A dependence vanishing theorem for sequences generated by Weyl transformation

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博士論文

A dependence vanishing theorem for sequences generated by Weyl transformation

(Weyl変換により生成される確率過程の従属性消滅定理)

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Contents

Introduction

Let us develop a real number x into the binary expansion, take the sum of the first m digits under the decimal point, divide the sum by 2, and denote the remainder by $X^{(m)}(x)$. Sugita [5] proposed that, when α is irrational and m is large, we can use the sequence $X^{(m)}(x), X^{(m)}(x+\alpha), X^{(m)}(x+2\alpha), \ldots$ as a pseudo-random number and showed the following theorem. Let $X_n^{(m)}$ be the $\{0,1\}$-valued function on $[0,1)^2$ defined by $X_n^{(m)}(x,\alpha) = X^{(m)}(x+n\alpha)$.

Theorem A. *For any normal number $\alpha \in [0,1)$, the process $\{X_n^{(m)}(\cdot,\alpha)\}_{n=0}^{\infty}$ on $([0,1), \mathcal{B}, P)$ converges in law to $\{0,1\}$-valued fair i.i.d. when $m \to \infty$ where P is the Lebesgue measure on $[0,1)$.*

Note that the process $\{X_n^{(m)}(\,\cdot\,,\alpha)\}_{n=0}^{\infty}$ is generated by the α-rotation or Weyl transformation and has strong dependence. Theorem A claims that the dependence vanish when $m \to \infty$.

We say that α is "good" when the process $\{X_n^{(m)}(\cdot,\alpha)\}_{n=0}^{\infty}$ on $([0,1), \mathcal{B}, P)$ converges in law to $\{0,1\}$-valued fair i.i.d. when $m \to \infty$. It is easy to see that any rational number is not "good". Sugita [5] conjectured that any irrational number α is "good".

Since the proof of Theorem A in Sugita [5] is very complicated, Sugita [6] tried to give a simple proof based on ergodic theory. He showed the following theorem, which has a simple proof but its assertion is weaker than Theorem A.

Theorem B. *The process $\{X_n^{(m)}\}_{n=0}^{\infty}$ on $([0,1)^2, \mathcal{B}([0,1)^2), P)$ converges in law to $\{0,1\}$-valued fair i.i.d. when $m \to \infty$ where P is the Lebesgue measure on $[0,1)^2$.*

Takanobu [7] studied on this theorem in detail. We see an outline of these result in Chapter 1.

In Chapter 2, we modify the idea of Sugita [6] and prove a theorem asserting that almost every α is "good". Although the theorem is stronger than Theorem B, it does not decide whether each given α is "good" or not since α is regarded as a random variable.

In Chapter 3, we give an alternative proof of Theorem A. The method is an application of a standard technique of Markov chain. I believe that the method is enough simple to understand the nature of the phenomenon.

Let us recall Theorem A and see that any normal number α is "good". There exist enough normal numbers, indeed, the set of all normal numbers in $[0,1)$ have Lebesgue measure 1. We can even construct some normal numbers, e.g. $0.1\,10\,11\,100\,101\,110\,111\ldots$. But we do not know any concrete number, π, e, $\sqrt{2}$, $\sqrt{3}$, and so on, is normal or not, and hence we can not know what α is "good" in practice. For the purpose of application, it is necessary to know at least one "good" α, and his conjecture is fulfill this need if it is proved affirmatively.

In Chapter 3, we prove that any irrational α is "good" and give the affirmative answer to the conjecture. We apply the technique of Markov chain from a different point of view and prove the final result.

Notation

We introduce some notation for following chapters. Let $b \geq 2$ be a natural number, $d^{(m)}(x)$ be the m-th digit of $x \geq 0$ in decimal part of its base-b expansion, and $X_n^{(m)}$ be the $\{0,\ldots,b-1\}$-valued function on $[0,1)^2$ defined by

$$X_n^{(m)}(x,\alpha) = \sum_{k=1}^{m} d^{(k)}(x+n\alpha) \pmod{b}.$$

Let us identify $\{0,\ldots,b-1\}$ with the group $\mathbb{Z}/b\mathbb{Z}$.

We say that a measure μ on $[0,1)$ is a Bernoulli measure iff $\{d^{(m)}\}_{m=1}^{\infty}$ is an i.i.d. with respect to μ and that a Bernoulli measure μ is non-degenerate iff $\mu(d^{(1)} = s) \neq 0$ for all $s \in \mathbb{Z}/b\mathbb{Z}$. Note that the Lebesgue measure is a non-degenerate Bernoulli measure.

For real number $x \geq 0$, let $\lfloor x \rfloor$ be the integral part of x i.e., $\lfloor x \rfloor := \max\{n \in \mathbb{Z} \mid x \geq n\}$.

Chapter 1

Skew product and Sugita's proof

In this chapter, we see a sketch of proof of the following result by Sugita [6]:

Theorem B. *The process* $\{X_n^{(m)}\}_{n=0}^{\infty}$ *on* $([0,1)^2, \mathcal{B}([0,1)^2), P)$ *converges in law to* $\{0,1\}$*-valued fair i.i.d. when* $m \to \infty$ *where* P *is the Lebesgue measure on* $([0,1)^2, \mathcal{B}([0,1)^2))$ *and* $b = 2$.

By harmonic analysis on $(\mathbb{Z}/2\mathbb{Z})^n$, it is sufficient to prove the Theorem B to see

$$\int \exp(i\pi \sum_{l=0}^{n-1} a_l X_l^{(m)}(x,\alpha))\, dx d\alpha \to 0 \quad (m \to \infty)$$

for any $n \in \mathbb{N}$ and $a_l \in \mathbb{Z}/2\mathbb{Z}$ $(l < n)$. This is equivalent to

$$\int \exp(i\pi \sum_{i=0}^{k-1} X_{l'_i}^{(m)}(x,\alpha))\, dx d\alpha \to 0 \quad (m \to \infty)$$

for any $k \in \mathbb{N}$ and $0 \le l'_0 < l'_1 < \cdots < l'_{k-1}$. By taking $l_i = l'_i - l'_0$, the shift invariance of Lebesgue measure implies that it is also equivalent to

$$\int \exp(i\pi \sum_{i=0}^{k-1} X_{l_i}^{(m)}(x,\alpha))\, dx d\alpha = \int \exp(i\pi \sum_{i=0}^{k-1} X_{l'_i - l'_0}^{(m)}(x,\alpha))\, dx d\alpha \to 0 \quad (m \to \infty)$$

for $0 = l_0 < l_1 < \cdots < l_{k-1}$.

Note that $\exp(i\pi X_l^{(m)}(x+1/2,\alpha)) = -\exp(i\pi X_l^{(m)}(x,\alpha))$ by the definition of $X_l^{(m)}$. Then the shift invariance of Lebesgue measure implies

$$\int \exp(i\pi \sum_{i=0}^{k-1} X_{l_i}^{(m)}(x,\alpha))\, dxd\alpha = 0$$

for any $0 = l_0 < l_1 < \cdots < l_{k-1}$ and m if k is odd.

If $k = 2$, since $X_{l_0}^{(m)}$ does not depend on α,

$$\begin{aligned} &\int \exp(i\pi \sum_{i=0}^{k-1} X_{l_i}^{(m)}(x,\alpha))\, dxd\alpha \\ &= \int\int \exp(i\pi X_{l_1}^{(m)}(x,\alpha))d\alpha \exp(i\pi X_{l_0}^{(m)}(x,\alpha))\, dx = 0. \end{aligned}$$

From now on, we fix even $k \geq 4$ and $0 = l_0 < l_1 < \cdots < l_{k-1}$ and let $f(x,\alpha) := \exp(i\pi \sum_{i=0}^{k-1} d^{(1)}(x + l_i\alpha))$. Then we need is to show that

$$\int \prod_{j=1}^{m} f(2^{j-1}x, 2^{j-1}\alpha)\, dxd\alpha \to 0 \quad (m \to \infty).$$

We introduce a skew product T_f by

$$\begin{aligned} T_f : [0,1)^2 \times \{-1,1\} &\ni (x,\alpha,\epsilon) \\ \mapsto \ &(2x - \lfloor 2x \rfloor, 2\alpha - \lfloor 2\alpha \rfloor, \epsilon f(x,\alpha)) \in [0,1)^2 \times \{-1,1\}. \end{aligned}$$

Let us define a measure μ on $([0,1)^2 \times \{-1,1\}, \mathcal{B}([0,1)^2) \times 2^{\{-1,1\}})$ by $\mu := P \times \frac{1}{2}(\delta_{-1} + \delta_1)$. Then T_f is a measure preserving transformation with respect to μ and

$$\int \prod_{j=1}^{m} f(2^{j-1}x, 2^{j-1}\alpha)\, dxd\alpha = \int \Phi \cdot (\Phi \circ T_f^m)\, d\mu$$

where $\Phi(x,\alpha,\epsilon) = \epsilon$.

Sugita [6] showed the ergodicity of T_f and, in a similar way, the ergodicity of $T_f \times T_f$ which implies the weak mixing property of T_f.

We introduce a Markov Chain $Y^{(m)}$ which can describe Φ. Let $\{E_1, \ldots, E_J\}$ be the connected component of $[0,1)^2 \setminus \{\text{discontinuity of } f\}$ and $Y(x,\alpha) := j$ for $(x,\alpha) \in E_j$ where we regarded $[0,1)^2$ as the two dimensional torus. Then there exist $\tilde{\Phi}$ such that $\Phi = \tilde{\Phi} \circ Y$ a.e. and $Y^{(m)} := Y \circ T_f^m$ is a Markov Chain.

Then the ergodicity or weak mixing property of T_f implies the irreducibility or aperiodicity of $Y^{(m)}$, respectively. Therefore the Markov Chain $Y^{(m)}$ converge to equilibrium hence $Y^{(m)}$ is strongly mixing. Thus

$$\int \Phi\cdot(\Phi\circ T_f^m)\, d\mu = \int (\tilde{\Phi}\circ Y)\cdot(\tilde{\Phi}\circ Y\circ T_f^m)\, d\mu \to (\int \tilde{\Phi}\circ Y\, d\mu)^2 = (\int \Phi\, d\mu)^2 = 0.$$

which complete the proof.

If T_f itself is strongly mixing, we immediately have that

$$\int \Phi \cdot (\Phi \circ T_f^m)\, d\mu \to (\int \Phi\, d\mu)^2 = 0.$$

Takanobu [7] show this mixing property by a cancellation method used in Theorem A.

In the view of ergodic theory, stronger property of T_f is essentially proved in Sugita [6]. By noting that $\bigcup_k T_f^{-k}\sigma(Y)$ generates $\mathcal{B}([0,1)^2) \times 2^{\{-1,1\}}$, we have T_f is a Markov transformation. This is a kind of powerful property. Indeed, the weak mixing property is equivalent to the weak Bernoulli property for any Markov transformation. Thus we can see that T_f is a weak Bernoulli transformation and hence strongly mixing.

As we have surveyed, the Markov property has big weight in this method, and it is more natural to begin our proof with a Markov Chain instead of the skew product. In the next chapter we discuss from this point of view.

Chapter 2

Markov Chain and a result for almost every α

2.1 Result

In this Chapter, we show that the set of "good" αs has measure 1 for any non-degenerate Bernoulli measure μ, i.e.,

Theorem 1. *Let μ and ν be non-degenerate Bernoulli measures. For μ-*a.e. *α, the process $\{X_n^{(m)}(\cdot,\alpha)\}_{n=0}^{\infty}$ on $([0,1),\nu)$ converges in law to the $\{0,\dots,b-1\}$-valued fair i.i.d. when $m\to\infty$, i.e., for all $n\in\mathbb{N}$ and s_0, ..., $s_{n-1}\in\{0,\dots,b-1\}$,*

$$\nu\big(X_0^{(m)}(\,\cdot\,,\alpha)=s_0,\dots,X_{n-1}^{(m)}(\,\cdot\,,\alpha)=s_{n-1}\big)\longrightarrow\frac{1}{b^n}\qquad(m\to\infty).$$

2.2 Proof of the Theorem

Let $\Omega:=[0,1)^3$ and $P:=\nu\times\nu\times\mu$.

We define two $\{0,\dots,b-1\}^n$-valued processes $\{\mathbf{X}_1^{(m)}\}_{m=1}^{\infty}$ and $\{\mathbf{X}_2^{(m)}\}_{m=1}^{\infty}$ on (Ω,P) as

$$\mathbf{X}_j^{(m)}(x_1,x_2,\alpha):=\sum_{k=1}^{m}(d^{(k)}(x_j),d^{(k)}(x_j+\alpha),\dots,d^{(k)}(x_j+(n-1)\alpha)).$$

Note that, for any x_1 and x_2, the $\{0,\dots,b-1\}^n$-valued functions $\mathbf{X}_1^{(m)}(\cdot,x_2,\cdot)$ $\mathbf{X}_2^{(m)}(x_1,\,\cdot\,,\,\cdot\,)$, and $(X_0^{(m)},\dots,X_{n-1}^{(m)})$ are equal on $[0,1)^2$.

To prove Theorem 1, we show that

$$\nu\big(\mathbf{X}_1^{(m)}(\,\cdot\,,x_2,\alpha,)=\mathbf{s}\big)\longrightarrow\frac{1}{b^n}\quad\mu\text{-a.e. }\alpha\qquad(m\to\infty)$$

for any $\mathbf{s}\in\{0,\dots,b-1\}^n$.

Thus, it is sufficient to show that

$$\sum_{m=1}^{\infty}\int\left\{\nu\big(\mathbf{X}_1^{(m)}(\,\cdot\,,x_2,\alpha)=\mathbf{s}\big)-\frac{1}{b^n}\right\}^2\mu(d\alpha)<\infty. \tag{2.1}$$

Let β be the base-b transformation on $[0,1)$, i.e., $\beta x:=bx-\lfloor bx\rfloor$. We define a $\mathbb{Z}^{2n+1}$-valued process $\mathbf{Z}^{(m)}$;

$$\mathbf{Z}^{(m)}=(Z_{1,0}^{(m)},\dots,Z_{1,n-1}^{(m)},Z_{2,0}^{(m)},\dots,Z_{2,n-1}^{(m)},Z_3^{(m)})$$
$$Z_{j,l}^{(m)}(x_1,x_2,\alpha):=\lfloor b(\beta^{m-1}x_j+l\beta^{m-1}\alpha)\rfloor,$$
$$Z_3^{(m)}(x_1,x_2,\alpha):=\lfloor b\beta^{m-1}\alpha\rfloor.$$

We will prove the following proposition in the next section.

Proposition 2. $\{(\mathbf{Z}^{(m)},\mathbf{X}_1^{(m)},\mathbf{X}_2^{(m)})\}_{m=1}^{\infty}$ *is an irreducible and aperiodic Markov chain, and its stationary initial distribution* $\{\pi_{\mathbf{u},\mathbf{s},\mathbf{t}}\}$ *satisfies*

$$\pi_{\mathbf{u},\mathbf{s},\mathbf{t}}=P(\mathbf{Z}^{(1)}=\mathbf{u})\frac{1}{b^{2n}}.$$

Now, let us show the formula (2.1). By noting that $\mathbf{X}_1^{(m)}(\,\cdot\,,x_2,\,\cdot\,)$ and $\mathbf{X}_2^{(m)}(x_1,\,\cdot\,,\,\cdot\,)$ are identically distributed, that $\mathbf{X}_1^{(m)}$ does not depend on x_2 and $\mathbf{X}_2^{(m)}$ does not depend on x_1, and that $\mathbf{X}_1^{(m)}$ and $\mathbf{X}_2^{(m)}$ are independent when α is fixed, we have

$$\begin{aligned}
&\int\left\{\nu\big(\mathbf{X}_1^{(m)}(\,\cdot\,,x_2,\alpha)=\mathbf{s}\big)-\frac{1}{b^n}\right\}^2\mu(d\alpha)\\
&=\int\nu\big(\mathbf{X}_1^{(m)}(\,\cdot\,,x_2,\alpha)=\mathbf{s}\big)\nu\big(\mathbf{X}_2^{(m)}(x_1,\,\cdot\,,\alpha)=\mathbf{s}\big)\mu(d\alpha)-\frac{1}{b^{2n}}\\
&\quad-2\frac{1}{b^n}\Big\{\int\nu\big(\mathbf{X}_1^{(m)}(\,\cdot\,,x_2,\alpha)=\mathbf{s}\big)\mu(d\alpha)-\frac{1}{b^n}\Big\}\\
&=\int(\nu\times\nu)\big(\mathbf{X}_1^{(m)}(\,\cdot\,,\,\cdot\,,\alpha)=\mathbf{s}\big)(\nu\times\nu)\big(\mathbf{X}_2^{(m)}(\,\cdot\,,\,\cdot\,,\alpha)=\mathbf{s}\big)\mu(d\alpha)-\frac{1}{b^{2n}}\\
&\quad-2\frac{1}{b^n}\Big\{\int(\nu\times\nu)\big(\mathbf{X}_1^{(m)}(\,\cdot\,,\,\cdot\,,\alpha)=\mathbf{s}\big)\mu(d\alpha)-\frac{1}{b^n}\Big\}\\
&=P\big(\mathbf{X}_1^{(m)}=\mathbf{s},\mathbf{X}_2^{(m)}=\mathbf{s}\big)-\frac{1}{b^{2n}}-2\frac{1}{b^n}\Big\{P\big(\mathbf{X}_1^{(m)}=\mathbf{s}\big)-\frac{1}{b^n}\Big\}.
\end{aligned}$$

By Proposition 2, we have

$$\sum_{\mathbf{u}} \pi_{\mathbf{u},\mathbf{s},\mathbf{s}} = \frac{1}{b^{2n}}, \qquad \sum_{\mathbf{u},\mathbf{t}} \pi_{\mathbf{u},\mathbf{s},\mathbf{t}} = \frac{1}{b^n}.$$

Therefore, by noting the following theorem, we see that there exist $C > 0$ and $\rho < 1$ such that

$$\int \left\{ \nu \left(\mathbf{X}_1^{(m)}(\,\cdot\,, x_2, \alpha) = \mathbf{s} \right) - \frac{1}{2^n} \right\}^2 \mu(d\alpha) \leq C\rho^m,$$

i.e. the summand in (2.1) converges to 0 in exponential order and is summable in m. □

Theorem C. (Billingsley [1, Theorem 8.9]) *For an irreducible and aperiodic Markov chain which have a finite state space and transition probability $p_{ij}^{(m)}$, there exists a stationary distribution $\{\pi_i\}$ such that*

$$|p_{ij}^{(m)} - \pi_j| \leq A\rho^m$$

for some $A > 0$, and $0 \leq \rho < 1$.

