



Fiber surfaces and plumbing Hopf bands

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博士論文

Fiber surfaces and plumbing Hopf bands

(ファイバー曲面とホップバンドのプラミング)

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Preface

This thesis is written under the subject “Fiber surfaces and plumbing Hopf bands” to be submitted for the degree of Doctor at Kobe University. A knot is a simple closed curve in three-dimensional space. The aim of knot theory will be to distinguish knots, to investigate geometric properties of knots and to see something of the way they interact with three-dimensional topology. The major part of my research interests is on the last one. I have interests in a geometrical structure of the complement of a knot in three-sphere, especially when it arrows a structure of surface-bundle on the circle. The knot which complement possesses such a property is called a fibered knot, and has been studied by many knot theorists.

When I was in the first year of graduate school, I found one question in “Low dimensional topology problems” edited by H. Goda and T. Kitano:

“What kinds of knots become fibered knots through one crossing change?”

The *crossing change* is one of the most elementary operations in the knot theory. As its name suggests, it is an operation to change the sign of a crossing to the opposite on a diagram of a knot.

This question was hard to approach it directly, so I decided to begin at the case where a resulting fibered knot has an alternating diagram i.e., one can trace the knot on its diagram, encountering positive sign crossings and negative sign crossings

alternately. A link which has an alternating diagram is called an alternating link, and has been known that it has many good properties by works due to K. Murasugi and others. One of Murasugi's works showed that fibered alternating link has the fiber surface which constructed it from some Hopf bands, which is the most simple fiber surface excepting a disk, by gluing them together in a certain manner (this operation is called *plumbing*). This suggests that a fiber surface for a fibered alternating link has a decomposition into Hopf bands, which represents a structure of the surface. So I thought of seeing a change of this structure of the fiber surface through one crossing change. I tried to see a structure of the resulting surface which is very similar to that of a fibered alternating link and this would lead me to characterize the boundary knot.

As a result, I failed in this approach. In general, a crossing change on a fiber surface distorts the structure of the surface, and the surface has no longer a structure with respect to plumbing. However, through a lot of experiments of plumbing and deplumbing fiber surfaces, I came to think that such a decomposition of a fiber surface is very useful to understand a geometrical structure of the complement of a fibered knot. Actually D. Gabai showed that Murasugi sum, a generalization of plumbing, is a natural operation in a geometrical sence. Since this time, I have wanted to know the law which controls the way how the fiber surfaces are decomposed into more simple fiber surfaces with respect to plumbing or Murasugi sum.

In this thesis, I will present my work, done from 1996 to 1999.

To carry my research forward and to prepare the document of this thesis, suggestions from Professor Yasutaka Nakanishi was very helpful. I would like to express my deep gratitude to him. I also owe my thanks to Professor Hiroshi Goda for his kindness to always take care of me and my research, and to the members of Kobe

Topology Seminar on Saturdays at Kobe University, who were encouraging supports of my research. I have often joined Low Dimensional Topology Seminar at Osaka University, and have stimulated so much by the members of the seminar. Professor Makoto Sakuma and Doctor Mikami Hirasawa especially gave me many suggestions. I would like to thank them. Finally I want to thank Takuji Nakamura, who is my roommate and we, I think, are in friendly rivalry. He was indispensable in my research works.

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Chapter 1

Introduction

We live in 3-dimensional space. So it is a natural desire to want to know which one of possible 3-dimensional manifolds could be a model for the universe. To answer this question, we must know each 3-manifolds well. But of course, there are so many different possible 3-manifolds. Thus it is worthy beginning on our investigation at more simple 3-manifolds. We choose closed, orientable, connected 3-manifolds.

We have a method to obtain a closed, orientable, connected 3-manifold from a pair of S^3 and a knot, which is called *Dehn surgery*. This is the operation of drilling out a tubular neighborhood of the knot and then gluing in a solid torus so that its meridian curve goes to a (p, q) -curve on the torus boundary of the complement of the knot. We could also perform Dehn surgery on a link in S^3 . By this operation we expect to get a lot of different closed 3-manifolds. Surprisingly enough, every closed, orientable, connected 3-manifold comes from Dehn surgery on a link in S^3 . (This was proved by W. R. B. Lickorish [18] and A. Wallace, working independently and using entirely different methods.) This is the fundamental connection between knots and links and 3-manifolds. If we could really understand the complement of knots and

links, we would really understand all closed, orientable, connected 3-manifolds. But yet there is many different knot or link. So we will attend to investigation the fibered knot or link.

The topic of fibered knots and links was first discussed by L. Neuwirth [25] and J. R. Stallings [28]. Earlier results of K. Murasugi [21, 22] and R. Crowell [5] on alternating links were systematized and extended in a paper by Stallings [28], where he showed that a knot K is fibered if and only if the commutator subgroup of $\pi_1(S^3 - K)$ is finitely generated or, equivalently, free.

In the geometric view, Murasugi [24] and Stallings [29] have given techniques for constructing fibered knots and links:

The first construction is called “plumbing” which is a way to create a new Seifert surface, i.e., a bounded, oriented surface in S^3 , by pasting two Seifert surfaces along a disk on each surface. The second one, we call it “Stallings twist” in this paper, involves a ± 1 Dehn surgery on a circle C in a fiber surface F which is unknotted in S^3 and have a certain condition of the Seifert linking of C . We can say that this paper describes a study of these operations.

Murasugi [24] showed the following theorem:

Theorem 1.0.1. *A fiber surface of a fibered alternating link may be obtained from disk by plumbing some Hopf bands.*

A *Hopf band* is an annulus with one full-twist in S^3 , which is the most simple fiber surface in S^3 except for a disk. Since a class of 2-bridge links is included in a class of alternating links, the fiber surface of a fibered 2-bridge link also has the property above. More precisely, we can obtain the fiber surface of a fibered 2-bridge link by plumbing some Hopf bands each other in a line. In Chapter 3, we will consider a

relationship between this structure of the fiber surface of a fibered 2-bridge link and the Conway notation of the link.

Stallings showed that the surface obtained by plumbing two fiber surfaces is also fiber surface. This fact has been generalized and strengthened by D. Gabai as we mention below.

The *Murasugi sum* of Seifert surfaces, a generalized operation of plumbing, was introduced originally by Murasugi in order to estimate the degree of the Alexander polynomial of alternating links, and it has been playing important roles in the studies of Seifert surfaces and links. Gabai [7, 9] showed that Murasugi sum is natural operation by giving the following theorems. Considering that R is a surface embedded in S^3 which is a Murasugi sum of two compact oriented surfaces R_1 and R_2 , we have;

Theorem 1.0.2. *R is a minimal genus Seifert surface in S^3 if and only if both R_1 and R_2 are minimal genus Seifert surfaces.*

Theorem 1.0.3. *R is a fiber surface in S^3 if and only if both R_1 and R_2 are fiber surfaces.*

Gabai introduced the notion of a *sutured manifold* in [6] and investigated the properties of Seifert surfaces by looking directly at the complement of the surface, which is just a sutured manifold, in [11, 12, 13]. It has been a powerful tool in 3-manifold theory. (The solution of the Property R conjecture is one of the most significant consequences.) In [10] he demonstrated the techniques of sutured manifolds in the geometrical proof of Theorem 1.0.1. In Chapter 4, we will extend this proof to a class of links which are called *almost alternating links*.

The theory of sutured manifolds is reported synthetically in [27].

In [16], J. Harer showed the following theorem:

Theorem 1.0.4. *Every fibered link in S^3 is obtained from the unknot by a sequence of plumbings, deplumbings (converse operation of plumbing) and Stallings twists.*

Then he asked whether the twisting operation can be omitted or not, that is, any Stallings twist can be realized by plumbings and deplumbings or not. This question has attracted my interest so much. In Chapter 5, we will give a partial affirmative answer of this question.

Our terminology and notations used in common throughout all chapters are given in Chapter 2. And in Appendix we partially answer Adams' open question in [2, p.140]. We give an almost alternating diagram for a knot which had not been known to be almost alternating or not.

Chapter 2

Notation and definitions

2.1 Definitions

The *exterior* $E(L)$ of a link L in S^3 is the closure of $S^3 - N(L; S^3)$. If R is a Seifert surface for L , we may assume $R \cap E(L)$ is homeomorphic to R , and often abbreviate $R \cap E(L)$ as R .

We say that a loop in a surface is *essential* if it is not null-homotopic in the surface. Let F be a surface in a connected 3-manifold M . A disk D in M is called a *compressing disk* for F in M if $D \cap F = \partial D$ and ∂D is essential in F .

Let F be a compact proper surface in a compact connected 3-manifold M . For F not a sphere or disk, we say that F is *compressible* in M if there exists a compressing disk for F in M . Otherwise, we say that F is *incompressible* in M .

2.2 Fibered knots and links

A knot or Link L in S^3 is called *fibered* if its complement is a Seifert surface bundle over S^1 . We can construct a exterior of a fibered link from a Seifert surface F times the interval I by gluing $F \times \{0\}$ to $F \times \{1\}$ via a homeomorphism $h : F \rightarrow F$. The Seifert surface F is called a *fiber surface* and the (isotopy class of the) homeomorphism h is called a *monodromy map* for F .

The most simple fibered knot is the trivial knot. The fiber surface for it is a disk. A *Hopf band*, which is a full-twisted annulus in S^3 , is one of simple fiber surfaces. Gabai suggests a general approach for deciding whether a given oriented link is fibred in [10].

2.3 Murasugi sum

A surface $R (\subset S^3)$ is a *2n-Murasugi sum* of two surfaces R_1 and R_2 in S^3 if the following conditions are satisfied;

1. $R = R_1 \cup_{\Delta} R_2$, where Δ is a $2n$ -gon, i.e., $\partial\Delta = \mu_1 \cup \nu_1 \cup \dots \cup \mu_n \cup \nu_n$ (possibly $n = 1$), where μ_i (resp. ν_i) is an arc properly embedded in R_1 (resp. R_2).
2. There exist 3-balls B_1 and B_2 in S^3 such that:
 - (i) $B_1 \cup B_2 = S^3$, $B_1 \cap B_2 = \partial B_1 = \partial B_2 = S^2$: a 2-sphere,
 - (ii) $R_1 \subset B_1, R_2 \subset B_2$ and $R_1 \cap S^2 = R_2 \cap S^2 = \Delta$.

We also say that R has a *2n-Murasugi decomposition* into R_1 and R_2 . The 2-Murasugi sum is known as the connected sum, and the 4-Murasugi sum is known as the *plumbing*.

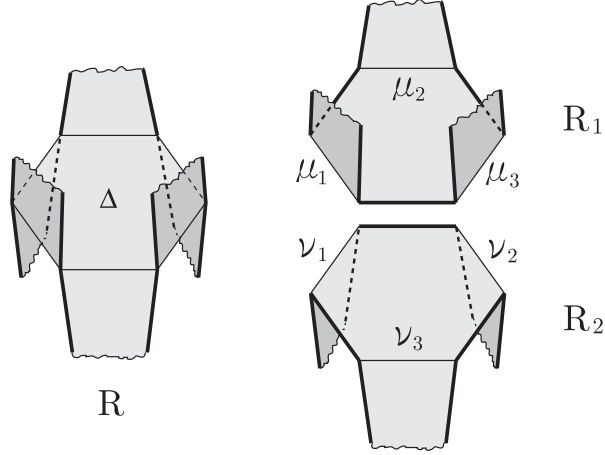


Figure 2.1

We call a fiber surface F a *Hopf plumbing* if F can be constructed successively plumbing a finite number of Hopf bands to a disk, and call F a *stable Hopf plumbing* if F can be constructed by plumbing and deplumbing a finite number of Hopf bands to a disk.

2.4 Sutured manifolds

A *sutured manifold* (M, γ) is a codimension-0 submanifold M of S^3 together with a set $s(\gamma) \subset \partial M$ of pairwise disjoint, embedded, oriented, simple closed curves (called *sutures*) separating ∂M into oriented regions $R_+(\gamma)$ and $R_-(\gamma)$ i.e., $\partial M = R_+(\gamma) \cup R_-(\gamma)$, $s(\gamma) = R_+(\gamma) \cap R_-(\gamma)$ and the oriented boundary of $R_\pm(\gamma) = s(\gamma)$. (The definition of sutured manifold in [8, 6] is more general than that given here, which is enough for our needs.) Figure 2.3 shows examples of sutured manifolds.

We say that a sutured manifold (M, γ) is a *product sutured manifold* if (M, γ) is

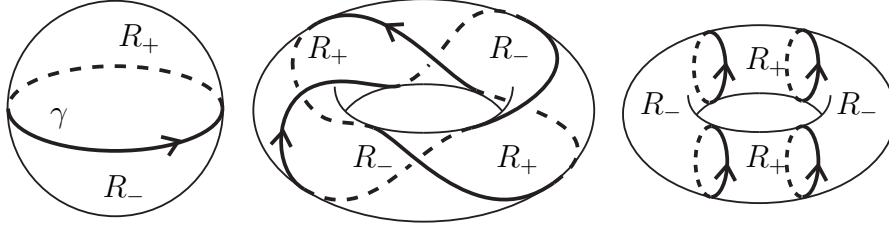


Figure 2.2

homeomorphic to $(R \times I, \partial R \times I)$ with $R_+(\gamma) = R \times \{1\}$, $R_-(\gamma) = R \times \{0\}$, where R is a compact oriented surface with no closed components and I is the unit interval $[0, 1]$.

Let R be a Seifert surface for L in S^3 . The product sutured manifold $(M, \gamma) = (R \times I, \partial R \times I)$ is called the sutured manifold *obtained* from R and the sutured manifold $(N, \delta) = (E(L) - \text{Int}M, \partial E(L) - \text{Int}\gamma)$ is the *complementary* sutured manifold for R (or for (M, γ)).

Note that R is a fiber surface if and only if the complementary sutured manifold for R is a product sutured manifold.

A *product decomposition* is a sutured manifold decomposition

$$(M_1, \gamma_1) \xrightarrow{B} (M_2, \gamma_2),$$

where B is a disk properly embedded in M_1 such that $B \cap s(\gamma_1) = (2 \text{ points})$, $M_2 = M_1 - \text{Int}N(B)$ and that $s(\gamma_2)$ is obtained by extending $s(\gamma_1) - \text{Int}N(B)$ in the natural way (Figure 2.3 (a)). The disk B is called a *product disk*.

Dually, *C-product decomposition* is the operation

$$(M_1, \gamma_1) \xrightarrow{E} (M_2, \gamma_2),$$

where E is a disk properly embedded in $S^3 - \text{Int}M_1$ such that $E \cap s(\gamma_1) = (2 \text{ points})$, M_2 is obtained from M_1 by attaching the 2-handle $N(E)$ and that $s(\gamma_2)$ is obtained by extending $s(\gamma_1) - \text{Int}N(E)$ in the natural way (Figure 2.3 (b)). The disk E is called a *C-product disk*.

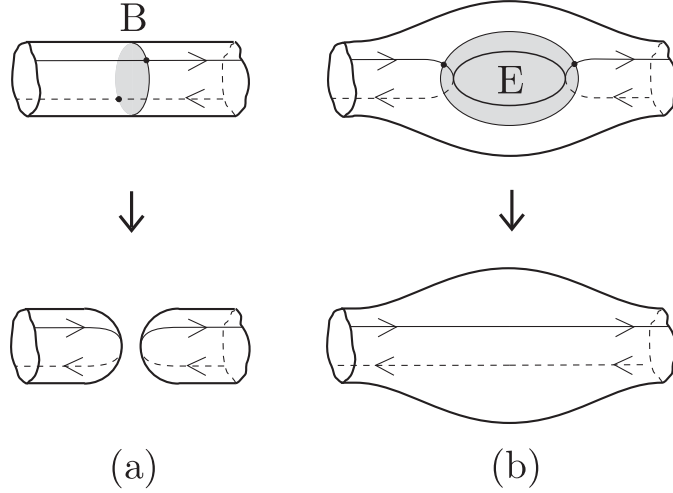


Figure 2.3

Let R be a Seifert surface for a link L . We say that R has a *product decomposition* if there exists a sequence of *C-product decompositions*

$$(R \times I, \partial R \times I) = (M_0, \gamma_0) \xrightarrow{E_1} (M_1, \gamma_1) \xrightarrow{E_2} \cdots \xrightarrow{E_p} (M_p, \gamma_p),$$

where the complementary sutured manifold for (M_p, γ_p) is a union of 3-balls each with a single suture.

As a criterion to detect a fiber surface, Gabai has shown the following:

Theorem 2.4.1 ([10, Theorem 1.9]). *Let L be an oriented link in S^3 , and R a Seifert surface for L . Then, L is a fibered link with fiber R if and only if R has a product decomposition.*

Chapter 3

Fibered 2-bridge links

The Conway notation was introduced by Conway [4] as a notation for a rational tangle. This notation is very useful to treat tangles systematically. In this chapter, we introduce a *labeled Conway notation* for a 2-bridge link L , in order to represent a structure of a Seifert surface S obtained by applying Seifert's algorithm to the diagram for L represented by a Conway notation C . This new notation can be constructed by labeling $\mathfrak{H}$ or $\mathfrak{T}$ to each term a_i of C according to how to span the part of the surface S corresponding to a_i . For convenience, we divide it into a number of *blocks*. This corresponds to a Murasugi decomposition of the Seifert surface. We will make fibered 2-bridge links a particular study, then we have

Theorem 3.0.2. *A 2-bridge link L is a fibered link if and only if L has a labeled Conway notation such that each block is the following type.*

$$\left| \begin{array}{ccc} 1, & m, & 1 \\ \mathfrak{T}, & \mathfrak{H}, & \mathfrak{T} \end{array} \right| \quad (m \geq 0)$$

We note that a 2-bridge link is an arborescent link. The notion of an arborescent link was first defined by Conway [4], who called it an algebraic link. Montesinos [20] constructed the method to define these links from weighted trees. By this construction, an arborescent link possesses a weighted tree representation, in particular, a 2-bridge link has a weighted path representation. To prove our theorem, first we represent the fiber surface of a fibered 2-bridge link by a labeled Conway notation, second we obtain a path representation of the fiber surface from its labeled Conway notation, and last we apply the following proposition to the path representation.

