Some results on gaps in P(ω)/fin

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Some results on gaps in $\mathcal{P}(\omega)/\mathrm{fin}$

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Dedicated to Yuuko

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Contents

Chapter 1

Introduction

1.1 Introduction to set theory

For a set X, we write $|X|$ for the cardinality of X, and $\mathcal{P}(X)$ for the power set of X, that is the collection of all subsets of X. (Under the axiom of choice, any set has a cardinality. Throughout this dissertation, we always consider *Zermelo-Fraenkel set theory with Choice* ZFC.) Cantor proved that $|X|$ is less than $|\mathcal{P}(X)|$ for any set X. It follows that the size of the continuum $\mathfrak{c}$, which is equal to the size of the collection of all subsets of the natural numbers, is necessarily larger than the size of the collection of the natural numbers, denoted by ω. *The Continuum Hypothesis* CH says that the size of the continuum is just $\aleph_1$, which is the least uncountable cardinal. It has been studied whether or not the Continuum Hypothesis can be deduced from ZFC, but this problem has been solved by Kurt Gödel and Paul Cohen. Gödel has proved that we cannot prove the negation of the Continuum Hypothesis from ZFC and Cohen has proved that we cannot prove the Continuum Hypothesis if ZFC is assumed consistent. This means that the Continuum Hypothesis is *independent* from ZFC. We should note that to understand such results, we necessarily need to formalize mathematics. For this reason, we consider mathematics in ZFC or some other formal system.

To prove the independence of CH, Gödel has introduced the notion of *the constructible universe*, and Cohen has introduced *the forcing method.* The forcing method is often used to show that some statement cannot be deduced from ZFC assuming that ZFC is consistent.

Set theory of the reals is one of the areas of set theory. In ZFC, we can identify the real numbers with the subsets of the natural numbers, the infinite sequences of the natural numbers, and so on. The following are aims of studying set theory of the reals: to investigate what statements about the real numbers may be proved from ZFC and to do research on the combinatorics of the real numbers. By continuing these studies, we hope to find a new axiom which is consistent with ZFC and decides the cardinality of the continuum. (We believe this is in contrast to Gödel's program, which is that the size of the continuum should be decided from "strong axioms of infinity", so-called "large cardinal axioms". [19])

We often consider *cardinal invariants* which are cardinals expressing some combinatorial structure. For example, we define a relation $\leq^*$ on functions on the natural numbers as follows:

$$f \leq^* g \iff \exists m \in \omega \forall n \in \omega (n \geq m \implies f(n) \leq g(n))$$

for functions f and g. And we give one example of a cardinal invariant $\mathfrak{b}$:

$$\mathfrak{b} := \min \{|\mathcal{F}|; \mathcal{F} \subseteq \omega^\omega \ \& \ \forall g \in \omega^\omega \exists f \in \mathcal{F}(f \not\leq^* g)\},$$

where ω^ω denotes the set of all functions from ω into ω. By a diagonal argument, we can show from ZFC that $\mathfrak{b}$ is an uncountable cardinal and clearly $\mathfrak{b}$ is less than or equal to the size of the continuum. The author has studied some cardinal invariants, e.g. the cofinality of the strong measure zero ideal [52] and the covering number of the Marczewski ideal [53].

Theorem ([52]). *It is consistent with* ZFC *that the cofinality of the strong measure zero ideal is less than the power of the continuum.*

Theorem ([53]). *In the extension with finite support iteration of Hechler forcing of length κ, where κ is a regular cardinal, the covering number of the Marczewski ideal is $\aleph_1$.*

I think that finding lots of statements on the reals consistent with ZFC as above gives us clues on new axioms deciding the continuum.

In this thesis, we concentrate on one topic in set theory: gaps in $\mathcal{P}(\omega)/\text{fin}$. This has been studied by many mathematicians in many areas. In particular, the study of gaps from the set theoretic aspect has grown up. We will especially look at the relationship between gaps and forcing extensions.

1.2 Introduction to gaps

Gaps, which are the main topic of this thesis, had been studied in areas other than set theory, for example topology or algebra. Gaps had been first studied by du Bois-Reymond in the 1870's [14]. And in the first half of 20th century, Hausdorff and Rothberger had studied gaps from the set theoretic point of view. The important results due to Hausdorff and Rothberger are the existence of (ω_1, ω_1)-gaps and $(\omega, \mathfrak{b})$-gaps ([22], [40]). After that, in the 1970's Kunen has studied gaps and their destructibility by forcing.

It follows from Kunen's research that the gap discovered by Hausdorff still forms a gap in any forcing extension preserving $\aleph_1$. Moreover, it follows that any (ω_1, ω_1)-gap is indestructible under $\mathsf{MA}_{\aleph_1}(\text{ccc})$. Around the same time, Laver has built a forcing notion which generically adds a destructible gap by finite approximations and has proved that it is consistent with ZFC that $\mathfrak{c} > \aleph_1$ and every linear ordering of cardinality $\leq \mathfrak{c}$ is embeddable in ω^ω. The study of the relationship between gaps and forcing extensions has two fields: *the existence of a destructible gap* and *gap spectra under Martin's Axiom.*

The first field has been particularly studied by Todorčević, Farah, Hirschorn and others ([2], [24], [25], [46]). Todorčević has constructed a destructible gap from a diamond sequence and from adding a Cohen real (see chapter 3), and Hirschorn has studied the existence of a destructible gap in the extension with random forcing.

The other field was firstly studied by Kunen, and later by Kamo, Rabus, Scheepers, Todorčević, Woodin and others ([9], [31], [33], [39], [41], [45]). Todorčević introduced the Open Coloring Axiom and proved that the axiom decides the value of the bounding number to $\aleph_2$. To show this, he thought of the gap spectrum under the Open Coloring Axiom. We notice that Martin's Axiom plus the Open Coloring Axiom decides the size of the continuum. This is one example of deciding the continuum from new axioms consistent with ZFC using an argument on gaps. And Woodin has used a gap to do research on the Automatic Continuity Hypothesis ([9], [51]). He proved that the Automatic Continuity Hypothesis is consistent with ZFC by forcing Martin's Axiom and a stronger statement on gaps.

1.2.1 Parametrized diamond principles

The diamond principle $\diamondsuit$ introduced by Ronald Jensen is the following statement:

$$\begin{aligned}&\exists \langle A_\alpha; \alpha < \omega_1\rangle \text{ with } A_\alpha \subseteq \alpha \text{ such that}\\ &\quad \forall X \subseteq \omega_1, \{\alpha < \omega_1; X \cap \alpha = A_\alpha\} \text{ is stationary.}\end{aligned}$$

In [10], Devlin and Shelah presented the following axiom, called the weak diamond principle, which is formulated as a weaker version of the diamond principle:

$$\begin{aligned}&\forall F : 2^{<\omega_1} \to 2 \exists g : \omega_1 \to 2 \forall f : \omega_1 \to 2,\\ &\quad \{\alpha < \omega_1; F(f \restriction \alpha) = g(\alpha)\} \text{ is stationary,}\end{aligned}$$

which is equivalent to the inequality $2^{\aleph_0} < 2^{\aleph_1}$. Parametrized diamond principles have been designed by Moore, Hrušák and Džamonja as weak diamond principles parametrized by cardinal invariants in the following form:

Definition 1.1 ([38]). *1. An invariant is a triple (A, B, E) such that*

- *both A and B have cardinality at most $\mathfrak{c}$,*
- *$E \subseteq A \times B$ is a relation,*
- *$\forall a \in \exists b \in B(\langle a, b\rangle \in E)$, and*
- *$\forall b \in B \exists a \in A(\langle a, b\rangle \notin E)$.*

2. For an invariant (A, B, E), define its evaluation $\langle A, B, E\rangle$ by

$$\langle A, B, E\rangle := \min\{|X|; X \subseteq B \;\&\; \forall a \in A \exists b \in X(\langle a, b\rangle \in E)\}.$$

3. For an invariant $\langle A, B, E\rangle$, define the diamond principle for $\langle A, B, E\rangle$ as follows:

$$\begin{aligned}\diamondsuit(A, B, E) \equiv \forall \text{ Borel } F : 2^{<\omega_1} \to A \exists g : \omega_1 \to B \forall f : \omega_1 \to 2,\\ \{\alpha < \omega_1; \langle F(f \restriction \alpha), g(\alpha)\rangle \in E\} \text{ is stationary.}\end{aligned}$$

In this notation, $\mathfrak{b} = \langle \omega^\omega, \omega^\omega, \not\geq^*\rangle$ and the covering number of the meager ideal is equal to the evaluation of the invariant $\langle \omega^\omega, \mathcal{M}, \in\rangle$. To simplify, we write $\diamondsuit(\mathfrak{b})$ instead of $\diamondsuit(\omega^\omega, \omega^\omega, \not\geq^*)$, etc. In [38], there are many applications of parametrized diamond principles. In particular, a Suslin tree has been constructed from $\diamondsuit(\mathsf{non}(\mathcal{M}))$. The author constructs a destructible gap from the same parametrized diamond principle, which is one of the main results of the thesis. This answers Question 5.8 in [38].

1.2.2 Martin's Axiom

Martin's Axiom MA is the following statement:

> For any forcing notion $\mathbb{P}$ with the countable chain condition and κ many dense subsets $\{\mathcal{D}_\alpha; \alpha < \kappa\}$ in $\mathbb{P}$ for any $\kappa < \mathfrak{c}$, there exists a filter G in $\mathbb{P}$ which meets $\mathcal{D}_\alpha$ for all $\alpha < \kappa$.

Solovay and Tennenbaum had proved that this can be forced by a finite support iteration of forcing notions with the countable chain condition. There are many variations of this axiom. For a collection $\mathcal{P}$ of forcing notions and a cardinal $\kappa \leq \mathfrak{c}$, $\mathsf{MA}_\kappa(\mathcal{P})$ is the following statement:

> For any forcing notion $\mathbb{P} \in \mathcal{P}$ with the countable chain condition and λ many dense subsets $\{\mathcal{D}_\alpha; \alpha < \lambda\}$ in $\mathbb{P}$ for any $\lambda \leq \kappa$, there exists a filter G in $\mathbb{P}$ which meets all $\mathcal{D}_\alpha$ for all $\alpha < \lambda$.

In this notation, Martin's Axiom is the statement $\bigwedge_{\kappa<\mathfrak{c}} \mathsf{MA}_\kappa(\text{ccc})$ and *the Proper Forcing Axiom* PFA is the statement $\mathsf{MA}_{\aleph_1}(\text{proper})$. Scheepers has studied the gap spectra under several versions of Martin's Axiom and provided some problems ([41]).

Under Martin's Axiom, if there exists a (κ, λ)-gap, then it is necessarily satisfied that either $\kappa = \lambda = \omega_1$ or $\kappa = \mathfrak{c}$ or $\lambda = \mathfrak{c}$. Moreover if κ is a regular cardinal less than $\mathfrak{c}$ and not ω_1, then there exists a $(\kappa, \mathfrak{c})$-gap. But we cannot deduce the existence of $(\omega_1, \mathfrak{c})$-gaps and $(\mathfrak{c}, \mathfrak{c})$-gaps from Martin's Axiom. In fact, Kunen proved that both of the following statements are consistent with $\mathsf{ZFC} + \mathsf{MA} + \neg\,\mathsf{CH}$:

- Both $(\omega_1, \mathfrak{c})$-gaps and $(\mathfrak{c}, \mathfrak{c})$-gaps exist.
- Neither $(\omega_1, \mathfrak{c})$-gaps nor $(\mathfrak{c}, \mathfrak{c})$-gaps exist.

The following statement, which answers a question addressed in Scheepers' survey article (Problem 13 in [41]), is the other main result of this thesis:

> *It is consistent with* ZFC *that Martin's Axiom holds and there are* $(\mathfrak{c}, \mathfrak{c})$*-gaps but no* $(\omega_1, \mathfrak{c})$*-gaps.*

⋆ ⋆ ⋆

Our notation is quite standard. In particular, we refer to [3] and [34] for undefined symbols and notation.

Chapter 2

Preliminaries on gaps

The notion of gap can be defined for any transitive order. But whenever we consider gaps on the real numbers, we do not need to care what order we think of. At first, we explain such results.

Definition 2.1. *Let $\langle X, \lhd \rangle$ be a transitive order, i.e. for x, y and z in X, if $x \lhd y$ and $y \lhd z$, then $x \lhd z$ holds. And let $\mathcal{A}$ and $\mathcal{B}$ be subsets of X.*

1. *$(\mathcal{A}, \mathcal{B})$ is called a pregap if $a \lhd b$ holds for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$.*

2. *$(\mathcal{A}, \mathcal{B})$ is called separated if there exists $c \in X$ such that $a \lhd c \lhd b$ for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$. This c is called a separation or an interpolation of this pregap.*

3. *$(\mathcal{A}, \mathcal{B})$ is called a gap if it is a pregap and not separated.*

4. *If $\mathrm{ot}(\mathcal{A}, \lhd) = \kappa$ and $\mathrm{ot}(\mathcal{B}, \rhd) = \lambda$, then $(\mathcal{A}, \mathcal{B})$ is called linear, and a (κ, λ)-pregap (or a (κ, λ)-gap) if it is a pregap (or gap).*

In the thesis, we write a pregap in many forms: $(\mathcal{A}, \mathcal{B})$, $\langle a_\alpha, b_\beta; \alpha < \kappa, \beta < \lambda \rangle$ or $\langle a_\alpha, b_\alpha; \alpha < \kappa \rangle$. If we write $\langle a_\alpha, b_\beta; \alpha < \kappa, \beta < \lambda \rangle$, we necessarily mean a (κ, λ)-pregap with this order, i.e.

$$\forall \xi < \xi' < \kappa \forall \eta < \eta' < \lambda (a_\xi \lhd a_{\xi'} \lhd b_{\eta'} \lhd b_\eta \ \& \ a_{\xi'} \not\lhd a_\xi \ \& \ b_\eta \not\lhd b_{\eta'}).$$

Examples of transitive orders.

$\langle \omega^\omega, \prec \rangle$: $f \prec g \ :\Longleftrightarrow \ \lim_{n \to \infty} g(n) - f(n) = +\infty$.

$\langle \omega^\omega, \ll \rangle$: $f \ll g \;:\Longleftrightarrow\; \forall^\infty n (f(n) < g(n))$.

$\langle \omega^\omega, \leq^* \rangle$: $f \leq^* g \;:\Longleftrightarrow\; \forall^\infty n (f(n) \leq g(n))$.

$\langle 2^\omega, \leq^* \rangle$: $f \leq^* g \;:\Longleftrightarrow\; \forall^\infty n (f(n) \leq g(n))$.

$\langle \mathcal{P}(\omega), \subseteq^* \rangle$: $a \subseteq^* b \;:\Longleftrightarrow\; a \setminus b$ is finite.

The gap spectrum $\mathsf{Gap}(X, \lhd)$ for a transitive order $\langle X, \lhd \rangle$ is the collection of pairs of infinite regular cardinals (κ, λ) whose type of gaps exists.

Lemma 2.2 (Folklore). *Let κ and λ be infinite regular cardinals. If a (κ, λ)-gap exists in one of the structures $\langle \omega^\omega, \prec \rangle$, $\langle \omega^\omega, \ll \rangle$, $\langle \omega^\omega, \leq^* \rangle$, $\langle 2^\omega, \leq^* \rangle$ and $\langle \mathcal{P}(\omega), \subseteq^* \rangle$, it also exists in all of those structures.*

Proof. Identifying infinite subsets of ω with their characteristic functions, it is trivial to see $\mathsf{Gap}(2^\omega, \leq^*) = \mathsf{Gap}(\mathcal{P}(\omega), \subseteq^*)$. Other trivial cases are

- $\mathsf{Gap}(\omega^\omega, \prec) \subseteq \mathsf{Gap}(\omega^\omega, \ll) \subseteq \mathsf{Gap}(\omega^\omega, \leq^*)$
- $\mathsf{Gap}(2^\omega, \leq^*) \subseteq \mathsf{Gap}(\omega^\omega, \leq^*)$.

Assume $\langle a_\alpha, b_\beta ; \alpha < \kappa, \beta < \lambda \rangle$ is a gap in $\langle \omega^\omega, \leq^* \rangle$. For $x \in \omega^\omega$, we let

$$p(x) := \{ \langle i, n \rangle \, ; i < x(n) \} \subseteq \omega \times \omega.$$

Then $\langle p(a_\alpha), p(b_\beta) ; \alpha < \kappa, \beta < \lambda \rangle$ forms a gap in $\langle \mathcal{P}(\omega^2), \subseteq^* \rangle$ which is equivalent to $\langle \mathcal{P}(\omega), \subseteq^* \rangle$ and $\langle 2^\omega, \leq^* \rangle$. Thus $\mathsf{Gap}(\omega^\omega, \leq^*) \subseteq \mathsf{Gap}(2^\omega, \leq^*)$.

Assume $\langle a_\alpha, b_\beta ; \alpha < \kappa, \beta < \lambda \rangle$ is a gap in $\langle \mathcal{P}(\omega), \subseteq^* \rangle$. For $x \in [\omega]^\omega$, let

$$\widetilde{x}(n) := \sum_{i \in x \cap n} 2^i.$$

Then $\left\langle \widetilde{a_\alpha}, \widetilde{b_\beta} ; \alpha < \kappa, \beta < \lambda \right\rangle$ is also a gap in $\langle \omega^\omega, \prec \rangle$, which completes the proof of $\mathsf{Gap}(\omega^\omega, \leq^*) \subseteq \mathsf{Gap}(\omega^\omega, \prec)$. □

Therefore we can concentrate on just one structure. In this thesis, we adopt the following framework to consider gaps because it simplifies. For a Boolean algebra, we can consider the notions of pregaps and gaps as follows.

Definition and Notation 2.3. *Let a and b be elements of $\mathcal{P}(\omega)$, and $\mathcal{A}$ and $\mathcal{B}$ subsets of $\mathcal{P}(\omega)$.*

1. *$a \perp b$ denotes that $a \cap b$ is finite.*

2. *$\mathcal{A}^\perp := \{c \subseteq \omega; \forall a \in \mathcal{A}(a \perp c)\}$.*

3. *$(\mathcal{A}, \mathcal{B})$ is called a pregap if for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$, $a \perp b$, i.e. $\mathcal{B} \subseteq \mathcal{A}^\perp$.*

4. *A pregap $(\mathcal{A}, \mathcal{B})$ is separated if there is $c \in \mathcal{A}^\perp$ such that $b \subseteq^* c$ for all $b \in \mathcal{B}$. This c is called a separation or an interpolation of this pregap.*

5. *$(\mathcal{A}, \mathcal{B})$ is called a gap if it is a pregap and not separated.*

6. *If $\mathrm{ot}(\mathcal{A}, \subseteq^*) = \kappa$ and $\mathrm{ot}(\mathcal{B}, \subseteq^*) = \lambda$, then $(\mathcal{A}, \mathcal{B})$ is called a linear gap, and a (κ, λ)-pregap (or a (κ, λ)-gap) if it is a pregap (or gap).*

If $(\mathcal{A}, \mathcal{B})$ is a gap in the above sense, then $(\mathcal{A}, \{\omega \setminus b; b \in \mathcal{B}\})$ is a gap in the structure $\langle \mathcal{P}(\omega), \subseteq^* \rangle$. When we let $\langle a_\alpha, b_\beta; \alpha < \kappa, \beta < \lambda \rangle$ be a pregap in the above sense, it means that

$$\forall \xi < \xi' < \alpha \forall \eta < \eta' < \lambda (a_\xi \subseteq^* a_{\xi'} \ \& \ b_\eta \subseteq^* b_{\eta'} \ \& \ a_\xi \perp b_\eta).$$

By the above definition, it is trivial that (κ, λ)-gaps exist iff (λ, κ)-gaps exist. For a (κ, λ)-(pre)gap, if $\kappa = \lambda$, it is called symmetric, otherwise, it's called asymmetric. The following proposition is an easy test whether a given symmetric pregap $\langle a_\alpha, b_\alpha; \alpha < \kappa \rangle$ forms a gap or not.

Proposition 2.4 (Folklore). *Assume that $\langle a_\alpha, b_\alpha; \alpha < \kappa \rangle$ forms a pregap satisfying that $a_\alpha \cap b_\alpha$ is empty for every $\alpha < \kappa$. Then the following statements are equivalent:*

1. *$\langle a_\alpha, b_\alpha; \alpha < \kappa \rangle$ is a gap.*

2. *$\forall X \in [\kappa]^\kappa \exists \alpha \neq \beta \in X((a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \emptyset)$.*

Proof. If it is not a gap, letting c be its interpolation, there are $n \in \omega$, $s, t \subseteq n$ and $X \in [\kappa]^\kappa$ such that

- $\forall \alpha \in X(a_\alpha \cap n = s \ \& \ b_\alpha \cap n = t)$ (so $s \cap t$ is empty),
- $\forall \alpha \in X(a_\alpha \cap c \subseteq n \ \& \ b_\alpha \setminus n \subseteq c)$.

Then for any distinct α and β in X, $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset$.

Conversely, assume there exists $X \in [\kappa]^\kappa$ such that for any α and β in X, $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha)$ is empty. Let $c := \bigcup_{\alpha \in X} a_\alpha$, then c separates $\langle a_\alpha, b_\alpha; \alpha \in X \rangle$. Therefore it also separates $\langle a_\alpha, b_\alpha; \alpha < \kappa \rangle$. □

This gives a hint of a condition on indestructible gaps. (See next chapter.)

2.1 Classical results

In this section, we review three classical results. All of them had been proved before modern set theory appeared.

Lemma 2.5 (Hadamard, [21]). *There are no (ω, ω)-gaps.*

Proof. Assume that $\langle a_n, b_n; n \in \omega \rangle$ forms a pregap. Then we can recursively construct an increasing sequence $\langle k_n; n \in \omega \rangle$ of natural numbers such that

- $(a_n \cap b_n) \smallsetminus k_n$ is empty for all $n \in \omega$,
- $a_i \smallsetminus k_n \subseteq a_n$ and $b_i \smallsetminus k_n \subseteq b_n$ for all $n \in \omega$ and $i < n$, and
- $b_n \cap (k_{n+1} \smallsetminus k_n)$ is not empty for infinitely many $n \in \omega$.

Letting $c := \bigcup_{n \in \omega} b_n \cap (k_{n+1} \smallsetminus k_n)$ which is an infinite subset of ω, c separates $\langle a_n, b_n; n \in \omega \rangle$, i.e. $a_n \perp c$ and $b_n \subseteq^* c$ for all $n \in \omega$. □

Theorem 2.6 (Rothberger, [40]). *There exist $(\omega, \mathfrak{b})$-gaps.*

Proof. For $a \in [\omega]^\omega$, let $\widetilde{a}$ be a function on ω such that

$$\widetilde{a}(n) := \sum_{i \in a \cap n} 2^i$$

for every $n \in \omega$. For a pregap $\langle a_\alpha, b_n; \alpha < \kappa, n < \omega \rangle$ satisfying that $b_n \subseteq b_{n+1}$ for all $n \in \omega$ and a function $f \in \omega^\omega$ with $f \leq^* \widetilde{b_n^{\mathsf{c}}}$ for all $n \in \omega$, where b^{c} is a complement of b ($b^{\mathsf{c}} := \omega \smallsetminus b$), define

$$h_f(n) := \max \left\{ k \in \omega; \widetilde{b_n^{\mathsf{c}}}(k) \leq f(k) \right\}$$

for every $n \in \omega$. Then the following statements are equivalent:

(i) $\langle a_\alpha, b_n; \alpha < \kappa, n < \omega \rangle$ forms a gap, and

(ii) $\{h_{\widetilde{a_\alpha}}; \alpha < \kappa\}$ is unbounded (with respect to $\leq^*$ in ω^ω).

This says that there always exists $(\omega, \mathfrak{b})$-gaps because we can always find an increasing chain in $(\omega^\omega, \leq^*)$ of length $\mathfrak{b}$.

Assume (i) holds and a function $h \in \omega^\omega$ dominates all $h_{\widetilde{a_\alpha}}$, $\alpha < \kappa$. Without loss of generality, we may assume that $n < h(n) < h(n+1)$ holds for all $n \in \omega$. Define $p \in \omega^\omega$ by

$$p(n) := \widetilde{b_m^{\mathsf{c}}}(n) \text{ where } m = \min\{k \in \omega; h(k+1) > j\}$$

for every $n \in \omega$.

Claim 1. *$p \leq^* \widetilde{b_m}^{c}$ for any $m \in \omega$.*

For $m \in \omega$ and $n > h(m+1)$, letting $k \in \omega$ be so that $p(n) = \widetilde{b_k}^{c}(n)$, since $m < k$ holds, $p(n) \leq \widetilde{b_m}^{c}(n)$.

Claim 2. *$\widetilde{a_\alpha} \leq^* p$ for any $\alpha < \kappa$.*

For $\alpha < \kappa$, there is $N \in \omega$ such that $h_{\widetilde{a_\alpha}}(n) \leq h(n)$ for all $n \geq N$. Then for all $n \geq h(N)$, letting $l = \max\{k \in \omega; n \geq h(k)\}$ $(\geq N)$, $n \geq h(l) > h_{\widetilde{a_\alpha}}(l)$ hold and $p(n) = \widetilde{b_{l+1}}^{c}(n)$. So $\widetilde{a_\alpha}(n) < \widetilde{b_l}^{c}(n) \leq \widetilde{b_{l+1}}^{c}(n) = p(n)$.

Define $c \in [\omega]^\omega$ by

$$k \in c \iff p(k) > \sum_{i<k} 2^i.$$

Then this c separates $\langle a_\alpha, b_n; \alpha < \kappa, n < \omega \rangle$. (We note that for $a, b \in [\omega]^\omega$, $a \perp b \iff a \subseteq^* b^c \iff \widetilde{a} \leq^* \widetilde{b^c}$.)

Assume that (i) doesn't hold and let $c \in [\omega]^\omega$ separate $\langle a_\alpha, b_n; \alpha < \kappa, n < \omega \rangle$. Then we can define $h_{\widetilde{c}}$ which dominates all $h_{\widetilde{a_\alpha}}$, $\alpha < \kappa$. □

From the above proof, we can conclude

Corollary 2.7.

$$\mathfrak{b} = \min \{\kappa; \exists \ (\omega, \kappa)\text{-}gaps\ \}.$$

□

And this proof gives us a condition on destructibility of Rothberger gaps. (See also next chapter.)

Theorem 2.8 (Hausdorff, [22]). *There exist (ω_1, ω_1)-gaps.*

Proof. By recursion on $\alpha < \omega_1$, we construct a_α and b_α in $[\omega]^\omega$ so that $a_\alpha \cap b_\alpha$ is empty, $\omega \setminus (a_\alpha \cup b_\alpha)$ is infinite and letting

$$C(a, \langle b_\xi; \xi < \alpha \rangle, n) := \{\xi < \alpha; a_\alpha \cap b_\xi \subseteq n\}$$

for any $n \in \omega$, $a \in [\omega]^\omega$ and an increasing sequence $\langle b_\xi; \xi < \alpha \rangle$, $C(a_\alpha, \langle b_\xi; \xi < \alpha \rangle, n)$ is finite for all $n \in \omega$.

At first, we put $a_0 = b_0 = \emptyset$. And given a_α and b_α, we let $a_{\alpha+1}$ and $b_{\alpha+1}$ be any a and b such that $a_\alpha \subseteq a$, $b_\alpha \subseteq b$, and both $a \setminus a_\alpha$, $b \setminus b_\alpha$ and $\omega \setminus (a \cup b)$ are infinite. Then since $C(a_\alpha, \langle b_\xi; \xi < \alpha \rangle, n)$ is finite, $C(a_{\alpha+1}, \langle b_\xi; \xi < \alpha + 1 \rangle, n)$ is also finite for any $n \in \omega$.

In the limit stage, assuming that we have already constructed $\langle a_\xi, b_\xi; \xi < \alpha \rangle$, there exists its interpolation c_0. By the inductive hypothesis, $C(c_0, \langle b_\xi; \xi < \beta \rangle, n)$, which is a subset of $C(a_\beta, \langle b_\xi; \xi < \beta \rangle, n)$, is finite for every $\beta < \alpha$ and $n \in \omega$. By recursion on $i \in \omega$, we construct $\{c_i \in \omega$ such that

- $\langle c_i, b_\xi; i \in \omega, \xi < \alpha \rangle$ forms a pregap, and
- $C(c_{i+1}, \langle b_\xi; \xi < \alpha \rangle, i)$ is finite for all $i \in \omega$.

Given c_i, if $C(c_i, \langle b_\xi; \xi < \alpha \rangle, i+1)$ is finite, then we let $c_{i+1} := c_i$. Otherwise, we notice that $C(c_i, \langle b_\xi; \xi < \alpha \rangle, i+1)$ has the order type ω and its suprimum is α. (Since $C(c_i, \langle b_\xi; \xi < \alpha \rangle, i+1) \setminus \xi$ is infinite, in particular non-empty for every $\xi < \alpha$, $C(c_i, \langle b_\xi; \xi < \alpha \rangle, i+1)$ is cofinal in α. If its order type is larger than ω, then there is $\beta \in C(c_i, \langle b_\xi; \xi < \alpha \rangle, i+1)$ such that $C(c_i, \langle b_\xi; \xi < \beta \rangle, i+1)$ is infinite, which is a contradiction.) Let $\langle \xi_n; n \in \omega \rangle$ be an increasing enumeration of $C(c_i, \langle b_\xi; \xi < \alpha \rangle, i+1)$. We find a strictly increasing sequence $\langle n_l; l \in^\omega \rangle$ of natural numbers such that

$$c_i \cap b_{\xi_j} \subseteq n_l \ \&\ b_{\xi_j} \cap b_{\xi_k} \subseteq n_l \ \&\ b_{\xi_l} \cap (n_{l+1} \setminus n_l) \neq \emptyset$$

for all $j < k \leq l$. Then letting

$$c_{i+1} := c_i \cup \bigcup_{j \in \omega} (b_{\xi_j} \cap n_{j+1} \setminus n_j),$$

c_{i+1} is as desired. After constructing $\langle c_i; i \in \omega \rangle$, we take a separation a_α of $\langle c_i, b_\xi; i \in \omega, \xi < \alpha \rangle$, which means that

$$\forall i \in \omega \forall \xi < \alpha (c_i \subseteq^* a_\alpha \ \&\ a \perp b_\xi),$$

and next we pick an interpolation b_α of $\langle a_\alpha, b_\xi; \xi < \alpha \rangle$, which completes the construction.

All we have to do is to check that $\langle a_\alpha, b_\alpha; \alpha < \omega_1 \rangle$ forms a gap. Assume there exists a separation d of this pregap. Then by the counting argument, we can find $A \in [\omega]^{\omega_1}$ and $n \in \omega$ so that $a_\alpha \cap d \subseteq n$ and $b_\alpha \subseteq d \cup n$ hold for every $\alpha \in A$. Let $\alpha \in A$ be such that $A \cap (\alpha + 1)$ is infinite, then since

$$a_\alpha \cap b_\xi \subseteq (a_\alpha \cap d) \cup (b_\xi \setminus d) \subseteq n$$

for all $\xi \in A$, $C(a_\alpha, \langle b_x i; \xi < \alpha \rangle, n)$ is not finite which is a contradiction. □

If an (ω_1, ω_1)-gap $\langle a_\alpha, b_\alpha; \alpha < \omega_1 \rangle$ satisfies that

$$\{\xi < \alpha; a_\alpha \cap b_\xi \subseteq n\} \text{ is finite for all } n \in \omega \text{ and } \alpha < \omega_1,$$

then it is called that *Hausdorff gap*. It can be proved that any Hausdorff gap is indestructible (Lemma 3.7), but by a recent work of Hirschorn [26], the reverse may not be true.

Chapter 3

Forcing destructibility and survivability of gaps

3.1 Basic forcing notions on gaps

In this chapter, we consider the relationship between gaps and forcing extensions. For a gap $(\mathcal{A}, \mathcal{B})$ and a forcing notion $\mathbb{P}$, we call $(\mathcal{A}, \mathcal{B})$ $\mathbb{P}$-survivable if it still forms a gap in the extension with $\mathbb{P}$, otherwise it is called $\mathbb{P}$-destructible. In [41], Scheepers researched destructibility of gaps under forcing notions having a stronger condition than the countable chain condition. We also consider survivability under forcing notions not having the ccc like Sacks forcing or Laver forcing.

