Affine Weyl group symmetry of the Garnier system

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博　士　論　文

Affine Weyl group symmetry of the Garnier system

(ガルニエ系のアフィンワイル群対称性)

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Contents

1 Introduction

For the sixth Painvevé equation P_{VI}, the symmetry structure is well-known ([1], [5]). Furthermore, the τ-functions for P_{VI} satisfy the various bilinear relations ([4], [5], [6]). But, such properties are not clarified completely for the Garnier system which is extension of P_{VI} to several variables. In this paper, we show that the Garnier system in n-variables ($n \geq 2$) has affine Weyl group symmetry of type $B^{(1)}_{n+3}$. We also formulate the τ-functions for the Garnier system (or the Schlesinger system of rank 2) on the root lattice $Q(C_{n+3})$ and present the relations of three types, Toda equations, Hirota-Miwa equations and bilinear differential equations, which are satisfied by those τ-functions.

Consider a Fuchsian differential equation on $\mathbb{P}^1(\mathbb{C})$

$$\frac{d^2w}{dz^2} + P_1(z,t)\,\frac{dw}{dz} + P_2(z,t)\,w = 0, \tag{1.1}$$

with regular singularities $z = t_1, \ldots, t_n$, 0, 1, ∞, apparent singularities $z = \lambda_1, \ldots, \lambda_n$ and the Riemann scheme

$$\begin{pmatrix} z = t_i & z = 0 & z = 1 & z = \infty & z = \lambda_j \\ 0 & 0 & 0 & \rho & 0 \\ \theta_i & \kappa_0 & \kappa_1 & \rho + \kappa_\infty & 2 \end{pmatrix} \quad i, j = 1, \ldots, n, \tag{1.2}$$

assuming that the Fuchs relation

$$\sum_{i=1}^{n} \theta_i + \kappa_0 + \kappa_1 + \kappa_\infty + 2\,\rho = 1 \tag{1.3}$$

is satisfied. The monodromy preserving deformations of the equation (1.1) with the scheme (1.2) is described as the following completely integrable Hamiltonian system:

$$\frac{\partial \lambda_j}{\partial t_i} = \frac{\partial \mathcal{H}_i}{\partial \mu_j}, \quad \frac{\partial \mu_j}{\partial t_i} = -\frac{\partial \mathcal{H}_i}{\partial \lambda_j} \quad (i, j = 1, \ldots, n), \tag{1.4}$$

where

$$\mu_j = \operatorname*{Res}_{z=\lambda_j} P_2(z,t)\,dz \quad (j = 1, \ldots, n) \tag{1.5}$$

and $\mathcal{H}_i$ $(i = 1, \ldots, n)$ are the rational functions in (λ, μ) defined by

$$\mathcal{H}_i = -\operatorname*{Res}_{z=t_i} P_2(z,t)\,dz. \tag{1.6}$$

By the canonical transformation

$$x_i = \frac{t_i}{t_i - 1}, \quad q_i = \frac{t_i \prod_{j=1}^{n} (t_i - \lambda_j)}{\prod_{j=1, j\neq i}^{n+2} (t_i - t_j)} \quad (i = 1, \ldots, n). \tag{1.7}$$

the system (1.4) is transformed into the Hamiltonian system

$$\frac{\partial q_j}{\partial x_i} = \frac{\partial K_i}{\partial p_j}, \quad \frac{\partial p_j}{\partial x_i} = -\frac{\partial K_i}{\partial q_j} \quad (i, j = 1, \ldots, n), \tag{1.8}$$

with *polynomial* Hamiltonians K_i $(i = 1, \ldots, n)$. The Hamiltonians K_i are given explicitly by

$$\begin{aligned} x_i(x_i - 1)\, K_i = q_i &\left(\rho + \sum_{j=1}^{n} q_j p_j\right)\left(\rho + \kappa_\infty + \sum_{j=1}^{n} q_j p_j\right) \\ &- \sum_{j=1, j\neq i}^{n} S_{ij} q_i p_i (q_j p_j - \theta_j) - \sum_{j=1, j\neq i}^{n} S_{ji} q_i (q_j p_j - \theta_j)\, p_j \\ &- \sum_{j=1, j\neq i}^{n} S^*_{ij} (q_i p_i - \theta_i)\, p_i q_j - \sum_{j=1, j\neq i}^{n} S_{ij} (q_i p_i - \theta_i)\, q_j p_j \\ &+ x_i p_i (q_i p_i - \theta_i) - (x_i + 1)(q_i p_i - \theta_i)\, q_i p_i \\ &+ (\kappa_1 x_i + \kappa_0 - 1)\, q_i p_i, \end{aligned} \tag{1.9}$$

where

$$S_{ij} = \frac{x_i (x_j - 1)}{x_j - x_i}, \quad S^*_{ij} = \frac{x_i (x_i - 1)}{x_i - x_j}. \tag{1.10}$$

We call the Hamiltonian system (1.8) with the Hamiltonians (1.9) *the Garnier system.*

As is known in [1], the Garnier system is derived from the Schlesinger system, through a transformation which takes a system of linear differential equations into the Fuchsian differential equation (1.1). Then the independent and dependent variables of the Garnier system are expressed as the rational functions in the independent and dependent variables of the Schlesinger system. Furthermore, we can identify the τ-functions for the Garnier system with those for the Schlesinger system. Therefore we first investigate the symmetries and the properties of the τ-functions for the Schlesinger system. After that, we apply the obtained results to the Garnier system.

In Section 2, we describe the transformations of three types, permutation of the points, sign changes of exponents and Schlesinger transformations, acting on the Schlesinger system. In Section 3, we introduce the τ-functions for the Schlesinger system on the root lattice $Q(C_{n+3})$. We also show in Section 3 that those τ-functions satisfy the relations of three types, Toda equations, Hirota-Miwa equations and bilinear differential equations. In Section 4, we show that the Garnier system has affine Weyl group symmetry of type $W(B^{(1)}_{n+3})$. We also show in Section 4 that the τ-functions formulated on the root lattice $Q(C_{n+3})$ satisfy the relations of three types, Toda equations, Hirota-Miwa equations and bilinear differential equations.