Proposition 3.0.3 ([11]). *Suppose $L = \partial S$ where S is the plumbed surface constructed as specified by the tree T . Then S is the fiber of a fibered link L if and only if the weight at each vertex is ± 2 .*

The organization of this chapter is as follows: In Section 3.1, we review the definition of an arborescent link. In Section 3.2, we define a labeled Conway notation. In Section 3.3, we construct an algorithm to directly get a tree representation of a 2-bridge link from its labeled Conway notation. In Section 3.4, we prove the main theorem. In Section 3.5, we consider the topology of the subspace of $\mathbb{Q}$ which consists of rational numbers corresponding to fibered 2-bridge links.

3.1 Preliminaries

We can obtain a 2-bridge link from a rational tangle by closing it as shown in Figure 3.1. Thus we can regard that a Conway notation for a rational tangle represents the diagram of a corresponding 2-bridge link.

Definition 3.1.1. An arborescent link is the boundary of a surface constructed by

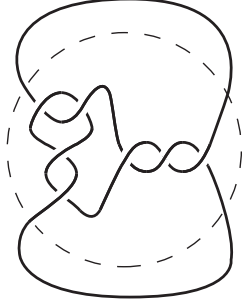


Figure 3.1

plumbing (Murasugi sum along a 4-gon) as specified by a weighted tree.

Construction 3.1.2. Let T be a weighted tree, and v the vertex on T which has a weight w and adjoining k 1-cells, for $w \in \mathbb{Z}$ and $k \in \mathbb{N}$. To the vertex v , we associate a w -half-twisted band which has k squares labeled $v_1, v_2, \dots, v_k$ where the v_i is the vertex in a cyclic order which shares an edge with v (Figure 3.2). We define positive half-twisted as in Figure 3.3(a), and negative ones as in Figure 3.3(b).

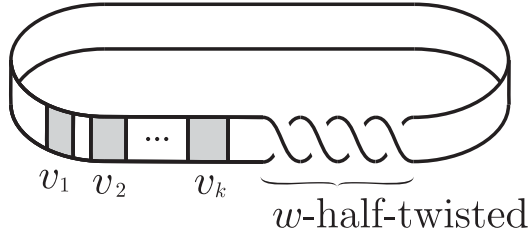


Figure 3.2

In the same way, we obtain the bands from all vertices of T . Finally, if v_i, v_j are adjacent vertices then plumb corresponding bands by gluing the squares v_i and v_j together. Let S be the surface we obtain above. We call T a *tree representation* for the link $L = \partial S$. We identify T with the tree obtained by changing the signs of all

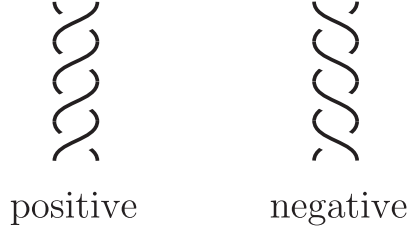


Figure 3.3

weights of T .

Remark 3.1.3. A 2-bridge link is an arborescent link which possesses a path, that is, a tree such that no vertex adjoins more than two edges.

3.2 Labeled Conway notation for 2-bridge link

In this section we define a *labeled Conway notation* which represents a structure of a Seifert surface of a 2-bridge link.

Let L be a 2-bridge link, $C = \{a_1, a_2, \dots, a_n\}$ a Conway notation for L , D the diagram for L which C represents, and S a Seifert surface obtained by applying Seifert's algorithm to D .

We put a label $\mathfrak{H}$ or a label $\mathfrak{T}$ to each integer a_i ($i = 1, 2, \dots, n$). If the integral tangle t_i on D which corresponds to a_i spans a twisted band on S then we give a_i the label $\mathfrak{T}$. Otherwise we give the label $\mathfrak{H}$ (Figure 3.4).

We denote by l_i the label of a_i ($i = 1, 2, \dots, n$), and for convenience we assume $l_0 = l_{n+1} = \mathfrak{T}$.

Labeling Rule. The labels observe the following rules.

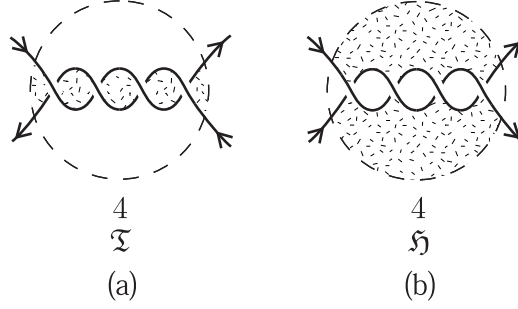


Figure 3.4

1. The labels adjacent to a label $\mathfrak{H}$ are only $\mathfrak{T}$.
2. The labels adjacent to a label $\mathfrak{T}$ satisfy the following conditions:

Suppose $l_k = \mathfrak{T}$ ($1 \leq k \leq n$).

- (a) If a_k is odd, the two labels l_{k-1}, l_{k+1} are distinct.
- (b) If a_k is even, the two labels l_{k-1}, l_{k+1} are same.

Lemma 3.2.1. *A Conway notation C such that each integer has a label $\mathfrak{H}$ or $\mathfrak{T}$ realizes a Seifert surface S if and only if the labels observe Labeling Rule.*

Proof. We can easily show ‘if’ part by practically spanning a surface in each case of Labeling Rule. Thus we show ‘only if’ part.

We note that if a_i has a label $\mathfrak{T}$, the four legs of the integral tangle t_i corresponding to a_i have the orientations as in Figure 3.4(a), and that otherwise as in Figure 3.4(b).

We will show the lemma by using a 4-plat presentation for a 2-bridge link. Thus we can depict integral tangles t_{i-1}, t_i, t_{i+1} as in Figure 3.5(a),(b), and can determine the orientation of the top strand of Figure 3.5(a),(b) as shown without loss of generality. In the following, we investigate the case of Figure 3.5(a) only, but we can do similarly

in the case of Figure 3.5(b).

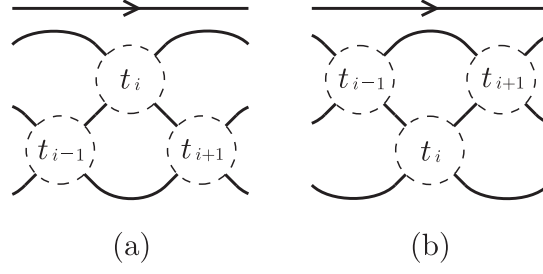


Figure 3.5

First, we suppose a label of a_i ($i = 2, 3, \dots, n-1$) is $\mathfrak{H}$. As in Figure 3.6, we can determine the orientation of the strand x from the orientations of four legs of t_i . Then from orientations of the legs of t_{i-1}, t_{i+1} , we see that labels of a_{i-1} and a_{i+1} are $\mathfrak{T}$.

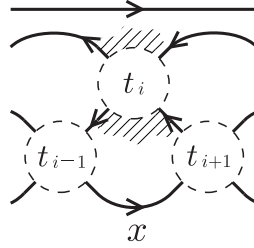


Figure 3.6

Second, we suppose a label of a_i ($i = 2, 3, \dots, n-1$) is $\mathfrak{T}$ and the strand y has the orientation as in Figure 3.7. If a_i is odd, we have orientations of the four legs of t_i as in Figure 3.7, and have that of the strand x . Then we have $l_{i-1} = \mathfrak{T}$, and $l_{i+1} = \mathfrak{H}$. If a_i is even, we have $l_{i-1} = l_{i+1} = \mathfrak{T}$ similarly. When we suppose the strand y has the alternate orientation, we have that if a_i is odd, $l_{i-1} = \mathfrak{H}$ and $l_{i+1} = \mathfrak{T}$, and that if a_i is even, $l_{i-1} = l_{i+1} = \mathfrak{T}$.

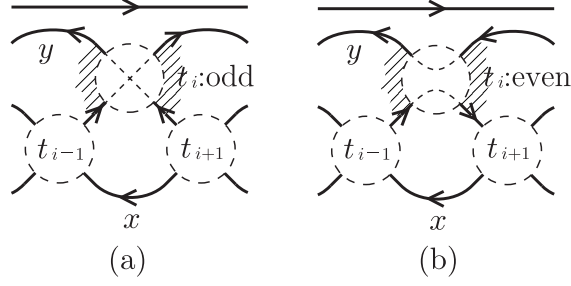


Figure 3.7

Last, we check labels of a_1 and a_n . Suppose l_1 is $\mathfrak{H}$, from orientations of the legs of t_i we have $l_2 = \mathfrak{T}$ (Figure 3.8(a)). Suppose l_1 is $\mathfrak{T}$. If a_i is odd, we have $l_2 = \mathfrak{H}$. If a_i is even, we have $l_2 = \mathfrak{T}$ (Figure 3.8(b),(c)). Since $l_0 = \mathfrak{T}$, a label of a_1 also observes Labeling Rule. In the case of a_n , we can have similar arguments.

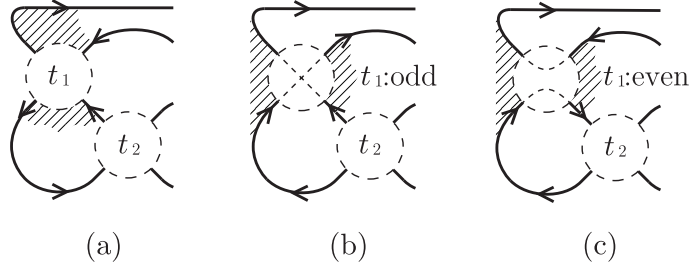


Figure 3.8

□

Definition 3.2.2. A *labeled Conway notation* is a Conway notation such that each integer has a label $\mathfrak{H}$ or $\mathfrak{T}$ and all labels observe Labeling Rule. We denote it by $\{(a_1, l_1), (a_2, l_2), \dots, (a_n, l_n)\}$ or

$$\begin{array}{ccccccc} a_1, & a_2, & \dots, & a_n \\ l_1, & l_2, & \dots, & l_n \end{array}.$$

Remark 3.2.3. When we determine whether a label of a_1 is $\mathfrak{H}$ or $\mathfrak{T}$, the following labels $l_2, l_3, \dots, l_n$ are determined inductively by using Labeling Rule. For example, if a_1 is an odd number and we determine a label of a_1 is $\mathfrak{T}$, according to Labeling Rule the label of a_2 is $\mathfrak{H}$. (Note that the label of a_0 is $\mathfrak{T}$.) But there is a case where a label of a_n which is detected with this method does not observe Labeling Rule. By a brief observation we have the following facts: If L is a knot, the diagram D obtained from the Conway notation has exactly one orientation, and then in either case of $l_1 = \mathfrak{H}$ or of $l_1 = \mathfrak{T}$, all labels observe Labeling Rule, and we obtain exactly one labeled Conway notation, which represents a Seifert surface obtained from the (oriented) diagram D by applying Seifert's algorithm. If L is a two-component link, in both cases of $l_1 = \mathfrak{H}$ and $l_1 = \mathfrak{T}$, all labels observe Labeling Rule, and two labeled Conway notations respectively represent two Seifert surfaces obtained from D_1 and D_2 , where D_1 and D_2 are two oriented diagrams obtained from D .

The following is a way to know a part of the labels of C .

Lemma 3.2.4. *Suppose $l_k = \mathfrak{T}$.*

1. *If a_{k+1} is even, then we have $l_{k+2} = \mathfrak{T}$.*
2. *If a_{k+1} is odd, and if a_{k+s} is the next odd number, then $l_{k+s+1} = \mathfrak{T}$.*

Proof. **Case 1.** a_{k+1} is even.

By Labeling Rule, the labels l_k, l_{k+1}, l_{k+2} appear as

$$\begin{array}{ccccccc} \dots & a_k, & a_{k+1}, & a_{k+2}, & \dots & & \\ & \mathfrak{T}, & \mathfrak{H} \text{ or } \mathfrak{T}, & \mathfrak{T}, & & & \end{array},$$

and we have $l_{k+2} = \mathfrak{T}$.

Case 2. a_{k+1} is odd.

By Labeling Rule, the labels $l_k, l_{k+1}, \dots, l_{k+s+1}$ appear as

$$\begin{array}{ccccccccccc} \dots & a_k, & a_{k+1}, & a_{k+2}, & \dots & a_{k+s-1}, & a_{k+s}, & a_{k+s+1}, & \dots & & \\ & \mathfrak{T}, & \mathfrak{H} \text{ or } \mathfrak{T}, & \mathfrak{T} \text{ or } \mathfrak{H}, & & \mathfrak{T} \text{ or } \mathfrak{H}, & \mathfrak{H} \text{ or } \mathfrak{T}, & \mathfrak{T}, & & & \end{array},$$

and we have $l_{k+s+1} = \mathfrak{T}$.

□

Remark 3.2.5. The label l_0 must be $\mathfrak{T}$. So starting from l_0 and skipping some parts, we can determine some terms which have a label $\mathfrak{T}$ by Lemma 3.2.4.

3.3 Tree representation of 2-bridge link

In this section we show a relationship between two representations for a structure of a Seifert surface of a 2-bridge link, a labeled Conway notation and a tree representation.

A 2-bridge link has a Conway notation $\{a_1, a_2, \dots, a_n\}$ such that $a_i > 0$ ($i = 1, 2, \dots, n$). Note that the diagram D obtained from such a notation is alternating, so a Seifert surface obtained by applying Seifert's algorithm to D is of minimal genus. In the remaining of this paper, we always choose a Conway notation which consists of positive integers.

As in Remark 3.1.3, a 2-bridge link has a path representation with weighted vertices associated to twisted bands. In the following, we will present the way to directly construct a tree representation of a 2-bridge link from its labeled Conway notation.

We need some preparations.

Let L be a 2-bridge link, $C = \{(a_1, l_1), (a_2, l_2), \dots, (a_n, l_n)\}$ its labeled Conway notation, and P a weighted path representing a minimal genus Seifert surface obtained from C .

If the label $\mathfrak{T}$ repeats twice on C , by cutting between two $\mathfrak{T}$'s, divide C into a number of *blocks* $B_1, B_2, \dots, B_m$. Let a block B_{2k-1} have a positive sign $+$, and B_{2k} a negative sign $-$ ($k \in \mathbb{N}$).

To the term (a_i, l_i) in a block B_{2k-1} (B_{2k} resp.), if $l_i = \mathfrak{H}$, we associate the weighted path p_i which has $a_i - 1$ vertices weighted 2 (-2 resp.), and if $l_i = \mathfrak{T}$, we associate the weighted path p_i which has exactly one vertex weighted $a_j + s$ ($-a_j - m$ resp.), where s is the number of labels $\mathfrak{H}$ which adjacent to l_j , i.e. $s = 0, 1, 2$ (see Figure 3.9).

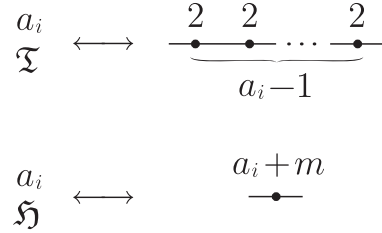


Figure 3.9

Lemma 3.3.1. *We can construct P from $\{p_i\}_{i=1,2,\dots,n}$ by connecting p_i and p_{i+1} with an edge in a line.*

Proof. We show an isotopy which transforms the Seifert surface S into a surface specified by a weighted path, starting from the part of S corresponding to a_1 and continuing to $a_2, a_3, \dots, a_n$ one by one.

We perform the first step of the isotopy. There are two cases of l_1 :

Case 1. $l_1 = \mathfrak{H}$.

The part of the surface S corresponding to a_1 appears as in Figure 3.10(a). We can slide the first half-twisted band b (in Figure 3.10(a)) so that we can see that a Hopf band which is summed by a plumbing on the next band (Figure 3.10(b)). By

performing such an isotopy from the left side one by one, we obtain the surface such that $a_1 - 1$ half-twisted bands pile on the last half-twisted band by plumbing (Figure 3.10(c)).

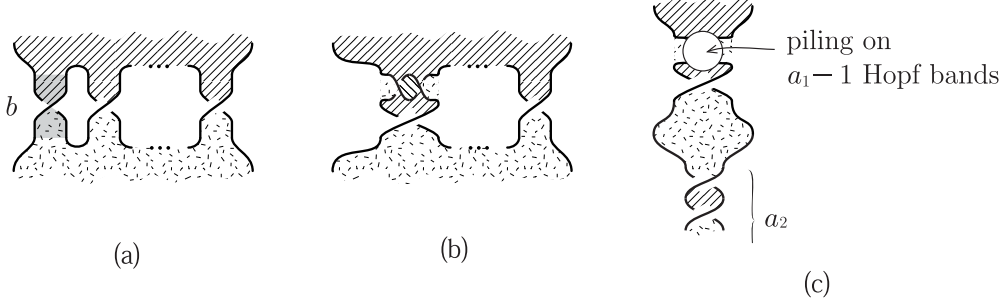


Figure 3.10

Since the label of a_2 is $\mathfrak{T}$, the part of S which corresponds to a_2 is an a_2 -half-twisted band. So we have an $a_2 + 1$ -half-twisted band which $a_1 - 1$ Hopf bands pile on.

