimes \mathbb{C}^m), \nabla_{\pm}) \text{ and } H^r(\Gamma(T, \mathcal{D}^{\cdot} \otimes \mathbb{C}^m), \nabla_{\pm})$$

are isomorphic (resp. cohomology groups expressed by the global sections of $\mathcal{E}^{\cdot}_c$ and $\mathcal{D}^{\cdot}_c$). Hence, any cocycle ξ in $\Gamma(T, \mathcal{D}^r \otimes \mathbb{C}^m)$ can be deformed to a smooth cocycle $\xi' \in \Gamma(T, \mathcal{E}^r \otimes \mathbb{C}^m)$ modulo $\nabla_{\pm}\Gamma(T, \mathcal{D}^{r-1} \otimes \mathbb{C}^m)$. We call ξ' the regularization of ξ and denote it by $\mathrm{reg}(\xi)$.

We define intersection numbers among elements of $H^r_c(T, \mathrm{Ker}\,\nabla_+)$ and $H^r(T, \mathrm{Ker}\,\nabla_-)$.

Definition 1.2.1 (cf. [12], [9, p.224]). For $[\xi] \in H^r(\Gamma(T, \mathcal{E}^{\cdot}_c \otimes \mathbb{C}^m), \nabla_+)$ and $[\eta] \in H^r(\Gamma(T, \mathcal{E}^{\cdot} \otimes \mathbb{C}^m), \nabla_-)$, the integral

$$[\xi] \cdot [\eta] := \int_T S(\xi, \eta)$$

is called *the intersection number* of the cocycles $[\xi]$ and $[\eta]$.

The intersection numbers depend only on cohomology classes. See [12, 1.5] for the detail.

Let us define twisted homology groups and intersection numbers among cycles. Once the duality between $\mathrm{Ker}\,\nabla^*_+$ and $\mathrm{Ker}\,\nabla^*_-$ is given explicitly by $S^* = {}^tS^{-1}$ as in Lemma 1.2.1, we may just follow [12] to define the intersection number.

Let Δ be a p-dimensional oriented smooth simplex in T. We denote by $C_p(T, \mathrm{Ker}\,\nabla^*_-)$ the space of formal finite sums of $\Delta \otimes u^-_\Delta$ where $u^-_\Delta \in \lim_{\Delta \subset U}(\mathrm{Ker}\,\nabla^*_-)(U)$ and by $C^{lf}_p(T, \mathrm{Ker}\,\nabla^*_+)$ the space of formal locally finite sums of $\Delta \otimes u^*_\Delta$ where $u^+_\Delta \in \lim_{\Delta \subset U}(\mathrm{Ker}\,\nabla^*_+)(U)$. The pairing between a p-chain and a vector valued p-form φ is given by the linear extension of the following pairing between $\Delta \otimes u^+_\Delta$ and φ :

$$(\varphi, \Delta \otimes u^+_\Delta) = \int_\Delta (u^+_\Delta, \varphi).$$

Here $(\cdot,\ \cdot)$ is the standard inner product.

We define the boundary operator $\partial_{\nabla^*_\pm}$ by the $\mathbb{C}$-linear extension of the boundary operator

$$\partial_{\nabla^*_\pm}\left(\Delta \otimes u^\pm_\Delta\right) = (\partial\Delta) \otimes (u^\pm_\Delta)_{|\partial\Delta}.$$

Then, we have the twisted Stokes theorem $(\nabla_\pm \varphi, \sigma) = (\varphi, \partial_{\nabla^*_\pm}\sigma)$ for $\sigma \in C_p(T, \mathrm{Ker}\,\nabla^*_\pm)$ (resp. $\sigma \in C^{lf}_p(T, \mathrm{Ker}\,\nabla^*_\pm)$) and $\varphi \in \mathcal{E}^{p-1} \otimes \mathbb{C}^m$ (resp. $\varphi \in \mathcal{E}^{p-1}_c \otimes \mathbb{C}^m$).

The homology groups of the complexes $C_{\cdot}(T, \operatorname{Ker}\nabla^*_{\pm})$ and $C^{lf}_{\cdot}(T, \operatorname{Ker}\nabla^*_{\pm})$ are denoted by $H_{\cdot}(T, \operatorname{Ker}\nabla^*_{\pm})$ and $H^{lf}_{\cdot}(T, \operatorname{Ker}\nabla^*_{\pm})$.

We can regard twisted cycles as a current. In fact, for $\sigma \in C^{lf}_r(T, \operatorname{Ker}\nabla^*_+)$, the functional

$$F_\sigma \ : \ \varphi \longmapsto (\varphi, \sigma), \quad \varphi \in \Gamma(T, \mathcal{E}^r_c) \otimes \mathbb{C}^m$$

defines a vector valued current of degree r. We denote by $\langle F_\sigma, \varphi\rangle$ the evaluation by φ. For $\psi \in \mathcal{E}^{r-1}_c \otimes \mathbb{C}^m$, we have

$$\begin{aligned}
\langle \nabla^*_+ F_\sigma, \psi\rangle &= \langle dF_\sigma - {}^t\Omega_+ F_\sigma, \psi\rangle \\
&= (-1)^{r+1}\langle F_\sigma, d\psi\rangle - (-1)^r\langle F_\sigma, \Omega_+\psi\rangle \\
&= (-1)^{r+1}(\nabla_+\psi, \sigma) \\
&= (-1)^{r+1}(\partial_{\nabla^*_+}\sigma, \psi) = 0.
\end{aligned}$$

Therefore, we have a morphism

$$H^{lf}_r(T, \operatorname{Ker}\nabla^*_+) \ni [\sigma] \longmapsto [F_\sigma] \in H^r(\Gamma(T, \mathcal{D}^{\cdot} \otimes \mathbb{C}^m), \nabla^*_+) = H^r(T, \nabla^*_+).$$

Similarly, we have

$$H_r(T, \operatorname{Ker}\nabla^*_-) \ni [\tau] \longmapsto [F_\tau] \in H^r_c(\Gamma(T, \mathcal{D}^{\cdot}_c \otimes \mathbb{C}^m), \nabla^*_-) = H^r_c(T, \nabla^*_-).$$

Let us take a cycle

$$\gamma \in C^{lf}_r(T, \operatorname{Ker}\nabla^*_+)$$

and a cocycle

$$\varphi \in \operatorname{Ker}(\nabla_+ \ : \ \Gamma(T, \mathcal{E}^r_c \otimes \mathbb{C}^m) \to \Gamma(T, \mathcal{E}^{r+1}_c \otimes \mathbb{C}^m)),$$

the pairing

$$(\varphi, \gamma) = \sum_\Delta \int_\Delta (u^+_\Delta, \varphi), \quad \gamma = \sum \Delta \otimes u^+_\Delta, \ u^+_\Delta \in \operatorname{Ker}\nabla^*_+(\Delta)$$

is called *the hypergeometric function*. By virtue of the twisted Stokes theorem, the value of the pairing depends only on the cohomology and homology classes.

If we regard γ as a current F_γ, then the hypergeometric function (φ, γ) can be regarded as the pairing of elements of $\operatorname{Ker}\nabla_+$ and $\operatorname{Ker}\nabla^*_+$; for

$$\varphi \in \operatorname{Ker}(\nabla_+ \ : \ \Gamma(T, \mathcal{E}^r_c \otimes \mathbb{C}^m) \to \Gamma(T, \mathcal{E}^{r+1}_c \otimes \mathbb{C}^m))$$

and

$$F_\gamma \in \operatorname{Ker}(\nabla^*_+ \ : \ \Gamma(T, \mathcal{D}^r \otimes \mathbb{C}^m) \to \Gamma(T, \mathcal{D}^{r+1}_c \otimes \mathbb{C}^m)),$$

(φ, γ) is $\langle F_\gamma, \varphi\rangle$.

Example 1.2.1. Let us consider the function

$$I(\alpha, \beta, \gamma) = \int_{s\geq 0, t\geq 0, s+t\leq 1} s^\alpha t^\beta (1-s-t)^\gamma \frac{dsdt}{st(1-s-t)}.$$

The integral can be written as an iteration of single integrals;

$$I(\alpha,\beta,\gamma)=\int_0^1 u^+(\alpha,\beta,\gamma;t)\frac{dt}{t(1-t)},$$

where

$$\begin{aligned} u^+(t) &= t^\beta\int_0^{1-t} s^\alpha(1-s-t)^\gamma\frac{(1-t)ds}{s(1-s-t)} \\ &= t^\beta(1-t)^{\alpha+\gamma}\int_0^1 \xi^\alpha(1-\xi)^\gamma\frac{d\xi}{\xi(1-\xi)}. \end{aligned}$$

The function $u^+(t)$ satisfies $\nabla_+^* u^+=0$ where

$$\nabla_+^*=d-\left(\frac{\beta}{t}-\frac{\alpha+\gamma}{1-t}\right)dt.$$

We regard $\frac{dt}{(1-t)t}$ as an element of $H^1(\mathbb{C}\setminus\{0,1\},\nabla_+)$ and we have

$$I(\alpha,\beta,\gamma)=\left(\frac{dt}{t(1-t)},\mathrm{reg}([0,1]\otimes u^+(t))\right).$$

Definition 1.2.2. For $[\sigma]\in H_r^{lf}(T,\mathrm{Ker}\,\nabla_+^*)$ and $[\tau]\in H_r(T,\mathrm{Ker}\,\nabla_-^*)$,

$$[\sigma]\cdot[\tau]:=[\mathrm{reg}(F_\sigma)]\cdot[\mathrm{reg}(F_\tau)]=\int_T S^*(\mathrm{reg}(F_\sigma),\mathrm{reg}(F_\tau))$$

is called *the intersection number of the cycles* $[\sigma]$ and $[\tau]$.

When $H_r^{lf}(T,\mathrm{Ker}\,\nabla_+^*)\simeq H_r(T,\mathrm{Ker}\,\nabla_+^*)$ and $H_r^{lf}(T,\mathrm{Ker}\,\nabla_-^*)\simeq H_r(T,\mathrm{Ker}\,\nabla_-^*)$ hold, we denote by "reg" the isomorphism from the locally finite twisted homology group to the twisted homology group. Under this assumption, we can define intersection numbers on $H_r^{lf}(T,\mathrm{Ker}\,\nabla_+^*)\times H_r^{lf}(T,\mathrm{Ker}\,\nabla_-^*)$, $H_r(T,\mathrm{Ker}\,\nabla_+^*)\times H_r^{lf}(T,\mathrm{Ker}\,\nabla_-^*)$ and $H_r(T,\mathrm{Ker}\,\nabla_+^*)\times H_r(T,\mathrm{Ker}\,\nabla_-^*)$ through the isomorphism reg.

Let $\varphi_\mu^\pm$ and $\gamma_\mu^\pm$ be bases of $H^r(T,\mathrm{Ker}\,\nabla_\pm)$ and $H_r(T,\mathrm{Ker}\,\nabla_\pm^*)$ respectively. We assume that $H^r(T,\mathrm{Ker}\,\nabla_\pm)\simeq H_c^r(T,\mathrm{Ker}\,\nabla_\pm)$ and $H_r(T,\mathrm{Ker}\,\nabla_\pm^*)\simeq H_r^{lf}(T,\mathrm{Ker}\,\nabla_\pm^*)$. We also assume that the dimension of these homology and cohomology groups as $\mathbb{C}$-vector spaces is s. We define four $s\times s$-matrices

$$P_+=(\varphi_\mu^+,\gamma_\nu^+)_{\mu\nu},\ P_-=(\varphi_\mu^-,\gamma_\nu^-)_{\mu\nu},\ I_{ch}=([\varphi_\mu^+]\cdot[\varphi_\nu^-])_{\mu\nu},\ I_h=([\gamma_\mu^+]\cdot[\gamma_\nu^-])_{\mu\nu}.$$

If these matrices is well-defined, then we have the following twisted period relation under our definition of intersection numbers.

Theorem 1.2.1 ([3]).

$$I_h={}^tP_+{}^tI_{ch}^{-1}P_-.$$

We can prove this theorem in a similar way of the proof given by [3].

Chapter 2

Monodromy of hypergeometric function ${}_pF_{p-1}$

2.1 Construction of bases of a twisted homology group

In this section we consider a construction of bases of a twisted homology group $H_1(T, \operatorname{Ker} \nabla^*)$.

Let $p_1, \ldots, p_n$ be linear forms in $\mathbb{C}^\ell$ and D a hyperplane arrangement defined by $\prod_{i=1}^n p_i = 0$ in $\mathbb{C}^\ell$. We suppose that the hyperplane arrangement D is generic. Then, for a derivation $\nabla : \tau \mapsto d\tau - \sum_{i=1}^n \alpha_i (dp_i/p_i) \wedge \tau$, the dimension of $H_n(\mathbb{C}^\ell - D, \operatorname{Ker} \nabla)$ is equal to $\binom{n-1}{\ell}$ under a certain condition on the parameters $\alpha_1, \ldots, \alpha_n$ (see e.g. M. Kita [10] Theorem 2). We prove a similar proposition to this theorem later when the rank is more than one and the dimension is equal to one.

Now let x_i $(1 \le i \le n)$ be distinct n points in $\mathbb{C}$ and set $T = \mathbb{C} \setminus \{x_1, \ldots, x_n\}$. We denote by $S^* f$ the analytic continuation of a section f of $\operatorname{Ker} \nabla^*$ along any loop S inside T.

Theorem 2.1.1. *Let S_i be a loop around the point x_i for each i. If there exists S_{i_0} such that $\det(S_{i_0}^* - \mathrm{id}) \neq 0$, then $\dim_{\mathbb{C}} H_1(T, \operatorname{Ker} \nabla^*) = m(n-1)$.*

Proof. Without loss of generality, we can assume that $\det(S_1^* - \mathrm{id}) \neq 0$. Let P_i be the base point of the loop S_i and $\overline{P_1 P_i}$ the oriented segment from P_1 to P_i in T. We define the triangulation K of $\tilde{T} = (\bigcup_{i=1}^n S_i) \cup \left(\bigcup_{i=2}^n \overline{P_1 P_i}\right)$ by choosing two consecutive points Q_i and R_i for each S_i as in the following figure and by regarding S_i as the sum of three simplices $\overline{P_i Q_i}$, $\overline{Q_i R_i}$, and $\overline{R_i P_i}$. Since $\tilde{T}$ is a deformation retract of T, $H_1(\tilde{T}, \operatorname{Ker} \nabla^*) \simeq H_1(T, \operatorname{Ker} \nabla^*)$. Let U be a simply connected open set in T which contains $\bigcup_{i=2}^n \overline{P_1 P_i} \cup \bigcup_{i=1}^n \overline{P_i Q_i}$. Any bounded cycle σ of $C_1(K, \operatorname{Ker} \nabla^*)$ is written as

$$\sigma = \sum_{i=1}^n (\overline{P_i Q_i} \otimes v_i + \overline{Q_i R_i} \otimes v_i' + \overline{R_i P_i} \otimes v_i'') + \sum_{i=2}^n \overline{P_1 P_i} \otimes w_i,$$

where $\{w_i, v_i\}$ is a set of sections of $\operatorname{Ker} \nabla^*$ on U, $\{v_i'\}$ is a set of sections on a neighborhood U_i' of the path $\overline{P_i Q_i}$, and $\{v_i''\}$ is a set of sections on a neighborhood U_i'' of the path $\overline{Q_iR_i}$. Since

$$\begin{aligned}\partial_\nabla^*\sigma &= \sum_{i=1}^n (Q_i \otimes (v_i - v_i') + R_i \otimes (v_i' - v_i''))\\ &\qquad + \sum_{i=2}^n P_i \otimes (v_i'' - v_i + w_i) + P_1 \otimes (v_1'' - v_1 - \sum_{i=2}^n w_i)\\ &= 0,\end{aligned}$$

v_i' is the analytic continuation of v_i along the path $\overline{P_iQ_i}$ starting from P_i and v_i'' is the analytic continuation of v_i' along the path $\overline{Q_iR_i}$ starting from Q_i; that is, $v_i'' = S_i^* v_i$. Moreover, it holds that $w_i = -(S_i^* - \mathrm{id})v_i$ for each $i = 2, \dots, n$ and that

$$v_1 = (S_1^* - \mathrm{id})^{-1}\left(\sum_{i=2}^n w_i\right)$$

by the assumption of the theorem. Thus, any bounded cycle c depends only on a set $\{v_2, \dots, v_n\}$. Since $\dim_{\mathbb{C}} \Gamma(U, \mathrm{Ker}\,\nabla^*) = m$, it holds that $\dim_{\mathbb{C}} H_1(K, \mathrm{Ker}\,\nabla^*) = m(n-1)$. □

The base given in the proof of Theorem 2.1.1 may contain cycles in the form

$$S_1 \otimes R_1 f + \overline{P_1P_i} \otimes f + S_i \otimes R_2 f \qquad ((S_i^* - \mathrm{id})^n f = 0), \tag{2.1}$$

where R_1 and R_2 are constant matrices and f is a section of $\Gamma(U, \mathrm{Ker}\,\nabla^*)$. We cannot compute monodromy groups in the case that there is cycles in the form (2.1) by using our method explained in latter sections of this paper. So, we add technical conditions to construct a base which contains no cycles in the form (2.1).

Under the assumption of Theorem 2.1.1, we assume the condition that, for each i such that $\det(S_i^* - \mathrm{id}) = 0$, there exists a decomposition $\Gamma(U, \mathrm{Ker}\,\nabla^*) = \Gamma_i' \oplus \Gamma_i''$ so that $\det(S_i^*|_{\Gamma_i'} - \mathrm{id}|_{\Gamma_i'}) \neq 0$ and that $S_i^*|_{\Gamma_i''} = \mathrm{id}|_{\Gamma_i''}$. Now assume that $\det(S_1^* - \mathrm{id}) \neq 0$ as in the proof of Theorem 2.1.1 and construct a set $\{\sigma_{ij}\}$ of the bounded cycles as follows. From the proof of Theorem 2.1.1, any bounded cycle depends only on a set $\{v_2, \dots, v_n\}$. If $\det(S_i^* - \mathrm{id}) \neq 0$, then we take a basis $\{u_{ij} \mid 1 \leq j \leq m\}$ of $\Gamma(U, \mathrm{Ker}\,\nabla^*)$ and define $v_i^{(j)} = -(S_i^* - \mathrm{id})^{-1}u_{ij}$. For each set $\{0, \dots, \overset{i}{\breve{v}_i^{(j)}}, \dots, 0\}$, we define a cycle σ_{ij}, that is,

$$\sigma_{ij} = S_1 \otimes (S_1^* - \mathrm{id})^{-1}u_{ij} + \overline{P_1P_i} \otimes u_{ij} - S_i \otimes (S_i^* - \mathrm{id})^{-1}u_{ij} \quad (1 \leq j \leq m). \tag{2.2}$$

If $\det(S_i^* - \mathrm{id}) = 0$, then let $\{u_{ij} \mid 1 \leq j \leq \dim_{\mathbb{C}} \Gamma_i'\}$ be a basis of Γ_i' and $\{u_{ij} \mid \dim_{\mathbb{C}} \Gamma_i' < j \leq m\}$ a basis of Γ_i''. We define

$$v^{(j)} = \begin{cases} -(S_i^*|_{\Gamma_i'} - \mathrm{id}|_{\Gamma_i'})^{-1}u_{ij} & (1 \leq j \leq \dim_{\mathbb{C}} \Gamma_i'),\\ u_{ij} & (\dim_{\mathbb{C}} \Gamma_i' < j \leq m).\end{cases}$$

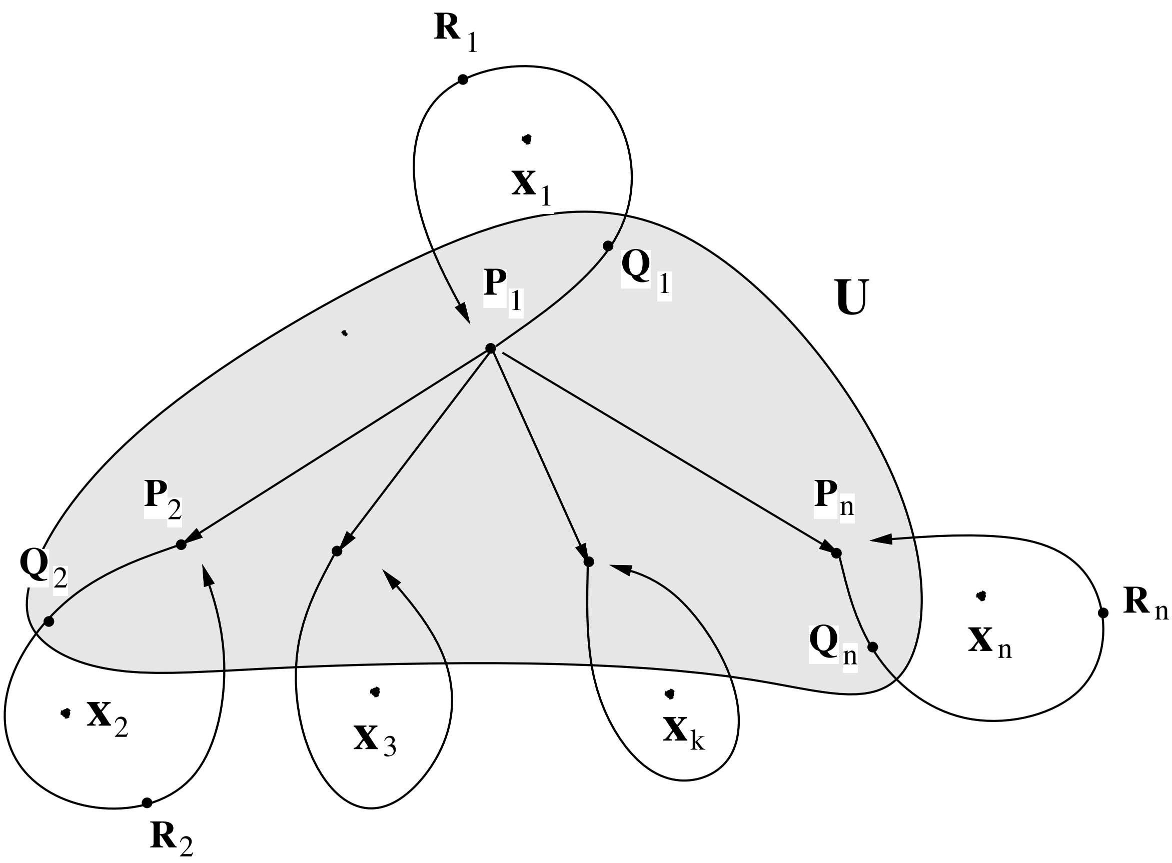


Figure 2.1: $\tilde{T}$ and U

Then, for each set $\{0,\dots,v_i^{(j)},\dots,0\}$, we define a cycle σ_{ij}, that is,

$$\begin{aligned}\sigma_{ij} &= S_1 \otimes (S_1^* - \mathrm{id})^{-1}u_{ij} + \overline{P_1P_i} \otimes u_{ij} && (2.3)\\ &\qquad - S_i \otimes (S_i^*|_{\Gamma_i'} - \mathrm{id}\,|_{\Gamma_i'})^{-1}u_{ij} \qquad (1 \le j \le \dim_{\mathbb{C}} \Gamma_i'), \\ \sigma_{ij} &= S_i \otimes u_{ij} \qquad (\dim_{\mathbb{C}} \Gamma_i' < j \le m). && (2.4)\end{aligned}$$

From the proof of Theorem 2.1.1, clearly, the set $\{\sigma_{ij} \mid 2 \le i \le n,\ 1 \le j \le m\}$ is a basis of $H_1(T, \mathrm{Ker}\,\nabla^*)$. Thus, we have the following theorem.