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2 Schlesinger system

We consider a system of linear differential equations for a vector of unknown functions $\vec{w} = {}^t(w_1, w_2)$

$$\partial_z \vec{w} = \sum_{j=1}^{n+2} \frac{A_j(t)}{z - t_j} \vec{w}, \quad \partial_{t_i} \vec{w} = -\frac{A_i(t)}{z - t_i} \vec{w} \quad (i = 1, \dots, n), \tag{2.11}$$

on $\mathbb{P}^1(\mathbb{C})$ with parameters $t = (t_1, \dots, t_n)$, where $\partial_z = \partial/\partial z$, $\partial_{t_i} = \partial/\partial t_i$ and $t_{n+1} = 0$, $t_{n+2} = 1$. The Schlesinger system (of rank 2) is the system of total differential equations obtained as the compatibility condition for (2.11). It is expressed by

$$\begin{aligned} dA_j &= \sum_{i=1, i\neq j}^{n+2} [A_i, A_j]\, d\log(t_j - t_i) \quad (j = 1, \dots, n+2), \\ dG_j &= \sum_{i=1, i\neq j}^{n+2} A_i G_j\, d\log(t_j - t_i) \qquad (j = 1, \dots, n+2), \end{aligned} \tag{2.12}$$

where

$$A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}, \quad G_j = \begin{pmatrix} -d_j & b_j \\ c_j & d_j \end{pmatrix} \begin{pmatrix} f_j & 0 \\ 0 & g_j \end{pmatrix} \quad (j = 1, \dots, n+2). \tag{2.13}$$

We always assume that the following conditions are satisfied:

(i) $\det A_j = 0$ and $\mathrm{tr} A_j = \theta_j \notin \mathbb{Z} \quad (j = 1, \dots, n+2)$,

(ii) The matrices A_j $(j = 1, \dots, n+2)$ satisfy the following relation:

$$A_\infty := -\sum_{i=j}^{n+2} A_j = \begin{pmatrix} \rho & 0 \\ 0 & \rho + \theta_{n+3} \end{pmatrix}, \quad \theta_{n+3} \notin \mathbb{Z}, \quad \rho = -\frac{1}{2} \sum_{j=1}^{n+3} \theta_j. \tag{2.14}$$

The matrices G_j $(j = 1, \dots, n+2)$ are obtained as follows. Let Y be a fundamental solution of the system (2.11). We can expand Y at $x = t_j$ $(j = 1, \dots, n+2)$ into the power series

$$Y = G_j \sum_{k=0}^{\infty} Y_k^{(j)} (x - t_j)^k (x - t_j)^{L_j}, \quad Y_0^{(j)} = I_2, \tag{2.15}$$

where

$$L_j = G_j^{-1} A_j\, G_j = \begin{pmatrix} 0 & 0 \\ 0 & \theta_j \end{pmatrix} \quad (j = 1, \dots, n+2) \tag{2.16}$$

are diagonal matrices.

The Schlesinger system is invariant under the action of the following transformations of three types. They are associated with (1) permutation of the points $t_1, \dots, t_{n+2}, t_{n+3} = \infty$, (2) sign change of the exponents $\theta_1, \dots, \theta_{n+3}$, and (3) shifting exponents by integers (*Schlesinger transformations*). In this section, we describe those transformations.

2.1 Permutation of the points

The action of the symmetric group $\mathfrak{S}_{n+3}$ on the set of the points $t_1, \dots, t_n, t_{n+1} = 0, t_{n+2} = 1, t_{n+3} = \infty$ can be lifted to transformations of the independent and dependent variables. Denoting the adjacent transpositions by $\sigma_1 = (12), \dots, \sigma_{n+2} = (n+2, n+3)$, we describe the action of those σ_k on the variables t_i $(i = 1, \dots, n)$ and a_j, b_j, c_j, d_j, f_j, g_j $(j = 1, \dots, n+2)$. In the following, we use the matrix notations

$$w(A_j) = \begin{pmatrix} w(a_j) & w(b_j) \\ w(c_j) & w(d_j) \end{pmatrix} \tag{2.17}$$

and

$$w(G_j) = \begin{pmatrix} -w(d_j) & w(b_j) \\ w(c_j) & w(d_j) \end{pmatrix} \begin{pmatrix} w(f_j) & 0 \\ 0 & w(g_j) \end{pmatrix}, \tag{2.18}$$

for a transformation w of the dependent variables. For $k = 1, \dots, n-1$,

$$\sigma_k(t_i) = t_{\sigma_k(i)}, \quad \sigma_k(A_j) = A_{\sigma_k(j)}, \quad \sigma_i(G_j) = G_{\sigma_i(j)}. \tag{2.19}$$

We remark that σ_n, σ_{n+1} and σ_{n+2} are derived from Mëbius transformations on $\mathbb{P}^1(\mathbb{C})$. The transformation σ_n is derived from the transformation $x \to (x - t_n) / (1 - t_n)$:

$$\begin{aligned} &\sigma_n(t_i) = \frac{t_i - t_n}{1 - t_n} \quad (i \neq n), \quad \sigma_n(t_n) = \frac{-t_n}{1 - t_n}, \\ &\sigma_n(A_j) = (1 - t_n)^{A_\infty} A_{\sigma_n(j)} (1 - t_n)^{-A_\infty}, \\ &\sigma_n(G_j) = (1 - t_n)^{A_\infty} G_{\sigma_n(j)} (1 - t_n)^{L_{\sigma_n(j)}}, \end{aligned} \tag{2.20}$$

Similarly, the transformation σ_{n+1} is derived from the transformation $x \to 1 - x$:

$$\sigma_{n+1}(t_i) = 1 - t_i, \quad \sigma_{n+1}(A_j) = A_{\sigma_{n+1}(j)}, \quad \sigma_{n+1}(G_j) = G_{\sigma_{n+1}(j)}. \tag{2.21}$$

Finally, the transformation σ_{n+2} derived from the transformation $x \to 1/x$:

$$\begin{aligned}
&\sigma_{n+2}(t_i) = \frac{t_i}{t_i - 1}, \\
&\sigma_{n+2}(A_j) = G_{n+2}^{-1}\, A_j\, G_{n+2} && (j \neq n+2), \\
&\sigma_{n+2}(A_{n+2}) = G_{n+2}^{-1}\, L_{n+3}\, G_{n+2}, \\
&\sigma_{n+2}(G_j) = G_{n+2}^{-1}\, G_j\, (t_j - 1)^{\rho I_2 + 2L_j} && (j \neq n+2), \\
&\sigma_{n+2}(G_{n+2}) = G_{n+2}^{-1},
\end{aligned} \tag{2.22}$$

where

$$L_{n+3} = \begin{pmatrix} 0 & 0 \\ 0 & \theta_{n+3} \end{pmatrix}. \tag{2.23}$$

The action of each σ_k on the parameters θ_j is given by

$$\sigma_k(\theta_j) = \theta_{\sigma_k(j)} \quad (j = 1, \ldots, n+3). \tag{2.24}$$

Remark 2.1 *The transformations σ_k $(k = 1, \ldots, n+2)$ is determined with the following ambiguity. σ_n has the ambiguity for*

$$e\,(l_1\rho), \quad e\,(l_2(\rho+\theta_{n+3})), \quad e\,(m_j\theta_j) \quad (l_1, l_2, m_j \in \mathbb{Z},\ j = 1, \ldots, n+2) \tag{2.25}$$

and σ_{n+2} has the ambiguity for

$$e\,(l\rho), \quad e\,(m_j(\rho + 2\,\theta_j)) \quad (l, m_j \in \mathbb{Z},\ j = 1, \ldots, n+1), \tag{2.26}$$

where

$$e\,(\lambda) = exp\,(2\pi i\lambda). \tag{2.27}$$

2.2 Sign change of exponents

Let Y be a fundamental solution of system (2.11). We consider the gauge transformations

$$\begin{aligned}
&r_k(Y) = (x - t_k)^{-\theta_k} Y \quad (k = 1, \ldots, n+2), \\
&r_{n+3}(Y) = WY,
\end{aligned} \tag{2.28}$$

where

$$W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.29}$$

Each r_k acts on the parameters θ_j as follows:

$$r_k(\theta_j) = (-1)^{\delta_{jk}} \theta_j \quad (j = 1, \ldots, n+3), \tag{2.30}$$

where δ_{jk} stands for the Kronecker's delta, and can be lifted to transformations of the dependent variables. We describe the action of r_k on the dependent variables a_j, b_j, c_j, d_j, f_j, g_j $(j = 1, \ldots .n + 2)$. For $k = 1, \ldots, n + 2$,

$$\begin{aligned} r_k(A_k) &= A_k - \theta_k I_2, & r_k(A_j) &= A_j & (j \neq k), \\ r_k(G_k) &= G_k, & r_k(G_j) &= (t_j - t_k)^{-\theta_k} G_j & (j \neq k). \end{aligned} \tag{2.31}$$