Case 2. $l_1 = \mathfrak{T}$.

we have two cases of l_2 :

Case 2.1. $l_2 = \mathfrak{H}$.

By sliding the a_1 -half-twisted band b appears as in Figure 3.11(a), we can isotope S into the surface which appears as in Figure 3.11(b).

Case 2.2. $l_2 = \mathfrak{T}$.

It is easy to see that the a_1 -half-twisted band is on the a_2 -half-twisted band by plumbing.

Now we could move the part of the surface S corresponding to a_1 onto the one corresponding to a_2 . We can apply such an isotopy to the part of S corresponding to a_2 , and further to $a_3, a_4, \dots$, then finally we obtain a surface which is specified by a

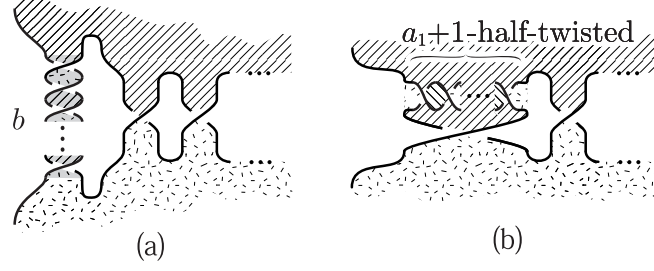


Figure 3.11

path. We can see that if a_i has a label $\mathfrak{H}$ then it yields $a_i - 1$ Hopf bands, and that if a_i has a label $\mathfrak{T}$ then it yields an $a_i + s$ -half-twisted band, where s is the number of labels $\mathfrak{H}$ adjacent to l_i .

□

3.4 Proof of Theorem 3.0.2

Proof of Theorem 3.0.2. Let L be a fibered 2-bridge link, and $C = \{(a_1, l_1), (a_2, l_2), \dots, (a_n, l_n)\}$ a labeled Conway notation for L . By Proposition 3.0.3, the vertices of a path which represents a fibered 2-bridge link must have weights ± 2 . If the label of a_k is $\mathfrak{H}$, by Lemma 3.3.1, a_k yields vertices with weight ± 2 . So a_k can take all positive integers. If the label of a_k is $\mathfrak{T}$, by Lemma 3.3.1, a_k yields a vertex with weight $a_k + s$, where s is the number of labels $\mathfrak{H}$ adjacent to l_k . We have $a_k + s = 2$, and $a_k > 0$, therefore we have $(a_k, s) = (2, 0)$ or $(1, 1)$. Thus C must have blocks of the following types only;

$$\left| \begin{array}{c} 2 \\ \mathfrak{T} \end{array} \right|, \left| \begin{array}{ccc} 1, & m, & 1 \\ \mathfrak{T}, & \mathfrak{H}, & \mathfrak{T} \end{array} \right| (m \in \mathbb{N}), \left| \begin{array}{cc} a_1, & 1 \\ \mathfrak{H}, & \mathfrak{T} \end{array} \right|, \left| \begin{array}{cc} 1, & a_n \\ \mathfrak{T}, & \mathfrak{H} \end{array} \right|.$$

Note that exchanging blocks on a labeled Conway notation as

$$\begin{array}{c} \left| \begin{array}{c} 2 \\ \mathfrak{T} \end{array} \right| \longleftrightarrow \left| \begin{array}{ccc} 1, & 0, & 1 \\ \mathfrak{T}, & \mathfrak{H}, & \mathfrak{T} \end{array} \right| \\ \\ \left| \begin{array}{cc} a_1, & 1 \\ \mathfrak{H}, & \mathfrak{T} \end{array} \right| \longleftrightarrow \left| \begin{array}{ccc} 1, & (a_1 - 1), & 1 \\ \mathfrak{T}, & \mathfrak{H}, & \mathfrak{T} \end{array} \right| \\ \\ \left| \begin{array}{cc} 1, & a_n \\ \mathfrak{T}, & \mathfrak{H} \end{array} \right| \longleftrightarrow \left| \begin{array}{ccc} 1, & (a_n - 1), & 1 \\ \mathfrak{T}, & \mathfrak{H}, & \mathfrak{T} \end{array} \right| \end{array}$$

does not change the link type. Then the proof is complete. $\square$

By Theorem 3.0.2, we have the following Corollary.

Corollary 3.4.1. *A link L is not a fibered link if and only if its labeled Conway notation divided into blocks as listed below.*

- i) $\left| \begin{array}{c} a_k \\ \mathfrak{T} \end{array} \right|$ for $a_k \geq 4$,
- ii) $\left| \begin{array}{ccc} a_k, & a_{k+1}, & a_{k+2} \\ \mathfrak{T}, & \mathfrak{H}, & \mathfrak{T} \end{array} \right|$ for $a_k \geq 3$ or $a_{k+2} \geq 3$,
- iii) $\left| \begin{array}{cccc} \dots, & a_k, & a_{k+1}, & a_{k+2}, & \dots \\ & \mathfrak{H}, & \mathfrak{T}, & \mathfrak{H} \end{array} \right|$.

By Lemma 3.2.4 and Theorem 3.0.2, we have the following theorem, which shows that we can detect that a 2-bridge link is non-fibered by looking at a part of its Conway notation.

Theorem 3.4.2. *A 2-bridge link L is non-fibered, if its Conway notation $C = \{a_1, a_2, \dots, a_n\}$ satisfies one of the following conditions;*

1) *Suppose the label of a_k is $\mathfrak{T}$. One of the following cases occurs;*

a) *a_{k+1} is even, and $a_{k+2} \geq 3$.*

b) *a_{k+1} is odd, and suppose the next odd number is a_{k+l} , then $a_{k+l+1} \geq 3$.*

2) *There is an integer $i \in \{1, 2, \dots, n-1\}$ such that $a_i \geq 3$, $a_{i+1} \geq 3$.*

3.5 Applications

In this section, we give some applications, and at the end of this section, we raise an open question.

Application 1. By Theorem 3.4.2 and Remark 3.2.5, we have that the 2-bridge links which have Conway notations listed below are non-fibered links.

1. $\{a, b, \dots\}$ for $a \geq 2$ and $b \geq 3$.
2. $\{a, 1, c, \dots\}$ for odd a and $c \geq 3$.
3. $\{a, 2, c, \dots\}$ for odd a and $b \geq 3$.
4. $\{a, 1, c, d, \dots\}$ for $a \geq 2$, $c \geq 3$, and $d \geq 3$.

In each the case, we can figure out the ranges of the slope

$$q = a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \dots}}}$$

for 2-bridge links with such Conway notations.

1. $a < q \leq a + \frac{1}{3}$ ($a = 2, 3, 4, \dots$).
2. $a + \frac{3}{4} < q < a + 1$ ($a = 1, 3, 5, 7, \dots$).
3. $a + \frac{3}{7} < q < a + \frac{1}{2}$ ($a = 3, 5, 7, \dots$).
4. $a + \frac{13}{17} < q < a + 1$ ($a = 2, 3, 4, \dots$).

Let L be a 2-bridge link with a slope q . If q is in one of the above ranges of rational numbers, then we can see that L is a non-fibered link.

Application 2. We consider the following question:

Question 1. Is there a convergent sequence of points on $\mathbb{Q}$ such that each value of points is the slope of a fibered 2-bridge link, and that its accumulating point is irrational number?

The answer of this question is “yes”.

Let $\{a_n\}$ be a sequence of rational numbers such that a_k is the slope of the 2-bridge link L_k which has a Conway notation

$$\{2, \underbrace{1, 1, \dots, 1}_k, 2\}$$

($k = 1, 2, \dots, n$). We consider the labels for this Conway notation. If $k \equiv 0 \pmod{3}$, then the labels are

$$\left| \begin{array}{c} 2, \\ \mathfrak{T} \end{array} \right| \left| \begin{array}{ccc} 1, & 1, & 1, \\ \mathfrak{T}, & \mathfrak{H}, & \mathfrak{T} \end{array} \right| \left| \begin{array}{ccc} 1, & 1, & 1, \\ \mathfrak{T}, & \mathfrak{H}, & \mathfrak{T} \end{array} \right| \dots \left| \begin{array}{ccc} 1, & 1, & 1, \\ \mathfrak{T}, & \mathfrak{H}, & \mathfrak{T} \end{array} \right| \left| \begin{array}{c} 2 \\ \mathfrak{T} \end{array} \right|.$$

If $k \equiv 1 \pmod{3}$, then L_k is a 2-component link. So the labels are

$$\left| \begin{array}{c} 2, \\ \mathfrak{T} \end{array} \right| \left| \begin{array}{ccc} 1, & 1, & 1, \\ \mathfrak{T}, & \mathfrak{H}, & \mathfrak{T} \end{array} \right| \left| \begin{array}{ccc} 1, & 1, & 1, \\ \mathfrak{T}, & \mathfrak{H}, & \mathfrak{T} \end{array} \right| \dots \left| \begin{array}{ccc} 1, & 1, & 1, \\ \mathfrak{T}, & \mathfrak{H}, & \mathfrak{T} \end{array} \right| \left| \begin{array}{cc} 1, & 2 \\ \mathfrak{T}, & \mathfrak{H} \end{array} \right|$$

and

$$\left| \begin{array}{cc} 2, & 1, \\ \mathfrak{H}, & \mathfrak{T} \end{array} \right| \left| \begin{array}{ccc} 1, & 1, & 1, \\ \mathfrak{T}, & \mathfrak{H}, & \mathfrak{T} \end{array} \right| \left| \begin{array}{ccc} 1, & 1, & 1, \\ \mathfrak{T}, & \mathfrak{H}, & \mathfrak{T} \end{array} \right| \dots \left| \begin{array}{ccc} 1, & 1, & 1, \\ \mathfrak{T}, & \mathfrak{H}, & \mathfrak{T} \end{array} \right| \left| \begin{array}{c} 2 \\ \mathfrak{T} \end{array} \right|.$$

If $k \equiv 2 \pmod{3}$, then

$$\left| \begin{array}{cc} 2, & 1, \\ \mathfrak{H}, & \mathfrak{T} \end{array} \right| \left| \begin{array}{ccc} 1, & 1, & 1, \\ \mathfrak{T}, & \mathfrak{H}, & \mathfrak{T} \end{array} \right| \left| \begin{array}{ccc} 1, & 1, & 1, \\ \mathfrak{T}, & \mathfrak{H}, & \mathfrak{T} \end{array} \right| \dots \left| \begin{array}{ccc} 1, & 1, & 1, \\ \mathfrak{T}, & \mathfrak{H}, & \mathfrak{T} \end{array} \right| \left| \begin{array}{cc} 1, & 2 \\ \mathfrak{T}, & \mathfrak{H} \end{array} \right|.$$

In any case, L_k is a fibered 2-bridge link by Theorem 3.0.2. The value a_k is represented by a Fibonacci sequence $\{b_k\} = \{2, 3, 5, 8, 13, \dots\}$ as

$$a_k = 2 + \frac{b_k}{b_{k+1}}.$$

Then we have

$$\lim_{k \rightarrow \infty} a_k = \frac{3 + \sqrt{5}}{2}.$$

The following question is still open.

Question 2. Is there a convergent sequence of rational numbers which are slopes of fibered 2-bridge links such that its accumulating point is a rational number which is also a slope of a fibered 2-bridge link?

Chapter 4

Almost alternating diagrams and fiber surfaces in S^3

Let R be a Seifert surface obtained by applying Seifert's algorithm to a connected diagram D for a link L . In this chapter, letting D be *almost alternating*, we give a practical algorithm to determine whether L is a fibered link and R is a fiber surface. It is an extension of Theorem 1.0.1, which is restated in Theorem 4.1.1, and we show that this is not always true if D is *2-almost alternating*. We further show that L is a fibered link and R is a fiber surface for L if and only if R is a Hopf plumbing.

4.1 Statements of results

We say that a Seifert surface R ($\subset S^3$) *desums* into a union of surfaces if R is a Murasugi sum of them. In [10], Gabai gave a geometric proof to the following theorem (also known by [24] and [28]).

Theorem 4.1.1 ([10, Theorem 5.1]). *Let L be an oriented link with an alternating diagram D . L is a fibered link if and only if the surface R obtained by applying Seifert's algorithm to D is connected and (obviously) desums into a union of Hopf bands.*

By using Propositions 4.5.1 and 4.5.2, the ‘only if’ part of Theorem 4.1.1 can be strengthened as in the following theorem.

Theorem 4.1.2. *Let L be an oriented alternating link, and R a Seifert surface obtained by applying Seifert's algorithm to an alternating diagram for L . Then the following conditions are equivalent: (1) L is a fibered link. (2) R is a Hopf plumbing. (3) R can be deformed into a disk by successively cutting one of a pair of ‘parallel bands’ (defined in Section 4.5).*

In [2], C. Adams et al. generalized the concept of alternating links and introduced the concept of almost alternating links. A diagram D in S^2 is said to be *almost alternating* (resp. *2-almost alternating*) if D becomes an alternating diagram after one crossing change (resp. two crossing changes). A link L in S^3 is said to be *almost alternating* if L is not alternating but admits an almost alternating diagram. If D is an almost alternating diagram, the specific crossing to change is called the *dealternator* and we call the other crossings the *alternators*.

In this paper, we extend Theorems 4.1.1 and 4.1.2 to almost alternating links. Note that almost alternating diagrams, however, do not always yield a minimal genus Seifert surface via Seifert's algorithm. Our first result is as follows:

Theorem 4.1.3. *Let D be an almost alternating diagram, and R a Seifert surface obtained by applying Seifert's algorithm to D . Then, R is a fiber surface if and only if R is connected and desums into a union of Hopf bands.*

In Section 4.5, we show a stronger version, as given below, of Theorem 4.1.3, by using Corollary 4.1.5 obtained from the arguments in the proof of Theorem 4.1.3.

Theorem 4.1.4. *Let R be a Seifert surface obtained by applying Seifert's algorithm to an almost alternating diagram. Then, R is a fiber surface if and only if R is a Hopf plumbing.*

Corollary 4.1.5 below gives a practical algorithm to determine whether or not a given almost alternating diagram yields a fiber surface via Seifert's algorithm. We say that a diagram D is *nested* if D has a Seifert circle which contains another circle in both of its complementary region. Otherwise we say D is *unnested*.

Corollary 4.1.5. *Let D be an almost alternating diagram and R a Seifert surface obtained from D by Seifert's algorithm. Then R is a fiber surface if and only if R is connected and desums into a union of Hopf bands by repeating the following decompositions;*

- (1) *a Murasugi decomposition along a nested Seifert circle,*
- (2) *a prime decomposition, and*
- (3) *Murasugi decompositions of type (A) and (B) in Figure 4.1, where each decomposition yields Seifert surfaces with first Betti numbers smaller than that of R .*

Proof. In the proof of Theorem 4.1.3, we explicitly show how we can desum such R into surfaces of smaller first Betti numbers. All necessary decompositions are covered in the above three cases. □

In [16], J. Harer proved Theorem 1.0.4 and raised the question we mentioned in Introduction. So it is worthy presenting the following partial affirmative answer as a corollary, which immediately follows from Theorems 4.1.2 and 4.1.4.

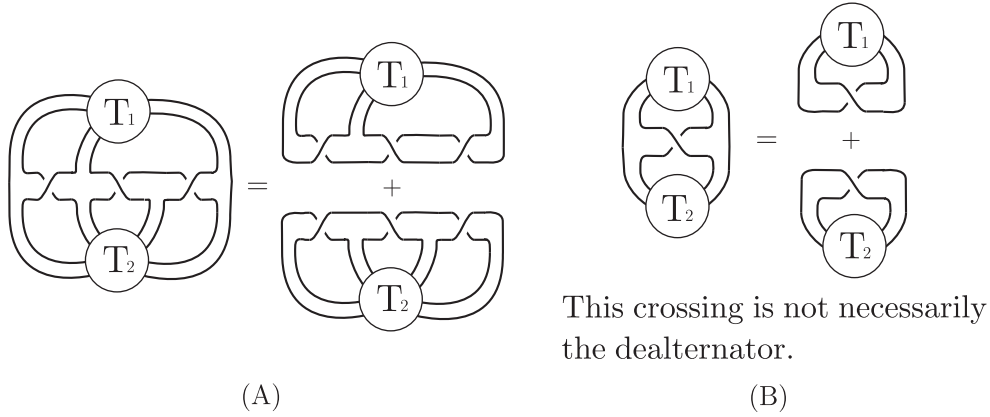


Figure 4.1

Corollary 4.1.6. *Let R_1 and R_2 be any fiber surface obtained by applying Seifert's algorithm to an alternating or almost alternating diagram. Then R_1 and R_2 can be changed into each other by plumbing and deplumbing Hopf bands.*

We say that a Hopf band B is *positive* (resp. *negative*) if the linking number of ∂B is 1 (resp. -1). By the following fact together with an observation of the way fiber surfaces deplumb in the proof of Theorem 4.1.3 (see Section 4.4) and Theorem 4.1.4 (see Section 4.5), we have the following corollary:

Corollary 4.1.7. *Let D be an unnested almost alternating diagram such that the sign of the dealternator is negative. Suppose that the surface R obtained from D by Seifert's algorithm is a fiber surface. Then R is a plumbing of positive Hopf bands.*

Fact 4.1.8. *Suppose that a diagram D is unnested. Then D is alternating (resp. almost alternating) if and only if all the crossings of D have the same sign (resp. the same sign except exactly one crossing).*

This chapter is organized as follows; Section 4.2 is for preliminaries. In Section

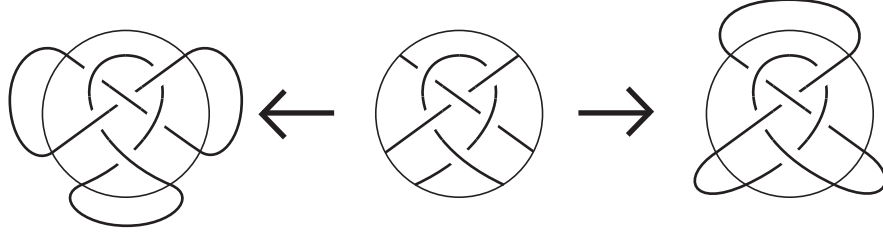


Figure 4.2

4.3, we give an example for our theorem. We also show that our theorem can not be extended to 2-almost alternating diagrams. In Sections 4.4 and 4.5, we prove Theorems 4.1.3 and 4.1.4 respectively.