Remark: We can estimate the order of convergence. Let ρ' be as $\rho < \rho'^2 < 1$ and

$$M(\alpha) := \left\{ \sum_{m=1}^{\infty} \left\{ \nu \left(\mathbf{X}_1^{(m)}(\,\cdot\,, x_2, \alpha) = \mathbf{s} \right) - \frac{1}{2^n} \right\}^2 \left(\frac{1}{\rho'^2} \right)^m \right\}^{1/2}.$$

Then, M is square integrable and

$$\left| \nu \left(\mathbf{X}_1^{(m)}(\,\cdot\,, x_2, \alpha) = \mathbf{s} \right) - \frac{1}{2^n} \right| \leq M(\alpha)\rho'^m \qquad \mu\text{-a.e. } \alpha.$$

2.3 Proof of Lemmas

In this section, we prove Proposition 2.

Lemma 1. $\{\mathbf{Z}^{(m)}\}_{m=1}^{\infty}$ *is an irreducible Markov chain.*

Proof. By the definition of $\mathbf{Z}^{(m)}$, we have

$$\mathbf{Z}^{(m)}(x_1, x_2, \alpha) = \mathbf{Z}^{(1)}(\beta^{m-1}x_1, \beta^{m-1}x_2, \beta^{m-1}\alpha).$$

Therefore we can define $\operatorname{Im}\mathbf{Z}$ as $\operatorname{Im}\mathbf{Z} := \operatorname{Im}\mathbf{Z}^{(m)}$.

Let us calculate the probability of a cylinder set $A := \{\mathbf{Z}^{(k)} = \mathbf{u}^{(k)}, 1 \leq k \leq m\}$ for $\mathbf{u}^{(k)} = (u_{1,0}^{(k)}, \ldots, u_{1,n-1}^{(k)}, u_{2,0}^{(k)}, \ldots, u_{2,n-1}^{(k)}, u_3^{(k)}) \in \operatorname{Im}\mathbf{Z}$ such that $P(A) \geq 0$.

First, we put

$$A' := \left\{ Z_{1,0}^{(k)} = u_{1,0}^{(k)}, Z_{2,0}^{(k)} = u_{2,0}^{(k)}, Z_3^{(k)} = u_3^{(k)} \text{ for } 1 \leq k < m, \mathbf{Z}^{(m)} = \mathbf{u}^{(m)} \right\}$$

and show $A = A'$. $A \subset A'$ is clear. Note the definition of $Z_{j,l}^{(k)}$ and $bx = \beta x + \lfloor bx \rfloor$. Then we have that, for any $\omega = (x_1, x_2, \alpha) \in \Omega$,

$$\begin{aligned} Z_{j,l}^{(k-1)}(\omega) &= \lfloor b(\beta^{k-2}x_j + l\beta^{k-2}\alpha) \rfloor \\ &= \lfloor \beta^{k-1}x_j + \lfloor b\beta^{k-2}x_j \rfloor + l\beta^{k-1}\alpha + l\lfloor b\beta^{k-2}\alpha \rfloor \rfloor \\ &= \lfloor \beta^{k-1}x_j + l\beta^{k-1}\alpha \rfloor + \lfloor b\beta^{k-2}x_j \rfloor + l\lfloor b\beta^{k-2}\alpha \rfloor \\ &= \lfloor \frac{1}{b} Z_{j,l}^{(k)}(\omega) \rfloor + Z_{j,0}^{(k-1)}(\omega) + lZ_3^{(k-1)}(\omega), \end{aligned}$$

Because $P(A) > 0$ or $A \neq \phi$, we have

$$u_{j,l}^{(k-1)} = \lfloor \frac{1}{b} u_{j,l}^{(k)} \rfloor + u_{j,0}^{(k-1)} + lu_3^{(k-1)}$$

for $1 < k \leq m$. Therefore $\mathbf{Z}^{(m)} = \mathbf{u}^{(m)}$, $Z_{j,0}^{(m-1)} = u_{j,0}^{(m-1)}$, and $Z_3^{(m-1)} = u_3^{(m-1)}$ imply $\mathbf{Z}^{(m-1)} = \mathbf{u}^{(m-1)}$. Thus, in the same way, we have that $\mathbf{Z}^{(k)}(\omega) = \mathbf{u}^{(k)}$ for all $k \leq m$ if $\omega \in A'$, i.e., $A \supset A'$.

Note that $Z_{j,0}^{(k)}(x_1, x_2, \alpha) = \lfloor b\beta^{(k-1)}x_j \rfloor = d^{(k)}(x_j)$ and $Z_3^{(k)}(x_1, x_2, \alpha) = d^{(k)}(\alpha)$. Therefore, by the independence of $\{(d^{(k)}(x_1), d^{(k)}(x_2), d^{(k)}(\alpha))\}_{k=1}^{\infty}$ and the fact that $\mathbf{Z}^{(m)}$ can be written as a function on $\{(d^{(k)}(x_1), d^{(k)}(x_2), d^{(k)}(\alpha))\}_{k=m}^{\infty}$, we have

$$P(A) = P(A') = P(\mathbf{Z}^{(m)} = \mathbf{u}^{(m)}) \prod_{k=1}^{m-1} P \begin{pmatrix} d^{(k)}(x_1) = u_{1,0}^{(k)} \\ d^{(k)}(x_2) = u_{2,0}^{(k)} \\ d^{(k)}(\alpha) = u_3^{(k)} \end{pmatrix}.$$

Since $P(B) > 0$ by the assumption $P(A) > 0$, in the same way as A, we can show the following:

$$P(B) = P\big(\mathbf{Z}^{(m+1)} = \mathbf{u}^{(m+1)}, \mathbf{Z}^{(m)} = \mathbf{u}^{(m)}\big) \prod_{k=1}^{m-1} P \begin{pmatrix} d^{(k)}(x_1) = u_{2,0}^{(k)} \\ d^{(k)}(x_1) = u_{2,0}^{(k)} \\ d^{(k)}(\alpha) = u_3^{(k)} \end{pmatrix}.$$

Therefore we have shown

$$P(B \mid A) = P\big(\mathbf{Z}^{(m+1)} = \mathbf{u}^{(m+1)} \mid \mathbf{Z}^{(m)} = \mathbf{u}^{(m)}\big),$$

i.e., $\{\mathbf{Z}^{(m)}\}_{m=1}^{\infty}$ is a Markov chain. The mixing property of the transformation $(x_1, x_2, \alpha) \mapsto (\beta x_1, \beta x_2, \beta\alpha)$, which is a Cartesian product of mixing transformations, implies irreducibility. $\square$

Let ι be the canonical surjection $\mathbb{Z} \to \mathbb{Z}/b\mathbb{Z}$. We define two $(\mathbb{Z}/b\mathbb{Z})^n$-valued functions $\widehat{\mathbf{d}}_1$ and $\widehat{\mathbf{d}}_2$ on $\operatorname{Im}\mathbf{Z}$ as

$$\widehat{\mathbf{d}}_j(u_{1,0}, \ldots, u_{1,n-1}, u_{2,0}, \ldots, u_{2,n-1}, u_3) := (\iota(u_{j,0}), \ldots, \iota(u_{j,n-1})),$$

By noting that

$$\begin{aligned} (\iota \circ Z_{j,l}^{(m)})(x_1, x_2, \alpha) &= \iota(\lfloor b(\beta^{m-1}x_j + l\beta^{m-1}\alpha)\rfloor) \\ &= d^{(1)}(\beta^{m-1}x_j + l\beta^{m-1}\alpha) \\ &= d^{(m)}(x_j + l\alpha), \end{aligned}$$

we have

$$\mathbf{X}_j^{(m)} = \sum_{k=1}^{m} \widehat{\mathbf{d}}_j \circ \mathbf{Z}^{(k)} \qquad (\operatorname{mod} b). \tag{2.2}$$

We introduce some additional notation:

$$G := (\mathbb{Z}/b\mathbb{Z})^{2n}$$
$$\varphi : \operatorname{Im}\mathbf{Z} \ni \mathbf{u} \mapsto (\widehat{\mathbf{d}}_1(\mathbf{u}), \widehat{\mathbf{d}}_2(\mathbf{u})) \in G$$
$$W^{(m)} := (\mathbf{X}_1^{(m)}, \mathbf{X}_2^{(m)}).$$

Let 0_G be the unit element of the finite group G. By the definitions of $\mathbf{X}_1^{(m)}$, $\mathbf{X}_2^{(m)}$, $\widehat{\mathbf{d}}_1$, and $\widehat{\mathbf{d}}_2$, we have that $W^{(m)} = \sum_{k=1}^{m} \varphi(\mathbf{Z}^{(k)})$.

Lemma 2. $\{(\mathbf{Z}^{(m)}, W^{(m)})\}_{m=1}^{\infty}$ *is a Markov chain.*

Proof. For $\mathbf{u}^{(k)} = (u_{1,0}^{(k)}, \dots, u_{1,n-1}^{(k)}, u_{2,0}^{(k)}, \dots, u_{2,n-1}^{(k)}, u_3^{(k)}) \in \operatorname{Im} \mathbf{Z}$ and $g^{(k)} \in G$, we put

$$\widetilde{A} := \{(\mathbf{Z}^{(k)}, W^{(k)}) = (\mathbf{u}^{(k)}, g^{(k)}),\ 1 \le k \le m\},$$
$$\widetilde{B} := \{(\mathbf{Z}^{(k)}, W^{(k)}) = (\mathbf{u}^{(k)}, g^{(k)}),\ 1 \le k \le m+1\}.$$

Let us calculate the conditional probability of $\widetilde{B}$ that $\widetilde{A}$ has occurred. We put $g^{(0)} := 0_G$, $\delta_g^g := 1$, and $\delta_h^g := 0$ when $g \neq h$. Because $W^{(k)} = \varphi(Z^{(k)}) + W^{(k-1)}$ and $W^{(0)} = 0$, we have

$$\begin{aligned} P(\widetilde{A}) &= P\big(\mathbf{Z}^{(k)} = \mathbf{u}^{(k)}, g^{(k)} = \varphi(\mathbf{u}^{(k)}) + g^{(k-1)}, 1 \le k \le m\big) \\ &= P\big(\mathbf{Z}^{(k)} = \mathbf{u}^{(k)}, 1 \le k \le m\big) \prod_{k=1}^{m} \delta_{g^{(k)}}^{\varphi(\mathbf{u}^{(k)}) + g^{(k-1)}}. \end{aligned}$$

Similarly, we can prove the following:

$$P(\widetilde{B}) = P\big(\mathbf{Z}^{(k)} = \mathbf{u}^{(k)}, 1 \le k \le m+1\big) \prod_{k=1}^{m+1} \delta_{g^{(k)}}^{\varphi(\mathbf{u}^{(k)}) + g^{(k-1)}}.$$

Thus, when $P(\widetilde{A}) > 0$,

$$\begin{aligned} P(\widetilde{B} \mid \widetilde{A}) &= P\big(\mathbf{Z}^{(m+1)} = \mathbf{u}^{(m+1)} \mid \mathbf{Z}^{(k)} = \mathbf{u}^{(k)}, 1 \le k \le m\big) \delta_{g^{(m+1)}}^{\varphi(\mathbf{u}^{(m+1)}) + g^{(m)}} \\ &= P\big(\mathbf{Z}^{(2)} = \mathbf{u}^{(m+1)} \mid \mathbf{Z}^{(1)} = \mathbf{u}^{(m)}\big) \delta_{g^{(m+1)}}^{\varphi(\mathbf{u}^{(m+1)}) + g^{(m)}} \\ &= P\big((\mathbf{Z}^{(2)}, W^{(2)}) = (\mathbf{u}^{m+1}, g^{(m+1)}) \mid (\mathbf{Z}^{(1)}, W^{(1)}) = (\mathbf{u}^{(m)}, g^{(m)})\big). \end{aligned} \tag{2.3}$$

□

Let $p^{(k)}\big((\mathbf{u}, g), (\mathbf{u}', g')\big)$ be the k-step transition probability from $(\mathbf{u}, g)$ to $(\mathbf{u}', g')$ of the Markov chain $\{(\mathbf{Z}^{(m)}, W^{(m)})\}_{m=1}^{\infty}$, and $p_{\mathbf{Z}}^{(k)}(\mathbf{u}, \mathbf{u}')$ be the k-step transition probability from $\mathbf{u}$ to $\mathbf{u}'$ of $\{\mathbf{Z}^{(m)}\}_{m=1}^{\infty}$. Then, the formula (2.3) can be simply written by

$$p^{(1)}\big((\mathbf{u}, g), (\mathbf{u}', g')\big) = p_{\mathbf{Z}}^{(1)}(\mathbf{u}, \mathbf{u}') \delta_{\varphi(\mathbf{u}') + g}^{g'}. \tag{2.4}$$

Especially the transition probability is determined on only by $\mathbf{u}$, $\mathbf{u}'$, and $h = g' - g$, i.e.,

$$p^{(1)}\big((\mathbf{u}, g), (\mathbf{u}', g + h)\big) = p^{(1)}\big((\mathbf{u}, 0_G), (\mathbf{u}', h)\big).$$

Thus we can easily show

$$p^{(k)}\big((\mathbf{u}, g), (\mathbf{u}', g + h)\big) = p^{(k)}\big((\mathbf{u}, 0_G), (\mathbf{u}', h)\big).$$

For $\mathbf{u} \in \operatorname{Im} \mathbf{Z}$, we put

$$H_{\mathbf{u}} := \{h \mid \text{There exists a } k \text{ such that } p^{(k)}\big((\mathbf{u}, 0_G), (\mathbf{u}, h)\big) > 0\}.$$

Note, by the irreducibility of $\{\mathbf{Z}^{(m)}\}_{m=1}^{\infty}$, $H_{\mathbf{u}}$ is not empty.

Lemma 3. *$H_{\mathbf{u}}$ is a subgroup of G.*

Proof. For $h \in H_{\mathbf{u}}$ and $m \in \mathbb{N}$,

$$\begin{aligned} p^{(mk)}\big((\mathbf{u}, 0_G), (\mathbf{u}, mh)\big) &\geq \prod_{j=0}^{m-1} p^{(k)}((\mathbf{u}, jh), (\mathbf{u}, (j+1)h)) \\ &= \{p^{(k)}\big((\mathbf{u}, 0_G), (\mathbf{u}, h)\big)\}^m. \end{aligned}$$

Therefore there exists a k such that $p^{(mk)}\big((\mathbf{u}, 0_G), (\mathbf{u}, mh)\big) > 0$, i.e., $mh \in H_{\mathbf{u}}$. Thus, since G is finite, $H_{\mathbf{u}}$ is a subgroup of G. □

Lemma 4. *$H_{\mathbf{u}}$ does not depend on $\mathbf{u}$.*

Proof. We show that $H_{\mathbf{u}} \subset H_{\mathbf{u}'}$ for any $\mathbf{u}, \mathbf{u}' \in \operatorname{Im} \mathbf{Z}$. By the irreducibility of $\{\mathbf{Z}^{(m)}\}_{m=1}^{\infty}$, there exist k_1, $k_2 \in \mathbb{N}$ and g_1, $g_2 \in G$ such that

$$p^{(k_1)}\big((\mathbf{u}, 0_G), (\mathbf{u}', g_1)\big) > 0, \qquad p^{(k_2)}\big((\mathbf{u}', 0_G), (\mathbf{u}, g_2)\big) > 0.$$

For all $h \in H_{\mathbf{u}}$, there exists a k_h such that

$$p^{(k_h)}\big((\mathbf{u}, 0_G), (\mathbf{u}, h)\big) > 0.$$

Therefore

$$\begin{aligned} &p^{(k_1+k_2+k_h)}\big((\mathbf{u}', 0_G), (\mathbf{u}', g_1 + g_2 + h)\big) \\ &\geq p^{(k_2)}\big((\mathbf{u}', 0_G), (\mathbf{u}, g_2)\big) p^{(k_h)}\big((\mathbf{u}, g_2), (\mathbf{u}, g_2 + h)\big) \\ &\qquad\qquad \times p^{(k_1)}\big((\mathbf{u}, g_2 + h), (\mathbf{u}', g_2 + h + g_1)\big) \\ &= p^{(k_2)}\big((\mathbf{u}', 0_G), (\mathbf{u}, g_2)\big) p^{(k_h)}\big((\mathbf{u}, 0_G), (\mathbf{u}, h)\big) p^{(k_1)}\big((\mathbf{u}, 0_G), (\mathbf{u}', g_1)\big) \\ &> 0, \end{aligned}$$

i.e., $g_1 + g_2 + h \in H_{\mathbf{u}'}$. In the same way, we have $g_1 + g_2 \in H_{\mathbf{u}'}$. Thus $H_{\mathbf{u}} \subset H_{\mathbf{u}'} - (g_1 + g_2) = H_{\mathbf{u}'}$. □

Now, we put $H := H_{\mathbf{u}}$ and $D := \{\mathbf{u} \in \operatorname{Im} \mathbf{Z} \mid p_{\mathbf{Z}}^{(1)}(\mathbf{u}, \mathbf{u}) > 0\}$. Then,

Lemma 5. *$\{(\mathbf{Z}^{(m)}, W^{(m)})\}_{m=1}^{\infty}$ is irreducible when $\varphi(D)$ generates G.*

Proof. By formula (2.4), for $\mathbf{u} \in D$,

$$p^{(1)}\big((\mathbf{u}, 0_G), (\mathbf{u}, \varphi(\mathbf{u}))\big) = p_{\mathbf{Z}}^{(1)}(\mathbf{u}, \mathbf{u})\delta_{\varphi(\mathbf{u})+0_G}^{\varphi(\mathbf{u})} = p_{\mathbf{Z}}^{(1)}(\mathbf{u}, \mathbf{u}) > 0,$$

i.e., $\varphi(D) \subset H_{\mathbf{u}} = H$. Since $\varphi(D)$ generates G and H is a subgroup, we have $H = G$.