For $k = n + 3$,

$$r_{n+3}(A_j) = W A_j W, \quad r_{n+3}(G_j) = W G_j. \tag{2.32}$$

The independent variables t_i $(i = 1, \ldots, n)$ are invariant under the action of each r_k.

Remark 2.2 *The transformations* r_k $(k = 1, \ldots, n + 3)$ *is determined with the ambiguity for*

$$e\,(l_j \theta_j) \quad (l_j \in \mathbb{Z},\, j = 1, \ldots, n + 2). \tag{2.33}$$

2.3 Schlesinger transformations

In this subsection, we construct the Schlesinger transformations, following the manner of [3]. For each vector of integers $\mu = (\mu_1, \ldots, \mu_{n+3}) \in \mathbb{Z}^{n+3}$ with $\mu_1 + \ldots + \mu_{n+3} \in 2\,\mathbb{Z}$, we consider the gauge transformation

$$T_\mu(Y) = R_\mu Y, \tag{2.34}$$

which acts on the parameters $\theta_1, \ldots, \theta_{n+3}$ by

$$T_\mu(\theta_j) = \theta_j - \mu_j \quad (j = 1, \ldots, n + 3), \tag{2.35}$$

where R_μ is a 2×2 matrix of functions which are rational in the independent variables x and t_i $(i = 1, \ldots, n)$. Then, for each μ, R_μ is determined up to multiplication by a scalar matrix and the gauge transformation T_μ can be lifted to a birational transformation (called the Schlesinger transformation) of the dependent variables.

The group of the Schlesinger transformations is generated by the transformations T_k $(k = 1, \ldots, n + 2)$, such that

$$\begin{aligned} &T_k(\theta_k) = \theta_k + 1, \quad T_k(\theta_{k+1}) = \theta_{k+1} - 1, \quad T_k(\theta_j) = \theta_j \\ &\qquad\qquad (j = 1 \ldots, n + 3,\, j \neq k, k + 1), \end{aligned} \tag{2.36}$$

and T_{n+3}, such that

$$\begin{aligned} &T_{n+3}(\theta_j) = \theta_j & &(j = 1 \ldots, n + 1), \\ &T_{n+3}(\theta_{n+2}) = \theta_{n+2} + 1, & &T_{n+3}(\theta_{n+3}) = \theta_{n+3} + 1. \end{aligned} \tag{2.37}$$

Now the action of T_k on the variables a_j, b_j, c_j, d_j, f_j, g_j $(j = 1, \dots .n+2)$ is described as follows. For $k = 1, \dots, n+1$,

$$
\begin{aligned}
T_k(A_k) &= A_{k+1} + \frac{(1+\theta_k-\theta_{k+1})\,R_k}{t_k - t_{k+1}} + \sum_{j=1,\, j\neq k,k+1}^{n+2} \frac{R_k^* A_j R_k}{(t_k-t_j)(t_k-t_{k+1})},\\
T_k(A_{k+1}) &= A_k - \frac{(1+\theta_k-\theta_{k+1})\,R_k}{t_k - t_{k+1}} - \sum_{j=1,\, j\neq k,k+1}^{n+2} \frac{R_k A_j R_k^*}{(t_{k+1}-t_j)(t_k-t_{k+1})},\\
T_k(A_j) &= A_j - \frac{R_k^* A_j R_k}{(t_k-t_j)(t_k-t_{k+1})} + \frac{R_k A_j R_k^*}{(t_{k+1}-t_j)(t_k-t_{k+1})}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j \neq k, k+1),\\
T_k(G_k) &= \frac{R_k^* G_k}{t_k - t_{k+1}} + \frac{G_k E_2}{t_k - t_{k+1}} + \sum_{j=1,\, j\neq k}^{n+2} \frac{R_k^* G_k E_1 G_k^{-1} A_j G_k E_2}{(1+\theta_k)(t_k-t_{k+1})(t_k-t_j)},\\
T_k(G_{k+1}) &= -R_k G_{k+1} - G_{k+1} E_2 - \sum_{j=1,\, j\neq k}^{n+2} \frac{R_k G_{k+1} E_2 G_k^{-1} A_j G_k E_1}{(1-\theta_k)(t_k-t_j)},\\
T_k(G_j) &= \left(I_2 + \frac{R_k}{t_{k+1}-t_j}\right) G_j \qquad (j \neq k, k+1).
\end{aligned}
\tag{2.38}
$$

where

$$
\begin{aligned}
R_k &= \frac{t_k - t_{k+1}}{b_k a_{k+1} + d_k b_{k+1}} \begin{pmatrix} b_k \\ d_k \end{pmatrix} \begin{pmatrix} a_{k+1} & b_{k+1} \end{pmatrix},\\
R_k^* &= (t_k - t_{k+1})\, I_2 - R_k,\\
E_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}
\tag{2.39}
$$

For $k = n+2$,

$$\begin{aligned}
T_{n+2}(A_{n+2}) &= R_{n+2}A_{n+2}E_1 + E_2A_{n+2}R^*_{n+2} + E_2R^*_{n+2} \\
&\qquad - \sum_{j=1}^{n+1} \frac{R_{n+2}A_jR^*_{n+2}}{t_j - 1}, \\
T_{n+2}(A_j) &= (t_j - 1)\, E_2A_jE_1 + R_{n+2}A_jE_1 + E_2A_jR^*_{n+2} \\
&\qquad + \frac{R_{n+2}A_jR^*_{n+2}}{t_j - 1} \qquad (j \neq n+2), \\
T_{n+2}(G_{n+2}) &= R_{n+2}G_{n+2} + E_2G_{n+2}E_2 \\
&\qquad + \sum_{j=1,\, j\neq k}^{n+2} \frac{R_{n+2}G_{n+2}E_1G^{-1}_{n+2}A_jG_{n+2}E_2}{(1+\theta_{n+2})(t_{n+2} - t_j)}, \\
T_{n+2}(G_j) &= \{(t_j - 1)\, E_2 + R_{n+2}\}\, G_j \qquad (j \neq n+2),
\end{aligned} \tag{2.40}$$

where

$$\begin{aligned}
R_{n+2} &= \frac{1}{(1-\theta_{n+3})\, d_{n+2}} \begin{pmatrix} 1\theta_{n+3} \\ \mathcal{X}^* \end{pmatrix} \begin{pmatrix} d_{n+2} & -b_{n+2} \end{pmatrix}, \\
R^*_{n+2} &= \frac{1}{(1-\theta_{n+3})\, d_{n+2}} \begin{pmatrix} b_{n+2} \\ d_{n+2} \end{pmatrix} \begin{pmatrix} -\mathcal{X}^* & 1-\theta_{n+3} \end{pmatrix}, \\
\mathcal{X}^* &= \sum_{j=1}^{n+2} t_j\, c_j.
\end{aligned} \tag{2.41}$$