Theorem 2.1.2. *Suppose that* $\det(S_1^* - \mathrm{id}) \neq 0$ *and that, if* $\det(S_i^* - \mathrm{id}) = 0$ *for some* i, *then there exists a decomposition* $\Gamma(U, \mathrm{Ker}\,\nabla^*) = \Gamma_i' \oplus \Gamma_i''$ *so that* $\det(S_i^*|_{\Gamma_i'} - \mathrm{id}\,|_{\Gamma_i'}) \neq 0$ *and that* $S_i^*|_{\Gamma_i''} = \mathrm{id}\,|_{\Gamma_i''}$. *Then* $\{\sigma_{ij} \mid 2 \le i \le n,\ 1 \le j \le m\}$ *is a basis of* $H_1(T, \mathrm{Ker}\,\nabla^*)$.

We give a few remarks. First, due to Theorem 2.1.2, the homology group $H_1(T, \mathrm{Ker}\,\nabla^*)$ has a natural decomposition. Let $\mathrm{Pr}(H_1(T, \mathrm{Ker}\,\nabla^*))$ be the subspace of $H_1(T, \mathrm{Ker}\,\nabla^*)$ spanned by σ_{ij} of (2.2) and (2.3). And let $\mathrm{Deg}(H_1(T, \mathrm{Ker}\,\nabla^*))$ be the subspace spanned by σ_{ij} of (2.4). Then $H_1(T, \mathrm{Ker}\,\nabla^*) = \mathrm{Pr}(H_1(T, \mathrm{Ker}\,\nabla^*)) \oplus \mathrm{Deg}(H_1(T, \mathrm{Ker}\,\nabla^*))$. We call $\mathrm{Pr}(H_1(T, \mathrm{Ker}\,\nabla^*))$ *the primary part* and $\mathrm{Deg}(H_1(T, \mathrm{Ker}\,\nabla^*))$ *the degenerate part*. We explain implicit meaning of this decomposition in Theorem 3.2.1.

Second, we recall the notion of regularization. Let (x_1, x_j) be the oriented polygonal arc $\overline{x_1P_1P_ix_j}$ which includes neither end points. The analytic continuation of $u \in \Gamma(U, \mathrm{Ker}\,\nabla^*)$ starting up from P_1 along $\overline{P_1P_ix_j}$ and along $\overline{P_1x_1}$ are also sections of $\mathrm{Ker}\,\nabla^*$. We write those by u again. Under Theorem 2.1.2, we have a linear map "reg" from $H_1^{lf}(T, \mathrm{Ker}\,\nabla^*)$ to $H_1(T, \mathrm{Ker}\,\nabla^*)$ by

$$\mathrm{reg}((x_1, x_j) \otimes u_{ij}) = \sigma_{ij} \quad \text{and} \quad \mathrm{reg}(S_i \otimes u_{ij}) = \sigma_{ij},$$

where $\{\sigma_{ij}\}$ is the basis of $H_1(T, \mathrm{Ker}\,\nabla^*)$ as above. The map "reg" is an isomorphism of $\mathbb{C}$-vector spaces and was called the regularization in Chapter 1. Moreover, the twisted cycle $\sigma = \mathrm{reg}((x_1, x_j) \otimes u)$ depends on the base point as follows. Let P_k' be a point on the path (x_1, x_j) and assume that there exists $v \in \Gamma(U, \mathrm{Ker}\,\nabla^*)$ such that $({S'}_k^* - \mathrm{id})v = u$, where S_k' is a loop around x_k with the base point P_k'. Then there clearly exists a decomposition of cycles as

$$\mathrm{reg}((x_1, x_j) \otimes u) = \mathrm{reg}((x_1, x_k) \otimes u) + \mathrm{reg}((x_k, x_j) \otimes u),$$

where (x_k, x_j) is the polygonal arc $\overline{x_kP_k'P_jx_j}$.

2.2 Smooth deformation

We call

$$X_n = \mathbb{C}^n \setminus \bigcup_{i \neq j} \{(x_1, \dots, x_n) \mid x_i - x_j = 0\}$$

the pure braid space of dimension n. The fibre of the canonical projection

$$p : X_{n+1} \ni (x, t) = (x_1, \dots, x_n, t) \longmapsto (x_1, \dots, x_n) = x \in X_n,$$

is denoted by $T(x)$

We give an $m \times m$-matrix-valued holomorphic 1-form $\Omega(x_1, \ldots, x_n, t)$ on X_{n+1} such that $\nabla_{x,t}\nabla_{x,t} = 0$, where $\nabla_{x,t}u = d_{x,t}u - \Omega \wedge u$ and $d_{x,t}$ is the exterior differentiation with respect to both $x_1, \ldots, x_n$, and t. Let $\mathcal{S} = \operatorname{Ker} \nabla_{x,t}$. The pull back $\Omega_x^*(t) = \iota^*\Omega(x,t)$ by the natural inclusion $\iota : T(x) \hookrightarrow X_{n+1}$ is an $m \times m$-matrix-valued holomorphic 1-form on $T(x)$. Put $\mathcal{L}_x = \operatorname{Ker} \nabla_x$, where $\nabla_x u = d_t u - \Omega_x^*(t) \wedge u$. Then we have $\mathcal{S}|_{T(x)} = \mathcal{L}_x$.

Let $\pi_1(X_n, a)$ be the fundamental group of X_n with a base point a. If we can smoothly deform a base of $H_1(T(x), \mathcal{L}_x)$ along an element γ of $\pi_1(X_n, a)$, then we get the monodromy representation of $\pi_1(X_n, a)$:

$$\pi_1(X_n, a) \ni \gamma \longmapsto \gamma^* \in \operatorname{Aut}(H_1(T(x), \mathcal{L}_x)).$$

In the actual computation, we need to define rigorously the notion of smooth deformation. Although we defined twisted homology groups basing on a smooth triangulation of the base space, it is more convenient to regard twisted cycles as singular chains to define smooth deformation. We say that a mapping $\triangle$ is a singular simplex in the fibre $T(x)$ if $\triangle : [0,1] \to T(x)$ is a C^∞ mapping. Let f be a germ of $\mathcal{L}_x$ at $\triangle(0)$. We write the formal pair of $\triangle$ and f by $\triangle \otimes f$ and call it the singular chain.

Suppose that K is a triangulation of the image $\triangle([0,1])$ and that $f_i \in \Gamma(U_i, \mathcal{L}_x)$ is analytic continuation of f along the image $\triangle([0,1])$ in $T(x)$, where U_i is a neighborhood of a simplex $\triangle_i'$ of K. We identify the singular chain $\triangle \otimes f(t)$ with the chain $\sum_{\triangle_i'} \triangle_i' \otimes f_i(t)$.

We next define the smooth deformation of paths and the continuation of sections of a locally constant sheaf.

Definition 2.2.1. Suppose that a path $\gamma(s) : [0,1] \to X_n$ and a singular simplex $\triangle_0(u)$ in $T(\gamma(0))$ are given. A singular simplex $\triangle_1(u)$ in $T(\gamma(1))$ is called a *smooth deformation* of $\triangle_0(u)$ along γ if there exists a smooth mapping $F : [0,1] \times [0,1] \ni (s,u) \longmapsto F(s,u) \in X_{n+1}$ satisfying the following conditions:

1. $F(s, \cdot) : [0,1] \to T(\gamma(s)) \subset \mathbb{C}$ is a singular chain in the fibre $T(\gamma(s))$ for all $0 \le s \le 1$.
2. $\triangle_0(u) = F(0,u)$ and $\triangle_1(u) = F(1,u)$.

We denote $F(s,u)$ by $\triangle_s(u)$.

Definition 2.2.2. Let $\gamma(s)$ be a path in X_n and $\triangle_1$ a smooth deformation of $\triangle_0$ along γ. Put $\xi(s) = (\gamma(s), \triangle_s(0))$, i.e., ξ is a lift of γ to a path in X_{n+1}. Suppose that f_i $(i = 0, 1)$ be a germ of $\mathcal{L}_{\gamma(0)}$ at $\triangle_i(0)$. f_1 is called the *continuation* of f_0 along ξ if, over a simply connected open set $U \subset X_{n+1}$ which contains ξ, there exists $g(s,t) \in \Gamma(U, \mathcal{S})$ which satisfies the following condition: for $i = 0$ and $i = 1$, f_i is equal to the germ $[g(s,t)|_{T(\gamma(i))}]$ of $\mathcal{L}_{\gamma(i)}$ at $\triangle_i(0)$ where $g(s,t)|_{T(\gamma(i))}$ is a section of $\mathcal{L}_{\gamma(i)}$ in a neighborhood of $\triangle_i(0)$ in $T(\gamma(i))$.

We can now define the smooth deformation of $\triangle_s(u) \otimes f_s(t)$.

Definition 2.2.3. Let $\gamma(s)$ be a path in X_n and $\triangle_1$ a smooth deformation of $\triangle_0$ along γ. $\triangle_1 \otimes f_1$ is called the *smooth deformation* of $\triangle_0 \otimes f_0$ along γ if f_1 is the continuation of f_0 along the path $\xi(s) = (\gamma(s), \triangle_s(0))$ in X_{n+1}.

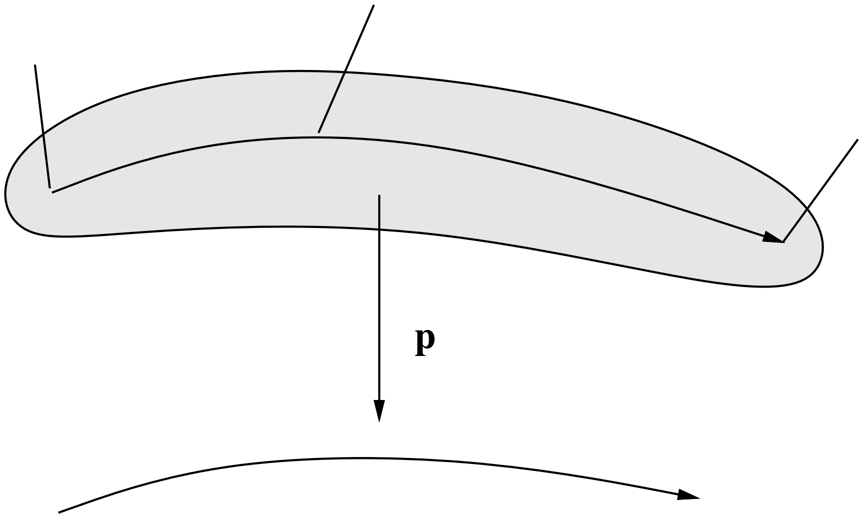


Figure 2.2: the lift ξ of γ

The finite composition of smooth deformations defined above is also called a smooth deformation.

Example 2.2.1. Put $n = 2$ and we regard each configuration of 2 points in $\mathbb{C}$ as a point in X_2. Let γ be a path in X_2 such that the point x_2 encircles the fixed point x_1 in the positive direction. Suppose that $\triangle$ is a singular chain that encircles the point x_2 in the positive direction. In this case, the smooth deformation of $\triangle$ along the path γ is again $\triangle$. Let us consider the lift ξ of γ in the (x_1, x_2, t)-space X_3. ξ is the path such that the points t and x_2 encircle the point x_1 in the positive direction. Consider the twisted chain $\triangle \otimes \mathrm{Log}(t - x_1)$. Since the analytic continuation of the function $\mathrm{Log}(t - x_1)$ along the path ξ is $\mathrm{Log}(t - x_1) + 2\pi\sqrt{-1}$, the smooth deformation of $\triangle \otimes \mathrm{Log}(t - x_1)$ is $\triangle \otimes \mathrm{Log}(t - x_1) + \triangle \otimes 2\pi\sqrt{-1}$.

2.3 The generalized hypergeometric function $_pF_{p-1}$

The generalized hypergeometric function

$$ {}_pF_{p-1}(a_1, \ldots, a_p; b_2, \ldots, b_p; z) = \sum_{n=0}^{\infty} \frac{(a_1; n)(a_2; n) \cdots (a_p; n)}{(b_2; n)(b_3; n) \cdots (b_p; n)n!} z^n $$

has a single integral representation

$$ \frac{\Gamma(\alpha_{3p-2})\Gamma(\alpha_{3p-1})}{\Gamma(\alpha_{3p-2} + \alpha_{3p-1})} {}_pF_{p-1}(a_1, \ldots, a_p; b_2, \ldots, b_p; z) $$
$$ = z^{\alpha_{3p}} \int_0^z t^{\alpha_{3p-2}} (z - t)^{\alpha_{3p-1}} {}_{p-1}F_{p-2}(a_1, \ldots, a_{p-1}; b_2, \ldots, b_{p-1}; t) \frac{z dt}{t(z - t)}, $$

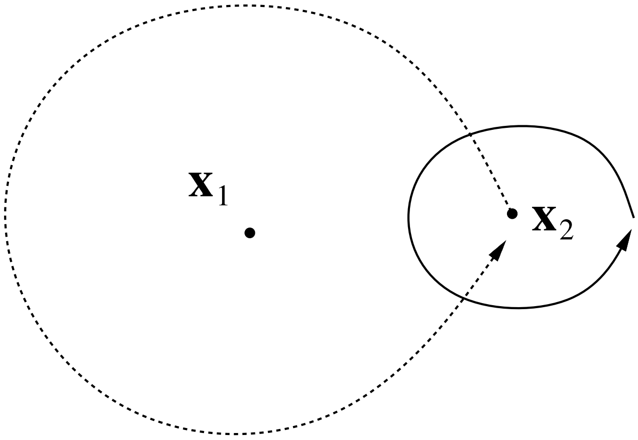


Figure 2.3: $\triangle \otimes \mathrm{Log}(t - x_1)$ and γ

where we use the notation: $\alpha_{3j-2} = a_j$, $\alpha_{3j-1} = b_j - a_j$, $\alpha_{3j} = -b_j$ and put $b_1 = 1$ for simplicity. Let $c_i = \exp(2\pi\sqrt{-1}\alpha_i)$. Since $\alpha_{3j-2} + \alpha_{3j-1} + \alpha_{3j} = 0$, we have $c_{3j-2}c_{3j-1}c_{3j} = 1$.

We consider the 1-dimensional subspace $X_3' = \{(0,1,z) \mid z \neq 0, 1\}$ of the braid space X_3 and the projection

$$X_4 \supset X_4' = \{(0,1,z,t) \mid z, t \neq 0, 1,\ z \neq t\} \xrightarrow{\pi} X_3' \subset X_3.$$

Then the function $t^{\alpha_{3p-2}}(z-t)^{\alpha_{3p-1}}{}_{p-1}F_{p-2}(a;b;t)$ defines a locally constant sheaf $\mathcal{S}$ of rank $p-1$ on X_4' and we regard the generalized hypergeometric function ${}_pF_{p-1}$ as a pairing of an element of $H_1(\pi^{-1}(z), \mathcal{S}_z)$ and an element of $H^1(\pi^{-1}(z), \mathcal{S}_z^*)$.

Thus, we can consider twisted cycles on the 1-dimensional space with a locally constant sheaf of rank $p-1$ instead of twisted cycles on $(p-1)$-dimensional space with a locally constant sheaf of rank one and to compute monodromy groups by utilizing the general arguments given in Section 2.2.

In this section, we consider two cases ${}_2F_1$ and ${}_3F_2$ to clarify our idea. The computation in the general case will be given in the next section.

Example 2.3.1. We review a computation of the monodromy for the Gauss hypergeometric function. For simplicity we rewrite the integration of Example 2.2 by the transformation $t \to s/x$ and $\alpha \leftrightarrow \beta$.

$$u(s/x) = s^{\beta}(x-s)^{\gamma-\beta}(1-s)^{\alpha} \times x^{-\gamma}.$$

Namely, the Gauss hypergeometric function has the integral representation

$${}_2F_1(a_1, a_2; b_2; x) = \frac{\Gamma(\alpha_4 + \alpha_5)}{\Gamma(\alpha_4)\Gamma(\alpha_5)} x^{\alpha_6} \int_0^x s^{\alpha_4}(x-s)^{\alpha_5}(1-s)^{\alpha_2 - 1} \frac{xds}{s(x-s)}.$$

We define a locally constant sheaf which corresponds to this integration as follows. We put the holomorphic 1-form

$$\Omega = \alpha_4 \frac{ds}{s} + \alpha_5 \frac{d(x-s)}{x-s} + (\alpha_2 - 1)\frac{d(1-s)}{1-s}$$

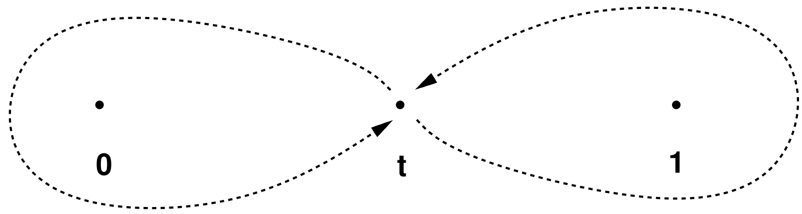


Figure 2.4: $\gamma_1, \gamma_2 \in \pi_1(\mathbb{C} \setminus \{0,1\}, x)$

over $T(x) = \mathbb{C} \setminus \{0, x, 1\}$ and define the covariant derivation

$$\nabla^* : \tau \longmapsto d\tau - \Omega \wedge \tau.$$

Suppose that α_2, α_4, and α_5 are not integers. Let $U = \{s \in \mathbb{C} \mid \operatorname{Im} s < 0\}$ be the lower half plane and u a branch of $s^{\alpha_4}(x-s)^{\alpha_5}(1-s)^{\alpha_1}$ in U. We should notice that $s^{\alpha_4}(x-s)^{\alpha_5}(1-s)^{\alpha_1}$ is multi-valued in a neighborhood of x. Therefore, we identify x with the point x_1 of Theorem 2.1.2. Let $\nu_1 = \operatorname{reg}((x,1) \otimes u)$ and $\nu_2 = \operatorname{reg}((0,x) \otimes u)$. Then $\{\nu_1, \nu_2\}$ is the basis of $H_1(T(x), \operatorname{Ker} \nabla^*)$ which corresponds to the compact chambers of the arrangement $\{s \mid s(x-s)(1-s) = 0\}$ in $\mathbb{R}$.

The monodromy group acts on the basis $\{\nu_1, \nu_2\}$ of $H_1(T(x), \operatorname{Ker} \nabla^*)$ as

$$\gamma_1^* : (\nu_1, \nu_2) \longmapsto (\nu_1, \nu_2) \frac{1}{c_6} \begin{pmatrix} c_6 & 0 \\ c_5 c_6 - 1 & 1 \end{pmatrix},$$
$$\gamma_2^* : (\nu_1, \nu_2) \longmapsto (\nu_1, \nu_2) \begin{pmatrix} c_2 c_5 & -(c_2 - 1) \\ 0 & 1 \end{pmatrix}.$$

Here, $\gamma_1(s)$ is the closed curve around the point 0 in positive direction and $\gamma_2(s)$ is the closed curve around the point 1 in positive direction as drawn below.