Separating gaps by forcing

From Lemma 2.5, any gap is $\mathbb{P}$-destructible for some $\mathbb{P}$ collapsing an enough larger cardinal to $\aleph_0$. So as usual, when we think of $\mathbb{P}$-destructibility, we assume $\mathbb{P}$ does not collapse any cardinals. A gap is called indestructible if it is $\mathbb{P}$-survivable for any forcing notion $\mathbb{P}$ collapsing no cardinals. Kunen and Laver have defined the following forcing notions both of which generically adds an interpolation and have similar conditions.

Definition 3.1 (Kunen). *For a pregap $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\beta; \alpha < \kappa, \beta < \lambda \rangle$, define*

$$\begin{aligned}\mathsf{Kunen}(\mathcal{A}, \mathcal{B}) := \big\{ \langle p, q \rangle \,; p \subseteq \kappa \times \omega, q \subseteq \lambda \times \omega\text{: \textit{finite partial functions}} \\ \& \ \forall \alpha \in \mathrm{dom}(p) \forall \beta \in \mathrm{dom}(q) \big((a_\alpha \setminus p(\alpha)) \cap (b_\beta \setminus q(\beta)) = \emptyset \big) \big\},\end{aligned}$$

$$\langle p_0, q_0\rangle \leq \langle p_1, q_1\rangle \iff p_0 \supseteq p_1 \;\&\; q_0 \supseteq q_1.$$

Definition 3.2 (Laver). *For a pregap* $(\mathcal{A},\mathcal{B}) = \langle a_\alpha, b_\beta; \alpha < \kappa, \beta < \lambda\rangle$, *define*

$$\mathsf{Laver}(\mathcal{A},\mathcal{B}) := \{\langle L, R, s\rangle \in [\kappa]^{<\omega} \times [\lambda]^{<\omega} \times 2^{<\omega}; \forall\alpha \in L \forall\beta \in R (a_\alpha \cap b_\beta \subseteq |s|)\},$$

$$\begin{aligned}\langle L_0, R_0, s_0\rangle \leq \langle L_1, R_1, s_1\rangle \iff & L_0 \supseteq L_1 \;\&\; R_0 \supseteq R_1 \;\&\; s_0 \supseteq s_1 \\ & \& \;\forall\alpha \in L_1 \forall\beta \in R_1 \big(a_\alpha \cap (|s_0| \setminus |s_1|) \subseteq {s_0}^{-1}\{1\} \cap (|s_0| \setminus |s_1|) \\ & \qquad \& \; {s_0}^{-1}\{1\} \cap b_\beta \cap (|s_0| \setminus |s_1|) = \emptyset\big).\end{aligned}$$

As we see below, both of them have similar properties. But I don't know whether both are equivalent or not. $\mathsf{Kunen}(\mathcal{A},\mathcal{B})$ adds an interpolation *indirectly*, but on the other hand, $\mathsf{Laver}(\mathcal{A},\mathcal{B})$ adds an interpolation *directly.*

Lemma 3.3 ([33],[41]). *If a pregap* $(\mathcal{A},\mathcal{B})$ *does not form a gap, then* $\mathsf{Kunen}(\mathcal{A},\mathcal{B})$ *has strong property K, and* $\mathsf{Laver}(\mathcal{A},\mathcal{B})$ *has a* σ*-centered dense subset (so it also has the strong property K).*

Proof. Let $c \in [\omega]^\omega$ interpolate it, i.e. $a \perp c$ and $b \subseteq^* c$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Let

$$\begin{aligned}\mathsf{C}_K(s) := \big\{&\langle p, q\rangle \in \mathsf{Kunen}(\mathcal{A},\mathcal{B}); \forall\alpha \in \mathrm{dom}(p) \forall\beta \in \mathrm{dom}(q), \\ &(a_\alpha \setminus p(\alpha)) \cap (s^{-1}\{1\} \cup (c \setminus |s|)) = \emptyset \;\&\; b_\beta \setminus q(\beta) \subseteq s^{-1}\{1\} \cup (c \setminus |s|)\big\},\end{aligned}$$

and

$$\mathsf{C}_L(s) := \big\{\langle L, R, s\rangle \in \mathsf{Laver}(\mathcal{A},\mathcal{B}); \forall\alpha \in L \forall\beta \in R \big(a_\alpha \cap c,\; b_\beta \setminus c \subseteq |s|\big)\big\},$$

for any $s \in 2^{<\omega}$. Then $\bigcup_{s \in 2^{<\omega}} \mathsf{C}_K(s) = \mathsf{Kunen}(\mathcal{A},\mathcal{B})$ and $\bigcup_{s \in 2^{<\omega}} \mathsf{C}_L(s)$ is dense in $\mathsf{Laver}(\mathcal{A},\mathcal{B})$. If $\langle p.q\rangle$ and $\langle p', q'\rangle$ are in $\mathsf{C}_K(s)$ and $p \restriction \mathrm{dom}(p) = p' \restriction \mathrm{dom}(p')$ and $q \restriction \mathrm{dom}(q) = q' \restriction \mathrm{dom}(q')$, then $\langle p.q\rangle$ and $\langle p', q'\rangle$ are compatible. So it follows from a Δ-system argument that $\bigcup_{s \in 2^{<\omega}} \mathsf{C}_K(s)$ has strong property K. We note that each $\mathsf{C}_L(S)$ is σ-centered. □

Lemma 3.4 ([41]). *Let* $(\mathcal{A},\mathcal{B}) = \langle a_\alpha, b_\beta; \alpha < \kappa, \beta < \lambda\rangle$ *be a* (κ,λ)*-gap (assuming that* $\kappa \leq \lambda$*).*

- *If* $\kappa < \lambda$, *then both* $\mathsf{Kunen}(\mathcal{A},\mathcal{B})$ *and* $\mathsf{Laver}(\mathcal{A},\mathcal{B})$ *have the strong property K.*

- *If $\kappa = \lambda > \omega_1$, then both* $\mathsf{Kunen}(\mathcal{A},\mathcal{B})$ *and* $\mathsf{Laver}(\mathcal{A},\mathcal{B})$ *have the property K.*

Proof. If $\kappa = \omega$, then $\mathsf{Kunen}(\mathcal{A},\mathcal{B})$ has strong property K, and $\mathsf{Laver}(\mathcal{A},\mathcal{B})$ has a σ-centered dense subset as above:

$$\mathsf{Kunen}(\mathcal{A},\mathcal{B}) = \bigcup_{\substack{E \in [\omega]^{<\omega} \\ p \in \omega^E}} \mathsf{D}_K(E,p) \text{ where}$$

$$\mathsf{D}_K(E,p) := \{\langle p,q\rangle \in \mathsf{Kunen}(\mathcal{A},\mathcal{B}); q \subseteq \lambda \times \omega \text{ is a finite partial function}\},$$

$$\mathsf{Laver}(\mathcal{A},\mathcal{B}) = \bigcup_{\substack{L \in [\omega]^{<\omega} \\ s \in \omega^{<\omega}}} \mathsf{D}_L(L,s) \text{ where}$$

$$\mathsf{D}_L(L,s) := \{\langle L,R,s\rangle \in \mathsf{Laver}(\mathcal{A},\mathcal{B}); R \in [\lambda]^{<\omega}\}.$$

So we may assume that $\kappa \geq \omega_1$.

Case of $\mathsf{Kunen}(\mathcal{A},\mathcal{B})$. Take any $\{\langle p_\alpha, q_\alpha\rangle\,; \alpha < \mu\} \subseteq \mathsf{Kunen}(\mathcal{A},\mathcal{B})$, where μ is uncountable and regular. To show the lemma, we have only to consider the following two cases:

- $\circ$ $\mu < \lambda$. Find $\delta < \lambda$ with $\bigcup_{\alpha<\mu} \mathrm{dom}(q_\alpha) \subseteq \delta$. We denote $\langle b_\beta; \beta < \delta\rangle$ by $\mathcal{B} \restriction \delta$. Then $\{\langle p_\alpha, q_\alpha\rangle\,; \alpha < \mu\} \subseteq \mathsf{Kunen}(\mathcal{A}, \mathcal{B} \restriction \delta)$ which has the strong property K because $(\mathcal{A}, \mathcal{B} \restriction \delta)$ doesn't form a gap.

- $\circ$ $\kappa < \lambda \leq \mu$. Find $I \in [\mu]^\mu$, $p \in [\kappa \times \omega]^{<\omega}$ and such that $p_\alpha = p$ for any $\alpha \in I$ and $\{q_\alpha; \alpha \in I\}$ forms a Δ-system. Then $\{\langle p_\alpha, q_\alpha\rangle\,; \alpha \in I\}$ is pairwise compatible.

Case of $\mathsf{Laver}(\mathcal{A},\mathcal{B})$. Take any $\{\langle L_\alpha, R_\alpha, s_\alpha\rangle\,; \alpha < \mu\} \subseteq \mathsf{Laver}(\mathcal{A},\mathcal{B})$, where μ is uncountable and regular. We have only to consider the following two cases as above:

- $\circ$ $\mu < \lambda$. Find $\delta < \lambda$ with $\bigcup_{\alpha<\mu} R_\alpha \subseteq \delta$. Then $\{\langle L_\alpha, R_\alpha, s_\alpha\rangle\,; \alpha < \mu\} \subseteq \mathsf{Laver}(\mathcal{A}, \mathcal{B} \restriction \delta)$ which is σ-centered.

- $\circ$ $\kappa < \lambda \leq \mu$. Find $I \in [\mu]^\mu$, $L \in [\kappa]^{<\omega}$ and $s \in \omega^{<\omega}$ such that $L_\alpha = L$ and $s_\alpha = s$ for any $\alpha \in I$. Then $\{\langle L_\alpha, R_\alpha, s_\alpha\rangle\,; \alpha \in I\}$ is pairwise compatible.

□

Theorem 3.5 (Kunen [33], Todorčević [46], Woodin [51]). *For an (ω_1, ω_1)-pregap $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\beta; \alpha, \beta < \omega_1\rangle$ with $a_\alpha \cap b_\alpha = \emptyset$ for every $\alpha < \omega_1$, the following statements are equivalent:*

1. *$(\mathcal{A}, \mathcal{B})$ is $\mathbb{P}$-destructible for some forcing notion $\mathbb{P}$ which does not collapse $\aleph_1$,*

2. *$\forall X \in [\omega_1]^{\omega_1} \exists \alpha \neq \beta \in X\ ((a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset)$,*

3. *the partition $[\mathcal{A} \otimes \mathcal{B}]^2 = L_0 \dot{\cup} L_1$, where*
$$[\mathcal{A} \otimes \mathcal{B}]^2 := \{\langle a, b\rangle \in \mathcal{A} \times \mathcal{B}; a \cap b = \emptyset\},$$
$$\{\langle a_0, b_0\rangle, \langle a_1, b_1\rangle\} \in L_0 \iff (a_0 \cap b_1) \cup (a_1 \cap b_0) = \emptyset,$$
is Suslin, i.e. for every uncountable collection $\mathcal{F}$ of finite L_0-homogeneous subsets of $[\mathcal{A} \otimes \mathcal{B}]^2$, there are $F \neq G \in \mathcal{F}$ so that $F \cup G$ is also L_0-homogeneous,

4. *$\mathsf{Kunen}(\mathcal{A}, \mathcal{B})$ has the countable chain condition,*

5. *$\mathsf{Laver}(\mathcal{A}, \mathcal{B})$ has the countable chain condition.*

Proof. Both $4 \implies 1$ and $5 \implies 1$ are trivial. Proofs of $\neg 4 \implies \neg 1$ and $\neg 5 \implies \neg 1$ are the same way: Assume $X \subseteq \mathsf{Kunen}(\mathcal{A}, \mathcal{B})$ is an uncountable antichain. Then $\Vdash_{\mathbb{P}}$ " $\check{X}$ is also an uncountable antichain " because $\mathbb{P}$ preserves $\aleph_1$. By Lemma 3.3,
$$\Vdash_{\mathbb{P}} \text{" } \mathsf{Kunen}(\mathcal{A}, \mathcal{B}) \text{ doesn't have the ccc, so } (\mathcal{A}, \mathcal{B}) \text{ is a gap ".}$$

For a proof of $4 \implies 2$, let $X \subseteq \omega_1$ be uncountable. For each $\alpha \in X$, we let $e_\alpha = \langle p_\alpha, q_\alpha\rangle$ be a condition of $\mathsf{Kunen}(\mathcal{A}, \mathcal{B})$ such that $p_\alpha := \{\langle \alpha, 0\rangle\}$ and $q_\alpha := \{\langle \alpha, 0\rangle\}$. Then we consider the collection $\{e_\alpha\ \alpha \in X\}$. By the ccc-ness of $\mathsf{Kunen}(\mathcal{A}, \mathcal{B})$, there are distinct α and β in X so that e_α and e_β are compatible, which means that $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha)$ is empty.

For a proof of $2 \implies 3$, we assume that there exists an uncountable collection $\{F_\alpha; \alpha < \omega_1\}$ of finite L_0-homogeneous subsets of $\mathcal{A} \otimes \mathcal{B}$ such that for any distinct α and β in ω_1, $F_\alpha \cup F_\beta$ is not L_0-homogeneous. For each $\alpha < \omega_1$, we fix an ordinal $\xi_\alpha < \omega_1$ such that $a \subseteq^* a_{\xi_\alpha}$ and $b \subseteq^* b_{\xi_\alpha}$ for all $\langle a, b\rangle \in F_\alpha$ and all ξ_α are distinct ordinals. By shrinking if need, we may assume that there exists $n \in \omega$ such that

- $\forall\alpha < \omega_1 \forall \langle a, b\rangle \in F_\alpha(a \setminus n \subseteq a_{\xi_\alpha} \ \&\ b \setminus n \subseteq b_{\xi_\alpha})$,
- $\forall\alpha, \beta < \omega_1(a_{\xi_\alpha} \cap n = a_{\xi_\beta} \cap n \ \&\ b_{\xi_\alpha} \cap n = b_{\xi_\beta} \cap n)$.

Let $X := \{\xi_\alpha; \alpha < \omega_1\}$. We note that X is L_1-homogeneous, which means that for any distinct $\xi, \eta \in X$, $(a_\xi \cap b_\eta) \cup (a_\eta \cap b_\xi)$ is not empty, i.e. $\neg 2$ holds: If $(a_{\xi_\alpha} \cap b_{\xi_\beta}) \cup (a_{\xi_\beta} \cap b_{\xi_\alpha})$ is empty for some $\alpha \neq \beta \in \omega_1$, then for any $\langle a, b\rangle \in F_\alpha$ and $\langle a', b'\rangle \in F_\beta$,

$$(a \cap b') \cup (a' \cap b) \subseteq n \cup \left((a_{\xi_\alpha} \cap b_{\xi_\beta}) \cup (a_{\xi_\beta} \cap b_{\xi_\alpha})\right) = \emptyset.$$

(We should remember that $a_\alpha \cap b_\alpha = \emptyset$ for all $\alpha < \omega_1$.) It follows that $F_\alpha \cup F_\beta$ is L_0-homogeneous, which is a contradiction.

For a proof of $3 \implies 5$, assume that there exists an uncountable antichain $X = \{p_\alpha; \alpha < \omega_1\} \subseteq \mathsf{Laver}(\mathcal{A}, \mathcal{B})$. Letting $p_\alpha := \langle L_\alpha, R_\alpha, s_\alpha\rangle$, as the argument above, we can find and assume that there are $\xi_\alpha < \omega_1$ for $\alpha < \omega_1$ and $n \in \omega$ such that

- $\forall\alpha < \omega_1(n > |s_\alpha|)$
- $\forall\alpha < \omega_1 \forall\gamma \in L_\alpha \forall\delta \in R_\alpha(a_\gamma \setminus n \subseteq a_{\xi_\alpha} \ \&\ b_\delta \setminus n \subseteq b_{\xi_\alpha})$,
- $\forall\alpha, \beta < \omega_1(a_{\xi_\alpha} \cap n = a_{\xi_\beta} \cap n \ \&\ b_{\xi_\alpha} \cap n = b_{\xi_\beta} \cap n)$,
- $\forall\alpha, \beta < \omega_1(\{a_\gamma \cap n; \gamma \in L_\alpha\} = \{a_\gamma \cap n; \gamma \in L_\beta\} \ \&\ \{b_\gamma \cap n; \gamma \in R_\alpha\} = \{b_\gamma \cap n; \gamma \in R_\beta\})$.

Since $\{p_\alpha; \alpha < \omega_1\}$ is pairwise incompatible, by conditions above, $\left((a_{\xi_\alpha} \cap b_{\xi_\beta}) \cup (a_{\xi_\beta} \cap b_{\xi_\alpha})\right) \setminus n$ is non-empty. Let $F_\alpha := \{\xi_\alpha\}$ for every $\alpha < \omega_1$, then all F_α are L_0-homogeneous but any union $F_\alpha \cup F_\beta$ with $\alpha \neq \beta$ is L_1-homogeneous, which leads that $\neg 3$ holds. □

Corollary 3.6 (Kunen [33], Todorčević [46], Woodin [51]). *For an (ω_1, ω_1)-pregap $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\beta; \alpha, \beta < \omega_1\rangle$ with $a_\alpha \cap b_\alpha = \emptyset$ for every $\alpha < \omega_1$, the following statements are equivalent:*

1. *$(\mathcal{A}, \mathcal{B})$ is indestructible,*
2. *$\exists X \in [\omega_1]^{\omega_1} \forall\alpha \neq \beta \in X\, ((a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \emptyset)$,*
3. *$\mathcal{A} \otimes \mathcal{B}$ has an uncountable K_0-homogeneous subsets in the open partition $[\mathcal{A} \otimes \mathcal{B}]^2 = K_0 \dot\cup K_1$;*

$$\{\langle a_0, b_0\rangle, \langle a_1, b_1\rangle\} \in K_0 \iff (a_0 \cap b_1) \cup (a_1 \cap b_0) \neq \emptyset,$$

4. Kunen$(\mathcal{A}, \mathcal{B})$ *doesn't have the countable chain condition,*

5. Laver$(\mathcal{A}, \mathcal{B})$ *doesn't have the countable chain condition.*

□

2 $\Longrightarrow$ 1 is trivial from Lemma 2.4. If an (ω_1, ω_1)-gap has a condition 2, then $(\mathcal{A} \restriction X, \mathcal{B} \restriction X) = \langle a_\alpha, b_\alpha; \alpha \in X \rangle$ is also a gap and still forms a gap in any forcing extension collapsing no cardinals because $(\mathcal{A} \restriction X, \mathcal{B} \restriction X)$ still satisfies a condition 2 of Lemma 2.4.

Lemma 3.7 **([41]).** *Every Hausdorff gap is indestructible.*

Proof. Assume that $\langle a_\alpha, b_\alpha; \alpha < \omega_1 \rangle$ is Hausdorff gap, i.e. the set

$$\{\xi < \alpha; a_\alpha \cap b_\xi \subseteq n\}$$

is finite for any $n \in \omega$ and $\alpha < \omega_1$. We define a function $f : \omega_1 \to [\omega_1]^{<\omega}$ with

$$f(\alpha) := \{\xi < \alpha; a_\alpha \cap b_\xi = \emptyset\}.$$

(It is well defined.) Now we use the following combinatorial theorem:

Theorem 3.8 (Lázár,see e.g. [27]**).** *Let κ be a regular cardinal and λ a (finite or infinite) cardinal with $\lambda < \kappa$. Assume that A is a set with size κ and a function $K : A \to \mathcal{P}(A)$ satisfies that $|K(x)| < \lambda$ for every $x \in A$. Then there is a subset $B \subseteq A$ of the cardinality κ which is independent with respect to K, i.e. for any distinct x and y in B, $x \notin K(y)$.* ⊣

Let X be an uncountable subset of ω_1 with respect to f, which satisfies the condition 2 of Corollary 3.6. □

Freezing gaps by forcing

We have a forcing notion which *freezes* a given (ω_1, ω_1)-gap, that is, which makes it indestructible. As seen above, if a type of a gap is not (ω_1, ω_1), then it can be destroyed by a forcing notion with the countable chain condition. As seen in the next section, it is consistent with ZFC that there exists an destructible (ω_1, ω_1)-gap. In general, $(\mathcal{A}, \mathcal{B})$ is called *a destructible gap* if it has a type (ω_1, ω_1) and is not indestructible. As seen above, if an (ω_1, ω_1)-gap is destructible, then it can be destroyed by a ccc forcing notion.

Definition 3.9 (Kunen). *For an (ω_1, ω_1)-pregap $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\beta; \alpha, \beta < \omega_1 \rangle$, define*

$$\mathsf{Freezing}_{\mathsf{Kunen}}(\mathcal{A}, \mathcal{B}) := \big\{\sigma \in [\omega_1]^{<\omega}; \{\langle\{\langle\alpha, 0\rangle\}, \{\langle\alpha, 0\rangle\}\rangle\,; \alpha \in \sigma\} \text{ is an antichain in } \mathsf{Kunen}(\mathcal{A}, \mathcal{B})\big\},$$

$$\sigma_0 \leq \sigma_1 \iff \sigma_0 \supseteq \sigma_1.$$

Definition 3.10 (Woodin). *For an (ω_1, ω_1)-pregap $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\beta; \alpha, \beta < \omega_1 \rangle$, define*

$$\begin{aligned}\mathsf{Freezing}_{\mathsf{Woodin}}(\mathcal{A}, \mathcal{B}) := \big\{\langle f, g\rangle\,; f, g \subseteq \omega_1 \times 2^{<\omega}\text{: finite partial functions}\\ \&\ \mathrm{dom}(f) = \mathrm{dom}(g)\ \&\ \forall\alpha \in \mathrm{dom}(f)\forall\beta \in \mathrm{dom}(g),\\ \text{either } \big((a_\alpha \setminus |f(\alpha)|) \cup f(\alpha)^{-1}[\{1\}]\big) \cap \big((b_\beta \setminus |g(\beta)|) \cup g(\beta)^{-1}[\{1\}]\big)\\ \text{or } \big((a_\beta \setminus |f(\beta)|) \cup f(\beta)^{-1}[\{1\}]\big) \cap \big((b_\alpha \setminus |g(\alpha)|) \cup g(\alpha)^{-1}[\{1\}]\big)\\ \text{is non-empty}\big\},\end{aligned}$$

$$\langle f_0, g_0\rangle \leq \langle f_1, g_1\rangle \iff f_0 \supseteq g_0\ \&\ f_1 \supseteq g_1.$$

Lemma 3.11 (Kunen, [33], [41]). *For an (ω_1, ω_1)-pregap $(\mathcal{A}, \mathcal{B})$, the following statements are equivalent:*

1. *$(\mathcal{A}, \mathcal{B})$ forms a gap,*
2. *$\mathsf{Freezing}_{\mathsf{Kunen}}(\mathcal{A}, \mathcal{B})$ has the countable chain condition,*
3. *$\mathsf{Freezing}_{\mathsf{Woodin}}(\mathcal{A}, \mathcal{B})$ has the countable chain condition.*

Proof. For a proof of $1 \implies 2$, let $X \subseteq \mathsf{Freezing}_{\mathsf{Kunen}}(\mathcal{A}, \mathcal{B})$ be uncountable. Without loss of generality, we may assume that X forms a Δ-system with root ρ and there exists $n \in \omega$ such that

$$\forall\sigma \in X \forall\xi < \eta \in \sigma \setminus \rho(a_\xi \setminus n \subseteq a_\eta \setminus n\ \&\ b_\xi \setminus n \subseteq b_\eta \setminus n).$$

For $\sigma \in X$, let $\gamma_\sigma := \min(\sigma \setminus \rho)$. Since X is uncountable,

$$\langle a_{\gamma_\sigma} \setminus n, b_{\gamma_\sigma} \setminus n; \sigma \in X\rangle$$

is equivalent to $(\mathcal{A}, \mathcal{B})$ hence it is a gap. By Lemma 2.4, there are distinct σ and τ in X so that $(a_{\gamma_\sigma} \cap b_{\gamma_\tau}) \cup (a_{\gamma_\tau} \cap b_{\gamma_\sigma})$ is not empty. It follows that $\sigma \cup \tau$ is a condition of $\mathsf{Freezing}_{\mathsf{Kunen}}(\mathcal{A}, \mathcal{B})$, i.e. σ and τ are compatible.

For a proof of 2 $\implies$ 3, we assume that

$$X := \{p_\alpha; \alpha < \omega_1\} \subseteq \mathsf{Freezing}_{\mathsf{Woodin}}(\mathcal{A}, \mathcal{B})$$

is an uncountable antichain. Let $p_\alpha := \langle f_\alpha, g_\alpha \rangle$ for $\alpha < \omega_1$. Without loss of generality, we may assume that

- $\{\mathrm{dom}(f_\alpha); \alpha < \omega_1\}$ forms Δ-system with root $A \subseteq \omega_1$,
- $\forall \alpha, \beta < \omega_1 (f_\alpha \restriction A = f_\beta \restriction A \ \&\ g_\alpha \restriction A = g_\beta \restriction A)$, and
- letting $\gamma_\alpha := \min(\mathrm{dom}(f_\alpha) \setminus A)$ for $\alpha < \omega_1$, there exists $n \in \omega$ such that for all $\alpha < \omega_1$,

$$\bigcup_{\delta \in \mathrm{dom}(f_\alpha)} f(\delta) \leq n \ \&\ \forall \delta \in \mathrm{dom}(f_\alpha)\,(a_{\gamma_\alpha} \setminus n \subseteq a_\delta \setminus n)$$

and for any $\alpha, \beta < \omega_1$,

$$a_{\gamma_\alpha} \cap n = a_{\gamma_\beta} \cap n \ \&\ b_{\gamma_\alpha} \cap n = b_{\gamma_\beta} \cap n.$$

Moreover we may assume that all γ_α are pairwise distinct. Since X is pairwise incompatible, for any distinct ξ and η in ω_1, there are $\alpha \in \mathrm{dom}(f_\xi)$ and $\beta \in \mathrm{dom}(f_\eta)$ such that either

$$\big((a_\alpha \setminus |f(\alpha)|) \cup f(\alpha)^{-1}[\{1\}]\big) \cap \big((b_\beta \setminus |g(\beta)|) \cup g(\beta)^{-1}[\{1\}]\big)$$

or

$$\big((a_\beta \setminus |f(\beta)|) \cup f(\beta)^{-1}[\{1\}]\big) \cap \big((b_\alpha \setminus |g(\alpha)|) \cup g(\alpha)^{-1}[\{1\}]\big)$$

is empty. It follows that $(a_{\gamma_\alpha} \cap b_{\gamma_\beta}) \cup (a_{\gamma_\beta} \cap b_{\gamma_\alpha})$ is also empty. Let $Y := \{\{\gamma_\alpha\}; \alpha < \omega_1\} \subseteq \mathsf{Freezing}_{\mathsf{Kunen}}(\mathcal{A}, \mathcal{B})$ which is an uncountable antichain.

For a proof of 3 $\implies$ 1, assume that $(\mathcal{A}, \mathcal{B})$ is separated. Let c be its separation, then there are $n \in \omega$ and $X \in [\omega_1]^{\omega_1}$ so that $a_\alpha \cap c \subseteq n$ and $c \setminus n \subseteq b_\alpha \setminus n$ for every $\alpha \in X$. For each $\alpha \in X$, we let $p_\alpha := \big\langle \{\langle \alpha, n \rangle\}, \{\langle \alpha, n \rangle\} \big\rangle$ and consider $\{p_\alpha; \alpha \in X\}$ which is an uncountable antichain. $\square$

Adding gaps by forcing

We present a forcing notion adding an embedding map from some partially ordered set $\langle E, <_E \rangle$ into $\langle \mathcal{P}(\omega), \subseteq^* \rangle$ by finite approximations (see [15] and [41]). A forcing notion adding an embedding map by finite approximations is Hechler's forcing notion without rank (see [7], [23]).

Definition 3.12 ([15], [41], etc). *Let $\langle E, <_E \rangle$ be a partial order. $\mathcal{H}(\langle E, <_E \rangle)$ consists of conditions $p = \langle F_p, n_p, f_p \rangle$, where $F_P \in [E]^{<\omega}$, $n_p \in \omega$ and f_p is a function from $F_p \times n_p$ into 2. For conditions $p, q \in \mathcal{H}(\langle E, <_E \rangle)$, define $p \leq q$ if*

- *$F_p \supseteq F_q$, $n_p \geq n_q$ and $f_p \supseteq f_q$, and*
- *For $a, b \in F_q$, if $a <_E b$ then $f_p(a, i) \leq f_q(b, i)$ for every $i \in [n_q, n_p)$.*

We list basic theorems below. For their proofs, see [15].

Theorem 3.13. 1 (Folklore). *$\mathcal{H}(\langle E, <_E \rangle)$ has the countable chain condition.*

2 (Farah). *A pair $\langle a_\alpha, b_\beta; \alpha < \kappa, \beta < \lambda \rangle$ of subsets of E is called tight (κ, λ)-gap if $\langle a_\alpha, b_\beta; \alpha < \kappa, \beta < \lambda \rangle$ forms a gap in $\langle E, <_E \rangle$ and is tight, i.e. there are no $c \in E$ such that one of the following happens:*

- *$a_\alpha <_E c$ and $b_\beta \not<_E c$ for all $\alpha < \kappa$ and $\beta < \lambda$, or*
- *$c \not<_E a_\alpha$ and $c <_E b_\beta$ for all $\alpha < \kappa$ and $\beta < \lambda$.*

If $\langle E, <_E \rangle$ has a tight (κ, λ)-gap, then $\mathcal{H}(\langle E, <_E \rangle)$ adds a (κ, λ)-gap in $\langle \mathcal{P}(\omega), \subseteq^* \rangle$.

3 (Farah). *If $\kappa > \mathfrak{c}$ and $\mathcal{H}(\langle E, <_E \rangle)$ adds a (κ, λ)-gap in $\langle \mathcal{P}(\omega), \subseteq^* \rangle$, then either a (κ, λ)-gap or a (λ, κ)-gap exists in $\langle E, <_E \rangle$.* □

3.2 Destructibility and survivability under several forcing notions

In the last section, we saw basic forcing notions one type of which adds a separation of a given gap, the other type of which forces a given (ω_1, ω_1)-gap to be indestructible. In this section, we consider other forcing notions and investigate whether they destroy gaps. To do this, we must distinguish two types of gaps: Rothberger gaps and non-Rothberger gaps.

Definition 3.14. *1. A subset $\mathcal{A}$ of $\mathcal{P}(\omega)$ is called σ-directed if for every countable subset X of $\mathcal{A}$ there exists its upper bound $a \in \mathcal{A}$, i.e. $x \subseteq^* a$ for all $x \in X$.*

2. A pregap $(\mathcal{A}, \mathcal{B})$ is called Rothberger if either $\mathcal{A}$ or $\mathcal{B}$ is not σ-directed. In particular, if $(\mathcal{A}, \mathcal{B})$ is linear, then the order type of either $\mathcal{A}$ or $\mathcal{B}$ has countable cofinality. If a pregap is not Rothberger, then it is called non-Rothberger gap.

The proof of Theorem 2.6 gives a condition of a destructibility of Rothberger gap.

Corollary 3.15. *Under the notation of Theorem 2.6, for a forcing notion $\mathbb{P}$ and a gap $\langle a_\alpha, b_n; \alpha < \kappa, n \in \omega \rangle$ with $b_{n+1} \subseteq b_n$ for every $n \in \omega$, the following statements are equivalent:*

(i) $\langle a_\alpha, b_n; \alpha < \kappa, n \in \omega \rangle$ does not form a gap in the extension with $\mathbb{P}$, and

(ii) $\mathbb{P}$ adds a real which dominates all $h_{\widetilde{a_\alpha}}$, $\alpha < \kappa$. □

Thus adding a dominating real, for example Laver forcing or Mathias forcing (both of which are not ccc), collapses any Rothberger gap in the ground model.

In the rest of this section, we will consider destructibility of non-Rothberger gaps. At first, we will present two proofs using facts of basic forcing notions in the previous section.