For $k = n+3$,

$$\begin{aligned}
T_{n+3}(A_{n+2}) &= R_{n+3}A_{n+2}E_2 + E_1A_{n+2}R^*_{n+3} + E_1R^*_{n+3} \\
&\qquad - \sum_{j=1}^{n+1} \frac{R_{n+3}A_jR^*_{n+3}}{t_j - 1}, \\
T_{n+3}(A_j) &= (t_j - 1)\, E_1A_jE_2 + R_{n+3}A_jE_2 + E_1A_jR^*_{n+3} \\
&\qquad + \frac{R_{n+3}A_jR^*_{n+3}}{t_j - 1} \qquad (j \neq n+2), \\
T_{n+3}(G_{n+2}) &= R_{n+3}G_{n+2} + E_1G_{n+2}E_2 \\
&\qquad + \sum_{j=1,\, j\neq k}^{n+2} \frac{R_{n+3}G_{n+2}E_1G^{-1}_{n+2}A_jG_{n+2}E_2}{(1+\theta_{n+2})(t_{n+2} - t_j)}, \\
T_{n+3}(G_j) &= \{(t_j - 1)\, E_1 + R_{n+3}\}\, G_j \qquad (j \neq n+2),
\end{aligned} \tag{2.42}$$

where

$$
\begin{aligned}
R_{n+3} &= \frac{1}{(1+\theta_{n+3})\, b_{n+2}} \begin{pmatrix} \mathcal{X} \\ 1+\theta_{n+3} \end{pmatrix} \begin{pmatrix} -d_{n+2} & b_{n+2} \end{pmatrix}, \\
R^*_{n+3} &= \frac{1}{(1+\theta_{n+3})\, b_{n+2}} \begin{pmatrix} b_{n+2} \\ d_{n+2} \end{pmatrix} \begin{pmatrix} 1+\theta_{n+3} & -\mathcal{X} \end{pmatrix}, \\
\mathcal{X} &= \sum_{j=1}^{n+2} t_j\, b_j.
\end{aligned}
\tag{2.43}
$$

The independent variables t_i $(i = 1, \ldots, n)$ are invariant under the action of each T_k.

Remark 2.3 *The group of the Schlesinger transformations generated by* T_k $(k = 1, \ldots, n+3)$ *is isomorphic to the root lattice* $Q(C_{n+3})$ *of type* C_{n+3}*. The commutativity between two arbitrary Schlesinger transformations is obtained from the uniqueness of the Schlesinger transformations* ([3]).

3 τ-Functions on the root lattice

In this section, we formulate the τ-functions for the Schlesinger system on the root lattice $Q(C_{n+3})$ and show that they satisfy Toda equations, Hirota-Miwa equations and bilinear equations.

For each solution of the Schlesinger system, we introduce the τ-functions τ_μ ($\mu \in \mathbb{Z}^{n+3}$, $\mu_1 + \ldots + \mu_{n+3} \in 2\,\mathbb{Z}$) satisfying the Pfaffian systems

$$d \log \tau_\mu = \sum_{i=1}^{n} T_\mu(H_i)\, dt_i, \tag{3.44}$$

where

$$H_i = \sum_{j=1, j\neq i}^{n+2} \frac{1}{t_i - t_j} \left(\mathrm{tr} A_i A_j + B_{ij}\right) \quad (i = 1, \ldots, n) \tag{3.45}$$

are the Hamiltonians and

$$B_{ij} = -\frac{1}{2}\,\theta_i\theta_j + \frac{\theta_i^2 + \theta_j^2}{2\,(n+1)} - \frac{\sum_{i=1}^{n+3} \theta_i^2}{2\,(n+1)(n+2)}. \tag{3.46}$$

We also define the action of the transformations σ_k, r_l and T_μ on τ_0, such is consistent with the action of those on the Hamiltonians H_i ($i = 1, \ldots, n$). For each $\mu \in \mathbb{Z}^{n+3}$ with $\mu_1 + \ldots + \mu_{n+3} \in 2\,\mathbb{Z}$, the action of T_μ on τ_0 is defined by

$$T_\mu(\tau_0) = \tau_\mu \tag{3.47}$$

and the action of σ_k, r_l on τ_ν is defined by

$$\begin{aligned} \sigma_k(\tau_\nu) &= \tau_{\sigma_k(\nu)} \quad (k = 1, \ldots, n+2), \\ r_l(\tau_\nu) &= \tau_{r_l(\nu)} \quad (l = 1, \ldots, n+3), \end{aligned} \tag{3.48}$$

where

$$\begin{aligned} \sigma_k(\nu) &= (\nu_{\sigma_k(1)}, \ldots, \nu_{\sigma_k(n+3)}), \\ r_l(\nu) &= (\nu_1, \ldots, \nu_{l-1}, -\nu_l, \nu_{l+1}, \ldots, \nu_{n+3}). \end{aligned} \tag{3.49}$$

In the next subsection, we describe the action of the transformations σ_k, r_l and T_μ on the Hamiltonians, which is obtained from the action of those on the independent and dependent variables.

3.1 Symmetries for Hamiltonians

First, for each $\mu \in \mathbb{Z}^{n+3}$ with

$$\mu_1 + \ldots + \mu_{n+3} \in 2\,\mathbb{Z}, \quad \mu_1^2 + \ldots + \mu_{n+3}^2 = 2, \tag{3.50}$$

we describe the action of the Schlesinger transformation T_μ on the Hamiltonians H_i ($i = 1, \dots, n$). For each $k, l = 1, \dots, n+3$ with $k \neq l$, we denote the Schlesinger transformations satisfying the condition (3.50) by

$$T_{k,l} = T_{\mathbf{e}_k+\mathbf{e}_l}, \quad T_{k,-l} = T_{\mathbf{e}_k-\mathbf{e}_l}, \quad T_{-k,-l} = T_{-\mathbf{e}_k-\mathbf{e}_l}, \tag{3.51}$$

where

$$\begin{aligned} \mathbf{e}_1 &= (1,0,0,\dots,0,0), \\ \mathbf{e}_2 &= (0,1,0,\dots,0,0), \\ &\vdots \\ \mathbf{e}_{n+3} &= (0,0,0,\dots,0,1). \end{aligned} \tag{3.52}$$