For all $(\mathbf{u}, g)$ and $(\mathbf{u}', g')$, by the irreducibility of $\{\mathbf{Z}^{(m)}\}_{m=1}^{\infty}$, there exist $h \in G$ and $k \in \mathbb{N}$ such that

$$p^{(k)}\big((\mathbf{u}, 0_G), (\mathbf{u}', h)\big) > 0,$$

and by $H = G$, there exist $k' \in \mathbb{N}$ such that

$$p^{(k')}\big((\mathbf{u}', 0_G), (\mathbf{u}', g - g' - h)\big) > 0.$$

Thus,we have

$$\begin{aligned} p^{(k+k')}\big((\mathbf{u}, g), (\mathbf{u}', g')\big) &\geq p^{(k)}\big((\mathbf{u}, g), (\mathbf{u}', g + h)\big)p^{(k')}\big((\mathbf{u}', g + h), (\mathbf{u}', g')\big) \\ &= p^{(k)}\big((\mathbf{u}, 0_G), (\mathbf{u}', h)\big)p^{(k')}\big((\mathbf{u}', 0_G), (\mathbf{u}', g - g' - h)\big) \\ &> 0. \end{aligned}$$

□

Lemma 6. *$\{(\mathbf{Z}^{(m)}, W^{(m)})\}_{m=1}^{\infty}$ is irreducible.*

Proof. We show that $\varphi(D)$ generates G. For $1 \leq n_1, n_2 \leq n$, let us define $\mathbf{e}_{n_1,n_2} \in \mathbb{Z}^{2n+1}$ by the following:

$$\mathbf{e}_{n_1,n_2} := (\overbrace{\underbrace{b-1, \dots, b-1}_{n_1}, b, \dots, b}^{n}, \overbrace{\underbrace{b-1, \dots, b-1}_{n_2}, b, \dots, b}^{n}, 0).$$

Then, by the definition of $\varphi = (\widehat{\mathbf{d}}_1, \widehat{\mathbf{d}}_2)$, we have that

$$\varphi(\mathbf{e}_{n_1,n_2}) = ((\underbrace{b-1, \dots, b-1}_{n_1}, 0, \dots, 0), (\underbrace{b-1, \dots, b-1}_{n_2}, 0, \dots, 0))$$

and $\{\varphi(\mathbf{e}_{n_1,n_2})\}_{n_1,n_2=1}^{n}$ generates G. Therefore, it is sufficient to show that $\{\mathbf{e}_{n_1,n_2}\}_{n_1,n_2=1}^{n} \subset D$.

By the definitions of $\mathbf{Z}^{(1)}$ and $\mathbf{e}_{n_1,n_2}$, we have that

$$\mathbf{Z}^{(1)}(x_1,x_2,\alpha) = \mathbf{e}_{n_1,n_2} \Leftrightarrow \begin{cases} 1-\frac{1}{b} \le x_j + l\alpha < 1 & \text{for } 0 \le l < n_j \\ 1 \le x_j + l\alpha < 1+\frac{1}{b} & \text{for } n_j \le l < n \\ 0 < \alpha < \frac{1}{b} \end{cases}$$
$$\Leftrightarrow \begin{cases} 1-\frac{1}{b} \le x_j < 1-(n_j-1)\alpha \\ 1-n_j\alpha \le x_j < 1-(n-1)\alpha+\frac{1}{b} & \text{if } n_j < n \\ 0 < \alpha < \frac{1}{b}. \end{cases}$$

For the sake of simplicity, we investigate a stronger condition. By replacing the condition $0 < \alpha < \frac{1}{b}$ with $0 < \alpha < \frac{1}{b(n-1)}$, we have that

$$\begin{aligned} &\left\{(x_1,x_2,\alpha) \mid \mathbf{Z}^{(1)} = \mathbf{e}_{n_1,n_2}\right\} \\ &\quad \supset \left\{(x_1,x_2,\alpha) \,\middle|\, \begin{array}{l} 1-n_j\alpha \le x_j < 1-(n_j-1)\alpha \\ 0 < \alpha < \frac{1}{b(n-1)} \end{array} \right\}. \end{aligned}$$

Therefore $\left\{(x_1,x_2,\alpha) \mid \mathbf{Z}^{(1)} = \mathbf{e}_{n_1,n_2}\right\}$ has non-empty interior, and hence its Bernoulli measure is positive, and thereby $\mathbf{e}_{n_1,n_2} \in \operatorname{Im}\mathbf{Z}$. In a similar fashion, we can easily show that

$$\begin{aligned} &\left\{(x_1,x_2,\alpha) \mid \mathbf{Z}^{(1)} = \mathbf{Z}^{(2)} = \mathbf{e}_{n_1,n_2}\right\} \\ &\quad \supset \left\{(x_1,x_2,\alpha) \,\middle|\, \begin{array}{l} 1-n_j\alpha \le x_j < 1-(n_j-1)\alpha \\ 0 < \alpha < \frac{1}{b^2(n-1)} \end{array} \right\} \end{aligned}$$

Thus $p_{\mathbf{Z}}^{(1)}(\mathbf{e}_{n_1,n_2},\mathbf{e}_{n_1,n_2}) = P\left(\mathbf{Z}^{(1)} = \mathbf{Z}^{(2)} = \mathbf{e}_{n_1,n_2}\right) > 0$, i.e., $\mathbf{e}_{n_1,n_2} \in D$. ☐

Lemma 7. *$\{(\mathbf{Z}^{(m)}, W^{(m)})\}_{m=1}^{\infty}$ is aperiodic.*

Proof. Because we showed the irreducibility, it is enough to show that there exists at least one aperiodic state. Let $\mathbf{0} := (0,\ldots,0) \in \mathbb{Z}^{2n+1}$. By simple calculation, we have

$$\left\{(x_1,x_2,\alpha) \mid \mathbf{Z}^{(1)} = \mathbf{Z}^{(2)} = \mathbf{0}\right\} = \left\{(x_1,x_2,\alpha) \,\middle|\, \begin{array}{l} 0 \le x_j < \frac{1}{b^2}-(n-1)\alpha \\ 0 \le \alpha < \frac{1}{b^2(n-1)} \end{array} \right\}.$$

In the same way as in the proof of Lemma 6 one shows that $\mathbf{0} \in D$. Note that $\varphi(\mathbf{0}) = 0_G$, thus

$$\begin{aligned} p^{(1)}\big((\mathbf{0},g),(\mathbf{0},g)\big) &= p_{\mathbf{Z}}^{(1)}(\mathbf{0},\mathbf{0})\delta_g^{\varphi(\mathbf{0})+g} \\ &= p_{\mathbf{Z}}^{(1)}(\mathbf{0},\mathbf{0}) > 0. \end{aligned}$$

□

Lemma 8. *Let* $\mathbf{u} \in \operatorname{Im} \mathbf{Z}$ *and* $g \in G$. *Then, the stationary initial distribution* $\pi_{\mathbf{u},g}$ *of the Markov chain* $\{(\mathbf{Z}^{(m)}, W^{(m)})\}_{m=1}^{\infty}$ *is given as follows:*

$$\pi_{\mathbf{u},g} := P(\mathbf{Z}^{(1)} = \mathbf{u})\frac{1}{b^{2n}}.$$

Proof. Let us verify the stationarity of $\pi_{\mathbf{u},g}$:

$$\begin{aligned}
&\sum_{\mathbf{u},g} \pi_{\mathbf{u},g} P^{(1)}\big((\mathbf{u},g),(\mathbf{u}',g')\big) \\
&= \sum_{\mathbf{u},g} P(\mathbf{Z}^{(1)} = \mathbf{u})\frac{1}{b^{2n}} P(\mathbf{Z}^{(2)} = \mathbf{u}' \mid \mathbf{Z}^{(1)} = \mathbf{u})\delta_{g'}^{\varphi(\mathbf{u}')+g} \\
&= \sum_{\mathbf{u}} P(\mathbf{Z}^{(1)} = \mathbf{u})\frac{1}{b^{2n}} P(\mathbf{Z}^{(2)} = \mathbf{u}' \mid \mathbf{Z}^{(1)} = \mathbf{u}) \\
&= P(\mathbf{Z}^{(2)} = \mathbf{u}')\frac{1}{b^{2n}} = \pi_{\mathbf{u}',g'}.
\end{aligned}$$

Because the Markov chain is irreducible and aperiodic, the stationary initial distribution is unique. Thus, we have completed the proof. □

2.4 Absolutely continuous measures

In this section we show that the Theorem 1 is valid when ν is replaced with a measure ν' which is absolutely continuous with respect to a Bernoulli measure ν. Let h be the density function of ν' with respect to ν. It is sufficient to show that

$$\nu'\big(\mathbf{X}_1^{(m)}(\,\cdot\,, x_2, \alpha) = \mathbf{s}\big) = \int 1_{\{\mathbf{X}_1^{(m)}(x_1,x_2,\alpha)=\mathbf{s}\}} h(x_1)\nu(dx_1) \to \frac{1}{b^n} \quad \mu\text{-a.e. } \alpha. \tag{2.5}$$

Let h_i be a $\mathcal{F}(d^{(1)},\ldots,d^{(i)})$-measurable simple functions such that $\int |h_i - h| d\nu \longrightarrow 0$. For $A \in \mathcal{F}_\infty := \bigcup_{i=1}^{\infty} \mathcal{F}(d^{(1)},\ldots,d^{(i)})$, in a similar fashion to the proof in section 2, we have that

$$
\begin{aligned}
&\int \Big\{\nu\big(\mathbf{X}_1^{(m)}(\,\cdot\,, x_2, \alpha) = \mathbf{s}, A\big) - \frac{\nu(A)}{b^n}\Big\}^2 \mu(d\alpha) \\
&= \int 1_{\{\mathbf{X}_1^{(m)}=\mathbf{s}\}} 1_{A\times[0,1)\times[0,1)} 1_{\{\mathbf{X}_2^{(m)}=\mathbf{s}\}} 1_{[0,1)\times A\times[0,1)} dP - \frac{\nu(A)^2}{b^{2n}} \\
&\quad - 2\frac{\nu(A)}{b^n}\Big\{\int 1_{\{\mathbf{X}_1^{(m)}=\mathbf{s}\}} 1_{A\times[0,1)\times[0,1)} dP - \frac{\nu(A)}{b^n}\Big\} \\
&= P\big(\mathbf{X}_1^{(m)} = \mathbf{s}, \mathbf{X}_2^{(m)} = \mathbf{s}, A\times A\times[0,1)\big) - \frac{\nu(A)^2}{b^{2n}} \\
&\quad - 2\frac{\nu(A)}{b^n}\Big\{P\big(\mathbf{X}_1^{(m)} = \mathbf{s}, A\times[0,1)\times[0,1)\big) - \frac{\nu(A)}{b^n}\Big\},
\end{aligned}
$$

and this converges to 0 in exponential order (cf. Billingsley [2, Example 19.3.]). Thus there exists a $D_A \subset [0,1)$ such that $\mu(D_A) = 1$ and for all $\alpha \in D_A$,

$$\nu\big(\mathbf{X}_1^{(m)}(\,\cdot\,, x_2, \alpha) = \mathbf{s}, A\big) - \frac{\nu(A)}{b^n} \longrightarrow 0.$$

Let $D := \bigcap_{A\in\mathcal{F}_\infty} D_A$. Note that $\mathcal{F}_\infty$ is countable. Then $\mu(D) = 1$ and for all $\alpha \in D$,

$$\int 1_{\{\mathbf{X}_1^{(m)}(x_1,x_2,\alpha)=\mathbf{s}\}} 1_{\{x_1\in A\}} \nu(dx_1) - \frac{\nu(A)}{b^n} \longrightarrow 0 \quad \text{for } A \in \mathcal{F}_\infty.$$

Therefore, because h_i is a simple function, we have that, when $m \to \infty$,

$$\int 1_{\{\mathbf{X}_1^{(m)}(x_1,x_2,\alpha)=\mathbf{s}\}} h_i(x_1)\nu(dx_1) - \frac{\int h_i d\nu}{b^n} \longrightarrow 0 \quad \mu\text{-a.e. } \alpha.$$

For all $\varepsilon > 0$, there exists an N_ε such that $\int |h_i - h| d\nu < \varepsilon$ if $i > N_\varepsilon$. Thus,

$$
\begin{aligned}
&\Big|\int 1_{\{\mathbf{X}_1^{(m)}(x_1,x_2,\alpha)=\mathbf{s}\}} h(x_1)\nu(dx_1) - \frac{1}{b^n}\Big| \\
&= \Big|\int 1_{\{\mathbf{X}_1^{(m)}(x_1,x_2,\alpha)=\mathbf{s}\}} h_i(x_1)\nu(dx_1) - \frac{\int h_i d\nu}{b^n}\Big| \\
&\quad + \Big|\int 1_{\{\mathbf{X}_1^{(m)}(x_1,x_2,\alpha)=\mathbf{s}\}} (h_i(x_1) - h(x_1))\nu(dx_1)\Big| + \frac{1}{b^n}\Big|1 - \int h_i d\nu\Big| \\
&\le \Big|\int 1_{\{\mathbf{X}_1^{(m)}(x_1,x_2,\alpha)=\mathbf{s}\}} h_i(x_1)\nu(dx_1) - \frac{\int h_i d\nu}{b^n}\Big| + 2\varepsilon.
\end{aligned}
$$

We conclude the proof of the formula (2.5).

Chapter 3

Approximation by Markov Chains and a result for normal α

3.1 Result

We assume that P is a measure on $[0,1)$ such that $\{d^{(i)}\}_i$ is independent with respect to it, and that

$$\liminf_{i} \min_{0\le\varsigma<b} P(d^{(i)}=\varsigma)>0.$$

Our main result is the following:

Theorem 3. *Any normal number α to base b is "good", i.e., the process $\{X_n^{(m)}(\,\cdot\,,\alpha)\}_{n=0}^{\infty}$ on $([0,1),\mathcal{B}([0,1)),P)$ converges in law to $\{0,\dots,b-1\}$-valued fair i.i.d. when $m\to\infty$.*

Sugita [5] actually showed Theorem 3 in case $b=2$ and P is the Lebesgue measure. In fact, we show the following stronger statement:

Proposition 4. *For each $n\in\mathbb{N}$, let $A_n\subset[0,1)$ be the set of all $\alpha\in[0,1)$ whose base-b expansion contains a finite sequence*

$$\underbrace{0\cdots0}_{k+\kappa}\underbrace{\underbrace{0\cdots0}_{\kappa+2}1\underbrace{0\cdots0}_{\kappa+2}1\cdots\underbrace{0\cdots0}_{\kappa+2}1}_{M}\underbrace{0\cdots0}_{k+\kappa}$$

infinity many times for every k, where $\kappa := \min\{j \in \mathbb{N} \mid b^j \geq n-1\}$ and $M := n(b-1)(b^{\kappa+2} + (b-1)b^{\kappa})$. Then any $\alpha \in \cap_{n=1}^{\infty} A_n$ is "good", i.e., the process $\{X_n^{(m)}(\cdot, \alpha)\}_{n=0}^{\infty}$ on $([0,1), P)$ converges in law to $\{0, \ldots, b-1\}$-valued fair i.i.d. when $m \to \infty$.

3.2 Proof of the Theorem

We prove Proposition 5.

Proof. To show Proposition 5, it is sufficient to see

$$P((X_0^{(m)}(\,\cdot\,,\alpha), \ldots, X_{n-1}^{(m)}(\,\cdot\,,\alpha)) = \sigma) \to b^{-n} \qquad (m \to \infty) \tag{3.1}$$

for any n and $\sigma \in \{0, \ldots, b-1\}^n =: \Sigma$. Therefore we fix n and $\alpha \in A_n$ from now on, and define $\mathbf{X}^{(m)}$ by

$$\mathbf{X}^{(m)} := (X_0^{(m)}(\,\cdot\,,\alpha), \ldots, X_{n-1}^{(m)}(\,\cdot\,,\alpha)).$$

Let us consider m as a new time parameter. Then, we can see that for a certain increasing sequence $\{m_i\}_i$, the $\{\mathbf{X}^{(m_i)}\}_i$ is 'almost' a strong irreducible Markov chain whose unique stationary distribution is the uniform distribution on Σ. From this observation, (3.1) will be derived.