Lemma 3.16 ([41]). *Assume that $(\mathcal{A}, \mathcal{B})$is a (κ, λ)-gap where $\kappa, \lambda \geq \omega_1$ and $\mathbb{P}$ is a σ-centered forcing notion. Then $(\mathcal{A}, \mathcal{B})$ is $\mathbb{P}$-survivable.*

Proof. By Lemma 3.3, $\mathsf{Laver}(\mathcal{A}, \mathcal{B})$ is not σ-centered (in the ground model). Since $\mathbb{P} * \mathsf{Laver}(\mathcal{A}, \mathcal{B})$ and $\mathbb{P} \times \mathsf{Laver}(\mathcal{A}, \mathcal{B})$ are equivalent and now $\mathbb{P}$ is σ-centered, $\mathsf{Laver}(\mathcal{A}, \mathcal{B})$ is not σ-centered in the extension with $\mathbb{P}$. Therefore by Lemma 3.3 again, $(\mathcal{A}, \mathcal{B})$ is still a gap in the extension with $\mathbb{P}$. □

Lemma 3.17 ([41]). *Assume that $(\mathcal{A}, \mathcal{B})$is an (ω_1, ω_1)-gap and $\mathbb{P}$ has the productive ccc. Then $(\mathcal{A}, \mathcal{B})$ is $\mathbb{P}$-survivable.*

Proof. By Lemma 3.11, $\mathsf{Freezing}_{\mathsf{Kunen}}(\mathcal{A},\mathcal{B})$ has the ccc. Since

$$\mathbb{P} * \mathsf{Freezing}_{\mathsf{Kunen}}(\mathcal{A},\mathcal{B}) \cong \mathbb{P} \times \mathsf{Freezing}_{\mathsf{Kunen}}(\mathcal{A},\mathcal{B})$$

also has the ccc, $\mathsf{Freezing}_{\mathsf{Kunen}}(\mathcal{A},\mathcal{B})$ still has the ccc in the extension with $\mathbb{P}$, hence $(\mathcal{A},\mathcal{B})$ still forms a gap. □

The following lemma can be adopted for forcing notions having other variations of the property K. For example, if $\mathbb{P}$ has the strong property K for cardinality $\kappa \geq \aleph_1$, any (κ,κ)-gap is $\mathbb{P}$-survivable.

Lemma 3.18 ([41]). *Assume that $(\mathcal{A},\mathcal{B})$ is an (ω_1,ω_1)-gap and $\mathbb{P}$ has the property K. Then $(\mathcal{A},\mathcal{B})$ is $\mathbb{P}$-survivable.*

Proof. Assume $\langle a_\alpha, b_\alpha; \alpha < \omega_1\rangle$ is a gap and there is a $\mathbb{P}$-name $\dot{x}$ such that

$$\Vdash \text{“ } \dot{x} \text{ separates } \langle a_\alpha, b_\alpha; \alpha < \omega_1\rangle \text{ . ”}$$

For $\alpha < \omega_1$, we can find $p_\alpha \in \mathbb{P}$ and $n_\alpha \in \omega$ such that

$$p_\alpha \Vdash \text{“ } \check{a_\alpha} \cap \dot{x} \subseteq \check{n_\alpha} \ \& \ \check{b_\alpha} \setminus \check{n_\alpha} \subseteq \dot{x} \text{ ”}.$$

By the property K of $\mathbb{P}$, there is an uncountable X of ω_1 such that $\{p_\alpha; \alpha \in X\}$ is linked and all n_α for $\alpha \in X$ are equal to n. Then for any $\alpha, \beta \in X$, $((a_\alpha \cup a_\beta) \cap (b_\alpha \cup b_\beta)) \setminus n$ is empty. Thus $\bigcup_{\alpha \in X} b_\alpha \setminus n$ separates $\langle a_\alpha, b_\alpha; \alpha < \omega_1\rangle$ which is a contradiction. □

From now on we will look at survivability of non-Rothberger gaps. Then we will use the following notation.

Notation 3.19. *Let $\mathbb{P}$ be a forcing notion, $p \in \mathbb{P}$ and $\dot{x}$ a $\mathbb{P}$ name for a real (an infinite subset of natural numbers). We let*

$$u_{\mathbb{P}}(\dot{x},p)(\text{ or } u(\dot{x},p)) := \left\{k \in \omega; p \Vdash \text{“ } \check{k} \in \dot{x} \text{ ”}\right\},$$

and

$$v_{\mathbb{P}}(\dot{x},p)(\text{ or } v(\dot{x},p)) := \left\{k \in \omega; \exists q \leq p(q \Vdash \text{“ } \check{k} \in \dot{x} \text{ ”})\right\}.$$

We remark that the following statements hold:

- If $p \leq q$, then $u(\dot{x},p) \supseteq u(\dot{x},q)$ and $v(\dot{x},p) \subseteq v(\dot{x},q)$.
- $p \Vdash \text{“ } \check{u}(\dot{x},p) \subseteq \dot{x} \subseteq \check{v}(\dot{x},p) \text{ ”}$.

- $\Vdash$" $\dot{x} = \bigcup_{r \in \dot{G}} \check{u}(\dot{x}, r) = \bigcap_{r \in \dot{G}} \check{v}(\dot{x}, r)$ ".

Lemma 3.20 **([41]).** *Let $\mathbb{P}$ be a forcing notion of size $\leq \kappa$ and $(\mathcal{A}, \mathcal{B})$ a gap such that either $\mathcal{A}$ or $\mathcal{B}$ is $< \kappa^+$-directed, i.e. for any subfamily $\mathcal{F}$ of $\mathcal{A}$ (or $\mathcal{B}$) of size $< \kappa^+$, there is a $c \in \mathcal{A}$ (or $\mathcal{B}$) so that $x \subseteq^* c$ for all $x \in c$. Then $(\mathcal{A}, \mathcal{B})$ is $\mathbb{P}$-survivable.*

Proof. Assume that $\mathcal{A}$ is $< \kappa^+$-directed and $(\mathcal{A}, \mathcal{B})$ does not form a gap in the extension with $\mathbb{P}$, i.e. there exists a $\mathbb{P}$-name $\dot{x}$ such that

$$\Vdash\text{“ } \forall a \in \mathcal{A} \forall b \in \mathcal{B} (a \perp \dot{x} \ \&\ b \subseteq^* \dot{x}) \text{ ”}.$$

By the above remark, we note that for all conditions p of $\mathbb{P}$, $v(\dot{x}, p) \notin \mathcal{A}^\perp$. (Because if $v(\dot{x}, p) \in \mathcal{A}^\perp$ for some $p \in \mathbb{P}$, then $b \not\subseteq^* v(\dot{x}, p)$ holds for some $b \in \mathcal{B}$ since $(\mathcal{A}, \mathcal{B})$ forms a gap. Hence then $p \Vdash$ " $\check{b} \not\subseteq^* \dot{x}$ " which is a contradiction.) So for $p \in \mathbb{P}$, we can find $a_p \in \mathcal{A}$ with $v(\dot{x}, p) \perp a_p$. By the $< \kappa^*$-directedness of $\mathcal{A}$, there exists $a \in \mathcal{A}$ such that $a_p \subseteq^* a$ for all $p \in \mathbb{P}$. Then for all $p \in \mathbb{P}$, $v(\dot{x}, p) \not\perp a$ holds. Therefore $\Vdash$" $\dot{x} \not\perp \check{a}$ " holds because if not, i.e. for some $q \in \mathbb{P}$ which forces that" $\dot{x} \cap \check{a} \subseteq \check{n}$ for some $n \in \omega$, then it follows that $v(\dot{x}, q) \perp a$ which is a contradiction. □

Lemma 3.21 **([41]).** *Any σ-linked forcing notion does not collapse any non-Rothberger gap.*

Proof. Suppose that $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\beta; \alpha < \kappa, \beta < \lambda \rangle$ is a gap with $\omega < \kappa \leq \lambda$ regular, $\mathbb{P}$ is σ-linked, and $\dot{c}$ is a $\mathbb{P}$-name for a real such that

$$\Vdash\text{“ } \forall a \in \mathcal{A} \forall b \in \mathcal{B} (\check{a} \perp \dot{c} \ \&\ \check{b} \subseteq^* \dot{c}) \text{ ”}.$$

There are $l \in \omega$, $X \in [\kappa]^\kappa$ and $p_\gamma \in \mathbb{P}$ for each $\gamma \in X$ such that for all $\gamma \in X$,

$$p_\gamma \Vdash\text{“ } \check{a}_\gamma \cap \dot{c} \subseteq \check{l} \text{ ”}.$$

After that, for each $\gamma \in X$ we choose $m^\gamma \geq l$, $Y \in [\lambda]^\lambda$ and $q_\xi^\gamma \leq p_\gamma$ for every $\xi \in Y$ such that for all $\xi \in Y$,

$$q_\xi^\gamma \Vdash\text{“ } \check{b}_\xi \setminus m^\gamma \subseteq \dot{c} \text{ ”}.$$

Then there are $X' \in [X]^\kappa$ and $n \geq l$ with $m^\gamma \leq n$ for every $\gamma \in X'$.

Let $\mathbb{P} = \bigcup_{k \in \omega} P_k$, where all P_k is σ-linked. For each $\gamma \in X'$, we can find $k_\gamma \in \omega$ and $Y'_\gamma \in [Y]^\lambda$ such that $q_\xi^\gamma \in P_{k_\gamma}$ for any $\xi \in Y'_\gamma$. Then there is

$X'' \in [X']^\kappa$ and $k \in \omega$ with $k = k^\gamma$ for all $\gamma \in X''$. Then for any $\gamma, \delta \in X''$ and $\xi \in Y'_\gamma$, p_δ and q^γ_ξ are compatible because both q^γ_ξ and q^δ_ξ are in P_k, i.e. both of them are compatible and $q^\delta_\xi \leq p_\delta$. It follows that for every $\delta \in X''$ and $\xi \in \bigcup_{\gamma\in X''} Y'_\gamma$, $a_\delta \cap b_\xi \subseteq n$. Then we let $d = \omega \setminus \left(\bigcup_{\gamma\in X''} a_\gamma \cup n\right)$ which separates the gap, so it contradicts. □

In [35], Laflamme has pointed out that Mathias forcing and Matet forcing do not collapse any (ω_1, ω_1)-gap by the argument from Baumgartner (see also [11]). We give two proofs: for Sacks forcing and Laver forcing. Both proofs are similar to Baumgartner's argument, but the proof for Sacks forcing is simpler than the original one. Both of them consist of trees, so to explain the proofs below, we give some notation.

Definition and Notation 3.22. *1. For a subset T of $2^{<\omega}$ (or $\omega^{<\omega}$), T is called a tree if it is closed under taking initial segments.*

2. *For a tree $T \subseteq 2^{<\omega}$ (or $\omega^{<\omega}$) and $\sigma \in 2^{<\omega}$ (or $\omega^{<\omega}$), let*

$$T_\sigma := \{t \in T; t \subseteq \sigma \land \sigma \subseteq t\},$$

where $[\sigma] := \{x \in 2^\omega; \sigma \subseteq x\}$ (or $[\sigma] := \{x \in \omega^\omega; \sigma \subseteq x\}$) and

$$[T] := \{f \in 2^\omega (\text{or } \omega^\omega); \forall n \in \omega(f \restriction n \in T)\},$$

where $f \restriction n = f \cap (n \times \omega)$.

3. *For a tree $T \subseteq 2^{<\omega}$, $\sigma \in T$ is called a splitting node if both $\sigma^\frown\langle 0\rangle$ and $\sigma^\frown\langle 1\rangle$ are in T. For a tree $T \subseteq \omega^{<\omega}$, $\sigma \in T$ is called a splitting node if $\sigma^\frown\langle k\rangle \in T$ for at least two k's in ω. In both of cases, $\mathsf{split}_n(T)$ denotes the set of n-th splitting nodes and $\mathsf{Split}(T)$ denotes $\bigcup_{n\in\omega} \mathsf{split}_n(T)$. For all $t \in \mathsf{Split}(T)$, $\mathsf{succ}(T, t)$ denotes the set $\{k \in \omega; t^\frown\langle k\rangle \in T\}$.*

4. *A tree $T \subseteq 2^{<\omega}$ is called perfect if $\forall\sigma \in T\exists\tau, \nu \in T(\sigma \subseteq \tau, \nu \land \tau \perp \nu)$, where $\tau \perp \nu$ means that $\tau \not\subseteq \nu$ and $\nu \not\subseteq \tau$, that is, τ and ν are incomparable.*

   Sacks forcing $\mathbb{S}$ consists of perfect trees on $2^{<\omega}$, ordered by inclusion.

5. *A tree $T \subseteq \omega^{<\omega}$ is called Laver if there exists a node $s \in T$, called its stem $\mathsf{stem}(T)$, such that*

(a) for all $t \in T$, either $s \subseteq t$ or $t \subseteq t$, and

(b) for all $t \in T$ with $s \subseteq t$, $\mathsf{succ}(T, t)$ is infinite.

Laver forcing $\mathbb{L}$ consists of Laver trees on $\omega^{<\omega}$, ordered by inclusion.

Lemma 3.23. *Sacks forcing does not collapse any gap.*

Proof. By Corollary 3.15, Sacks forcing does not collapse any Rothberger gap because Sacks forcing is ω^ω-bounding.

In this proof (and the proof of the next Lemma) we identify $[\omega]^\omega$ with 2^ω, i.e. identify an infinite subset of ω with a characteristic function. Let $(\mathcal{A}, \mathcal{B})$ be a gap and both $\mathcal{A}$ and $\mathcal{B}$ are σ-directed and assume that $\mathbb{S}$ collapses it, i.e. there are $T \in \mathbb{S}$ and an $\mathbb{S}$-name $\dot{x}$ such that $T \Vdash$ "$\dot{x}$ separates $(\check{\mathcal{A}}, \check{\mathcal{B}})$". For $t \in T$, write $T_t := \{s \in T; s \subseteq t \text{ or } t \subseteq s\}$. By strengthening T if need, we may assume that there is an order preserving $\varphi : 2^{<\omega} \to 2^{<\omega}$ such that

$$T \Vdash \text{“} \dot{x} = \bigcap_{t \in \mathsf{Split}(T) \cap \bigcap \dot{G}} \varphi(t) \text{”},$$

that is, T_t decides the value of $\dot{x} \restriction |\varphi(t)|$ to $\varphi(t)$ for every $t \in \mathsf{Split}(T)$.

Claim 3. *There are $T' \leq T$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that for any $n \in \omega$ and $t \in \mathsf{split}_n(T')$, either $(a \cap u(\dot{x}, T_t)) \setminus n$ or $b \setminus (v(\dot{x}, T_t) \cup n)$ is not empty.*

Proof of claim. Since $\mathsf{Split}(T)$ is countable and both $\mathcal{A}$ and $\mathcal{B}$ are σ-directed, there are $a \in \mathcal{A}$ and $b \in \mathcal{B}$ so that for every $t \in \mathsf{Split}(T)$, either $a \not\perp v(\dot{x}, T_t)$ or $b \not\subseteq^* v(\dot{x}, T_t)$ holds. We note that

$$a \not\perp v(\dot{x}, T_t) \iff \exists^\infty k \exists s \supseteq t (k \in a \cap u(\dot{x}, T_s))$$

and

$$b \not\subseteq^* v(\dot{x}, T_t) \iff \exists^\infty k (k \in b \setminus v(\dot{x}, T_t)).$$

Recursively, we can construct $t_\sigma \in \mathsf{Split}(T)$ and $k_\sigma \in \omega$ for each $\sigma \in 2^{<\omega}$ such that

- if $\sigma \subsetneq \tau$, then $t_\sigma \subsetneq t_\tau$,
- $k_\sigma > |\sigma|$, and
- either $k_\sigma \in a \cap u(\dot{x}, T_{t_\sigma})$ or $k_\sigma \in b \setminus u(\dot{x}, T_{t_\sigma})$.

Let T' be the downward closure of all T_σ's, which is an extension of T and as desired. (We notice that $\mathsf{split}_n(T') = \{t_\sigma; \sigma \in 2^n\}$ for all $n \in \omega$.) $\dashv$

Therefore by strengthening T if need again, we may assume that there are $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that for every $n \in \omega$ and $t \in \mathsf{split}_n(T)$ either $(a \cap u(\dot{x}, T_t)) \smallsetminus n$ or $b \smallsetminus (v(\dot{x}, T_t) \cup n)$ is not empty.

Take a stronger condition $S \leq T$ and $n \in \omega$ such that $S \Vdash$ " $\check{a} \cap \dot{x} \subseteq \check{n}$ & $\check{b} \smallsetminus \check{n} \subseteq \dot{x}$ ". Take any $t \in S \cap \mathsf{split}_n(T)$, then either $(a \cap u(\dot{x}, T_t)) \smallsetminus n$ or $b \smallsetminus (v(\dot{x}, T_t) \cup n)$ is not empty. Look at S_t which leads us to a contradiction. □

We notice that the above proof can be adopted for other non-ccc forcing notions: Miller forcing, Silver forcing and their countable support iterations. So we have the following statement:

Corollary 3.24. *Assume that the ground model satisfies* CH. *Then in its extensions with countable support iterations of Sacks forcing, Silver forcing or Miller forcing, the gap spectra are* $\{(\omega, \omega_1),\ (\omega_1, \omega),\ (\omega_1, \omega_1)\}$.

Proof. In the ground model, there are only three types of gaps as the corollary. Since both of these forcing notions and these iterations preserve unbounded families, it follows from a standard Löwenhime-Skolem argument. □

Lemma 3.25. *Laver forcing does not collapse any non-Rothberger gap.*

Proof. Assume $(\mathcal{A}, \mathcal{B})$ be a gap, both $\mathcal{A}$ and $\mathcal{B}$ are σ-directed and there are $T \in \mathbb{L}$ and an $\mathbb{L}$-name $\dot{x}$ such that $T \Vdash$" $\dot{x}$ splits $(\check{\mathcal{A}}, \check{\mathcal{B}})$ ".

Without loss of generality, we may assume that T consists of strictly increasing finite sequences of natural numbers. For $t \in \mathsf{Split}(T)$ and $n \in \omega$,

$$T_{t,n} := \bigcup \left\{ T_{t^\frown \langle k \rangle}; k \in \mathsf{Succ}(T, t)\ \&\ k \geq n \right\}.$$

Claim 4. *There is an extension T' of T such that for all $k \in \omega$ and an extension $S \leq T'$, if S decides " $\check{k} \in \dot{x}$ ", then there exists $m \in \omega$ so that $T'_{\mathsf{stem}(S), m}$ decides " $\check{k} \in \dot{x}$ ",*

Proof of claim. For a given $T \in \mathbb{L}$ and $t \in T$ with $t \supseteq \mathsf{stem}(T)$, we can find $S \leq_0 T$ such that for any $k \in \omega$ there exists $n \in \omega$ so that $S_{t,n}$ decides " $\check{k} \in \dot{x}$ ". Because we can construct $\langle a_i; i \in \omega \rangle \subseteq \mathsf{succ}(T, t)$ and $T \geq_0 T_0 \geq_0 T_1 \geq_0 \cdots$ such that

- $a_i = \min(\mathsf{succ}(T_i, t))$ for every $i \in \omega$,

- $a_i < a_{i+1}$ for every $i \in \omega$, and
- T_i decides "$\check{k} \in \dot{x}$".

Then $S := \bigcup_{n\in\omega}(T_n)_{t^\frown\langle a_n\rangle}$ is as desired.

Using this, we construct a fusion sequence $\langle T_n; n \in \omega\rangle$ such that

- $T_0 \geq_0 T_0 \geq_1 T_1 \geq_2 T_2 \geq_3 \cdots$, and
- for every $n \in \omega$, $t \in \mathsf{split}_n(T_n)$ and $k \in \omega$, there exists $m \in \omega$ so that $T_{t,m}$ decides "$\check{k} \in \dot{x}$".

Letting T' be a fusion of $\langle T_n; n \in \omega\rangle$ which is as desired. ⊣

So without loss of generality, we may assume that T satisfies the above claim.

Claim 5. *There is an extension T' of T such that for all $t \in \mathsf{Split}(T')$, $F_t := \bigcup_{m\in\omega} u(T_{t,m}, \dot{x})$ is finite.*

Proof of claim. Assume that for any $S \leq T$ we can find $t \in S$ with F_t finite. Since there are only countably many different F_t's and both $\mathcal{A}$ and $\mathcal{B}$ are σ-directed, there are $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that either $a \not\perp F_t$ or $b \not\subseteq^* F_t$ for any possible $t \in T$ with F_t infinite. Let $S \leq T$ and $n \in \omega$ be such that

$$S \Vdash \text{"}\, \check{a} \cap \dot{x} \subseteq \check{n} \ \&\ \check{b} \setminus n \subseteq \dot{x} \,\text{"},$$

and let $t \in S$ satisfy that F_t infinite. Then we have two cases: $a \not\perp F_t$ or $b \not\subseteq^* F_t$.

If $a \not\perp F_t$, then there exists $k \in a \cap F_t$. Then there is $m \in \omega$ such that

$$T_{t,m} \Vdash \text{"}\, \check{k} \in \dot{x} \,\text{"},$$

hence

$$S_{t,m} \Vdash \text{"}\, \check{k} \in (\check{a} \cap \dot{x}) \setminus \check{n} \,\text{"},$$

which is a contradiction.

If $b \not\subseteq^* F_t$, The for all $m \in \omega$,

$$T_{t,m} \not\Vdash \text{"}\, \check{k} \in \dot{x} \,\text{"}.$$

Since

$$S \Vdash \text{"}\, \check{k} \in \check{b} \setminus \check{n} \subseteq \dot{x} \,\text{"},$$

by the condition of Claim 4, there is $m \in \omega$ such that

$$T_{\mathsf{stem}(S),m} \Vdash \text{“ } \check{k} \in \dot{x} \text{ ”},$$

which is a contradiction because S and $T_{\mathsf{stem}(T),m}$ are compatible. $\dashv$

So without loss of generality, we may assume that T also satisfies the above claim.

Claim 6. *There is an extension T' of T such that for all $t \in \mathsf{Split}(T')$, $T'_t \Vdash$ “ $\check{F}_t \subseteq \dot{x}$ ”.*

Proof of claim. We can construct a fusion sequence $\langle T_n; n \in \omega \rangle$ such that for every $n \in \omega$ and $t \in \mathsf{split}_n(T_n)$,

$$(T_n)_t \Vdash \text{“ } \check{F}_t \subseteq \dot{x} \text{ ”}.$$

This can be done because all F_t are finite. Then its fusion is as desired. $\dashv$

So without loss of generality again, we may also assume that T satisfies the above claim.

Claim 7. *There is an extension T' of T such that for all $t \in \mathsf{Split}(T')$,*

either (i) $\forall m \in \mathsf{succ}(T', t)(F_{t^\frown\langle m \rangle} = F_t)$

or (ii) $\exists a \in \mathcal{A} \exists b \in \mathcal{B} \forall n \in \omega \exists m \in \mathsf{succ}(T', t)$: either $(a \cap F_{t^\frown\langle m \rangle}) \setminus n \neq \emptyset$ or $b \setminus (F_{t^\frown\langle m \rangle} \cup n) \neq \emptyset$.

Proof of claim. We recursively construct a fusion sequence $\langle t_n; n \in \omega \rangle$ with $T_0 := T$ such that every $t \in \mathsf{split}_{n+1}(T_n)$ satisfies either (i) or (ii) for all $n \in \omega$ as follows:

Having constructed T_n, we let $\mathsf{split}_{n+1}(T_n) = \{s_i; i \in \omega\}$. We consider following two cases.

Case 1. Assume

$$I := \left\{ i \in \omega; \bigcup_{m \in \mathsf{succ}(t_n, s_i)} F_{s_i^\frown\langle m \rangle} \text{ is finite} \right\}$$

is infinite. For each $i \in I$ and $k \in \bigcup_{m \in \mathsf{succ}(t_n, s_i)} F_{s_i^\frown\langle m \rangle}$, let

$$B_{i,k} := \left\{ m \in \mathsf{succ}(T_n, s_i); k \in F_{s_i^\frown\langle m \rangle} \right\},$$

and let

$$A_i := \mathsf{succ}(T_n, s_i) \setminus \bigcup \left\{ B_{i,k}; k \in \bigcup_{m \in \mathsf{succ}(t_n, s_i)} F_{s_i \frown \langle m \rangle} \ \& \ B_{i,k} \text{ is finite} \right\}.$$

By the condition of Claim 6, $F_{s_i} \subseteq \bigcup_{m \in A_i} F_{s_i \frown \langle m \rangle}$. And if $k \in F_{s_i \frown \langle m \rangle}$ for some $m \in A_i$, then also $m \in B_{i,k}$, hence $B_{i,k}$ is infinite. So

$$S := \bigcup_{m \in B_{i,k}} (T_n)_{s_i \frown \langle m \rangle}$$

is an extension of T and forces that" $\check{k} \in \dot{x}$ ". Thus, by the condition of Claim 4, there is $l \in \omega$ such that $T_{\mathsf{stem}(S),l}$ forces it. Now $\mathsf{stem}(S) = s_i$, so it follows that $k \in F_{s_i}$. Therefore, it can be concluded that $\bigcup_{m \in A_i} F_{s_i \frown \langle m \rangle} = F_{s_i}$, that is $F_{s_i} = F_{s_i \frown \langle m \rangle}$ holds for all $m \in A_i$. Let

$$T_{n+1} := \bigcup_{i \in I} \bigcup_{m \in A_i} (T_n)_{s_i \frown \langle m \rangle}.$$

Case 2. Assume I is finite. Let $l \in \omega$ satisfy that for all $i \geq l$, $\bigcup_{m \in \mathsf{succ}(t_n, s_i)} F_{s_i \frown \langle m \rangle}$ is infinite. Then we can find $a \in \mathcal{A}$ and $c \in \mathcal{B}$ such that either $a \not\perp \bigcup_{m \in \mathsf{succ}(t_n, s_i)} F_{s_i \frown \langle m \rangle}$ or $b \not\subseteq^* \bigcup_{m \in \mathsf{succ}(t_n, s_i)} F_{s_i \frown \langle m \rangle}$ for all $i \geq l$. For each $i \geq l$, we choose $A_i \in [\mathsf{succ}(T_n, s_i)]^\omega$ so that for any $j \in \omega$ and the j-th element m of A_i, there exists $k \geq j$ such that k is in either $a \cap F_{s_i \frown \langle m \rangle}$ or $b \setminus F_{s_i \frown \langle m \rangle}$. Let

$$T_{n+1} := \bigcup_{i \geq l} \bigcup_{m \in A_i} (T_n)_{s_i \frown \langle m \rangle}.$$

Let T' be its fusion which is as desired. ⊣

Since T is a countable set and both $\mathcal{A}$ and $\mathcal{B}$ are σ-centered, there exist $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that for $t \in \mathsf{Split}(T)$ satisfying (ii), a pair $\langle a, b \rangle$ witnesses it for this t.

Take $S \leq T$ and $n \in \omega$ such that $S \Vdash$" $\check{a} \cap \dot{x} \subseteq \check{n}$ & $b \setminus \check{n} \subseteq \dot{x}$ ". If all $t \in \mathsf{Split}(S)$ satisfy (i), then $S \Vdash$" $\dot{x} \subseteq \check{F}_t$" which is a contradiction because F_t is finite. So there is $t \in \mathsf{Split}(S)$ satisfying (ii). Take $m \in \mathsf{succ}(T, t)$ such that either $(a \cap F_{t \frown \langle m \rangle}) \setminus n \neq \emptyset$ or $b \setminus (F_{t \frown \langle m \rangle} \cup n) \neq \emptyset$. Look at $S_{t \frown \langle m \rangle, l}$ for some large enough $l \in \omega$ which leads us to a contradiction. □

The above proof can be adopted for Miller forcing, Mathias forcing, and their countable support iterations.

Corollary 3.26. *Assume that the ground model satisfies* CH. *Then in its extensions with countable support iterations of Laver forcing or Mathias forcing, the gap spectra is* $\{(\omega, \omega_2),\ (\omega_2, \omega),\ (\omega_1, \omega_1)\}$.

Proof. By an argument similar to Corollary 3.24 and remarks of Theorem 2.6 and Corollary 3.15. □

3.3 Destructible gaps

In [2], Abraham and Todorčević pointed out that a destructible gap is analogous to a Suslin tree. An ω_1-tree is called an Aronszajn tree if it has no uncountable chains, and called a Suslin tree if it has no uncountable chains and antichains. Let $\langle a_\alpha, b_\alpha; \alpha < \omega_1\rangle$ be an (ω_1, ω_1)-pregap with $a_\alpha \cap b_\alpha$ empty for every $\alpha < \omega_1$. Letting α and β in ω_1 be called compatible when $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha)$ is empty, $\langle a_\alpha, b_\alpha; \alpha < \omega_1\rangle$ forms a gap if there are no uncountable chains in ω_1 under this compatibility relation, and it is a destructible gap if there are no uncountable chains and antichains in ω_1. So there are similar theorems about them, e.g. both of Suslin trees and destructible gaps can be constructed from the diamond principle and by adding a Cohen real, and both of them do not exist under Martin's Axiom.

In [10], Devlin and Shelah presented an axiom called the weak diamond principle, which is formulated as a weaker version of the diamond principle and which is equivalent to the inequality $2^{\aleph_0} < 2^{\aleph_1}$. Parametrized diamond principles have been designed by Moore, Hrušák and Džamonja in [38] as weak diamond principles parametrized by cardinal invariants. Parametrized diamond principles can be deduced from the diamond principle and some of them are really weaker than the original one.

The diamond principle has many applications, e.g. it implies the Continuum Hypothesis, the existence of an Ostaszewski space, a Suslin tree or a destructible gap. Parametrized diamond principles also have many applications. For example, Suslin trees can be constructed from the diamond principle for the uniformity of the meager ideal $\diamondsuit(\mathsf{non}(\mathcal{M}))$. But it was left open which parametrized diamond principle can be used to construct a destructible gap. The main result in this section says that, as for a Suslin tree, a destructible gap can also be constructed from the diamond principle for the uniformity of the meager ideal.

3.3.1 Remarks on Suslin trees and destructible gaps

In [2], Abraham and Todorčević pointed out that there are many analogies between (ω_1, ω_1)-gaps and Aronszajn trees. In this section, we look at such things.

Definition 3.27. *1. An ω_1-tree is a tree of height ω_1 such that all levels are countable.*

2. An Aronszajn tree is an ω_1-tree having no uncountable chains.

3. A Suslin tree is an Aronszajn tree having no uncountable antichains.

The following tables show characteristics of trees and (ω_1, ω_1)-gaps.

	$T \subseteq {\omega_1}^{<\omega_1}$: ω_1-tree
Aronszajn tree	$\forall X \in [T]^{\omega_1} \exists s \neq t \in X (s \not\subseteq t \ \& \ t \not\subseteq s)$
Suslin tree	T is Aronszajn tree and $\forall X \in [T]^{\omega_1} \exists s \neq t \in X (s \subseteq t \text{ or } t \subseteq s)$

	$\langle a_\alpha, b_\alpha; \alpha < \omega_1 \rangle$: (ω_1, ω_1)-pregap with $a_\alpha \cap b_\alpha = \emptyset$
a gap	$\forall X \in [\omega_1]^{\omega_1} \exists \alpha \neq \beta \in X ((a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \emptyset)$
a destructible gap	$\langle a_\alpha, b_\alpha; \alpha < \omega_1 \rangle$ is a gap and $\forall X \in [\omega_1]^{\omega_1} \exists \alpha \neq \beta \in X ((a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset)$

As seen above, if we define compatibility of α and β by

$$(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset,$$

then $\langle a_\alpha, b_\alpha; \alpha < \omega_1 \rangle$ should be called a gap when there are no uncountable chains, and a destructible gap when there are no uncountable chains and no uncountable antichains when we consider a chain as a pairwise compatible set, and an antichain as a pairwise incompatible. But in the sense of trees, compatibility agree with comparability, on the other hand, this is not the case in the sense of gaps. This is why constructing gaps is more difficult than constructing trees.

We list classical results about ω_1-trees:

Theorem 3.28. 1 (Baumgartner–Malitz–Reinhardt). *Under $\mathsf{MA}_{\aleph_1}(ccc)$, every Aronszajn tree is special.*

2 (Shelah, Todorčević). *Cohen reals add Suslin trees.*

3 (Jensen, Moore-Hrušák-Džamonja). $\diamondsuit$ *implies the existence of Suslin trees. Moreover* $\diamondsuit(\mathsf{non}(\mathcal{M}))$ *implies the existence of Suslin trees.*

Special trees are not Suslin at all. Similar statements as above are true for gaps. It follows form Lemma 3.11 that every (ω_1, ω_1)-gaps is indestructible under $\mathsf{MA}_{\aleph_1}(\text{ccc})$. Other theorems will prove in the rest of this chapter.

3.3.2 Todorčević's results on destructible gaps

In this section, we present two classical results due to Todorčević similar to 2 and 3 of Theorem 3.28.

Theorem 3.29 (Todorčević, [46]). *Let* $\langle a_\alpha, b_\alpha; \alpha < \omega_1\rangle$ *be a gap with* $a_\alpha \cap b_\alpha$ *empty for every* $\alpha < \omega_1$. *and* c *a Cohen real over the ground model* $\mathbf{V}$. *Then in* $\mathbf{V}[c]$, $\langle a_\alpha \cap c, b_\alpha \cap c; \alpha < \omega_1\rangle$ *is a destructible gap.*

Proof. Let $p \in \mathbb{C}$ and $\dot{X}$ a $\mathbb{C}$-name for an uncountable subset of ω_1.