We remark

$$T_k = T_{k,-(k+1)} \quad (k = 1, \dots, n+2), \quad T_{n+3} = T_{n+2,n+3}, \tag{3.53}$$

and

$$\begin{aligned} &T_{k,l}(\theta_k) = \theta_k + 1, && T_{k,l}(\theta_l) = \theta_l + 1, && T_{k,l}(\theta_j) = \theta_j && (j \neq k,l), \\ &T_{k,-l}(\theta_k) = \theta_k + 1, && T_{k,-l}(\theta_l) = \theta_l - 1, && T_{k,-l}(\theta_j) = \theta_j && (j \neq k,l), \\ &T_{-k,-l}(\theta_k) = \theta_k - 1, && T_{-k,-l}(\theta_l) = \theta_l - 1, && T_{-k,-l}(\theta_j) = \theta_j && (j \neq k,l). \end{aligned} \tag{3.54}$$

Then, for each $k, l = 1, \dots, n+2$, the action of the transformation $T_{k,l}$ on the Hamiltonians H_i ($i = 1, \dots, n$) is described as follows:

$$\begin{aligned} T_{k,l}(H_i) &= H_i - \frac{\mathrm{tr} A_i R_{k,l}}{(t_i - t_k)(t_i - t_l)} + \frac{\Delta_k^i}{t_i - t_k} + \frac{\Delta_l^{-i}}{t_i - t_l} \\ &\quad + \sum_{j=1, j\neq i}^{n+2} \frac{\Delta_{k,l}}{t_i - t_j} \qquad (i \neq k,l), \\ T_{k,l}(H_k) &= H_k - \sum_{j=1, j\neq k,l}^{n+2} \frac{\mathrm{tr} A_i R_{k,l}}{(t_k - t_j)(t_k - t_l)} - \frac{(n-1)(1+\theta_k+\theta_l)}{2\,(n+1)(t_k - t_l)} \\ &\quad + \sum_{j=1, j\neq k,l}^{n+2} \frac{\Delta_k^j}{t_k - t_j} + \sum_{j=1, j\neq k}^{n+2} \frac{\Delta_{k,l}}{t_k - t_j}, \\ T_{k,l}(H_l) &= H_l - \sum_{j=1, j\neq k,l}^{n+2} \frac{\mathrm{tr} A_i R_{k,l}}{(t_l - t_j)(t_l - t_k)} - \frac{(n-1)(1+\theta_k+\theta_l)}{2\,(n+1)(t_l - t_k)} \\ &\quad + \sum_{j=1, j\neq k,l}^{n+2} \frac{\Delta_k^{-j}}{t_l - t_j} + \sum_{j=1, j\neq l}^{n+2} \frac{\Delta_{k,l}}{t_l - t_j}, \end{aligned} \tag{3.55}$$

where

$$
\begin{aligned}
&R_{k,l} = \frac{t_k - t_l}{b_k d_l - d_k b_l} \begin{pmatrix} b_k \\ d_k \end{pmatrix} \begin{pmatrix} -d_l & b_l \end{pmatrix}, \\
&\Delta_k^j = -\frac{\theta_j}{2} + \frac{1 + 2\,\theta_k}{2\,(n+2)}, \qquad \Delta_k^{-j} = \frac{\theta_j}{2} + \frac{1 + 2\,\theta_k}{2\,(n+2)}, \\
&\Delta_{k,l} = -\frac{1 + \theta_k + \theta_l}{(n+1)(n+2)}.
\end{aligned} \tag{3.56}
$$

For each $k, l = 1, \ldots, n+2$, the action of the transformation $T_{k,-l}$ on the Hamiltonians H_i $(i = 1, \ldots, n)$ is described as follows:

$$
\begin{aligned}
T_{k,-l}(H_i) &= H_i - \frac{\mathrm{tr} A_i R_{k,-l}}{(t_i - t_k)(t_i - t_l)} + \frac{\Delta_k^i}{t_i - t_k} + \frac{\Delta_{-l}^{-i}}{t_i - t_l} \\
&\quad + \sum_{j=1, j \neq i}^{n+2} \frac{\Delta_{k,-l}}{t_i - t_j} \qquad (i \neq k, l), \\
T_{k,-l}(H_k) &= H_k - \sum_{j=1, j \neq k,l}^{n+2} \frac{\mathrm{tr} A_i R_{k,-l}}{(t_k - t_j)(t_k - t_l)} - \frac{(n-1)(1 + \theta_k - \theta_l)}{2\,(n+1)(t_k - t_l)} \\
&\quad + \sum_{j=1, j \neq k,l}^{n+2} \frac{\Delta_k^j}{t_k - t_j} + \sum_{j=1, j \neq k}^{n+2} \frac{\Delta_{k,-l}}{t_k - t_j}, \\
T_{k,-l}(H_l) &= H_l - \sum_{j=1, j \neq k,l}^{n+2} \frac{\mathrm{tr} A_i R_{k,-l}}{(t_l - t_j)(t_l - t_k)} - \frac{(n-1)(1 + \theta_k - \theta_l)}{2\,(n+1)(t_l - t_k)} \\
&\quad + \sum_{j=1, j \neq k,l}^{n+2} \frac{\Delta_{-l}^{-j}}{t_l - t_j} + \sum_{j=1, j \neq l}^{n+2} \frac{\Delta_{k,-l}}{t_l - t_j},
\end{aligned} \tag{3.57}
$$

where

$$
\begin{aligned}
&R_{k,-l} = -\frac{t_k - t_l}{b_k a_l + d_k b_l} \begin{pmatrix} b_k \\ d_k \end{pmatrix} \begin{pmatrix} a_l & b_l \end{pmatrix}, \\
&\Delta_{k,-l} = -\frac{1 + \theta_k - \theta_l}{(n+1)(n+2)}, \qquad \Delta_{-k}^{-j} = \frac{\theta_j}{2} + \frac{1 - 2\,\theta_k}{2\,(n+2)}.
\end{aligned} \tag{3.58}
$$

For each $k, l = 1, \dots, n+2$, the action of the transformation $T_{-k,-l}$ on the Hamiltonians H_i $(i = 1, \dots, n)$ is described as follows:

$$\begin{aligned}
T_{-k,-l}(H_i) &= H_i - \frac{\mathrm{tr} A_i R_{-k,-l}}{(t_i - t_k)(t_i - t_l)} + \frac{\Delta^i_{-k}}{t_i - t_k} + \frac{\Delta^{-i}_{-l}}{t_i - t_l} \\
&\quad + \sum_{j=1, j\neq i}^{n+2} \frac{\Delta_{-k,-l}}{t_i - t_j} \qquad (i \neq k, l), \\
T_{-k,-l}(H_k) &= H_k - \sum_{j=1, j\neq k,l}^{n+2} \frac{\mathrm{tr} A_i R_{-k,-l}}{(t_k - t_j)(t_k - t_l)} - \frac{(n-1)(1-\theta_k-\theta_l)}{2\,(n+1)(t_k - t_l)} \\
&\quad + \sum_{j=1, j\neq k,l}^{n+2} \frac{\Delta^j_{-k}}{t_k - t_j} + \sum_{j=1, j\neq k}^{n+2} \frac{\Delta_{-k,-l}}{t_k - t_j}, \\
T_{-k,-l}(H_l) &= H_l - \sum_{j=1, j\neq k,l}^{n+2} \frac{\mathrm{tr} A_i R_{-k,-l}}{(t_l - t_j)(t_l - t_k)} - \frac{(n-1)(1-\theta_k-\theta_l)}{2\,(n+1)(t_l - t_k)} \\
&\quad + \sum_{j=1, j\neq k,l}^{n+2} \frac{\Delta^{-j}_{-l}}{t_l - t_j} + \sum_{j=1, j\neq l}^{n+2} \frac{\Delta_{-k,-l}}{t_l - t_j},
\end{aligned} \tag{3.59}$$

where

$$\begin{aligned}
R_{-k,-l} &= \frac{t_k - t_l}{a_k b_l - b_k a_l} \begin{pmatrix} b_k \\ -a_k \end{pmatrix} \begin{pmatrix} a_l & b_l \end{pmatrix}, \\
\Delta_{-k,-l} &= -\frac{1-\theta_k-\theta_l}{(n+1)(n+2)}, \qquad \Delta^j_{-k} = -\frac{\theta_j}{2} + \frac{1-2\,\theta_k}{2\,(n+2)}.
\end{aligned} \tag{3.60}$$