4.2 Preliminaries

In this section, we observe how alternating and almost alternating diagrams behave when the Seifert surfaces obtained by Seifert's algorithm are Murasugi sums.

Concerning alternating and almost alternating tangles, we can confirm the following facts.

Fact 4.2.1. *Suppose that a link diagram D is a tangle sum of two tangle diagrams D_1 and D_2 . If D is alternating, then both D_1 and D_2 are alternating. And if D is almost alternating, then one of them, say, D_1 is alternating and D_2 is almost alternating, except for the case where the tangle containing the dealternator has only one crossing.*

Fact 4.2.2. *By connecting neighboring strands running out of an alternating (resp. almost alternating) tangle diagram, we obtain an alternating (resp. almost alternating) link diagram. See Figure 4.2.*

Then by these two facts, we can confirm the following propositions. Let R be a Seifert surface obtained by applying Seifert's algorithm to a diagram D .

Proposition 4.2.3. *If an almost alternating diagram D is a connected sum of two reduced diagrams, then one of them, say, D_1 is alternating and the other, say, D_2 is almost alternating. The Seifert surface R is a 2-Murasugi sum of R_1 and R_2 , where R_i is obtained from D_i .*

Proposition 4.2.4. *Suppose that an almost alternating diagram D has a nested Seifert circle C . Then, along the disk bounded by C , R is a Murasugi sum of R_1 and R_2 , where R_1 (resp. R_2) is obtained from an alternating (resp. almost alternating) diagram.*

Proposition 4.2.5. *Suppose that R desums into two surfaces R_1 and R_2 as illustrated in Figure 4.1, where the left figures in (A) and (B) are both almost alternating. Then R_i ($i = 1, 2$) is obtained from an alternating or almost alternating diagram.*

4.3 Examples

In this section, we present some examples. Example 4.3.1 is for Theorems 4.1.3 and 4.1.4. Example 4.3.2 shows that our results do not extend to 2-almost alternating diagrams. For the names of knots, refer to Rolfsen's book [26].

Example 4.3.1. Figure 4.3 (a) depicts an almost alternating diagram for the knot 10_{151} , together with a fiber surface R obtained by Seifert's algorithm. We can observe that R desums into a union of Hopf bands and is a Hopf plumbing.

Example 4.3.2. Let R be the Seifert surface obtained by applying Seifert's algorithm

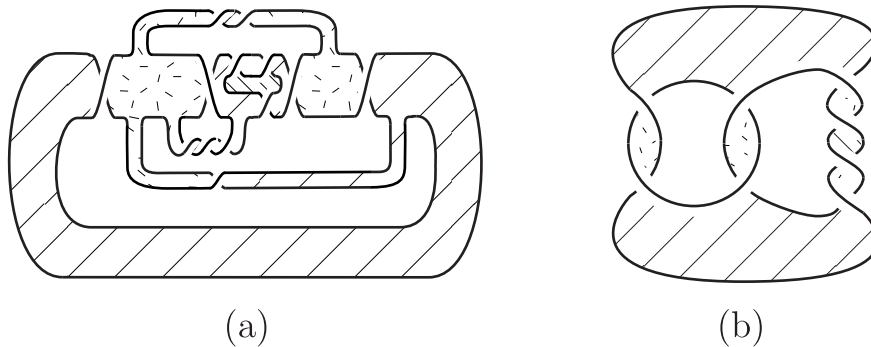


Figure 4.3

to the oriented pretzel link diagram of type $(2, -2, 2p)$ as in Figure 4.3 (b), where $p \neq 0$. R is a fiber surface but does not desum into a union of Hopf bands.

We note that this example has been known in [10] as a fiber surface for a link which does not admit a non-trivial Murasugi sum. There is also an example of a fiber surface for a non-trivial atoroidal knot which does not admit a non-trivial Murasugi sum. See [14, p.27] for how to construct an example.

Remark 4.3.3. The 2-almost alternating diagram in Example 4.3.2 yields a fiber surface which does not desum into a union of Hopf bands. Therefore, without evading this problem, Theorem 4.1.3 or 4.1.4 does not extend to 2-almost alternating diagrams. We also note that the link $P(2, -2, 2p)$ has an alternating diagram on a standard torus embedded in S^3 meeting the definition of the toroidally alternating link [2, p. 145].

Example 4.3.4. Figure 4.4 depicts 2-almost alternating diagrams for the knots 9_{42} , 9_{44} and 9_{45} , from which we can obtain fiber surfaces by Seifert's algorithm. Those surfaces are not Hopf plumbings. We can show this by the following proposition and direct calculations of the genera and the Conway polynomials of these knots.

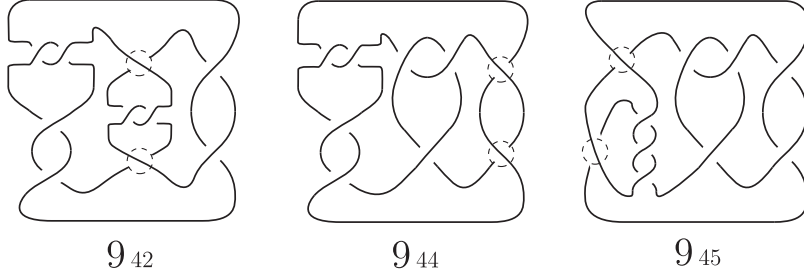


Figure 4.4

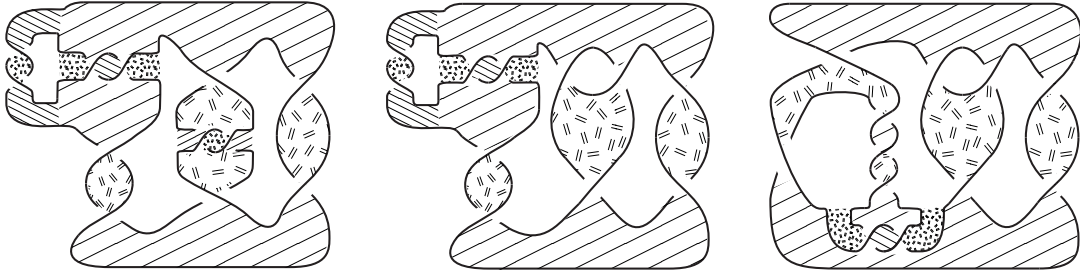


Figure 4.5

Proposition 4.3.5 ([19, Theorem 3]). *If a fibered knot K of genus 2 can be constructed by plumbing Hopf bands, then the Conway polynomial $\nabla_K(z)$ of K satisfies the following;*

$$\nabla_K(z) \neq \begin{cases} 1 + c_1 z^2 + z^4 & \text{for } c_1 \equiv 0 \pmod{4}, \\ 1 + c_1 z^2 - z^4 & \text{for } c_1 \equiv 2 \pmod{4}. \end{cases}$$

Remark 4.3.6. By using the technique of plumbing and deplumbing in [19, p. 167], we can show that the surfaces in Examples 4.3.2 and 4.3.4 are stable Hopf plumbings (i.e., become Hopf plumbings when Hopf bands are plumbed). Figure 4.5 shows where to plumb a Hopf band to show that.

4.4 Proof of Theorem 4.1.3

Since the ‘if’ part is shown by Theorem 1.0.3, we show the ‘only if’ part. Let D be an almost alternating diagram for a link $L (\subset S^3)$ on the *level 2-sphere* S^2 and let R be a Seifert surface obtained by applying Seifert’s algorithm to D .

Note that if a diagram D is unnested, then the surfaces obtained by applying Seifert’s algorithm to D are isotopic to one another. We say that a Seifert surface R is *flat* if R is obtained from an unnested diagram and thus lies in S^2 except in the neighborhood of each crossing.

Suppose that R is a fiber surface. Since any fiber surface is connected, we can assume that D is connected. Suppose D is nested. Then, by Theorem 1.0.3, R desums into fiber surfaces R_1 and R_2 . Moreover, by Proposition 4.2.4, one of them, say, R_1 is obtained from an alternating diagram and R_2 from an almost alternating diagram. By Theorem 4.1.1, R_1 desums into a union of Hopf bands. Therefore, we may assume that D is unnested. Similarly, by Proposition 4.2.3, we may assume that D is prime, and in particular, reduced.

Now we prove the theorem by induction on the first Betti number β_1 of R , where R is a fiber surface obtained by applying Seifert’s algorithm to a connected unnested prime almost alternating diagram D . If $\beta_1 = 1$, then R is an unknotted annulus and D has n crossings which are of the same sign except exactly one crossing. Note that R is a fiber surface if and only if $n = 4$, in which case R is a Hopf band. Hence we have the conclusion. Then we assume that the theorem holds when $\beta_1(R) < k$ and prove the theorem for R with $1 \leq \beta_1(R) = k$.

In order to do that, we use Gabai’s technique in [10] using sutured manifold theory. Let (M, γ) be the sutured manifold obtained from R . We identify $s(\gamma)$ as L . Then

it follows from Theorem 2.4.1 that there exists a C -product disk for (M, γ) , i.e., $S^3 - \text{Int}M$ so that $E \cap L = (2 \text{ points})$. In the following, the existence of the disk E enables us to find where we can desum R into surfaces R_1 and R_2 obtained by the algorithm with smaller first Betti numbers. In each case, it is easy to confirm that D_i ($i = 1, 2$) is an alternating or almost alternating diagram, that is, they satisfy the assumption of the induction (see Corollary 4.1.5 and Proposition 4.2.5).

We assume that $|E \cap S^2|$ is minimal among all such disks. Further, we may assume by isotopy that $\partial E \cap L$ occurs only in small neighborhoods of the crossings of D . Similarly, we can assume that $\partial E \cap S^2$ occurs only in small neighborhoods of the crossings. For convenience, we say that $\partial E \cap L$ and $\partial E \cap S^2$ occur at the crossings.

Case A. $E \cap S^2 = \emptyset$.

If $\partial E \cap L$ occurs at one crossing, then E is boundary parallel, a contradiction. Thus, we suppose that $\partial E \cap L$ occurs at two crossings. If both crossings are alternators (see Figure 4.6), we see that R is a plumbing of flat surfaces, one of which is obtained from an unnested almost alternating diagram and has a first Betti number smaller than k . If one crossing is the dealternator, we also see that R is a plumbing of surfaces, one of which is compressible. This contradicts Theorem 1.0.3, because any fiber surface is incompressible.

Case B. $E \cap S^2 \neq \emptyset$.

Label the crossings with $\mathfrak{a}_0, \mathfrak{a}_1, \dots, \mathfrak{a}_{w-1}$ so that the dealternator has $\mathfrak{a}_0$. By standard innermost circle argument, we may assume, by the minimality of $|E \cap S^2|$, that $E \cap S^2$ consists of arcs. Let α be an arc of $E \cap S^2$. By assumption, each endpoint of α lies in a neighborhood of a crossing and hence is accordingly labeled. Then the label of α is a pair $(\mathfrak{a}_i, \mathfrak{a}_j)$ of the labels of $\partial\alpha$. The two points of $\partial E \cap L$ are also labeled according to the crossings at which $\partial E \cap L$ occurs.

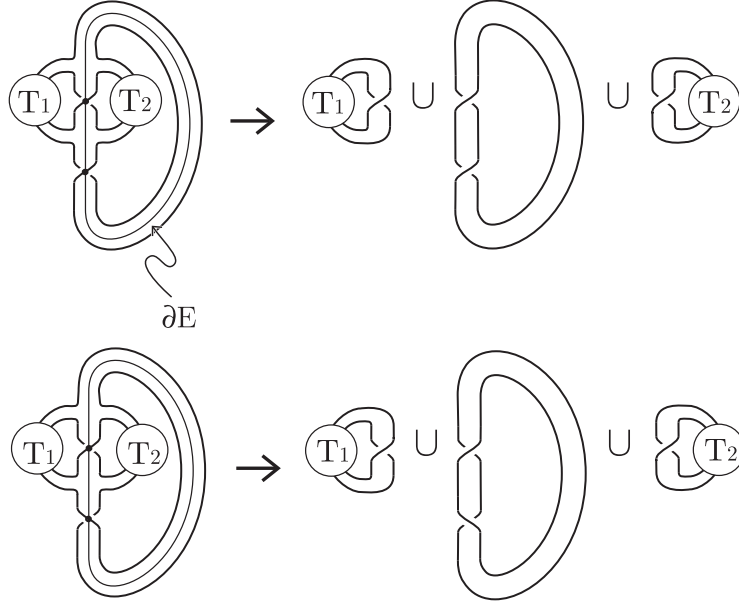


Figure 4.6

Lemma 4.4.1. *For any arc α of $E \cap S^2$ with label $(\mathbf{a}_i, \mathbf{a}_j)$, we have $i \neq j$.*

Proof. If both of the endpoints of α occur at the same crossing $\mathbf{a}_i$, we can observe that one of the two cases in Figure 4.7 occurs. In Figure 4.7 (a), D is non-prime. In Figure 4.7 (b), there exists an arc α' of $E \cap S^2$ in $S^2 - \text{Int}M$ such that the endpoints of α' occur at the same crossing $\mathbf{a}_i$, and that α' cuts off a disk H from $S^2 - \text{Int}M$ with $\text{Int}H \cap (E \cap S^2) = \emptyset$. We can surgery E along H so that we obtain two disks E_1, E_2 properly embedded in $S^3 - \text{Int}M$. Since both endpoints of α' are in $R \times \{1\}$ ($\subset \partial M$) (or $R \times \{-1\}$), one of them, say E_1 , intersects L twice. Since E is non-boundary-parallel, so is E_1 or E_2 . If E_2 is, then it yields a compressing disk for R , a contradiction. Hence E_1 is a non-boundary-parallel C -product disk with $|E_1 \cap S^2| < |E \cap S^2|$, a contradiction. $\square$

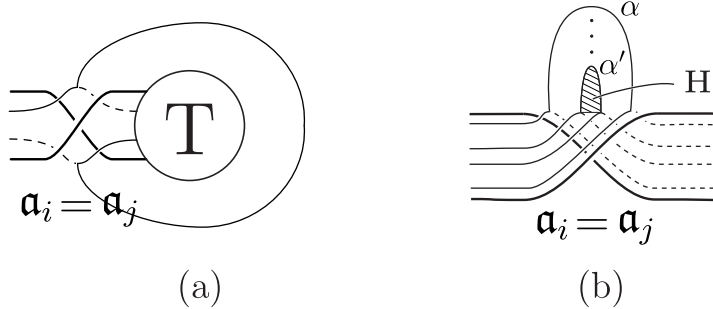


Figure 4.7

We look at an outermost disk $F \subset E$ (i.e., F is the closure of a component of $E - S^2$ such that $F \cap S^2$ is connected).

Lemma 4.4.2. *Let α be an outermost arc of $E \cap S^2$ with label $(\mathbf{a}_i, \mathbf{a}_j)$, cutting an outermost disk F off E . Then we may assume that $i \neq j$ and that i or $j = 0$ if $F \cap L = \emptyset$ or (a point).*

Proof. By Lemma 4.4.1, we have $i \neq j$. Suppose $i \neq 0$ and $j \neq 0$. If $|F \cap L| = 0$, R is non-prime, a contradiction (Figure 4.8 (a)). If $|F \cap L| = 1$, then either $|E \cap S^2|$ is not minimal, or R is a plumbing (Figures 4.8 (b) and (c)). $\square$

Concerning outermost disks, we have two cases.

Case B-1. *There exists an outermost disk F with $F \cap L = \{a \text{ point}\}$.*

Let α be the arc $F \cap S^2 (\subset E)$. By Lemma 4.4.2, we assume that the label of α is $(\mathbf{a}_0, \mathbf{a}_j)$, where $j \neq 0$. Let $\mathbf{a}_k$ be the label of the point of $\partial E \cap L$ on F . Then we have three cases; **Subcase 1:** $k = 0$, **Subcase 2:** $k = j$, and **Subcase 3:** $k \neq 0$ and $k \neq j$. In Subcases 1 and 2, D is non-prime (Figure 4.9 (a)). In Subcase 3, R is a plumbing or we can isotope E so that the outermost disk of Case B-1 is replaced by an outermost disk of Case B-2 (Figure 4.9 (b)).