Claim 8. *There are* $q \leq p$ *and* $Y \in [\omega_1]^{\omega_1}$ *such that*

$$q \Vdash \text{“}\, \check{Y} \subseteq \dot{X} \,\text{”}.$$

Proof of claim. By recursion on $\gamma < \omega_1$, we can construct $q_\gamma \leq p$ and $o_\gamma < \omega_1$ such that

- if $\gamma < \gamma' < \omega_1$, then $o_\gamma < o_{\gamma'}$, and
- $q_\gamma \Vdash$“ $\check{o}_\gamma \in \dot{X}$ ”.

We choose $I \in [\omega_1]^{\omega_1}$ and $q \in \mathbb{C}$ so that $q_\gamma = q$ for all $\gamma \in I$. Let $Y := \{o_\gamma; \gamma \in I\}$. ⊣

By shrinking Y if need, there are $s, t \in 2^{\mathrm{dom}(q)}$ with $a_\alpha \cap \mathrm{dom}(q) = s$ and $b_\alpha \cap \mathrm{dom}(q) = t$ for every $\alpha \in Y$. (We notice that then $s \cap t$ is empty.) Since Y is uncountable and $\langle a_\alpha \setminus \mathrm{dom}(q), b_\alpha \setminus \mathrm{dom}(q); \alpha \in Y\rangle$ is also a gap, we can find distinct ξ and $\eta \in Y$ with

$$u := \big((a_\xi \cap b_\eta) \cup (a_\eta \cap b_\xi)\big) \setminus \mathrm{dom}(q) \text{ is non-empty.}$$

Let $r \leq q$ be with $r(i) = 1$ for some $i \in u$. Then

$$r \Vdash \text{“ } \left((\check{a_\xi} \cap \check{b_\eta}) \cup (\check{a_\eta} \cap \check{b_\xi})\right) \cap \dot{c} \neq \emptyset \text{ ”}.$$

Therefore we can conclude that

$$\Vdash \text{“ } \left\langle \check{a_\alpha} \cap \dot{c}, \check{b_\alpha} \cap \dot{c}; \alpha < \omega_1 \right\rangle \text{ forms a gap ”}.$$

Let $r \leq q$ be with $r(i) = 0$ for all $i \in u$. Then

$$r \Vdash \text{“ } \left((\check{a_\xi} \cap \check{b_\eta}) \cup (\check{a_\eta} \cap \check{b_\xi})\right) \cap \dot{c} = \emptyset \text{ ”}.$$

Therefore we can also conclude that

$$\Vdash \text{“ } \left\langle \check{a_\alpha} \cap \dot{c}, \check{b_\alpha} \cap \dot{c}; \alpha < \omega_1 \right\rangle \text{ is destructible ”}.$$

□

Theorem 3.30 (Todorčević, see e.g. [12]**).** *$\diamondsuit$ implies the existence of destructible gaps.*

Proof. Let $\langle D_\alpha; \alpha < \omega_1\rangle$ and $\langle \mathcal{D}_\alpha; \alpha < \omega_1\rangle$ be diamond sequences on ω_1 and $[\omega_1]^{<\omega}$ respectively, i.e.

- $D_\alpha \subseteq \alpha$ for all $\alpha < \omega_1$ and for all $E \subseteq \omega_1$, $\{\alpha \in \omega_1; E \cap \alpha = D_\alpha\}$ is stationary on ω_1, and
- $\mathcal{D}_\alpha \subseteq [\alpha]^{<\omega}$ for all $\alpha < \omega_1$ and for all $\mathcal{E} \subseteq [\omega_1]^{<\omega}$, $\{\alpha \in \omega_1; \mathcal{E} \cap [\alpha]^{<\omega} = \mathcal{D}_\alpha\}$ is stationary on ω_1.

We recursively construct $\langle a_\alpha, b_\alpha; \alpha < \omega_1\rangle$ such that

1. $a_\alpha \cap b_\alpha = \emptyset$ and $\omega \setminus (a_\alpha \cup b_\alpha)$ is infinite for all $\alpha < \omega_1$,
2. if $\beta \leq \alpha < \omega_1$, then both $a_\beta \subseteq^* a_\alpha$ and $b_\beta \subseteq^* b_\alpha$,
3. for a limit ordinal $\beta < \alpha$, if $\mathcal{D}_\beta$ satisfies that

$$(*)_\beta \quad \forall \gamma < \beta \exists \delta \in D_\beta((a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) \neq \emptyset),$$

then for all $l \in \omega$ and $\delta \in D_\beta$ such that

$$\left((a_\alpha \setminus l) \cap b_\delta\right) \cup \left(a_\delta \cap (b_\alpha \setminus l)\right) \neq \emptyset,$$

4. for a limit ordinal $\beta < \alpha$, if $\mathcal{D}_\beta$ satisfies that

$$(**)_\beta \quad \forall F \in [\beta]^{<\omega}(A_F \cap B_F = \emptyset \Rightarrow \exists G \in \mathcal{D}_\beta(A_{F\cup G} \cap B_{F\cup G} = \emptyset)),$$

(where $A_F := \bigcup_{\gamma\in F} a_\gamma$ and $B_F := \bigcup_{\gamma\in F} b_\gamma$) then for any $F \in [\alpha]^{<\omega}$ and $H, I, J, K \in [\omega]^{<\omega}$ with

$$(A_G \cup H \cup (a_\alpha \setminus I)) \cap (B_G \cup J \cup (b_\alpha \setminus K)) = \emptyset,$$

there is $G \in \mathcal{D}_\beta$ such that

$$(A_{F\cup G} \cup H \cup (a_\alpha \setminus I)) \cap (B_{F\cup G} \cup J \cup (b_\alpha \setminus K)) = \emptyset,$$

5. if α is a limit ordinal and $\beta < \alpha$, then for any H and $J \in [\omega]^{<\omega}$ with $H \cap J = \emptyset$ and any $i > \max(H \cup J)$, we can find $\xi < \alpha$ such that

$$a_\xi \cap i = H \ \& \ a_\xi \setminus i = a_\beta \setminus i \ \&$$

$$b_\xi \cap i = J \ \& \ b_\xi \setminus i = b_\beta \setminus i.$$

Construction. Let $a_0 := \{3n; n \in \omega\}$ and $b_0 := \{3n+1; n \in \omega\}$. For a limit ordinal $\alpha < \omega_1$, we construct $a_{\alpha+n}$ and $b_{\alpha+n}$ satisfying conditions 1,2 and 5. (For example, letting $\langle\langle H_n, J_n\rangle; n \in \omega\rangle$ enumerate $\{\langle H, J,\rangle \in [\omega]^{<\omega} \times [\omega]^{<\omega}; H \cap J = \emptyset\}$, we put

$$a_{\alpha+n} := H \cup (a_\alpha \setminus J)$$

and

$$b_{\alpha+n} := J \cup (b_\alpha \setminus H).)$$

Then conditions 3 and 4 also holds for every $\alpha + n$, $n \in \omega$:

For 3. Let $\beta < \alpha$ $(\alpha+n)$ be a limit ordinal and suppose that $(*)_\beta$ holds. We can find $k \in \omega$ so that $a_{\alpha+n} \setminus k = a_\alpha \setminus k$ and $b_{\alpha+n} \setminus k = b_\alpha \setminus k$. By the inductive hypothesis, for any $l \geq k$, we can find δ such that

$$\big((a_\alpha \setminus l) \cap b_\delta\big) \cup \big(a_\delta \cap (b_\alpha \setminus l)\big) \neq \emptyset.$$

Then this δ is also a witness for $\alpha + n$.

For 4. Let $\beta < \alpha$ $(\alpha + n)$ be a limit ordinal and suppose that $(**)_\beta$ holds. Moreover assume that $(A_F \cup H \cup (a_{\alpha+n} \setminus I)) \cap (B_F \cup J \cup (b_{\alpha+n} \setminus K)) = \emptyset$ where $F \in [\alpha + n]^{<\omega}, L, M \in [\omega]^{<\omega}$, $a_{\alpha+n} = (a_\alpha \cup L) \setminus M$ and $b_{\alpha+n} = (b_\alpha \cup M) \setminus L$. Then $(A_F \cup (H \cup (L \setminus (I \cup M))) \cup (a_\alpha \setminus (I \cup M))) \cap (B_F \cup (J \cup (M \setminus (K \cup L))) \cup (b_\alpha \setminus (K \cup L))) = \emptyset$, so there exists $G \in \mathcal{D}_\beta$ such that

$$(A_{F\cup G} \cup \underbrace{(H \cup (L \setminus (I \cup M))) \cup (a_\alpha \setminus (I \cup M))}_{H\cup(a_{\alpha+n}\setminus I)}$$
$$\cap(B_{F\cup G} \cup \underbrace{(J \cup (M \setminus (K \cup L))) \cup (b_\alpha \setminus (K \cup L))}_{J\cup(b_{\alpha+n}\setminus K)} = \emptyset.$$

Let $\alpha \in \omega_1$ be a limit ordinal. At first we enumerate

$$(\mathsf{Lim} \cap \alpha) \times [\alpha]^{<\omega} \times ([\omega]^{<\omega})^4 = \{\langle \beta_k, F_k, H_k, I_k, J_k, K_k \rangle ; k \in \omega\}.$$

To construct a_α and b_α, we construct a cofinal sequence $\langle \zeta_k; k \in \omega \rangle$ of α and natural numbers i_k, j_k, with $i_k < j_k < i_{k+1}$ as follows, then we define $a_\alpha := \bigcup_{k\in\omega} a_{\zeta_k}$ and $b_\alpha := \bigcup_{k\in\omega} b_{\zeta_k}$:

Assume that we have already constructed ζ_j, $j < k$ and let $\beta = \beta_k$, $F = F_k$, $H = H_k$, $I = I_k$, $J = J_k$ and $K = K_k$. (We put $\zeta_{-1} = 0$, $i_{-1} = j_{-1} = 0$.) If $(**)_\beta$ does not hold or $(A_F \cup H \cup (a_{\zeta_{k-1}} \setminus I)) \cap (B_F \cup J \cup (b_{\zeta_{k-1}} \setminus K)) \neq \emptyset$, then let $\zeta_k := \zeta_{k-1}$ and $j_k := j_{k-1}$. And moreover if $(*)_\beta$ holds, then by the inductive hypothesis for 3, there is $\delta \in D_\beta$ such that

$$\big((a_{\zeta_{k-1}} \setminus i_{k-1}) \cap b_\delta\big) \cup \big(a_\delta \cap (b_{\zeta_{k-1}} \setminus i_{k-1})\big) \neq \emptyset.$$

We let $i_k > i_{k-1}, J_{k-1}$ be so that

$$\Big(\big((a_{\zeta_{k-1}} \setminus i_{k-1}) \cap b_\delta\big) \cup \big(a_\delta \cap (b_{\zeta_{k-1}} \setminus i_{k-1})\big)\Big) \cap i_k \neq \emptyset.$$

If $(*)_\beta$ does not hold, then let $i_k := i_{k-1}$.

Assume that $\mathcal{D}_\beta$ satisfies $(**)_\beta$ and $(A_F \cup H \cup (a_{\zeta_{k-1}} \setminus I)) \cap (B_F \cup J \cup (b_{\zeta_{k-1}} \setminus K)) = \emptyset$. If $(*)_\beta$ holds, then we choose $i_k > i_{k-1}, j_{k-1}$ such that

- $\forall j < k - 1(a_{\zeta_j} \setminus i_k \subseteq a_{\zeta_{k-1}} \ \&\ b_{\zeta_j} \setminus i_k \subseteq b_{\zeta_{k-1}})$,
- $|i_k \setminus (a_{\zeta_{k-1}} \cup b_{\zeta_{k-1}})| \geq k$, and

- $\Big(\big((a_{\zeta_{k-1}} \setminus i_{k-1}) \cap b_\delta\big) \cup \big(a_\delta \cap (b_{\zeta_{k-1}} \setminus i_{k-1})\big)\Big) \cap i_k \neq \emptyset$ holds for some $\delta \in D_\beta$.

If $(*)_\beta$ does not hold, then choose $i_k > i_{k-1}, j_{k-1}$ such that

- $\forall j < k-1 (a_{\zeta_j} \setminus i_k \subseteq a_{\zeta_{k-1}} \; \& \; b_{\zeta_j} \setminus i_k \subseteq b_{\zeta_{k-1}})$, and
- $|i_k \setminus (a_{\zeta_{k-1}} \cup b_{\zeta_{k-1}})| \geq k$.

By the condition 4 for $\beta < \max(F \cup \{\zeta_{k-1}\})$, F, H, I, J and K, there exists $G_k = G \in \mathcal{D}_\beta$ such that

$$(A_{F \cup G} \cup H \cup (a_{\zeta_{k-1}} \setminus I)) \cap (B_{F \cup G} \cup J \cup (b_{\zeta_{k-1}} \setminus K)) = \emptyset.$$

Let $\eta_k = \eta > \max(F \cup G \cup \{\zeta_{k-1}\})$ be large enough less than α. Then we choose $j_k > i_k, \max(H \cup I \cup J \cup K)$ such that

- $(A_{F \cup G} \cup a_{\zeta_{k-1}}) \setminus j_k \subseteq a_{\eta_k}$ and $(B_{F \cup G} \cup b_{\zeta_{k-1}}) \setminus j_k \subseteq b_{\eta_k}$.

After that, using the condition 5, we choose $\zeta_k < \alpha$ such that

- $a_{\zeta_k} \cap i_k = a_{\zeta_{k-1}} \cap i_k$, $a_{\zeta_k} \cap [i_k, j_k) = (A_{F \cup G} \cup a_{\zeta_{k-1}}) \cap [i_k, j_k)$ and $a_{\zeta_k} \setminus i_k = a_{\eta_k} \setminus i_k$, and
- $b_{\zeta_k} \cap i_k = b_{\zeta_{k-1}} \cap i_k$, $b_{\zeta_k} \cap [i_k, j_k) = (B_{F \cup G} \cup b_{\zeta_{k-1}}) \cap [i_k, j_k)$ and $b_{\zeta_k} \setminus i_k = b_{\eta_k} \setminus i_k$.

which completes the construction.

We check it works:

1. If there exists $m \in a_\alpha \cap b_\alpha$, then choosing $k \in \omega$ with $m < i_k$, it leads a contradiction because

$$m \in a_\alpha \cap b_\alpha \cap i_k = a_{\zeta_k} \cap b_{\zeta_k} \cap i_k \subseteq a_{\zeta_k} \cap b_{\zeta_k} = \emptyset.$$

   So $a_\alpha \cap b_\alpha$ is empty. And since

$$\forall k \in \omega \left(|i_k \setminus (a_{\zeta_k} \cup b_{\zeta_k})| \geq k \; \& \; a_\alpha \cap i_k = a_{\zeta_k} \cap i_k \; \& \; b_\alpha \cap i_k = b_{\zeta_k} \cap i_k\right),$$

   $\omega \setminus (a_\alpha \cup b_\alpha)$ is infinite.

2. It follows from the fact that $a_\alpha = \bigcup_{k \in \omega} a_{\zeta_k}$ and $\langle \zeta_k ; k \in \omega \rangle$ is cofinal in α.

3. It follows from the construction.

4. Assume $\beta < \alpha$ is a limit ordinal, $\mathcal{D}_\beta$ satisfies $(**)_\beta$ and $F \in [\alpha]^{<\omega}$, $H, I, J, K \in [\omega]^{<\omega}$ satisfy that

$$(A_F \cup H \cup (a_\alpha \setminus I)) \cap (B_F \cup J \cup (b_\alpha \setminus K)) = \emptyset.$$

Let $k \in \omega$ be such that $\beta_k = \beta$, $F_k = F$, $H_k = H$, $I_k = I$, $J_k = J$ and $K_k = K$. Since $a_{\zeta_{k-1}} \subseteq a_\alpha$ and $b_{\zeta_{k-1}} \subseteq b_\alpha$,

$$(A_F \cup H \cup (a_{\zeta_{k-1}} \setminus I)) \cap (B_F \cup J \cup (b_{\zeta_{k-1}} \setminus K)) = \emptyset.$$

So, letting $G := G_k$,

$$(A_{F\cup G} \cup H \cup (a_{\zeta_{k-1}} \setminus I)) \cap (B_{F\cup G} \cup J \cup (b_{\zeta_{k-1}} \setminus K)) = \emptyset.$$

Now we have

$$a_\alpha \setminus i_k \supseteq a_{\zeta_k} \setminus i_k \supseteq (A_{F\cup G} \cup a_{\zeta_{k-1}}) \setminus i_k \ \&\ b_\alpha \cap i_k = b_{\zeta_k - 1} \cap i_k.$$

Thus if there is an $m \in A_G \cap (b_\alpha \setminus K)$, then

$$\begin{aligned} \text{either} \quad & m \in (A_G \cap i_k) \setminus K \\ \text{or} \quad & m \in A_G \setminus i_k \subseteq a_\alpha, \end{aligned}$$

both of which leads contradictions. So $A_G \cap (b_\alpha \setminus K)$ is empty, similarly we have $B_G \cap (a_\alpha \setminus I)$ empty which conclude

$$(A_{F\cup G} \cup H \cup (a_\alpha \setminus I)) \cap (B_{F\cup G} \cup J \cup (b_\alpha \setminus K)) = \emptyset.$$

5. We have nothing to do.

From now on we check that $\langle a_\alpha, b_\alpha; \alpha < \omega_1 \rangle$ forms a gap, i.e. for all uncountable $X \subseteq \omega_1$, there exist distinct $\gamma, \delta \in X$ with $(a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma)$ non-empty:

Assume that $X \subseteq \omega_1$ is uncountable and pairwise compatible with the maximal sense, i.e.

$$\forall \gamma, \delta \in X((a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) = \emptyset), \text{ and}$$

$$\forall \gamma \in \omega_1 \exists \delta \in X((a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) \neq \emptyset).$$

Then we define $f : \omega_1 \to \omega_1$ such that

$$\forall \alpha \in \omega_1 \forall \gamma < \alpha \exists \delta \in X \cap f(\alpha)((a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) \neq \emptyset).$$

We note that $\{\alpha \in \omega_1; \forall \beta < \alpha(f(\beta) < \alpha)\}$ is club on ω_1, so there exists α in this set with $X \cap \alpha = D_\alpha$.

Take $\gamma \in X \setminus (\alpha + 1)$, and then we can find $l \in \omega$ with

$$a_{\alpha+1} \setminus l \subseteq a_\gamma \;\&\; b_{\alpha+1} \setminus l \subseteq b_\gamma.$$

Then by the condition 3, since now $(*)_\alpha$ holds, there is $\delta \in D_\alpha$ such that

$$\big((a_{\alpha+1} \setminus l) \cap b_\delta\big) \cup \big(a_\delta \cap (b_{\alpha+1} \setminus l)\big) \neq \emptyset.$$

Then

$$(a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) \neq \emptyset,$$

which is a contradiction because both γ and δ are in X.

By the similar argument, we show that $\langle a_\alpha, b_\alpha; \alpha < \omega_1 \rangle$ is destructible. To prove this, we denote

$$\mathsf{C} := \{F \in [\omega_1]^{<\omega}; A_F \cap B_F = \emptyset\}.$$

Then it suffices to show that for all uncountable $\mathcal{A} \subseteq \mathsf{C}$, there exist distinct $F, G \in \mathcal{A}$ with $A_{F\cup G} \cap B_{F\cup G}$ empty:

Assume that there exists an uncountable $\mathcal{A} \subseteq \mathsf{C}$ so that for any distinct $F, G \in \mathcal{A}$ with $A_{F\cup G} \cap B_{F\cup G}$ nonempty. Without loss of generality, we may assume that $\mathcal{A}$ is maximal with this property, i.e. $\mathcal{A}$ satisfies that

$$\forall F \neq G \in \mathcal{A}(A_{F\cup G} \cap B_{F\cup G} \neq \emptyset), \text{ and}$$

$$\forall F \in \mathsf{C} \exists G \in \mathcal{A}(A_{F\cup G} \cap B_{F\cup G} = \emptyset).$$

We note that

$$\{\alpha \in \mathcal{I}; \forall F \in \mathsf{C} \cap [\alpha]^{<\omega} \exists G \in \mathcal{A} \cap [\alpha]^{<\omega} \text{ so that } A_{F\cup G} \cap B_{F\cup G} = \emptyset\}$$

is club on ω_1. Thus there is α in this set with $\mathcal{A} \cap [\alpha]^{<\omega} = \mathcal{D}_\alpha$. It follows that

$$\forall F \in \mathsf{C} \cap [\alpha]^{<\omega} \exists G \in \mathcal{D}_\alpha \text{ so that } A_{F\cup G} \cap B_{F\cup G} = \emptyset,$$

i.e. $(**)_\alpha$ holds.

Take any $F \in \mathcal{A} \setminus [\alpha]^{<\omega}$, then by the construction (applying the condition 4 for $\alpha < \max(F)$, $F \setminus \{\max(F)\}$, $\emptyset$, $\emptyset$, $\emptyset$ and $\emptyset$), we can find $G \in \mathcal{D}_\alpha$ such that $A_{F\cup G} \cap B_{F\cup G} = \emptyset$ which is a contradiction because both F and G are in $\mathcal{A}$.

Therefore $\langle a_\alpha, b_\alpha \rangle_{\alpha\in\omega_1}$ is a destructible gap. □

3.3.3 The diamond principle for the uniformity of the meager ideal implies the existence of a destructible gap

In this section, we prove the title of this section, which answers a question addressed in [38], Question 5.8.

Theorem 3.31. $\diamondsuit(\mathsf{non}(\mathcal{M}))$ *implies the existence of destructible gaps.*

Proof. We recall that $\diamondsuit(\mathsf{non}(\mathcal{M}))$ is the following statement:

$$\forall \text{ Borel } \mathcal{F} : 2^{<\omega_1} \to \mathcal{M} \exists g : \omega_1 \to (\omega{\times}{\uparrow}\omega)^\omega \forall f : \omega_1 \to 2,$$
$$\{\alpha \in \omega_1 ; \mathcal{F}(f \restriction \alpha) \not\ni g(\alpha)\} \text{ is stationary.}$$

Here for any countable ordinal ε, we let

$$(\varepsilon{\times}{\uparrow}\omega)^{<\omega} := \{x \in (\varepsilon \times \omega)^{<\omega} ; \forall m < n(\mathsf{proj}_1(x(m)) < \mathsf{proj}_1(x(n)))\},$$

and consider $\mathcal{M}$ as the set of codes of F_σ-meager sets in $(\omega{\times}{\uparrow}\omega)^\omega$.

$$\mathsf{clnwd} := \{T \subseteq (\omega{\times}{\uparrow}\omega)^{<\omega} ; \forall s \in T \forall n < |s| (s \restriction n \in T) \;\&\; \forall s \in T \exists t \in T (s \subsetneq t) \;\&\; \forall s \in T \exists t \supseteq s (t \notin T)\}$$

is the set of codes of closed nowhere dense sets. $2^{((\omega\times\uparrow\omega)^{<\omega})}$ is the product space of 2 which has the discrete topology. clnwd is Borel and in this paper, consider

$$\mathcal{M} := \left\{ \bigcup_{n\in\omega} T_n ; \langle T_n ; n \in \omega \rangle \subseteq \mathsf{clnwd} \right\}.$$

We call that a function $\mathcal{F} : 2^{<\omega_1} \to \mathcal{M}$ is Borel when $\mathcal{F} \restriction \alpha$ is a Borel function for every $\alpha \in \omega_1$.

For each $\alpha \in \omega_1$, define

$$
\begin{aligned}
\mathsf{Pregap}^\alpha := \Big\{ & f \in 2^{\alpha\times\omega\times 2}; \\
& \forall\gamma < \alpha \forall n \in \omega (f(\gamma, n, 0) = 1 \Rightarrow f(\gamma, n, 1) = 0) \\
& \& \ \forall\gamma < \alpha \forall m \in \omega \exists n \geq m (f(\gamma, n, 0) = f(\gamma, n, 1) = 0) \\
& \& \ \forall\gamma < \alpha \forall m \in \omega \exists n \geq m (f(\gamma, n, 0) = 1) \\
& \& \ \forall\gamma < \alpha \forall m \in \omega \exists n \geq m (f(\gamma, n, 1) = 1) \\
& \& \ \forall\gamma \leq \delta < \alpha \exists M \in \omega \forall n \geq M (f(\gamma, n, 0) = 1 \Rightarrow f(\delta, n, 0) = 1 \\
& \quad \& \ f(\gamma, n, 1) = 1 \Rightarrow f(\delta, n, 1) = 1) \Big\},
\end{aligned}
$$

which is Borel in $2^{\alpha\times\omega\times 2}$ (which is equivalent to the Cantor space 2^ω). For $f \in \mathsf{Pregap}^\alpha$ and $\xi < \alpha$, we let $a_\xi := \{n \in \omega; f(\xi, n, 0) = 1\}$ and $b_\xi := \{n \in \omega; f(\xi, n, 1) = 1\}$. Then $\langle a_\xi, b_\xi \rangle_{\xi<\alpha}$ is really an (α, α)-pregap in $\mathcal{P}(\omega)/fin$ with $\omega \setminus (a_\xi \cup b_\xi)$ infinite for all $\xi < \alpha$. We call that a pregap $\langle a_\xi, b_\xi \rangle_{\xi<\alpha}$ admits any finite change if

$$
\begin{aligned}
& \forall\beta < \alpha \text{ with } \beta = \eta + k \text{ for some } \eta \in \mathsf{Lim} \cap \alpha \text{ and } k \in \omega \\
& \forall H, J \in [\omega]^{<\omega} \text{ with } H \cap J = \emptyset \ \forall i > \max(H \cup J) \exists n \in \omega \\
& \qquad \text{such that} \quad a_{\eta+n} \cap i = H \quad \& \quad a_{\eta+n} \setminus i = a_\beta \setminus i \\
& \qquad\qquad\qquad \& \quad b_{\eta+n} \cap i = J \quad \& \quad b_{\eta+n} \setminus i = b_\beta \setminus i.
\end{aligned}
$$

We fix bijections $\pi_\alpha : \omega \to \alpha$ for all $\alpha < \omega_1$ and for a pregap $\langle a_\xi, b_\xi \rangle_{\xi<\alpha}$ which admits any finite change, recursively define a function $T\left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}\right) (= T) : (\omega\times\!\uparrow\!\omega)^{<\omega} \to (\alpha\times\!\uparrow\!\omega)^{<\omega}$ by

$$
\begin{aligned}
T(\langle\langle n,p\rangle\rangle) :=\ & \langle\langle \pi_\alpha(n), p\rangle\rangle\,,\\
T(t^\frown\langle\langle 0,p\rangle\rangle) :=\ & T(t)^\frown\langle\langle\gamma,p\rangle\rangle\\
& \text{where } \gamma\in\alpha\setminus\{\mathsf{proj}_0\,(T(t)(i))\,;i<k\} \text{ such that}\\
& a_{\mathsf{proj}_0(T(t)(|t|-1))}\cap \mathsf{proj}_1(T(t)(|t|-1)) = a_\gamma\cap\mathsf{proj}_1(T(t)(|t|-1))\\
& \&\ a_{\mathsf{proj}_0(T(t)(|t|-1))}\subsetneq a_\gamma\\
& \&\ b_{\mathsf{proj}_0(T(t)(|t|-1))}\cap \mathsf{proj}_1(T(t)(|t|-1)) = b_\gamma\cap\mathsf{proj}_1(T(t)(|t|-1))\\
& \&\ b_{\mathsf{proj}_0(T(t)(|t|-1))}\subsetneq b_\gamma \text{ and}\\
& \text{for any } \delta\in\alpha\setminus\{\mathsf{proj}_0(T(t)(i));i<k\} \text{ with}\\
& a_{\mathsf{proj}_0(T(t)(|t|-1))}\cap \mathsf{proj}_1(T(t)(|t|-1)) = a_\delta\cap\mathsf{proj}_1(T(t)(|t|-1))\\
& \&\ a_{\mathsf{proj}_0(T(t)(|t|-1))}\subsetneq a_\delta\\
& \&\ b_{\mathsf{proj}_0(T(t)(|t|-1))}\cap \mathsf{proj}_1(T(t)(|t|-1)) = b_\delta\cap\mathsf{proj}_1(T(t)(|t|-1))\\
& \&\ b_{\mathsf{proj}_0(T(t)(|t|-1))}\subsetneq b_\delta,\\
& \pi_\alpha(\gamma)<\pi_\alpha(\delta) \text{ holds,}\\
T(t^\frown\langle\langle m,p\rangle\rangle) :=\ & T(t)^\frown\langle\langle\gamma,p\rangle\rangle\\
(m\geq 1)\quad & \text{where } \gamma\in\alpha\setminus\{\mathsf{proj}_0(T(t)(i)),\\
& \qquad\qquad \mathsf{proj}_0(T(t^\frown\langle\langle l,p\rangle\rangle)(|t|));i<k,l<m\}\\
& \text{such that}\\
& a_{\mathsf{proj}_0(T(t)(|t|-1))}\cap \mathsf{proj}_1(T(t)(|t|-1)) = a_\gamma\cap\mathsf{proj}_1(T(t)(|t|-1))\\
& \&\ a_{\mathsf{proj}_0(T(t)(|t|-1))}\subsetneq a_\gamma\\
& \&\ b_{\mathsf{proj}_0(T(t)(|t|-1))}\cap \mathsf{proj}_1(T(t)(|t|-1)) = b_\gamma\cap\mathsf{proj}_1(T(t)(|t|-1))\\
& \&\ b_{\mathsf{proj}_0(T(t)(|t|-1))}\subsetneq b_\gamma \text{ and}\\
& \text{for any } \delta\in\alpha\setminus\{\mathsf{proj}_0(T(t)(i))\\
& \qquad\qquad ,\mathsf{proj}_0(T(t^\frown\langle\langle l,p\rangle\rangle)(|t|));i<k,l<m\}\\
& \text{with}\\
& a_{\mathsf{proj}_0(T(t)(|t|-1))}\cap \mathsf{proj}_1(T(t)(|t|-1)) = a_\delta\cap\mathsf{proj}_1(T(t)(|t|-1))\\
& \&\ a_{\mathsf{proj}_0(T(t)(|t|-1))}\subsetneq a_\delta\\
& \&\ b_{\mathsf{proj}_0(T(t)(|t|-1))}\cap \mathsf{proj}_1(T(t)(|t|-1)) = b_\delta\cap\mathsf{proj}_1(T(t)(|t|-1))\\
& \&\ b_{\mathsf{proj}_0(T(t)(|t|-1))}\subsetneq b_\delta,\\
& \pi_\alpha(\gamma)<\pi_\alpha(\delta) \text{ holds.}
\end{aligned}
$$

(That is, γ is the m-th element with respect to the order by π_α satisfying that
$a_{\mathsf{proj}_0(T(t)(|t|-1))}\cap \mathsf{proj}_1(T(t)(|t|-1)) = a_\gamma\cap\mathsf{proj}_1(T(t)(|t|-1))$
$\&\ a_{\mathsf{proj}_0(T(t)(|t|-1))}\subsetneq a_\gamma$
$\&\ b_{\mathsf{proj}_0(T(t)(|t|-1))}\cap \mathsf{proj}_1(T(t)(|t|-1)) = b_\gamma\cap\mathsf{proj}_1(T(t)(|t|-1))$
$\&\ b_{\mathsf{proj}_0(T(t)(|t|-1))}\subsetneq b_\gamma$.)