For each $k = 1, \dots, n+2$, the action of the transformation $T_{k,n+3}$ on the Hamiltonians H_i $(i = 1, \dots, n)$ is described as follows:

$$\begin{aligned}
T_{k,n+3}(H_i) &= H_i + \frac{1}{t_i - t_k} \left(a_i + b_i \frac{d_k}{b_k} \right) \\
&\quad + \frac{\Delta^i_k}{t_i - 1} + \sum_{j=1, j\neq i}^{n+2} \frac{\Delta_{k,n+3}}{t_i - t_j} \qquad (i \neq k), \\
T_{k,n+3}(H_k) &= H_k + \sum_{j=1, j\neq k}^{n+2} \frac{1}{t_k - t_j} \left(a_j + b_j \frac{d_k}{b_k} \right) \\
&\quad + \sum_{j=1, j\neq k}^{n+2} \frac{\Delta^j_k + \Delta_{k,n+3}}{t_k - t_j}.
\end{aligned} \tag{3.61}$$

For each $k = 1, \dots, n+2$, the action of transformation $T_{k,-(n+3)}$ on the Hamiltonians H_i $(i = 1, \dots, n)$ is described as follows:

$$\begin{aligned}
T_{k,-(n+3)}(H_i) &= H_i + \frac{1}{t_i - t_k}\left(d_i + c_i\,\frac{a_k}{c_k}\right) \\
&\quad + \frac{\Delta_k^i}{t_i - 1} + \sum_{j=1, j\neq i}^{n+2} \frac{\Delta_{k,-(n+3)}}{t_i - t_j} \qquad (i \neq k), \\
T_{k,-(n+3)}(H_k) &= H_k + \sum_{j=1, j\neq k}^{n+2} \frac{1}{t_k - t_j}\left(d_j + c_j\,\frac{a_k}{c_k}\right) \\
&\quad + \sum_{j=1, j\neq k}^{n+2} \frac{\Delta_k^j + \Delta_{k,-(n+3)}}{t_k - t_j},
\end{aligned} \tag{3.62}$$

For each $k = 1, \dots, n+2$, the action of the transformation $T_{n+3,-k}$ on the Hamiltonians H_i $(i = 1, \dots, n)$ is described as follows:

$$\begin{aligned}
T_{n+3,-k}(H_i) &= H_i + \frac{1}{t_i - t_k}\left(a_i - b_i\,\frac{a_k}{b_k}\right) \\
&\quad + \frac{\Delta_{-k}^i}{t_i - 1} + \sum_{j=1, j\neq i}^{n+2} \frac{\Delta_{n+3,-k}}{t_i - t_j} \qquad (i \neq k), \\
T_{n+3,-k}(H_k) &= H_k + \sum_{j=1, j\neq k}^{n+2} \frac{1}{t_k - t_j}\left(a_j - b_j\,\frac{a_k}{b_k}\right) \\
&\quad + \sum_{j=1, j\neq k}^{n+2} \frac{\Delta_{-k}^j + \Delta_{n+3,-k}}{t_k - t_j},
\end{aligned} \tag{3.63}$$

For each $k = 1, \dots, n+2$, the action of the transformation $T_{-k,-(n+3)}$ on the Hamiltonians H_i $(i = 1, \dots, n)$ is described as follows:

$$\begin{aligned}
T_{-k,-(n+3)}(H_i) &= H_i + \frac{1}{t_i - t_k}\left(d_i - c_i\,\frac{d_k}{c_k}\right) \\
&\quad + \frac{\Delta_{-k}^i}{t_i - 1} + \sum_{j=1, j\neq i}^{n+2} \frac{\Delta_{-k,-(n+3)}}{t_i - t_j} \qquad (i \neq k), \\
T_{-k,-(n+3)}(H_k) &= H_k + \sum_{j=1, j\neq k}^{n+2} \frac{1}{t_k - t_j}\left(d_j - c_j\,\frac{d_k}{c_k}\right) \\
&\quad + \sum_{j=1, j\neq k}^{n+2} \frac{\Delta_{-k}^j + \Delta_{-k,-(n+3)}}{t_k - t_j},
\end{aligned} \tag{3.64}$$

For the other $\mu \in \mathbb{Z}^{n+3}$ with $\mu_1 + \ldots + \mu_{n+3} \in 2\,\mathbb{Z}$, the action of T_μ on the Hamiltonians, which is not described in this paper, is similarly obtained from the action of T_μ on the dependent variables.

Next, we describe the action of the transformations σ_k $(k = 1, \ldots, n+2)$ and r_l $(l = 1, \ldots, n+3)$ on the Hamiltonians. Since the Hamiltonians H_i $(i = 1, \ldots, n)$ are invariant under the action of each σ_k and r_l, for $k', l' = 1, \ldots, n+3$ with $k' \neq l'$, we obtain

$$\begin{aligned} \sigma_k T_{k',l'}(H_i) &= T_{\sigma_k(k'),\sigma_k(l')}(H_i), \\ \sigma_k T_{k',-l'}(H_i) &= T_{\sigma_k(lk'),-\sigma_k(l')}(H_i), \\ \sigma_k T_{-k',-l'}(H_i) &= T_{-\sigma_k(k'),-\sigma_k(l')}(H_i) \end{aligned} \tag{3.65}$$

and

$$\begin{aligned} r_l T_{k',l'}(H_i) &= T_{k',l'}(H_i), & r_l T_{k',l}(H_i) &= T_{k',-l}(H_i), \\ r_l T_{k',-l'}(H_i) &= T_{k',-l'}(H_i), & r_l T_{k',-l}(H_i) &= T_{k',l}(H_i), \\ r_l T_{-k',-l'}(H_i) &= T_{-k',-l'}(H_i), & r_l T_{-k',-l}(H_i) &= T_{l,-k'}(H_i). \end{aligned} \tag{3.66}$$

Such action of σ_k and r_l is naturally extended to each $\mu \in \mathbb{Z}^{n+3}$ with $\mu_1 + \ldots + \mu_{n+3} \in 2\,\mathbb{Z}$ as follows:

$$\sigma_k T_\mu(H_i) = T_{\sigma_k(\mu)}(H_i), \quad r_l T_\mu(H_i) = T_{r_l(\mu)}(H_i), \tag{3.67}$$

where

$$\begin{aligned} \sigma_k(\mu) &= (\mu_{\sigma_k(1)}, \ldots, \mu_{\sigma_k(n+3)}), \\ r_l(\mu) &= (\mu_1, \ldots, \mu_{l-1}, -\mu_l, \mu_{l+1}, \ldots, \mu_{n+3}). \end{aligned} \tag{3.68}$$

We remark that the action of σ_k and r_l on the Hamiltonians $T_\mu(H_i)$ is determined without having ambiguity.