We use two lemmas. Lemma 9 claims that $P(\mathbf{X}^{(m)} = \sigma)$ is 'almost' a Markov kernel on Σ. By the assumption of measure P, we can find $p > 0$ and $\bar{m}$ as $\inf_{i \geq \bar{m}} \min_{\varsigma} P(d^{(i)} = \varsigma) \geq p$. Let $\equiv$ mean $\bmod b$ equality for any component on Σ.

Lemma 9. *Let $m \geq \bar{m}$, $m' \geq m + k + \kappa$, and $d^{(i)}(\alpha) = 0$ for $m < i \leq m'$. Then, for any $\sigma' \in \Sigma$,*

$$\left| P(\mathbf{X}^{(m')} = \sigma') - \sum_{\sigma \in \Sigma} P(\mathbf{X}^{(m)} = \sigma) P(\mathbf{X}^{(m')} - \mathbf{X}^{(m)} \equiv \sigma' - \sigma) \right| \leq 2(1-p)^k.$$

Lemma 10 claims 'strong irreducibility'. Let $*$ denote any one of 0, 1, $\ldots$, $b-1$.

Lemma 10. *There exists $\varepsilon > 0$ such that*

$$\min_{\sigma} P(\mathbf{X}^{(\hat{m})} - \mathbf{X}^{(m)} \equiv \sigma' - \sigma) \geq \varepsilon$$

for any $\sigma' \in \Sigma$, $k \geq 2$, $m \geq \bar{m}$ *such that*

$$\alpha = 0.\underbrace{*\cdots*}_{m}\underbrace{0\cdots0}_{k+\kappa}\underbrace{\underbrace{0\cdots0}_{\kappa+2}1\underbrace{0\cdots0}_{\kappa+2}1\cdots\underbrace{0\cdots0}_{\kappa+2}1}_{M}\underbrace{0\cdots0}_{k+\kappa}*\cdots,$$

and $\widehat{m} := m + k + \kappa + M(\kappa+3)$.

Now, we show (3.1) by using Lemma 9 and Lemma 10. Let $m_i \geq \bar{m}$ be as

$$\alpha = 0.\underbrace{*\cdots*}_{m_i}\underbrace{0\cdots0}_{k_i+\kappa}\underbrace{\underbrace{0\cdots0}_{\kappa+2}1\underbrace{0\cdots0}_{\kappa+2}1\cdots\underbrace{0\cdots0}_{\kappa+2}1}_{M}\underbrace{0\cdots0}_{k_i+\kappa}*\cdots,$$

$\widehat{m}_i := m_i + k_i + \kappa + M(\kappa+3)$, and $E^{(j)}(\sigma) := P(\mathbf{X}^{(j)} = \sigma) - b^{-n}$. Then, by Lemma 9, for $m' \geq \widehat{m}_i + k_i + \kappa$,

$$\left| E^{(m')}(\sigma') - \sum_{\sigma\in\Sigma} E^{(\widehat{m}_i)}(\sigma) P(\mathbf{X}^{(m')} - \mathbf{X}^{(\widehat{m}_i)} \equiv \sigma' - \sigma) \right| \leq 2(1-p)^{k_i}.$$

Therefore, because $P(\mathbf{X}^{(m')} - \mathbf{X}^{(\widehat{m}_i)} \equiv \sigma' - \sigma) \geq 0$,

$$\left| E^{(m')}(\sigma') \right| \leq \sum_{\sigma\in\Sigma} \left| E^{(\widehat{m}_i)}(\sigma) \right| P(\mathbf{X}^{(m')} - \mathbf{X}^{(\widehat{m}_i)} \equiv \sigma' - \sigma) + 2(1-p)^{k_i}.$$

Note that $\sum_{\sigma\in\Sigma} P(\mathbf{X}^{(m')} - \mathbf{X}^{(\widehat{m}_i)} \equiv \sigma' - \sigma) = 1$. Thus

$$\max_{\sigma\in\Sigma} |E^{(m')}(\sigma)| \leq \max_{\sigma\in\Sigma} |E^{(\widehat{m}_i)}(\sigma)| + 2(1-p)^{k_i}. \tag{3.2}$$

Again, by Lemma 9

$$\left| E^{(\widehat{m}_i)}(\sigma') - \sum_{\sigma\in\Sigma} E^{(m_i)}(\sigma) P(\mathbf{X}^{(\widehat{m}_i)} - \mathbf{X}^{(m_i)} \equiv \sigma' - \sigma) \right| \leq 2(1-p)^{k_i}.$$

Noting $\varepsilon \sum_{\sigma\in\Sigma} E^{(m_i)}(\sigma) = 0$ and that $P(\mathbf{X}^{(\widehat{m}_i)} - \mathbf{X}^{(m_i)} \equiv \sigma' - \sigma) - \varepsilon \geq 0$ by Lemma 10, we have

$$\left| E^{(\widehat{m}_i)}(\sigma') \right| \leq \sum_{\sigma\in\Sigma} \left| E^{(m_i)}(\sigma) \right| (P(\mathbf{X}^{(\widehat{m}_i)} - \mathbf{X}^{(m_i)} \equiv \sigma' - \sigma) - \varepsilon) + 2(1-p)^{k_i}.$$

Thus

$$\max_{\sigma\in\Sigma}|E^{(\widehat{m}_i)}(\sigma)| \le (1-b^n\varepsilon)\max_{\sigma\in\Sigma}|E^{(m_i)}(\sigma)| + 2(1-p)^{k_i}. \qquad (3.3)$$

By the assumption of α, we can define $\{m_i, k_i\}$ as $m_{i+1} \ge \widehat{m}_i + k_i + \kappa$ and $k_i \ge i\log((1-b^n)\varepsilon 2^{-1})/\log(1-p)$. Therefore, by (3.2) and (3.3), we have

$$\begin{aligned}
\max_{\sigma\in\Sigma}|E^{(m_{i+1})}(\sigma)| &\le (1-b^n\varepsilon)\max_{\sigma\in\Sigma}|E^{(m_i)}(\sigma)| + 4(1-p)^{k_i} \\
&\le (1-b^n\varepsilon)^i\Big(\max_{\sigma\in\Sigma}|E^{(m_1)}(\sigma)| + 4\sum_{j=1}^{i}(1-b^n\varepsilon)^{-j}(1-p)^{k_j}\Big) \\
&\le (1-b^n\varepsilon)^i\Big(\max_{\sigma\in\Sigma}|E^{(m_1)}(\sigma)| + 4\sum_{j=1}^{i}\frac{1}{2^j}\Big) \\
&= (1-b^n\varepsilon)^i\Big(\max_{\sigma\in\Sigma}|E^{(m_1)}(\sigma)| + 4\Big). \qquad (3.4)
\end{aligned}$$

Since $k_i \to \infty$ when $i \to \infty$, by (3.2) and (3.4), for $m \ge \widehat{m}_{i+1} + k_{i+1} + \kappa$, we have

$$\max_{\sigma\in\Sigma}|E^{(m)}(\sigma)| \le \max_{\sigma\in\Sigma}|E^{(m_{i+1})}(\sigma)| + 2(1-p)^{k_{i+1}} \to 0 \quad (i\to\infty)$$

□

3.3 Proof of Lemmas

Let the symbol $\lfloor\,\cdot\,\rfloor^m$ be the number which is rounded down to the m-th digit and $\langle\,\cdot\,\rangle_m$ be $\cdot - \lfloor\,\cdot\,\rfloor^m$, i.e. $\lfloor\,\cdot\,\rfloor^m = \cdot - \sum_{j=m+1}^{\infty} b^{-j}d^{(j)}(\,\cdot\,)$ and $\langle\,\cdot\,\rangle_m = \sum_{j=m+1}^{\infty} b^{-j}d^{(j)}(\,\cdot\,)$. For $m < m'$, define $\langle\!\lfloor\,\cdot\,\rfloor\!\rangle_m^{m'}$ by $\langle\!\lfloor\,\cdot\,\rfloor\!\rangle_m^{m'} := \lfloor\langle\,\cdot\,\rangle_m\rfloor^{m'} = \langle\lfloor\,\cdot\,\rfloor^{m'}\rangle_m = \sum_{j=m+1}^{m'} b^{-j}d^{(j)}(\,\cdot\,)$.

The main idea of Lemma 9 is as follows. The dependence of $\mathbf{X}^{(m)}$ and $\mathbf{X}^{(m')} - \mathbf{X}^{(m)}$ is caused by the carry at m-th digit which arises from the addition $x + l\alpha$. Therefore, intuitively, the 'dependence' is 'little' if $\langle\alpha\rangle_m$ is enough less.

Proof of Lemma 9. Let

$$A := \{x \in [0,1) \mid \langle x\rangle_m < \frac{b^k-1}{b^{m+k}}\}.$$

Since $\langle\!\lfloor \alpha \rfloor\!\rangle_m^{m+k+\kappa} = 0$ by the assumption of α, m and κ, we have

$$l\langle\alpha\rangle_m = l(\langle\!\lfloor \alpha \rfloor\!\rangle_m^{m+k+\kappa} + \langle\alpha\rangle_{m+k+\kappa}) < \frac{l}{b^{m+k+\kappa}} \le \frac{1}{b^{m+k}}\frac{n-1}{b^\kappa} \le \frac{1}{b^{m+k}} \quad (3.5)$$

for $l \le n-1$. Therefore, $\langle x\rangle_m + l\langle\alpha\rangle_m < 1/b^m$ for $x \in A$, and hence no carry arises from the addition $x + l\alpha$ at m-th digit, i.e., $\lfloor \langle x\rangle_m + l\langle\alpha\rangle_m \rfloor^m = 0$. Thus $\lfloor x + l\alpha \rfloor^m = \lfloor x \rfloor^m + l\lfloor \alpha \rfloor^m + \lfloor \langle x\rangle_m + l\langle\alpha\rangle_m \rfloor^m = \lfloor x \rfloor^m + l\lfloor \alpha \rfloor^m = \lfloor x + l\lfloor \alpha \rfloor^m \rfloor^m$, i.e., $\mathbf{X}^{(m)}(\,\cdot\,, \alpha) = \mathbf{X}^{(m)}(\,\cdot\,, \lfloor \alpha \rfloor^m)$ on A. We let $\triangle$ denote the symmetric difference. Then

$$\{\mathbf{X}^{(m)} = \sigma\} \triangle \{\mathbf{X}^{(m)}(\,\cdot\,, \lfloor \alpha \rfloor^m) = \sigma\} \subset A^c.$$

Note that $\mathbf{X}^{(m)}(\,\cdot\,, \lfloor \alpha \rfloor^m)$ depend only on $d^{(1)}, \dots, d^{(m)}$ and that $B_\sigma := \{\mathbf{X}^{(m')} - \mathbf{X}^{(m)} \equiv \sigma' - \sigma\}$ depend only on $d^{(m+1)}, d^{(m+2)}, \dots$. By the independence of base-b expansion,

$$\begin{aligned}
&\left| P(\mathbf{X}^{(m')} = \sigma') - \sum_{\sigma\in\Sigma} P(\mathbf{X}^{(m)} = \sigma) P(\mathbf{X}^{(m')} - \mathbf{X}^{(m)} \equiv \sigma' - \sigma) \right| \\
\le\ & \sum_{\sigma\in\Sigma} \left| P(\{\mathbf{X}^{(m')} = \sigma'\} \cap B_\sigma) - P(\mathbf{X}^{(m)} = \sigma) P(B_\sigma) \right| \\
\le\ & \sum_{\sigma\in\Sigma} \left| P(\{\mathbf{X}^{(m')} = \sigma'\} \cap B_\sigma) - P(\{\mathbf{X}^{(m)}(\,\cdot\,, \lfloor \alpha \rfloor^m) = \sigma\} \cap B_\sigma) \right| \\
& + \sum_{\sigma\in\Sigma} \left| P(\{\mathbf{X}^{(m)}(\,\cdot\,, \lfloor \alpha \rfloor^m) = \sigma\} \cap B_\sigma) - P(\mathbf{X}^{(m)} = \sigma) P(B_\sigma) \right| \\
=\ & \sum_{\sigma\in\Sigma} \left| P(\mathbf{X}^{(m)} = \sigma, \mathbf{X}^{(m')} - \mathbf{X}^{(m)} \equiv \sigma' - \sigma) - P(\{\mathbf{X}^{(m)}(\,\cdot\,, \lfloor \alpha \rfloor^m) = \sigma\} \cap B_\sigma) \right| \\
& + \sum_{\sigma\in\Sigma} P(B_\sigma) \left| P(\mathbf{X}^{(m)}(\,\cdot\,, \lfloor \alpha \rfloor^m) = \sigma) - P(\mathbf{X}^{(m)} = \sigma) \right| \\
\le\ & \sum_{\sigma\in\Sigma} P(A^c \cap B_\sigma) + \sum_{\sigma\in\Sigma} P(B_\sigma) P(A^c) = 2P(A^c)
\end{aligned}$$

By the definition of A,

$$\begin{aligned}
P(A^c) &= P(0 < \forall i \le k,\ d^{(m+i)}(x) = b-1) \\
&= \prod_{i=1}^{k} P(d^{(m+i)}(x) = b-1) \le (1-p)^k.
\end{aligned}$$

□

To prove Lemma 10 we use following lemma. For $u \in \mathbb{N}$, let

$$Y(u) := \sum_{i=1}^{\kappa+3} d^{(i)}\left(\frac{u}{b^{\kappa+3}}\right) \pmod b, \quad \mathbf{Y}(u) := (Y(u), Y(u+1), \ldots, Y(u+n-1)).$$

Lemma 11. *For any $\sigma \in \Sigma$, there exist $0 \le \varsigma < b$ and $0 \le u_i < b^{\kappa+3} - b^{\kappa+1}$ such that*

$$\sigma \equiv \varsigma \mathbf{1} + \sum_{i=1}^{M} \mathbf{Y}(u_i).$$

Now we can see the proof of Lemma 10.

Proof of Lemma 10. Let $\widetilde{m}_{-1} := m$, $\widetilde{m}_i := m+k+\kappa+i(\kappa+3)$ for $0 \le i \le M$ and

$$A_{\varsigma,u_1,\cdots,u_M} := \left\{ x \;\middle|\; \sum_{i=\widetilde{m}_{-1}+1}^{\widetilde{m}_0} d^{(i)}(x) = \varsigma,\ \langle\!\lfloor x \rfloor\!\rangle_{\widetilde{m}_0}^{\widetilde{m}_M+1} = \sum_{i=1}^{M} \frac{u_i}{b^{\widetilde{m}_i}} \right\}.$$

First, by the independence of $\{d^{(i)}\}_i$, we have

$$\begin{aligned} P(A_{\varsigma,u_1,\cdots,u_M}) &= P\left(\sum_{i=\widetilde{m}_{-1}+1}^{\widetilde{m}_0} d^{(i)}(x) = \varsigma\right) \prod_{j=\widetilde{m}_0+1}^{\widetilde{m}_M+1} P\left(d^{(j)}(x) = d^{(j)}\left(\sum_{i=1}^{M} \frac{u_i}{b^{\widetilde{m}_i}}\right)\right) \\ &\ge p^{M(\kappa+3)+1} \sum_{\varsigma'} P\left(\sum_{i=\widetilde{m}_{-1}+1}^{\widetilde{m}_0-1} d^{(i)}(x) = \varsigma'\right) P\left(d^{(\widetilde{m}_0)} = \varsigma - \varsigma'\right) \\ &\ge p^{M(\kappa+3)+2} \sum_{\varsigma'} P\left(\sum_{i=\widetilde{m}_{-1}+1}^{\widetilde{m}_0-1} d^{(i)}(x) = \varsigma'\right) = p^{M(\kappa+3)+2} =: \varepsilon. \end{aligned}$$

Note that $\varepsilon > 0$ does not depend on $0 \le \varsigma < b$ or $0 \le u_i < b^{\kappa+3} - b^{\kappa+1}$. Thus, by Lemma 11, it is sufficient to prove Lemma 10 to see that

$$\left\{ x \;\middle|\; \mathbf{X}^{(\widetilde{m})}(x) - \mathbf{X}^{(m)}(x) \equiv \varsigma\mathbf{1} + \textstyle\sum_{i=1}^{M} \mathbf{Y}(u_i) \right\} \supset A_{\varsigma,u_1,\cdots,u_M} \tag{3.6}$$

for any $0 \le \varsigma < b$ and $0 \le u_i < b^{\kappa+3} - b^{\kappa+1}$.

We will see that $\mathbf{X}^{(\widetilde{m}_{i-1})} - \mathbf{X}^{(\widetilde{m}_i)}$ is determined only by $\langle\!\lfloor x \rfloor\!\rangle_{\widetilde{m}_{i-1}}^{\widetilde{m}_i}$ on $A_{\varsigma,u_1,\cdots,u_M}$. Note the assumption of k, α, and m, i.e., $k \ge 2$ and

$$\alpha = 0.\underbrace{*\cdots*}_{m}\underbrace{0\cdots0}_{k+\kappa}\underbrace{\underbrace{0\cdots0}_{\kappa+2}1\underbrace{0\cdots0}_{\kappa+2}1\cdots\underbrace{0\cdots0}_{\kappa+2}1}_{M}\underbrace{0\cdots0}_{k+\kappa}*\cdots.$$

Since $\langle\lfloor\alpha\rfloor\rangle_{\tilde{m}_i}^{\tilde{m}_i+\kappa+2}=0$, we have

$$l\langle\alpha\rangle_{\tilde{m}_i}<\frac{1}{b^{\tilde{m}_i+2}}\quad\text{for } 0\le i\le M$$

in the same way as (3.5).