Put

$$\omega_1 = \frac{ds}{s} - \frac{d(x-s)}{x-s} = \frac{x ds}{s(x-s)},$$
$$\omega_2 = -(\alpha_4 x + (\alpha_6 + 1)s) \frac{x ds}{s(x-s)^2}.$$

Then we have $x\frac{d}{dx}\left(x^{\alpha_6} \int u\omega_1\right) = x^{\alpha_6} \int u\omega_2$. We already defined the pairing $(\nu_j, \omega_k) = \int_{\nu_j} u\omega_k$. To define the covariant derivation for the case ${}_3F_2$ in the next example, we rely on that the pairing $x^{\alpha_6} \begin{pmatrix} (\nu_j, \omega_1) \\ (\nu_j, \omega_2) \end{pmatrix}$ satisfies the formula:

$$d_x \left\{ x^{\alpha_6} \begin{pmatrix} (\nu_j, \omega_1) \\ (\nu_j, \omega_2) \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & -\alpha_6 - 1 \end{pmatrix} \frac{dx}{x} + \begin{pmatrix} 0 & 0 \\ -\alpha_1 \alpha_4 & \alpha_2 + \alpha_5 - 2 \end{pmatrix} \frac{d(1-x)}{1-x} \right\} \left\{ x^{\alpha_6} \begin{pmatrix} (\nu_j, \omega_1) \\ (\nu_j, \omega_2) \end{pmatrix} \right\}.$$

The formula is also induced by the differential equation

$$\left\{ x\frac{d}{dx}\left(x\frac{d}{dx} + b_2 - 1\right) - x\left(x\frac{d}{dx} + a_1\right)\left(x\frac{d}{dx} + a_2\right) \right\} y = 0.$$

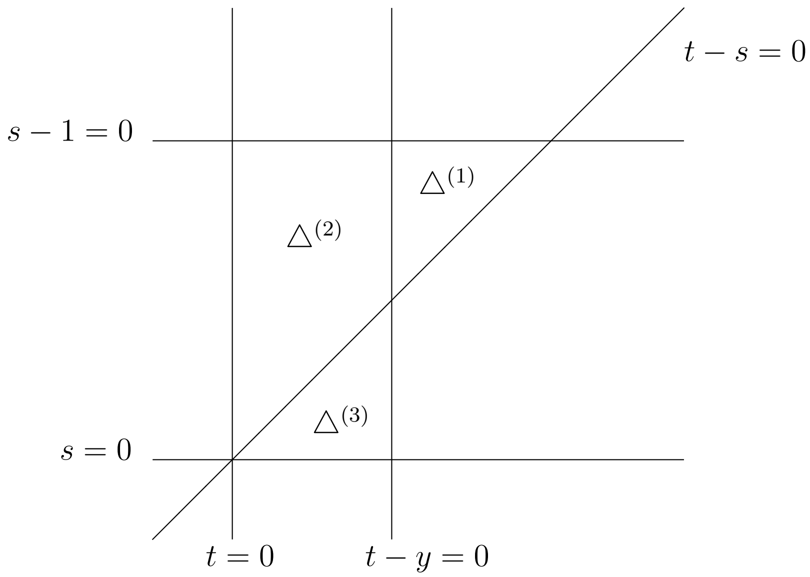


Figure 2.5: the arrangement $D = \{st(t-y)(t-s)(s-1) = 0\}$

Example 2.3.2. As we have seen, $_3F_2$ has an integral representation

$$\frac{\Gamma(\alpha_7)\Gamma(\alpha_8)}{\Gamma(\alpha_7+\alpha_8)}{}_3F_2(a_1,a_2,a_3;b_2,b_3;z) = z^{\alpha_9}\int_0^z t^{\alpha_7}(z-t)^{\alpha_8}{}_2F_1(a_1,a_2;b_2;t)\frac{zdt}{t(z-t)}.$$

We put the matrix-valued holomorphic 1-form

$$\Omega = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha_6-1 \end{pmatrix}\frac{dt}{t} + \begin{pmatrix} 0 & 0 \\ -\alpha_1\alpha_4 & \alpha_2+\alpha_5-2 \end{pmatrix}\frac{d(1-t)}{1-t} + \begin{pmatrix} \alpha_7 & 0 \\ 0 & \alpha_7 \end{pmatrix}\frac{dt}{t} + \begin{pmatrix} \alpha_8 & 0 \\ 0 & \alpha_8 \end{pmatrix}\frac{d(z-t)}{z-t}$$

over $T(z) = \mathbb{C} \setminus \{0, z, 1\}$ and define the covariant derivation

$$\nabla^* : \tau \longmapsto d\tau - \Omega \wedge \tau.$$

Suppose that $\alpha_7, \alpha_7 - \alpha_6, \alpha_8$, and $\alpha_2 + \alpha_5$ are not integers. For the pairing $\begin{pmatrix} (\nu_j, \omega_1) \\ (\nu_j, \omega_2) \end{pmatrix}$ of Example 2.3.1, put $q_j = t^{\alpha_6+\alpha_7}(z-t)^{\alpha_8}\begin{pmatrix} (\nu_j, \omega_1) \\ (\nu_j, \omega_2) \end{pmatrix}$; it is clear that $\nabla^* q_j = 0$.

Now we consider the arrangement $D = \{st(z-t)(1-s)(t-s) = 0\}$. By the definition of ν_1, the twisted cycle $\mathrm{reg}((z,1)\otimes q_1)$ corresponds to $\triangle^{(1)} \otimes t^{\alpha_7}(z-t)^{\alpha_8}u(t,s)$. Here $\triangle^{(1)}$ is the compact chamber in D enclosed by the hyperplanes $\{z-t=0, t-s=0, 1-s=0\}$. In the same way, let $\triangle^{(2)}$ be the compact chamber enclosed by the hyperplanes $\{t=0, z-t=0, t-s=0, 1-s=0\}$ and $\triangle^{(3)}$ the compact chamber enclosed by the hyperplanes $\{z-t=0, t-s=0, s=0, t=0\}$. Then $\mathrm{reg}((0,z)\otimes q_j)$ corresponds to $\triangle^{(j+1)} \otimes t^{\alpha_7}(z-t)^{\alpha_8}u(t,s)$ $(j=1,2)$ (Figure 2.5).

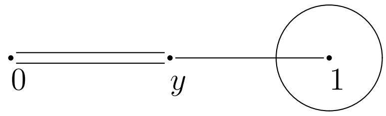


Figure 2.6: $\{\sigma_1, \dots, \sigma_4\}$

Define four bounded twisted cycles:

$$\begin{aligned}
\sigma_1 &= \mathrm{reg}((z,1)\otimes q_1),\\
\sigma_2 &= \mathrm{reg}((0,z)\otimes q_1),\\
\sigma_3 &= \mathrm{reg}((0,z)\otimes q_2),\\
\sigma_4 &= S^1_\varepsilon(1)\otimes\{(c_2-1)q_1-(c_2c_5-1)q_2\},
\end{aligned}$$

where $S^1_\varepsilon(1)$ is the circle of the radius ε with center 1. The set $\{\sigma_1, \dots, \sigma_4\}$ is a basis of $H_1(T(z), \operatorname{Ker}\nabla^*)$ by Theorem 2.1.2. Moreover, we notice that $\{\sigma_1, \sigma_2, \sigma_3\}$ spans the primary part of $H_1(T(z), \operatorname{Ker}\nabla^*)$ and corresponds to the compact chambers in D.

For the same $\gamma_1, \gamma_2 \in \pi_1(\mathbb{C}\setminus\{0,1\}, z)$ as in Example 2.3.1, the monodromy group acts on the primary part as

$$\gamma_1^* : (\sigma_1,\sigma_2,\sigma_3) \longmapsto (\sigma_1,\sigma_2,\sigma_3)\frac{1}{c_9}\begin{pmatrix} c_9 & 0 & 0\\ c_8c_9-c_6 & c_6 & 0\\ 1-c_5c_6 & c_5c_6-1 & 1\end{pmatrix},$$

$$\gamma_2^* : (\sigma_1,\sigma_2,\sigma_3) \longmapsto (\sigma_1,\sigma_2,\sigma_3)\begin{pmatrix} c_2c_5c_8 & 1-c_2c_5 & c_2-1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix};$$

This fact in a more general setting will be proved by using an induction in the next section.

2.4 The monodromy group of ${}_pF_{p-1}$

We put the $(p-1)\times(p-1)$-matrix-valued holomorphic 1-form

$$\Omega = \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1\\ 0 & -B_1 & \cdots & -B_{p-2}\end{pmatrix}\frac{dt}{t} + \begin{pmatrix} 0 & \cdots & 0\\ \vdots & & \vdots\\ 0 & & 0\\ A_0 & \cdots & A_{p-2}\end{pmatrix}\frac{d(1-t)}{1-t}$$
$$+\begin{pmatrix}\alpha_{3p-2} & & & \\ & \alpha_{3p-2} & & \\ & & \ddots & \\ & & & \alpha_{3p-2}\end{pmatrix}\frac{dt}{t} + \begin{pmatrix}\alpha_{3p-1} & & & \\ & \alpha_{3p-1} & & \\ & & \ddots & \\ & & & \alpha_{3p-1}\end{pmatrix}\frac{d(z-t)}{z-t}$$

over $T(z) = \mathbb{C} \setminus \{0, z, 1\}$, where A_i and B_i are defined as follows:

$$A_0 + A_1 X + \cdots + A_{p-2} X^{p-2} = X \prod_{k=2}^{p-1} (X + b_k - 1) - \prod_{k=1}^{p-1} (X + a_k),$$

$$B_1 X + \cdots + B_{p-2} X^{p-2} + X^{p-1} = X \prod_{k=2}^{p-1} (X + b_k - 1).$$

Relative to the covariant derivation $\nabla^* : \tau \longmapsto d\tau - \Omega \wedge \tau$, $\operatorname{Ker} \nabla^*$ is a locally constant sheaf on $T(z)$.

Moreover, we introduce the notation:

$$S_{ij} = \begin{cases} (-1)^{i+j} \left(c_{3p-3i+3} + \dfrac{\delta_{ij} - 1}{c_{3p-3i+4}} \right) & (j \le i \le p), \\ 0 & (\text{otherwise}), \end{cases}$$

$$T_i = (-1)^{i+1} \left(\prod_{l=0}^{p-i} c_{3l+2} - 1 \right),$$

where δ_{ij} is Kronecker's delta and define a $p \times p$ lower triangular matrix M_1 and a $p \times p$ upper triangular matrix M_2 as follows:

$$M_1 = \begin{pmatrix} S_{11} & & 0 \\ \vdots & \ddots & \\ S_{p,1} & \cdots & S_{p,p} \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} + \begin{pmatrix} T_1 & \cdots & T_p \\ & & \\ & 0 & \end{pmatrix}.$$

Theorem 2.4.1. *Suppose that*

$$\alpha_{3k-1},\ \alpha_{3k-2} - \alpha_{3j},\ \sum_{i=1}^{k} \alpha_{3i-1} \notin \mathbb{Z} \quad \textit{for all } 1 \le j \le k < p. \tag{2.5}$$

Let $A = H_1(\mathbb{C} \setminus \{0, z, 1\}, \operatorname{Ker} \nabla^*)$. Then $\dim_{\mathbb{C}} A = 2p-2$. The twisted homology group A can be decomposed as $A = \Pr(A) \oplus \operatorname{Deg}(A)$ where both $\Pr(A)$ and $\operatorname{Deg}(A)$ are invariant under the action of the monodromy group. Then the same loops $\gamma_1, \gamma_2 \in \pi_1(\mathbb{C} \setminus \{0, 1\}, z)$ as in Example 2.3.1 *act on* $\Pr(A)$ *as*

$$\gamma_1^* : (\nu_1, \ldots, \nu_p) \longmapsto (\nu_1, \ldots, \nu_p) \frac{1}{c_{3p}} M_1,$$

$$\gamma_2^* : (\nu_1, \ldots, \nu_p) \longmapsto (\nu_1, \ldots, \nu_p) M_2.$$

Proof. It is clear that each part of A is invariant under the action of the monodromy group. In order to find the monodromy matrix on $\Pr(A)$, we argue by induction on p. In order to distinguish the twisted homology group A on each step of induction, we denote by $A(p)$ the twisted homology group A with parameter p and use the similar notation for S_{ij}, T_i, M_i, and ∇^*. We will prove $\dim_{\mathbb{C}} A(p+1) = 2p$ and construct a basis of $A(p+1)$ from a basis of $A(p) = H_1(\mathbb{C} \setminus \{0, z, 1\}, \operatorname{Ker} \nabla^*(p))$.

For the initial step, we assume $p = 2$; in this case, the theorem is equivalent to Example 2.3.1, that is,

$$\begin{pmatrix} c_6 & 0 \\ c_5c_6 - 1 & 1 \end{pmatrix} = \begin{pmatrix} S_{11}(2) & 0 \\ S_{21}(2) & S_{22}(2) \end{pmatrix}, \quad \begin{pmatrix} c_2c_5 & -(c_2-1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+T_1(2) & T_2(2) \\ 0 & 1 \end{pmatrix}.$$

Next we assume that the assertion is true for p and proceed to $p+1$. The assumption implies that

$$\gamma_1^* : (\nu_1, \dots, \nu_p) \longmapsto (\nu_1, \dots, \nu_p)\frac{1}{c_{3p}}M_1(p),$$
$$\gamma_2^* : (\nu_1, \dots, \nu_p) \longmapsto (\nu_1, \dots, \nu_p)M_2(p).$$

We can take p-dimensional vector-valued rational 1-forms $\omega_1, \dots, \omega_p$ with the poles at $0, z$, and 1 such that $\nabla^*(p+1)q_j = 0$ where

$$q_j = t^{\alpha_{3p}+\alpha_{3p+1}}(z-t)^{\alpha_{3p+2}-1}\begin{pmatrix} (\nu_j, \omega_1) \\ \vdots \\ (\nu_j, \omega_p) \end{pmatrix}, \quad (j = 1, \dots, p).$$

In fact, we construct ω_k's by the condition:

$$\omega_1 = \begin{pmatrix} \frac{zdt}{t(z-t)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad z\frac{d}{dz}\left(z^{\alpha_{3p}}(\nu_j, \omega_k)\right) = z^{\alpha_{3p}}(\nu_j, \omega_{k+1}) \quad (1 \le j \le p,\ 1 \le k < p).$$

The explicit forms of ω_k are very complicated and not written down here. However, we should notice that for any p-dimensional vector ω of rational 1-forms with poles at $0, z$, and 1, the action of the monodromy group on (ν_j, ω) is decided only by ν_j and not by ω.

With this chain of 1-forms, we check the assumption of Theorem 2.1.1. Let $S^1(0)$, $S^1(1)$, and $S^1(z)$ be loops around the points 0, 1, and z, respectively. Since $S^1(1)^* : (q_1, \dots, q_p) \mapsto c_{3p+2}(q_1, \dots, q_p)$, we have $\det(S^1(1)^* - \mathrm{id}) \neq 0$ by the assumption (2.5). Hence, the assumption of Theorem 2.1.1 is satisfied and we get $\dim_{\mathbb{C}} A(p+1) = 2p$.

Since $S^1(0)^* : (q_1, \dots, q_p) \mapsto (q_1, \dots, q_p)c_{3p+1}M_1(p)$, the eigenvalues of $S^1(0)^*$ are $\{c_{3k+3}c_{3p+1} \mid 1 \le k < p\}$. By the assumption (2.5), $c_{3k+3}c_{3p+1} \neq 1$ $(1 \le k < p)$, that is, $\det(S^1(0)^* - \mathrm{id}) \neq 0$. Moreover, since $S^1(z)^* : (q_1, \dots, q_p) \mapsto (q_1, \dots, q_p)M_2(p)$, the eigenvalues of $M_2(p)$ are $\{1 + T_1(p), 1, \dots, 1\}$. Thus $\det(S^1(z)^* - \mathrm{id}) = 0$. On the other hand, since $T_1(p) \neq 0$ by (2.5), we can see that the matrix $M_2(p)$ is diagonalizable. Therefore, the assumption of Theorem 2.1.2 is satisfied.

We identify $\{q_i\}$ with a set of sections of $\operatorname{Ker} \nabla^*(p+1)$ on the lower half plane $\{t \in \mathbb{C} \mid \operatorname{Im} t < 0\}$ and define $2p$ bounded cycles as

$$\begin{aligned} \sigma_1 &= \mathrm{reg}((z,1) \otimes q_1(t)), \\ \sigma_{j+1} &= \mathrm{reg}((0,z) \otimes q_j(t)) && (j = 1, \dots, p), \\ \sigma_{p+k} &= S^1(z) \otimes \{T_k(p)q_1(t) - T_1(p)q_k(t)\} && (k = 2, \dots, p), \end{aligned}$$

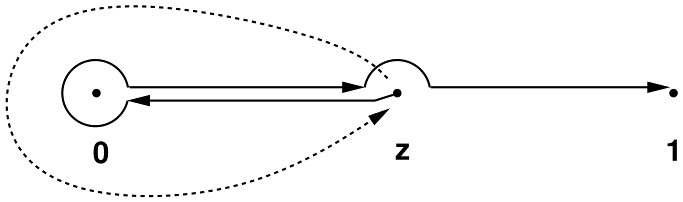


Figure 2.7: The smooth deformation of σ_1 under the action of γ_1^*.

where $T_k(p)q_1 - T_1(p)q_k$ are distinct eigenvectors of $M_2(p)$ with eigenvalue 1. The set of these cycles is a basis of $A(p+1)$ by Theorem 2.1.2. Note that $\Pr(A(p+1))$ is spanned by $\sigma_1, \ldots, \sigma_{p+1}$, and $\mathrm{Deg}(A(p+1))$ by $\sigma_{p+2}, \ldots, \sigma_{2p}$. As we have seen, $\sigma_1, \ldots, \sigma_{p+1}$ correspond to compact chambers of the arrangement D.

First, we will trace the action of γ_1^* on $\sigma_1, \ldots, \sigma_{p+1}$. Recalling the process of decomposing cycles by referring to Section 2.1, we obtain the action

$$\begin{aligned}
\gamma_1^* : \mathrm{reg}((z,1)\otimes q_1) \longmapsto\ & \mathrm{reg}((z,1)\otimes q_1) + \mathrm{reg}((0,z)\otimes(c_{3p+2}q_1)) \\
& + \mathrm{reg}\left((z,0)\otimes\left(c_{3p+2}c_{3p+1}c_{3p}\frac{1}{c_{3p}}\sum_{i=1}^{p} S_{i,1}(p)q_i\right)\right) \\
=\ & \sigma_1 + \frac{1}{c_{3p+3}c_{3p+1}}\sigma_2 - \frac{1}{c_{3p+3}}\sum_{j=2}^{p+1} S_{j-1,1}(p)\sigma_j \\
=\ & \frac{1}{c_{3p+3}}\left\{c_{3p+3}\sigma_1 + \left(\frac{1}{c_{3p+1}} - S_{11}(p)\right)\sigma_2 + \sum_{j=3}^{p+1}(-S_{j-1,1}(p))\sigma_j\right\} \\
=\ & \frac{1}{c_{3p+3}}\sum_{j=1}^{p+1} S_{j,1}(p+1)\sigma_j,
\end{aligned}$$

where we used the identity

$$S_{j,1}(p+1) = \begin{cases} c_{3p+3} & (j=1), \\ \frac{1}{c_{3p+1}} - S_{11}(p) & (j=2), \\ -S_{j-1,1}(p) & (j=3,\ldots,p+1). \end{cases}$$

Similarly, by tracing the smooth deformation of the path in detail, we see that

$$\begin{aligned}
\gamma_1^* : \mathrm{reg}((0,z)\otimes q_j) \longmapsto\ & \mathrm{reg}\left((0,z)\otimes\left\{c_{3p+1}c_{3p}\frac{1}{c_{3p}}\sum_{i=1}^{p} S_{i,j}(p)(c_{3p+2}q_i)\right\}\right) \\
=\ & \frac{1}{c_{3p+3}}\sum_{i=1}^{p} S_{i,j}(p)\sigma_{i+1} \\
=\ & \frac{1}{c_{3p+3}}\sum_{i=1}^{p} S_{i+1,j+1}(p+1)\sigma_{i+1},
\end{aligned}$$

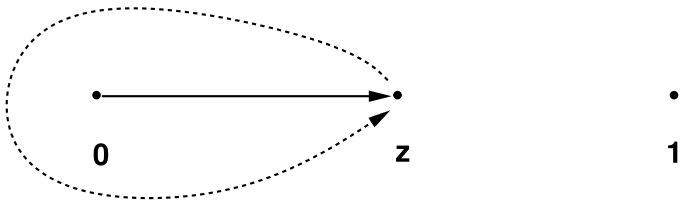


Figure 2.8: The smooth deformation of σ_{j+1} under the action of γ_1^*.

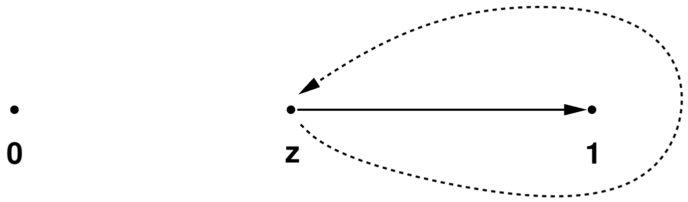


Figure 2.9: The smooth deformation of σ_1 under the action of γ_2^*.

where we used the identity $S_{ij}(p) = S_{i+1,j+1}(p+1)$. Thus γ_1^* acts on $\Pr(A)$ as

$$\begin{aligned}(\sigma_1, \dots, \sigma_{p+1}) \longmapsto (\sigma_1, \dots, \sigma_{p+1}) \frac{1}{c_{3p+3}} &\begin{pmatrix} S_{11}(p+1) & & & 0 \\ S_{21}(p+1) & S_{22}(p+1) & & \\ \vdots & & \ddots & \\ S_{p+1,1}(p+1) & S_{p+1,2}(p+1) & \cdots & S_{p+1,p+1}(p+1) \end{pmatrix} \\ = (\sigma_1, \dots, \sigma_{p+1}) \frac{1}{c_{3p+3}} M_1(p+1).\end{aligned}$$

Second, we will trace the action of γ_2^* on $\sigma_1, \dots, \sigma_{p+1}$. In a similar way as above, shown as in Figure 2.9, by tracing the smooth deformation of the path in detail, we see that

$$\gamma_2^* : \mathrm{reg}((z,1) \otimes q_1) \longmapsto \mathrm{reg}((z,1) \otimes \{(1 + T_1(p))(c_{3p+2} q_1)\}) = (1 + T_1(p+1))\sigma_1$$

by using $(1 + T_1(p))c_{3p+2} = 1 + T_1(p+1)$.
Moreover, as in Figure 2.10, we see that

$$\begin{aligned}\gamma_2^* : \mathrm{reg}((0,z) \otimes q_j) \longmapsto\ & \mathrm{reg}((0,z) \otimes q_j) + \mathrm{reg}((z,1) \otimes q_j) + \mathrm{reg}((1,z) \otimes (T_j(p)q_1 + q_j)) \\ &= \sigma_{j+1} - T_j(p)\sigma_1 \\ &= \sigma_{j+1} + T_{i+1}(p+1)\sigma_1,\end{aligned}$$

where we used the relation $-T_i(p) = T_{i+1}(p+1)$.

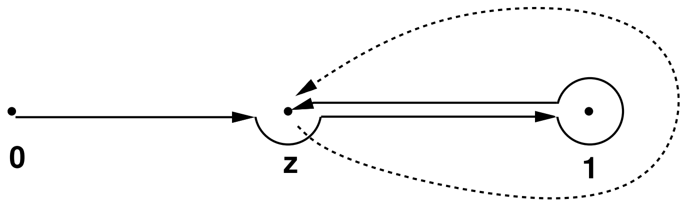


Figure 2.10: The smooth deformation of σ_{j+1} under the action of γ_2^*.