3.2 Toda equations

In this subsection, we present the Toda equations for the Schlesinger transformations T_k $(k = 1, \ldots, n+3)$. We consider the Hamiltonians

$$\widetilde{H}_i = H_i - \sum_{j=1, j\neq i}^{n+2} \frac{B_{ij}}{t_i - t_j} = \sum_{j=1, j\neq i}^{n+2} \frac{\mathrm{tr} A_i A_j}{t_i - t_j} \quad (i = 1, \ldots, n). \tag{3.69}$$

Then from the theorem about τ-quotient given in [3], we obtain the following lemma.

Lemma 3.1 *The Hamiltonian $\widetilde{H}_i$ $(i = 1, \dots, n)$ satisfy the following equations:*

$$T_k(\widetilde{H}_i) + T_k^{-1}(\widetilde{H}_i) - 2\,\widetilde{H}_i = \partial_{t_i} \log \frac{(G_{k+1}^{-1} G_k)_{22} (G_k^{-1} G_{k+1})_{22}}{(t_k - t_{k+1})^2} \qquad (k = 1, \dots, n+1), \tag{3.70}$$
$$T_{n+2}(\widetilde{H}_i) + T_{n+2}^{-1}(\widetilde{H}_i) - 2\,\widetilde{H}_i = \partial_{t_i} \log (G_{n+2})_{22} (G_{n+2}^{-1})_{22},$$
$$T_{n+3}(\widetilde{H}_i) + T_{n+3}^{-1}(\widetilde{H}_i) - 2\,\widetilde{H}_i = \partial_{t_i} \log (G_{n+2})_{12} (G_{n+2}^{-1})_{21},$$

where $(G_j)_{kl}$ stands for the (k, l)-component of the 2×2-matrix G_j.

We remark that the action of T_k $(k = 1, \dots, n+3)$ on $\widetilde{H}_i$ $(i = 1, \dots, n)$ is obtained from the action of those on the dependent variables. Furthermore, we obtain the following lemma.

Lemma 3.2 *The Hamiltonians $\widetilde{H}_i$ $(i = 1, \dots, n)$ satisfy the following equations:*

$$\partial_{t_k}(\widetilde{H}_{k+1}) = \frac{\mathrm{tr} A_k A_{k+1}}{(t_k - t_{k+1})^2} \qquad (k = 1, \dots, n-1),$$
$$\partial_{t_n}\left(\sum_{i=1}^{n} (t_i - 1) \widetilde{H}_i \right) = \frac{\mathrm{tr} A_n A_{n+1}}{t_n^2},$$
$$(X_1 + 1)\left(\sum_{i=1}^{n} t_i \widetilde{H}_i \right) = -\mathrm{tr} A_{n+1} A_{n+2} - \sum_{i=1}^{n+2} \sum_{j=1, j \neq i}^{n+2} \frac{B_{ij}}{2},$$
$$(X_2 + 1)\left(\sum_{i=1}^{n} t_i \widetilde{H}_i \right) = \theta_{n+3}\, d_{n+2} + \theta_{n+2}\, (\rho + \theta_{n+2}) - \sum_{i=1}^{n+2} \sum_{j=1, j \neq i}^{n+2} \frac{B_{ij}}{2}, \tag{3.71}$$

where

$$X_1 = \sum_{i=1}^{n} (t_i - 1)\, \partial_{t_i}, \quad X_2 = \sum_{i=1}^{n} t_i (t_i - 1)\, \partial_{t_i}, \quad \rho = -\frac{1}{2} \sum_{j=1}^{n+3} \theta_j. \tag{3.72}$$

Proof The first equation of (3.71) is obtained by a direct computation. The second equation of (3.71) is obtained by using

$$\sum_{i=1}^{n} (t_i - 1) \widetilde{H}_i = -\sum_{j=1, j \neq n+1}^{n+2} \frac{\mathrm{tr} A_j A_{n+1}}{t_j} + \sum_{i=1}^{n+2} \sum_{j=1, j \neq i}^{n+2} \mathrm{tr} A_i A_j \tag{3.73}$$

and

$$\sum_{i=1}^{n+2}\sum_{j=1,j\neq i}^{n+2}\mathrm{tr}A_iA_j=-\sum_{i=1}^{n+2}\sum_{j=1,j\neq i}^{n+2}B_{ij}. \tag{3.74}$$

The third equation of (3.71) is obtained by using (3.74),

$$\sum_{i=1}^{n} t_i\widetilde{H}_i=\sum_{j=1}^{n+1}\frac{\mathrm{tr}A_jA_{n+2}}{t_j-1}+\sum_{i=1}^{n+2}\sum_{j=1,j\neq i}^{n+2}\frac{\mathrm{tr}A_iA_j}{2} \tag{3.75}$$

and

$$(X_1+1)\left(\sum_{j=1}^{n+1}\frac{\mathrm{tr}A_jA_{n+2}}{t_j-1}\right)=-\mathrm{tr}A_{n+1}A_{n+2}. \tag{3.76}$$

The fourth equation of (3.71) is obtained by using (3.74), (3.75) and

$$(X_2+1)\left(\sum_{j=1}^{n+1}\frac{\mathrm{tr}A_jA_{n+2}}{t_j-1}\right)=\theta_{n+3}\,d_{n+2}+\theta_{n+2}\,(\rho+\theta_{n+2}). \tag{3.77}$$

□

From Lemma 3.1, Lemma 3.2 and the following identities:

$$\begin{aligned}(G_{k+1}^{-1}G_k)_{22}(G_k^{-1}G_{k+1})_{22}&=-\frac{\mathrm{tr}A_kA_{k+1}}{\theta_k\theta_{k+1}}\quad(k=1,\dots,n+1),\\ (G_{n+2})_{22}(G_{n+2}^{-1})_{22}&=\frac{d_{n+2}}{\theta_{n+2}},\\ (G_{n+2})_{12}(G_{n+2}^{-1})_{21}&=\frac{a_{n+2}}{\theta_{n+2}},\end{aligned} \tag{3.78}$$

we obtain

$$T_k(\widetilde{H}_i)+T_k^{-1}(\widetilde{H}_i)-2\,\widetilde{H}_i=\partial_{t_i}\log\mathit{\Gamma}_k\quad(k=1,\dots,n+3), \tag{3.79}$$