We note that for $x \in (\omega \times \uparrow\omega)^\omega$, letting

$$a\left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}, x\right) := \bigcup_{k\in\omega} a_{\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)\left(x\restriction(k+1)\right)(k)\right)}$$

$$\left(= \bigcup_{k\in\omega}\left(a_{\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)\left(x\restriction(k+1)\right)(k)\right)} \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)\left(x\restriction(k+1)\right)(k)\right)\right)\right),$$

and

$$b\left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}, x\right) := \bigcup_{k\in\omega} b_{\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)\left(x\restriction(k+1)\right)(k)\right)}$$

$$\left(= \bigcup_{k\in\omega}\left(b_{\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)\left(x\restriction(k+1)\right)(k)\right)} \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)\left(x\restriction(k+1)\right)(k)\right)\right)\right).$$

$a\left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}, x\right) \cap b\left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}, x\right)$ is empty. For $\alpha \in \omega_1$, a pregap $\langle a_\xi, b_\xi \rangle_{\xi<\alpha} \in \mathsf{Pregap}^\alpha$ which admits any finite change and a subset $\mathcal{A} \subseteq \alpha$, define

$$
\begin{aligned}
\mathcal{D}_\alpha\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \mathcal{A}\right) := \Big\{ & x \in (\omega\times\uparrow\omega)^\omega; \\
\textbf{either}\ \ & \exists\beta<\alpha\forall k\in\omega,\ \mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(x\restriction(k+1))(k)\right) < \beta, \\
\textbf{or}\ \ & \exists n\in\omega\forall k\geq n, \\
& k\in\left(a_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)\left(x\restriction(k+1)\right)(k)\right)} \cup b_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)\left(x\restriction(k+1)\right)(k)\right)}\right) \\
& \qquad \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(x\restriction(k+1))(k)\right), \\
\textbf{or}\ \ & \exists n\in\omega\forall\gamma\in\mathcal{A}\forall k\in\omega, \\
& \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(x\restriction(k+1))(k)\right) < n \\
& \text{or}\ \left(\left(\left(a_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)\left(x\restriction(k+1)\right)(k)\right)} \setminus n\right)\cap b_\gamma\right)\right. \\
& \qquad \left.\cup\left(a_\gamma\cap\left(b_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)\left(x\restriction(k+1)\right)(k)\right)}\setminus n\right)\right)\right) \\
& \qquad \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(x\restriction(k+1))(k)\right) = \emptyset, \\
\textbf{or}\ \ & \exists m\in\omega\exists K,L\subseteq m \text{ such that } \exists\delta\in\mathcal{A} \text{ with } a_\delta\cap K = b_\delta\cap L=\emptyset \text{ and} \\
& \forall\gamma\in\mathcal{A}\forall k\in\omega,\ \text{either}\ \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(x\restriction(k+1))(k)\right) < m \\
& \text{or}\ a_\gamma\cap K\neq\emptyset\ \text{or}\ a_\gamma\setminus m\not\subseteq a_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)\left(x\restriction(k+1)\right)(k)\right)} \\
& \text{or}\ b_\gamma\cap L\neq\emptyset\ \text{or}\ b_\gamma\setminus m\not\subseteq b_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)\left(x\restriction(k+1)\right)(k)\right)} \Big\}.
\end{aligned}
$$

We note that $\mathcal{D}_\alpha\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \mathcal{A}\right)$ $(= \mathcal{D}_\alpha)$ is in $\mathcal{M}$ if $\mathcal{A}$ is non-empty, i.e. a countable union of closed nowhere dense in $(\omega\times\uparrow\omega)^\omega$ and $\mathcal{D}_\alpha$ is a Borel function:

$$
\begin{aligned}
&\mathcal{D}_\alpha\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \mathcal{A}\right)\\
&= \bigcup_{\beta<\alpha} \overbrace{\bigcap_{k\in\omega}\left\{x \in (\omega\times\uparrow\omega)^\omega; T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(x\restriction(k+1))(k) < \beta\right\}}^{X_\beta}\\
&\cup \bigcup_{n\in\omega} \overbrace{\bigcap_{k\geq n}\left\{x \in (\omega\times\uparrow\omega)^\omega; k \in \left(a_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)\left(x\restriction(k+1)\right)(k)\right)}\right.\right.}^{Y_n}\\
&\quad \left.\left.\cup\, b_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)\left(x\restriction(k+1)\right)(k)\right)}\right) \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(x\restriction(k+1))(k)\right)\right\}\\
&\cup \bigcup_{n\in\omega} \overbrace{\bigcap_{\gamma\in\mathcal{A}} \bigcap_{k\in\omega}\left\{x \in (\omega\times\uparrow\omega)^\omega; \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(x\restriction(k+1))(k)\right) < n\right.}^{Z_n}\\
&\qquad \text{or } \left(\left(\left(a_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)\left(x\restriction(k+1)\right)(k)\right)} \setminus n\right) \cap b_\gamma\right)\right.\\
&\qquad\qquad \left.\cup \left(a_\gamma \cap \left(b_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)\left(x\restriction(k+1)\right)(k)\right)} \setminus n\right)\right)\right)\\
&\qquad\qquad\qquad \left.\cap\, \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(x\restriction(k+1))(k)\right) = \emptyset\right\}\\
&\cup \bigcup_{\substack{m\in\omega,\\ K,L\subseteq m \text{ with}\\ \exists\gamma\in\mathcal{A} \text{ so that}\\ a_\gamma\cap K=\emptyset\\ \&\, b_\gamma\cap L=\emptyset}} \overbrace{\bigcap_{\substack{\gamma\in\mathcal{A} \text{ with}\\ a_\gamma\cap K=\emptyset\\ \&\, b_\gamma\cap L=\emptyset}} \bigcap_{k\in\omega}}^{W_{m,K,L}}\\
&\qquad \left\{x \in (\omega\times\uparrow\omega)^\omega; a_\gamma \setminus m \not\subseteq a_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)\left(x\restriction(k+1)\right)(k)\right)}\right.\\
&\qquad\qquad \left.\text{or } b_\gamma \setminus m \not\subseteq b_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)\left(x\restriction(k+1)\right)(k)\right)}\right\}.
\end{aligned}
$$

It can be proved that all X_β, Y_n, Z_n and $W_{m,K,L}$ are closed in $(\omega\times\uparrow\omega)^\omega$.

X_β is nowhere dense in $(\omega\times\uparrow\omega)^\omega$: Given $s \in (\omega\times\uparrow\omega)^{<\omega}$, we let $l \in \omega$ be such that $\pi_\alpha(l) >> \beta$ (i.e. $\exists\varepsilon \in \mathsf{Lim} \cap \alpha(\beta < \varepsilon \leq \pi_\alpha(l))$ and find $\gamma < \alpha$ such that

$$\begin{aligned}
& a_{\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)} \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right) \\
& \quad = a_\gamma \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right) \\
& \& \ a_{\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)} \subseteq a_\gamma \\
& \& \ b_{\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)} \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right) \\
& \quad = b_\gamma \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right) \\
& \& \ b_{\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)} \subseteq b_\gamma \\
& \& \ a_{\pi_\alpha(l)} =^* a_\gamma \ \& \ b_{\pi_\alpha(l)} =^* b_\gamma.
\end{aligned}$$

We note that $\gamma > \beta$. Then there exists $m \in \omega$ (and large enough $q \in \omega$) with

$$\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s^\frown\langle\langle m, q\rangle\rangle)(|s|)\right) = \gamma.$$

Let $t := s^\frown\langle\langle m, q\rangle\rangle$, then

$$[t] := \{x \in (\omega\times\uparrow\omega)^\omega ; t \subseteq x\},$$

which is a (basic) open set in $(\omega\times\uparrow\omega)^\omega$ contained in $[s]$, does not meet X_β.

Y_n is nowhere dense in $(\omega\times\uparrow\omega)^\omega$: Given $s \in (\omega\times\uparrow\omega)^{<\omega}$, (without loss, assuming that $|s| \geq n$,) take $l, k \in \omega$ with

$$\pi_\alpha(l) >> \mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)$$

and $k > |s|+1$, $\mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)$ such that

$$\begin{aligned}
& a_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)} \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right) \\
&\quad = a_{\pi_\alpha(l)} \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right) \\
&\ \&\ a_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)} \subseteq a_{\pi_\alpha(l)} \\
&\ \&\ b_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)} \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right) \\
&\quad = b_{\pi_\alpha(l)} \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right) \\
&\ \&\ b_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)} \subseteq b_{\pi_\alpha(l)} \\
&\ \&\ k \notin a_{\pi_\alpha(l)} \cup b_{\pi_\alpha(l)}.
\end{aligned}$$

Letting $t := s^\frown\langle\langle\pi_\alpha(l), k+1\rangle\rangle$, $[t]$ does not meet Y_n because for any $x \in (\omega\times\uparrow\omega)^\omega$ with $t \subseteq x$, since

$$|t| < k < \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(x\restriction(k+1))(k)\right),$$

$$\begin{aligned}
&\left(a_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)(x\restriction(k+1))(k)\right)} \cup b_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)(x\restriction(k+1))(k)\right)}\right) \cap (k+1) \\
&\qquad = (a_{\pi_\alpha(l)} \cup b_{\pi_\alpha(l)}) \cap (k+1).
\end{aligned}$$

Z_n is nowhere dense in $(\omega\times\uparrow\omega)^\omega$: Given $s \in (\omega\times\uparrow\omega)^{<\omega}$, we take any $\gamma \in \mathcal{A}$ and find $\delta < \alpha$ and $p > n$, $\mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)$ such that

$$\begin{aligned}
& a_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)} \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right) \\
&\quad = a_\delta \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right) \\
&\ \&\ a_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)} \subsetneq a_\delta \\
&\ \&\ b_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)} \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right) \\
&\quad = b_\delta \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right) \\
&\ \&\ b_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)} \subsetneq b_\delta \\
&\ \&\ (a_\delta \setminus n) \cap b_\gamma \cap p \neq \emptyset \ \text{ or } (b_\delta \setminus n) \cap a_\gamma \cap p \neq \emptyset.
\end{aligned}$$

Then
$$\forall i < |s| \left(\mathsf{proj}_0 \left(T \left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}\right) (s)(i)\right) \neq \gamma\right)$$
holds. And then there exists $m \in \omega$ with
$$\mathsf{proj}_0 \left(T \left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}\right) (s^\frown\langle\langle m,p\rangle\rangle)(|s|)\right) = \delta.$$
Letting $t := s^\frown\langle\langle m,p\rangle\rangle$, $[t]$ does not meet Z_n.

$W_{m,K,L}$ **is nowhere dense in** $(\omega\times\uparrow\omega)^\omega$**:** Let $\gamma \in \mathcal{A}$ be such that
$$a_\gamma \cap K = \emptyset \text{ or } b_\gamma \cap L = \emptyset.$$

(Note that $W_{m,K,L}$ is defined when such a γ exists.)

Given $s \in (\omega\times\uparrow\omega)^{<\omega}$, we find $\beta < \alpha$ with $\mathsf{proj}_0 \left(T \left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}\right) (s)(|s|-1)\right)$, $\gamma << \beta$ and $n \geq m$, $\mathsf{proj}_1 \left(T \left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}\right) (s)(|s|-1)\right)$ such that
$$\left(a_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)} \cup a_\gamma\right) \setminus n \subseteq a_\beta$$
$$\& \left(b_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)} \cup b_\gamma\right) \setminus n \subseteq b_\beta.$$

After that, we find $\delta < \alpha$ such that
$$\begin{aligned}
&a_\delta \cap n = \left(a_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)} \cup (a_\gamma \setminus m)\right) \cap n \\
&\& \ a_\delta \setminus n = a_\beta \setminus n \\
&\& \ b_\delta \cap n = \left(b_{\mathsf{proj}_0\left(T\left(\langle a_\xi,b_\xi\rangle_{\xi<\alpha}\right)(s)(|s|-1)\right)} \cup (b_\gamma \setminus m)\right) \cap n \\
&\& \ b_\delta \setminus n = b_\beta \setminus n.
\end{aligned}$$

Then there exists $l \in \omega$ with
$$\mathsf{proj}_0 \left(T \left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}\right) (s^\frown\langle\langle l,n\rangle\rangle)(|s|)\right) = \delta.$$
and we let $t := s^\frown\langle\langle l,n\rangle\rangle$, then $[t]$ does not meet $W_{m,K,L}$.

$\mathcal{D}_\alpha$ **is Borel:** For $s \in (\omega\times\uparrow\omega)^{<\omega}$, $\langle a_\xi, b_\xi \rangle_{\xi<\alpha} \in \mathsf{Pregap}^\alpha$ and $\mathcal{A} \subseteq \alpha$, define the following predicates:

$\mathsf{P}_1\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \beta, s\right) \equiv \forall k<|s|,\ \mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(k)\right)<\beta.$

$\mathsf{P}_2\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, n, s\right) \equiv \forall k<|s|$ with $k \geq n$,

$$k \in \left(a_{\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(k)\right)} \cup b_{\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(k)\right)}\right) \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(k)\right).$$

$\mathsf{P}_3\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \mathcal{A}, n, s\right) \equiv \forall\gamma \in \mathcal{A}\forall k<|s|,$

$\mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(k)\right)<n$

or

$$\left(\left(\left(a_{\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(k)\right)} \setminus n\right) \cap b_\gamma\right) \cup\left(a_\gamma \cap\left(b_{\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(k)\right)} \setminus n\right)\right)\right) \cap \mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(k)\right)=\emptyset.$$

$\mathsf{P}_4\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \mathcal{A}, m, K, L, s\right) \equiv \forall\gamma \in \mathcal{A}\forall k<|s|,$

$\mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(k)\right)<m$ or $a_\gamma \cap K \neq \emptyset$ or $b_\gamma \cap L \neq \emptyset$
or $a_\gamma \setminus m \not\subseteq a_{\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(k)\right)}$ or $b_\gamma \setminus m \not\subseteq b_{\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(k)\right)}$.

For $s \in (\omega\times\uparrow\omega)^{<\omega}$ and $\left\langle\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \mathcal{A}\right\rangle \in 2^{(\alpha\times\omega\times 2)\times\alpha}$ such that $\langle a_\xi, b_\xi\rangle_{\xi<\alpha}$ admits any finite change,

$\left\langle\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \mathcal{A}\right\rangle \in \mathcal{D}_\alpha{}^{-1}[[s]] \iff \exists x$ with $s \subseteq x \in \mathcal{D}_\alpha\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \mathcal{A}\right)$

$\iff$ **either** $\exists\beta<\alpha$ such that $\mathsf{P}_1\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \beta, s\right)$
and $\forall t \supseteq s$ with $\mathsf{P}_1\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \beta, t\right)$ $\exists u \supsetneq t$ with $\mathsf{P}_1\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \beta, u\right)$,

or $\exists n \in \omega$ such that $\mathsf{P}_2\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, n, s\right)$
and $\forall t \supseteq s$ with $\mathsf{P}_2\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, n, t\right)$ $\exists u \supsetneq t$ with $\mathsf{P}_2\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, n, u\right)$,

or $\exists n \in \omega$ such that $\mathsf{P}_3\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \mathcal{A}, n, s\right)$
and $\forall t \supseteq s$ with $\mathsf{P}_3\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \mathcal{A}, n, t\right)$ $\exists u \supsetneq t$ with $\mathsf{P}_3\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \mathcal{A}, n, u\right)$,

or $\exists m \in \omega \exists K, L \subseteq m$ such that $\exists \delta \in \mathcal{A}$ with $a_\delta \cap K = b_\delta \cap L = \emptyset$, $\mathsf{P}_4\left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}, \mathcal{A}, m, K, L, s\right)$ and $\forall t \supseteq s$ with $\mathsf{P}_4\left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}, \mathcal{A}, m, K, L, t\right)$ $\exists u \supsetneq t$ with $\mathsf{P}_4\left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}, \mathcal{A}, m, K, L, u\right)$.

This is a Borel set in $2^{(\alpha\times\omega\times 2)\times\alpha}$. To see it, we see the following basic relations. For $u \in (\omega\times\uparrow\omega)^{<\omega}$,

$$\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}\right)(u)(0)\right) = \beta \iff \pi_\alpha(u(0)) = \beta.$$

Here we define a formula with parameters $\xi, \eta < \alpha$ and $M \in \omega$:

$$\Phi(\xi, \eta, M) \equiv \text{“ } a_\xi \cap M = a_\eta \cap M \ \& \ a_\xi \subseteq a_\eta \ \& \ b_\xi \cap M = b_\eta \cap M \ \& \ b_\xi \subseteq b_\eta \text{”}.$$

Then for $u \in (\omega\times\uparrow\omega)^{<\omega}$, $k \leq |u|$ and $\langle \beta_i; i < k\rangle \subseteq \alpha$ with $\beta_i \neq \beta_j$ for any distinct $i, j < k$,

$$\begin{aligned}
&\forall i < k \left(\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}\right)(u)(i)\right) = \beta_i\right) \\
\iff\ & \pi_\alpha(u(0)) = \beta_0 \\
& \& \ \forall i < k \text{ with } i \geq 1 \Big(\beta_i \text{ is } (\mathsf{proj}_0(u(i)) + 1)\text{-th element } \gamma \\
& \qquad\qquad \text{satisfying that } \Phi(\beta_{i-1}, \gamma, \mathsf{proj}_1(u(i-1)))\Big) \\
\iff\ & \pi_\alpha(u(0)) = \beta_0 \\
& \& \ \forall i < k \text{ with } i \geq 1,\ \Phi(\beta_{i-1}, \beta_i, \mathsf{proj}_1(u(i-1))) \\
& \& \ \exists I \in [\pi_\alpha{}^{-1}(\beta_i) \setminus \{\pi_\alpha{}^{-1}(\beta_j); j < i\}]^{\mathsf{proj}_0(u(i))} \\
& \text{so that } \forall l \in I\Big(\Phi(\beta_{i-1}, \pi_\alpha(l), \mathsf{proj}_1(u(i-1)))\Big) \text{ and} \\
& \forall m \in (\pi_\alpha{}^{-1}(\beta_i) \setminus I)\left(\neg\Phi(\beta_{i-1}, \pi_\alpha(m), \mathsf{proj}_1(u(i-1)))\right),
\end{aligned}$$

which is a Borel relation. And we notice that for $u \in (\omega\times\uparrow\omega)^{<\omega}$ and $k \leq |u|$

$$\mathsf{proj}_1\left(T\left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}\right)(u)(k)\right) = \mathsf{proj}_1(u(k)).$$

Therefore, for example, we can see that the following statement is Borel:

$$\begin{aligned}&\mathsf{P}_3\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \mathcal{A}, n, s\right)\\ &\iff \bigcap_{\gamma\in\alpha}\ \bigcap_{\substack{k<|s|\ \&\\ \mathsf{proj}_1(s(k))\geq n}}\ \bigcap_{\langle\beta_i; i<k+1\rangle\in\alpha^{k+1}}\\ &\left(\gamma\notin\mathcal{A} \text{ or } \exists i<k+1\left(\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)(s)(i)\right)\neq\beta_i\right)\right.\\ &\left.\left.\text{or } \left(\left(\left(a_{\beta_k}\setminus n\right)\cap b_\gamma\right)\cup\left(a_\gamma\cap\left(b_{\beta_k}\setminus n\right)\right)\right)\cap\mathsf{proj}_1(s(k))=\emptyset\right)\right).\end{aligned}$$

We stop here explaining that $\mathcal{D}_\alpha$ is a Borel function.

We notice that for all $x\in(\omega\times\uparrow\omega)^\omega$ with $x\notin\mathcal{D}_\alpha\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \mathcal{A}\right)$,

- $a\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, x\right)$ and $b\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, x\right)$ separate $\langle a_\xi, b_\xi\rangle_{\xi<\alpha}$,
- $\omega\setminus\left(a\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, x\right)\cup b\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, x\right)\right)$ is infinite,
- $\forall n\in\omega\exists\gamma\in\mathcal{A}$ so that
$$\left(\left(a\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, x\right)\setminus n\right)\cap b_\gamma\right)\cap\left(a_\gamma\cap\left(b\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, x\right)\setminus n\right)\right)\neq\emptyset,$$
- if $\mathcal{A}$ satisfies that
$$\forall K, L\in[\omega]^{<\omega}\exists\gamma\in\mathcal{A} \text{ so that } a_\gamma\cap K=\emptyset \text{ and } b_\gamma\cap L=\emptyset,$$
then $\forall m\in\omega\forall K, L\subseteq m\exists\gamma\in\mathcal{A}$ so that $a_\gamma\cap K=\emptyset$, $a_\gamma\setminus m\subseteq a\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, x\right)$, $b_\gamma\cap L=\emptyset$, and $b_\gamma\setminus m\subseteq b\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, x\right)$.

For sets A and B, we write $A\dot\cup B$ for a disjoint union of A and B. We define a Borel function $\mathcal{F}:\bigcup_{\alpha<\omega_1}2^{\left((\alpha\times\omega\times2)\dot\cup\alpha\right)}\to\mathcal{M}$ by

$$\mathcal{F}\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \mathcal{A}\right)=\begin{cases}\mathcal{D}_\alpha\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, \mathcal{A}\right) & \text{if } \langle a_\xi, b_\xi\rangle_{\xi<\alpha}\in\mathsf{Pregap}^\alpha \\ & \text{admits any finite change and} \\ & \mathcal{A} \text{ is not empty,}\\ \emptyset & \text{otherwise.}\end{cases}$$

By $\diamondsuit(\mathsf{non}(\mathcal{M}))$, there exists a function $g : \omega_1 \to (\omega\times\uparrow\omega)^\omega$ such that for all $f : (\omega_1 \times \omega \times 2)\dot\cup\omega_1 \to 2$, $\{\alpha \in \omega_1; \mathcal{F}(f \restriction \alpha) \not\ni g(\alpha)\}$ is stationary (where $f \restriction \alpha := f \cap (((\alpha \times \omega \times 2)\dot\cup\alpha) \times 2))$. Then we can construct $\langle a_\alpha, b_\alpha\rangle_{\alpha\in\omega_1}$ by recursion on $\alpha \in \omega_1$ such that

1. $a_\alpha \cap b_\alpha = \emptyset$ and $\omega \setminus (a_\alpha \cup b_\alpha)$ is infinite,

2. $\forall\beta < \alpha(a_\beta \subseteq^* a_\alpha \;\&\; b_\beta \subseteq^* b_\alpha)$,

3. if α is a limit ordinal, then $\langle a_\xi, b_\xi\rangle_{\xi<\alpha}$ admits any finite change, recall that for any $\beta < \alpha$ with $\beta = \eta + k$ for some $\eta \in \mathsf{Lim} \cap \alpha$ and $k \in \omega$, $H, J \in [\omega]^{<\omega}$ with $H \cap J = \emptyset$ and $i > \max(H \cup J)$ there exists $n \in \omega$ so that
$$\begin{aligned} & a_{\eta+n} \cap i = H \quad \& \quad a_{\eta+n} \setminus i = a_\beta \setminus i \\ \& \quad & b_{\eta+n} \cap i = J \quad \& \quad b_{\eta+n} \setminus i = b_\beta \setminus i, \text{ and} \end{aligned}$$

4. if α is a limit ordinal and
$$\left\{\mathsf{proj}_0\left(T\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}\right)\left(g(\alpha)\restriction(k+1)\right)(k)\right); k \in \omega\right\},$$
is cofinal in α, then let
$$a_\alpha := a\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, g(\alpha)\right) \text{ and } b_\alpha := b\left(\langle a_\xi, b_\xi\rangle_{\xi<\alpha}, g(\alpha)\right),$$
otherwise let a_α and b_α be any elements in $[\omega]^{<\omega}$ such that $a_\alpha \cap b_\alpha = \emptyset$, $\omega \setminus (a_\alpha \cup b_\alpha)$ is infinite and both a_α and b_α separate $\langle a_\xi, b_\xi\rangle_{\xi<\alpha}$, i.e. $a_\xi \subsetneq^* a_\alpha$ and $b_\xi \subsetneq^* b_\alpha$ for all $\xi < \alpha$.

From now on we check that for all uncountable $\mathcal{A} \subseteq \omega_1$, there exist distinct $\gamma, \delta \in \mathcal{A}$ with $(a_\gamma \cap b_\delta) \cap (a_\delta \cap b_\gamma)$ non-empty, i.e. $\langle a_\alpha, b_\alpha\rangle_{\alpha\in\omega_1}$ forms a gap:

Assume that $\mathcal{A} \subseteq \omega_1$ is uncountable such that for any distinct $\gamma, \delta \in \mathcal{A}$ $(a_\gamma \cap b_\delta) \cap (a_\delta \cap b_\gamma)$ is empty. Without loss of generality, we may assume that $\mathcal{A}$ is maximal with this property, i.e. $\mathcal{A}$ satisfies that
$$\forall\gamma \neq \delta \in \mathcal{A}((a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) = \emptyset), \text{ and}$$
$$\forall\gamma \in \omega_1 \exists\delta \in \mathcal{A}((a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) \neq \emptyset).$$

We note that

$$\{\alpha \in \mathsf{Lim} \cap \omega_1; \forall \gamma \in \alpha \exists \delta \in \mathcal{A} \cap \alpha((a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) \neq \emptyset)\}$$

is club on ω_1. Thus there is α in this set with

$$\mathcal{F}\left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}, \mathcal{A} \cap \alpha\right) \not\ni g(\alpha).$$

It follows that, in particular

- $\langle a_\xi, b_\xi \rangle_{\xi<\alpha}$ admits any finite change,
- $a_\alpha = a\left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}, g(\alpha)\right)$ and $b_\alpha = b\left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}, g(\alpha)\right)$, and
- $\forall n \in \omega \exists \gamma \in \mathcal{A}$ so that

$$((a_\alpha \setminus n) \cap b_\gamma) \cup (a_\gamma \cap (b_\alpha \setminus n)) \neq \emptyset.$$

Take any $\gamma \in \mathcal{A} \setminus \alpha$, then there exists $n \in \omega$ such that $a_\alpha \setminus n \subseteq a_\gamma$ and $b_\alpha \setminus n \subseteq b_\gamma$. Then we can find $\delta \in \mathcal{A}$ such that

$$((a_\alpha \setminus n) \cap b_\delta) \cup (a_\delta \cap (b_\alpha \setminus n)) \neq \emptyset.$$

Then

$$(a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) \neq \emptyset,$$

which is a contradiction because both γ and δ are in $\mathcal{A}$.

By the argument as above, we can show that for all uncountable $\mathcal{B} \subseteq \omega_1$, there exist distinct $\gamma, \delta \in \mathcal{B}$ with $(a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma)$ empty, i.e. $\langle a_\alpha, b_\alpha \rangle_{\alpha \in \omega_1}$ is destructible:

Assume that there exists an uncountable $\mathcal{B} \subseteq \alpha$ so that

$$\forall \gamma \neq \delta \in \mathcal{B}((a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) \neq \emptyset), \text{ and}$$

$$\forall \gamma \in \omega_1 \exists \delta \in \mathcal{B}((a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) = \emptyset).$$

We note that

$$\{\alpha \in \mathsf{Lim} \cap \omega_1; \forall \gamma \in \alpha \exists \delta \in \mathcal{B} \cap \alpha((a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) = \emptyset)\}$$

is club on ω_1. Thus there is α in this set with

$$\mathcal{F}\left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}, \mathcal{B} \cap \alpha\right) \not\ni g(\alpha).$$

It follows that, in particular

- $\langle a_\xi, b_\xi \rangle_{\xi<\alpha}$ admits any finite change,
- $a_\alpha = a\left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}, g(\alpha)\right)$ and $b_\alpha = b\left(\langle a_\xi, b_\xi \rangle_{\xi<\alpha}, g(\alpha)\right)$, and
- $\forall m \in \omega \forall K, L \subseteq m \exists \gamma \in \mathcal{B} \in \omega$ so that $a_\gamma \cap K = \emptyset$, $a_\gamma \setminus m \subseteq a_\alpha$, $b_\gamma \cap L = \emptyset$, and $b_\alpha \setminus m \subseteq b_\alpha$.

  (Since $\langle a_\alpha, b_\alpha \rangle_{\alpha\in\omega_1}$ admits any finite change and $\forall \gamma \in \alpha \exists \delta \in \mathcal{B}((a_\gamma \cap b_\delta) \cap (a_\delta \cap b_\gamma) = \emptyset)$, $\forall m \in \omega \forall K, L \subseteq m \exists \delta \in \mathcal{B}(a_\delta \cap K = \emptyset \ \& \ b_\delta \cap L = \emptyset)$ holds.)

Take any $\gamma \in \mathcal{B} \setminus \alpha$, then there exists $m \in \omega$ such that $a_\alpha \setminus m \subseteq a_\gamma$ and $b_\alpha \setminus m \subseteq b_\gamma$. Then we can find $\delta \in \mathcal{B}$ such that $a_\delta \cap b_\gamma \cap m = \emptyset$, $a_\delta \setminus m \subseteq a_\alpha$, $b_\delta \cap a_\gamma \cap m = \emptyset$, and $b_\delta \setminus m \subseteq b_\alpha$. Then

$$(a_\gamma \cap b_\delta) \cup (a_\delta \cap b_\gamma) = \emptyset,$$

which is a contradiction because both γ and δ are in $\mathcal{B}$.

Therefore $\langle a_\alpha, b_\alpha \rangle_{\alpha\in\omega_1}$ is a destructible gap. □

Chapter 4

Gap spectra under Martin's Axiom

In this chapter, we study specific types of gaps in $\mathcal{P}(\omega)/\text{fin}$, namely $(\omega_1, \mathfrak{c})$-gaps and $(\mathfrak{c}, \mathfrak{c})$-gaps. As seen in previous chapters, it is one of the classical results that there always exist some types of gaps: (ω_1, ω_1)-gaps and $(\omega, \mathfrak{b})$-gaps. Concerning the existence of (κ, λ)-gaps, we know that it is consistent with ZFC that if there exists a (κ, λ)-gap where κ and λ are regular cardinals with $\kappa \leq \lambda$, then either ($\kappa = \omega$ and $\lambda = \mathfrak{c}$) or $\kappa = \lambda = \omega_1$. In the last section of this chapter, we give one result on the existence of (κ, λ)-gaps under Martin's Axiom for regular cardinals κ and λ. This subject has also been studied in the past.

It is one of the classical results that any (ω, ω)-pregap is separated. And if κ and λ are regular cardinals so that κ or λ is not ω_1, then for any (κ, λ)-gap $(\mathcal{A}, \mathcal{B})$ there is a ccc forcing notion which forces that $(\mathcal{A}, \mathcal{B})$ is separated. Therefore under Martin's Axiom if $(\mathcal{A}, \mathcal{B})$ is an (κ, λ)-gap, then $\kappa = \lambda = \omega_1$ or $(\kappa = \mathfrak{c} \vee \lambda = \mathfrak{c})$ holds. In fact, we know that there is a $(\omega, \mathfrak{b})$-gap, so under MA there always exists an $(\omega, \mathfrak{c})$-gap. In [22], Hausdorff has proved that there always exists an (ω_1, ω_1)-gap. In particular, his proof gives that there exists an indestructible (ω_1, ω_1)-gap, hence under MA, there exists an (ω_1, ω_1)-gap. (In fact under MA, every (ω_1, ω_1)-gap is indestructible.) Moreover, we can prove the following proposition.

Proposition 4.1 (Kunen, [33]). *Assume Martin's Axiom and that there exist (κ, λ)-gaps where $\omega_1 < \kappa \leq \lambda$, then $\lambda = \mathfrak{c}$. Conversely, under Martin's Axiom if κ is regular less than $\mathfrak{c}$ and not ω_1, then there exists a $(\kappa, \mathfrak{c})$-gap.*

Proof. The first statement follows from Lemma 3.4.

For the second statement, we fix an increasing chain $\langle a_\alpha; \alpha < \kappa \rangle \subseteq [\omega]^\omega$. Let $\mathcal{B} := \mathcal{A}^\perp$, recall that

$$\mathcal{B} := \{b \subseteq \omega; \forall a \in \mathcal{A}(b \perp a)\}.$$

By Martin's Axiom and $\kappa < \mathfrak{c}$, $\mathcal{B}$ is not empty. We choose any unbounded increasing sequence $\langle b_\beta; \beta < \lambda \rangle$ of members of $\mathcal{B}$. Then $\langle a_\alpha, b_\beta; \alpha < \kappa, \beta < \lambda \rangle$ forms a gap and by Lemma 3.4 and $\kappa \neq \omega_1$, this λ must be just $\mathfrak{c}$. $\square$

So the remaining problems are about the existence of $(\omega_1, \mathfrak{c})$-gaps and $(\mathfrak{c}, \mathfrak{c})$-gaps under MA.

In [33], Kunen has proved that the following statements are consistent with ZFC:

1. $\mathsf{MA} + \exists(\mathfrak{c}, \mathfrak{c})$-gaps $+ \exists(\omega_1, \mathfrak{c})$-gaps, and

2. $\mathsf{MA} + \neg\exists(\mathfrak{c}, \mathfrak{c})$-gaps $+ \neg\exists(\omega_1, \mathfrak{c})$-gaps.

Todorčević has proved that under the Proper Forcing Axiom, if a (κ, λ)-gap exists for regular cardinals $\kappa \leq \lambda$, then either $\kappa = \omega < \lambda = \omega_2 (= \mathfrak{b} = \mathfrak{c})$ or $\kappa = \lambda = \omega_1$, i.e. there are no $(\omega_1, \mathfrak{c})$-gaps and $(\mathfrak{c}, \mathfrak{c})$-gaps under PFA.