Let $x\in A_{\varsigma,u_1,\cdots,u_M}$. Since $\langle\lfloor x\rfloor\rangle_{\tilde{m}_i}^{\tilde{m}_{i+1}}=u_{i+1}/b^{\tilde{m}_{i+1}}$ for $0\le i<M$ and $\langle\lfloor x\rfloor\rangle_{\tilde{m}_M}^{\tilde{m}_M+1}=0$, we have

$$\begin{aligned}\langle x\rangle_{\tilde{m}_i} &= \langle\lfloor x\rfloor\rangle_{\tilde{m}_i}^{\tilde{m}_{i+1}}+\langle x\rangle_{\tilde{m}_{i+1}}\\ &< \frac{u_{i+1}+1}{b^{\tilde{m}_{i+1}}}\le\frac{b^{\kappa+3}-b^{\kappa+1}}{b^{\tilde{m}_{i+1}}}=\frac{1}{b^{\tilde{m}_i}}-\frac{1}{b^{\tilde{m}_i+2}}\quad\text{for } 0\le i<M,\\ \langle x\rangle_{\tilde{m}_M} &= \langle\lfloor x\rfloor\rangle_{\tilde{m}_M}^{\tilde{m}_M+1}+\langle x\rangle_{\tilde{m}_M+1}\le\frac{1}{b^{\tilde{m}_M+1}}<\frac{1}{b^{\tilde{m}_M}}-\frac{1}{b^{\tilde{m}_M+2}}.\end{aligned}$$

Thus, no carry arises from the addition $x+l\alpha$ at $\tilde{m}_i$-th digit, i.e. $\lfloor\langle x\rangle_{\tilde{m}_i}+l\langle\alpha\rangle_{\tilde{m}_i}\rfloor^{\tilde{m}_i}=0$ for $0\le i\le M$. Therefore

$$\langle\lfloor x+l\alpha\rfloor\rangle_{\tilde{m}_{i-1}}^{\tilde{m}_i}=\langle\lfloor x\rfloor^{\tilde{m}_i}+l\lfloor\alpha\rfloor^{\tilde{m}_i}\rangle_{\tilde{m}_{i-1}}=\langle\langle\lfloor x\rfloor\rangle_{\tilde{m}_{i-1}}^{\tilde{m}_i}+l\langle\lfloor\alpha\rfloor\rangle_{\tilde{m}_{i-1}}^{\tilde{m}_i}\rangle_{\tilde{m}_{i-1}}.$$

Thus

$$\begin{aligned}X_l^{(\tilde{m}_i)}(x)-X_l^{(\tilde{m}_{i-1})}(x) &= \sum_{j=\tilde{m}_{i-1}+1}^{\tilde{m}_i} d^{(j)}(\langle\lfloor x+l\alpha\rfloor\rangle_{\tilde{m}_{i-1}}^{\tilde{m}_i})\\ &= \sum_{j=\tilde{m}_{i-1}+1}^{\tilde{m}_i} d^{(j)}(\langle\lfloor x\rfloor\rangle_{\tilde{m}_{i-1}}^{\tilde{m}_i}+l\langle\lfloor\alpha\rfloor\rangle_{\tilde{m}_{i-1}}^{\tilde{m}_i}).\end{aligned}$$

Since $\langle\lfloor\alpha\rfloor\rangle_{\tilde{m}_{i-1}}^{\tilde{m}_i}=1/b^{\tilde{m}_i}$ for $1\le i\le M$ and $\langle\lfloor\alpha\rfloor\rangle_{\tilde{m}_{-1}}^{\tilde{m}_0}=0$,

$$\begin{aligned}X_l^{(\tilde{m}_i)}(x)-X_l^{(\tilde{m}_{i-1})}(x) &= \sum_{j=\tilde{m}_{i-1}+1}^{\tilde{m}_i} d^{(j)}(\frac{u_i+l}{b^{\tilde{m}_i}})=Y(u_i+l)\quad\text{for } 1\le i\le M,\\ X_l^{(\tilde{m}_0)}(x)-X_l^{(\tilde{m}_{-1})}(x) &= \sum_{j=\tilde{m}_{-1}+1}^{\tilde{m}_0} d^{(j)}(\langle\lfloor x\rfloor\rangle_{\tilde{m}_{-1}}^{\tilde{m}_0})=\varsigma.\end{aligned}$$

Therefore

$$\mathbf{X}^{(\tilde{m})}(x)-\mathbf{X}^{(m)}(x)\equiv\varsigma\mathbf{1}+\sum_{i=1}^{M}\mathbf{Y}(u_i).$$

Now we have the inclusion relation (3.6) and complete the proof of Lemma 10. $\square$

Finally, we see Lemma 11.

Proof. (Proof of Lemma 11) We begin with some properties of the function Y. Let $J := (b-1)b^\kappa + b^{\kappa+2}$ and

$$s_j := \sum_{i=1}^{J} Y(j+i).$$

Then, we have that for $0 \le j < b^\kappa$,

$$\begin{aligned} s_j - s_{j-1} &= \sum_{i=1}^{J} Y(j+i) - \sum_{i=1}^{J} Y(j-1+i) \\ &= Y(j+J) - Y(j) \\ &= \sum_{h=1}^{\kappa+3} \big(d^{(h)}(\frac{j}{b^{\kappa+3}} + \frac{b-1}{b^3} + \frac{1}{b^1}) - d^{(h)}(\frac{j}{b^{\kappa+3}})\big) \\ &= \sum_{h=4}^{\kappa+3} \big(d^{(h)}(\frac{j}{b^{\kappa+3}}) - d^{(h)}(\frac{j}{b^{\kappa+3}})\big) + b - 1 + 1 = 0 \pmod b \end{aligned}$$

and that

$$\begin{aligned} s_{b^\kappa} - s_{b^\kappa-1} &= \sum_{i=1}^{J} Y(b^\kappa+i) - \sum_{i=1}^{J} Y(b^\kappa-1+i) \\ &= Y(b^\kappa+J) - Y(b^\kappa) \\ &= \sum_{h=1}^{\kappa+3} \big(d^{(h)}(b^{-2}+b^{-1}) - d^{(h)}(b^{-3})\big) = 1 \pmod b \end{aligned}$$

Thus, we have $s := s_{-1} = \cdots = s_{b^\kappa-1} \pmod b$ and $s_{b^\kappa} = s+1 \pmod b$.

Then, for $1 \le l \le n$,

$$\begin{aligned} \sum_{i=1}^{J} \mathbf{Y}(b^\kappa - l + i) &\equiv \sum_{i=1}^{J} (Y(b^\kappa - l + i), \ldots, Y(b^\kappa - l + n - 1 + i)) \\ &\equiv (s_{b^\kappa-l}, \ldots, s_{b^\kappa-l+n-1}) \\ &\equiv (\underbrace{s, \ldots, s}_{l}, s+1, s_{b^\kappa+1}, \ldots, s_{b^\kappa-l+n-1}). \end{aligned}$$

Therefore,

$$-s\mathbf{1}+\sum_{i=1}^{J}\mathbf{Y}(b^{\kappa}-l+i)\equiv(\underbrace{0,\ldots,0}_{l},1,s_{b^{\kappa}+1}-s,\ldots,s_{b^{\kappa}-l+n-1}-s)\equiv:\sigma_l.$$

Let $\sigma_0:=\mathbf{1}$. Then, for any $\sigma\in\Sigma$, there exist $M_l\geq 0$ such that $\sigma\equiv M_0\sigma_0+\cdots+M_{n-1}\sigma_{n-1}$ and $M_0+\cdots+M_{n-1}\leq n(b-1)$. Therefore, since $\sigma_n\equiv\mathbf{0}$, $\sigma\equiv M_0\sigma_0+\cdots+M_{n-1}\sigma_{n-1}+M_n\sigma_n$ and $M_1+\cdots+M_n=n(b-1)$ where $M_n:=n(b-1)-(M_1+\cdots+M_{n-1})$. Thus, we have

$$\begin{aligned}\sigma &\equiv M_0\mathbf{1}+\sum_{l=1}^{n}M_l(-s\mathbf{1}+\sum_{i=1}^{J}\mathbf{Y}(b^{\kappa}-l+i))\\ &\equiv (M_0-s\sum_{l=1}^{n}M_l)\mathbf{1}+\sum_{l=1}^{n}M_l\sum_{i=1}^{J}\mathbf{Y}(b^{\kappa}-l+i)\end{aligned}$$

Let

$$\begin{aligned}\varsigma &:= M_0-s\sum_{l=1}^{n}M_l \pmod b\\ u_{J\sum_{1\leq l'<l}M_{l'}+jJ+i} &:= b^{\kappa}-l+i\end{aligned}$$

for $1\leq i\leq J$, $1\leq l\leq n$, and $0\leq j<M_l$. Then, since $M=(b^{\kappa+2}+(b-1)b^{\kappa})n(b-1)=J\sum_{l=1}^{n}M_l$, we have

$$\sigma\equiv\varsigma\mathbf{1}+\sum_{i=1}^{M}\mathbf{Y}(u_i)$$

and $0\leq u_i\leq b^{\kappa}+J-1=b^{\kappa+1}+b^{\kappa+2}-1<b^{\kappa+3}-b^{\kappa+1}$. □

Chapter 4

Nonhomogeneous Markov Chain and a result for irrational α

4.1 Result: Main Theorem

In this chapter, we apply the technique of Markov chain from a different point of view and prove the final result.

We assume that μ is a probability measure on $([0,1), \mathcal{B}([0,1)))$ such that $\{d^{(j)}\}_{j>0}$ is independent random variable under μ with

$$\liminf_{j\to\infty} \min_{0\le s<b} \mu(d^{(j)} = s) > 0.$$

Theorem 5. *If b is prime and α is irrational, then the distribution of the process $\{X_l^{(m)}\}_{l=0}^{\infty}$ on $([0,1), \mu)$ converges weakly to the distribution of $\mathbb{Z}/b\mathbb{Z}$-valued fair i.i.d. when m tends to ∞.*

When $b = 2$ and μ is the Lebesgue measure, Theorem 5 reduce to the conjectured result. We use the condition on b only in Section 4.3 to show 'strong irreducibility'.

4.2 Proof of the Main Theorem

To prove Theorem 5, it is sufficient to verify the convergence of any finite dimensional distribution, i.e., to verify

$$\int_{[0,1)} f(X_0^{(m)}, \ldots, X_n^{(m)})\, d\mu \to \frac{1}{b^{n+1}} \sum_{\mathbf{e} \in (\mathbb{Z}/b\mathbb{Z})^{n+1}} f(\mathbf{e}) \tag{4.1}$$

for any n and complex function f on $(\mathbb{Z}/b\mathbb{Z})^{n+1}$.

We fix n and define $(\mathbb{Z}/b\mathbb{Z})^{n+1}$-valued functions $\mathbf{d}^{(j)}$ and $\mathbf{X}^{(m)}$ by

$$\begin{aligned} \mathbf{d}^{(j)}(\omega) &:= (d^{(j)}(\omega), d^{(j)}(\omega+\alpha), \ldots, d^{(j)}(\omega+n\alpha)), \\ \mathbf{X}^{(m)} &:= (X_0^{(m)}, X_1^{(m)}, \ldots, X_n^{(m)}) \equiv \sum_{j=1}^{m} \mathbf{d}^{(j)} \qquad \text{for } m \geq 1, \end{aligned}$$

and $\mathbf{X}^{(0)} := \mathbf{0} \in (\mathbb{Z}/b\mathbb{Z})^{n+1}$.

Since $\{d^{(j)}\}_j$ is an independent process, $\{X_0^{(m)}\}_m = \{\sum_{1\leq j\leq m} d^{(j)}\}_m$ is a Markov chain. But, since $\{\mathbf{d}^{(j)}\}_j$ is not independent, we can not easily see that $\{\mathbf{X}^{(m)}\}_m = \{\sum_{1\leq j\leq m} \mathbf{d}^{(j)}\}_m$ is a Markov chain. Thus, we introduce a symbol $\mathbf{Z}^{(m)}$ to manage the dependence among $\{\mathbf{d}^{(j)}\}_j$. Define $\{0, \ldots, 2b-1\}^n$-valued function $\mathbf{Z}^{(m)} = (Z_1^{(m)}, \ldots, Z_n^{(m)})$ by

$$\begin{aligned} Z_l^{(m)}(\omega) &:= \lfloor b^m\omega - b\lfloor b^{m-1}\omega\rfloor + b^m l\alpha - b\lfloor b^{m-1} l\alpha\rfloor\rfloor \\ &= \lfloor b^m(\omega + l\alpha)\rfloor - b\big(\lfloor b^{m-1}\omega\rfloor + \lfloor b^{m-1} l\alpha\rfloor\big). \end{aligned}$$

Then, we have that

$$\begin{aligned} Z_l^{(m)}(\omega) &= d^{(m)}(\omega + l\alpha) + b\big(\lfloor b^{m-1}(\omega + l\alpha)\rfloor - \lfloor b^{m-1}\omega\rfloor - \lfloor b^{m-1} l\alpha\rfloor\big) \\ &\equiv d^{(m)}(\omega + l\alpha), \end{aligned}$$

and $\mathbf{d}^{(m)} \equiv (d^{(m)}, \mathbf{Z}^{(m)})$. Furthermore, we can say that $\mathbf{Z}^{(1)}, \mathbf{Z}^{(2)}, \ldots$ is a backward Markov chain, i.e., $\ldots, \mathbf{Z}^{(2)}, \mathbf{Z}^{(1)}$ is a Markov chain:

Proposition 6. *The process $\{\mathbf{Z}^{(m)}\}_{m=\infty}^{1}$ is a Markov chain.*

Proof. By the definition of $d^{(j)}$,

$$\sum_{j=m'}^{m-1} b^{m-1-j} d^{(j)}(x) = \lfloor b^{m-1}x\rfloor - b^{m-m'}\lfloor b^{m'-1}x\rfloor$$

for $m' < m$.

Thus,

$$
\begin{aligned}
& Z_l^{(m')}(\omega) \\
= \ & \left\lfloor b^{m'-m} \lfloor b^m(\omega + l\alpha) \rfloor \right\rfloor - b\left(\lfloor b^{m'-1}\omega \rfloor + \lfloor b^{m'-1} l\alpha \rfloor \right) \\
= \ & \left\lfloor b^{m'-m} \lfloor b^m(\omega + l\alpha) \rfloor - b\left(\lfloor b^{m'-1}\omega \rfloor + \lfloor b^{m'-1} l\alpha \rfloor \right) \right\rfloor \\
= \ & \left\lfloor b^{m'-m} \left(Z_l^{(m)}(\omega) + b\left(\lfloor b^{m-1}\omega \rfloor + \lfloor b^{m-1} l\alpha \rfloor \right) \right) - b\left(\lfloor b^{m'-1}\omega \rfloor + \lfloor b^{m'-1} l\alpha \rfloor \right) \right\rfloor \\
= \ & \left\lfloor b^{m'-m} Z_l^{(m)}(\omega) + \sum_{j=m'}^{m-1} b^{m'-j} \left(d^{(j)}(\omega) + d^{(j)}(l\alpha) \right) \right\rfloor \qquad (4.2) \\
= \ & \left\lfloor \sum_{j=m'}^{\infty} b^{m'-j} \left(d^{(j)}(\omega) + d^{(j)}(l\alpha) \right) \right\rfloor \qquad (4.3)
\end{aligned}
$$

where $1 \le l \le n$.

Let us define $\Phi^{(m,m')}$ a random transformation on $\{0, \ldots, 2b-1\}^n$ by

$$
\begin{aligned}
\phi_l^{(m,m')}(\omega; z) &= \left\lfloor b^{m'-m} z + \sum_{m' \le j < m} b^{m'-j} \left(d^{(j)}(\omega) + d^{(j)}(l\alpha) \right) \right\rfloor \quad \text{and} \\
\Phi^{(m,m')}(\omega; z_1, \ldots, z_n) &= \left(\phi_1^{(m,m')}(\omega; z_1), \ldots, \phi_n^{(m,m')}(\omega; z_n) \right)
\end{aligned}
$$

where $z, z_1, \ldots, z_n \in \{0, \ldots, 2b-1\}$.

Then, we have that $\mathbf{Z}^{(m')}(\omega) = \Phi^{(m,m')}(\omega; \mathbf{Z}^{(m)}(\omega))$ by (4.2), that $\omega \mapsto \Phi^{(m,m')}(\omega; \cdot)$ is $\sigma(d^{(m')}, \ldots, d^{(m-1)})$-measurable, and that $\mathbf{Z}^{(m)}$ is $\sigma(d^{(m)}, d^{(m+1)}, \ldots)$-measurable by (4.3). Thus, the independence of $d^{(j)}$ implies the Markov property. $\square$

In heuristic words, we can say that "$\left\{ (\mathbf{X}^{(\infty)} - \mathbf{X}^{(m-1)}, \mathbf{Z}^{(m)}) \right\}_{m=\infty}^{1}$ is a Markov chain":

Proposition 7. *For any $M \in \mathbb{N}$, the $(\mathbb{Z}/b\mathbb{Z})^{n+1} \times \{0, \ldots, 2b-1\}^n$-valued process $\{(\mathbf{X}^{(M)} - \mathbf{X}^{(m-1)}, \mathbf{Z}^{(m)})\}_{m=M,\ldots,1}$ is a Markov chain with the transition probability*

$$
\widetilde{P}^{(m,m')}_{(\mathbf{e},\mathbf{z}),(\mathbf{e}',\mathbf{z}')} = \mu\Big(\sum_{m' \le j < m} (d^{(j)}, \Phi^{(m,j)}(\mathbf{z})) \equiv \mathbf{e}' - \mathbf{e}, \Phi_{m,m'}(\mathbf{z}) = \mathbf{z}' \Big)
$$

where $1 \le m' < m \le M$, $\mathbf{e}, \mathbf{e}' \in (\mathbb{Z}/b\mathbb{Z})^{n+1}$ and $\mathbf{z}, \mathbf{z}' \in \{0, \ldots, 2b-1\}^n$.