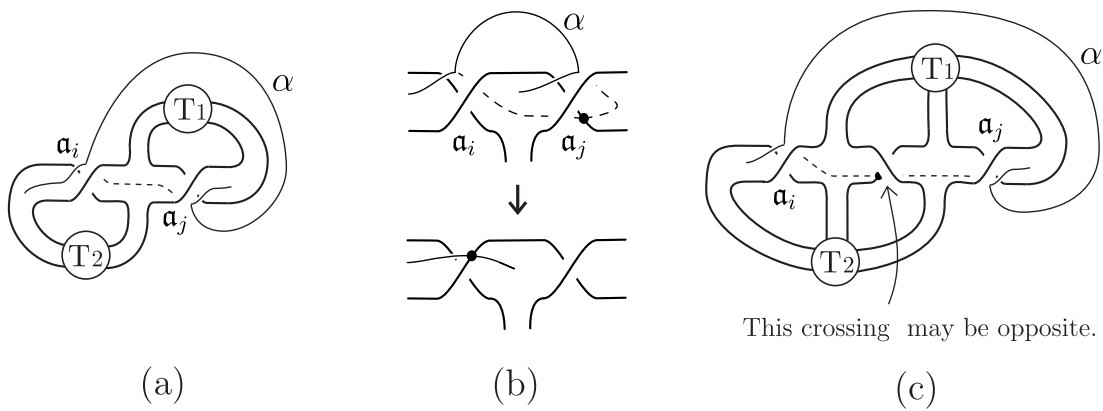


Figure 4.8

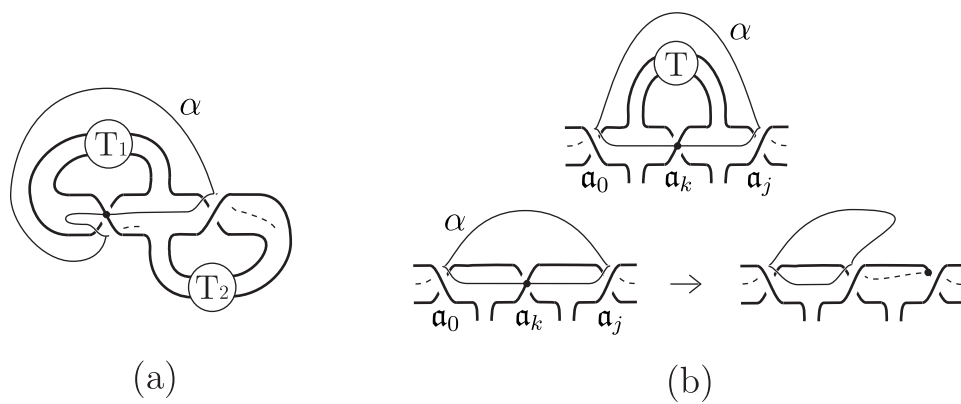


Figure 4.9

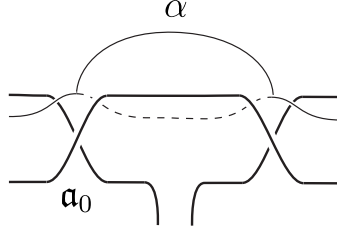


Figure 4.10

Lemma 4.4.3. *We may assume that there exists no outermost disk of Case B-1.*

Proof. If the latter situation of Subcase 3 above occurs, we can view the above isotopy of E as sliding a point of $\partial E \cap L$ out of F . Hence by repeating the above isotopies at most twice, we may eliminate outermost disks of Case B-1. $\square$

Case B-2. *There exists an outermost disk F with $F \cap L = \emptyset$.*

By Lemma 4.4.2, we may assume that $\alpha = F \cap S^2$ appears as in Figure 4.10. We note that outermost disks of this kind are typically found in the complementary sutured manifold for the fiber surface in Figure 4.3 (b), which is obtained from a 2-almost alternating diagram. The rest of the proof really depends on the almost-alternatingness of D .

Lemma 4.4.4. *For any arc β of $\partial E - (E \cap S^2)$, if $\beta \cap L = \emptyset$, then the endpoints of β have different labels.*

Proof. Suppose that the two endpoints of β have the same label. Then β appears as in Figure 4.11 and we can isotope E to a C -product disk E' such that $|E' \cap S^2| = |E \cap S^2| - 1$, a contradiction. $\square$

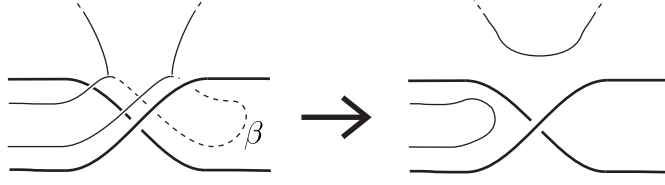


Figure 4.11

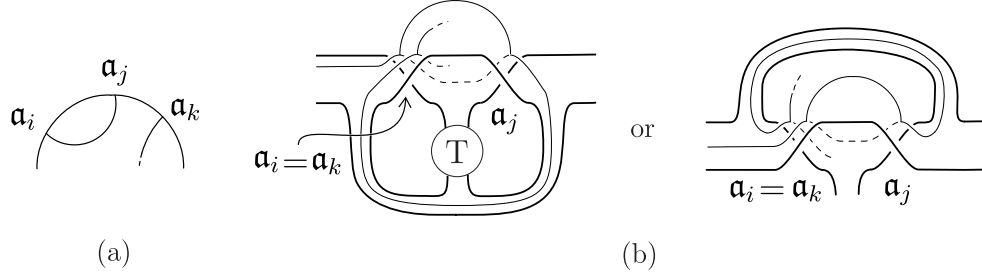


Figure 4.12

Lemma 4.4.5. *Suppose that E locally appears as in Figure 4.12 (a), i.e., $\mathfrak{a}_i$, $\mathfrak{a}_j$ and $\mathfrak{a}_k$ are the labels of points of $(\partial E \cap S^2) \cup (\partial E \cap L)$ sequential in ∂E such that the former two points are connected by an outermost arc of $E \cap S^2$ and the last is a point of $\partial E \cap S^2$. Then i, j, k are mutually different.*

Proof. By Lemma 4.4.2, we have $i \neq j$ and i or $j = 0$. Suppose $i = k$. Then we can find a compressing disk for R in Figure 4.12 (b), a contradiction. By Lemma 4.4.4, we have $j \neq k$. $\square$

Lemma 4.4.6. *We may assume that the following situation never occurs; The disk E locally appears as in Figure 4.13 (a), i.e., $\mathfrak{a}_i$, $\mathfrak{a}_j$, $\mathfrak{a}_k$ and $\mathfrak{a}_l$ are the labels of points of $(\partial E \cap S^2) \cup (\partial E \cap L)$ sequential in ∂E such that the former two points and the latter two are respectively connected by outermost arcs α_1 and α_2 of $E \cap S^2$.*

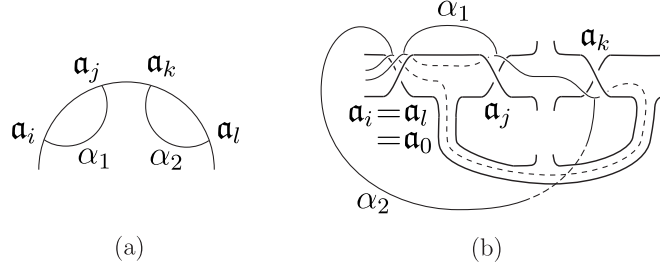


Figure 4.13

Proof. By Lemmas 4.4.2, 4.4.4 and 4.4.5, we may assume that $i = l = 0$ and $j \neq k$. Then we obtain the conclusion, since in Figure 4.13 (b) α_2 can not coexist with the arc of $\partial E - S^2$ connecting $\mathbf{a}_k$ and $\mathbf{a}_l$. $\square$

Lemma 4.4.7. *Suppose that E locally appears as in Figure 4.14 (a), i.e., $\mathbf{a}_i, \mathbf{a}_j$ and $\mathbf{a}_k$ are the labels of points of $(\partial E \cap S^2) \cup (\partial E \cap L)$ sequential in ∂E such that the former two points are connected by an outermost arc of $E \cap S^2$ and the third point is of $\partial E \cap L$. Then we may assume that i, j, k are mutually different.*

Proof. By Lemma 4.4.2, we have $i \neq j$, and i or $j = 0$. If $k = i$, then R is compressible, a contradiction (see Figure 4.14 (b)). If $k = j$, we can reduce $|E \cap S^2|$ by isotopy, a contradiction (see Figure 4.14 (c)). $\square$

Lemma 4.4.8. *Let $\mathbf{a}_l$ be the label of the point x of $\partial E \cap L$. Suppose that the two points adjacent to x in ∂E are points of $\partial E \cap S^2$. Then the two adjacent points do not have the same label except for the case where they are both $\mathbf{a}_l$.*

Proof. Suppose that the two points have the same label $\mathbf{a}_i (\neq \mathbf{a}_l)$. By Lemma 4.4.1, we may assume that they are not connected by an arc of $E \cap S^2$. Then we can find a C -product disk E' in Figure 4.15 such that $|E' \cap S^2| = 0$, a contradiction. $\square$

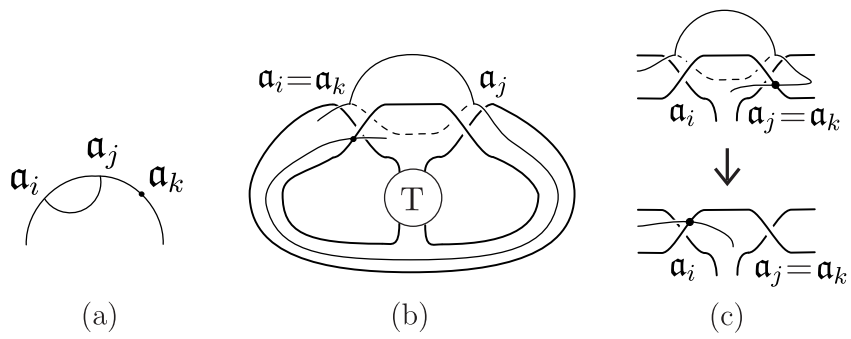
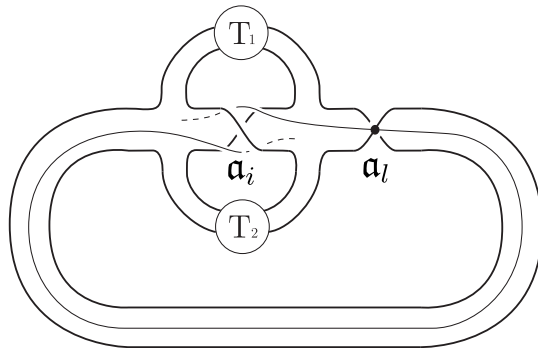


Figure 4.14



The crossing a_l may be opposite.

Figure 4.15

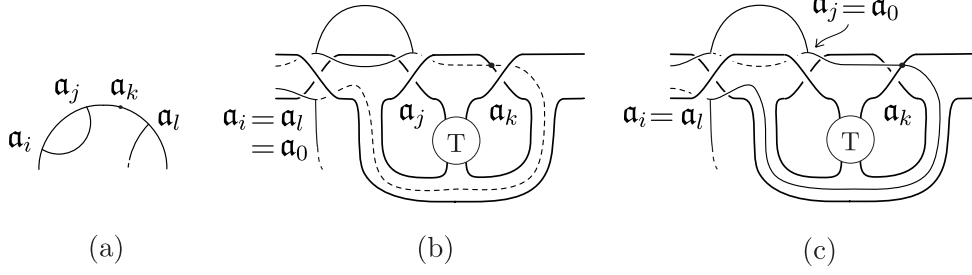


Figure 4.16

Similarly we have the following lemma.

Lemma 4.4.9. *Suppose that E locally appears as in Figure 4.16(a), i.e., $\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k$ and $\mathbf{a}_l$ are the labels of points of $(\partial E \cap S^2) \cup (\partial E \cap L)$ sequential in ∂E such that the former two points are connected by an outermost arc of $E \cap S^2$, the third point is of $\partial E \cap L$ and that the fourth is of $\partial E \cap S^2$. If $k \neq l$, then i, j, k and l are mutually different.*

Proof. By Lemma 4.4.7, we may assume that i, j, k are mutually different. Then by Lemma 4.4.8, we have $l \neq j$. Suppose $l \neq k$ and $l = i$. Then by Lemma 4.4.2, $\mathbf{a}_i = \mathbf{a}_l = \mathbf{a}_0$ (Figure 4.16 (b)) or $\mathbf{a}_j = \mathbf{a}_0$ (c). In either case, we can find a C -product disk E' such that $|E' \cap S^2| = 0$, a contradiction. $\square$

Lemma 4.4.10. *We may assume that the following situation never occurs; The disk E locally appears as in Figure 4.17 (a), i.e., $\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k, \mathbf{a}_l$ and $\mathbf{a}_m$ are the labels of points of $(\partial E \cap S^2) \cup (\partial E \cap L)$ sequential in ∂E such that the first two points and the last two points are respectively connected by an outermost arc of $E \cap S^2$, and that the third point is of $\partial E \cap L$.*

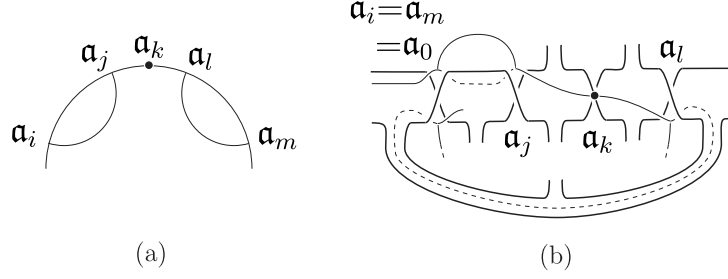


Figure 4.17

Proof. By Lemma 4.4.7, k, l and m are mutually different and hence by Lemma 4.4.9, i, j, k and l are mutually different. By Lemma 4.4.2, i or $j = 0$ and l or $m = 0$, and hence $m = 0$, and by symmetry, we have $i = 0$. Then a_i, a_j, a_k, a_l and a_m appear as in Figure 4.17 (b), where we can find a C -product disk E' such that $|E \cap S^2| > |E' \cap S^2| = 1$, a contradiction. We note that $E' \cap L$ occurs at a_k and a_l . $\square$

An arc ε of $E \cap S^2$ is said to be of *level 2* if it is not outermost and, for one component E_1 of $E - \varepsilon$, $E_1 \cap S^2$ is a union of outermost arcs in $E \cap S^2$. Suppose that there is no arc of level 2. Then by Lemmas 4.4.3, 4.4.6 and 4.4.10, we see that $E \cap S^2$ consists of only one arc α such that one component of $E - \alpha$ contains the two points of $\partial E \cap L$. Let (a_0, a_j) be the label of α , and let a_k and a_l be the labels of the two points of $\partial E \cap L$, where a_0, a_j, a_k and a_l appear in this order in ∂E . If $l = k$, then we can isotope E so that $E \cap L = \emptyset$ and we have a compressing disk for R , for E is not boundary parallel, a contradiction. Hence by Lemma 4.4.7, we can assume that $j, k, l, 0$ are mutually different. In this case, R desums into three surfaces R_1, R_2 and R_3 obtained by applying Seifert's algorithm to the almost alternating diagrams D_1, D_2 and D_3 respectively (Figure 4.18).

Hence we assume that there is an arc of level 2. Then by Lemmas 4.4.3, 4.4.6 and

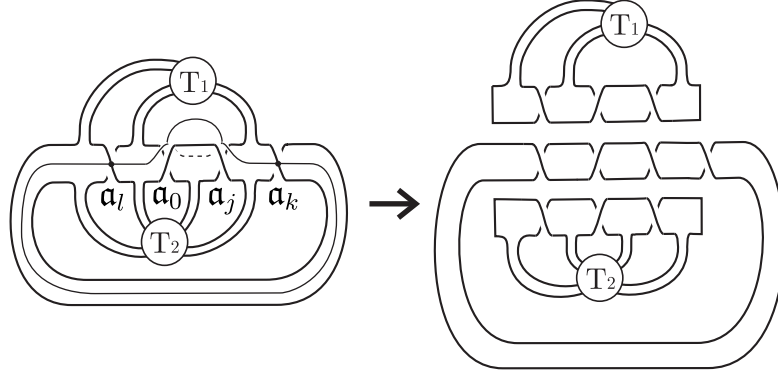


Figure 4.18

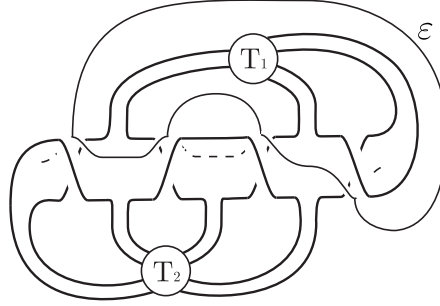


Figure 4.19

4.4.10, we see that there exists an arc ε of level 2 such that one disk E_1 cut by ε off E contains one outermost arc of $E \cap S^2$ and satisfies one of the following conditions;

- (*) $E_1 \cap L = \emptyset$,
- (**) $E_1 \cap L = \text{a point}$.

If E_1 satisfies (*), by Lemmas 4.4.1 and 4.4.5, all four labels of points of $E_1 \cap S^2$ are mutually different. Then, we can see that D is non-prime or R is a plumbing (Figure 4.19).