The main theorem of this chapter is the following theorem which gives a positive answer to one of the open problems in [41] (Problem 13):

Theorem 4.2. *It is consistent with* ZFC *that Martin's Axiom holds and there are* $(\mathfrak{c}, \mathfrak{c})$*-gaps but no* $(\omega_1, \mathfrak{c})$*-gaps.*

As we see from the above, we cannot distinguish $(\omega_1, \mathfrak{c})$-gaps and $(\mathfrak{c}, \mathfrak{c})$-gaps as types of gaps by only considering Kunen's results and the result under PFA. Theorem 4.2 says that under MA, $(\omega_1, \mathfrak{c})$-gaps and $(\mathfrak{c}, \mathfrak{c})$-gaps can be distinguished.

This chapter consists of three parts. In the first section, we will consider the gap spectrum under $\mathsf{MA} + \mathsf{OCA}$. This is also due to Todorčević.

In next section, we will prove Kunen's theorems and in the last section, we will show the Theorem 4.2. To prove it, we will use proper forcing notions with models as side conditions, a method which is also due to Todorčević.

4.1 Martin's Axiom plus the Open Coloring Axiom

The Open Coloring Axiom first appeared in [1]. After that, in [45], Todorčević presented another form of the Open Coloring Axiom, which is different from the one due to Abraham, Rubin and Shelah, but now, Todorčević's Open Coloring Axiom is more popular than the other. (Recently, it was proved that both of them decides the power of the continuum. [37]) In [45], Todorčević has proved that under the Proper Forcing Axiom, if a (κ, λ)-gap exists for regular cardinals $\kappa \leq \lambda$, then either $\kappa = \omega < \lambda = \omega_2 (= \mathfrak{b} = \mathfrak{c})$ or $\kappa = \lambda = \omega_1$, i.e. there are no $(\omega_1, \mathfrak{c})$-gaps and $(\mathfrak{c}, \mathfrak{c})$-gaps under PFA (see also [5]). In [45], this was shown in the following way. PFA implies the Open Coloring Axiom and $\mathsf{OCA} + \mathfrak{c} = \aleph_2$ implies the conclusion. At first, we show that OCA implies that $\mathfrak{b} = \aleph_2$. Then since MA implies that $\mathfrak{b} = \mathfrak{c}$, it completes the proof.

Definition 4.3 (Todorčević, [45]). *The Open Coloring Axiom* OCA *is the following statement:*

> *Let X be a subset of the real line. For any open partition $[X]^2 = C_0 \dot\cup C_1$ (i.e. C_0 is a open subset of $[X]^2$), either there exists an uncountable C_0-homogeneous subsets or X can be decomposed by countably many C_1-homogeneous subsets.*

Definition 4.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $\mathcal{P}(\omega)$.*

1. $\mathcal{B}^+ := \{c \subseteq \omega; \exists b \in \mathcal{B}(c \subseteq^* b)\}$.

2. $\mathcal{A} \otimes \mathcal{B} := \{\langle a, b\rangle \in \mathcal{A} \times \mathcal{B}; a \cap b = \emptyset\}$.

3. (Todorčević [45]) *Coloring:* $[\mathcal{A} \otimes \mathcal{B}]^2 = K_0 \dot\cup K_1$, *where*

$$\{\langle a, b\rangle, \langle a', b'\rangle\} \in K_0 :\iff (a \cap b') \cup (a' \cap b) \neq \emptyset.$$

4. *A pregap $(\mathcal{A}, \mathcal{B})$ is countably separated if there is a sequence $\langle c_n; n \in \omega\rangle$ of elements of $\mathcal{P}(\omega)$ such that for all $\langle a, b\rangle \in \mathcal{A} \times \mathcal{B}$ there is an $n \in \omega$ with $a \perp c_n$ and $b \subseteq^* c_n$.*

$\mathcal{P}(\omega)$ is identified with the Cantor space. Now we fix a recursive linear order $<_{\mathcal{P}(\omega)}$ in $\mathcal{P}(\omega)$ and then we identify $[\mathcal{A} \otimes \mathcal{B}]^2$ with the topological space $\{\langle a, b\rangle \in \mathcal{P}(\omega) \times \mathcal{P}(\omega); a <_{\mathcal{P}(\omega)} b\}$. Then we notice that $[\mathcal{A} \otimes \mathcal{B}]^2 =$

$(\mathcal{A} \times \mathcal{B} \cup \mathcal{B} \times \mathcal{A}) \cap \{\langle a, b\rangle \in \mathcal{P}(\omega) \times \mathcal{P}(\omega); a <_{\mathcal{P}(\omega)} b\}$ and K_0 is open in this topology. For $\mathcal{A} \subseteq \mathcal{P}(\omega)$, $\mathcal{A}$ is called σ-directed if for every countable subset X of $\mathcal{A}$, there is $a \in \mathcal{A}$ so that for all $x \in X$, $x \subseteq^* a$.

The next two propositions are basic facts about this coloring.

Proposition 4.5 (Folklore, [17]). *For a pregap $(\mathcal{A}, \mathcal{B})$, $\mathcal{A} \otimes \mathcal{B}$ can be decomposed by countably many K_1-homogeneous subsets iff $(\mathcal{A}, \mathcal{B})$ is countably separated. So if $\mathcal{A} \otimes \mathcal{B}$ can be decomposed by countably many K_1-homogeneous subsets and both $\mathcal{A}$ and $\mathcal{B}$ are σ-directed, then $(\mathcal{A}, \mathcal{B})$ does not form a gap.*

Proof. It suffices to consider the following translations:

$$X \subseteq \mathcal{P}(\omega) \times \mathcal{P}(\omega) \Longrightarrow c_X := \bigcup_{\langle a,b\rangle \in X} b,$$

$$c \in \mathcal{P}(\omega) \Longrightarrow X_c := \{\langle a, b\rangle \in \mathcal{P}(\omega) \times \mathcal{P}(\omega); a \cap c = \emptyset \;\&\; b \subseteq c\}.$$

Then if $X \subseteq \mathcal{A} \otimes \mathcal{B}$ is K_1-homogeneous, then any $\langle a, b\rangle \in X$ is separated by c_X and if c separates $\langle a, b\rangle \in \mathcal{A} \otimes \mathcal{B}$, then $\langle a, b\rangle \in X_c$.

For the last statement, we assume that $(\mathcal{A}, \mathcal{B})$ is countably separated by $\langle c_n; n \in \omega\rangle$ and both $\mathcal{A}$ and $\mathcal{B}$ are σ-directed. For each n and $k \in \omega$, we let

$$C_{n,k} := \{\langle a, b\rangle \in \mathcal{A} \otimes \mathcal{B}; a \cap c_n \subseteq k \;\&\; b \setminus k \subseteq c_n\}.$$

Then $\mathcal{A} \otimes \mathcal{B} = \bigcup_{n,k\in\omega} C_{n,k}$.

Claim 9. *For some n and $k \in \omega$, $C_{n,k}$ is cofinal in $\mathcal{A} \otimes \mathcal{B}$, i.e. for any $\langle a, b\rangle \in \mathcal{A} \otimes \mathcal{B}$, there is $\langle a', b'\rangle \in C_{n,k}$ such that $a \subseteq^* a'$ and $b \subseteq^* b'$.*

Proof of claim. Assume not, then for each $\langle n, k\rangle \in \omega^2$, we can find $\langle a_{n,k}, b_{n,k}\rangle \in \mathcal{A} \otimes \mathcal{B}$ such that either $a_{n,k} \not\subseteq^* a$ or $b_{n,k} \not\subseteq^* b$ for every $\langle a, b\rangle \in C_{n,k}$. Since both $\mathcal{A}$ and $\mathcal{B}$ are σ-directed, there are $a \in \mathcal{A}$ and $b \in \mathcal{B}$ so that $a_{n,k} \subseteq^* a$ and $b_{n,k} \subseteq^* b$ hold for every $\langle n, k\rangle \in \omega^2$. But then $\langle a, b\rangle \notin \bigcup_{n,k\in\omega} C_{n,k}$ which is a contradiction. ⊣

Assume that $C_{n,k}$ is cofinal in $\mathcal{A} \otimes \mathcal{B}$ and let $c := \bigcup_{\langle a,b\rangle \in C_{n,k}} b \setminus k$. Then c separates $C_{n,k}$ and so separates $(\mathcal{A}, \mathcal{B})$. □

Proposition 4.6 (Folklore, [17]). *If $(\mathcal{A}, \mathcal{B})$ is a pregap and $\mathcal{A} \otimes \mathcal{B}$ has an uncountable K_0-homogeneous subset, then $(\mathcal{A}, \mathcal{B})$ forms a gap, in particular if $(\mathcal{A}, \mathcal{B})$ is linear, then it forms an (ω_1, ω_1)-gap.*

Proof. It follows from Proposition 2.4. □

From now on, we show that the gap spectrum is decided by Martin's Axiom plus the Open Coloring Axiom. For the argument below, I refer to [18].

Lemma 4.7 ([18]). *Under the Open Coloring Axiom, any uncountable $X \subseteq \mathcal{P}(\omega)$ contains an uncountable antichain with respect to $\subseteq$.*

Proof. We consider the following open partition $[X]^2 = I_0 \dot{\cup} I_1$ where

$$\{x, y\} \in I_0 \iff x \not\subseteq y \;\&\; y \not\subseteq x.$$

By OCA, it follows that either

(a) there exists an uncountable I_0-homogeneous subset Y

or

(b) X can be decomposed by countably many I_1-homogeneous subsets Y_n, $n \in \omega$.

In the case (a), Y is an uncountable antichain and in the case (b), some Y_n is an uncountable chain but this case have never occurred. □

Lemma 4.8 (Todorčević, [6], [45]). *Under the Open Coloring Axiom, $\mathfrak{b} > \aleph_1$.*

Proof. Assume that $\mathfrak{b} = \aleph_1$. Then there exists an unbounded and strictly increasing sequence $\langle f_\alpha; \alpha < \omega_1 \rangle$ with respect to $\leq^*$ and all f_α is also strictly increasing.

Claim 10. *For any unbounded and strictly increasing sequence $\langle f_\alpha; \alpha < \kappa \rangle$ with every f_α strictly increasing and $\kappa \geq \aleph_1$, and any $I \in [\kappa]^\kappa$, there are $\alpha < \beta \in I$ with $f_\alpha \leq f_\beta$ everywhere.*

Proof of claim. Without loss of generality, we may assume that $I = \kappa$. For $s \in S := \{t \in \omega^{<\omega}; t \subseteq f_\alpha \text{ for some } \alpha < \kappa\}$, let $\alpha_s := \min\{\gamma < \kappa; s \subseteq f_\gamma\}$ and let $\beta := \sup\{\alpha_s; s \in S\}$. Then $\beta < \kappa$ (because κ has an uncountable cofinality and S is countable) and for $s \in S$, $f_{\alpha_s} \leq^* f_\beta$. We can find cofinal $I \subseteq \kappa$, $l \in \omega$ and $t \in \omega^n$ such that for all $\gamma \in I$, $t \subseteq f_\gamma$ and for every $k \geq l$, $f_\beta(k) \leq f_\gamma(k)$. Since $\{f_\gamma; \gamma \in I\}$ is unbounded, there exists $m \in \omega$ so that for all $\alpha < \kappa$ and $k \in \omega$, we can find $\gamma \in I$ with $\alpha \leq \gamma$ such that $k \leq f_\gamma(m)$. We may assume that this m is such a least number. By the minimality of m, there are $t' \in \omega^m$ and $\delta < \kappa$ with $\beta \leq \delta$ such that $f_\gamma \restriction m \leq t'$ everywhere for $\gamma \in I$ with $\delta \leq \gamma$. Then we choose $t'' \in \omega^m$ and $I' \subseteq I$ so that $t'' \subseteq f_\gamma$ for any $\gamma \in I'$ and $\{f_\gamma(m); \gamma \in I'\}$ is unbounded in ω. After that, we pick $n \geq m$ with $f_{\alpha_{t''}} \restriction [n, \omega) \leq f_\beta \restriction [n, \omega)$ everywhere and take $\varepsilon \in I'$ with $f_\varepsilon(m) \geq f_{\alpha_{t''}}(n)$. Then we can check that $f_{\alpha_{t''}} \leq f_\varepsilon$ everywhere:

If $k < m$, then $f_{\alpha_{t''}}(k) = t''(k) = f_\varepsilon(k)$.

If $k \in n \setminus m$, then $f_{\alpha_{t''}}(k) \leq f_{\alpha_{t''}}(n) \leq f_\varepsilon(m) \leq f_\varepsilon(k)$.

If $k \geq n$, then $f_{\alpha_{t''}}(k) \leq f_\beta(k) \leq f_\varepsilon(k)$. ⊣

For $\alpha < \omega_1$, let $x_\alpha := \{\langle n, m\rangle \in \omega \times \omega; f_\alpha(n) \leq m\}$. And we put $X := \{x_\alpha; \alpha < \omega_1\}$. By Lemma 4.7, X contains an uncountable antichain Y. Then there are distinct x_α and x_β in Y, letting $\alpha < \beta$, such that $f_\alpha \leq f_\beta$. But then $x_\alpha \supseteq x_\beta$ holds, which is a contradiction. □

Corollary 4.9. *Under the Open Coloring Axiom, there are no (ω, ω_1)-gaps.* □

Lemma 4.10 (Todorčević, [6], [45]). *Under the Open Coloring Axiom, if (κ, λ)-gap exists and both κ and λ have uncountable cofinality, then $\kappa = \lambda = \omega_1$.*

Proof. Assume that $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\beta; \alpha < \kappa, \beta < \lambda\rangle$ forms a gap. We consider the open coloring $\mathcal{A} \otimes \mathcal{B} = K_0 \dot{\cup} K_1$ in the definition of 4.4. By OCA, either $\mathcal{A} \otimes \mathcal{B}$ has an uncountable K_0-homogeneous or $\mathcal{A} \otimes \mathcal{B}$ can be decomposed by countably many k_1-homogeneous subsets. So by propositions 4.5 and 4.6, $(\mathcal{A}, \mathcal{B})$ is an (ω_1, ω_1)-gap. □

Lemma 4.11 (Todorčević, [6], [45]). *Under the Open Coloring Axiom, $\mathfrak{b} = \aleph_2$.*

Proof. Assume $\mathfrak{b} > \aleph_2$. Fixing a strictly increasing sequence $\mathcal{A} := \langle f_\alpha; \alpha < \omega_2\rangle \subseteq \omega^\omega$, we take a maximal linearly ordered set $\mathcal{C} \subseteq \omega^\omega$ with respect to $\leq^*$ containing $\mathcal{A}$. We choose a coinitial subset $\mathcal{B}$ of $\{g \in \mathcal{C}; \forall f \in \mathcal{A}(f \leq^* g)\}$. By Corollary 2.7, the order type of $\mathcal{B}$ has an uncountable cofinality. But this contradicts to Lemma 4.10. □

Theorem 4.12 (Todorčević, [6], [45]). *Under Martin's Axiom and the Open Coloring Axiom, if there exists a (κ, λ)-gap, then either $\kappa = \omega$ & $\lambda = \mathfrak{b}$ or $\kappa = \lambda = \omega_1$. (It is also holds under the Proper Forcing Axiom.)*

Proof. Under the Martin's Axiom, $\mathfrak{b} = \mathfrak{c}$ holds, so the theorem follows from 4.9, 4.10 and 4.11. □

4.2 Kunen's unpublished work

This section consists of Kunen's unpublished note [33]. We already have tools to prove the theorems.

Theorem 4.13 (Kunen, [33]). *It is consistent with* ZFC *that Martin's Axiom holds and there exist both $(\mathfrak{c}, \mathfrak{c})$-gaps and $(\omega_1, \mathfrak{c})$-gaps.*

Proof. Without loss of generality, we may assume that the ground model $\mathbf{V}$ has both $(\mathfrak{c}, \mathfrak{c})$-gaps and $(\omega_1, \mathfrak{c})$-gaps and $\mathfrak{c} = \kappa = 2^{<\kappa}$ is a regular cardinal larger than ω_1. (See section 3.1.) Let $\mathbb{P}$ be a finite support iteration of ccc-forcing notions of size $< \kappa$ of length κ to force MA by a standard book-keeping argument. Then in $\mathbf{V}^{\mathbb{P}}$, Martin's Axiom holds, $\mathfrak{c} = \kappa$ and both $(\mathfrak{c}, \mathfrak{c})$-gaps and $(\omega_1, \mathfrak{c})$-gaps survive by Lemma 3.20. □

Theorem 4.14 (Kunen, [33]). *It is consistent with* ZFC *that Martin's Axiom holds and there exist neither $(\mathfrak{c}, \mathfrak{c})$-gaps nor $(\omega_1, \mathfrak{c})$-gaps.*

Proof. Assume that the ground model $\mathbf{V}$ satisfies $\diamondsuit_\kappa(\mathsf{Lim} \cap \kappa)$, i.e. there exists a sequence $\langle D_\alpha; \alpha \in \mathsf{Lim} \cap \kappa \rangle$ such that

1. $D_\alpha \subseteq \alpha$ for all $\alpha \in \mathsf{Lim} \cap \kappa$, and
2. $\{\alpha \in \kappa; E \cap \alpha = D_\alpha\}$ is stationary on κ for every $E \subseteq \kappa$,

and κ is larger than ω_1. We construct a finite support iteration $\left\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha; \alpha < \kappa \right\rangle$ of length κ as follows:

Basic and successor stages. Construct to force MA by a standard book-keeping argument.

Limit stages. If D_α codes a $\mathbb{P}_\alpha$-name $(\dot{\mathcal{A}}, \dot{\mathcal{B}})$ for an (ω_1, ω_1)-gap, then let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\mathsf{Freezing}_{\mathsf{Kunen}}(\mathcal{A}, \mathcal{B})$, otherwise let $\dot{\mathbb{Q}}_\alpha$ be a trivial forcing notion.

We note that in the extension with $\mathbb{P}_\kappa$, Martin's Axiom holds and $\mathfrak{c} = \kappa$.

Assume that in the extension, there exists a (κ, κ)-gap $\langle a_\alpha, b_\alpha; \alpha < \kappa \rangle$. From now on we lead a contradiction using a standard Löwenhime-Skolem argument. (In the case of the existence of (ω_1, κ)-gaps, we can lead a contradiction by the same way.)

Claim 11. $\{\alpha < \kappa; \langle a_\gamma, b_\gamma; \gamma < \alpha \rangle$ *forms a gap in* $\mathbf{V}^{\mathbb{P}_\alpha}\}$ *is* ω_1*-club.*

Proof of claim. ω_1-closedness is trivial because if $\langle \gamma_\xi; \xi < \omega_1 \rangle$ is strictly increasing in κ and $\delta := \sup_{\xi<\omega_1} \gamma_\xi$, then

$$2^\omega \cap \mathbf{V}^{\mathbb{P}_\delta} = \bigcup_{\xi<\omega_1} 2^\omega \cap \mathbf{V}^{\mathbb{P}_{\gamma_\xi}}.$$

Taking $\alpha < \kappa$, we construct $\langle \gamma_\xi; \xi < \omega_1 \rangle \subseteq \kappa$ by recursion on $\xi < \omega_1$ such that

- $\alpha \le \gamma_0$ and $\gamma_\xi \le \gamma_\eta$ for $\xi \le \eta < \omega_1$,
- in $\mathbf{V}^{\mathbb{P}_{\gamma_{\xi+1}}}$, any $x \in [\omega]^\omega \cap \mathbf{V}^{\mathbb{P}_{\gamma_\xi}}$ does not separate $\langle a_\eta, b_\eta; \eta < \gamma_{\xi+1} \rangle$.

Then $\sup_{\xi<\omega_1} \gamma_\xi$ is in that set. ⊣

Therefore we have an $\alpha < \kappa$ such that D_α codes $\langle a_\gamma, b_\gamma; \gamma < \alpha \rangle$ and is is a gap in the extension with $\mathbb{P}_\alpha$. By the construction, $\mathbb{Q}_\alpha$ forces $\langle a_\gamma, b_\gamma; \gamma < \alpha \rangle$ to be indestructible. But in fact, $\langle a_\gamma, b_\gamma; \gamma < \alpha \rangle$ is separated by e.g. $b_{\alpha+1}$ in the extension with $\mathbb{P}_\kappa$ which is a contradiction. □

4.3 Distinguishing types of gaps in $\mathcal{P}(\omega)/\mathbf{fin}$

In this section, we prove Theorem 4.2. The main idea of this proof is the same as Kunen's proof, i.e. forcing (ω_1, ω_1)-gaps to be indestructible as far as possible. One of the differences is that to freeze (ω_1, ω_1)-gaps, we use non-ccc proper forcing notions, namely the method with models as side conditions. Forcing notions with models as side conditions are one of the types of proper forcing notions which has been introduced by Stevo Todorčević ([45]). A condition of a forcing notion of this type consists of two parts: a working part D and a side part $\mathcal{N}$ which is a finite $\in$-chain of countable elementary submodels of some large enough structure $H(\theta)$. To define such a forcing notion, we always require that $\mathcal{N}$ *separates* D, i.e.

$$\forall x \neq y \in D \exists N \in \mathcal{N}(\{x,y\} \cap N \text{ has exactly one element}).$$

(See also [32], [55].) Todorčević used the method to show that the conjecture (S) is true under the Proper Forcing Axiom. We use an open coloring in the proof, but we cannot simply apply the Open Coloring Axiom (see section 4.3.1). Our freezing forcing notions are designed based on these facts.

Before giving the proof of Theorem 4.2, we show the following theorem which is the motivation for the proof of the previous theorem:

Theorem 4.15. *Under the Proper Forcing Axiom, there exists a forcing notion which forces that Martin's Axiom holds and there are* $(\mathfrak{c},\mathfrak{c})$*-gaps but no* $(\omega_1,\mathfrak{c})$*-gaps.*

(After proving this theorem, we consider removing PFA. This strategy was used in e.g. [8].)

In [48], Todorčević has proved that under PFA there exists a σ-closed, ω_2-Baire forcing notion which adds a non-trivial almost coinciding family. Since we can construct a $(\mathfrak{c},\mathfrak{c})$-gap from a non-trivial almost coinciding family ([13]), it adds a $(\mathfrak{c},\mathfrak{c})$-gap, hence it is enough to prove that it does not add $(\omega_1,\mathfrak{c})$-gaps (Main Lemma 4.17, section 4.3.1).

In section 4.3.1, we prove Theorem 4.15, and in section 4.3.2, we prove Theorem 4.2.

4.3.1 Proof of Theorem 4.15

For subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{P}(\omega)$ and subsets $X, Y \subseteq \mathcal{A} \otimes \mathcal{B}$, we write

$$X \star Y := \{\{x,y\} \in [\mathcal{A} \otimes \mathcal{B}]^2; x \in X \ \&\ y \in Y \ \&\ x \neq y\}.$$

For an (ω_1,ω_1)-gap $(\mathcal{A},\mathcal{B})$, "*freezing* $(\mathcal{A},\mathcal{B})$ " means adding an uncountable K_0-homogeneous subset of $\mathcal{A} \otimes \mathcal{B}^+$ by some forcing notion.

Suppose that PFA holds in the ground model $\mathbf{V}$. Then $\mathfrak{c} = \aleph_2$ and there is a decreasing sequence $\langle X_\alpha; \alpha < \omega_2 \rangle$ of elements of $\mathcal{P}(\omega)$ which is a generator of an ultrafilter, i.e.

1. $\forall \alpha < \beta < \omega_2 (X_\beta \subseteq^* X_\alpha)$

2. $\forall Y \subseteq \omega \exists \alpha < \omega_1 (X_\alpha \subseteq^* Y \vee X_\alpha \subseteq^* \omega \setminus Y)$

Let $\mathcal{U}$ be the ultrafilter generated by $\langle X_\alpha; \alpha < \omega_2 \rangle$, and $\mathcal{U}^*$ the dual ideal of $\mathcal{U}$. We define a forcing notion $\mathbb{P}(\mathcal{U})(=\mathbb{P}) := \bigcup_{X \in \mathcal{U}^*} 2^X$, for conditions f, g in $\mathbb{P}$ $f \leq_{\mathbb{P}} g$ iff $g \subseteq^* f$. And we let $\mathbb{P}'(\mathcal{U})(=\mathbb{P}') := \bigcup_{X \in \mathcal{U}^*} 2^X$ be Grigorieff forcing ([20]), i.e. for conditions f, g in $\mathbb{P}'$, $f \leq_{\mathbb{P}'} g$ iff $g \subseteq f$. ($\mathbb{P}$ and $\mathbb{P}'$ have the same underlying set. The only difference is the ordering, but $\mathbf{1}(=\emptyset)$ is the strongest condition in both $\mathbb{P}$ and $\mathbb{P}'$.) We must note that $\mathbb{P}'(\mathcal{U})$ is proper if $\mathcal{U}$ is a fat p-filter (by Shelah, see [42]). Now, since $\mathcal{U}$ satisfies the properties of fat-ness and p-filter, $\mathbb{P}'(\mathcal{U})$ is a proper forcing notion.

The following proposition is very similar to [48] and [50].

Proposition 4.16. *$\mathbb{P}$ is σ-closed, ω_2-Baire and adds an (ω_2, ω_2)-gap (under* PFA*).*

Proof. (For the first two statements, see [48]) σ-closed-ness is trivial. Assume that $D_\alpha \subseteq \mathbb{P}$ is open dense in $\mathbb{P}$ for $\alpha < \omega_1$ and $f \in \mathbb{P}$. We note that $D'_\alpha := \bigcap_{\gamma < \alpha} D_\gamma$ is also open and dense in $\mathbb{P}'$ for $\alpha < \omega_1$. By $\mathsf{MA}_{\aleph_1}(\mathbb{P}')$, we can find a filter $G \subseteq \mathbb{P}'$ which meets all D'_α for $\alpha < \omega_1$, say $f_\alpha \in G \cap D'_\alpha$ and $f \in G$. Since the length of the generating sequence of $\mathcal{U}$ is ω_2, there is an $X \in \mathcal{U}^*$ with $\mathrm{dom}(f_\alpha) \subseteq^* X$ for all $\alpha < \omega_1$. By the counting argument, there are $n \in \omega$ and $I \in [\omega_1]^{\omega_1}$ such that $\mathrm{dom}(f_\alpha) \subseteq X \cup n$ for every $\alpha \in I$. Since G is a filter in $\mathbb{P}'$, $\bigcup_{\alpha \in I} f_\alpha$ is a function. So we can find $g \in 2^{X \cup n} \subseteq \mathbb{P}$ with $g \subseteq \bigcup_{\alpha \in I} f_\alpha$, then $g \in \bigcap_{\alpha \in I} D'_\alpha$, i.e. $g \in \bigcap_{\alpha < \omega_1} D_\alpha$ and g is an extension of f (in both $\mathbb{P}$ and $\mathbb{P}'$).

For the last statement, let G be a $\mathbb{P}$-generic filter over $\mathbf{V}$. Then we may take a condition $f_\alpha \in G \cap 2^{X_\alpha}$ for every $\alpha < \omega_2$, and let $a_\alpha := \{n \in X_\alpha; f_\alpha(n) = 0\}$ and $b_\alpha := \{n \in X_\alpha; f_\alpha(n) = 1\}$. We show that $(\{a_\alpha; \alpha < \omega_2\}, \{b_\alpha; \alpha < \omega_2\})$ is an (ω_2, ω_2)-gap in $\mathbf{V}[G]$. By the σ-closedness of $\mathbb{P}$ (so $\mathbb{P}$ adds no new reals), it suffices to show that

$$\forall c \in [\omega]^\omega \cap \mathbf{V} \forall p \in \mathbb{P} \exists q \leq p \exists \alpha < \omega_2 \left(q \Vdash \text{“ } \check{c} \text{ does not separate } \left\langle \dot{a}_\alpha, \dot{b}_\alpha \right\rangle \text{ ”} \right).$$

Given such c and p, we have cases:

$c \cap \mathrm{dom}(p)$ *is finite.* Then we can find $\alpha < \omega_2$ and $q \leq p$ which forces that “ $\check{c} \subseteq^* \dot{a}_\alpha$ ”.

$c \cap p^{-1}\{0\}$ *is infinite.* Then we can find $\alpha < \omega_2$ and $q \leq p$ which forces that “ $\dot{a}_\alpha \not\subseteq^* \check{c}$ ”.

$c \cap p^{-1}\{1\}$ *is infinite.* Then we can find $\alpha < \omega_2$ such that p forces that "$\check{c} \not\perp \dot{b}_\alpha$".

□

Therefore in the extension with $\mathbb{P}$ over $\mathbf{V}$, there are no new reals and MA holds. The next lemma completes the proof of theorem 4.15.

Freezing gaps by a non-ccc proper forcing

Main Lemma 4.17. $\mathbb{P}$ *adds no* (ω_1, ω_2)*-gaps (under* PFA*).*

We devote the rest of the section to prove Lemma 4.17, so this subsection completes the proof of Theorem 4.15.

Assume that in the extension with $\mathbb{P}$, there exists an (ω_1, ω_2)-gap, whose $\mathbb{P}$-name is $(\dot{\mathcal{A}}, \dot{\mathcal{B}})$, i.e.

$$\Vdash_{\mathbb{P}} \text{“ } (\dot{\mathcal{A}}, \dot{\mathcal{B}}) \text{ is an } (\omega_1, \omega_2)\text{-gap”}.$$

Since $\mathbb{P}$ is ω_2-Baire in $\mathbf{V}$, there are $\mathcal{A} \in \mathbf{V}$ and $f \in \mathbb{P}$ such that $f \Vdash_{\mathbb{P}}$ "$\dot{\mathcal{A}} = \check{\mathcal{A}}$". So by the homogeneity of $\mathbb{P}$, without loss of generality, we may assume that $\Vdash_{\mathbb{P}}$"$\dot{\mathcal{A}} = \check{\mathcal{A}}$".

We recall that

$$\Vdash_{\mathbb{P}} \text{“ } \dot{\mathcal{B}}^+ = \{c \subseteq \omega; \exists b \in \dot{\mathcal{B}}(c \subseteq^* b)\} \text{ ”}.$$

Then

$$\Vdash_{\mathbb{P}} \text{“ } (\check{\mathcal{A}}, \dot{\mathcal{B}}^+) \text{ also forms a gap ”}.$$

In $\mathbf{V}$, for all $f \in \mathbb{P}$, let $\mathcal{B}^+(f) := \{b \subseteq \omega; \exists g \leq_{\mathbb{P}} f(g \Vdash_{\mathbb{P}} \text{“ } \check{b} \in \dot{\mathcal{B}}^+ \text{ ”})\}$. It is trivial that for conditions f, g in $\mathbb{P}$, $f \Vdash_{\mathbb{P}}$"$\mathcal{B}^+(f) \supseteq \dot{\mathcal{B}}^+$" and if $f \leq_{\mathbb{P}} g$, then $\mathcal{B}^+(f) \subseteq \mathcal{B}^+(g)$.

Proposition 4.18. *For every* $f \in \mathbb{P}$, $\mathcal{A} \otimes \mathcal{B}^+(f)$ *is not a union of countably many* K_1*-homogeneous subsets.*

Proof. Assume not, i.e. $\mathcal{A} \otimes \mathcal{B}^+(f) = \bigcup_{n \in \omega} \mathcal{H}_n$ and every $\mathcal{H}_n$ is K_1-homogeneous. For $n \in \omega$, let $c_n := \bigcup_{\langle a,b \rangle \in \mathcal{H}_n} b$. Let G be a $\mathbb{P}$-generic filter over $\mathbf{V}$ containing f. We show that $\{c_n; n \in \omega\}$ separates $(\mathcal{A}, \dot{\mathcal{B}}^+[G])$ in $\mathbf{V}[G]$.

To prove this, taking a pair $a \in \mathcal{A}$ and $b \in \dot{\mathcal{B}}^+[G]$ with $a \cap b = \emptyset$, we pick $n \in \omega$ with $\langle a, b\rangle \in \mathcal{H}_n$. Since $b \in \dot{\mathcal{B}}^+[G]$, $f \in G$ and $\mathbb{P}$ does not add new reals, b is in $\mathcal{B}^+(f)$. Hence $a \cap c_n = \emptyset$ and $b \subseteq c_n$.