Proof. Since $\mathbf{Z}^{(j)} = \Phi^{(m,j)}(\mathbf{Z}^{(m)})$, we have that $\mathbf{d}^{(j)} \equiv (d^{(j)}, \Phi^{(m,j)}(\mathbf{Z}^{(m)}))$.

Let B_m be any $\sigma(\{(\mathbf{X}^{(M)} - \mathbf{X}^{(j-1)}, \mathbf{Z}^{(j)})\}_{j=M,\dots,m})$-measurable set. Since $\mathbf{Z}^{(j)}$ is $\sigma(d^{(j)}, d^{(j+1)}, \dots)$-measurable, $\mathbf{X}^{(M)} - \mathbf{X}^{(m-1)} \equiv \sum_{j=m}^{M} \mathbf{Z}^{(j)}$ and B_m are $\sigma(d^{(m)}, d^{(m+1)}, \dots)$-measurable. Noting that $\Phi^{(m,m')}$ is $\sigma(d^{(m')}, \dots, d^{(m-1)})$-measurable, we see that the independence of $d^{(j)}$ implies

$$
\begin{aligned}
& \mu\big(\mathbf{X}^{(M)} - \mathbf{X}^{(m'-1)} \equiv \mathbf{e}', \mathbf{Z}^{(m')} = \mathbf{z}' \,\big|\, \{\mathbf{X}^{(M)} - \mathbf{X}^{(m-1)} \equiv \mathbf{e}, \mathbf{Z}^{(m)} = \mathbf{z}\} \cap B_m\big) \\
= \; & \mu\Big(\sum_{m' \le j < m} \mathbf{d}^{(j)} \equiv \mathbf{e}' - \mathbf{e}, \mathbf{Z}^{(m')} = \mathbf{z}' \,\Big|\, \{\mathbf{X}^{(M)} - \mathbf{X}^{(m-1)} \equiv \mathbf{e}, \mathbf{Z}^{(m)} = \mathbf{z}\} \cap B_m\Big) \\
= \; & \mu\Big(\sum_{m' \le j < m} (d^{(j)}, \Phi^{(m,j)}(\mathbf{z})) \equiv \mathbf{e}' - \mathbf{e}, \Phi^{(m,m')}(\mathbf{z}) = \mathbf{z}' \\
& \qquad \Big|\, \{\mathbf{X}^{(M)} - \mathbf{X}^{(m-1)} \equiv \mathbf{e}, \mathbf{Z}^{(m)} = \mathbf{z}\} \cap B_m\Big) \\
= \; & \mu\Big(\sum_{m' \le j < m} (d^{(j)}, \Phi^{(m,j)}(\mathbf{z})) \equiv \mathbf{e}' - \mathbf{e}, \Phi^{(m,m')}(\mathbf{z}) = \mathbf{z}'\Big).
\end{aligned}
$$

□

Note that $\widetilde{P}^{(m,m')}_{(\mathbf{e},\mathbf{z}),(\mathbf{e}',\mathbf{z}')}$ dose not depend on M.

To give a rough sketch of our proof, let us temporarily assume that 'the Markov chain $\{(\mathbf{X}^{(\infty)} - \mathbf{X}^{(m-1)}, \mathbf{Z}^{(m)})\}_{m=\infty,\dots,1}$ is strongly irreducible', i.e., for an $\epsilon > 0$ and any $N \in \mathbb{N}$, there exists a pair (m, m') with $N \le m' < m$ such that

$$
\min_{\mathbf{e},\mathbf{z},\mathbf{e}',\mathbf{z}'} \widetilde{P}^{(m,m')}_{(\mathbf{e},\mathbf{z}),(\mathbf{e}',\mathbf{z}')} \ge \epsilon. \tag{4.4}
$$

Then, by a standard method of Markov chain, we have that, for any complex function g on $(\mathbb{Z}/b\mathbb{Z})^{n+1} \times \{0, \dots, 2b-1\}^n$,

$$
(\widetilde{P}^{(M,1)} g)(\mathbf{e}, \mathbf{z}) := \sum_{\mathbf{e}',\mathbf{z}'} \widetilde{P}^{(M,1)}_{(\mathbf{e},\mathbf{z}),(\mathbf{e}',\mathbf{z}')} g(\mathbf{e}', \mathbf{z}') \to \frac{1}{b^{n+1}(2b)^n} \sum_{\mathbf{e}',\mathbf{z}'} g(\mathbf{e}', \mathbf{z}') \quad (M \to \infty).
$$

Let $\varpi : (\mathbb{Z}/b\mathbb{Z})^{n+1} \times \{0, \dots, 2b-1\}^n \to (\mathbb{Z}/b\mathbb{Z})^{n+1}$ be the projection. Then, by Proposition 7,

$$\begin{aligned}
\int_{[0,1)} f(\mathbf{X}^{(M)})\, d\mu &= \int_{(\mathbb{Z}/b\mathbb{Z})^{n+1}\times\{0,\ldots,2b-1\}^n} (f\circ\varpi)\, d\mu^{(\mathbf{X}^{(M)},\mathbf{Z}^{(1)})} \\
&= \int_{(\mathbb{Z}/b\mathbb{Z})^{n+1}\times\{0,\ldots,2b-1\}^n} \widetilde{P}^{(M,1)}(f\circ\varpi)\, d\mu^{(\mathbf{X}^{(M)}-\mathbf{X}^{(M-1)},\mathbf{Z}^{(M)})} \\
&\to \frac{\sum_{\mathbf{e},\mathbf{z}}(f\circ\varpi)(\mathbf{e},\mathbf{z})}{b^{n+1}(2b)^n} = \frac{1}{b^{n+1}}\sum_{\mathbf{e}} f(\mathbf{e}) \qquad (M\to\infty),
\end{aligned}$$

hence we have the convergence (4.1).

Unfortunately, the assumption (4.4) is not true. But, $f\circ\varpi$ dose not depend on the second variable, and hence 'strong irreducibility on $(\mathbb{Z}/b\mathbb{Z})^{n+1}$' is enough for our purpose.

Since the 'state space' $(\mathbb{Z}/b\mathbb{Z})^{n+1}$ is still too big to show 'strong irreducibility', we use following formulation.

For $\mathbf{e}=(e_0,\ldots,e_n)$, $\widehat{\mathbf{e}}=(\widehat{e}_0,\ldots,\widehat{e}_n)\in(\mathbb{Z}/b\mathbb{Z})^{n+1}$, let $\mathbf{e}\cdot\widehat{\mathbf{e}}$ be the inner product, i.e.,$\mathbf{e}\cdot\widehat{\mathbf{e}} := \sum_{l=0}^n e_l\widehat{e}_l$.

Then, for any complex function f on $(\mathbb{Z}/b\mathbb{Z})^{n+1}$,

$$\begin{aligned}
f(\mathbf{e}) &= \sum_{\widehat{\mathbf{e}}\in(\mathbb{Z}/b\mathbb{Z})^{n+1}} \widehat{f}(\widehat{\mathbf{e}})\exp\left(\frac{i2\pi\mathbf{e}\cdot\widehat{\mathbf{e}}}{b}\right), \quad \text{where} \\
\widehat{f}(\widehat{\mathbf{e}}) &= \frac{1}{b^{n+1}}\sum_{\mathbf{e}\in(\mathbb{Z}/b\mathbb{Z})^{n+1}} f(\mathbf{e})\exp\left(\frac{-i2\pi\mathbf{e}\cdot\widehat{\mathbf{e}}}{b}\right),
\end{aligned}$$

by the Plancherel's Theorem (cf. H. Dym - H. P. McKean[3]).

Since $\mathbf{X}^{(m)}\cdot\mathbf{0}$ is constant 0,

$$\begin{aligned}
f(\mathbf{X}^{(m)}) &= \sum_{\widehat{\mathbf{e}}\in(\mathbb{Z}/b\mathbb{Z})^{n+1}\setminus\{\mathbf{0}\}} \widehat{f}(\widehat{\mathbf{e}})\exp\left(\frac{i2\pi\mathbf{X}^{(m)}\cdot\widehat{\mathbf{e}}}{b}\right) + \widehat{f}(\mathbf{0}) \\
&= \sum_{\widehat{\mathbf{e}}\in(\mathbb{Z}/b\mathbb{Z})^{n+1}\setminus\{\mathbf{0}\}} \widehat{f}(\widehat{\mathbf{e}})\exp\left(\frac{i2\pi\mathbf{X}^{(m)}\cdot\widehat{\mathbf{e}}}{b}\right) + \frac{1}{b^{n+1}}\sum_{\mathbf{e}\in(\mathbb{Z}/b\mathbb{Z})^{n+1}} f(\mathbf{e}).
\end{aligned}$$

Thus, it is sufficient to see (4.1) to see that,

$$\int_{[0,1)} \exp\left(\frac{i2\pi\mathbf{X}^{(m)}\cdot\widehat{\mathbf{e}}}{b}\right) d\mu \to 0 \tag{4.5}$$

for any $\widehat{\mathbf{e}}\in(\mathbb{Z}/b\mathbb{Z})^{n+1}\setminus\{\mathbf{0}\}$. We fix $\widehat{\mathbf{e}}$ from now on, and see 'strong irreducibility' of the following Markov chain to show (4.5).

Lemma 12. *For any $M \in \mathbb{N}$, the $(\mathbb{Z}/b\mathbb{Z}) \times \{0, \ldots, 2b-1\}^n$-valued process $\{((\mathbf{X}^{(M)} - \mathbf{X}^{(m-1)}) \cdot \widehat{\mathbf{e}}, \mathbf{Z}^{(m)})\}_{m=M,\ldots,1}$ is a Markov chain with the transition probability*

$$P^{(m,m')}_{(z,\mathbf{z}),(z',\mathbf{z}')} = \mu\Big(\sum_{m' \le j < m} (d^{(j)}, \Phi^{(m,j)}(\mathbf{z})) \cdot \widehat{\mathbf{e}} \equiv z' - z, \Phi^{(m,m')}(\mathbf{z}) = \mathbf{z}' \Big)$$

for $z, z' \in \mathbb{Z}/b\mathbb{Z}$, $\mathbf{z}, \mathbf{z}' \in \{0, \ldots, 2b-1\}^n$, and $1 \le m' < m \le M$.

Proof. The same way as that of Proposition 7. □

For $0 < m' < m$ and any complex function g on $(\mathbb{Z}/b\mathbb{Z}) \times \{0, \ldots, 2b-1\}^n$, define a function $P^{(m,m')}g$ on $(\mathbb{Z}/b\mathbb{Z}) \times \{0, \ldots, 2b-1\}^n$ by

$$(P^{(m,m')}g)(z, \mathbf{z}) := \sum_{z', \mathbf{z}'} P^{(m,m')}_{(z,\mathbf{z}),(z',\mathbf{z}')} g(z', \mathbf{z}'),$$

and $\| \cdot \|_\infty$ be the max norm, i.e., $\|g\|_\infty := \max_{z,\mathbf{z}} |g(z, \mathbf{z})|$.

Then, by Lemma 12,

$$\begin{aligned} & \left| \int_{[0,1)} \exp\left(\frac{i2\pi \mathbf{X}^{(m)} \cdot \widehat{\mathbf{e}}}{b} \right) d\mu \right| \\ = & \left| \int_{(\mathbb{Z}/b\mathbb{Z}) \times \{0,\ldots,2b-1\}^n} h \, d\mu^{(\mathbf{X}^{(m)} \cdot \widehat{\mathbf{e}}, \mathbf{Z}^{(1)})} \right| \\ = & \left| \int_{(\mathbb{Z}/b\mathbb{Z}) \times \{0,\ldots,2b-1\}^n} P^{(m,1)} h \, d\mu^{((\mathbf{X}^{(m)} - \mathbf{X}^{(m-1)}) \cdot \widehat{\mathbf{e}}, \mathbf{Z}^{(m)})} \right| \\ \le & \ \|P^{(m,1)} h\|_\infty, \end{aligned}$$

where $h(z, \mathbf{z}) := \exp(i2\pi z/b)$ for $(z, \mathbf{z}) \in (\mathbb{Z}/b\mathbb{Z}) \times \{0, \ldots, 2b-1\}^n$. Let

$$\begin{aligned} \mathcal{G}_0 &:= \left\{ g : (\mathbb{Z}/b\mathbb{Z}) \times \{0, \ldots, 2b-1\}^n \to \mathbb{C} \ \middle| \ \textstyle\sum_{z,\mathbf{z}} g(z, \mathbf{z}) = 0 \right\} \\ \|P^{(m,m')}\|_{\mathcal{G}_0} &:= \max_{g \in \mathcal{G}_0, \|g\|_\infty \ne 0} \frac{\|P^{(m,m')} g\|_\infty}{\|g\|_\infty}. \end{aligned}$$

Then $h \in \mathcal{G}_0$. Similarly as (4.4), we temporarily assume that 'the Markov chain $\{((\mathbf{X}^{(\infty)} - \mathbf{X}^{(m-1)}) \cdot \widehat{\mathbf{e}}, \mathbf{Z}^{(m)})\}_{m=\infty,\ldots,1}$ is strong irreducible', i.e., for an $\epsilon > 0$ and any $N \in \mathbb{N}$, there exists a pair (m, m') with $N \le m' < m$ such that

$$\min_{z, \mathbf{z}, z', \mathbf{z}'} P^{(m,m')}_{(z,\mathbf{z}),(z',\mathbf{z}')} \ge \epsilon \tag{4.6}$$

Then, we have that $\|P^{(m,m')}\|_{\mathcal{G}_0} \le 1 - b(2b)^n\epsilon$, and hence

$$\begin{aligned} \left| \int_{[0,1)} \exp\left(\frac{i2\pi \mathbf{X}^{(M)} \cdot \widehat{\mathbf{e}}}{b} \right) d\mu \right| &\le \|P^{(M,1)}h\|_\infty \\ &\le \|P^{(M,1)}\|_{\mathcal{G}_0} \|h\|_\infty \to 0 \qquad (M \to \infty). \end{aligned}$$

Unfortunately, (4.6) is again not true. But, since h dose not depend on the second variable, it is enough for our purpose to estimate the operator norm $\|P^{(m,m')}\|_{\mathcal{G}}$ defined by

$$\begin{aligned} \mathcal{G} &:= \left\{ g \in \mathcal{G}_0 \mid \textstyle\sum_z g(z,\mathbf{z}) = 0 \text{ for any } \mathbf{z} \in \{0,\ldots,2b-1\}^n \right\}, \\ \|P^{(m,m')}\|_{\mathcal{G}} &:= \max_{g \in \mathcal{G}, \|g\|_\infty \neq 0} \frac{\|P^{(m,m')}g\|_\infty}{\|g\|_\infty}. \end{aligned}$$

Clearly $h \in \mathcal{G}$ and $\|P^{(m,m')}h\|_\infty \le \|P^{(m,m')}\|_{\mathcal{G}} \|h\|_\infty$. The following lemma is to iterate the estimate of $\|P^{(m,m')}\|_{\mathcal{G}}$.

Lemma 13. *$\mathcal{G}$ is $P^{(m,m')}$-invariant, i.e., $P^{(m,m')}\mathcal{G} \subset \mathcal{G}$.*

Proof. For any $g \in \mathcal{G}$,

$$\begin{aligned} &\sum_z (P^{(m,m')}g)(z,\mathbf{z}) \\ &= \sum_{z,z',\mathbf{z}'} \mu\Big(\sum_{m' \le j < m} (d^{(j)}, \Phi^{(m,j)}(\mathbf{z})) \cdot \widehat{\mathbf{e}} \equiv z' - z, \Phi^{(m,m')}(\mathbf{z}) = \mathbf{z}' \Big) g(z',\mathbf{z}') \\ &= \sum_{z',\mathbf{z}'} \mu(\Phi^{(m,m')}(\mathbf{z}) = \mathbf{z}') g(z',\mathbf{z}') \\ &= \sum_{\mathbf{z}'} \mu(\Phi^{(m,m')}(\mathbf{z}) = \mathbf{z}') \sum_{z'} g(z',\mathbf{z}') = 0. \end{aligned}$$

□

The following lemma claims that 'strong irreducibility on $\mathbb{Z}/b\mathbb{Z}$' is enough for our purpose.