Thus we have:

Lemma 4.4.11. *We may assume that there is no arc of level 2 which cuts a disk E_1 off E such that E_1 contains only one (outermost) arc of $E \cap S^2$ and that $E_1 \cap L = \emptyset$.*

In what follows, we assume that there exists an arc ε of level 2 which cuts off E a disk E_1 containing one outermost arc of $E \cap S^2$ and satisfying (**).

By Lemma 4.4.3, we may suppose that E_1 appears as in Figure 4.20 (a) with labels $\mathfrak{a}_i, \mathfrak{a}_j, \mathfrak{a}_k, \mathfrak{a}_l$ and $\mathfrak{a}_m$.

Lemma 4.4.12. *All five labels in E_1 are mutually different.*

Proof. By Lemma 4.4.5, i, j, k are mutually different. By Lemma 4.4.7, $l \neq k$ and $l \neq j$. We see $l \neq i$, for if not, R appears as in Figure 4.20 (b) or (c), and in either case, R is compressible, a contradiction. Now we have seen that i, j, k, l are mutually different. Next suppose $m = l$. Then R appears as in Figure 4.20 (d) or (e). In Figure 4.20 (d), R is a plumbing or we can isotope E to reduce $|E \cap S^2|$. In Figure 4.20 (e), R is a Murasugi sum or we can isotope R so that D becomes an alternating diagram and the result follows from Theorem 4.1.1. Hence we can assume $m \neq l$ and by Lemma 4.4.9, we see that j, k, l, m are mutually different and by Lemma 4.4.1, $m \neq i$. $\square$

Lemma 4.4.13. *We may assume $\mathfrak{a}_j = \mathfrak{a}_0$.*

Proof. If not, $\mathfrak{a}_k = \mathfrak{a}_0$ by Lemma 4.4.2. Then R is a 6-Murasugi sum as in Figure 4.21. $\square$

Lemma 4.4.14. *Let ε and E_1 be as above. Then there is no arc ε' of $E \cap S^2$ as in Figure 4.22 (a) which cuts a disk E_2 off E with the following conditions:*

1. $E_1 \subset E_2$,

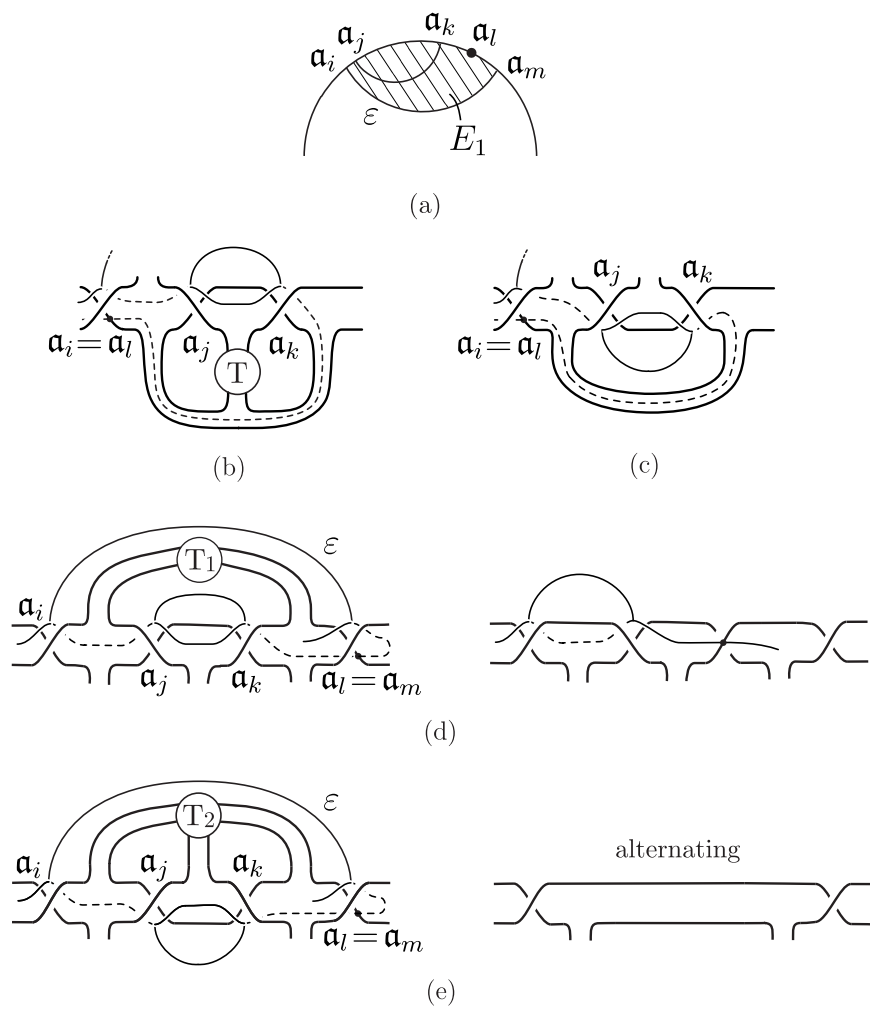


Figure 4.20

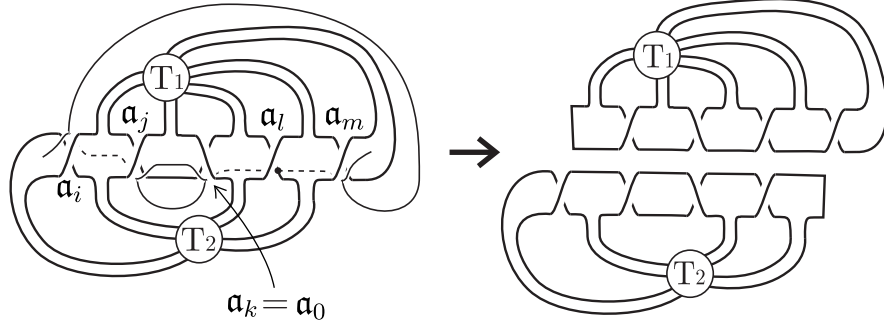


Figure 4.21

$$2. (\text{Int} E_2 - E_1) \cap (E \cap S^2) = \emptyset,$$

$$3. E_2 \cap L = E_1 \cap L = (1 \text{ point}).$$

Proof. By Lemmas 4.4.12 and 4.4.13, we may assume that E_1 appears as in Figure 4.22 (b). Recall that R is flat. Suppose that we have a disk E_2 as in Figure 4.22 (a). Then the arc ε' lies in some region of $S^2 - N(R)$. Hence, considering the orientation of R , we see that one of the following occurs;

- (1) The point ① is bounded by the same Seifert circle as one of the points ③ and ⑤,
- (2) The point ⑥ is bounded by the same Seifert circle as one of the points ② and ④.

In each case, we can find a C -product disk E' such that $|E \cap S^2| > |E' \cap S^2| = 0$ or 1 , a contradiction. $\square$

Lemma 4.4.15. *Let E_1 be as above. Then the following situation never occurs; The disk E locally appears as in Figure 4.23, i.e., there is an outermost disk F such that $\partial E - (E_1 \cup F)$ has a component β which contains no point of $(\partial E \cap S^2) \cup (\partial E \cap L)$.*

Proof. Suppose that there exists such a disk F . Let α be an arc in $E \cap S^2$ which cuts F off E , and $(\mathbf{a}_s, \mathbf{a}_t)$ the label of α where $\mathbf{a}_s$ is the label of an endpoint of β . First we

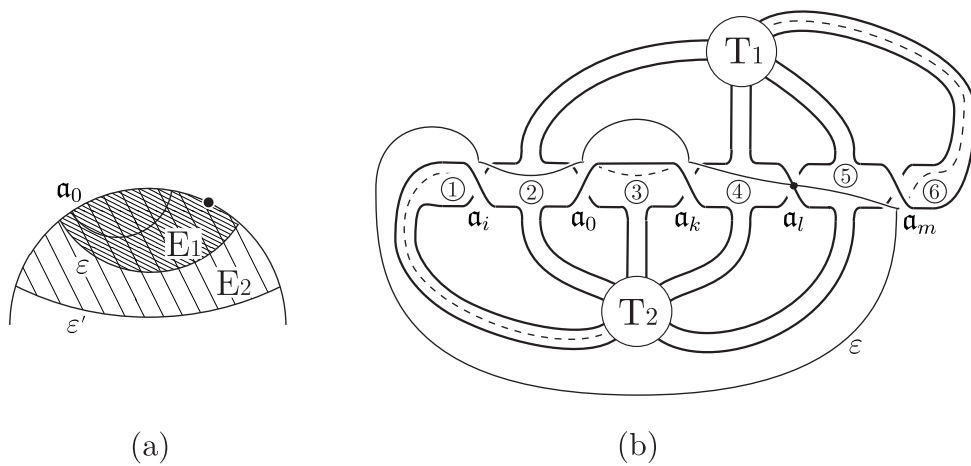


Figure 4.22

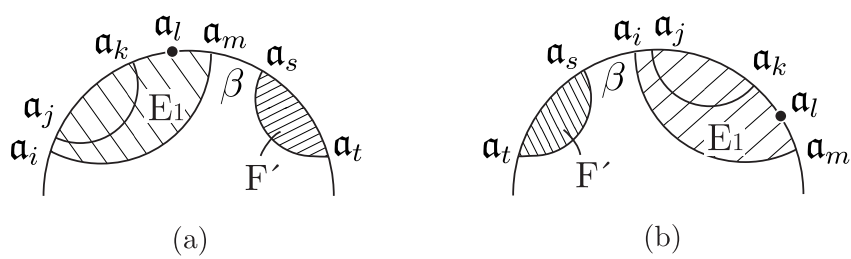


Figure 4.23

examine the case where E appears as in Figure 4.23 (a). If $s = i$ or 0 , we can find a C -product disk E' such that $|E' \cap S^2| = 1$, a contradiction (Figure 4.24 (a)). By Lemma 4.4.2, we have $s = 0$ or $t = 0$, and hence $t = 0$. If $s = k$ or l , we can find a C -product disk E' such that $|E' \cap S^2| = 0$, a contradiction (see Figure 4.24 (b)). By Lemma 4.4.4, we have $s \neq m$. Then we see that R locally appears as in Figure 4.24 (c). It is impossible that ∂E runs toward the dealternator $\mathfrak{a}_0(= \mathfrak{a}_t)$ after passing through $\mathfrak{a}_s$ because of the orientation of R .

Second, we examine the case where E locally appears as in Figure 4.23 (b). We can do this by a similar way to in the previous case. By Lemma 4.4.4, $s \neq i$. If $s = 0$ or k , we can find a C -product disk E' such that $|E' \cap S^2| = 0$, a contradiction. By Lemma 4.4.2, we have $s = 0$ or $t = 0$, and hence $t = 0$. If $s = l$ or m , we can find a C -product disk E' such that $|E' \cap S^2| = 1$, a contradiction. Then we see that it is impossible that ∂E runs toward the dealternator $\mathfrak{a}_0(= \mathfrak{a}_t)$ before passing through $\mathfrak{a}_s$. See Figure 4.25. $\square$

Let $E'_1 = E - E_1$. Then $E'_1 \cap L$ is exactly one point, say, x . By Lemma 4.4.3, $E'_1 \cap (E \cap S^2) \neq \emptyset$. By Lemmas 4.4.6 and 4.4.11, any arc of $E'_1 \cap (E \cap S^2)$ which does not separate ε and x is outermost in E'_1 . By Lemma 4.4.15, at least one of $E'_1 \cap (E \cap S^2)$ separates ε and x . Among such separating arcs, let α be the one closest to ε . Then by Lemma 4.4.15 again, the subdisk of E between ε and α contains no arc of $E'_1 \cap (E \cap S^2)$. However, this contradicts Lemma 4.4.14. This completes the proof. $\square$

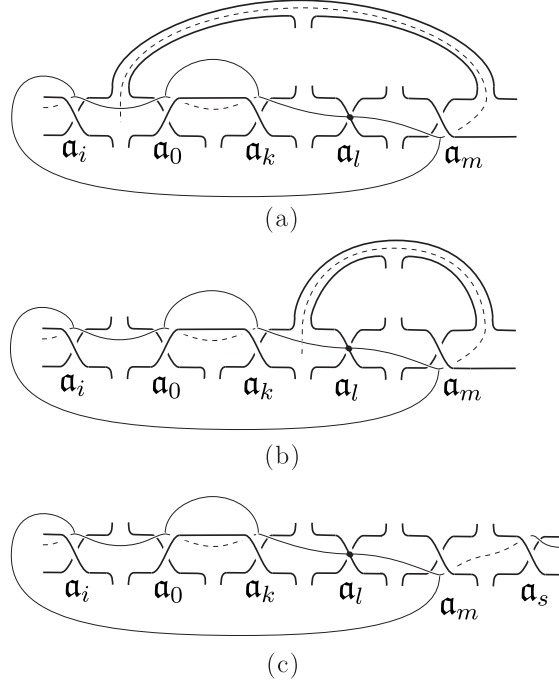


Figure 4.24

4.5 Proof of Theorem 4.1.4

In this section, we prove Theorem 4.1.4. Recall that a Seifert surface R obtained by Seifert's algorithm is a union of *Seifert disks* and *Seifert bands*.

Definition. Let R be a Seifert surface obtained by Seifert's algorithm. We say that two Seifert bands B_1 and B_2 of R are *parallel* if they connect the same two Seifert disks.

The following is a case where we can deplumb a Hopf band from a fiber surface:

Proposition 4.5.1. *Let R be a fiber surface obtained by Seifert's algorithm. Suppose that R has a pair of parallel bands B_1 and B_2 . Then, we can deplumb a Hopf band*

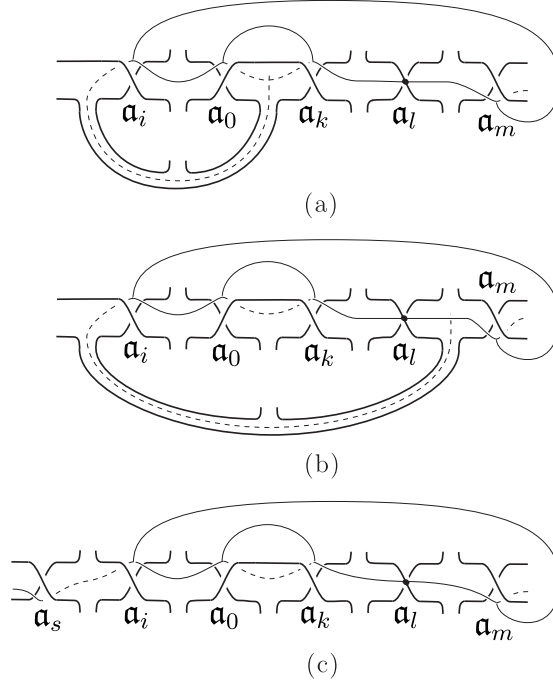


Figure 4.25

from R . Moreover, we have the following;

- (1) the parallel bands are of the same sign, and
- (2) for each $i = 1, 2$, we can cut the band B_i by deplumbing a Hopf band from R , i.e., R is a plumbing of $R - B_i$ and a Hopf band.

Proof. We denote by L the link ∂R . We may assume that the Seifert circles, say, C_1 and C_2 connected by B_1 and B_2 bound mutually disjoint Seifert disks on the level 2-sphere S^2 .

First, suppose that the pair of parallel bands are of the same sign. We may assume that they appear as in Figure 4.26 (a). We explicitly show that R is a plumbing of a Hopf band and the surface $R - B_1$. (The case $i = 2$ is similarly done.) Move L by

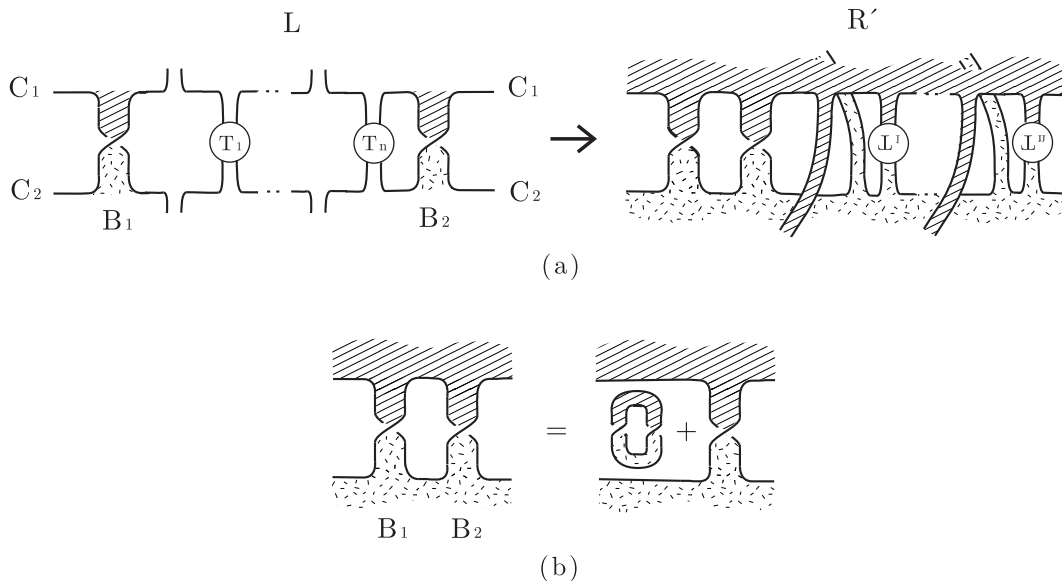


Figure 4.26

isotopy as in Figure 4.26 (a) and let R' be the surface as depicted. Apparently the Euler characteristic $\chi(R)$ is equal to $\chi(R')$. Hence by the uniqueness of fiber surfaces, we see that R is isotopic to R' . Now we can deplumb a Hopf band from R' as in Figure 4.26 (b). Then by retracing the above isotopy, we obtain the conclusion.