Since both $\mathcal{A}$ and $\mathcal{B}^+(f)$ are σ-directed, $(\mathcal{A}, \dot{\mathcal{B}}^+[G])$ is separated, which is a contradiction. □

We will find $\tilde{f} \in \mathbb{P}$ and $\mathcal{X} \subseteq \mathcal{A} \otimes \mathcal{B}^+(1)$ such that

1. $\mathcal{X}$ is uncountable and K_0-homogeneous, and
2. for all $\langle a, b\rangle \in \mathcal{X}$, $\tilde{f} \Vdash_{\mathbb{P}}$“ $\check{b} \in \dot{\mathcal{B}}^+$ ”,

which completes the proof of Main Lemma 4.17, because then

$$\tilde{f} \Vdash_{\mathbb{P}}\text{“ } \check{\mathcal{X}} \text{ forms an } (\omega_1, \omega_1)\text{-indestructible gap in } (\check{\mathcal{A}}, \dot{\mathcal{B}}^+) \text{ ”},$$

which is a contradiction.

In fact we can get an uncountable K_0-homogeneous subset of $\mathcal{A} \otimes \mathcal{B}^+(1)$ applying OCA. But now we need the condition $\tilde{f}$ as above to get a contradiction. To get the desired objects, we consider the extension by the following forcing notion $\mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$ (which is *the freezing forcing* with models as side conditions). It is similar to forcing notions in [45]. The difference is that in this definition both $\mathbb{P}$ and $\mathbb{P}'$ appear, in particular we use the proper-ness of $\mathbb{P}'$ to define it.

Definition 4.19. *A condition of $\mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$ is a triple $p = \langle f_p, X_p, \mathfrak{N}_p\rangle$ satisfying the following statements:*

(a) *f_p is a member of $\bigcup_{X \in \mathcal{U}^*} 2^X$,*

(b) *X_p is a finite K_0-homogeneous subset of $\mathcal{A} \otimes \mathcal{B}^+(1)$,*

(c) *$\mathfrak{N}_p$ is a finite $\in$-chain of countable elementary submodels of $H(\mathfrak{c}^+)(= H(\aleph_3))$ containing everything we need for our discussion, e.g. $\mathcal{A}$, $\dot{\mathcal{B}}$, $\mathcal{U}$, etc $\cdots$ (i.e. $\mathfrak{N}_p$ can be enumerated by $\{N_i; i < n\}$ such that for all $i < n-1$, $N_i \in N_{i+1}$ and N_i is an elementary submodel of N_{i+1} (say $N_i \prec N_{i+1}$)),*

(d) *for any $x\ (= \langle a_x, b_x\rangle) \in X_p$, $f_p \Vdash_{\mathbb{P}}$“ $\check{b}_x \in \dot{\mathcal{B}}^+$ ”,*

(e) *for any* $x, y \in X_p$ *with* $x \neq y$ *there exists* $N \in \mathfrak{N}_p$ *so that* $|N \cap \{x, y\}| = 1$, *(define* $x \lhd y : \iff \exists N \in \mathfrak{N}_p (x \in N \,\&\, y \notin N)$*),*

(f) *for all* $N \in \mathfrak{N}_p$, f_p *is* $(N, \mathbb{P}')$*-generic, and*

(g) *for every* $x \in X_p$ *and* $N \in \mathfrak{N}_p$ *with* $x \notin N$,

$$f_p \Vdash_{\mathbb{P}'} \text{“} \forall Y \in \check{N}[\dot{G}](Y \subseteq \check{\mathcal{A}} \otimes \mathcal{B}^{+}\check{(1)} \ \&\ Y \star Y \subseteq K_1 \Rightarrow \check{x} \notin Y)\text{”}.$$

For conditions $p, q \in \mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$,

$$p \leq_{\mathbb{Q}(\mathcal{A},\dot{\mathcal{B}},\mathcal{U})} q : \iff f_p \supseteq f_q \text{ (i.e. } f_p \leq_{\mathbb{P}'} f_q) \ \&\ X_p \supseteq X_q \ \&\ \mathfrak{N}_p \supseteq \mathfrak{N}_q.$$

We note that $\mathfrak{N}_p$ is an element of $H(\aleph_3)$ because every element of $\mathfrak{N}_p$ is a countable subset of $H(\aleph_3)$ and $\mathfrak{N}_p$ is finite. We must show that $\mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$ adds desired objects. To show this, we need two lemmata:

Lemma 4.20. *Under* $\mathfrak{m}(\mathbb{P}') = \mathfrak{c} = \aleph_2$ *(in particular under* PFA*),*

$\Vdash_{\mathbb{P}'}$ “$\check{\mathcal{A}} \otimes \mathcal{B}^{+}\check{(1)}$ *is not a union of countably many* K_1*-homogeneous subsets* ”.

More explicitly, for any $\mathbb{P}'$*-names* $\dot{X}_n$ *for* K_1*-homogeneous subsets,* $n \in \omega$ *and* $f \in \mathbb{P}'$*, there exist* $f' \leq_{\mathbb{P}'} f$ *and* $\langle a, b \rangle \in \mathcal{A} \otimes \mathcal{B}^{+}(1)$ *such that*

$$f' \Vdash \text{“} \check{b} \in \dot{\mathcal{B}}^{+} \ \&\ \langle \check{a}, \check{b} \rangle \notin \dot{X}_n \text{ for every } n \in \omega \text{ ”}.$$

Lemma 4.21. *For* $\mathcal{A}$, $\dot{\mathcal{B}}$ *and* $\mathcal{U}$, $\mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$ *is proper.*

Lemma 4.20 says that $\mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$ is non-atomic, and Lemma 4.21 is needed to apply PFA to $\mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$.

Proof of Lemma 4.20. For a $\mathbb{P}'$-name $\dot{c}$ for an infinite subset of ω and a condition $f \in \mathbb{P}'$, let $c(f) := \{k \in \omega; \exists g \leq_{\mathbb{P}'} f \ (g \Vdash_{\mathbb{P}'} \text{“} \check{k} \in \dot{c} \text{ ”})\}$. Then for a $\mathbb{P}'$-name $\dot{c}$ for an infinite subset of ω,

- for conditions $f, g \in \mathbb{P}'$ with $f \leq_{\mathbb{P}'} g$, $c(f) \subseteq c(g)$, and
- $\Vdash_{\mathbb{P}'}$“ $\dot{c} = \bigcap_{f \in \dot{G}} c\check{(f)}$ ”.

For $\mathbb{P}'$-name $\dot{c}$ for an infinite subset of ω and an infinite subset a of ω, let $D(\dot{c}, a) := \{f \in \mathbb{P}'; \forall g \leq_{\mathbb{P}'} f\ (c(g) \not\perp a) \text{ or } c(f) \perp a\}$. Then we note that $D(\dot{c}, a)$ is dense open in $\mathbb{P}'$.

To finish the proof of Lemma 4.20, by Proposition 4.5, it suffices to show that

$$\Vdash_{\mathbb{P}'} \text{“ } (\check{\mathcal{A}}, \check{\mathcal{B}^+(1)}) \text{ is not countably separated ”}.$$

We enumerate $\mathcal{A}$ by $\langle a_\xi; \xi < \omega_1 \rangle$ with $a_\xi \subseteq^* a_\eta$ for each $\xi \leq \eta < \omega_1$. Let $\langle \dot{c}_n; n \in \omega \rangle$ be countable $\mathbb{P}'$-names for infinite subsets of ω and $f \in \mathbb{P}'$. We'll find an extension f'' of f in $\mathbb{P}'$ so that

$$f'' \Vdash_{\mathbb{P}'} \text{“ } \langle \dot{c}_n; n \in \omega \rangle \text{ does not separete } (\check{\mathcal{A}}, \check{\mathcal{B}^+(1)}) \text{ ”},$$

which completes the proof.

By induction on $n \in \omega$, we construct $g_n \in \mathbb{P}'$ such that

- $g_0 = f$ and $g_{n+1} \leq_{\mathbb{P}'} g_n$, and
- for $i < n$, $t \in 2^n$ and almost all $\xi < \omega_1$ (i.e. all $\xi \geq \eta$ for some η), $t \cup g_n \restriction [n, \infty) \in D(\dot{c}_i, a_\xi)$.

Construction : $n \to n+1$. Let $\{t_j; j < N(= 2^{n+1})\}$ enumerate 2^{n+1} (the set of binary sequences of length n+1). We recursively define $h_j \in \mathbb{P}'$ for $j < N$ such that

- $h_{-1} = g_n$ and $h_{j+1} \leq_{\mathbb{P}'} h_j$
- for $i < n+1$ and almost all $\xi < \omega_1$, $t_j \cup h_j \restriction [n+1, \infty) \in D(\dot{c}_i, a_\xi)$.

Assume that we have already constructed h_{j-1}, and let $h_j^{(-1)} := t_j \cup h_{j-1} \restriction [n+1, \infty) \in \mathbb{P}'$. Since $\mathbb{P}'$ is proper, by PFA, we can find a filter G on $\mathbb{P}'$ containing $h_j^{(-1)}$, which meets all $D(\dot{c}_0, a_\xi)$ for $\xi < \omega_1$. We pick a condition $l_\xi \in G \cap D(\dot{c}_0, a_\xi)$ for every $\xi < \omega_1$. Since the generator of $\mathcal{U}$ is a decreasing sequence of length ω_2 again, there exists an $X \in \mathcal{U}^*$ such that $\mathrm{dom}(l_\xi) \subseteq^* X$ for all $\xi < \omega_1$. By a counting argument, there are an uncountable subset A of ω_1 and $n \in \omega$ such that $\mathrm{dom}(l_\xi) \subseteq X \cup n$ for all $\xi \in A$. Since G is a filter in $\mathbb{P}'$, $\bigcup_{\xi \in A} l_\xi$ is a function with the domain which is a subset of $X \cup n$. So we can find a condition $h_j^{(0)} \in \mathbb{P}'$ with the domain $X \cup n$ and $h_j^{(0)} \supseteq \bigcup_{\xi \in A} l_\xi$. (This $h_j^{(0)}$ is like a *fusion* of $\langle l_\xi; \xi < \omega_1 \rangle$ in $\mathbb{P}'$.) Then for almost all $\xi < \omega_1$,

$h_j^{(0)}$ is in $D(\dot{c}_0, a_\xi)$:
If $\exists\xi \in A\forall l \leq_{\mathbb{P}'} l_\xi(c_0(l) \not\perp a_\xi)$, then $\forall\eta \geq \xi\forall l \leq_{\mathbb{P}'} h_j^{(0)}(c_0(l) \not\perp a_\eta)$ because $\langle a_\xi; \xi < \omega_1\rangle$ is increasing and $h_j^{(0)}$ is an extension of l_ξ in $\mathbb{P}'$.
If not, i.e. $\forall\xi \in A(c_0(l_\xi) \perp a_\xi)$ (by $l_\xi \in D(\dot{c}_0, a_\xi)$), then $\forall\xi \in A(c_0(h_j^{(0)}) \perp a_\xi)$. Thus $\forall\xi < \omega_1(c_0(h_j^{(0)}) \perp a_\xi)$ because $\langle a_\xi; \xi < \omega_1\rangle$ is increasing again and A is uncountable in ω_1.

Repeating this argument $(n+1)$ many times, we get conditions $h_j^{(n+1)} \leq_{\mathbb{P}'} h_j^{(n)} \leq_{\mathbb{P}'} \cdots \leq_{\mathbb{P}'} h_j^{(0)}$ such that for $i \leq n$ and almost all $\xi < \omega_1$, $h_j^{(i)}$ is in $D(\dot{c}_i, a_\xi)$. Let $h_j := h_{j-1} \cup h_j^{(n+1)} \restriction [n+1, \infty) \in \mathbb{P}'$. Since any $D(\dot{c}, a)$ is open, h_j is as desired, and let $g_{n+1} := h_{N-1}$ which completes a construction.

Since the generator of $\mathcal{U}$ is a decreasing sequence of length ω_2, there is $X \in \mathcal{U}^*$ such that for any $n \in \omega$, $\text{dom}(g_n) \subseteq^* X$. Without loss of generality, we may assume that $\text{dom}(f) = \text{dom}(g_0) \subseteq X$. For all $i > 1$, we pick $m_i > i$ such that $\text{dom}(g_i)\setminus m_i \subseteq X$. Since $\mathcal{U}$ is an ultrafilter (in particular $\mathcal{U}$ is fat, see [42]), there is an increasing sequence $\langle l(n); n < \omega\rangle$ of natural numbers such that $X \cup \bigcup_{n\in\omega}[l(n), m_{l(n)}) \in \mathcal{U}^*$. Then let $f' := f \cup \bigcup_{n\in\omega}(g_{l(n)} \restriction [l(n), \infty))$, which is a condition of $\mathbb{P}'$. (This f' is also like a *fusion* of $\langle g_i; i < \omega\rangle$ in $\mathbb{P}'$.) Then for every $n \in \omega$, $i < l(n)$, $t \in 2^{l(n)}$ and almost all $\xi < \omega_1$,

$$t \cup f' \restriction [l(n), \infty) \in D(\dot{c}_i, a_\xi)$$

(since $t \cup f' \restriction [l(n), \infty) \leq_{\mathbb{P}'} t \cup g_{l(n)} \restriction [l(n), \infty) \in D(\dot{c}_i, a_\xi)$ and $D(\dot{c}_i, a_\xi)$ is open), i.e.

Case 1. for some $\xi < \omega_1$, $\forall h \leq_{\mathbb{P}'} t \cup f' \restriction [l(n), \infty)(c_i(h) \not\perp a_\xi)$
(denote such a ξ by $\xi(n, i, t)$) or,

Case 2. for all $\xi < \omega_1$, $c_i(t \cup f' \restriction [l(n), \infty)) \perp a_\xi$
(say $d(n, i, t) := c_i(t \cup f' \restriction [l(n), \infty))$, in this case $d(n, i, t)$ is in $\mathcal{A}^\perp$).

Let $\eta := \sup\{\xi(n, i, t); \langle n, i, t\rangle$ satisfies case 1$\}$ (which is $< \omega_1$) and $a := a_\eta$.

Let $\{d_j; j \in \omega\}$ enumerate $\{d(n, i, t); \langle n, i, t\rangle$ satisfies case 2$\}$. We recursively construct $g'_n \in \mathbb{P}$ and $b_n \subseteq \omega$ for $n \in \omega$ such that

- $g'_{n+1} \leq_{\mathbb{P}} g'_n$ and $b_n \subseteq^* b_{n+1}$,
- $g'_n \Vdash_{\mathbb{P}}$ " $\check{b}_n \in \dot{\mathcal{B}}^+$ ", and

- $b_n \not\subseteq^* d_n$.

Construction : $n \to n+1$. (Let $g_{-1} := \mathbf{1}$.) Since $d_{n+1} \in \mathcal{A}^\perp$ and $(\mathcal{A}, \mathcal{B}^+(g'_n))$ forms a gap (by Proposition 4.18), there is $b_{n+1} \in \check{\mathcal{B}}^+(g'_n)$ such that $b_{n+1} \not\subseteq^* d_{n+1}$. We pick $g'_{n+1} \leq_{\mathbb{P}} g'_n$ such that $g'_{n+1} \Vdash_{\mathbb{P}}$ " $\check{b}_{n+1} \in \dot{\mathcal{B}}^+$ " which completes the construction.

Since $\mathbb{P}$ is σ-closed, there exists a condition g_∞ of $\mathbb{P}$ which is an extension of every g'_n for $n \in \omega$.

Then there are $f'' \leq_{\mathbb{P}} g_\infty$ and $b' \subseteq \omega$ such that $f'' \Vdash_{\mathbb{P}}$ " $\check{b}' \in \dot{\mathcal{B}}^+$ " and $b_n \subseteq b'$ for all $n \in \omega$. Let $b := b' \setminus (a \cap b')$. Then $\langle a, b \rangle \in \mathcal{A} \otimes \mathcal{B}^+(\mathbf{1})$. We show that

$$f'' \Vdash_{\mathbb{P}'} \text{“ } \langle \dot{c}_n; n \in \omega \rangle \text{ does not separate } \langle \check{a}, \check{b} \rangle \text{ ”},$$

which completes the proof of Lemma 4.20.

Assume not, there are $i \in \omega$ and $g \leq_{\mathbb{P}'} f''$ such that $g \Vdash_{\mathbb{P}'}$" $\check{a} \perp \dot{c}_i$ & $\check{b} \subseteq^* \dot{c}_i$ ". Let $n \in \omega$ be such that $i < l(n)$ and $t := g \restriction |l(n)| \in 2^{l(n)}$. Then for almost all $\xi < \omega_1$, $t \cup f' \restriction [l(n), \infty) \in D(\dot{c}_i, a_\xi)$.
If $\langle n, i, t \rangle$ satisfies case 1, for any $h \leq_{\mathbb{P}'} g$, $c_i(h) \not\perp a$ holds, so $g \Vdash_{\mathbb{P}'}$" $\dot{c}_i \not\perp \check{a}$ ", which is a contradiction.
If $\langle n, i, t \rangle$ satisfies case 2, since $c_i(g) \subseteq c_i(t \cup f' \restriction [l(n), \infty)) = d(n, i, t) =: d_j$ and $b_j \subseteq^* b' =^* b$ and $b_j \not\subseteq^* d_j$, $b \not\subseteq^* c_i(g)$ holds, so $g \Vdash_{\mathbb{P}'}$" $\check{b} \not\subseteq \dot{c}_i$ ", which is a contradiction. □

Proof of Lemma 4.21. Let θ be a large enough regular cardinal, $M \prec H(\theta)$ a countable elementary submodel containing everything needed for our discussion, e.g. $\mathcal{A}$, $\dot{\mathcal{B}}$, $\mathcal{U}$, $H(\aleph_3)$ etc, and $p = \langle f_p, X_p, \mathfrak{N}_p \rangle \in M$ a condition of $\mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}) (= \mathbb{Q})$.

Since $\mathbb{P}'$ is proper, we can choose an extension $f_q \leq_{\mathbb{P}'} f_p$ such that f_q is $(M \cap H(\aleph_3), \mathbb{P}')$-generic. (We note that $M \cap H(\aleph_3)$ is an elementary submodel of $H(\aleph_3)$.) Then let $q := \langle f_q, X_p, \mathfrak{N}_p \cup \{M \cap H(\aleph_3)\} \rangle$, which is a condition of $\mathbb{Q}$. (We note that $\mathfrak{N}_p \subseteq M \cap H(\aleph_3)$, in fact $\mathfrak{N}_p \in M \cap H(\aleph_3)$ holds since $p \in M$ and $\mathfrak{N}_p \in H(\aleph_3)$.) Show that q is $(M, \mathbb{Q})$-generic, i.e. for every dense open subset $\mathcal{D} \in M$ in $\mathbb{Q}$ and an extension $r \in \mathbb{Q}$ of q there exists a condition $s \in \mathcal{D} \cap M$ such that r and s are compatible in $\mathbb{Q}$ (i.e. $\mathcal{D} \cap M$ is predense in $\mathbb{Q}$).

Taking such $\mathcal{D} \in M$ and $r \leq_{\mathbb{Q}} q$, without loss of generality, we may assume that r is in $\mathcal{D}$. Let $X_r \setminus M = \{x_i; i \leq n\}$ where $x_i \lhd x_{i+1}$ for $i < n$

and $N_0 := M \cap H(\aleph_3)$, and pick $N_i \in \mathfrak{N}_r$ such that $x_{i-1} \in N_i$ but $x_i \notin N_i$ for $1 \leq i \leq n$. We choose rational open intervals $U_i \subseteq \mathcal{A} \otimes \mathcal{B}^+(1)$ such that

- $x_i \in U_i$ for $i \leq n$,
- $U_i \cap U_j = \emptyset$ and $U_i \star U_j \subseteq K_0$ for every $i, j \leq n$ with $i \neq j$.

(We recall that K_0 is open, so this can be done.) We note that all rational open intervals are in any model of ZFC because those codes consists of finite elements. Let G be $\mathbb{P}'$-generic over $H(\theta)$ with $f_r \in G$.

Claim 12. *In $H(\theta)[G]$, there are rational open intervals $V_i^0, V_i^1 \subseteq U_i$ and $y_i \in V_i^1 \cap \mathbf{V}$ for $i \leq n$ and $s \in \mathcal{D} \cap \mathbf{V}$ such that*

1. *$x_i \in V_i^0$ for all $i \leq n$,*
2. *$V_i^0 \cap V_i^1 = \emptyset$ and $V_i^0 \star V_i^1 \subseteq K_0$ for all $i \leq n$,*
3. *$f_s \in G$,*
4. *$X_s = (X_r \cap M) \cup \{y_i; i \leq n\}$ and for any $x \in X_r \cap M$ and $i < j \leq n$, and $x \lhd y_i \lhd y_j$,*
5. *$\mathfrak{N}_s$ is an end extension of $\mathfrak{N}_r \cap M$.*

Proof. By induction on $i \leq n$, we construct rational open intervals $V_{n-i}^0, V_{n-i}^1 \subseteq U_{n-i}$, $y_{n-i}^{n-j} \in V_{n-j}^1 \cap \mathbf{V}$ for $j \leq i$ and $s_{n-j} \in \mathcal{D} \cap \mathbf{V}$ such that

1'. $x_{n-i} \in V_{n-i}^0$,

2'. $V_{n-i}^0 \cap V_{n-i}^1 = \emptyset$ and $V_{n-i}^0 \star V_{n-i}^1 \subseteq K_0$,

3'. $f_{s_{n-i}} \in G$,

4'. $X_{s_{n-i}} = (X_{s_{n-i}} \cap N_{n-i}) \cup \{y_{n-i}^{n-j}; j \leq i\}$ and for any $x \in X_{s_{n-i}} \cap N_{n-i}$ and $j < k \leq n$, $x \lhd y_{n-i}^{n-k} \lhd y_{n-i}^{n-j}$, and

5'. $\mathfrak{N}_{s_{n-i}}$ is an end extension of $\mathfrak{N}_{s_{n-i+1}} \cap N_{n-i}$.

Construction. Assume that we have already constructed $V_{n-i}^0, V_{n-i}^1, y_{n-i}^{n-j}, s_{n-j}$ for all $j < i$.

Let

$$Y_{n-i} := \{x \in U_{n-i} \cap \mathbf{V};\ \exists z_n \in V_n^1 \cap \mathbf{V} \cdots \exists z_{n-i+1}^1 \in V_{n-i+1}^1 \cap \mathbf{V} \exists s \in \mathcal{D} \cap \mathbf{V} \text{ s.t.}$$

- $f_s \in G$
- $X_s = (X_{s_{n-i+1}} \cap N_{n-i}) \cup \{x\} \cup \{z_{n-j}; j < i\}$
- $\forall z \in X_{s_{n-i+1}} \cap N_{n-i} \forall j < k \leq i (z \lhd x \lhd z_{n-k} \lhd z_{n-j})$
- $\mathfrak{N}_s$ is an end extension of $\mathfrak{N}_{s_{n-i+1}} \cap N_{n-i}$ $\}$.

Then $Y_{n-i} \in N_{n-i}[G]$ and $x_{n-i} \in Y_{n-i}$ by *3'*, *4'* and *5'*. Since $x_{n-i} \in Y_{n-i}$, by (**g**), Y_{n-i} is not K_1-homogeneous. Let $\overline{Y_{n-i}} := \{x \in Y_{n-i}; \exists y \in Y_{n-i} \setminus \{x\}(\{x,y\} \in K_0)\}$. Then $Y_{n-i} \setminus \overline{Y_{n-i}}$ is in $N_{n-i}[G]$ and K_1-homogeneous, hence x_{n-i} belongs to $\overline{Y_{n-i}}$ by (**g**) again. Therefore there exists $y_{n-i}^{n-i} \in Y_{n-i} \setminus \{x_{n-i}\}$ such that $\{x_{n-i}, y_{n-i}^{n-i}\}$ is in K_0. Then We take rational open intervals $V_{n-i}^0, V_{n-i}^1 \subseteq U_{n-i}$ such that $x_{n-i} \in V_{n-i}^0$, $V_{n-i}^0 \cap V_{n-i}^1 = \emptyset$ and $V_{n-i}^0 \star V_{n-i}^1 \subseteq K_0$. By $y_{n-i}^{n-i} \in Y_{n-i}$, there are $y_{n-i}^n \in V_n^1 \cap \mathbf{V}$, $\cdots$, $y_{n-i}^{n-i+1} \in V_{n-i+1}^1 \cap \mathbf{V}$ and $s_{n-i} \in \mathcal{D} \cap \mathbf{V}$ satisfying *3'*, *4'* and *5'*, which completes a construction. Put $y_i := y_0^i$ for $i \leq n$ and $s := s_0$, then these are as desired. $\dashv$

Since M is an elementary submodel of $H(\theta)$, $M[G]$ is an elementary submodel of $H(\theta)[G]$. So by the previous claim, there are $y_i \in V_i^1 \cap M[G] \cap \mathbf{V}$ for $i \leq n$ and $s \in \mathcal{D} \cap M[G] \cap \mathbf{V}$ satisfying *3*, *4* and *5* of the claim. Then we take a condition $g \in G$ which decides all values of V_i^0, V_i^1, y_i for all $i \leq n$ and s. By the separability of $\mathbb{P}'$, g is an extension of f_s in $\mathbb{P}'$. We may assume that $g \leq_{\mathbb{P}'} f_r$ because both g and f_r are in a filter G. Then we note that g is also a common extension of f_r and f_s in $\mathbb{P}$. By the construction, $\langle g, X_s \cup X_r, \mathfrak{N}_r \cup \mathfrak{N}_s \rangle$ is a condition of $\mathbb{Q}$ and a common extension of r and s. $\square$

To get $\tilde{f}$ and $\mathcal{X}$, we take any countable elementary submodel M of $H(\theta)$ containing $\mathcal{A}$, $\dot{\mathcal{B}}$, $\mathcal{U}$, $H(\aleph_3)$, etc. Let $M_0 := M \cap H(\aleph_3)$ and pick a $(M_0, \mathbb{P}')$-generic condition $f \in \mathbb{P}'$. We notice that $\mathcal{P}(\omega) \cap M_0 = \mathcal{P}(\omega) \cap M$. Then by Lemma 4.20,

$$f \Vdash_{\mathbb{P}'} \text{“} \check{\mathcal{A}} \otimes \check{\mathcal{B}^+}(1) \setminus \bigcup \{Y \in \check{M}_0[\dot{G}]; Y \subseteq \check{\mathcal{A}} \otimes \check{\mathcal{B}^+}(1) \ \&\ Y \star Y \subseteq K_1\} \neq \emptyset \text{”}.$$

So by Lemma 4.20, there are $x \in \mathcal{A} \otimes \mathcal{B}^+(1)$ and $g \leq_{\mathbb{P}'} f$ such that

$$g \Vdash_{\mathbb{P}'} \text{“} \check{b}_x \in \dot{\mathcal{B}}^+ \ \&\ \check{x} \notin \bigcup \{Y \in \check{M}_0[\dot{G}]; Y \subseteq \check{\mathcal{A}} \otimes \check{\mathcal{B}^+}(1) \ \&\ Y \star Y \subseteq K_1\} \text{”}.$$

Let $p := \langle g, \{x\}, \{M_0\}\rangle$ which is a condition of $\mathbb{Q}$ and we can show that p is $(M, \mathbb{Q})$-generic by the same argument as in the proof of Lemma 4.21. The following lemma indicates the density argument of $\mathbb{Q}$.

Lemma 4.22.

$$p \Vdash_{\mathbb{Q}} \text{“ } \dot{\mathcal{X}} := \bigcup\{X_q; q \in \dot{G}\} \textit{ is uncountable } K_0\textit{-homogeneous ”}.$$

Proof. It is trivial that $\Vdash_{\mathbb{Q}}$“ $\dot{\mathcal{X}}$ is K_0-homogeneous ”. From now on we show that $\Vdash_{\mathbb{Q}}$“ $\dot{\mathcal{X}}$ is uncountable ”.

Assume not, then there is $q \leq_{\mathbb{Q}} p$ so that $q \Vdash_{\mathbb{Q}}$“ $\dot{\mathcal{X}}$ is countable ”. Now $q \Vdash_{\mathbb{Q}}$“ $\check{x} \in \dot{\mathcal{X}}$ ”. Since $\Vdash_{\mathbb{Q}}$“ $\dot{G} \in \check{M}_0[\dot{G}]$ ”, $\Vdash_{\mathbb{Q}}$“ $\dot{\mathcal{X}} \in \check{M}_0[\dot{G}]$ ”. Since $q \Vdash_{\mathbb{Q}}$“ $\dot{\mathcal{X}}$ is countable ”, $q \Vdash_{\mathbb{Q}}$“ $\dot{\mathcal{X}} \subseteq \check{M}_0[\dot{G}]$ ”. So $q \Vdash_{\mathbb{Q}}$“ $\check{x} \in \dot{\mathcal{X}} \subseteq \check{M}_0[\dot{G}]$ ”. Now q is $(M, \mathbb{Q})$-generic, $q \Vdash_{\mathbb{Q}}$“ $\check{M}[\dot{G}] \cap \mathbf{V} = \check{M}$, so $\check{M}_0[\dot{G}] \cap \mathcal{P}(\omega) \cap \mathbf{V} = \check{M}[\dot{G}] \cap \mathcal{P}(\omega) \cap \mathbf{V} = \check{M} \cap \mathcal{P}(\omega) = \check{M}_0 \cap \mathcal{P}(\omega)$ ”. Thus $q \Vdash_{\mathbb{Q}}$“ $\check{x} \notin \check{M}_0[\dot{G}]$ ”, because of $x \notin M_0$, which is a contradiction. □

Applying PFA to $\mathbb{Q}$, we can get a filter $G \subset \mathbb{Q}$ such that $\mathcal{X}' = \bigcup\{X_p; p \in G\}$ is uncountable and K_0-homogeneous. Let $\{x_\alpha; \alpha < \omega_1\}$ list $\mathcal{X}'$. For each $\alpha < \omega_1$, we choose $p_\alpha \in G$ with $x_\alpha \in X_{p_\alpha}$. Then we take a fusion $\tilde{f}$ of $\langle f_{p_\alpha}; \alpha < \omega_1\rangle$, i.e. take $X \in \mathcal{U}^*$, a natural number n and an uncountable subset A of ω_1 such that $\mathrm{dom}(f_{p_\alpha}) \subseteq X \cup n$ for all $\alpha \in A$ and take a condition $\tilde{f} \in 2^{X \cup n}$ of $\mathbb{P}'$ with $\tilde{f} \supseteq \bigcup_{\alpha \in A} f_{p_\alpha}$. Then $\tilde{f}$ is an extension of f_{p_α} in $\mathbb{P}$ for all $\alpha \in A$, so $\tilde{f} \Vdash_{\mathbb{P}}$“ $\check{b}_{x_\alpha} \in \dot{\mathcal{B}}^+$ ”. Put $\mathcal{X} := \{x_\alpha; \alpha \in A\}$, then these are as desired so we finish the proof of Main Lemma 4.17.

4.3.2 Proof of Theorem 4.2

Redefinition of the freezing forcing

The key-point of the proof of Theorem 4.2 is same as the proof of Theorem 4.15. To prove Theorem 4.2, we use a countable support iteration instead of PFA. The problem is that in general $\mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$ collapses $\aleph_2$, so we cannot force by an iteration of $\mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$. To overcome this problem, we redefine the freezing forcing using the following objects:

Definition 4.23. *1. For a model N of* ZFC *(i.e. a model of sufficiently large fragments of* ZFC*), denote the transitive collapse of N by $\overline{N}$ and denote the unique isomorphism from N onto $\overline{N}$ by π_N, i.e. for $x \in N$, $\pi_N(x) := \{\pi_N(y); y \in N \,\&\, y \in x\}$. (This is defined by the $\in$-recursion.)*

2. *For $\mathcal{A}$, $\dot{\mathcal{B}}$ and $\mathcal{U}$, let*

$$\mathfrak{T}(\mathcal{A},\dot{\mathcal{B}},\mathcal{U}) := \left\{\overline{N}; N \prec H(\mathfrak{c}^+) \ \&\ N \textit{ is countable } \&\ \mathcal{A},\dot{\mathcal{B}},\mathcal{U} \in N\right\}.$$

3. *For $\mathcal{A}$, $\dot{\mathcal{B}}$, $\mathcal{U}$ and $M \in \mathfrak{T}(\mathcal{A},\dot{\mathcal{B}},\mathcal{U})$, let*

$$\mathfrak{M}_M := \left\{N \prec H(\mathfrak{c}^+);\ N \textit{ is countable } \&\ \mathcal{A},\dot{\mathcal{B}},\mathcal{U} \in N \& \ \overline{N} = M\right\}.$$

We note that

- for $x \in \mathcal{P}(\omega) \cap N$, $\pi_N(x) = x$, so $\mathcal{P}(\omega) \cap \overline{N} = \mathcal{P}(\omega) \cap N$,
- for $N, N' \in \mathfrak{M}_M$, N and N' are isomorphic and ${\pi_{N'}}^{-1} \circ \pi_N$ is an isomorphism from N onto N',
- for a countable elementary submodel N of $H(\mathfrak{c}^+)$, N is an element of $H(\mathfrak{c}^+)$, so $\mathfrak{M}_M \subseteq H(\mathfrak{c}^+)$ for each $M \in \mathfrak{T}(\mathcal{A},\dot{\mathcal{B}},\mathcal{U})$.