Therefore, this means that γ_2^* acts on $\Pr(A)$ as

$$(\sigma_1,\dots,\sigma_{p+1}) \longmapsto (\sigma_1,\dots,\sigma_{p+1}) \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} + \begin{pmatrix} T_1(p+1) & \cdots & T_{p+1}(p+1) \\ & 0 & \end{pmatrix} \right\}$$
$$= (\sigma_1,\dots,\sigma_{p+1})M_2(p+1).$$

Thus, Theorem 2.4.1 is proved. □

Corollary 2.4.1. *The monodromy matrices of solutions of the differential equation*

$$\left\{ z\frac{d}{dz}\prod_{k=2}^{p}\left(z\frac{d}{dz}+b_k-1\right) - z\prod_{k=1}^{p}\left(z\frac{d}{dz}+a_k\right)\right\} y = 0$$

which correspond to the compact chambers of the hyperplane arrangement

$$\left\{ (1-t_1)(z-t_{p-1})\prod_{i=1}^{p-2}(t_{i+1}-t_i)\prod_{i=1}^{p-1}t_i = 0 \right\}$$

are given by the matrices $M_1(p)$ and $M_2(p)$.

Example 2.4.1. We give a table of monodromy representation of $p = 2, 3$, and 4.
$p = 2$:

$$M_1 = \begin{pmatrix} c_6 & 0 \\ c_5c_6-1 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} c_2c_5 & 1-c_2 \\ 0 & 1 \end{pmatrix}.$$

$p = 3$:

$$M_1 = \begin{pmatrix} c_9 & 0 & 0 \\ c_8c_9-c_6 & c_6 & 0 \\ 1-c_5c_6 & c_5c_6-1 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} c_2c_5c_8 & 1-c_2c_5 & c_2-1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$p = 4$:

$$M_1 = \begin{pmatrix} c_{12} & 0 & 0 & 0 \\ c_{11}c_{12}-c_9 & c_9 & 0 & 0 \\ c_6-c_8c_9 & c_8c_9-c_6 & c_6 & 0 \\ c_5c_6-1 & 1-c_5c_6 & c_5c_6-1 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} c_2c_5c_8c_{11} & 1-c_2c_5c_8 & c_2c_5-1 & 1-c_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We remark $c_{3i} = \exp(-2\pi\sqrt{-1}b_i)$, $c_{3i-1} = \exp(2\pi\sqrt{-1}(b_i-a_i))$, and $c_{3i-2} = c_{3i-1}^{-1}c_{3i}^{-1} = \exp(2\pi\sqrt{-1}a_i)$.

Chapter 3

Computation of intersection numbers of twisted cycles

3.1 Another definition of intersection numbers of twisted cycles

Our definition of the intersection number of twisted cycles is a natural generalization of [12]. We can give a similar formula to evaluate intersection numbers of cycles for our case.

Let K_+ be a smooth triangulation of T and K_- the dual cell decomposition.

Theorem 3.1.1. *For*

$$\sigma = \sum c_\Delta \Delta \otimes u_\Delta^+ \in H_r^{lf}(K_+, \operatorname{Ker} \nabla_+^*)$$

and

$$\tau = \sum C_{\Delta'} \Delta' \otimes u_{\Delta'}^- \in H_r(K_-, \operatorname{Ker} \nabla_-^*),$$

the intersection number $[\sigma] \cdot [\tau]$ is given by

$$\sum_{\Delta, \Delta', \{v\} = \Delta \cap \Delta'} c_\Delta c_{\Delta'} S^*(u_\Delta^+, u_{\Delta'}^-) I_v(\Delta, \Delta')$$

where $I_v(\Delta, \Delta')$ is the topological intersection number of Δ and Δ' at v and S^* is the bilinear form defined by the matrix ${}^tS^{-1}$ which gives the pairing of $\operatorname{Ker} \nabla_+^*$ and $\operatorname{Ker} \nabla_-^*$.

Proof. Let δ_Δ be a delta r-current which has support on Δ. Then, we have

$$F_\sigma = \sum c_\Delta \delta_\Delta u_\Delta^+$$

and

$$F_\tau = \sum c_{\Delta'} \delta_{\Delta'} u_{\Delta'}^-.$$

We note that it is not always possible to take a wedge product for currents. For example, $\delta(t)\delta(t)$ is not well-defined. Since Δ and Δ' crosses transversally, the wedge product of currents $F_\sigma \wedge F_\tau$ can be defined. If we regard the operator $\nabla_\pm$ as an operator on the $2r$-dimensional real manifold T, it is holonomic at degree $r-1$ and hypo-elliptic on T; for any current F of degree r and G of degree $r-1$, if $\nabla_\pm G = F$ and F is smooth at the point p, then G is also smooth at the point p. Hence, when $\mathrm{reg}(F_\sigma) = F_\sigma + \nabla_+ G_\sigma$ and $\mathrm{reg}(F_\tau) = F_\tau + \nabla_- G_\tau$, the wedge product of G_σ and G_τ is well-defined. Therefore, we may compute the intersection number by evaluating the integral of currents

$$\int_T S^*(F_\sigma, F_\tau),$$

which is equal to

$$\sum_{\Delta,\Delta',\{v\}=\Delta\cap\Delta'} c_\Delta c_{\Delta'} S^*(u^+_\Delta, u^-_{\Delta'}) \int_T \delta_\Delta \wedge \delta_{\Delta'}.$$

□

Example 3.1.1. If $m=1$, then the formula above is nothing but Theorem 1.3 of [12].

Example 3.1.2. . This example is a continuation of Example 1.2.1. We are given two connections

$$\nabla^*_+ = d - \left(\frac{\beta}{t} - \frac{\alpha+\gamma}{1-t}\right) dt \quad \text{and} \quad \nabla^*_- = d + \left(\frac{\beta}{t} - \frac{\alpha+\gamma}{1-t}\right) dt.$$

The functions u^+ and

$$\begin{aligned} u^-(t) &= t^{-\beta}(1-t)\int_0^{1-t} s^{-\alpha}(1-s-t)^{-\gamma}\frac{ds}{s(1-s-t)} \\ &= t^{-\beta}(1-t)^{-\alpha-\gamma}\int_0^1 \xi^{-\alpha}(1-\xi)^{-\gamma}\frac{d\xi}{\xi(1-\xi)} \end{aligned}$$

belong to $\mathrm{Ker}\,\nabla^*_+$ and $\mathrm{Ker}\,\nabla^*_-$ respectively. Put $S = 1/\alpha + 1/\gamma$ and $S^* = S^{-1}$. Then, from the twisted period relation of the Beta function (or from a well-known identity of the Beta function), we have

$$u^+ \cdot S^* \cdot u^- = 2\pi i \frac{e^{2\pi i(\alpha+\gamma)} - 1}{(e^{2\pi i\alpha}-1)(e^{2\pi i\gamma}-1)} =: d_1.$$

Consider the twisted cycle $\sigma = (0,1)\otimes u^+$ which belong to $H^{lf}_1(\mathbb{C}\setminus\{0,1\}, \mathrm{Ker}\,\nabla^*_+)$. The regularization $\mathrm{reg}(\sigma)$ of σ is

$$\frac{1}{e^{2\pi i\beta}-1} C_0 \otimes u^+ + [0+\varepsilon, 1-\varepsilon]\otimes u^+ - \frac{1}{e^{2\pi i(\alpha+\gamma)}-1} C_1 \otimes u^+$$

where C_p is the circle with the radius ε and the center p. Put $\tau = (0,1)\otimes u^- \in H^{lf}_1(\mathbb{C}\setminus\{0,1\}, \mathrm{Ker}\,\nabla^*_-)$. By applying Theorem 3.1.1, we have

$$\begin{aligned} [\mathrm{reg}(\sigma)]\cdot[\tau] &= \frac{1}{e^{2\pi i\beta}-1}(-1)d_1 + (-1)d_1 - \frac{1}{e^{2\pi i(\alpha+\gamma)}-1} d_1 \\ &= 2\pi i \frac{1 - e^{2\pi i(\alpha+\beta+\gamma)}}{(e^{2\pi i\alpha}-1)(e^{2\pi i\beta}-1)(e^{2\pi i\gamma}-1)}. \end{aligned}$$

3.2 Example: generalized hypergeometric function ${}_3F_2$

The next example was the motivating example of this study. We uses the notations of Example 1.1.2.

Example 3.2.1 (Generalized hypergeometric function ${}_3F_2$). The generalized hypergeometric function ${}_3F_2$ admits a double integral representation that can be regarded as a pairing of a twisted cycle on a 2-dimensional space with the coefficients in a 1-dimensional locally constant sheaf and a twisted cocycle. Another integral representation is

$$\begin{aligned}&\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}y^{p+q-1}{}_3F_2(\alpha,\beta,p;\gamma,p+q;x)\\ &=\int_0^y t^{p-1}(y-t)^{q-1}F(\alpha,\beta,\gamma;y)dt\end{aligned}$$

(cf. [25, Section 3]), which can be proved by expanding the Gauss hypergeometric function in the integral into the series and using the formula $B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. We will regard the single integral representation above as an element of a period matrix associated to a locally constant sheaf of rank 2 and evaluate intersection numbers by applying our method.

Set

$$\begin{aligned}T&=\mathbb{C}\setminus\{0,y,1\}\\ \Omega&=\begin{pmatrix}0&0\\ \alpha&-\gamma\end{pmatrix}\frac{dt}{t}+\begin{pmatrix}0&\beta\\ 0&\alpha+\beta-\gamma\end{pmatrix}\frac{d(t-1)}{t-1}+\begin{pmatrix}p-1&0\\ 0&p-1\end{pmatrix}\frac{dt}{t}\\ &\quad+\begin{pmatrix}q-1&0\\ 0&q-1\end{pmatrix}\frac{d(y-t)}{y-t},\\ \nabla_+^*&=d-\Omega,\\ \nabla_-^*&=d+\Omega.\end{aligned}$$

The vector valued functions

$$\begin{aligned}q_i^+(t)&=t^{p-1}(y-t)^{q-1}p_i^+(t),\\ q_i^-(t)&=t^{-p+1}(y-t)^{-q+1}p_i^-(t)\end{aligned}$$

are in the kernels $\mathrm{Ker}\,\nabla_+^*$ and $\mathrm{Ker}\,\nabla_-^*$ respectively where the column vectors p_i^+ and p_i^- are defined in (1.2). The bilinear form S^* that gives the isomorphism between the dual of $\mathrm{Ker}\,\nabla_+^*$ and $\mathrm{Ker}\,\nabla_-^*$ is given by (1.1). Notice that the value of S^* can be given by the intersection number of cycles γ_i and γ_j :

$$S^*(q_i^+, q_j^-) = \text{reg}(\gamma_i) \cdot \gamma_j$$
$$= \begin{cases} -\frac{d_{i,i+1}}{d_i d_{i+1}}, & i = j \\ \frac{1}{d_j}, & j = i+1 \\ \frac{c_j}{d_j}, & j = i-1 \\ 0, & \text{other cases} \end{cases}$$

where we put

$$\beta_1 = \alpha, \beta_2 = \gamma - \alpha, \beta_3 = -\beta, \beta_4 = \beta - \gamma$$

and

$$c_i = e^{2\pi i \beta_i}, \quad d_i = c_i - 1, \quad d_{ij} = c_i c_j - 1$$

([3, Theorem 2] and [12, Theorem 2.1]).

Here we put

$$\beta_1 = \alpha_1, \beta_2 = b_2 - a_1, \beta_3 = -a_2, \beta_4 = -\alpha_5, p = \alpha_7, q = \alpha_8$$

We assume $1 < 1/x$. By looking at monodromy groups of the integrals of the period matrices, we can see that the functions p_i^+ and p_i^- have the following asymptotic behavior:

$$\begin{aligned} p_1^+ &= \text{ holomorphic function at } t = 0 \\ p_2^+ &= (1-t)^{\beta_2+\beta_3} \times (\text{meromorphic function at } t = 1\ ) \\ p_3^+ &= t^{\beta_3+\beta_4} \times (\text{meromorphic function at } t = 0\ ) \end{aligned}$$

and

$$\begin{aligned} p_1^- &= \text{ holomorphic function at } t = 0 \\ p_2^- &= (1-t)^{-\beta_2-\beta_3} \times (\text{ meromorphic function at } t = 1\ ) \\ p_3^- &= t^{-\beta_3-\beta_4} \times (\text{ meromorphic function at } t = 0\ ). \end{aligned}$$

Put $c_5 = e^{2\pi i p}, c_6 = e^{2\pi i q}$ and $d_{i,j,k,\ldots} = c_i c_j c_k \cdots - 1$. We assume $c_i \neq 1$. Define twisted cycles as follows:

$$\begin{aligned} \text{reg}(\sigma_1) &= \frac{1}{d_5} C_0 \otimes q_1^+ + [\varepsilon, y - \varepsilon] \otimes q_1^+ - \frac{1}{d_6} C_y \otimes q_1^+, \\ \text{reg}(\sigma_2) &= \frac{1}{d_6} C_y \otimes q_2^+ + [\varepsilon + y, 1 - \varepsilon] \otimes q_2^+ - \frac{1}{d_{2,3}} C_1 \otimes q_2^+, \\ \text{reg}(\sigma_3) &= \frac{1}{d_{3,4,5}} C_0 \otimes q_3^+ + [\varepsilon, y - \varepsilon] \otimes q_3^+ - \frac{1}{d_6} C_y \otimes q_3^+ \end{aligned}$$

and

$$\begin{aligned} \tau_1 &= (0, y) \otimes q_1^-, \\ \tau_2 &= (y, 1) \otimes q_2^-, \\ \tau_3 &= (0, y) \otimes q_3^- \end{aligned}$$

where C_p is the circle of the radius ε with the center p, then we have $\mathrm{reg}(\sigma_i) \in H_1(T, \mathrm{Ker}\,\nabla_+^*)$ and $\tau_i \in H_1^{lf}(T, \mathrm{Ker}\,\nabla_-^*)$.

Let us compute the self intersection number $\mathrm{reg}(\sigma_2) \cdot \tau_2$:

$$\begin{aligned}\mathrm{reg}(\sigma_2) \cdot \tau_2 &= \frac{-1}{d_6} S^*(q_2^+, q_2^-) - 1 \cdot S^*(q_2^+, q_2^-) - \frac{1}{d_{2,3}} S^*(q_2^+, q_2^-) \\ &= -\frac{d_{2,3,6}}{d_6 d_{2,3}} \cdot (-1) \cdot \frac{d_{2,3}}{d_2 d_3} \\ &= \frac{d_{2,3,6}}{d_2 d_3 d_6}. \end{aligned} \tag{3.1}$$

Evaluating other intersection numbers in a similar way, we get the following intersection matrix I_h' :

$$\begin{pmatrix} \frac{d_{5,6}d_{1,2}}{d_{1,2,5,6}} & \frac{1}{d_2 d_6} & 0 \\ \frac{c_2 c_6}{d_2 d_6} & \frac{d_{2,3,6}}{d_2 d_3 d_6} & \frac{c_3}{d_6 d_3} \\ 0 & \frac{c_6}{d_6 d_3} & \frac{d_{3,4,5,6} d_{3,4}}{d_{3,4,5} d_6 d_3 d_4} \end{pmatrix}.$$

Let us compare our result with [13]. We take an element $v \in \mathbb{C}^2$ such that $S((x, y), v) = x$. Clearly, $\nabla_+ v dt = 0$.

Put

$$u(t, s) = t^{\beta_5 - 1}(y - t)^{\beta_6 - 1} s^{\beta_1 - 1}(1 - s)^{\beta_2 - 1}(1 - ts)^{\beta_3}$$

and

$$\begin{aligned} D_1 &= \{(s, t) \mid 0 < s < 1, 0 < t < y\} \\ D_2 &= \{(s, t) \mid s > 1, t > y, s < 1/t\} \\ D_3 &= \{(s, t) \mid t < y, s > 1/t\}. \end{aligned}$$

Then the integral $(\mathrm{reg}(\sigma_i), vdt)$ is equal to

$$\int_{D_i} u(t, s) dt ds.$$

Thus, our 1-dimensional twisted cycles $\mathrm{reg}(\sigma_i)$ can be naturally regarded as an element of 2-dimensional twisted cycles; we consider the twisted homology group

$$H_2(X, \mathcal{L}^{-1})$$

where

$$\begin{aligned} X &= \{(s, t) \in \mathbb{C}^2 \mid t(y - t)s(1 - s)(1 - ts) \neq 0\} \\ \mathcal{L}^{\pm} &= \text{ the locally constant sheaf defined by } u^{\mp} \end{aligned}$$

and get the following embedding $i_{\pm}$.

Theorem 3.2.1. *1. Suppose that $\beta_i \notin \mathbb{Z}$ and $\sum \beta_i \notin \mathbb{Z}$. Then, there exists an injection*

$$i_{\pm} \;:\; H_2(X, \mathcal{L}^{\mp}) \longrightarrow H_1(\mathbb{C} \setminus \{0, y, 1\}, \mathrm{Ker}\,\nabla_{\pm}^*)$$

where $i_+(\mathrm{reg}(D_i) \otimes u(t, s)) = \mathrm{reg}(\sigma_i)$ and $i_-(D_i \otimes u^{-1}(t, s)) = \tau_i$.

2. *Under the same condition, the map*

$$j_\pm \ : \ H^2(X, \mathcal{L}^\pm) \longrightarrow H^1(\mathbb{C} \setminus \{0, y, 1\}, \mathrm{Ker}\, \nabla_\pm)$$

is an injection.

The statement 2 can be proved by using the natural isomorphisms $H_2(X, \mathcal{L}^\mp) \simeq H^2(X, \mathcal{L}^\pm)$ and $H_1(T, \mathrm{Ker}\, \nabla_+^*) \simeq H^1(T, \mathrm{Ker}\, \nabla_+)$ where $T = \mathbb{C}\setminus\{0, y, 1\}$ defined through the period matrix. We will give a different proof to the statement 2 by Orlik-Solomon algebras. The proof can be generalizable for "generic" locally constant sheaf $\mathcal{L}$.

Now, applying the transformation $s \to 1/s$, we get a similar arrangement of lines to that considered in [13, Figure 3.4]. We can see that $\mathrm{reg}(\sigma_i) \cdot \tau_j = \mathrm{reg}(D_i) \cdot D_j$ for $1 \leq i, j \leq 3$ by a suitable choice of branches. Note that a miracle cancellation is happening in (3.1) and $\mathrm{reg}(\sigma_2) \cdot \tau_2$ agrees with $\mathrm{reg}(D_2) \cdot D_2$, which is evaluated in [13]. In Section , it is shown that the two intersection numbers evaluated in $H^2(X, \mathcal{L}^\pm)$ and $H^1(T, \mathrm{Ker}\, \nabla_\pm)$ agree; for $[\xi] \in H^2(X, \mathcal{L}^+)$ and $[\eta] \in H^2(X, \mathcal{L}^-)$, $[\xi]\cdot[\eta] = [j_+(\xi)]\cdot[j_-(\eta)]$. Since we choose $j_\pm$ such that the period matrix defined as pairing of $H_2(X, \mathcal{L}^{\pm 1})$ and $H^2(X, \mathcal{L}^{\mp 1})$ is a submatrix of the period matrix obtained by the pairing $H_1(T, \mathrm{Ker}\, \nabla_\pm^*)$ and $H^1(T, \mathrm{Ker}\, \nabla_\pm)$. It follows from the twisted period relation that two intersection numbers of cycles agree; for $D_i \otimes u^{\pm 1} \in H_2(X, \mathcal{L}^{\pm 1})$,

$$[\mathrm{reg}(D_i \otimes u^{+1})] \cdot [D_j \otimes u^{-1}] = [i_+(D_i \otimes u^{+1})] \cdot [i_-(D_j \otimes u^{-1})].$$

Finally, we consider the projection

$$\pi \ : \ T' = \{(t, y) \in \mathbb{C}^2 \,|\, t \neq 0, y, 1, y \neq 0, 1\} \longrightarrow Y = \{y \in \mathbb{C} \,|\, y \neq 0, 1\}$$

and the families of locally constant sheaves on $\pi^{-1}(y)$: $\mathrm{Ker}\, \nabla_+$ and $\mathrm{Ker}\, \nabla_- \subset \mathcal{O}_{\pi^{-1}(y)} \times \mathbb{C}^2$. It follows form [4, p.105 Cor 6.11] and [11, Theorem 6] that

$$\dim H_1^{lf}(\pi^{-1}(y), \mathrm{Ker}\, \nabla_-^*) = \dim H^1(\pi^{-1}(y), \mathrm{Ker}\, \nabla_-) = 4.$$

It is shown in Example that the set $\{\mathrm{reg}(\sigma_1), \mathrm{reg}(\sigma_2), \mathrm{reg}(\sigma_2)\}$ is a basis of a monodromy invariant subspace of $H_1(\pi^{-1}(y), \mathrm{Ker}\, \nabla_+^*)$. The set defines a locally constant sheaf of rank 3 on Y by a period matrix, e.g.,

$$H_1 \ni \sigma \longmapsto ((\sigma, vdt), \frac{d}{dy}(\sigma, vdt), \frac{d^2}{dy^2}(\sigma, vdt)) \in \mathcal{O}_Y \otimes \mathbb{C}^3.$$

Intersection matrix I_h' is invariant under the monodromy matrices.

We explain an implicit meaning of statement 2 of Theorem 3.2.1. The injectivity is not a special property of ${}_3F_2$. It holds for "generic" locally constant sheaf of rank one on the complement of hyperplane arrangement. We consider $p + q$ hyperplanes

$$\ell_1(t) = t_1 - a_1, \ldots, \ell_p(t) = t_1 - a_p, \ell_{p+1}(t), \ldots, \ell_{p+q}(t)$$

where a_i are constants and $\ell_i(t)$, $i > p$ are not horizontal to the hyperplane $t_1 = 0$. We regard $\prod_{i=p+1}^{p+q} \ell_i(t) = 0$ as a hyperplane arrangement in $\mathbb{C}^{n-1}$ with a parameter t_1. Then, there exists a finite set of points $A = \{a_{p+1}, \cdots, a_{p+q'}\}$ such that the arrangement is not generic if and only if t_1 belongs to this set; for any i, the face lattice of the arrangement $\prod_{i=p+1}^{p+q} \ell_i(t) = 0$ in $\mathbb{C}^{n-1}$ is the same as far as t_1 keeps staying in one of the segments of $\mathbb{C} \setminus A$.

Put

$$\begin{aligned} X &= \mathbb{C}^n \setminus V\left(\prod_{i=1}^{p+q} \ell_i(t)\right), \\ X' &= X \setminus V\left(\prod_{i=p+1}^{p+q'} (t_1 - a_i)\right), \\ Z &= V\left(\prod_{i=p+1}^{p+q'} (t_1 - a_i)\right) \cap X, \\ T &= \mathbb{C} \setminus \{a_1, \ldots, a_p, a_{p+1}, \ldots, a_{p+q'}\}. \end{aligned}$$

The spaces X, X' and Z satisfy the relation

$$X \setminus Z = X'.$$

Let f be the projection

$$f \,:\, X' \ni (t_1, \ldots, t_n) \mapsto t_1 \in T$$

and g be a map which sends T to a point. We denote by Rf_* the right derived functor of f.