Next suppose that the pair of parallel bands are of the opposite signs, i.e., the twisting of B_1 is opposite. Then by the isotopy as implied by Figure 4.26 (a), we can find a compressing disk for R' , which contradicts the fact that fiber surfaces are of minimal genus and hence incompressible. $\square$

The following proposition assures that if a diagram D has a Seifert circle C which contains an alternating tangle diagram, then any Seifert surface obtained by applying Seifert's algorithm to D has parallel bands.

Proposition 4.5.2. *Suppose that a Seifert surface R obtained from an alternating*

diagram D is a fiber surface. Then R has parallel bands. Moreover, if D is reduced, then for any band B of R , there is a band B' of R which is parallel to B .

Proof. By untwisting R by isotopy if necessary, we may assume that D is reduced. Moreover, we may assume that D is unnested, because (1) by desumming along nested Seifert circles, we can decompose R into fiber surfaces obtained from unnested alternating diagrams, and (2) if one of the decomposed surfaces has parallel bands, then so does R . Suppose that a fiber surface R for a link L is obtained from a reduced unnested alternating diagram D . Then by [23] (or [3, Proposition 13.25]), L is a connected sum of $(2, n)$ -torus knots or links. Moreover the arguments in [23] shows that D is the ‘standard’ alternating diagram of a connected sum of $(2, n)$ -torus knots or links. Hence we obtain the conclusion. $\square$

Proof of Theorem 4.1.4. The ‘if’ part follows from Theorem 1.0.3. We show the ‘only if’ part, using Corollary 4.1.5, by induction on the first Betti number β_1 of R . If $\beta_1(R) = 1$, R is a Hopf band, and hence the theorem holds. Assume the theorem holds for such surfaces with $\beta_1 < k$, and let R be a Seifert surface with $\beta_1(R) = k$ obtained from an almost alternating diagram D . By untwisting R if necessary, we may assume that D is reduced. By Corollary 4.1.5, we know how R decomposes into Hopf bands. Hence by the following four lemmas, we will see that we can deplumb a Hopf band from R , in such a way that by deplumbing a Hopf band, we cut a band of R corresponding to an alternator. Therefore the deplumbed surface satisfies the assumption of induction so that we see that R is a Hopf plumbing. $\square$

Lemma 4.5.3. *If R desums along a nested Seifert circle, then we can cut a band of R by deplumbing a Hopf band from R .*

Proof. Suppose D is nested, i.e., there exists a Seifert circle C which contains another Seifert circle in both of its complementary regions in S^2 . Then R desums along C into two surfaces, say, R_1 and R_2 such that R_1 is obtained from an alternating diagram and R_2 from an almost alternating diagram (cf. Proposition 4.2.4). Note that by Theorem 1.0.3, both R_1 and R_2 are fibers. By Proposition 4.5.2, we see that R_1 has parallel bands and hence so does R . Then by Proposition 4.5.1, we can cut a band of R by deplumbing a Hopf band from R . $\square$

Lemma 4.5.4. *If R is a connected sum, then we can cut a band of R by deplumbing a Hopf band from R .*

Proof. Let R be a connected sum of R_1 and R_2 , where R_1 is obtained from an alternating diagram and R_2 from an almost alternating diagram by Proposition 4.2.3. Then by Theorem 1.0.3 and Proposition 4.5.2, R_1 has parallel bands, which are also parallel in R , and hence, by Proposition 4.5.1, we can cut a band of R by deplumbing a Hopf band from R . $\square$

Lemma 4.5.5. *If R admits a decomposition of type (A), then we can cut a band of R by deplumbing a Hopf band from R .*

Proof. Suppose that R admits a decomposition of type (A). Then we can deform R to R' by isotopy as depicted in Figure 4.27 (a), from which we can desum a fiber surface R_1 in Figure 4.27 (b). We can confirm that R_1 is obtained from an alternating diagram using Fact 4.2.2. By Proposition 4.5.2, R_1 has parallel bands. Though R' itself is not a surface obtained by Seifert's algorithm, we can apply the argument in the proof of Proposition 4.5.1, by regarding the inside of the dotted circle in Figure 4.27 (a) as a black box. Hence we can cut a band of R' (which is a band in the image

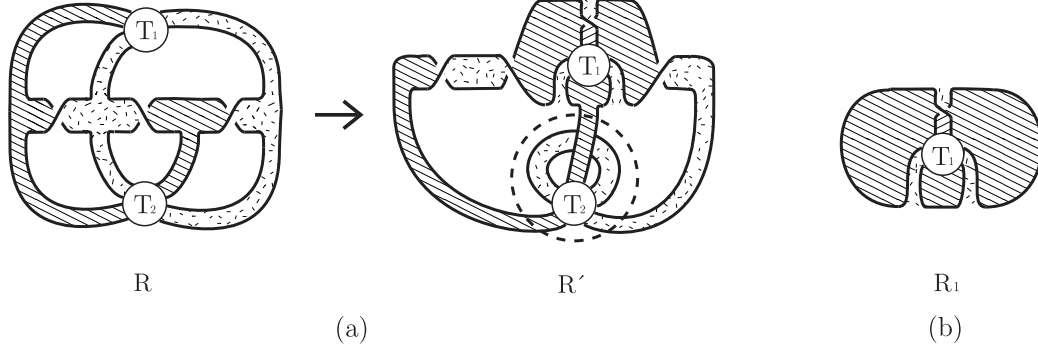


Figure 4.27

of R_1 in R') by deplumbing a Hopf band from R' . This corresponds to cutting a band of R by deplumbing a Hopf band from R . Note that we can confirm that the surface obtained from R by this cutting the band satisfies the assumption of induction. $\square$

Lemma 4.5.6. *If R admits a decomposition of type (B), then we can cut a band of R by deplumbing a Hopf band from R .*

Proof. According to whether the crossing visible in Figure 4.1 is an alternator or the dealternator, we have two cases. Let us call the former a decomposition of type (B1) and the latter of type (B2). Suppose that R admits a decomposition of type (B1). Then by the same way as in the proof of Lemma 4.5.4, we can cut a band of R by deplumbing a Hopf band from R .

Now assume that R does not admit a decomposition of type (B1). Then R deplumbs into R_1 and R_2 , which are both obtained from almost alternating diagrams (see Proposition 4.2.5). If R_1 or R_2 admits a decomposition of type (A), then we see, by the uniqueness of fiber surfaces, that R also admits a decomposition of type (A), and the claim follows from Lemma 4.5.5. Hence we assume that neither R_1 nor

R_2 admits a decomposition of type (A). Inductively, if we can do a decomposition of type (A) or (B1) in the process of desumming R into a union of Hopf bands, then we see that R also admits a decomposition of type (A) or (B1). So we assume that R desums into a union of Hopf bands using decompositions of type (B2) alone. Then by another inductive argument, we see that R is a pretzel surface of type $(1, -3, \dots, -3)$ or $(-1, 3, \dots, 3)$. In this case, obviously we can cut a band of R by deplumbing a Hopf band from R . $\square$

Chapter 5

Plumbing Hopf band and Stallings twist

J. Harer shows in [16] that every fiber surface in S^3 can be obtained from a disk by a sequence of the following operations:

1. Plumbing on a Hopf band.
2. Deplumbing a Hopf band.
3. Performing a Stallings twist.

Then he asks whether the operation 3 can be omitted entirely or not.

P. M. Melvin and H. R. Morton show in [19] that Burde's genus 2 knots $K(c_1, \pm 1)$ cannot be obtained by only the operation 1 from a disk when c_1 is even, but can be obtained by the operations 1 and 2 i.e., the fiber surfaces of $K(2k, \pm 1)$ are Stable Hopf plumbing, but are not Hopf plumbing. They prove that these knots can be

constructed by a sequence of the operations 1 and 2 and a particular Stallings twist, and that this particular twist can be realized by the operations 1 and 2.

In this chapter we will show that a certain type of Stallings twist which contains Melvin-Morton's particular twist can be realized by plumbing and deplumbing Hopf bands.

5.1 Results

Suppose c be a simple closed curve on a fiber surface F such that it is unknotted in S^3 . Let δ_1, δ_2 be ± 1 and suppose $\text{lk}(c, c^+) + \delta_1 = \delta_2$ where c^+ is a copy of c pushed in the positive normal direction to F and $\text{lk}(c, c^+)$ is the linking number of c and c^+ , computed with parallel orientations. Perform a surgery to S^3 using c as follows: Let A be an annular neighborhood of c in F and write N for the solid torus obtained by thickening A to one side of F . Remove N and reglue it by the map of Figure 5.1. This map sends a meridian of N to a meridian and a parallel copy of c along F . The surgery coefficient is then easily seen to be δ_2 . Since c is unknotted and $\delta_2 = \pm 1$ the resulting manifold is again S^3 . We call this operation a *Stallings twist along c* .

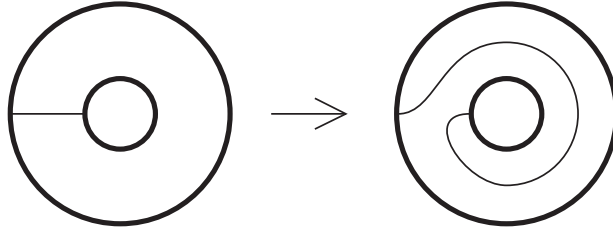


Figure 5.1

In the case where $\text{lk}(c, c^+) = 0$, since A is an untwisted annulus in S^3 , c bounds a

disk D such that the intersection of F and a regular neighborhood of ∂D in D is c . Since F is incompressible in $S^3 - \partial F$, we can isotope D so that $\text{Int}(D) \cap F$ is arcs. Let n be the minimal number of components of $\text{int}(D) \cap F$ up to isotopy of D . We call this twist a Stallings twist of type $(0, n)$, where the first number 0 denotes $\text{lk}(c, c^+)$.

Our main theorem in this chapter is here:

Theorem 5.1.1. *A Stallings twist of type $(0, 1)$ can be realized by plumbing and de-plumbing Hopf bands.*

Section 5.2 is for preliminaries and in Section 5.3 we prove the theorem above, finding C -product disks which has a close connection with plumbing a Hopf band or Stallings twist.

In Section 5.4, we define a *pretzel surface* and show the following theorem by applying main Theorem.

Theorem 5.1.2. *The pretzel surface $F(2n, -2, 2, -2, 2, \dots, -2, 2)$ ($n \neq 0$) is a stable Hopf plumbing, but not a Hopf plumbing.*

The first example of such fiber surfaces is a certain class of Burde's genus 2 knots mentioned above, and they have the same first Betti number 4. By Theorem 5.1.2, we obtain examples with an arbitrary high first Betti number.

5.2 Preliminaries

We prepare some symbols. If homeomorphisms h_1 and h_2 of topological space X are isotopic we write $h_1 \simeq h_2$. If two simple closed curves on X are isotopic, this will be written as $p \approx q$.

We recall the definitions of the complementary and product sutured manifolds for a Seifert surface F of an oriented link L in S^3 . In this chapter we adopt the following definitions of them for technical convenience.

Let N be a *blister neighborhood* of F , i.e., an embedded copy of $F \times [-1, 1]/\sim$ in S^3 , where $\sim$ pinches $\partial F \times [-1, 1]$, by collapsing each arc $x \times I$ to one point for each $x \in \partial F$; $F \times \{0\}$ and $\partial F \times \{0\}$ correspond to F and L respectively. We assume the orientation of the normal vector to F in S^3 coincides with the natural orientation of $[-1, 1]$. Then we call the pair (N, δ) with $\delta = L$ the *product sutured manifold* for F , and the image of $F \times \{+1\}$ and $F \times \{-1\}$ in ∂N are denoted by $R_+(\delta)$ and $R_-(\delta)$ respectively. The pair $(M, \gamma) = (\text{cl}(S^3 - N), L)$ is called the *complementary sutured manifold* for F , and the submanifolds $R_+(\delta)$ and $R_-(\delta)$ of ∂M are denoted by $R_-(\gamma)$ and $R_+(\gamma)$ respectively.

A disk D properly embedded in (M, γ) is called a *product disk* if ∂D intersects γ transversely in 2 points. For the product disk D , $\partial_+ D$, $\partial_- D$ means $p(D \cap R_+(\delta))$, $p(D \cap R_-(\delta))$ respectively. Let $\Delta_+ = \partial_+ D \times [0, 1]/\sim$ and $\Delta_- = \partial_- D \times [-1, 0]/\sim$ be disks embedded in $N = F \times [-1, 1]/\sim$. Put $\check{D} = D \cup \Delta_+ \cup \Delta_-$. Then $\check{D}$ is a singular disk in S^3 such that $\check{D} \cap F = \partial \check{D} = \partial_+ D \cup \partial_- D$; $\check{D}$ touches F in $\partial_\epsilon D$ from the ϵ -side of F . We call $\check{D}$ an *extended product disk*, and for the disk $\check{D}$, $\partial_+ \check{D}$, $\partial_- \check{D}$ means $(\Delta_+ \cap F)$, $(\Delta_- \cap F)$ respectively.

Proposition 5.2.1. *We can deplumb a Hopf band from a Seifert surface F if and only if F has an extended product disk D such that $\partial_- D$ intersects $\partial_+ D$ at the endpoints as illustrated in Figure 5.2(a).*

For the proof of this proposition, see [17, Section 2]. We say that a properly embedded arc A on F corresponds to a Hopf band if A and $h(A)$ intersects at the end

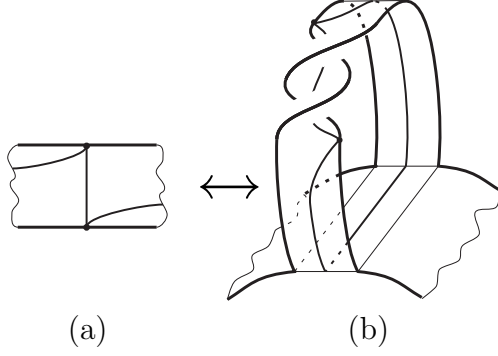


Figure 5.2

points as illustrated in Figure 5.2(a), where h is the monodromy map for F .

5.3 Proof of Theorem 5.1.1

In this section, we will give a proof of Theorem 5.1.1. First, we show the following lemma.

Lemma 5.3.1. *A fiber surface F has a simple closed curve c such that we can perform a Stallings twist of type $(0, 1)$ along it if and only if F has an extended product disk D such that $\partial_- D$ intersects $\partial_+ D$ at the endpoints as illustrated in Figure 5.3.*

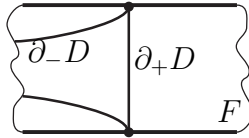


Figure 5.3

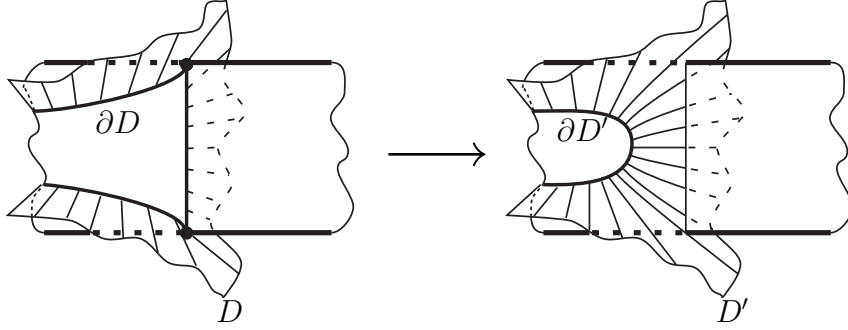


Figure 5.4

Proof. We firstly show the ‘if’ part. By an ambient isotopy of S^3 , we can obtain a disk D' from D such that $\partial D'$ is a simple closed curve on F , D' orthogonal to F along ∂D , and $\text{int}(D') \cap F$ is an arc which is isotopic rel ∂F to $\partial_+ D$ on F (see Figure 5.4). Thus the Stallings twist along $\partial D'$ is of type $(0, 1)$.

We must show the ‘only if’ part: Let (M, γ) be the complementary sutured manifold for F , E a disk which c bounds in S^3 and $A = E \cap M$. Since the twisting along c is of type $(0, 1)$, A is a properly embedded annulus in M . Note that one of the boundary components of A is on $R_-(\gamma)$ and the other intersects $\gamma = \partial F$ twice.

If A has a compressing disk, by compressing A along it, we obtain a compressing disk for F . This contradicts the fact that F is a fiber surface. So A is incompressible in M .

By Lemma 2.3 in [15], we have that A is boundary compressible toward $R_-(\gamma)$, that is, there exists a disk Δ in M such that $\Delta \cap A = \alpha$, which is an arc in $\partial \Delta$, and $\Delta \cap R_-(\gamma) = \beta$, which is an arc in $\partial \Delta$, with $\alpha \cap \beta = \partial \alpha = \partial \beta$, $\alpha \cup \beta = \partial \Delta$, and either α is essential in A or α is inessential in A and the boundaries of all disk components of $\text{cl}(A - \alpha)$ intersect γ . Then α is isotopic rel $\partial A - \gamma$ to one of the arcs

illustrated in Figure 5.5. By a boundary compressing toward $R_-(\gamma)$ along either arc, we obtain a properly embedded disk D' in M which intersects γ twice. This disk D' is an extended product disk satisfying the property mentioned in the conclusion of this lemma. This completes the proof of this lemma.