Lemma 14. *Let $m' < m$ and $\epsilon > 0$. Assume that, for any $z \in \mathbb{Z}/b\mathbb{Z}$ and $\mathbf{z} \in \{0,\ldots,2b-1\}^n$, there exists $\mathbf{z}'(z,\mathbf{z}) \in \{0,\ldots,2b-1\}^n$ such that*

$$\min_{z' \in \mathbb{Z}/b\mathbb{Z}} P^{(m,m')}_{(z,\mathbf{z}),(z',\mathbf{z}'(z,\mathbf{z}))} \ge \epsilon. \tag{4.7}$$

Then

$$\|P^{(m,m')}\|_{\mathcal{G}} \le 1 - b\epsilon.$$

Proof. By assumption, $P^{(m,m')}_{(z,\mathbf{z}),(z',\mathbf{z}')} - \epsilon 1_{\{\mathbf{z}'=\mathbf{z}'(\mathbf{z})\}} \geq 0$. Hence, for $g \in \mathcal{G}$,

$$\begin{aligned}
\left|(P^{(m,m')}g)(z,\mathbf{z})\right| &= \left|\sum_{z',\mathbf{z}'} P^{(m,m')}_{(z,\mathbf{z}),(z',\mathbf{z}')} g(z',\mathbf{z}')\right| \\
&= \left|\sum_{z',\mathbf{z}'} (P^{(m,m')}_{(z,\mathbf{z}),(z',\mathbf{z}')} - \epsilon 1_{\{\mathbf{z}'=\mathbf{z}'(z,\mathbf{z})\}}) g(z',\mathbf{z}') + \epsilon \sum_{z'} g(z',\mathbf{z}'(z,\mathbf{z}))\right| \\
&= \left|\sum_{z',\mathbf{z}'} (P^{(m,m')}_{(z,\mathbf{z}),(z',\mathbf{z}')} - \epsilon 1_{\{\mathbf{z}'=\mathbf{z}'(z,\mathbf{z})\}}) g(z',\mathbf{z}')\right| \\
&\leq \sum_{z,\mathbf{z}'} (P^{(m,m')}_{(z,\mathbf{z}),(z',\mathbf{z}')} - \epsilon 1_{\{\mathbf{z}'=\mathbf{z}'(z,\mathbf{z})\}}) \|g\|_\infty = (1-b\epsilon)\|g\|_\infty
\end{aligned}$$

Thus, we have $\|P^{(m,m')}g\|_\infty \leq (1-b\epsilon)\|g\|_\infty$. □

Assumption of following proposition can be regarded as the 'strong irreducibility on $\mathbb{Z}/b\mathbb{Z}$' and we show that in Section 4.3.

Proposition 8. *Let $\epsilon > 0$. Assume that, for any $N \in \mathbb{N}$, there exists a pair (m, m') with $N \leq m' < m$ satisfying the assumption of Lemma 14. Then $\|P^{(M,1)}\|_{\mathcal{G}} \to 0$ $(M \to \infty)$.*

Proof. By the assumption, we can take a sequence $m'_1 < \cdots \leq m'_j < m_j \leq m'_{j+1} < \cdots$ such that all (m_j, m'_j) satisfying assumption of Lemma 14.

By Lemma 13, we have $\|P^{(m,m')}\|_{\mathcal{G}} \leq \|P^{(m,m')}\|_{\mathcal{G}}\|P^{(m',m'')}\|_{\mathcal{G}}$ for $m > m' > m''$. Thus, thanks to that $\|P^{(m,m')}\|_{\mathcal{G}} \leq 1$ for any $m > m'$, Lemma 14 implies that

$$\begin{aligned}
\|P^{(M,1)}\|_{\mathcal{G}} &\leq \|P^{(M,m_{J(M)})}\|_{\mathcal{G}} \prod_{j=1}^{J(M)} \|P^{(m_j,m'_j)}\|_{\mathcal{G}}\|P^{(m'_j,m_{j-1})}\|_{\mathcal{G}} \\
&\leq \prod_{j=1}^{J(M)} \|P^{(m_j,m'_j)}\|_{\mathcal{G}} \leq (1-b\epsilon)^{J(M)} \to 0 \qquad (M \to \infty)
\end{aligned}$$

where $J(M) := \max\{j \mid M > m_j\}$ and $m_0 := 0$. □

4.3 Strong irreducibility

In this section, we show the assumption (4.7) of Proposition 8 for $\widehat{\mathbf{e}} = (\widehat{e}_0, \ldots, \widehat{e}_n) \in (\mathbb{Z}/b\mathbb{Z})^{n+1} \setminus \{\mathbf{0}\}$. We can assume $\widehat{e}_0 \not\equiv 0$ without loss of generality.

Proof. Let $l_0 := \min\{0 \le l \le n \mid \widehat{e}_l \not\equiv 0\}$, $\widehat{\mathbf{e}}' := (\widehat{e}'_0, \ldots, \widehat{e}'_n)$ such that

$$\widehat{e}'_l := \begin{cases} \widehat{e}_{l+l_0} & \text{if } l \le n - l_0 \\ 0 & \text{otherwise,} \end{cases}$$

and $\mu'(\,\cdot\,) := \mu(\,\cdot\, - l_0\alpha)$. Then, $\widehat{e}'_0 \not\equiv 0$ and

$$\begin{aligned} \int_{[0,1)} \exp\left(\frac{i2\pi \mathbf{X}^{(m)} \cdot \widehat{\mathbf{e}}}{b}\right) d\mu &= \int_{[0,1)} \exp\left(\frac{i2\pi}{b} \sum_{l=0}^{n} X^{(m)}(\omega + l_0\alpha + l\alpha)\widehat{e}'_l\right) \mu(d\omega) \\ &= \int_{[0,1)} \exp\left(\frac{i2\pi \mathbf{X}^{(m)} \cdot \widehat{\mathbf{e}}'}{b}\right) d\mu'. \end{aligned}$$

Since

$$\mu'(d^{(j)} = s) \ge \begin{cases} \mu(d^{(j)} \equiv s + d^{(j)}(l_0\alpha) + 1, d^{(j+1)} = 0) & \text{if } d^{(j+1)}(l_0\alpha) \ne 0 \\ \mu(d^{(j)} \equiv s + d^{(j)}(l_0\alpha), d^{(j+1)} \ne 0) & \text{if } d^{(j+1)}(l_0\alpha) = 0, \end{cases}$$

we have that $\liminf_{j\to\infty} \min_{1 \le s < b} \mu'(d^{(j)} = s) \ge \left(\liminf_{j\to\infty} \min_{1\le s<b} \mu(d^{(j)} = s)\right)^2 > 0.$ □

Thus, now we show the assumption of Proposition 4.7 for $\widehat{\mathbf{e}} = (\widehat{e}_0, \ldots, \widehat{e}_n)$ such that $\widehat{e}_0 \not\equiv 0$. To show that, we use the following proposition claims that any irrational number α has infinite number of 'irregular' digits and is shown in Section 4.4.

Proposition 9. *For any irrational number α and $n \in \mathbb{N}$, there exist infinitely many m satisfying the following condition:*
CONDITION A. *For any $1 \le l \le n$, there exist j_l and j'_l with $m - 3n - 1 \le j_l, j'_l < m$ such that*

$$0 < d^{(j_l)}(l\alpha) \quad \textit{and} \quad d^{(j'_l)}(l\alpha) < b - 1. \tag{4.8}$$

We begin with to show that the chain can visit a neighborhood.

Lemma 15. *Let m satisfy Condition A. Then, for any $\mathbf{z} \in \{0, \ldots, 2b-1\}^n$, there exist $m' \in \mathbb{N}$ with $m - 3n - b - 1 \le m' < m$, $\mathbf{z}' \in \{0, \ldots, 2b-1\}^n$ and $y \in \mathbb{Z}/b\mathbb{Z}$ such that*

$$\min_{z \in \mathbb{Z}/b\mathbb{Z}, z' \equiv z+y, z+y+1} P^{(m,m')}_{(z,\mathbf{z}),(z',\mathbf{z}')} \ge \left(\inf_{j > m-3n-b} \min_{0 \le s < b} \mu(d^{(j)} = s)\right)^{3n+b+1}.$$

Proof. Let $m' < m - 3n - 1$ and

$$\begin{aligned} \overline{A} &:= \{d^{(m')} = 0, d^{(j)} = b - 1 \text{ for } m' < j < m\} \\ \underline{A} &:= \{d^{(m')} = 1, d^{(j)} = 0 \text{ for } m' < j < m\}. \end{aligned}$$

Then $d^{(k)}$ and $\Phi^{(m,k)}$ are non random either on $\overline{A}$ or on $\underline{A}$ where $m' \leq k < m$. Let

$$\begin{aligned} \overline{y} &:\equiv \textstyle\sum_{m' \leq j < m} (d^{(j)}(\overline{\omega}), \Phi^{(m,j)}(\overline{\omega}; \mathbf{z})) \cdot \widehat{\mathbf{e}}, \qquad & \overline{\mathbf{z}}' &:= \Phi^{(m,m')}(\overline{\omega}; \mathbf{z}), \\ \underline{y} &:\equiv \textstyle\sum_{m' \leq j < m} (d^{(j)}(\underline{\omega}), \Phi^{(m,j)}(\underline{\omega}; \mathbf{z})) \cdot \widehat{\mathbf{e}}, \qquad & \underline{\mathbf{z}}' &:= \Phi^{(m,m')}(\underline{\omega}; \mathbf{z}) \end{aligned}$$

where $\overline{\omega} \in \overline{A}$ and $\underline{\omega} \in \underline{A}$. Then,

$$\begin{aligned} \overline{A} &\subset \Big\{ \sum_{m' \leq j < m} (d^{(j)}, \Phi^{(m,j)}(\mathbf{z})) \cdot \widehat{\mathbf{e}} \equiv \overline{y}, \Phi^{(m,m')}(\mathbf{z}) = \overline{\mathbf{z}}' \Big\} \\ \underline{A} &\subset \Big\{ \sum_{m' \leq j < m} (d^{(j)}, \Phi^{(m,j)}(\mathbf{z})) \cdot \widehat{\mathbf{e}} \equiv \underline{y}, \Phi^{(m,m')}(\mathbf{z}) = \underline{\mathbf{z}}' \Big\}. \end{aligned}$$

Since

$$\begin{aligned} \mu\Big(\sum_{m' \leq j < m} (d^{(j)}, \Phi^{(m,j)}(\mathbf{z})) \cdot \widehat{\mathbf{e}} \equiv \overline{y}, \Phi^{(m,m')}(\mathbf{z}) = \overline{\mathbf{z}}' \Big) &\leq P^{(m,m')}_{(z,\mathbf{z}),(z+\overline{y},\overline{\mathbf{z}}')}, \\ \mu\Big(\sum_{m' \leq j < m} (d^{(j)}, \Phi^{(m,j)}(\mathbf{z})) \cdot \widehat{\mathbf{e}} \equiv \underline{y}, \Phi^{(m,m')}(\mathbf{z}) = \underline{\mathbf{z}}' \Big) &\leq P^{(m,m')}_{(z,\mathbf{z}),(z+\underline{y},\underline{\mathbf{z}}')} \end{aligned}$$

and $\min\{\mu(\overline{A}), \mu(\overline{A})\} \geq (\inf_{j \geq m'} \min_{0 \leq s < b} \mu(d^{(j)} = s))^{m-m'}$, it is sufficient to show that there exists m' with $m - 3n - b - 1 \leq m' < m$ such that $\overline{\mathbf{z}}' = \underline{\mathbf{z}}'$ and $\overline{y} = \underline{y} + 1$.

Firstly, we show that $\overline{\mathbf{z}}' = \underline{\mathbf{z}}'$ for any m' with $m' < m - 3n - 1$. By the definition of $\phi_l^{(m,m')}$,

$$\begin{aligned}
\phi_l^{(m,m')}(\overline{\omega}; z) &= \lfloor b^{m'-m}z + \sum_{m' \le j < m} b^{m'-j}(d^{(j)}(\overline{\omega}) + d^{(j)}(l\alpha))\rfloor \\
&= \lfloor b^{m'-m}z + 1 - b^{m'-m+1} + \sum_{m' \le j < m} b^{m'-j}d^{(j)}(l\alpha)\rfloor \\
&= d^{(m')}(l\alpha) + \lfloor b^{m'-m}z + 1 - b^{m'-m+1} + \sum_{m' < j < m} b^{m'-j}d^{(j)}(l\alpha)\rfloor, \\
\phi_l^{(m,m')}(\underline{\omega}; z) &= \lfloor b^{m'-m}z + \sum_{m' \le j < m} b^{m'-j}(d^{(j)}(\underline{\omega}) + d^{(j)}(l\alpha))\rfloor \\
&= \lfloor b^{m'-m}z + 1 + \sum_{m' \le j < m} b^{m'-j}d^{(j)}(l\alpha)\rfloor \\
&= d^{(m')}(l\alpha) + 1 + \lfloor b^{m'-m}z + \sum_{m' < j < m} b^{m'-j}d^{(j)}(l\alpha)\rfloor.
\end{aligned}$$

By Proposition 9, we can verify, for $k < m - 3n - 1$,

$$b^{k-m+1} \le \sum_{k<j<m} b^{k-j}d^{(j)}(l\alpha) \le 1 - 2b^{k-m+1}.$$

Therefore,

$$\lfloor b^{k-m}z + 1 - b^{k-m+1} + \sum_{k<j<m} b^{k-j}d^{(j)}(l\alpha)\rfloor = 1, \tag{4.9}$$

$$\lfloor b^{k-m}z + \sum_{k<j<m} b^{k-j}d^{(j)}(l\alpha)\rfloor = 0. \tag{4.10}$$

Thus, we have that

$$\phi_l^{(m,m')}(\overline{\omega}; z) = d^{(m')}(l\alpha) + 1 = \phi_l^{(m,m')}(\underline{\omega}; z),$$

and hence $\overline{\mathbf{z}}' = \underline{\mathbf{z}}'$.

Secondly, we show that $\overline{y} = \underline{y} + 1$ for suitably chosen m'. For any m' and

k with $m' < k < m - 3n - 1$, we have

$$\begin{aligned}
\phi_l^{(m,k)}(\overline{\omega}; z) &= \lfloor b^{k-m} z + b - b^{k-m+1} + \sum_{k \le j < m} b^{k-j} d^{(j)}(l\alpha) \rfloor \\
&= d^{(k)}(l\alpha) + b - 1 + \lfloor b^{k-m} z + 1 - b^{k-m+1} + \sum_{k < j < m} b^{k-j} d^{(j)}(l\alpha) \rfloor \\
\phi_l^{(m,k)}(\underline{\omega}; z) &= \lfloor b^{k-m} z + \sum_{k \le j < m} b^{k-j} d^{(j)}(l\alpha) \rfloor \\
&= d^{(k)}(l\alpha) + \lfloor b^{k-m} z + \sum_{k \le j < m} b^{k-j} d^{(j)}(l\alpha) \rfloor .
\end{aligned}$$

Again, by (4.9) and (4.10), we have that

$$\phi_l^{(m,k)}(\overline{\omega}; z) = d^{(k)}(l\alpha) + b \equiv d^{(k)}(l\alpha) = \phi_l^{(m,k)}(\underline{\omega}; z)$$

and hence

$$\begin{aligned}
\overline{y} - \underline{y} &= \sum_{m-3n-1 \le j < m} \big((d^{(j)}(\overline{\omega}), \Phi^{(m,j)}(\overline{\omega}; \mathbf{z})) - (d^{(j)}(\underline{\omega}), \Phi^{(m,j)}(\underline{\omega}; \mathbf{z})) \big) \cdot \widehat{\mathbf{e}} \\
&\quad + \sum_{m' \le j < m-3n-1} (d^{(j)}(\overline{\omega}) - d^{(j)}(\underline{\omega})) \widehat{e}_0 \\
&= \sum_{m-3n-1 \le j < m} \big((d^{(j)}(\overline{\omega}), \Phi^{(m,j)}(\overline{\omega}; \mathbf{z})) - (d^{(j)}(\underline{\omega}), \Phi^{(m,j)}(\underline{\omega}; \mathbf{z})) \big) \cdot \widehat{\mathbf{e}} \\
&\quad - \widehat{e}_0 + (m - 3n - 2 - m')(b-1) \widehat{e}_0 .
\end{aligned}$$

Therefore, since $\mathbb{Z}/b\mathbb{Z}$ is a simple group and $\widehat{e}_0 \neq 0$, we can take m' with $m - 3n - b - 1 \le m' < m - 3n - 1$ and $\overline{y} = \underline{y} + 1$. □

At last, we can show that the chain can visit everywhere.

Lemma 16. *Let* $\epsilon := \left(\frac{\varepsilon^{3n+b+1}}{b(2b)^n} \right)^{b-1}$ *for* $\varepsilon := \frac{1}{2} \liminf_j \min_{1 \le s < b} \mu(d^{(j)} = s)$*. The assumption in Proposition 8 holds, i.e., for any* $N \in \mathbb{N}$*, there exists a pair* (m, m') *with* $N \le m' < m$ *satisfying* (4.7).

Proof. We can take N as $\inf_{N \le j} \min_{1 \le s < b} \mu(d^{(j)} = s) \ge \varepsilon$ with out any loss of generality.

By Proposition 9, we can take a monotone decreasing sequence $m^{(1)}, \ldots, m^{(b)} \ge N$ such that $m^{(j+1)} < m^{(j)} - 3n - b - 1$ and each $m^{(j)}$ satisfies Condition A.

Let $\mathbf{z} \in \{0, \ldots, 2b-1\}^n$.