In order to show injectivity, we utilize well-known facts in the sheaf theory.

Let $\mathcal{L}$ be a locally constant sheaf on X and by $\mathcal{L}'$ the restriction of $\mathcal{L}$ to X'. Then, by the definition of the cohomology group, we have $H^i(X', \mathcal{L}') = R^i(g \circ f)_* \mathcal{L}'$. The first fact we need is

$$R(g \circ f)_* \mathcal{L}' = Rg_* Rf_* \mathcal{L}'. \tag{3.2}$$

The second fact is the following short exact sequence:

$$0 \to H^i(X, \mathcal{L}) \to H^i(X', \mathcal{L}) \to H^{i-1}(Z, \mathcal{L}_{|Z}) \to 0 \tag{3.3}$$

for any "generic" locally constant sheaf. In this case, "generic" means that $\mathcal{L}$ satisfies $H^i(X, \mathcal{L}) \simeq H^i(A(\mathcal{A}), \omega)$, where $A(\mathcal{A})$ is the Orlik-Solomon algebra. We will prove the exact sequence (3.3) in Appendix A.1 using Orlik-Solomon algebras.

Proof. (*Proof of statement 2 of Theorem 3.2.1.*)

We apply a transformation $s \to 1/s$ and prove the theorem for

$$\begin{aligned} X &= \{(t, s) \,|\, st(t-y)(s-t)(s-1) \neq 0\}, \\ X' &= \{(t, s) \,|\, st(t-y)(s-t)(s-1)(t-1) \neq 0\}, \\ Z &= X \cap \{(t, s) \,|\, t - 1 = 0\}, \\ T &= \{t \,|\, t \neq 0, y, 1\}. \end{aligned}$$

Let $\mathcal{L}$ be the locally constant sheaf defined by

$$u^{-1}(1/s, t) = t^{-\beta_5+1}(y-t)^{-\beta_6+1} s^{-2+\beta_1+\beta_2+\beta_3}(s-1)^{-\beta_2+1}(s-t)^{-\beta_3}.$$

We have $H^1(Z, \mathcal{L}_{|Z}) = \mathbb{C}$, since $\mathcal{L}_{|Z}$ is the locally constant sheaf defined by

$$s^{-2+\beta_1+\beta_2+\beta_3}(s-1)^{-\beta_2-\beta_3+1}.$$

In this case, "generic" means that the following condition holds:

$$\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_1+\beta_2-\beta_5,$$
$$\beta_1+\beta_3-\beta_5-\beta_6, \beta_1-\beta_5-\beta_6, \beta_2+\beta_3, \beta_2+\beta_5,$$
$$\beta_2+\beta_6, \beta_3+\beta_6, \beta_4+\beta_6, \beta_5+\beta_6 \notin \mathbb{Z}$$

Then we have

$$0 \to H^2(X, \mathcal{L}) \to H^2(X', \mathcal{L}') \to H^1(Z, \mathcal{L}_{|Z}) = \mathbb{C} \to 0.$$

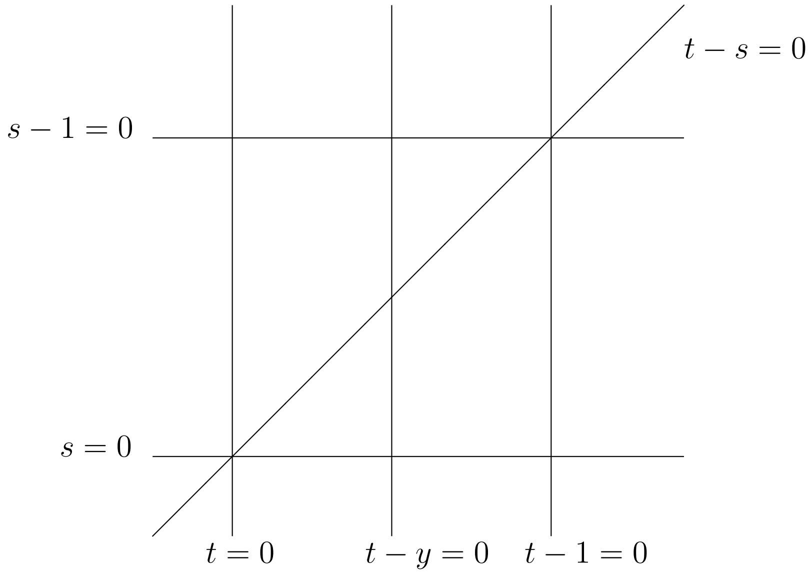
Since $R^i f_* \mathcal{L}' = 0$ for $i \neq 1$ and (3.2) holds under the condition above, we have

$$H^2(X', \mathcal{L}') = R^1 g_*(R^1 f_* \mathcal{L}') = H^1(T, R^1 f_* \mathcal{L}') = H^1(T, \operatorname{Ker} \nabla_+).$$

Hence, we have the exact sequence

$$0 \longrightarrow H^2(X, \mathcal{L}) \longrightarrow H^1(T, \operatorname{Ker} \nabla_+) \longrightarrow \mathbb{C} \longrightarrow 0,$$

which implies that $H^2(X, \mathcal{L}) \to H^1(T, \operatorname{Ker} \nabla_+)$ is injective. □



3.3 Example: Selberg integrals

Let $x_1, \dots, x_m$ be distinct real numbers as $x_1 < x_2 < \cdots < x_m$ and $\alpha_0, \alpha_1, \dots, \alpha_m$ complex numbers. In this section we consider twisted homology groups associated with the Selberg-type integral with dimension 2:

$$\int_{[x_i,x_j]\times[x_i,x_j]} \prod_{k=1}^{m} (s-x_k)^{\alpha_k}(t-x_k)^{\alpha_k}(s-t)^{\alpha_0} \frac{dsdt}{(s-x_i)(t-x_i)}$$
$$= \int_{x_i}^{x_j} \prod_{k=1}^{m} (t-x_k)^{\alpha_k} \left\{ \int_{x_i}^{x_j} \prod_{k=1}^{m} (s-x_k)^{\alpha_k}(s-t)^{\alpha_0} \frac{ds}{s-x_i} \right\} \frac{dt}{t-x_i}.$$

In order to treat this integral, we use a similar method of Example 3.2.

Set

$$
\begin{aligned}
T &= \left\{ t \in \mathbb{C} \;\middle|\; \prod_{k=1}^{m} (t - x_k) \neq 0 \right\}, \\
\Omega &= \frac{dt}{t - x_1} A_1 + \cdots + \frac{dt}{t - x_1} A_m, \\
A_j &= \alpha_j I_m + \alpha_1 E_{j1} + \cdots + \alpha_m E_{jm} + \alpha_0 E_{jj}, \\
\nabla_+^* &= d - \Omega, \\
\nabla_-^* &= d + \Omega.
\end{aligned}
$$

Here A_j are $m \times m$-matrices and E_{ij} are matrix elements at (i, j) component. The differential system $df - \Omega f = 0$ are given in [1].

We assume that the weight $\{\alpha_k\}$ satisfies the condition:

$$
\alpha_0, \alpha_1, \ldots, \alpha_m, \alpha_0 + 2\alpha_1, \ldots, \alpha_0 + 2\alpha_m \notin \mathbb{Z}. \tag{3.4}
$$

This is the special case of the condition [1, (C.1)].

We put $x_0 = t$ and denote by γ_i the open interval (x_i, x_{i+1}). Let $\omega_j = \frac{ds}{s - x_j}$. For $1 \leq i \leq m$ we define vector valued functions $p_i^{\pm}(t)$ on $\{t \in \mathbb{C} \mid \operatorname{Im} t < 0\}$ by

$$
p_i^{\pm}(t) = {}^t\left(\int_{\gamma_i} u^{\pm} \omega_1, \ldots, \int_{\gamma_i} u^{\pm} \omega_m \right),
$$

where

$$
u^{\pm}(s, t) = \prod_{k=0}^{m} (s - x_k)^{\pm \alpha_k}
$$

is a single valued function on the simply connected domain $\{(s, t) \in \mathbb{C}^2 \mid \operatorname{Im} s < \operatorname{Im} t < 0\}$.

Then vector valued functions

$$
q_i^{\pm}(t) = \prod_{k=1}^{m} (t - x_k)^{\pm \alpha_k} p_i^{\pm}(t) \qquad (1 \leq i \leq m)
$$

are solutions of the differential system $\nabla_{\pm}^* q^{\pm} = 0$. We put $q_{ij}^{\pm}(t) = \sum_{i \leq k < j} q_k^{\pm}(t)$.

Put

$$
c_j = \exp(2\pi\sqrt{-1}\alpha_j), \qquad d_j = c_j - 1, \qquad c_{jk\cdots} = c_j c_k \cdots, \qquad d_{jk\cdots} = c_{jk\cdots} - 1.
$$

The duality of $\operatorname{Ker} \nabla_+^*$ and $\operatorname{Ker} \nabla_-^*$ is written as

$$
\begin{aligned}
S^*(q_i^+, q_j^-) &= \operatorname{reg}(\gamma_i \otimes u) \cdot (\gamma_j \otimes u^{-1}) \\
&= \begin{cases}
-\frac{d_{i,i+1}}{d_i d_{i+1}}, & i = j \\
\frac{1}{d_j}, & j = i + 1 \\
\frac{c_j}{d_j}, & j = i - 1 \\
0. & \text{other cases}
\end{cases}
\end{aligned}
$$

Fix a real number t_0 as $x_m < t_0$. Then the analytic continuation of the functions $q_1^+(t), q_2^+(t), \ldots, q_m^+(t)$ along a simple closed curve around x_i of positive orientation at the base point t_0 are written as ${}^t(q_1^+, \ldots, q_m^+) \longmapsto \Gamma_i{}^t(q_1^+, \ldots, q_m^+)$, where Γ_i are $m \times m$ matrices which has eigenvalues $c_i, \ldots, c_i$ ($m-1$ times) and $c_i^2 c_0$. From the condition (3.4) we obtain that Γ_i and $D_i = \Gamma_i - I_m$ are regular. Therefore there exists the inverse map D_i^{-1}.

In easy way we obtain

$$D_k^{-1} q_{kl}^+ = \frac{d_{k,k}}{d_k d_{0,k,k}} q_{kl}^+ - \frac{c_k^2 d_0}{d_k d_{0,k,k}} q_{l,0}^+,$$
$$D_l^{-1} q_{kl}^+ = \frac{1}{d_l} q_{kl}^+ + \frac{c_l d_0}{d_l d_{0,l,l}} q_{l,0}^+$$

for $1 \leq k < l \leq m$. These formulae is used to evaluate intersection numbers.

In [1], symmetric domains of $X = \{(s,t) \in \mathbb{C}^2 \mid \prod_{k=1}^m (s-x_k)(t-x_k)(s-t) \neq 0\}$ with natural actions of elements of $\mathfrak{S}_2$ are treated as integral domains of Selberg-type integrals.

We define cycles $\tau_{ij}^{\pm}$ of $H_1^{lf}(T, \mathrm{Ker}\,\nabla_{\pm}^*)$ by

$$\tau_{ij}^{\pm} = (x_i, x_j) \otimes q_{ij}^{\pm}, \qquad (1 \leq i < j \leq m).$$

The cycle $\tau_{ij}^{\pm}$ corresponds to a symmetric domain $\{(s,t) \in \mathbb{R}^2 \mid x_1 \leq s, t \leq x_j\}$. We denote by $H_1^{lf}(T, \mathrm{Ker}\,\nabla_{\pm}^*)^{\mathfrak{S}_2}$ the subspace of $H_1^{lf}(T, \mathrm{Ker}\,\nabla_{\pm}^*)$ spanned by $\{\tau_{ij}^{\pm}\}$.

We are interested in the intersection form $H_1^{lf}(T, \mathrm{Ker}\,\nabla_{-}^*)^{\mathfrak{S}_2} \times H_1^{lf}(T, \mathrm{Ker}\,\nabla_{+}^*)^{\mathfrak{S}_2} \to \mathbb{C}$ for the bases $\{\tau_{ij}^{\pm}\}$ and explicitly evaluate the intersection number $\mathrm{reg}(\tau_{kl}^+) \cdot \tau_{ij}^-$ for the set of four natural numbers $\{i,j,k,l \mid i < j \text{ and } k < l\}$. Then the set $\{i,j,k,l\}$ has thirteen different configurations. For each configurations we discuss the intersection number.

Theorem 3.3.1. *The intersection number of $\tau_{ij}^- \in H_1(T, \mathrm{Ker}\,\nabla_-^*)$ with $\tau_{kl}^+ \in H_1(T, \mathrm{Ker}\,\nabla_+^*)$ has the following values:*

1. *If $i < j = k < l$, then* $\tau_{kl}^+ \cdot \tau_{ij}^- = \dfrac{c_{k,k} d_{0,0,k,k}}{d_k d_{0,k,k} d_{0,k}}$.

2. *If $i < k < j < l$, then* $\tau_{kl}^+ \cdot \tau_{ij}^- = 1$.

3. *If $i = k < j < l$, then* $\tau_{kl}^+ \cdot \tau_{ij}^- = \dfrac{d_{k,k}}{d_k^2 d_{0,k,k}}$.

4. *If $i = k < j = l$, then* $\tau_{kl}^+ \cdot \tau_{ij}^- = \dfrac{d_{0,0,k,k,l,l} d_{kl} + d_{0,k,l}(c_k + c_l)}{d_k d_l d_{0,k,k} d_{0,l,l}}$.

5. *If $i < k < j = l$, then* $\tau_{kl}^+ \cdot \tau_{ij}^- = \dfrac{d_{l,l}}{d_l^2 d_{0,l,l}}$.

6. *If $i = k < l < j$, then* $\tau_{kl}^+ \cdot \tau_{ij}^- = \dfrac{c_{0,k,k} d_{k,k}}{d_k^2 d_{0,k,k}}$.

7. *If $k < l = i < j$, then* $\tau_{kl}^+ \cdot \tau_{ij}^- = \dfrac{d_{l,0}}{d_l d_{0,l,l}}$.

8. *If* $k < i < l < j$, *then* $\tau_{kl}^{+} \cdot \tau_{ij}^{-} = 1$.

9. *If* $k < i < l = j$, *then* $\tau_{kl}^{+} \cdot \tau_{ij}^{-} = \dfrac{c_{0,l,l}d_{l,l}}{d_l^2 d_{0,l,l}}$.

10. *If otherwise, then* $\tau_{kl}^{+} \cdot \tau_{ij}^{-} = 0$.

Example 3.3.1. Set $m = 3$. The twisted homology group $H_1(T, \operatorname{Ker} \nabla_{-}^{*})^{\mathfrak{S}_2}$ (resp. $H_1(T, \operatorname{Ker} \nabla_{+}^{*})^{\mathfrak{S}_2}$) has a basis $\{\tau_{12}^{-}, \tau_{13}^{-}, \tau_{23}^{-}\}$ (resp. $\{\tau_{12}^{+}, \tau_{13}^{+}, \tau_{23}^{+}\}$). The intersection matrix of τ_{ij}^{-} with τ_{kl}^{+} is

$$\begin{pmatrix} \tau_{12}^{+} \cdot \tau_{12}^{-} & \tau_{13}^{+} \cdot \tau_{12}^{-} & \tau_{23}^{+} \cdot \tau_{12}^{-} \\ \tau_{12}^{+} \cdot \tau_{13}^{-} & \tau_{13}^{+} \cdot \tau_{13}^{-} & \tau_{23}^{+} \cdot \tau_{13}^{-} \\ \tau_{12}^{+} \cdot \tau_{23}^{-} & \tau_{13}^{+} \cdot \tau_{23}^{-} & \tau_{23}^{+} \cdot \tau_{23}^{-} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{d_{001122}d_{12}+d_{012}(c_1+c_2)}{d_1 d_2 d_{0,1,1} d_{0,2,2}} & \frac{d_{11}}{d_1^2 d_{0,1,1}} & \frac{c_2^2 d_{0022}}{d_2 d_{0,2,2} d_{02}} \\ \frac{c_0 c_1^2 d_{11}}{d_1^2 d_{0,1,1}} & \frac{d_{113300}d_{13}+d_{013}(c_1+c_3)}{d_1 d_3 d_{0,1,1} d_{0,3,3}} & \frac{d_{33}}{d_3^2 d_{0,3,3}} \\ \frac{d_{02}}{d_2 d_{0,2,2}} & \frac{c_0 c_3^2 d_{33}}{d_3^2 d_{0,3,3}} & \frac{d_{002233}d_{23}+d_{023}(c_2+c_3)}{d_2 d_3 d_{0,2,2} d_{0,3,3}} \end{pmatrix}$$

Chapter 4

Intersection numbers of Selberg-type integrals

4.1 Twisted cohomology groups

We assume that X is the complement of hyperplanes in $\mathbb{C}^r$. Let $h_1, \ldots, h_N$ be linear forms in $\mathbb{C}[x_1, \ldots, x_r]$ and set

$$X = \left\{ (x_1, \ldots, x_r) \in \mathbb{C}^r \;\middle|\; \prod_{i=1}^{N} h_i \neq 0 \right\},$$
$$\omega_+ = \omega_- = \lambda_1 \frac{dh_1}{h_1} + \cdots + \lambda_N \frac{dh_N}{h_N},$$
$$\nabla_+ = d + \omega_+, \qquad \nabla_- = d - \omega_-,$$

where $\lambda_1, \ldots, \lambda_N$ are sufficiently generic complex numbers so that all k-th twisted cohomology groups appearing in this section vanish, where k is not equal to the dimension of the base space X, and so that $H_c^k(X, \operatorname{Ker} \nabla_\pm)$ and $H^k(X, \operatorname{Ker} \nabla_\pm)$ agree.

We assume that we can choose linear forms $h'_1, \ldots, h'_n$ in $\mathbb{C}[y]$ such that there exists the projection

$$f : X \ni (x_1, \ldots, x_r) \longmapsto x_r \in T = \left\{ y \in \mathbb{C} \;\middle|\; \prod_{i=1}^{n} h'_i \neq 0 \right\}.$$

Then we have the natural inclusion $\iota : f^{-1}(y) \hookrightarrow X$, where $f^{-1}(y)$ is the fibre of y. We denote by ∇'_+ and ∇'_- covariant derivations $d + (\iota^* \omega_+)$ and $d - (\iota^* \omega_-)$ on $f^{-1}(y)$. Let $\{\sigma_1^+, \ldots, \sigma_m^+\}$ be a basis of $H^{r-1}(f^{-1}(y), \operatorname{Ker} \nabla'_+)$ and $\{\sigma_1^-, \ldots, \sigma_m^-\}$ a basis of $H_c^{r-1}(f^{-1}(y), \operatorname{Ker} \nabla'_-)$. Put $S = (S_{ij})_{i,j=1,\ldots,m}$ as

$$S_{ij} = \int_{f^{-1}(y)} \sigma_i^+ \wedge \sigma_j^-,$$

that is, it is an intersection matrix of twisted $(r-1)$-cocycles σ_i^+ and σ_j^-. We suppose that we can take the bases $\{\sigma_i^+\}$ and $\{\sigma_i^-\}$ so that the matrix S does not depend on y.

We take a suitable integral domain $\triangle$ and set

$$u^+(\triangle) = {}^t\left(\int_\triangle \prod_{k=1}^N h_k^{\lambda_k}\sigma_1^+, \dots, \int_\triangle \prod_{k=1}^N h_k^{\lambda_k}\sigma_m^+\right),$$
$$u^-(\triangle) = {}^t\left(\int_\triangle \prod_{k=1}^N h_k^{-\lambda_k}\sigma_1^-, \dots, \int_\triangle \prod_{k=1}^N h_k^{-\lambda_k}\sigma_m^-\right).$$

The functions $u^+(\triangle)$ (resp. $u^-(\triangle)$) satisfies a system of first order differential equation; let Ω_+, Ω_- be $m \times m$-matrix valued one forms which satisfy the following equations:

$$du^+ - {}^t\Omega_+ u^+ = 0, \qquad du^- + {}^t\Omega_- u^- = 0.$$

Here,the complete integrable condition $d\Omega_+ = \Omega_+ \wedge \Omega_+$ and $-d\Omega_- = \Omega_- \wedge \Omega_-$ are satisfied. In the sequel, we assume the condition (1.4). Put $\nabla''_+ = d + \Omega_+$ and $\nabla''_- = d - \Omega_-$. Then the locally constant sheaf $\operatorname{Ker}\nabla''_+$ is the dual sheaf of $\operatorname{Ker}\nabla''_-$. In Section , we will prove (1.4) for the Selberg-type arrangement using a recursive method.

We consider cohomology groups $H^1(T, \operatorname{Ker}\nabla''_+)$ and $H^1_c(T, \operatorname{Ker}\nabla''_-)$. In Definition 1.2.1, we defined the intersection number of the cocycles $\{\xi\}$ of $H^r(T, \operatorname{Ker}\nabla''_+)$ and η of $H^r_c(T, \operatorname{Ker}\nabla''_-)$ by the integral

$$[\xi] \cdot [\eta] := \int_T S(\xi, \eta).$$

We will see that the two intersection numbers evaluated in $H^r(X, \nabla_\pm)$ and $H^1(T, \operatorname{Ker}\nabla''_\pm)$ agree.

We suppose that we can choose bases $\{\tau^\pm\}$ of $H^r(X, \operatorname{Ker}\nabla_+)$ and $H^r_c(X, \operatorname{Ker}\nabla_-)$ such that $\tau^\pm$ can be expressed as

$$\tau^\pm = \sum_{i=1}^m \sigma_i^\pm \wedge \varphi_i^\pm, \qquad {}^t(\varphi_1^\pm, \dots, \varphi_m^\pm) \in H^1(T, \operatorname{Ker}\nabla''_\pm).$$

A small example of this decomposition is given in (**??**). For these bases $\{\tau^\pm\}$, we define the following $\mathbb{C}$-linear maps

$$R^+ : H^r(X, \operatorname{Ker}\nabla_+) \ni \tau^+ = \sigma_1^+ \wedge \varphi_1^+ + \cdots + \sigma_m^+ \wedge \varphi_m^+ \mapsto \begin{pmatrix} \varphi_1^+ \\ \vdots \\ \varphi_m^+ \end{pmatrix} \in H^1(T, \operatorname{Ker}\nabla''_+), \tag{4.1}$$

$$R^- : H^r_c(X, \operatorname{Ker}\nabla_-) \ni \tau^- = \sigma_1^- \wedge \varphi_1^- + \cdots + \sigma_m^- \wedge \varphi_m^- \mapsto \begin{pmatrix} \varphi_1^- \\ \vdots \\ \varphi_m^- \end{pmatrix} \in H^1_c(T, \operatorname{Ker}\nabla''_-). \tag{4.2}$$

Rewriting a multiple integral to an iterated integral, we get the following lemma.