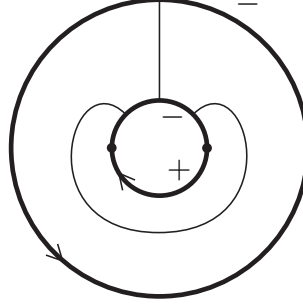


Figure 5.5

□

By the lemma above, we have only to show the following claim.

Claim. The Stallings twist along $\partial D'$ can be realized by plumbing and deplumbing Hopf bands.

Proof. Let p and q be the two points of $D \cap L$. Plumb on a Hopf band H along a in F as shown in Figure 5.6(ii). Then the boundary of an extended product disk D_H corresponding to H appears as in Figure 5.6(ii). Let r and s be the two points of $D_H \cap L$, and α an arc on F connecting $\partial_+ D$ and $\partial_+ D_H$ (see Figure 5.6(iii)). We obtain a disk D^* such that ∂D^* is on F and $\partial D^* \cap L = \{p, q, r, s\}$ by taking a connected sum of D and D_H along α . Since p and r in $\partial D^* \cap L$ can be canceled, we have the new extended product disk $\hat{D}$, which corresponds to a Hopf band (Figure 5.6(iv)). So

we can deplumb the Hopf band from the surface $F \cup H$. This operation is realized by cutting $F \cup H$ along $\partial_+ \hat{D}$, and we have the surface as in Figure 5.6(vi). $\square$

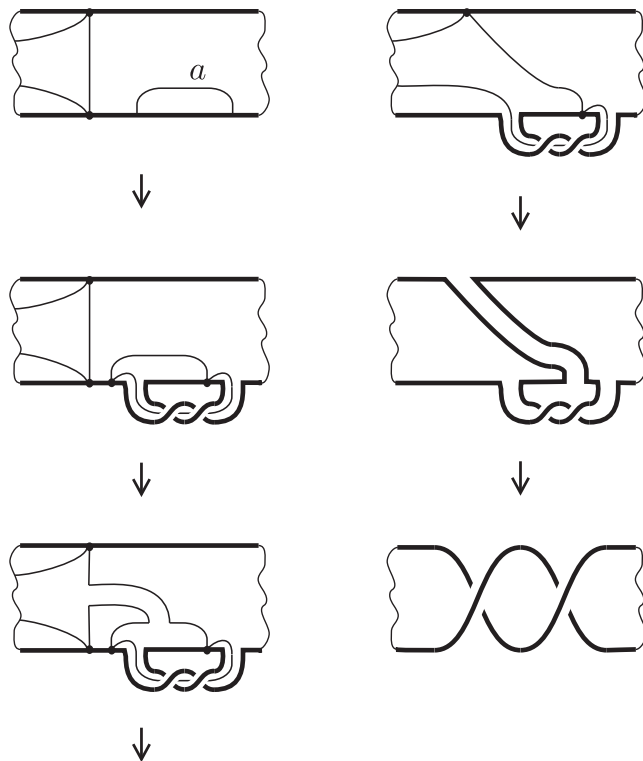


Figure 5.6

5.4 Application

For $(p_1, p_2, \dots, p_n)$ an n -tuple of integers, let us consider a surface $F(p_1, p_2, \dots, p_n)$ in S^3 , constructing of two disks joined by n twisted but unknotted vertical bands, where the i -th band in order has p_i half-twists, right- or left-handed according to the sign of p_i . We call the surface a *pretzel surface*.

In this section we prove Theorem 5.1.2. We shall abbreviate the pretzel surface

$$F(2n, \underbrace{-2, 2, -2, 2, \dots, -2, 2}_{2k})$$

to $F_{n,k}$. Note that the pretzel surface

$$F(2n, \underbrace{2, -2, 2, -2, \dots, 2, -2}_{2k})$$

is isotopic to $F_{n,k}$ in S^3 . The positive side of $F_{n,k}$ is a side of $F_{n,k}$ where the normal vectors point out. A normal vector on $F_{n,k}$ induces an orientation on ∂R using the rule that if an observer walking along ∂R on the positive side of $F_{n,k}$, sees $F_{n,k}$ to the left, then the observer follows the orientation of ∂R .

First we show that $F_{n,k}$ is a stable Hopf plumbing.

We can construct $F_{0,k}$ from a disk by plumbing negative Hopf bands and positive Hopf bands alternately, and let $a_1, a_2, \dots, a_{2k}$ be core curves of Hopf bands as in Figure 5.7.

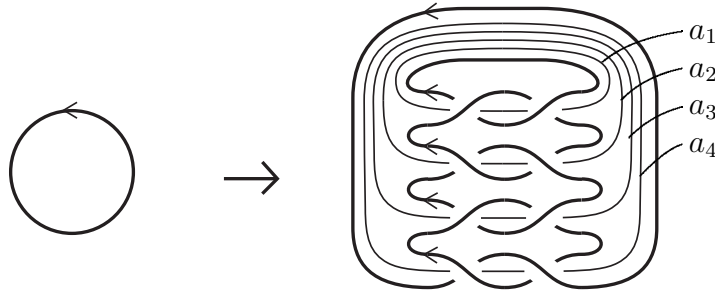


Figure 5.7

Let c_k be the simple loop on $F_{0,k}$ illustrated in Figure 5.8. Then we can easily see that c_k is an unknotted and untwisted loop of type $(0,1)$ and that the surface

obtained from $F_{0,k}$ by performing the Stallings twist along c_k n times is isotopic to the pretzel surface $F_{n,k}$.

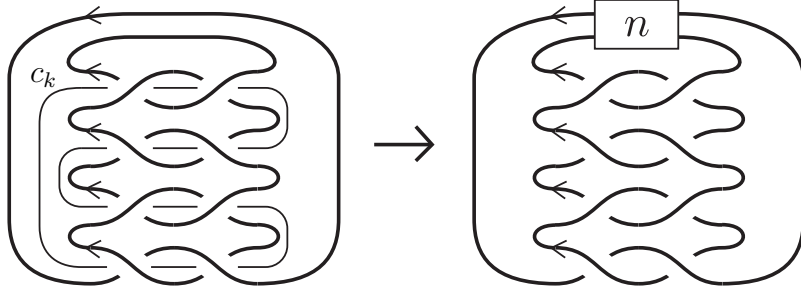


Figure 5.8

Thus, by Theorem 5.1.1, this twisting is realized by plumbing and unplumbing Hopf bands, and then we have that $F_{n,k}$ is a stable Hopf plumbing.

By the above construction of $F_{n,k}$, we see that a monodromy $h_{n,k}$ for this surface is isotopic to $D^+(a_{2k})D^-(a_{2k-1}) \cdots D^+(a_2)D^-(a_1)\{D^+(c_k)\}^n$, where $D^+(l)$ ($D^-(l)$ resp.) denotes a right-handed (left-handed resp.) Dehn twist along a simple closed curve l on $F_{n,k}$.

Now it remains to prove that $F_{n,k}$ is not a Hopf plumbing. It is enough to show that we cannot deplumb a Hopf band from $F_{n,k}$. By Proposition 5.2.1, this is equivalent to the claim below.

Claim. No properly embedded arc on $F_{n,k}$ corresponds to a Hopf band.

We shall use the induction on k . If $k = 1$, $F_{n,1}$ is a twice punctured disk. We have only three properly embedded arcs on $F_{n,1}$ up to an isotopy fixing endpoints of them. So, by checking their images of $h_{n,1}$, we can see that $F_{n,1}$ has no properly embedded

arc corresponds to a Hopf band.

For $k > 1$, we assume that the surface $F_{n,k-1}$ has no properly embedded arc corresponds to a Hopf band and prove the claim for $F_{n,k}$.

Let A be a properly embedded arc on $F_{n,k}$ and $b_0, b_1, \dots, b_{2k}$ boundary components of $F_{n,k}$ as shown in Figure 5.9. We suppose that $A \cap \partial F_{n,k} = A \cap (b_s \cup b_t)$, where $0 \leq s \leq t \leq 2k$. If $s = t$, A separates $F_{n,k}$, and A doesn't correspond a Hopf band. So we may assume that $s \neq t$.

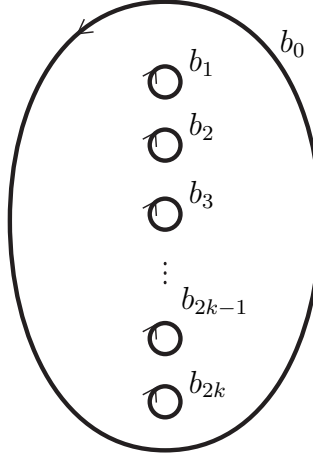


Figure 5.9

We take two successive boundary components b_i and b_{i+1} such that $A \cap (b_i \cup b_{i+1}) = \emptyset$. If $k \geq 3$, we can take such b_i and b_{i+1} for some i such that $1 \leq i \leq 2k-1$. If $k = 2$, we need $i = 0, 1, 2$ to obtain such b_i and b_{i+1} .

Let $\overline{F_{n,k}}$ be the surface which is obtained from $F_{n,k}$ by capping b_i and b_{i+1} with two disks.

Case 1. $i = 2, 3, \dots, 2k-1$.

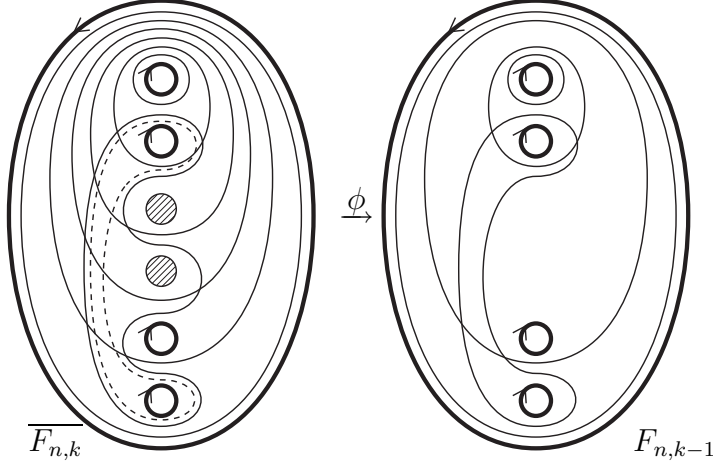


Figure 5.10

In $\overline{F_{n,k}}$, $a_{i+1} \approx a_i \approx a_{i-1}$ and $c_k \approx \overline{c_k}$, where $\overline{c_k}$ is the loop illustrated by the dotted curve in Figure 5.10.

We regard that h_k is a homeomorphism of $\overline{F_{n,k}}$, and we have

$$\begin{aligned}
 h_{n,k} &= D^+(a_{2k}) \cdots D^{\varepsilon_{i+1}}(a_{i+1}) D^{\varepsilon_i}(a_i) D^{\varepsilon_{i-1}}(a_{i-1}) \cdots D^-(a_1) \{D^+(c_k)\}^n \\
 &\simeq D^+(a_{2k}) \cdots D^{\varepsilon_{i-1}}(a_{i-1}) \{D^{\varepsilon_{i-1}}(a_{i-1})\}^{-1} D^{\varepsilon_{i-1}}(a_{i-1}) \\
 &\quad \cdots D^-(a_1) \{D^+(\overline{c_k})\}^n \\
 &\simeq D^+(a_{2k}) \cdots D^{\varepsilon_{i-1}}(a_{i-1}) \cdots D^-(a_1) \{D^+(\overline{c_k})\}^n,
 \end{aligned}$$

where ε_i is a sign $+$ when i is even, and a sign $-$ otherwise. There is a natural homeomorphism $\phi : \overline{F_{n,k}} \rightarrow F_{n,k-1}$ such that

$$\phi(b_j) = \begin{cases} b'_j & (1 \leq j \leq i-1) \\ b'_{j-2} & \text{otherwise} \end{cases},$$

where b'_j is the “ j -th” boundary component of $F_{n,k-1}$ in the same way as $F_{n,k}$, and

the following diagram is commutative;

$$\begin{array}{ccc}
\overline{F_{n,k}} & \xrightarrow{h_k} & \overline{F_{n,k}} \\
\phi \downarrow & \circlearrowleft & \downarrow \phi \\
F_{n,k-1} & \xrightarrow{h_{k-1}} & F_{n,k-1}
\end{array}$$

On $F_{n,k-1}$, $\phi(A)$ doesn't correspond to a Hopf band. Therefore A is also not on $F_{n,k}$.

Case 2. $i = 1$.

In $\overline{F_{n,k}}$, a_1 and a_2 are inessential loops, and $c_k \approx \overline{c_k}$, where $\overline{c_k}$ is the loop illustrated by the dotted curve in Figure 5.11.

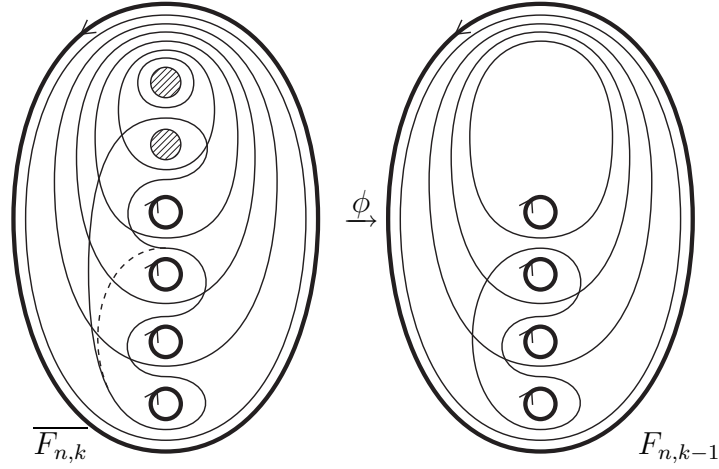


Figure 5.11

Then we can show that A doesn't correspond to a Hopf band by a similar way as in the case 1.

Case 3. $i = 0$.

Note that we need to study only the case where $k = 2$.

The surface $F_{n,2}$ and simple closed curves a_1, a_2, a_3, a_4 and c_2 , which represent a monodromy of $F_{n,2}$, may be illustrated as in Figure 5.12 (top). On $\overline{F_{n,2}}$, $a_1 \approx a_4$;

inessential loop, and c_2 is a loop which isotopic to b_3 . A twice punctured disk in S^3 with a monodromy $D^+(a_2)D^-(a_3)\{D^+(c_2)\}^n$ is a pretzel surface $F(2n, 2, -2)$, and it is isotopic to $F_{n,1}$ in S^3 . Thus the claim has proved. $\square$

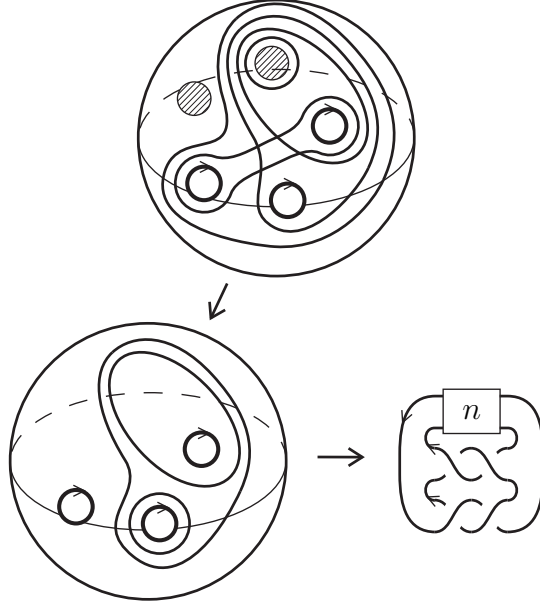


Figure 5.12

Appendix

In [2], Adams et al. showed that among non-alternating knots of 11 or fewer crossings and links of 10 or fewer crossings, all but five, listed below by Conway notation, are almost alternating.

- 2.-20.-2.20, 20.-2.-20.20, 20.-3.-20.2, 20.-21.-20.2, 9*.-2:-2.

Since then it has been an open question whether or not these five knots and links are almost alternating. (See also [1].)

Figure 5.13 depicts our newly found almost alternating diagram for the knot 9*.-2:-2.

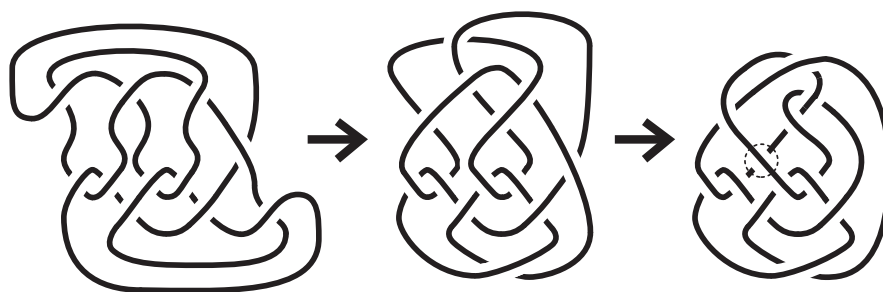


Figure 5.13

A Seifert surface obtained from the almost alternating diagram is a fiber surface and, as Theorem 4.1.4 assures, is a Hopf plumbing.

We note that all of the remaining four knots and links are also fibered, which have 2-almost alternating diagrams.

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