Then, by Lemma 15, there exist $m'^{(1)}$, $\mathbf{z}'^{(1)}$ and $y'^{(1)}$ with $m^{(2)} < m'^{(1)} < m^{(1)}$ such that

$$\min_{z \in \mathbb{Z}/b\mathbb{Z}} \min_{z' \equiv z+y'^{(1)}, z+y'^{(1)}+1} P^{(m^{(1)}, m'^{(1)})}_{(z,\mathbf{z}),(z',\mathbf{z}'^{(1)})} \geq \varepsilon^{3n+b+1}.$$

Since the cardinal number of the state space is $b(2b)^n$, there exist $y^{(1)}$ and $\mathbf{z}^{(2)}$ such that

$$P^{(m'^{(1)}, m^{(2)})}_{(y'^{(1)},\mathbf{z}'^{(1)}),(y^{(1)},\mathbf{z}^{(2)})} \geq \frac{1}{b(2b)^n}.$$

Noting that, by the definition, $P^{(k,k')}_{(w,\mathbf{w}),(w',\mathbf{w}')}$ is shift invariant on $\mathbb{Z}/b\mathbb{Z}$, i.e., $P^{(k,k')}_{(w,\mathbf{w}),(w',\mathbf{w}')} = P^{(k,k')}_{(w+v,\mathbf{w}),(w'+v,\mathbf{w}')}$ where $k > k'$, $w, w', v \in \mathbb{Z}/b\mathbb{Z}$ and $\mathbf{w}, \mathbf{w}' \in \{0, \ldots, 2b-1\}^n$, we see that, for any $z \in \mathbb{Z}/b\mathbb{Z}$,

$$\begin{aligned} P^{(m^{(1)}, m^{(2)})}_{(z,\mathbf{z}),(z+y^{(1)},\mathbf{z}^{(2)})} &\geq P^{(m^{(1)}, m'^{(1)})}_{(z,\mathbf{z}),(z+y'^{(1)},\mathbf{z}'^{(1)})} P^{(m'^{(1)}, m^{(2)})}_{(z+y'^{(1)},\mathbf{z}'^{(1)}),(z+y^{(1)},\mathbf{z}^{(2)})} \\ &= P^{(m^{(1)}, m'^{(1)})}_{(z,\mathbf{z}),(z+y'^{(1)},\mathbf{z}'^{(1)})} P^{(m'^{(1)}, m^{(2)})}_{(y'^{(1)},\mathbf{z}'^{(1)}),(y^{(1)},\mathbf{z}^{(2)})} \geq \frac{\varepsilon^{3n+b+1}}{b(2b)^n}. \end{aligned}$$

Since we can verify $P^{(m^{(1)}, m^{(2)})}_{(z,\mathbf{z}),(z+y^{(1)}+1,\mathbf{z}^{(2)})} \geq \frac{\varepsilon^{3n+b+1}}{b(2b)^n}$ in the same way,

$$\min_{z \in \mathbb{Z}/b\mathbb{Z}} \min_{z' \equiv z+y^{(1)}, z+y^{(1)}+1} P^{(m^{(1)}, m^{(2)})}_{(z,\mathbf{z}),(z',\mathbf{z}^{(2)})} \geq \frac{\varepsilon^{3n+b+1}}{b(2b)^n}.$$

Hence, inductively, we have that there exist $\mathbf{z}^{(3)}, \ldots, \mathbf{z}^{(b)}$ and $y^{(3)}, \ldots, y^{(b)}$ such that

$$\min_{z \in \mathbb{Z}/b\mathbb{Z}} \min_{z' \equiv z+y^{(j)}, z+y^{(j)}+1} P^{(m^{(j)}, m^{(j+1)})}_{(z,\mathbf{z}^{(j)}),(z',\mathbf{z}^{(j+1)})} \geq \frac{\varepsilon^{3n+b+1}}{b(2b)^n}$$

where $j = 1, \ldots, b-1$. Again, using the shift invariance of $P^{(k,k')}_{(w,\mathbf{w}),(w',\mathbf{w}')}$, we have that

$$\min_{z'=z+y^{(1)}+y^{(2)}, z+y^{(1)}+y^{(2)}+1, z+y^{(1)}+y^{(2)}+2} P^{(m^{(1)}, m^{(3)})}_{(z,\mathbf{z}),(z',\mathbf{z}^{(3)})} \geq \left(\frac{\varepsilon^{3n+b+1}}{b(2b)^n}\right)^2.$$

Repeat that to $m^{(b)}$ and let $m := m^{(1)}$, $m' := m^{(b)}$, $z^{(b)} := z + y^{(1)} + \cdots + y^{(b-1)}$ and $\mathbf{z}'(z, \mathbf{z}) := \mathbf{z}^{(b)}$. Then, we have

$$\min_{z'} P^{(m,m')}_{(z,\mathbf{z}),(z',\mathbf{z}'(z,\mathbf{z}))} = \min_{z'=z^{(b)}, \cdots, z^{(b)}+b-1} P^{(m^{(1)}, m^{(b)})}_{(z,\mathbf{z}),(z',\mathbf{z}^{(b)})} \geq \left(\frac{\varepsilon^{3n+b+1}}{b(2b)^n}\right)^{b-1} = \epsilon.$$

□

4.4 Proof of the Key Proposition

We use reduction to absurdity, i.e., we see that α is a rational number when only finite number of m satisfies (4.8). Let

$$N \;:=\; \max\left\{ m \;\middle|\; \begin{array}{l} \text{for any } 1 \le l \le n, \text{ there exist } m-3n-1 \le j_l < j_l' < m \\ \text{such that } d^{(j_l)}(l\alpha) \ne d^{(j_l')}(l\alpha) \end{array} \right\}.$$

Then, following lemma implies Proposition 9:

Lemma 17. *If N is finite, α is a rational number.*

We prepare two lemmas to see Lemma 17.

Lemma 18. *The finite sequence $d^{(m-2n-1)}(\alpha), \ldots, d^{(m-1)}(\alpha)$ is periodic and the period is at most n, when there exists $1 \le l \le n$ such that $d^{(m-3n-1)}(l\alpha) = \cdots = d^{(m-1)}(l\alpha)$.*

The proof is the calculation by writing of dividing $l\alpha$ by l .

Proof. Let r_j be the reminder of j-th decimal place, i.e.,

$$\begin{aligned} a_1 &:= \max\{0 \le a < b \mid la \le \lfloor bl\alpha \rfloor\}, \\ r_1 &:= \lfloor bl\alpha \rfloor - la_1, \\ a_j &:= \max\{0 \le a < b \mid la \le br_{j-1} + d^{(j)}(l\alpha)\} \quad \text{for } j > 1, \\ r_j &:= br_{j-1} + d^{(j)}(l\alpha) - la_j \quad \text{for } j > 1. \end{aligned}$$

Then r_j is determined by r_{j-1} and $d^{(j)}(l\alpha)$. Since $d^{(m-3n-1)}(l\alpha) = \cdots = d^{(m-1)}(l\alpha)$ and $0 \le r_j < l$, r_j must be periodic at $m-3n+l-2 \le j < m$ and the period bounded by l hence a_j is periodic at $m-3n+l-1 \le j < m$ and the period bounded by l. By the definition of a_j, we have $a_j = d^{(j)}(\alpha)$. □

Next lemma is about the least periods of a sequence and a its subsequence.

Lemma 19. *Let $\{a_j\}_{j=0}^{p-1}$ be a sequence and its least period be k $(2k \le p)$. The least period of a subsequence $\{a_j\}_{j=q}^{q+p'-1}$ $(0 \le q < q+p' \le p)$ is k when $p' \ge 2k$.*

Proof. Let k' be least period of $\{a_j\}_{j=q}^{q+p'-1}$. Then $k' \le k$. For any $1 \le l \le p-k'$, there exist $j \in \mathbb{Z}$ and $0 \le r < k$ such that $l-q = kj+r$. Since $r+k' < 2k \le p'$,

$$a_l = a_{q+kj+r} = a_{q+r} = a_{q+r+k'} = a_{q+kj+r+k'} = a_{l+k'}.$$

hence k' is a period of $\{a_j\}_{j=0}^{p-1}$ and $k' = k$. □

Now, we proof Lemma 17.

Proof. For any $m > N$, there exists $1 \le l_m \le n$ such that

$$d^{(m-3n-1)}(l_m\alpha) = \cdots = d^{(m-1)}(l_m\alpha).$$

Let $m := N+1$. Then, by Lemma 18, $\{d^{(j)}(\alpha)\}_{j=N-2n}^{N}$ is periodic. Let the least period be k_1. Similarly, $\{d^{(j)}(\alpha)\}_{j=N-2n+1}^{N+1}$ is periodic. Let the least period be k_2. Since $k_1, k_2 \le n$, Lemma 19 implies that both k_1 and k_2 are same as the least period of $\{d^{(j)}(\alpha)\}_{j=N-2n+1}^{N}$ hence $k_1 = k_2$. Thus, by the induction, $\{d^{(j)}(\alpha)\}_{j=N-2n}^{\infty}$ is periodic and α is a rational number. □

Chapter 5

Appendix

In this chapter we see some properties of nonhomogeneous Markov Chain.

5.1 Definition

Let Σ be at most countable set and $T = \mathbb{Z} \cap (a, b)$. We say that a Σ-valued process $\{X^{(m)}\}_{m \in T}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a *Markov Chain* if it satisfies the Markov property, i.e.,

$$\mathbb{P}(X^{(m)} = \sigma \mid X^{(k)}, k \in T, k \leq m - 1) = \mathbb{P}(X^{(m)} = \sigma \mid X^{(m-1)}).$$

For $m < m'$, let $P^{(m,m')} = (p^{(m,m')}_{\sigma,\sigma'})_{\sigma,\sigma' \in \Sigma}$ be as $p^{(m,m')}_{\sigma,\sigma'} := \mathbb{P}(X^{(m')} = \sigma' \mid X^{(m)} = \sigma)$. We call $P^{(m,m')}$ *the transition matrix from time m to time m'*. Denote $P^{(m)} := P^{(m,m+1)}$.

Proposition 10. *Let S be a set, $\{Y^{(m)}\}$ be an S-valued independent process, and $f^{(m)} : S \times \Sigma \to \Sigma$. Then the process $\{X^{(m)}\}$ defined by $X^{(1)} = f^{(1)}(Y^{(1)}, \sigma_0)$ and $X^{(m+1)} = f^{(m+1)}(Y^{(m+1)}, X^{(m)})$ is a Markov chain.*

Proof.

$$\begin{aligned}
& \mathbb{P}(X^{(1)} = x^{(1)}, \ldots, X^{(m)} = x^{(m)}) \\
= \; & \mathbb{P}(f^{(1)}(Y^{(1)}, \sigma_0) = x^{(1)}, \ldots, f^{(m)}(Y^{(m)}, x^{(m-1)}) = x^{(m)}) \\
= \; & \mathbb{P}(f^{(1)}(Y^{(1)}, \sigma_0) = x^{(1)}) \cdots \mathbb{P}(f^{(m)}(Y^{(m)}, x^{(m-1)}) = x^{(m)})
\end{aligned}$$

□

Proposition 11. *Let $\{X^{(m)}\}_{m\in T}$ be a Markov Chain with transition probability P. The Markov property valid for reversed time, i.e.,*

$$\mathbb{P}(X^{(m)} = \sigma \mid X^{(k)}, k \in T, k \geq m+1) = \mathbb{P}(X^{(m)} = \sigma \mid X^{(m+1)}).$$

Proof. Let us see probability of a cylinder set. For $m < m'$,

$$\mathbb{P}(X^{(m)} = \sigma^{(m)}, \dots, X^{(m')} = \sigma^{(m')}) = \mathbb{P}(X^{(m)} = \sigma^{(m)}) p^{(m)}_{\sigma^{(m)},\sigma^{(m+1)}} \cdots p^{(m'-1)}_{\sigma^{(m'-1)},\sigma^{(m')}}.$$

Therefore

$$\frac{\mathbb{P}(X^{(m)} = \sigma^{(m)}, \dots, X^{(m')} = \sigma^{(m')})}{\mathbb{P}(X^{(m+1)} = \sigma^{(m+1)}, \dots, X^{(m')} = \sigma^{(m')})} = \frac{\mathbb{P}(X^{(m)} = \sigma^{(m)}) P_{\sigma^{(m)},\sigma^{(m+1)}}}{\mathbb{P}(X^{(m)} = \sigma^{(m)})}.$$

Thus, by taking $m' = m+1$,

$$\frac{\mathbb{P}(X^{(m)} = \sigma^{(m)}, \dots, X^{(m')} = \sigma^{(m')})}{\mathbb{P}(X^{(m+1)} = \sigma^{(m+1)}, \dots, X^{(m')} = \sigma^{(m')})} = \frac{\mathbb{P}(X^{(m)} = \sigma^{(m)}, X^{(m+1)} = \sigma^{(m+1)})}{\mathbb{P}(X^{(m+1)} = \sigma^{(m+1)})}$$

and this is the Markov property for reversed time. □

For $m < m'$, let $\bar{P}^{(m',m)} = (\bar{p}^{(m',m)}_{\sigma',\sigma})_{\sigma',\sigma\in\Sigma}$ be as $\bar{p}^{(m',m)}_{\sigma',\sigma} := \mathbb{P}(X^{(m)} = \sigma \mid X^{(m')} = \sigma')$. We called $\bar{P}^{(m',m)}$ as *reversed-transition matrix from time m' to time m*. $\bar{P}^{(m')} := \bar{P}^{(m',m'-1)}$.

By the definitions, we have that

$$p^{(m)}_{\sigma,\sigma'} \mathbb{P}(X^{(m)} = \sigma) = \mathbb{P}(X^{(m)} = \sigma, X^{(m+1)} = \sigma') = \bar{p}^{(m+1)}_{\sigma',\sigma} \mathbb{P}(X^{(m+1)} = \sigma').$$

5.2 Convergence to equilibrium

Let $T := \mathbb{N}$. In this section, we show a convergence to equilibrium of non time homogeneous Markov Chain.

Theorem 12. *Let $\varepsilon > 0$ and π be a probability measure on Σ. If $\sum_\sigma \pi_\sigma p^{(m)}_{\sigma,\sigma'} = \pi_{\sigma'}$ for any m and there exists infinity many pairs (m, m') such that $\min_{\sigma,\sigma'\in\Sigma} p^{(m,m')}_{\sigma,\sigma'} \geq \varepsilon$, then $\lim_{m'\to\infty} p^{(m,m')}_{\sigma,\sigma'} = \pi_{\sigma'}$ for any m and σ.*

By the assumption, $\pi p^{(m,m')} = \pi p^{(m)} \cdots p^{(m'-1)} = \pi$, i.e., π is the eigenvector of the eigenvalue 1 of $p^{(m,m')}$. And by $\min_{\sigma,\sigma'\in\Sigma} p^{(m,m')}_{\sigma,\sigma'} \geq \varepsilon$, Perron-Frobenius Theorem implies the uniqueness of π.

Proposition 13. $\min_{\sigma,\sigma'\in\Sigma} p_{\sigma,\sigma'}^{(m,m')} \geq \varepsilon$ *and* $\sum_\sigma \pi_\sigma p_{\sigma,\sigma'}^{(m,m')} = \pi_{\sigma'}$ *implies that* $\|p^{(m,m')} f\|_{\max} \leq (1-\varepsilon)\|f\|_{\max}$ *for any complex valued function* f *on* Σ *where* $\|f\|_{\max} := \max_\sigma |f_\sigma - (\pi, f)|$.

Proof. Noting that $(\pi, p^{(m,m')} f) = (\pi p^{(m,m')}, f) = (\pi, f)$ and $\sum_{\sigma'} p_{\sigma,\sigma'}^{(m,m')} = 1$, We have

$$\begin{aligned} \|p^{(m,m')} f\|_{\max} &= \max_\sigma |(P^{(m,m')} f)_\sigma - (\pi, p^{(m,m')} f)| \\ &= \max_\sigma |(P^{(m,m')} f)_\sigma - (\pi, f)| \\ &= \max_\sigma |\sum_{\sigma'} p_{\sigma,\sigma'}^{(m,m')} (f_{\sigma'} - (\pi, f))| \\ &= \max_\sigma |\sum_{\sigma'} (p_{\sigma,\sigma'}^{(m,m')} - \varepsilon\pi_{\sigma'})(f_{\sigma'} - (\pi, f))| \\ &\leq \max_\sigma \sum_{\sigma'} (p_{\sigma,\sigma'}^{(m,m')} - \varepsilon\pi_{\sigma'})|f_{\sigma'} - (\pi, f)| \\ &\leq (1-\varepsilon) \max_{\sigma'} |f_{\sigma'} - (\pi, f)|. \end{aligned}$$

□

Now, we see the proof of Theorem 12.

Proof. Let $m < m_1 < m_1' \leq m_2 < \cdots$ be the sequence such that $\min_{\sigma,\sigma'\in\Sigma} p_{\sigma,\sigma'}^{(m_j,m_j')} \geq \varepsilon$ and J be as $J(m) := \max\{j \mid m_j' \leq m\}$.

It is easy to see that $\|p^{(m,m')} f\|_{\max} \leq \|f\|_{\max}$ for any $m < m'$. Therefore we have

$$\begin{aligned} \|p^{(m,m')} f\|_{\max} &= \|p^{(m,m_1)} p^{(m_1,m')} f\|_{\max} \\ &\leq \|p^{(m_1,m')} f\|_{\max} \\ &= \|p^{(m_1,m_1')} p^{(m_1',m')} f\|_{\max} \\ &\leq (1-\varepsilon)\|p^{(m_1',m')} f\|_{\max} \\ &\cdots \\ &\leq (1-\varepsilon)^{J(m')} \|p^{({m'}_{J(m')},m')} f\|_{\max} \\ &\leq (1-\varepsilon)^{J(m')} \|f\|_{\max} \end{aligned}$$

When $m' \to \infty$, we have $\|p^{(m,m')} f\|_{\max} \to 0$ since $J(m') \to \infty$. Let $f = \chi_{\sigma'}$ then we have $\max_\sigma |p_{\sigma,\sigma'}^{(m,m')} - \pi_{\sigma'}| \to 0$. □

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