Lemma 4.1.1. *The following diagram is commutative:*

$$\begin{array}{ccc} H^r(X, \operatorname{Ker} \nabla_+) \times H^r_c(X, \operatorname{Ker} \nabla_-) & \xrightarrow{I_X} & \mathbb{C} \\ {\scriptstyle R^+ \times R^-} \downarrow & & \| \\ H^1(T, \operatorname{Ker} \nabla''_+) \times H^1_c(T, \operatorname{Ker} \nabla''_-) & \xrightarrow[I_T]{} & \mathbb{C} \end{array}$$

Proof. Set

$$\tau^+ = \sigma_1^+ \wedge \varphi_1^+ + \cdots + \sigma_m^+ \wedge \varphi_m^+,$$
$$\tau^- = \sigma_1^- \wedge \varphi_1^- + \cdots + \sigma_m^- \wedge \varphi_m^-.$$

Then, we have

$$\begin{aligned} [\tau^+] \cdot [\tau^-] &= \int_X \tau^+ \wedge \tau^- \\ &= \int_X \left(\sum_{i=1}^m \sigma_i^+ \wedge \varphi_i^+ \right) \wedge \left(\sum_{j=1}^m \sigma_j^- \wedge \varphi_j^- \right) \\ &= \sum_{i=1}^m \sum_{j=1}^m \int_X \sigma_i^+ \wedge \varphi_i^+ \wedge \sigma_j^- \wedge \varphi_j^- \\ &= \sum_{i=1}^m \sum_{j=1}^m \int_T \left(\int_{f^{-1}(y)} \sigma_i^+ \wedge \sigma_j^- \right) \varphi_i^+ \wedge \varphi_j^- \\ &= \sum_{i=1}^m \sum_{j=1}^m \int_T s_{ij} \varphi_i^+ \wedge \varphi_j^- \\ &= \int_T S(\varphi^+, \varphi^-) \\ &= [R^+ \tau^+] \cdot [R^- \tau^-] \end{aligned}$$

□

We next explain a method to evaluate intersection numbers on $H^1(T, \operatorname{Ker} \nabla''_\pm)$. Put

$$\Omega_+ = \Omega_- = \frac{dh'_1}{h'_1} A_1 + \cdots + \frac{dh'_n}{h'_n} A_n,$$
$$A_\infty = -\sum_{k=1}^n A_k$$

where $A_1, \ldots, A_n \in M(m, \mathbb{C})$, and recall that

$$T = \left\{ y \in \mathbb{C} \;\middle|\; \prod_{i=1}^n h'_i \neq 0 \right\}.$$

Here $h'_1, \ldots, h'_n$ are linear forms in $\mathbb{C}[y]$.

Theorem 4.1.1. *Suppose that there exists* $A_1, \ldots, A_n, A_\infty \in M(m, \mathbb{C})$ *of which eigenvalues are not integers and so that* $A_1 + \cdots + A_n + A_\infty = 0$*. In addition, if the connections* $\nabla_+ = d + \Omega$ *and* $\nabla_- = d - \Omega$ *on* T *are written as*

$$\Omega = \frac{dh_1'}{h_1'} A_1 + \cdots + \frac{dh_n'}{h_n'} A_n,$$

then the intersection number of cocycles $\sigma = [\frac{dt}{t-x_i} v^+] \in H^1(T, \mathrm{Ker}\, \nabla_+'')$ *and* $\nu = [\frac{dt}{t-x_j} v^-] \in H^1(T, \mathrm{Ker}\, \nabla_-'')$ *is*

$$[\sigma] \cdot [\nu] = 2\pi\sqrt{-1} \times \left\{ \delta_{ij}\ S(A_i^{-1} v^+,\ v^-) + S(A_\infty^{-1} v^+,\ v^-) \right\},$$

where $v^\pm \in \mathbb{C}^m$.

Theorem 4.1.1 is a generalization of Theorem 2.1 ($k = 1$) in Matsumoto [15] to twisted cohomology groups with the coefficient sheaf of which rank is more than one. We omit the proof because it can be shown in a similar method with that explained in Section 4 of [15]. In Section 4.2, Theorem 4.1.1 and Lemma 4.1.1 are used to derive a recursion formula of intersection numbers for a basis of symmetric subspaces of twisted cohomology groups associated with Selberg-type integrals.

Remark. Let $b_1, \ldots, b_m$ be the eigenvalues of the matrix A_i. Then the eigenvalues of the local monodromy of the locally constant sheaf $\mathrm{Ker}\, \nabla_+''$ at the point t_i are $\exp(2\pi\sqrt{-1} b_1), \ldots, \exp(2\pi\sqrt{-}$ If none of $b_1, \ldots, b_m$ is an integer, then none of the eigenvalues is one. Therefore, if the hypothesis of Theorem 4.1.1 holds, then there exists the isomorphism $H^1(T, \mathrm{Ker}\, \nabla_+'') \simeq H_c^1(T, \mathrm{Ker}\, \nabla_+'')$. ([6, Lemma (1.6)])

Example 4.1.1. We consider twisted cohomology groups associated with the B-function

$$\int_0^1 x^a (1-x)^b \frac{dx}{x(1-x)}.$$

Set

$$X = \{x \in \mathbb{C} \mid x(1-x) \neq 0\},$$
$$\omega = a\frac{dx}{x} + b\frac{dx}{x-1},$$
$$\nabla_+ = d + \omega, \qquad \nabla_- = d - \omega$$

and assume that $a, b, a+b \notin \mathbb{Z}$. Then the dimension of $H^1(X, \mathrm{Ker}\, \nabla_-)$ is one and an intersection number on $H^1(X, \mathrm{Ker}\, \nabla_\pm)$ is given by

$$\left[\frac{dx}{x(1-x)}\right] \cdot \left[\frac{dx}{x(1-x)}\right] = 2\pi\sqrt{-1}\frac{a+b}{ab}.$$

Example 4.1.2. Let us illustrate our method under the setting of Example 1.2.1 and 3.1.2. We regards $\frac{dt}{t(1-t)}$ as a cocycle of $H^1(\mathbb{C} \setminus \{0,1\}, \mathrm{Ker}\, \nabla_\pm)$. Since

$$u^+(t) = t^\beta \int_0^{1-t} s^\alpha (1-s-t)^\gamma \frac{(1-t)ds}{s(1-s-t)},$$

we have a correspondence $\frac{dt}{t(1-t)}$ to $\frac{(1-t)ds}{s(1-s-t)} \wedge \frac{dt}{t(1-t)}$.

In Example 4.1.1, we have that the self intersection number of $\frac{(1-t)ds}{s(1-s-t)}$ is $S = 2\pi\sqrt{-1}\frac{\alpha+\gamma}{\alpha\gamma}$. In addition, we suppose

$$\alpha+\gamma, \beta, \alpha+\beta+\gamma \notin \mathbb{Z}. \tag{4.3}$$

From Theorem 4.1.1, the intersection pairing on $H^1(T, \operatorname{Ker}\nabla_\pm)$ is given by

$$\begin{aligned}
\left[\frac{dt}{t}\right]\cdot\left[\frac{dt}{t}\right] &= 2\pi\sqrt{-1}\left(\frac{1}{\beta}-\frac{1}{\alpha+\beta+\gamma}\right)S,\\
\left[\frac{dt}{t-1}\right]\cdot\left[\frac{dt}{t}\right] = \left[\frac{dt}{t}\right]\cdot\left[\frac{dt}{t-1}\right] &= 2\pi\sqrt{-1}\left(-\frac{1}{\alpha+\beta+\gamma}\right)S,\\
\left[\frac{dt}{t-1}\right]\cdot\left[\frac{dt}{t-1}\right] &= 2\pi\sqrt{-1}\left(\frac{1}{\alpha+\gamma}-\frac{1}{\alpha+\beta+\gamma}\right)S.
\end{aligned}$$

Since intersection forms on $H^1(T, \operatorname{Ker}\nabla_\pm)$ is bilinear,

$$\begin{aligned}
\left[\frac{dt}{t(1-t)}\right]\cdot\left[\frac{dt}{t(1-t)}\right] &= \left[\frac{dt}{t}-\frac{dt}{t-1}\right]\cdot\left[\frac{dt}{t}-\frac{dt}{t-1}\right]\\
&= \left[\frac{dt}{t}\right]\cdot\left[\frac{dt}{t}\right]-\left[\frac{dt}{t}\right]\cdot\left[\frac{dt}{t-1}\right]-\left[\frac{dt}{t-1}\right]\cdot\left[\frac{dt}{t}\right]+\left[\frac{dt}{t-1}\right]\cdot\left[\frac{dt}{t-1}\right]\\
&= 2\pi\sqrt{-1}\left(\frac{1}{\beta}+\frac{1}{\alpha+\gamma}\right)S\\
&= -4\pi^2\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma},
\end{aligned}$$

that is, the self intersection number of $\frac{ds\wedge dt}{st(1-s-t)}$ on $H^2(X, \operatorname{Ker}\nabla_\pm)$ is

$$\left[\frac{ds\wedge dt}{st(1-s-t)}\right]\cdot\left[\frac{ds\wedge dt}{st(1-s-t)}\right] = (2\pi\sqrt{-1})^2\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}.$$

4.2 Recursion formulae of intersection numbers with the Selberg-type integral.

In this section, using the method explained in previous sections, we study the intersection matrix of cohomology groups associated with the Selberg-type integral:

$$\int_C \prod_{1\le i<j\le n}(x_i-x_j)^\nu \prod_{i=1}^n\prod_{k=1}^m (x_i-t_k)^{\lambda_k} dx_1\cdots dx_n.$$

Set

$$\begin{aligned}
X = X(n,m) &= \left\{ (x_1,\dots,x_n)\in\mathbb{C}^n \;\middle|\; \prod_{1\le i<j\le n}(x_i-x_j)\prod_{i=1}^{n}\prod_{k=1}^{m}(x_i-t_k)\neq 0 \right\},\\
\Phi(n,m) &= \prod_{1\le i<j\le n}(x_i-x_j)^{\nu}\prod_{i=1}^{n}\prod_{k=1}^{m}(x_i-t_k)^{\lambda_k},\\
\omega &= \sum_{1\le i<j\le n}\nu\frac{dx_i-dx_j}{x_i-x_j}+\sum_{i=1}^{n}\sum_{k=1}^{m}\lambda_k\frac{dx_i}{x_i-t_k},\\
\nabla_{\pm}\tau &= d\tau\pm\omega\wedge\tau.
\end{aligned}$$

The cohomology group $H^n(X,\nabla_+)$ admits the natural action of $\mathfrak{S}_n$ by the change of indices of $x_1,\dots,x_n$. The symmetric subspace $H^n(X,\nabla_+)^{\mathfrak{S}_n}$ was studied in Aomoto [1] and Mimachi [17]. Our purpose is to derive recurrence relations of intersection numbers for a basis of the symmetric subspace. Our intersection matrix is expressed in terms of $n, m, \nu, \lambda_1,\dots,\lambda_m$. We will derive a recursion formula of intersection numbers with respect to n and m. We do not have explicit expressions of intersection numbers in general, but we can obtain the explicit formula of intersection numbers for small n and m using computer algebra systems.

In order to describe a basis of the symmetric subspace, we define some notations as follows. Put

$$\varphi_{(a_1a_2\cdots a_n)} = \left\{ \sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^{n}\frac{1}{x_{\sigma(i)}-t_{a_i}} \right\} dx_1\wedge\cdots\wedge dx_n.$$

This index $(a_1a_2\cdots a_n)$ is abbreviated as

$$(1^{k_1}2^{k_2}\cdots m^{k_m}) := (\underbrace{1\cdots1}_{k_1}\underbrace{2\cdots2}_{k_2}\cdots\underbrace{m\cdots m}_{k_m}).$$

We define the following finite set of indices:

$$\Xi_{n,m} = \{(1^{k_1}\cdots(m-1)^{k_{m-1}}) \mid k_1+k_2+\cdots+k_{m-1}=n\}.$$

The cardinal number of the set $\Xi_{n,m}$ is $\binom{n+m-2}{m-2}$. We regard $\Xi_{n,m}$ as a subset of $\Xi_{n,m+1}$ by $(1^{k_1}\cdots(m-1)^{k_{m-1}}) = (1^{k_1}\cdots(m-1)^{k_{m-1}}m^0)$.

We define the following maps:

$$\Xi_{n,m}\ni\xi=(1^{k_1}\cdots(m-1)^{k_{m-1}})\mapsto\xi_j:=(1^{k_1}\cdots j^{k_j-1}\cdots(m-1)^{k_{m-1}})\in\Xi_{n-1,m}, \tag{4.4}$$

$$\Xi_{n-1,m}\ni\xi=(1^{k_1}\cdots(m-1)^{k_{m-1}})\mapsto\xi^r:=(1^{k_1}\cdots r^{k_r+1}\cdots(m-1)^{k_{m-1}})\in\Xi_{n,m}, \tag{4.5}$$

$$\ell_j:\xi=(1^{k_1}\cdots(m-1)^{k_{m-1}})\mapsto\ell_j(\xi)=k_j. \tag{4.6}$$

Then we get the following relation between n-forms φ_ξ ($\xi\in\Xi_{n,m}$) and $(n-1)$-forms φ_{ξ_j} ($\xi_j\in\Xi_{n-1,m+1}$):

$$\varphi_\xi=\sum_{j=1}^{m-1}\varphi_{\xi_j}\wedge\ell_j(\xi)\frac{dx_n}{x_n-t_j}. \tag{4.7}$$

Let $\lambda_{m+1} = \lambda_{m+2} = \cdots = \lambda_{m+n-1} = \nu$. For any i such that $0 \leq i < n$ we put

$$\Lambda(n-i, m+i) = \left\{ k\lambda_j + \frac{k(k-1)}{2}\nu \;\middle|\; 1 \leq j \leq m+i, 1 \leq k \leq n-i \right\} \cup \left\{ -k\sum_{j=1}^{m+i}\lambda_j - \frac{k(k-1)}{2}\nu \;\middle|\; 1 \leq k \leq n-i \right\}.$$

Note that, if $\Lambda(n,m) \cap \mathbb{Z} = \emptyset$, then $\prod_{k=0}^{n-1}(\lambda_m + \frac{k}{2}\nu) \neq 0$.

Theorem 4.2.1 (Aomoto [1]). *Suppose the following conditions:*

1. $\Lambda(n,m) \cap \mathbb{Z}_{>0} = \emptyset$,
2. $\prod_{k=0}^{n-1}(\lambda_m + \frac{k}{2}\nu) \neq 0$.

Then the set $\{\varphi_\xi\}_{\xi\in\Xi_{n,m}}$ *is a basis of* $H^n(X(n,m), \nabla_+)^{\mathfrak{S}_n}$.

In order to evaluate intersection numbers for φ_ξ ($\xi \in \Xi_{n,m}$) by using the method explained in the previous section, we use the projection

$$f : X(n,m) \ni (x_1, \dots, x_n) \mapsto x_n \in Y = \left\{ y \in \mathbb{C} \;\middle|\; \prod_{k=1}^{m}(y - t_k) \neq 0 \right\}.$$

The fibre of $y \in Y$ is

$$f^{-1}(y) = \left\{ (x_1, \dots, x_{n-1}) \in \mathbb{C}^{n-1} \;\middle|\; \prod_{1\leq i<j\leq n-1}(x_i - x_j)\prod_{i=1}^{n-1}\prod_{k=1}^{m+1}(x_i - t_k) \neq 0 \right\} = X(n-1, m+1),$$

where $t_{m+1} = y$.

Since our purpose is to derive a recursion formula of intersection numbers, we assume that

$$\bigcup_{i=1}^{n}\Lambda(i,\ m+n-i) \cap \mathbb{Z} = \emptyset. \tag{4.8}$$

By virtue of Theorem 4.2.1, the set $\{\varphi_\xi \mid \xi \in \Xi_{n-1,m+1}\}$ is a basis of $H^{n-1}(f^{-1}(y), \nabla'_-)^{\mathfrak{S}_{n-1}}$ under the condition (4.8). For any $\xi \in \Xi_{n-1,m+1}$ we put $u_\xi = \int_\Gamma \Phi(n,m)\varphi_\xi$ where we assume that the integral domain Γ is invariant by the action of $\mathfrak{S}_n$ ([1]). The function u_ξ of y satisfies the ordinary differential equation:

$$\frac{d}{dy}u_\xi = \sum_{s=1}^{m}\sum_{r=1}^{m}\frac{\ell_s(\xi)(\lambda_r + \frac{\nu}{2}\ell_r(\xi))}{y - t_s}u_{\xi_s^r} + \sum_{s=1}^{m}\frac{\lambda_s + \frac{\nu}{2}\ell_s(\xi)}{y - t_s}u_\xi.$$

(see [17], Prop. 2.1.) Namely the $\binom{n+m-2}{m-1}$-dimensional vector valued function $U = (u_\xi)_{\xi\in\Xi_{n-1,m+1}}$ of y satisfies the system:

$$\frac{d}{dy}U = \left(\frac{1}{y-t_1}{}^tA_1 + \cdots + \frac{1}{y-t_m}{}^tA_m\right)U,$$

where $A_1, \dots, A_m$ are square matrices of size $\binom{n+m-2}{m-1}$ and all elements of $A_1, \dots, A_m$ are linear forms of $\lambda_1, \dots, \lambda_m, \nu$. We put

$$\Omega = \frac{dy}{y-t_1}A_1 + \cdots + \frac{dy}{y-t_m}A_m,$$
$$\nabla''_+ = d + \Omega, \qquad \nabla''_- = d - \Omega.$$

By Aomoto [1] Lemma 1.6, we can see that none of eigenvalues of matrices $A_1, \dots, A_m, A_\infty$ is a non-positive integer under the condition $\Lambda \cap \mathbb{Z}_{\le 0} = \emptyset$.

Remark. Under a suitable total order in $\Xi_{n-1,m+1}$, one of A_i can be expressed as a tridiagonal matrix. For example, A_1 is expressed as a lower tridiagonal matrix with respect to the lexicographic order in $\Xi_{n-1,m+1}$.

Example 4.2.1. In the case $n = 2$, $m = 3$, the coefficient matrices $A_1, \dots, A_3$ are written as

$$A_1 = \begin{pmatrix} 2\lambda_1+\nu & & \\ \lambda_2 & \lambda_1 & \\ \lambda_3 & & \lambda_1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \lambda_2 & \lambda_1 & \\ & 2\lambda_2+\nu & \\ & \lambda_3 & \lambda_2 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} \lambda_3 & & \lambda_1 \\ & \lambda_3 & \lambda_2 \\ & & 2\lambda_3+\nu \end{pmatrix}.$$

We prove the relation (1.4) for the Selberg-type arrangement.

Theorem 4.2.2. *Let S be the intersection matrix for the basis $\{\varphi_\xi\}_{\xi\in\Xi_{n,m}}$ of $H^n(X(n,m), \nabla_\pm)$. If the condition* (4.8) *holds, then S satisfies the relation* (1.4)*;*

$$S^t\Omega - S\Omega = 0.$$

Proof. We use an induction on the dimension n of X. If $n = 1$ then the theorem is clear. Assume $n > 1$. We identify S with an intersection matrix on $H^1(Y, \nabla''_\pm)$. From the condition (4.8) none of eigenvalues of each A_i are integers. Then we can apply Theorem 1.2.1;

$$I_h = {}^tP_+{}^tS^{-1}P_-.$$

From Theorem 4.2.3 for $n-1$ the matrix S does not depend on $y \in Y$ and, from the intersection theory of twisted homology groups, the intersection matrix I_h is also constant (cf. [12, Theorem 1.3]). Therefore

$$\begin{aligned} 0 = d({}^tP_+{}^tS^{-1}P_-) &= (d{}^tP_+){}^tS^{-1}P_- + {}^tP_+{}^tS^{-1}(dP_-) \\ &= (-{}^tP_+\Omega){}^tS^{-1}P_- + {}^tP_+{}^tS^{-1}({}^t\Omega P_-) \\ &= -{}^tP_+(\Omega{}^tS^{-1} - {}^tS^{-1t}\Omega)P_-. \end{aligned}$$

Hence $\Omega{}^tS^{-1} - {}^tS^{-1t}\Omega = 0$, that is, $S^t\Omega - S\Omega = 0$. □

In order to apply Theorem 4.1.1 to evaluate intersection numbers for the basis $\{\varphi_\xi\}$, we suppose that inverse matrices of $A_1, \cdots, A_m, A_\infty$ are expressed as

$$A_1^{-1} = (h^1_{\xi\xi'})_{\xi,\xi'\in\Xi_{n-1,m+1}}, \dots, A_m^{-1} = (h^m_{\xi\xi'})_{\xi,\xi'\in\Xi_{n-1,m+1}}, \; A_\infty^{-1} = (h^\infty_{\xi\xi'})_{\xi,\xi'\in\Xi_{n-1,m+1}}.$$

Note that we can get $h^i_{\xi\xi'}$ from A_i for given small n and m using computer algebra systems.

The following theorem gives a recursion formula in which the intersection form $S_{\xi\xi'}$ on $H^n(X(n,m), \nabla_\pm)$ are expressed in terms of the intersection form $S_{\pi\xi'_j}$ on $H^{n-1}(X(n-1, m+1), \nabla'_\pm)$.