The following partial order $\mathbb{Q}'(\mathcal{A},\dot{\mathcal{B}},\mathcal{U},f)$ is the new freezing forcing notion designed for the iteration with countable support. This is similar to the forcing notion due to Todorčević ([44]).

Definition 4.24. *For $\mathcal{A}$, $\dot{\mathcal{B}}$, $\mathcal{U}$ and $f \in \bigcup_{X \in \mathcal{U}^*} 2^X$, define $\mathbb{Q}'(\mathcal{A},\dot{\mathcal{B}},\mathcal{U},f)$ whose conditions p are triples $\langle f_p, X_p, \mathcal{N}_p\rangle$ such that*

(a') *f_p is a member of $\bigcup_{X \in \mathcal{U}^*} 2^X$ with $f_p \supseteq f$,*

(b) *X_p is a finite K_0-homogeneous subset of $\mathcal{A} \otimes \mathcal{B}^+(\mathbf{1})$,*
(recall that for $f \in \mathbb{P}(\mathcal{U})$, $\mathcal{B}^+(f) = \{b \subseteq \omega; \exists g \leq_{\mathbb{P}(\mathcal{U})} f(g \Vdash_{\mathbb{P}(\mathcal{U})} \text{“} \check{b} \in \dot{\mathcal{B}}^+ \text{”})\}$)

(c') *$\mathcal{N}_p$ is a function so that*

- (c1) $\mathrm{dom}(\mathcal{N}_p)$ *is a finite $\in$-chain of elements of $\mathfrak{T}(\mathcal{A},\dot{\mathcal{B}},\mathcal{U})$,*
- (c2) *for each $M \in \mathrm{dom}(\mathcal{N}_p)$, $\mathcal{N}_p(M)$ is a finite subset of $\mathfrak{M}_M$,*
- (c3) *for all M, $M' \in \mathrm{dom}(\mathcal{N}_p)$ with $M \in M'$ and $N \in \mathcal{N}_p(M)$, there exists $N' \in \mathcal{N}(M')$ with $N \in N'$ and $N \prec N'$,*

(d) *for any $x\ (= \langle a_x, b_x\rangle) \in X_p$, $f_p \Vdash_{\mathbb{P}(\mathcal{U})} \text{“} \check{b_x} \in \dot{\mathcal{B}}^+ \text{”}$,*

(e') *for any* $x, y \in X_p$ *with* $x \neq y$ *there exists* $M \in \mathrm{dom}(\mathcal{N}_p)$ *so that* $|M \cap \{x, y\}| = 1$,
(define $x \lhd y :\iff \exists M \in \mathrm{dom}(\mathcal{N}_p)(x \in M \,\&\, y \notin M)$*),*

(f') *for all* $N \in \bigcup \mathrm{ran}(\mathcal{N}_p)$, f_p *is* $(N, \mathbb{P}')$*-generic, and*

(g') *for every* $x \in X_p$, $M \in \mathrm{dom}(\mathcal{N}_p)$ *with* $x \notin M$ *and* $N \in \mathcal{N}_p(M)$,

$$f_p \Vdash_{\mathbb{P}'(\mathcal{U})} \text{“}\forall Y \in \check{N}[\dot{G}](Y \subseteq \check{\mathcal{A}} \otimes \check{\mathcal{B}^+}(\mathbf{1}) \;\&\; Y \star Y \subseteq K_1 \Rightarrow \check{x} \notin Y)\text{”}.$$

For conditions $p, q \in \mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$,
$p \leq_{\mathbb{Q}'(\mathcal{A},\dot{\mathcal{B}},\mathcal{U})} q :\iff f_p \supseteq f_q$ *(i.e.* $f_p \leq_{\mathbb{P}'} f_q$*)* $\&$ $X_p \supseteq X_q$ $\&$ $\mathrm{dom}(\mathcal{N}_p) \supseteq \mathrm{dom}(\mathcal{N}_q)$ $\&$ $\forall M \in \mathrm{dom}(\mathcal{N}_q)(\mathcal{N}_q(M) \subseteq \mathcal{N}_p(M))$.

By an argument similar to the one of Lemma 4.21, we show the following lemma.

Lemma 4.25. *For* $\mathcal{A}$, $\dot{\mathcal{B}}$, $\mathcal{U}$ *and* $f \in \bigcup_{X \in \mathcal{U}^*} 2^X$, $\mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}, f)$ *is proper.*

Proof. Let θ be a large enough regular cardinal, $H \prec H(\theta)$ a countable elementary submodel containing all relevant objects, e.g. $\mathcal{A}$, $\dot{\mathcal{B}}$, $\mathcal{U}$, $H(\mathfrak{c}^+)$ etc, and $p = \langle f_p, X_p, \mathcal{N}_p \rangle \in H$ a condition of $\mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}, f)(= \mathbb{Q}')$.

Since $\mathbb{P}'(\mathcal{U})(= \mathbb{P}')$ is proper, there is an extension $f_q \leq_{\mathbb{P}'} f_p$ such that f_q is $(H \cap H(\mathfrak{c}^+), \mathbb{P}')$-generic. Let $M_0 := \overline{H \cap H(\mathfrak{c}^+)}$ and

$$q := \left\langle f_q, X_p, \mathcal{N}_p \cup \left\{\left\langle M_0, \{H \cap H(\mathfrak{c}^+)\}\right\rangle\right\}\right\rangle.$$

Then q is a condition of $\mathbb{Q}'$. We show that q is $(H, \mathbb{Q}')$-generic.

Let $\mathcal{D} \in H$ be dense open in $\mathbb{Q}'$ and $r \leq_{\mathbb{Q}'} q$. We may assume that r is in $\mathcal{D}$. Let $X_r \setminus H = \{x_i; i \leq n\}$ where $x_i \lhd x_{i+1}$ for $i < n$ and take rational open intervals $U_i \subseteq \mathcal{A} \otimes \mathcal{B}^+(\mathbf{1})$ such that $x_i \in U_i$ for $i \leq n$, $U_i \cap U_j = \emptyset$ and $U_i \star U_j \subseteq K_0$ for every $i, j \leq n$ with $i \neq j$. Let $M_i \in \mathrm{dom}(\mathcal{N}_r)$ for $1 \leq i \leq n$ be such that $x_{i-1} \in M_i$ and $x_i \notin M_i$. And let $N_0 := H \cap H(\mathfrak{c}^+)$ and recursively pick $N_i \in \mathcal{N}_i(M_i)$ with $N_{i-1} \in N_i$ for $1 \leq i \leq n$. Let G be $\mathbb{P}'$-generic over $H(\theta)$ with $f_r \in G$. By the same argument as in the proof of claim 12, it is proved that in $H(\theta)[G]$, there are rational open intervals $V_i^0, V_i^1 \subseteq U_i$, $y_i \in V_i^1 \cap \mathbf{V}$ for $i \leq n$ and $s \in \mathcal{D} \cap \mathbf{V}$ such that

1. $x_i \in V_i^0$ for all $i \leq n$,
2. $V_i^0 \cap V_i^1 = \emptyset$ and $V_i^0 \star V_i^1 \subseteq K_0$ for all $i \leq n$,

3. $f_s \in G$,
4. $X_s = (X_r \cap H) \cup \{y_i; i \leq n\}$ and for any $x \in X_r \cap H$ and $i < j \leq n$, and $x \lhd y_i \lhd y_j$,
5. $\mathrm{dom}(\mathcal{N}_s)$ is an end extension of $\mathrm{dom}(\mathcal{N}_r) \cap H$ and for all $M \in \mathrm{dom}(\mathcal{N}_r) \cap H$, $\mathcal{N}_r(M) \cap H \subseteq \mathcal{N}_s(M)$.

By $H \prec H(\theta)$ and the genericity of f_r, we can find $g \in G$ which decides all values of $V_i^0, V_i^1 \subseteq U_i$, $y_i \in V_i^1 \cap H$ for $i \leq n$ and $s \in \mathcal{D} \cap H$ and is a common extension of f_r and f_s in $\mathbb{P}'$. It's enough to find a common extension of r and s.

To find it, let $\{L_i; i < l\}$ enumerate $\mathcal{N}_r(M_0)$ with $L_0 := H \cap H(\mathfrak{c}^+) = N_0$ and $\varphi_i := {\pi_{L_i}}^{-1}$ for $i < l$. We notice that for each $M \in \mathrm{dom}(\mathcal{N}_s) \setminus \mathrm{dom}(\mathcal{N}_r)$,

- $\varphi_i(M) \in L_i$ (because $M \in \overline{H \cap H(\mathfrak{c}^+)} = M_0$),
- $\overline{\varphi_i(M)} = M$ (because M is transitive), and
- $\varphi_i(M) \prec H(\mathfrak{c}^+)$ (because $\varphi_i(M)$ and N are isomorphic and $N \prec H(\mathfrak{c}^+)$ for $N \in \mathcal{N}_s(M)$).

We define a function $\mathcal{N}'$ with domain $\mathrm{dom}(\mathcal{N}_r) \cup \mathrm{dom}(\mathcal{N}_s)$ by:

$$\mathcal{N}'(M) := \begin{cases} \mathcal{N}_r(M) \cup \mathcal{N}_s(M) & \text{if } M \in \mathrm{dom}(\mathcal{N}_r) \cap H \\ \mathcal{N}_s(M) \cup \{\varphi_i(M); i < l\} & \text{if } M \in \mathrm{dom}(\mathcal{N}_s) \setminus \mathrm{dom}(\mathcal{N}_r) \\ \mathcal{N}_r(M) & \text{if } M \in \mathrm{dom}(\mathcal{N}_r) \setminus H \end{cases}$$

for every $M \in \mathrm{dom}(\mathcal{N}')$. Then it can be checked that $\langle g, X_r \cup X_s, \mathcal{N}' \rangle$ is a common extension of r and s if it is a condition of $\mathbb{Q}'$. To check $\langle g, X_r \cup X_s, \mathcal{N}' \rangle \in \mathbb{Q}'$, the only non-trivial requirement is that $\mathcal{N}'$ satisfies (c3), in particular the case that $M \in \mathrm{dom}(\mathcal{N}_r) \cap H$, $M' \in \mathrm{dom}(\mathcal{N}_s) \setminus \mathrm{dom}(\mathcal{N}_r)$ with $M \in M'$ and $N \in \mathcal{N}_r(M) \setminus H$. Then we can find $L_i \in \mathcal{N}_r(M_0)$ with $N \in L_i$, because of $r \in \mathbb{Q}'$ and $M \in M_0$. Then N is in $\varphi_i(M')$ and an elementary submodel of $\varphi_i(M')$, since $M \prec M' \prec M_0$ and $\varphi_i \restriction M = {\pi_M}^{-1} \subseteq {\pi_{M'}}^{-1} = \varphi_i \restriction M'$. □

If $\Vdash_{\mathbb{P}'(\mathcal{U})}$ " $\check{\mathcal{A}} \otimes \check{\mathcal{B}}^+(1)$ is not countably separated ", (by the argument similar to Lemma 4.22) $\mathcal{X}_G := \bigcup\{X_p; p \in G\}$ is uncountable. The biggest difference between $\mathbb{Q}(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U})$ and $\mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}, f)$ is that $\mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}, f)$ has a good chain condition. The following lemma says that it preserves cardinalities under CH.

Lemma 4.26. *For $\mathcal{A}$, $\dot{\mathcal{B}}, \mathcal{U}$ and $f \in \bigcup_{X \in \mathcal{U}^*} 2^X$, $\mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}, f)$ has the $\mathfrak{c}^+$-c.c.*

Proof. For conditions p and q in $\mathbb{Q}'(\mathcal{A}, \dot{\mathcal{B}}, \mathcal{U}, f)$, if $f_p = f_q$, $X_p = X_q$ and $\text{dom}(\mathcal{N}_p) = \text{dom}(\mathcal{N}_q)$, then $\langle f_p, X_p, \mathcal{N}'' \rangle$ is a common extension of p and q, where $\mathcal{N}''$ has the domain $\text{dom}(\mathcal{N}_p)$ and $\mathcal{N}''(M) = \mathcal{N}_p(M) \cup \mathcal{N}_q(M)$. Therefore $\{p \in \mathbb{Q}'; f_p = f \ \&\ X_p = X \ \&\ \text{dom}(\mathcal{N}_p) = \mathfrak{N}\}$ is centered for every $f \in \mathbb{P}$, $X \in [\mathcal{A} \otimes \mathcal{B}(1)]^{<\omega}$ and a finite $\in$-chain $\mathfrak{N}$ of countable transitive elementary submodels. □

Proof of Theorem 4.2

To prove Theorem 4.2, we assume that the ground model $\mathbf{V}$ is $\mathbf{L}$. Let S_0 and S_1 be stationary on ω_2 with $S_0 \cap S_1 = \emptyset$ and $S_0 \cup S_1 = \text{Cof}(\omega_1) \cap \omega_2$, where $\text{Cof}(\omega_1) = \{\alpha \in \mathbf{On}; \text{cf}(\alpha) = \omega_1\}$. Then $\mathbf{V}$ satisfies $\diamondsuit_{\omega_2}(S_1)$. Let $\{D_\alpha; \alpha \in S_1\}$ be a diamond sequence, i.e. for any subset E of ω_2, $\{\alpha \in S_1; E \cap \alpha = D_\alpha\}$ is stationary. We define a countable support iteration $\left\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha; \alpha < \omega_2 \right\rangle$ (and pick a $\mathbb{P}_\alpha$-generic filter $G \restriction \alpha$ over $\mathbf{V}$ for $\alpha < \omega_2$ recursively) as follows:

Stage 2α with $2\alpha \notin \text{Cof}(\omega_1)$. Construct an ultrafilter base $\langle X_\alpha; \alpha < \omega_2 \rangle$ (e.g. using a σ-centered Mathias forcing).

Stage $2\alpha + 1$. Construct to force MA by a book-keeping argument.

Stage $\alpha \in S_0$. Let $\mathbb{Q}_\alpha := \mathbb{P}'(\mathcal{U}(\langle X_\xi; \xi < \alpha \rangle))$, where $\mathcal{U}(\langle X_\xi; \xi < \alpha \rangle)$ is the ultrafilter generated by $\langle X_\xi; \xi < \alpha \rangle$. (We notice that

$$\Vdash_{\mathbb{P}_\alpha} \text{“} \mathcal{U}(\langle X_\xi; \xi < \alpha \rangle) \text{ is an ultrafilter ”}$$

if $\alpha < \omega_2$ has the cofinality ω_1.)

Stage $\alpha \in S_1$. If D_α codes some $\left\langle \dot{f}, \dot{\mathcal{A}}, \dot{\mathcal{B}} \right\rangle$, where

- $\dot{f}$ is a $\mathbb{P}_\alpha$-name for a condition of $\dot{\mathbb{P}}(\mathcal{U}(\langle X_\xi; \xi < \alpha \rangle))$,
- $\dot{\mathcal{A}}$ is a $\mathbb{P}_\alpha$-name for a family of infinite subsets of ω, and
- $\dot{\mathcal{B}}$ is a $\mathbb{P}_\alpha * \dot{\mathbb{P}}(\mathcal{U}(\langle X_\xi; \xi < \alpha \rangle))$-name for a family of infinite subsets of ω,

such that

$$\mathbf{V}[G\restriction\alpha] \models \text{“ } \dot{f}[G\restriction\alpha] \Vdash_{\mathbb{P}(\mathcal{U}(\langle X_\xi;\xi<\alpha\rangle))} \text{“ } (\dot{\mathcal{A}}[G\restriction\alpha], \dot{\mathcal{B}}[G\restriction\alpha]) \text{ forms an } (\omega_1,\alpha)\text{-gap ” ”},$$

then let $\mathbb{Q}_\alpha := \mathbb{Q}'(\dot{\mathcal{A}}[G\restriction\alpha], \dot{\mathcal{B}}[G\restriction\alpha], \mathcal{U}(\langle X_\xi;\xi<\alpha\rangle), \dot{f}[G\restriction\alpha])$. Otherwise, let $\mathbb{Q}_\alpha := \{\mathbf{1}\}$.

We write $G\restriction\omega_2$ by G.

We note that $\mathbb{P}_{\omega_2}$ is proper because proper-ness is closed under countable support iterations. So the following lemma indicates that it does not collapse cardinals. To show it, we use the following definition (see [42], [43] or [44]).

Definition 4.27 (Shelah). *For a forcing notion $\mathbb{X}$, $\mathbb{X}$ satisfies the $\aleph_2$-properness isomorphism condition ($\aleph_2$-pic) if for all (some) large enough regular cardinal θ, $\alpha<\beta<\omega_2$, countable elementary submodels N_α, N_β of $H(\theta)$ and a function $\pi : N_\alpha \to N_\beta$ satisfying that*

- $\alpha\in N_\alpha$, $\beta\in N_\beta$, $N_\alpha\cap\omega_2\subseteq\beta$, $N_\alpha\cap\alpha = N_\beta\cap\beta$, $\mathbb{X}\in N_\alpha\cap N_\beta$,
- *π is an isomorphism, $\pi(\alpha)=\pi(\beta)$, and $\pi\restriction(N_\alpha\cap N_\beta)$ is identity,*

if $p\in\mathbb{X}\cap N_\alpha$, then there exists an $(N_\alpha,\mathbb{X})$-generic condition q which is a common extension of p and $\pi(p)$ such that

$$q \Vdash_{\mathbb{X}} \text{“ } \pi''(\dot{G}\cap\check{N}_\alpha) = \dot{G}\cap\check{N}_\beta \text{ ”}.$$

Lemma 4.28. *$\mathbb{P}_{\omega_2}$ has the $\aleph_2$-c.c.*

Proof. Shelah has shown the following facts about the $\aleph_2$-pic (see [43]):

- Under CH, any $\aleph_2$-pic forcing notion has the $\aleph_2$-chain condition and preserves $\aleph_1$.
- Under CH, $\aleph_2$-pic-ness is closed under countable support iterations.
- If a forcing notion is proper and has size $\leq\aleph_1$, it has the $\aleph_2$-pic.

Therefore it suffices to show that all $\mathbb{Q}'(\mathcal{A},\dot{\mathcal{B}},\mathcal{U},f)$ have the $\aleph_2$-pic. (I refer to the proof of Lemma 6 in [44] for the argument below.)

Let θ be a large enough regular cardinal, and $\alpha < \beta < \omega_2$, countable elementary submodels N_α, N_β of $H(\theta)$ and a function $\pi : N_\alpha \to N_\beta$ satisfy the assumptions of $\aleph_2$-pic. And let $p \in \mathbb{Q}'(\mathcal{A},\dot{\mathcal{B}},\mathcal{U},f) \cap N_\alpha$. Because of $N_\alpha \cap \mathcal{P}(\omega) = N_\beta \cap \mathcal{P}(\omega)$ and $\overline{N_\alpha} = \overline{N_\beta}$, it is proved that $\pi(p)$ is a condition of $\mathbb{Q}'(\mathcal{A},\dot{\mathcal{B}},\mathcal{U},f)$, $f_p = f_{\pi(p)}$, $X_p = X_{\pi(p)}$, and $\mathrm{dom}(\mathcal{N}_p) = \mathrm{dom}(\mathcal{N}_{\pi(p)})$. So $\langle f_p, X_p, \mathcal{N}'\rangle$ is a common extension of p and $\pi(p)$, where $\mathrm{dom}(\mathcal{N}') = \mathrm{dom}(\mathcal{N}_p)$ and for $M \in \mathrm{dom}(\mathcal{N}_p)$, $\mathcal{N}'(M) = \mathcal{N}_p(M) \cup \mathcal{N}_{\pi(p)}(M)$. We put

$$q := \left\langle f_p, X_p, \mathcal{N}' \cup \left\{\left\langle \overline{N_\alpha \cap H(\mathfrak{c}^+)}, \{N_\alpha \cap H(\mathfrak{c}^+), N_\beta \cap H(\mathfrak{c}^+)\}\right\rangle\right\}\right\rangle.$$

As before, we can prove that q is also a condition of $\mathbb{Q}'(\mathcal{A},\dot{\mathcal{B}},\mathcal{U},f)$ and an $(N_\alpha, \mathbb{Q}'(\mathcal{A},\dot{\mathcal{B}},\mathcal{U},f))$-generic. So it is true that $q \Vdash_{\mathbb{Q}'(\mathcal{A},\dot{\mathcal{B}},\mathcal{U},f)}$ " $\pi''(\dot{G} \cap \check{N}_\alpha) = \dot{G} \cap \check{N}_\beta$ " because the compatibility in $\mathbb{Q}'(\mathcal{A},\dot{\mathcal{B}},\mathcal{U})$ is simply decided by f_p, X_p and $\mathrm{dom}(\mathcal{N}_p)$ for any $p \in \mathbb{Q}'(\mathcal{A},\dot{\mathcal{B}},\mathcal{U})$. □

In $\mathbf{V}[G]$, $\mathfrak{c} = \aleph_2$ and MA holds. By the standard Löwenheim-Skolem argument (see also [29]), since we iterate $\mathbb{P}'(\mathcal{U}(\langle X_\xi; \xi < \alpha\rangle))$ stationary many times, it follows that $\mathfrak{m}(\mathbb{P}'(\mathcal{U}(\langle X_\alpha; \alpha < \omega_2\rangle))) = \aleph_2$ in $\mathbf{V}[G]$, hence $\mathbb{P}(\mathcal{U}(\langle X_\alpha; \alpha < \omega_2\rangle))$ is ω_2-Baire. So it suffices to show that $\mathbb{P}(\mathcal{U}(\langle X_\alpha; \alpha < \omega_2\rangle))$ adds no (ω_1, ω_2)-gaps.

Assume not, i.e. in $\mathbf{V}$ there are $\mathbb{P}_{\omega_2}$-names $\dot{f}$, $\dot{\mathcal{A}}$ and a $\mathbb{P}_{\omega_2} * \dot{\mathbb{P}}(\mathcal{U}(\langle X_\alpha; \alpha < \omega_2\rangle))$-name $\dot{\mathcal{B}}$ such that $\dot{f}[G](= f) \in \mathbb{P}(\mathcal{U}(\langle X_\alpha; \alpha < \omega_2\rangle))$, $\dot{\mathcal{A}}[G](= \mathcal{A}) \subseteq \mathcal{P}(\omega)$ and

$$f \Vdash_{\mathbb{P}(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))} \text{“ } (\check{\mathcal{A}}, \dot{\mathcal{B}}[G]) \text{ forms an } (\omega_1, \omega_2)\text{-gap ”}.$$

We may consider $\dot{\mathcal{B}}[G]$ as a $\mathbb{P}(\mathcal{U}(\langle X_\alpha; \alpha < \omega_2\rangle))$-name for a function from ω_2 into $\mathcal{P}(\omega)$, i.e.

$$\begin{aligned} f \Vdash_{\mathbb{P}(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))} \text{“ } & \forall\alpha < \beta < \omega_2(\dot{\mathcal{B}}[G](\alpha) \subseteq^* \dot{\mathcal{B}}[G](\beta)) \\ & \& \ \forall a \in \check{\mathcal{A}} \forall \alpha < \omega_2(a \perp \dot{\mathcal{B}}[G](\alpha)) \text{ ”}. \end{aligned}$$

We note that $\mathbb{P}(\mathcal{U}(\langle X_\alpha; \alpha < \omega_2\rangle))$ does not add new reals.

Claim 13. $\mathcal{C}\left(\dot{\mathcal{A}},\dot{\mathcal{B}},\mathcal{U}\left(\left\langle \dot{X}_\alpha;\alpha<\omega_2\right\rangle\right),\dot{f}\right) := \Big\{\alpha\in\mathrm{Cof}(\omega_1)\cap\omega_2;$

$$\mathbf{V}[G\restriction\alpha]\models\text{“}\quad \begin{array}{l} f\in\mathbb{P}(\mathcal{U}(\langle X_\xi;\xi<\alpha\rangle)), \\ \mathcal{A}\subseteq\mathbf{V}[G\restriction\alpha],\ \textit{and} \\ f\Vdash_{\mathbb{P}(\mathcal{U}(\langle X_\xi;\xi<\alpha\rangle))}\text{“}\,(\check{\mathcal{A}},\dot{\mathcal{B}}[G]\restriction\alpha)\ \textit{forms an}\ (\omega_1,\alpha)\textit{-gap}\text{ ” ”}\Big\} \end{array}$$

is ω_1*-club.*

Proof. “ $\mathbf{V}[G\restriction\alpha]\models$ “ $f\in\mathbb{P}(\mathcal{U}(\langle X_\xi;\xi<\alpha\rangle))$ & $\mathcal{A}\subseteq\mathbf{V}[G\restriction\alpha]$ ” ” is upward closed with respect to α, and ω_1-closed-ness is trivial because for $\alpha\in\mathrm{Cof}(\omega_1)\cap\omega_2$,

$$\mathbf{V}[G\restriction\alpha]\cap 2^\omega=\bigcup_{\xi<\alpha}\mathbf{V}[G\restriction\xi]\cap 2^\omega.$$

So we check that it is unbounded.

We note that in $\mathbf{V}[G]$ for all $x\in\mathcal{A}^\perp$ and $g\leq_{\mathbb{P}(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))} f$, there are $y_{x,g}\in\mathcal{P}(\omega)$, $r_{x,g}\leq_{\mathbb{P}(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))} g$ and $\xi_{x,g}<\beta_{x,g}<\omega_2$ such that

- $x\not\subseteq^* y_{x,g}$,
- $y_{x,g}\in\mathbf{V}[G\restriction\beta_{x,g}]$, and
- $r_{x,g}\Vdash_{\mathbb{P}(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))}$ “ $\dot{\mathcal{B}}[G](\check{\xi}_{x,g})=\check{y}_{x,g}$ ”.

Taking $\alpha<\omega_2$, we recursively construct $\langle\gamma_\xi;\xi<\omega_1\rangle\subseteq\mathrm{Cof}(\omega_1)\cap\omega_2$ such that

- $\alpha\leq\gamma_0$ and $\gamma_\xi\leq\gamma_\eta$ for $\xi\leq\eta<\omega_1$,
- $\mathbf{V}[G\restriction\gamma_{\xi+1}]\models$“ $\forall x\in\mathcal{A}^\perp\cap\mathbf{V}[G\restriction\gamma_\xi]\forall g\in\mathbb{P}(\mathcal{U}(\langle X_\zeta;\zeta<\gamma_\xi\rangle))$ with $g\leq_{\mathbb{P}(\mathcal{U}(\langle X_\zeta;\zeta<\gamma_\xi\rangle))} f$ $(r_{x,g}\in\mathbb{P}(\mathcal{U}(\langle X_\zeta;\zeta<\gamma_{\xi+1}\rangle))$ & $\xi_{x,g}<\gamma_{\xi+1}$ ”, and
- if η is limit, then let $\gamma_\eta:=\sup_{\xi<\eta}\gamma_\xi$.

Then $\sup_{\xi<\omega_1}\gamma_\xi$ is in $\mathcal{C}\left(\dot{\mathcal{A}},\dot{\mathcal{B}},\mathcal{U}\left(\left\langle \dot{X}_\alpha;\alpha<\omega_2\right\rangle\right),\dot{f}\right)$. □

Since $\mathfrak{m}(\mathbb{P}'(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle)))=\aleph_2$, by Lemma 4.20, in $\mathbf{V}[G]$

$$f\Vdash_{\mathbb{P}'(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))}\text{“ }\check{\mathcal{A}}\otimes\left(\dot{\mathcal{B}}[G]^+(f)\right)^\vee$$
is not a union of countably many K_1-homogeneous subsets ”.

By Proposition 4.5, this is equivalent to

$$f \Vdash_{\mathbb{P}'(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))} \text{"} \left(\check{\mathcal{A}}, \left(\dot{\mathcal{B}}[G]^+(f)\right)^{\vee}\right) \text{ is not countably separated "}$$

(in $\mathbf{V}[G]$), i.e. for all $\mathbb{P}'(\mathcal{U}(\langle X_\alpha; \alpha < \omega_2\rangle))$-names $\vec{c} = \langle \dot{c}_n; n < \omega\rangle \in (\mathcal{A}^\perp)^\omega$ and $g \leq_{\mathbb{P}'(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))} f$, there are $y_{\vec{c},g}, z_{\vec{c},g} \in \mathcal{P}(\omega)$, $r_{\vec{c},g} \leq_{\mathbb{P}'(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))} g$ and $\xi_{\vec{c},g} < \beta_{\vec{c},g} < \omega_2$ such that

- $\langle z_{\vec{c},g}, y_{\vec{c},g}\rangle \in \mathcal{A} \otimes \dot{\mathcal{B}}[G]^+(f) \cap \mathbf{V}[G \restriction \beta_{\vec{c},g}]$, and
- $r_{\vec{c},g} \Vdash_{\mathbb{P}'(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))} \text{"} \dot{\mathcal{B}}[G](\check{\xi}_{\vec{c},g}) = \check{y}_{\vec{c},g} \;\&\; \forall n < \omega(\check{z}_{\vec{c},g} \not\perp \dot{c}_n \vee \check{y}_{\vec{c},g} \not\subseteq^* \dot{c}_n) \text{"}$.

So by an argument similar to the one of the previous claim,

$$\begin{aligned} \mathcal{C}'\left(\dot{\mathcal{A}}, \dot{\mathcal{B}}, \mathcal{U}\left(\left\langle \dot{X}_\alpha; \alpha < \omega_2\right\rangle\right), \dot{f}\right) := \Big\{ & \alpha \in \operatorname{Cof}(\omega_1) \cap \omega_2; \\ & \mathbf{V}[G\restriction\alpha] \models \text{"} f \in \mathbb{P}(\mathcal{U}(\langle X_\xi; \xi < \alpha\rangle)), \\ & \quad \mathcal{A} \subseteq \mathbf{V}[G\restriction\alpha], \text{ and} \\ & \quad f \Vdash_{\mathbb{P}'(\mathcal{U}(\langle X_\xi;\xi<\alpha\rangle))} \text{"} \left(\check{\mathcal{A}}, \left(((\dot{\mathcal{B}}\restriction\alpha)[G\restriction\alpha])^+(f)\right)^{\vee}\right) \\ & \quad \text{is not countably separated " "}\Big\} \end{aligned}$$

is ω_1-club. Thus by the diamond sequence, there exists

$$\alpha \in \mathcal{C}\left(\dot{\mathcal{A}}, \dot{\mathcal{B}}, \mathcal{U}\left(\left\langle \dot{X}_\alpha; \alpha < \omega_2\right\rangle\right), \dot{f}\right) \cap \mathcal{C}'\left(\dot{\mathcal{A}}, \dot{\mathcal{B}}, \mathcal{U}\left(\left\langle \dot{X}_\alpha; \alpha < \omega_2\right\rangle\right), \dot{f}\right)$$

such that D_α codes $\left\langle \dot{f}, \dot{\mathcal{A}}, \dot{\mathcal{B}}\restriction\alpha\right\rangle$. So

$$\mathbb{Q}_\alpha = \mathbb{Q}'(\mathcal{A}, (\dot{\mathcal{B}}\restriction\alpha)[G\restriction\alpha], \mathcal{U}(\langle X_\xi; \xi < \alpha\rangle), f)$$

and $G(\alpha)$ is $\mathbb{Q}_\alpha$-generic over $\mathbf{V}[G \restriction \alpha]$. Then $\mathcal{X}' := \bigcup\{X_p; p \in G(\alpha)\}$ is uncountable K_0-homogeneous. We note that for all $p \in G(\alpha)$, f_p is in $\mathbb{P}(\mathcal{U}(\langle X_\alpha; \alpha < \omega_2\rangle))$. So by the same argument at the end of the proof of Main Lemma 4.17, there are a fusion $\tilde{f}$ of $\langle f_p; p \in G(\alpha)\rangle$ and uncountable K_0-homogeneous $\mathcal{X} \subseteq \mathcal{A} \otimes \dot{\mathcal{B}}[G]^+(f)$ such that $\tilde{f} \Vdash_{\mathbb{P}(\mathcal{U}(\langle X_\alpha;\alpha<\omega_2\rangle))} \text{"} \check{b} \in \dot{\mathcal{B}}^+ \text{"}$ for all $\langle a, b\rangle \in \mathcal{X}$, which is a contradiction and completes the proof of Theorem 4.2.

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