Theorem 4.2.3. *Put* $S_{\xi\xi'} = [\varphi_\xi]\cdot[\varphi_{\xi'}]$. *Suppose the condition* (4.8). *Then, for any* $\xi, \xi' \in \Xi_{n,m}$, *we have*

$$S_{\xi\xi'} = 2\pi\sqrt{-1}\sum_{\pi\in\Xi_{n-1,m+1}}\left\{\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\ell_i(\xi)\ell_j(\xi')h^{\infty}_{\pi\xi_i}S_{\pi\xi'_j} + \sum_{i=1}^{m-1}\ell_i(\xi)\ell_i(\xi')h^{i}_{\pi\xi_i}S_{\pi\xi'_i}\right\}. \quad (4.9)$$

Proof. We use an induction on n.

In the case $n = 1$, since an intersection number $S_{\pi\xi'_j}$ in the right hand side of (4.9) is Kronecker's delta $\delta_{\pi\xi'_j}$, the theorem holds.

Next we assume $n > 2$. From the formula (4.7), we have the $\mathbb{C}$-linear map

$$H^n(X,\nabla_+)^{\mathfrak{S}_n} \ni \varphi_\xi \longmapsto \psi_\xi = \sum_{j=1}^{m-1}\frac{dy}{y-t_j}\ell_j(\xi)e_{\xi_j} \in H^1(Y,\nabla''_+), \quad (4.10)$$

where e_ξ is the vector whose ξ-th element is 1 and the other elements are 0.

Since this $\mathbb{C}$-linear map preserves intersection numbers, i.e. $[\varphi_\xi]\cdot[\varphi_{\xi'}] = [\psi_\xi]\cdot[\psi_{\xi'}]$, we may prove the theorem with respect to the right hand side.

By (4.8), none of eigenvalues of matrices $A_1,\dots,A_m,A_\infty$ is a non-positive integer, that is, it holds the hypothesis of Theorem 4.1.1. From the hypothesis of the induction, the intersection matrix for the basis $\{\varphi_\xi\}_{\xi\in\Xi_{n-1,m+1}}$ of $H^{n-1}(f^{-1}(y),\nabla'_-)^{\mathfrak{S}_{n-1}}$ is $S = (S_{\xi\xi'})_{\xi,\xi'\in\Xi_{n-1,m+1}}$. Since

$$A_i^{-1}e_\xi = \sum_{\pi\in\Xi_{n-1,m+1}} h^i_{\pi\xi}e_\pi,$$

we have

$$\begin{aligned} S(A_i^{-1}e_\xi,\ e_{\xi'}) &= \sum_{\pi\in\Xi_{n-1,m+1}} h^i_{\pi\xi}S(e_\pi, e_{\xi'}) \\ &= \sum_{\pi\in\Xi_{n-1,m+1}} h^i_{\pi\xi}S_{\pi\xi'}. \end{aligned}$$

From Theorem 4.1.1, for any $\xi,\xi'\in\Xi_{n-1,m+1}$, we have

$$\left[\frac{dy}{y-t_i}e_\xi\right]\cdot\left[\frac{dy}{y-t_j}e_{\xi'}\right] = 2\pi\sqrt{-1}\sum_{\pi\in\Xi_{n-1,m+1}}(\delta_{ij}h^i_{\pi\xi} + h^\infty_{\pi\xi})S_{\pi\xi'}. \quad (4.11)$$

From the formulae (4.10) and (4.11), the intersection numbers of $\{\psi_\xi\}_{\xi\in\Xi_n}$ are

$$\begin{aligned} S_{\xi\Xi'} &= [\psi_\xi]\cdot[\psi_{\xi'}] \\ &= 2\pi\sqrt{-1}\sum_{\pi\in\Xi_{n-1,m+1}}\left\{\sum_{i=1}^{m}\sum_{j=1}^{m}\ell_i(\xi)\ell_j(\xi')h^\infty_{\pi\xi_i}S_{\pi\xi'_j} + \sum_{i=1}^{m}\ell_i(\xi)\ell_i(\xi')h^i_{\pi\xi_i}S_{\pi\xi'_i}\right\}. \end{aligned}$$

□

4.3 Example: the case $n = 2$, $m = 3$.

Using Theorem 4.2.3, we evaluate intersection numbers in the case $n = 2$, $m = 3$. Set

$$\begin{aligned}
&X = \{(x, y) \in \mathbb{C}^2 \mid xy(x-1)(y-1)(x-t)(y-t)(x-y) \neq 0\},\\
&Y = \{y \in \mathbb{C} \mid y(y-1)(y-t) \neq 0\},\\
&\Phi = (xy)^{\lambda_1}((x-1)(y-1))^{\lambda_2}((x-t)(y-t))^{\lambda_3}(x-y)^{\nu},\\
&\omega = \lambda_1\left(\frac{dx}{x} + \frac{dy}{y}\right) + \lambda_2\left(\frac{dx}{x-1} + \frac{dy}{y-1}\right) + \lambda_3\left(\frac{dx}{x-t} + \frac{dy}{y-t}\right) + \nu\frac{dx-dy}{x-y}.
\end{aligned}$$

Then

$$\Xi_{1,4} = \{(1), (2), (3)\}.$$

Note that

$$\begin{aligned}
\Lambda(2,3) \cup \Lambda(1,4) = \{&\lambda_1, \lambda_2, \lambda_3, \nu, 2\lambda_1 + \nu, 2\lambda_2 + \nu, 2\lambda_3 + \nu,\\
&-(\lambda_1 + \lambda_2 + \lambda_3), -(\lambda_1 + \lambda_2 + \lambda_3 + \nu), -(2\lambda_1 + 2\lambda_2 + 2\lambda_3 + \nu)\}.
\end{aligned}$$

We assume that $(\Lambda(2,3) \cup \Lambda(1,4)) \cap \mathbb{Z} = \emptyset$.

The fibre of the projection $f : X \ni (x, y) \mapsto y \in Y$ is $f^{-1}(y) = \{x \in \mathbb{C} \mid x(x-1)(x-t)(x-y) \neq 0\}$.

We take a basis of $H^1(f^{-1}(y), \nabla'_+)$ as

$$\varphi_{(1)} = \frac{1}{x}, \quad \varphi_{(2)} = \frac{1}{x-1}, \quad \varphi_{(3)} = \frac{1}{x-t}.$$

We put $b = \lambda_1 + \lambda_2 + \lambda_3 + \nu$.

The intersection matrix of the basis $\{\varphi_{(1)}, \varphi_{(2)}, \varphi_{(3)}\}$ is

$$\begin{pmatrix} S_{(1),(1)} & S_{(1),(2)} & S_{(1),(3)} \\ S_{(2),(1)} & S_{(2),(2)} & S_{(2),(3)} \\ S_{(3),(1)} & S_{(3),(2)} & S_{(3),(3)} \end{pmatrix} = 2\pi\sqrt{-1}\left\{\begin{pmatrix} 1/\lambda_1 & & \\ & 1/\lambda_2 & \\ & & 1/\lambda_3 \end{pmatrix} - \frac{1}{b}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right\}$$

(e.g. Example 4.1.1).

We already gave A_1, A_2, A_3 in Example 4.2.1. Put

$$A_\infty = -(A_1 + A_2 + A_3) = -\begin{pmatrix} \lambda_1 + b & \lambda_1 & \lambda_1 \\ \lambda_2 & \lambda_2 + b & \lambda_2 \\ \lambda_3 & \lambda_3 & \lambda_3 + b \end{pmatrix}.$$

Then

$$A_1^{-1} = \begin{pmatrix} \frac{1}{2\lambda_1+\nu} & 0 & 0 \\ -\frac{\lambda_2}{(2\lambda_1+\nu)\lambda_1} & \frac{1}{\lambda_1} & 0 \\ -\frac{\lambda_3}{(2\lambda_1+\nu)\lambda_1} & 0 & \frac{1}{\lambda_1} \end{pmatrix}, \quad A_2^{-1} = \begin{pmatrix} \frac{1}{\lambda_2} & -\frac{\lambda_1}{(2\lambda_2+\nu)\lambda_2} & 0 \\ 0 & \frac{1}{2\lambda_2+\nu} & 0 \\ 0 & -\frac{\lambda_3}{(2\lambda_2+\nu)\lambda_2} & \frac{1}{\lambda_2} \end{pmatrix}, \quad A_3^{-1} = \begin{pmatrix} \frac{1}{\lambda_3} & 0 & -\frac{\lambda_1}{(2\lambda_3+\nu)\lambda_3} \\ 0 & \frac{1}{\lambda_3} & -\frac{\lambda_2}{(2\lambda_3+\nu)\lambda_3} \\ 0 & 0 & -\frac{1}{2\lambda_3+\nu} \end{pmatrix}$$

and

$$A_\infty^{-1} = \frac{1}{bc}\begin{pmatrix} \lambda_1 - c & \lambda_1 & \lambda_1 \\ \lambda_2 & \lambda_2 - c & \lambda_2 \\ \lambda_3 & \lambda_3 & \lambda_3 - c \end{pmatrix},$$

where $c = 2\lambda_1 + 2\lambda_2 + 2\lambda_3 + \nu$.

From Theorem 4.2.3, we have

$$
\begin{aligned}
S_{(11),(11)} &= 2\pi\sqrt{-1}\left\{4\sum_{\pi=1}^{3}(h_{\pi 1}^{\infty} + h_{\pi 1}^{1})S_{\pi 1}\right\},\\
S_{(12),(22)} &= 2\pi\sqrt{-1}\left\{2\sum_{\pi=1}^{3}(h_{\pi 1}^{\infty} + h_{\pi 2}^{\infty})S_{\pi 2} + 2\frac{1}{\lambda_2}S_{12}\right\},\\
S_{(12),(11)} &= 2\pi\sqrt{-1}\left\{2\sum_{\pi=1}^{3}(h_{\pi 1}^{\infty} + h_{\pi 2}^{\infty})S_{\pi 1} + 2\frac{1}{\lambda_1}S_{21}\right\},\\
S_{(12),(12)} &= 2\pi\sqrt{-1}\left\{\frac{1}{\lambda_2}S_{11} + \frac{1}{\lambda_1}S_{22} + \sum_{\pi=1}^{3}\{(h_{\pi 1}^{\infty} + h_{\pi 2}^{\infty})(S_{\pi 1} + S_{\pi 2})\}\right\},\\
S_{(22),(11)} &= 2\pi\sqrt{-1}\left\{4\sum_{\pi=1}^{3}h_{\pi 2}^{\infty}S_{\pi 1}\right\},\\
S_{(22),(22)} &= 2\pi\sqrt{-1}\left\{4\sum_{\pi=1}^{3}(h_{\pi 2}^{\infty} + h_{\pi 2}^{2})S_{\pi 2}\right\}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
[\varphi_{(11)}]\cdot[\varphi_{(11)}] &= -4\pi^2\frac{8(\lambda_2+\lambda_3)(\nu+2\lambda_2+2\lambda_3)}{\lambda_1(\nu+2\lambda_1)(\nu+\lambda_1+\lambda_2+\lambda_3)(\nu+2\lambda_1+2\lambda_2+2\lambda_3)},\\
[\varphi_{(11)}]\cdot[\varphi_{(12)}] &= 4\pi^2\frac{4(\nu+2\lambda_2+2\lambda_3)}{\lambda_1(\nu+\lambda_1+\lambda_2+\lambda_3)(\nu+2\lambda_1+2\lambda_2+2\lambda_3)},\\
[\varphi_{(11)}]\cdot[\varphi_{(22)}] &= -4\pi^2\frac{8}{(\nu+\lambda_1+\lambda_2+\lambda_3)(\nu+2\lambda_1+2\lambda_2+2\lambda_3)},\\
[\varphi_{(12)}]\cdot[\varphi_{(22)}] &= 4\pi^2\frac{4(\nu+2\lambda_1+2\lambda_3)}{\lambda_2(\nu+\lambda_1+\lambda_2+\lambda_3)(\nu+2\lambda_1+2\lambda_2+2\lambda_3)},\\
[\varphi_{(22)}]\cdot[\varphi_{(22)}] &= -4\pi^2\frac{8(\lambda_1+\lambda_3)(\nu+2\lambda_1+2\lambda_3)}{\lambda_2(\nu+2\lambda_2)(\nu+\lambda_1+\lambda_2+\lambda_3)(\nu+2\lambda_1+2\lambda_2+2\lambda_3)},\\
[\varphi_{(12)}]\cdot[\varphi_{(12)}] &= -4\pi^2\frac{2(\nu^2+(2\lambda_1+2\lambda_2+3\lambda_3)\nu+4\lambda_1\lambda_2+2\lambda_1\lambda_3+2\lambda_2\lambda_3+2\lambda_3^2)}{\lambda_1\lambda_2(\nu+\lambda_1+\lambda_2+\lambda_3)(\nu+2\lambda_1+2\lambda_2+2\lambda_3)}.
\end{aligned}
$$

Appendix A

Combinatorics for twisted cohomology groups

In this appendix, we give a brief overview on combinatorial description of twisted cohomology groups. We refer to the book [22] for combinatorics of arrangements and the theory of Orlik-Solomon algebras and the paper [7] for βnbc bases.

A.1 Orlik-Solomon algebras and βnbc

Let $h_1, \ldots, h_n$ be linear forms in $\mathbb{C}[x_1, \ldots, x_\ell]$ and H_i be the hyperplane in $\mathbb{C}^\ell$ defined by $h_i = 0$. The finite set $\mathcal{A} = \{H_1, \ldots, H_n\}$ is called a hyperplane arrangement. We put $M = \mathbb{C}^\ell \setminus \bigcup_{i=1}^n H_i$, that is, M is the complement of $\mathcal{A}$.

Definition A.1.1. Put $\omega_i = d \log h_i$. We define

$$A_k(\mathcal{A}) = \sum_{(i_1,\ldots,i_k)} \mathbb{C}\omega_{i_1} \wedge \cdots \wedge \omega_{i_k}, \qquad A(\mathcal{A}) = \bigoplus_{k=0}^{\ell} A_k(\mathcal{A}).$$

The algebra $A(\mathcal{A})$ is called *the Orlik-Solomon algebra*.

We use the followings of Orlik-Solomon algebras.

Definition A.1.2. We define a linear map $\nabla_\omega : A_k \to A_{k+1}$ by $\nabla_\omega \tau = \omega \wedge \tau$, where $\omega = \alpha_1\omega_1 + \cdots + \alpha_n\omega_n \in A_1(\mathcal{A})$

Clearly $\nabla_\omega \nabla_\omega = 0$, that is, $\{A(\mathcal{A}), \nabla_\omega\}$ is a cochain complex. We denote the p-th cohomology group by $H^p(A(\mathcal{A}), \nabla_\omega)$.

Assume that $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Let $U(x) = h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_n^{\alpha_n}$, that is, $U(x)$ is a multi-valued function. We denote by $\mathcal{L}$ a local constant sheaf of rank one over M which is determined by $U(x)$.

The following theorem is well-known for cohomology groups $H^k(M, \mathcal{L})$ in which we are interested.

Theorem A.1.1 ([5, p.558], [23, Corollary 10]). *Let $\mathcal{A}$ be a finite collection of hyperplanes, $A(\mathcal{A})$ the Orlik-Solomon algebra of $\mathcal{A}$, and M the complement of $\mathcal{A}$.*

Suppose that none of the exponents $\sum_{H_i \supset L} \alpha_i$ *lies in* $\mathbb{Z}_{>0}$ *for all* $L \in L(\mathcal{A}_\infty)$. *Then* $H^k(M, \mathcal{L}) \simeq H^k(A(\mathcal{A}), \nabla_\omega)$ *for* $0 \leq k \leq n$.

Since $H^k(A(\mathcal{A}), \nabla_\omega)$ has a more simple structure than $H^k(M, \mathcal{L})$, we can study $H^k(M, \mathcal{L})$ via the complex of the Orlik-Solomon algebra. As an application of the use of the Orlik-Solomon algebra, we give a proof of (3.3) by utilizing the Orlik-Solomon algebra.

Theorem A.1.2. *We assume that the exponent along the hyperplane* H_0 *is zero. Let* $\mathcal{A} = \{H_0, H_1, \ldots, H_m\}$, $\mathcal{A}' = \mathcal{A} \setminus \{H_0\}$, *and* $\mathcal{A}''$ *the restricted arrangement of* $\mathcal{A}'$ *to* H_0. *Then the following short exact sequence holds.*

$$0 \to H^k(A(\mathcal{A}'), \nabla_\omega) \to H^k(A(\mathcal{A}), \nabla_\omega) \to H^{k-1}(A(\mathcal{A}''), \nabla_{\omega_{\mathcal{A}''}}) \to 0. \qquad (exact)$$

Proof. From [22, Theorem 3.127], there is the following short exact sequence of $\mathbb{C}$-vector spaces:

$$0 \to A_k(\mathcal{A}') \xrightarrow{i} A_k(\mathcal{A}) \xrightarrow{j} A_{k-1}(\mathcal{A}'') \to 0.$$

Then $A_k(\mathcal{A}) = A_k(\mathcal{A}') \oplus \operatorname{coker} j$. We put $N_k = \operatorname{coker} j$. From a construction of the mapping j, we know that any elements of N_k has an expression $\omega_0 \wedge \eta$, where $\omega_0 = d \log h_0$. (see [22, p.97])

Since the exponent along H_0 is zero, we can see that $\omega \in A_1(\mathcal{A}')$. Then, $\nabla_\omega(A_k(\mathcal{A}')) \subset A_{k+1}(\mathcal{A}')$ holds. In addition, we know that $\nabla_\omega(N_k) \subset N_{k+1}$ because $\nabla_\omega(\omega_0 \wedge \eta) = \omega \wedge \omega_0 \wedge \eta$.

Therefore, there is a short exact sequence

$$0 \to H^k(A(\mathcal{A}'), \nabla_\omega) \to H^k(A(\mathcal{A}), \nabla_\omega) \to H^k(N, \nabla_\omega) \to 0.$$

In addition, we have that $H^k(N, \nabla_\omega) \simeq H^{k-1}(A(\mathcal{A}''), \nabla_{\omega_{\mathcal{A}''}})$.

Hence,

$$0 \to H^k(A(\mathcal{A}'), \nabla_\omega) \to H^k(A(\mathcal{A}), \nabla_\omega) \to H^{k-1}(A(\mathcal{A}''), \nabla_{\omega_{\mathcal{A}''}}) \to 0 \qquad \text{(exact)}$$

□

Let M be the complement of the arrangement $\mathcal{A}'$, $Z = M \cap H_0$, and $M' = M \setminus Z$. Namely, M' is the complement of $\mathcal{A}$. We assume that a locally constant sheaf $\mathcal{L}$ on M is "generic", i.e. the pair $(M, \mathcal{L})$ satisfies the hypothesis of Theorem A.1.1. Since the exponent along the hyperplane H_0 is zero, the pair $(M', \mathcal{L}')$ also satisfies the hypothesis of Theorem A.1.1. Then $H^k(M, \mathcal{L}) \simeq H^k(A(\mathcal{A}'), \nabla_\omega)$ and $H^k(M \setminus Z, \mathcal{L}) \simeq H^k(A(\mathcal{A}), \nabla_\omega)$. Hence, we obtain (3.3):

$$0 \to H^k(M, \mathcal{L}) \to H^k(M \setminus Z, \mathcal{L}) \to H^{k-1}(Z, \mathcal{L}_{|Z}) \to 0. \qquad \text{(exact)}$$

We shall consider how to take a basis of a cohomology group. In fact, Falk and Terao introduced βnbc bases of $H^\ell(A, \nabla_\omega)$ ([7]), which we are going to explain.

Firstly, we give terms of combinatorics of hyperplane arrangements to express βnbc.

Definition A.1.3. Let B be a subset of $\mathcal{A}$.
B is *dependent* $\underset{\text{def}}{\Longleftrightarrow} \cap B (= \bigcap_{H\in B} H) \neq \emptyset$ and $\operatorname{codim}(\cap B) < \#B$.
B is a *circuit* $\underset{\text{def}}{\Longleftrightarrow}$ B is a minimally dependent set.

We fix a linear order $\prec$ on the finite set $\mathcal{A}$. For example, $H_1 \prec H_2 \prec \cdots \prec H_n$.

Definition A.1.4. Let $B \subset \mathcal{A}$.
B is a *broken circuit* $\underset{\text{def}}{\Longleftrightarrow} \cap B \neq \emptyset$ and there exists $H \prec \min_{\prec}(B)$ such that $B \cup \{H\}$ is a circuit.

We define finite sets NBC and βNBC which depend on the fixed order $\prec$.

Definition A.1.5.

$$\begin{aligned} \mathrm{NBC} &= \{B \subset \mathcal{A} \mid \cap B \neq \emptyset \text{ and } B \text{ includes no broken circuit}\}, \\ \beta\mathrm{NBC} &= \{B \in \mathrm{NBC} \mid \text{For any } H \in B, \text{ there exists } H' \prec H \text{ such that } (B \setminus \{H\}) \cup \{H'\} \in \mathrm{NBC}\} \end{aligned}$$

Example A.1.1. Set $\ell = 2$, $n = 5$. Consider the hyperplane arrangement of Selberg-type

$$Q(\mathcal{A}) = x(x-1)y(y-1)(x-y);$$

$h_1 = x,\ h_2 = x-1,\ h_3 = y,\ h_4 = y-1,\ h_5 = x-y$. We fix a linear order $\prec$ as $H_1 \prec H_2 \prec H_3 \prec H_4 \prec H_5$. Then

$$\begin{aligned} \text{the set of circuits} &= \{(135), (245)\} \\ \text{the set of broken circuits} &= \{(35), (45)\} \\ \mathrm{NBC} &= \{(13), (14), (15), (23), (24), (25)\} \\ \beta\mathrm{NBC} &= \{(24), (25)\} \end{aligned}$$

βnbc depends on the fixed order $\prec$. For example, βNBC is the set $\{(14), (23)\}$ with respect to a linear order $\prec'$ defined by $H_5 \prec' H_1 \prec' H_2 \prec' H_3 \prec' H_4$.

Secondly, we explain the Folkman complex of $\mathcal{A}$.

Definition A.1.6 (Folkman Complex). We define a finite set

$$F(\mathcal{A}) = \{\cap B \mid B \subset \mathcal{A} \text{ and } \cap B \neq \emptyset\}$$

and a partial order on F by the reverse inclusion relation, that is,

$$X \leq Y \underset{\text{def}}{\Longleftrightarrow} X \supset Y \quad (\forall X, Y \in F).$$

We identify any linear ordered subset of F with a simplex and specially a maximal ordered subset with a facet. Then F is called *the Folkman complex*.

Example A.1.2. Under the hypothesis of Example A.1.1,

$$F(\mathcal{A}) = \{(1), (2), (3), (4), (5), (135), (23), (14), (245)\}$$

Finally, we introduce the βnbc basis due to [7].

Definition A.1.7. Let $B = \{H_{i_1}, \ldots, H_{i_\ell}\} \in \beta$NBC and assume that $H_{i_1} \prec \cdots \prec H_{i_\ell}$. We define a chain $\xi(B)$ of $C_\ell(F, \mathbb{C})$ as follows:

$$\xi(B) = (X_1, X_2, \ldots, X_\ell),$$

where $X_j = H_{i_j} \cap \cdots \cap H_{i_\ell}$.

Theorem A.1.3 (Bjöner-Ziegler). *The set $\{[\xi(B)^*] \mid B \in \beta NBC\}$ is a basis of $\tilde{H}^{\ell-1}(F, \mathbb{C})$.*

Theorem A.1.4 ([26]). *If none of the exponents $\displaystyle\sum_{H_i \supset L} \alpha_i$ is 0 for all dense $L \in L(\mathcal{A}_\infty)$, then*

$$\tilde{H}^{\ell-1}(F, \mathbb{C}) \simeq H^\ell(A(\mathcal{A}), \nabla_\omega)$$

This theorem is proved with the sheaf theory on the finite set $F \cup \{\mathbb{C}^\ell\}$.

Definition A.1.8. Let $B \in \beta$NBC and $\xi(B) = (X_1, X_2, \ldots, X_\ell)$. we define

$$\Xi(B) = (-1)^{\frac{\ell(\ell+1)}{2}} \omega(X_1) \wedge \cdots \wedge \omega(X_\ell),$$

where $\omega(X) = \sum_{H_i \leq X} \alpha_i d \log h_i$.

Theorem A.1.5 ([7]). *Under the hypothesis of Theorem A.1.4, the set $\{[\Xi(B)] \mid B \in \beta NBC\}$ is a basis of $H^\ell(A(\mathcal{A}), \nabla_\omega)$ and is called the βnbc basis.*

Example A.1.3. Since (245) is a circuit, the following holds.

$$\omega_4 \wedge \omega_5 - \omega_2 \wedge \omega_5 + \omega_2 \wedge \omega_4 = 0.$$

Then βnbc basis of $H^2(A(\mathcal{A}), \nabla_\omega)$ with respect to $\prec$ is

$$\begin{aligned}
\Xi(24) &= (\alpha_2\omega_2 + \alpha_4\omega_4 + \alpha_5\omega_5) \wedge \alpha_4\omega_4 \\
&= \alpha_{24}\omega_{24} + \alpha_{45}\omega_{54} \\
\Xi(25) &= (\alpha_2\omega_2 + \alpha_4\omega_4 + \alpha_5\omega_5) \wedge \alpha_5\omega_5 \\
&= \alpha_{25}\omega_{24} - (\alpha_{25} + \alpha_{45})\omega_{54},
\end{aligned}$$

where we put $\omega_{ij} = \omega_i \wedge \omega_j$ and $\alpha_{ij} = \alpha_i\alpha_j$.
Remark. The βnbc basis with respect to another order $\prec'$ is $\{\Xi(14) = \alpha_{14}\omega_{14},\ \Xi(23) = \alpha_{23}\omega_{23}\}$.

We can evaluate intersection numbers of βnbc bases using our method. For example, the self intersection number of βnbc base $\Xi(24)$ is

$$\begin{gathered}
[\Xi(24)] \cdot [\Xi(24)] = \\
\alpha_{14}(\alpha_1 + \alpha_2 + \alpha_5)^{-1}(\alpha_3 + \alpha_4 + \alpha_5)^{-1}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)^{-1} \\
(\alpha_{123} + \alpha_{125} + \alpha_{135} + \alpha_{145} + \alpha_{234} + 3\alpha_{235} + \alpha_{245} + \alpha_{345} + \alpha_{223} \\
+\alpha_{233} + \alpha_{225} + 2\alpha_{255} + \alpha_{335} + 2\alpha_{355} + \alpha_{455} + \alpha_{155} + \alpha_5^3)
\end{gathered}$$

A.2 An implementation for a computation of βnbc.

In Appendix $A.1$, we explained that a βnbc basis gives a basis of $H^\ell(M, \mathcal{L})$ under a "generic" condition for $\mathcal{L}$. In order to study twisted cohomology groups $H^\ell(M, \mathcal{L})$ and $H^1(T, \mathrm{Ker}\,\nabla)$, we need to obtain βnbc bases using a computer algebraic system.

In this section, we present an algorithm and an implementation to obtain βnbc bases. Our program is written in the computer algebraic system "Asir" which is developed by M. Noro in Fujitsu Laboratory. Asir is available from the URL `ftp://endeavor.fujitsu.co.jp/pub/isis/asir/` via the Internet. Our program is also available from the URL `http://www.math.kobe-u.ac.jp/HOME/ohara/`.

```
/* -*-C-*- */
/* ~/beta-nbc-asir/beta */

/* This program compute beta nbc bases for twisted cohomology groups
   with given arrangements.

   First, you give an arrangement of hyperplanes as follows:

   [1] Arr = [ [z-t, t, t-s, s, 1-s], [t,s] ];

   The data structure of arrangements as follows:
   [ (A list of linear forms), (A list of indeterminates) ].

   Second, you can get its incidence graph as follows:

   [2] IG = getIGraph(Arr);

   Third, give a total order on the hyperplanes and you can get the
   beta nbc for the incidence graph IG.

   [3] Order = [1,2,3,4,5];
   [4] B = getBetaNbcSet(IG, Order);

   Finally, you can get a basis of its twisted cohomology group.

   [5] BF = betaNbcSet2forms(B, IG, Order); */

/* These are samples of arrangemnts. */
Arr1 = [ [z-t, t, t-s, s, 1-s], [t,s] ]$
Arr2 = [ [z-u, u-t, t-s, 1-s, u, t, s], [u, t,s] ]$
Arr3 = [ [z-t, z-s, 1-t, 1-s, t-s, t, s], [t,s] ]$
Arr4 = [ [z-u, z-t, z-s, 1-u, 1-t, 1-s, u-t, t-s, u, t, s], [u, t,s] ]$
Arr5 = [ [s, t, s+t-z], [t,s] ]$

load("sets")$
load("igraph")$

#define hpPart(X) ((X)[0])

/* extract the family of intersectable sets with rank=Rank from an
incidence graph */
def igRank(IGraph, Rank) {
  return (--Rank < 0) ? ([]) : reverse(IGraph)[Rank];
}

def intersectable_p(Set, IGraph) {
```

```
  Len = length(IGraph);
  for (I = 0; I < Len; I++) {
    if (subset_of_subset_p(Set, IGraph[I])) {
      return 1;
    }
  }
  return 0;
}

def getDepSets(IGraph) {
  IGraph = reverse(IGraph);
  ASL = cdr(reverse(getAllSortedLists(length(car(IGraph)))));
  DepSets = [];
  while ((List = car(IGraph)) != [] && (SL = car(ASL)) != []) {
    while ((L = car(SL)) != []) {
      if (subset_of_subset_p(L, List)) {
        DepSets = cons(L, DepSets);
      }
      SL = cdr(SL);
    }
    IGraph = cdr(IGraph);
    ASL = cdr(ASL);
  }
  return DepSets;
}

def circuit1_p(Set, DepSets) {
  Len = length(DepSets);
  for (I = 0; I < Len; I++) {
    if (properSubSet_p(DepSets[I], Set)) {
      return 0;
    }
  }
  return 1;
}

def getCircuits(IGraph) {
  DepSets = getDepSets(IGraph);
  Len = length(DepSets);
  Circuits = [];
  for (I = 0; I < Len; I++) {
    if (circuit1_p(DepSets[I], DepSets)) {
      Circuits = cons(DepSets[I], Circuits);
    }
  }
  return Circuits;
```

```
}

def getIntersectables(Dimension, IGraph) {
  HSize = length(car(reverse(IGraph)));
  X = getSortedLists(Dimension, HSize);
  Len = length(X);
  Intersections = [];
  for (I = 0; I < Len; I++) {
    if (intersectable_p(X[I], IGraph)) {
      Intersections = cons(X[I], Intersections);
    }
  }
  return Intersections;
}

/* The following functions are used in computations of beta-nbc. */
def howSmallThisInOrder(This, Order) {
  Len = length(Order);
  for (I = 0; I < Len && Order[I] != This; I++)
    ;
  return I;
}

def minimalCounterWithOrder(List, Order) {
  Len = length(List);
  PrevNum = length(Order) + 1; Prev = -1;
  for (I = 0; I < Len; I++) {
    if ((Tmp = howSmallThisInOrder(List[I], Order)) < PrevNum) {
      Prev = I;
      PrevNum = Tmp;
    }
  }
  return Prev;
}

def listDeleteMinimalWithOrder(List, Order) {
  N = minimalCounterWithOrder(List, Order);
  return listDeleteNth(List, N);
}

/* the value is 1 when any elements of List are not contained in SupSet. */
def uncontained_p(List, SupSet) {
  while ((Set = car(List)) != []) {
    if (subset_p(Set, SupSet)) {
      return 0;
    }
```

```
    List = cdr(List);
  }
  return 1;
}

def number_of_planes(IGraph) {
  return length(car(reverse(IGraph)));
}

def rank_of_igraph(IGraph) {
  return length(IGraph);
}

def getNbcSet(IGraph, Order) {
  Circuits = getCircuits(IGraph);
  /* Len = length(Circuits); */
  BcSet = [];
  while ((Cir = car(Circuits)) != []) {
    BcSet = cons(listDeleteMinimalWithOrder(Cir, Order), BcSet);
    Circuits = cdr(Circuits);
  }

  /* BcSet = getBcSet(IGraph, Order); */
  Family_of_maxrank = car(IGraph);
  SL = getSortedLists(rank_of_igraph(IGraph), number_of_planes(IGraph));
  NbcSet = [];
  while ((L = car(SL)) != []) {
    if (subset_of_subset_p(L, Family_of_maxrank) && uncontained_p(BcSet, L)) {
      NbcSet = cons(L, NbcSet);
    }
    SL = cdr(SL);
  }
  return NbcSet;
}

def existLowerSetInListAtNth_p(Set, List, Order, N) {
  Ind = Set[N];
  while((Ind2 = car(Order)) != Ind) {
    Set2 = listSetObject(Set, Ind2, N);
    if (elementOfSet_p(sort(Set2), List)) {
      return 1;
    }
    Order = cdr(Order);
  }
  return 0;
}
```

```
def betaNbc_p(Nbc, NbcSet, Order) {
  Len = length(Nbc);
  for (I = 0; I < Len; I++) {
    if (!existLowerSetInListAtNth_p(Nbc, NbcSet, Order, I)) {
      return 0;
    }
  }
  return 1;
}

/* This is main module!! */
def getBetaNbcSet(IGraph, Order) {
  NbcSet = getNbcSet(IGraph, Order);
  Len = length(NbcSet);
  BetaNbcSet = [];
  for (I = 0; I < Len; I++) {
    if (betaNbc_p(NbcSet[I], NbcSet, Order)) {
      BetaNbcSet = cons(NbcSet[I], BetaNbcSet);
    }
  }
  return sort(BetaNbcSet);
}

/* get the support for indepence set in the fixed arrangement. */
def getSupport(Independence, IGraph) {
  IG2 = igRank(IGraph, length(Independence));
  Len = length(IG2);
  for (I=0; I<Len; I++) {
    if (subset_p(Independence, IG2[I])) {
      return IG2[I];
    }
  }
  return [];
}

/* We construct the list which represent a log-form for given beta-nbc. */
/* A ``list'' [[i,j],[k],[l]] means a 3-form (tau_i+tau_j)*tau_l*tau_l. */

def nbc2form(Nbc, IGraph, Order) {
  /* translate a nbc to its log-form */
  PrevSupport = [];
  Snbc = [];
  Form = [];
  Order = reverse(Order);
  Len = length(Order);
```

```
  for (I=0; I<Len; I++) {
    if (elementOfSet_p(Order[I], Nbc)) {
      Snbc = cons(Order[I], Snbc);
      /* reduce the log-form */
      Support = getSupport(Snbc, IGraph);
      Form = cons(setMinus(Support, PrevSupport), Form);
      PrevSupport = Support;
    }
  }
  return Form;
}

def betaNbcSet2forms(BetaNbcSet, IGraph, Order) {
  Forms = [];
  Len = length(BetaNbcSet);
  for (I=0; I<Len; I++) {
    Forms = cons(nbc2form(BetaNbcSet[I], IGraph, Order), Forms);
  }
  return reverse(Forms);
}

def listupBetaNbcs(Arrangement) {
  IGraph = getIGraph(Arrangement);
  HSize = length(hpPart(Arrangement));
  Orders = getOrders(HSize);
  MaxNumOrders = length(Orders);
  output("x")$
  print(["Incidence Graph:", IGraph, "--- Circuits", getCircuits(IGraph)]);
  for (I = 0; I < MaxNumOrders; I++) {
    Order = Orders[I];
    BetaNbcSet = getBetaNbcSet(IGraph, Order);

    print(["Order:", Order, "---  Beta NBC:", BetaNbcSet]);
  }
  output()$
}

end$
```

```
/* -*-C-*- */
/* ~/beta-nbc-asir/igraph */

/* This program generates the incidence graph for a given arrangement.

   The data structure of incidence graphs of arrangements as follws:
   [ (A list of the intersections with rank $m$),
     (A list of the intersections with rank $m-1$),
     ...,
     (A list of the intersections with rank $1$) ],
   where $m$ is the rank of a given arrangement.
   An intersection with rank $k$ is expressed as the set of its supporting
   hyperplanes.
 */

/* The data structure of arrangements as follows:
   [ (A list of linear forms), (A list of indefinite) ].
 */

/* Remark.
   This program run under the following assumption derived from
   the Groebner bases package of ``Asir'':
   The variables which are not included the list of indeterminates
   have ``generic'' value.

   For example, we assume that z is not equal to 0 with respect to the
   arrangement [ [s, t, s+t-z], [s,t] ].
 */

load("sets")$
load("gr")$

#define rankPart(X) ((X)[0])
#define setPart(X) ((X)[1])

#define hpPart(X) ((X)[0])
#define varPart(X) ((X)[1])
#define static

/* cf. List = [1,2,3,...] */
static def xlateSet2HSet(Set, Hyperplanes) {
  HSet = [];
  Len = length(Set);
  for (I = 0; I < Len; I++) {
    HSet = cons(Hyperplanes[Set[I]-1], HSet);
  }
```

```
  return HSet;
}

/* This function use Groebner package. */
static def getRankOfSet(Set, Arrangement) {
  Variables   = varPart(Arrangement);
  HSet = xlateSet2HSet(Set, hpPart(Arrangement));
  GBasis = gr(HSet, Variables, 0);
  if (length(GBasis) == 1 && dp_td(dp_ptod(car(GBasis), Variables)) == 0) {
    /* GBasis is near equal to [1] */
    return 0;
  }
  return length(GBasis);
}
static def xlateSListsWithRank(SLists, Arrangement) {
  SListsWithRank = [];
  while((Set = car(SLists)) != []) {
    Rank = getRankOfSet(Set, Arrangement);
    if(Rank != 0) {
      SListsWithRank = cons([Rank,Set], SListsWithRank);
    }
    SLists = cdr(SLists);
  }
  return SListsWithRank;
}

static def getIntersectionsList(Arrangement) {
  /* HSize is number of hyperplanes. */
  HSize = length(hpPart(Arrangement));
  RevASLists = reverse(getAllSortedLists(HSize));
  X = [];
  while((SLists = car(RevASLists)) != []) {
    SListsWithRank = xlateSListsWithRank(SLists, Arrangement);
    if (SListsWithRank == []) {
      return X;
    }
    X = append(SListsWithRank, X);
    RevASLists = cdr(RevASLists);
  }
  return X;
}

/* generate an incidence graph */
def getIGraph(Arrangement) {
  MaxRank = length(varPart(Arrangement));
  RevIGraph = [];
```

```
  for (I = 0; I < MaxRank; I++) {
    RevIGraph = cons([], RevIGraph);
  }
  IList = getIntersectionsList(Arrangement);
  while((Int = car(IList)) != []) {
    R = rankPart(Int) -1 ;
    X = RevIGraph[R];
    if (!subset_of_subset_p(setPart(Int), X)) {
      RevIGraph = listSetObject(RevIGraph, cons(setPart(Int), X), R);
    }
    IList = cdr(IList);
  }
  return reverse(RevIGraph);
}

end$
```

```
/* -*-C-*- */
/* ~/beta-nbc-asir/sets */

/* This program is a module treating a SET for Asir.  This module
   identify a SET with a list.  Namely, it ignores the order of
   elements in a list. */

load("gr")$

/* compute (SupSet - Subset) */
def setMinus(SupSet, SubSet) {
  LenSup = length(SupSet);
  X = [];
  for (I = 0; I < LenSup; I++) {
    if (!elementOfSet_p(SupSet[I], SubSet))
      X = cons(SupSet[I], X);
  }
  return reverse(X);
}

/* static */
/* Is given list sorted? */
def sortedList_p(List) {
  Len = length(List);
  for (I = 1; I < Len; I++) {
    if (List[I-1] > List[I]) {
      return 0;
    }
  }
  return 1;
}

/* deleting unsorted sets */
def chooseSortedLists(Lists) {
  Len = length(Lists);
  SortedLists = [];
  for (I = 0; I < Len; I++) {
    if (sortedList_p(Lists[I])) {
      SortedLists = cons(Lists[I], SortedLists);
    }
  }
  return SortedLists;
}

def getAllSortedLists(HSize) {
  ASLists = [];
```

```
  SLists = [ [] ];
  for (I = 0; I < HSize; I++) {
    SLists = getSubOrders(SLists, HSize);
    SLists = chooseSortedLists(SLists);
    ASLists = cons(SLists, ASLists);
  }
  return ASLists;
}

def getSortedListsWithDepth(HSize, Depth) {
  ASLists = [];
  SLists = [ [] ];
  for (I = 0; I < Depth; I++) {
    SLists = getSubOrders(SLists, HSize);
    SLists = chooseSortedLists(SLists);
    ASLists = cons(SLists, ASLists);
  }
  return ASLists;
}

def getIntersectables(Rank, IGraph) {
  HSize = length(car(reverse(IGraph)));
  X = getSortedLists(Rank, HSize);
  Len = length(X);
  Intersections = [];
  for (I = 0; I < Len; I++) {
    if (intersectable_p(X[I], IGraph)) {
      Intersections = cons(X[I], Intersections);
    }
  }
  return Intersections;
}

def getIntersectables2(Rank, IGraph) {
  Indep = getIndependentSets(IGraph);
  return igRank(Indep, Rank);
}

def getSortedLists(Rank, HSize) {
  Lists = [ [] ];
  for (I = 0; I < Rank; I++) {
    Lists = getSubOrders(Lists, HSize);
    Lists = chooseSortedLists(Lists);
  }
  return Lists;
}
```

```
def getSubOrders(SubOrders, HSize) {
  Len = length(SubOrders);
  X = [];
  for (I=0; I<Len; I++) {
    for (J=1; J<=HSize; J++) {
      if (elementOfSet_p(J, SubOrders[I]))
        continue;
      X = cons(cons(J, SubOrders[I]), X);
    }
  }
  return X;
}

def getOrders(HSize) {
  Orders = [ [] ];
  for (I=0; I<HSize; I++) {
    Orders = getSubOrders(Orders, HSize);
  }
  return sort(Orders);
}

/* set ''Object'' to N-th element of given list. */
def listSetObject(List, Object, N) {
  X = [];
  Len = length(List);
  for (I = 0; I < Len && I < N; I++) {
    X = cons(List[I], X);
  }
  X = cons(Object, X);
  for (I=N+1; I < Len; I++) {
    X = cons(List[I], X);
  }
  return reverse(X);
}

/* delete N-th element of given list. */
def listDeleteNth(List, N) {
  Len = length(List);
  X = [];
  for (I = 0; I < N; I++) {
    X = cons(List[I], X);
  }
  for (I=N+1; I < Len; I++) {
    X = cons(List[I], X);
  }
```

```
  return reverse(X);
}

/* Is ``Element'' included in ``Set''? */
def elementOfSet_p(Element, Set) {
  Len = length(Set);
  for (I = 0; I < Len; I++) {
    if (Element == Set[I])
      return 1;
  }
  return 0;
}

/* Is ``Set'' a proper subset of ``SupSet''? */
def properSubSet_p(Set, SupSet) {
  return (subset_p(Set, SupSet) && length(Set) < length(SupSet));
}

/* Is ``Set'' a subset of ``SupSet''? */
def subset_p(Set, SupSet) {
  Len = length(Set);
  for (I = 0; I < Len; I++) {
    if (!elementOfSet_p(Set[I], SupSet)) {
      return 0;
    }
  }
  return 1;
}

/* Is ``Set'' a subset of subset of ``SupSupSet''? */
def subset_of_subset_p(Set, SupSupSet) {
  Len = length(SupSupSet);
  for (I = 0; I < Len; I++) {
    if (subset_p(Set, SupSupSet[I])) {
      return 1;
    }
  }
  return 0;
}

/* This is a subfunction of ``sort()''. */
def sortSub(SortedList, Object) {
  X = [];
  while ((L = car(SortedList)) != [] && L < Object) {
    X = cons(L, X);
    SortedList = cdr(SortedList);
```

```
  }
  X = cons(Object, X);
  return append(reverse(X), SortedList);
}

/* This simple sorting function spends an O(n^2) time. */
def sort(List) {
  Sorted = [];
  while ((L = car(List)) != []) {
    Sorted = sortSub(Sorted, L);
    List = cdr(List);
  }
  return Sorted;
}

/* compute the sign of a permutation ``append(SortedList, [Object])''. */
def signSub(SortedList, Object) {
  X = [];
  while ((L = car(SortedList)) != [] && L < Object) {
    X = cons(L, X);
    SortedList = cdr(SortedList);
  }
  X = cons(Object, X);
  return [append(reverse(X), SortedList), length(SortedList)];
}

/* compute the sign of a permutation ``List'' */
def sign(List) {
  Sorted = [];
  Sign = 0;
  while ((L = car(List)) != []) {
    X = signSub(Sorted, L);
    Sorted = X[0];
    Sign += X[1];
    List = cdr(List);
  }
  return (Sign % 2) ? -1 : 1;
}

end$
```

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