where

$$
\begin{aligned}
\mathit{\Gamma}_k &= \partial_{t_k}(\widetilde{H}_{k+1}) \qquad (k=1,\ldots,n-1),\\
\mathit{\Gamma}_n &= \partial_{t_n}\left(\sum_{i=1}^{n}(t_i-1)\widetilde{H}_i\right),\\
\mathit{\Gamma}_{n+1} &= (X_1+1)\left(\sum_{i=1}^{n} t_i\widetilde{H}_i\right)+\sum_{i=1}^{n+2}\sum_{j=1,j\neq i}^{n+2}\frac{B_{ij}}{2},\\
\mathit{\Gamma}_{n+2} &= (X_2+1)\left(\sum_{i=1}^{n} t_i\widetilde{H}_i\right)+\sum_{i=1}^{n+2}\sum_{j=1,j\neq i}^{n+2}\frac{B_{ij}}{2}-\theta_{n+2}\,(\rho+\theta_{n+2}),\\
\mathit{\Gamma}_{n+3} &= (X_2+1)\left(\sum_{i=1}^{n} t_i\widetilde{H}_i\right)+\sum_{i=1}^{n+2}\sum_{j=1,j\neq i}^{n+2}\frac{B_{ij}}{2}-\theta_{n+2}\,(\rho+\theta_{n+2}+\theta_{n+3}),
\end{aligned}
\tag{3.80}
$$

Here we introduce the Hirota derivatives $\mathcal{D}_i$ $(i=1,\cdots,n)$ defined by

$$
P(\mathcal{D}_1,\cdots,\mathcal{D}_n)f\cdot g = P(\partial_{t_1},\cdots,\partial_{t_n})(f(s+t)g(s-t))\,\Big|_{t=0}, \tag{3.81}
$$

where $P(\mathcal{D}_1,\cdots,\mathcal{D}_n)$ is a polynopmial in the derivations $\mathcal{D}_i$ $(i=1,\ldots,n)$. By the definition, we obtain

$$
\begin{aligned}
\mathcal{D}_i\,\varphi\cdot\psi &= \partial_{t_i}(\varphi)\,\psi-\varphi\,\partial_{t_i}(\psi),\\
\mathcal{D}_i\mathcal{D}_j\,\varphi\cdot\psi &= \partial_{t_i}\partial_{t_j}(\varphi)\,\psi-\partial_{t_i}(\varphi)\,\partial_{t_j}(\psi)-\partial_{t_j}(\varphi)\,\partial_{t_i}(\psi)+\psi\,\partial_{t_i}\partial_{t_j}(\varphi)
\end{aligned}
\tag{3.82}
$$

and

$$
\begin{aligned}
\partial_{t_i}\log\frac{\varphi}{\psi} &= \frac{\mathcal{D}_i\,\varphi\cdot\psi}{\varphi\cdot\psi},\\
\partial_{t_i}\partial_{t_j}\log\varphi\,\psi &= \frac{\mathcal{D}_i\mathcal{D}_j\,\varphi\cdot\psi}{\varphi\cdot\psi}-\frac{\mathcal{D}_i\,\varphi\cdot\psi}{\varphi\cdot\psi}\frac{\mathcal{D}_j\,\varphi\cdot\psi}{\varphi\cdot\psi}.
\end{aligned}
\tag{3.83}
$$

Then, substituting (3.69) into (3.79), we obtain the Toda and Toda-like equations using those Hirota derivatives.

Theorem 3.3 *For the Schlesinger transformations* T_k $(k=1,\ldots,n+3)$, *the following Toda and Toda-like equations are satisfied:*

$$
\begin{aligned}
C_k\, T_k(\tau_0)\, T_k^{-1}(\tau_0) &= \mathcal{D}_k\mathcal{D}_{k+1}\,\tau_0\cdot\tau_0 - \frac{2\,B_{k,k+1}}{(t_k-t_{k+1})^2}\,\tau_0^2 \\
&\qquad\qquad\qquad\qquad (k=1,\cdots,n-1),\\
C_n\, T_n(\tau_0)\, T_n^{-1}(\tau_0) &= \sum_{i=1}^{n}(t_i-1)\,\mathcal{D}_i\mathcal{D}_n\,\tau_0\cdot\tau_0 + 2\,\partial_{t_n}(\tau_0)\,\tau_0 - \frac{2}{t_n^2}\,B_{n,n+1}\,\tau_0^2,\\
C_{n+1}\, T_{n+1}(\tau_0)\, T_{n+1}^{-1}(\tau_0) &= \sum_{i=1}^{n}\sum_{j=1}^{n}(t_i-1)\,t_j\,\mathcal{D}_i\mathcal{D}_j\,\tau_0\cdot\tau_0\\
&\quad + 2\sum_{i=1}^{n}(2\,t_i-1)\,\partial_{t_i}(\tau_0)\,\tau_0 + 2\,B_{n+1,n+2}\,\tau_0^2,\\
C_{n+2}\, T_{n+2}(\tau_0)\, T_{n+2}^{-1}(\tau_0) &= \sum_{i=1}^{n}\sum_{j=1}^{n}t_i(t_i-1)\,t_j\,\mathcal{D}_i\mathcal{D}_j\,\tau_0\cdot\tau_0 + 2\sum_{i=1}^{n}t_i^2\,\partial_{t_i}(\tau_0)\,\tau_0\\
&\quad + 2\left(\theta_{n+2}\,(\rho+\theta_{n+2}) + \sum_{j=1}^{n+1}B_{i,n+2}\right)\tau_0^2,\\
C_{n+3}\, T_{n+3}(\tau_0)\, T_{n+3}^{-1}(\tau_0) &= \sum_{i=1}^{n}\sum_{j=1}^{n}t_i(t_i-1)\,t_j\,\mathcal{D}_i\mathcal{D}_j\,\tau_0\cdot\tau_0 + 2\sum_{i=1}^{n}t_i^2\,\partial_{t_i}(\tau_0)\,\tau_0\\
&\quad + 2\left(\theta_{n+2}\,(\rho+\theta_{n+2}+\theta_{n+3}) + \sum_{j=1}^{n+1}B_{i,n+2}\right)\tau_0^2,
\end{aligned}
\tag{3.84}
$$

where

$$
\begin{aligned}
C_k &= \gamma_k\,(t_k-t_{k+1})^{-\frac{1}{2}}\prod_{j\neq k}^{n+2}(t_k-t_j)^{-\Delta_k^j}\prod_{j\neq k+1}^{n+2}(t_{k+1}-t_j)^{-\Delta_{-k+1}^{-j}}\\
&\quad\times\prod_{i=1}^{n+2}\prod_{j=1,j\neq i}^{n+2}(t_i-t_j)^{-\frac{1}{2}\Delta_{k,-(k+1)}}\qquad (k=1,\ldots,n+1),\\
C_{n+2} &= \gamma_{n+2}\prod_{j=1}^{n+1}(t_j-1)^{-\Delta_{n+2}^j}\prod_{i=1}^{n+2}\prod_{j=1,j\neq i}^{n+2}(t_i-t_j)^{-\frac{1}{2}\Delta_{n+2,-(n+3)}},\\
C_{n+3} &= \gamma_{n+2}\prod_{j=1}^{n+1}(t_j-1)^{-\Delta_{n+2}^j}\prod_{i=1}^{n+2}\prod_{j=1,j\neq i}^{n+2}(t_i-t_j)^{-\frac{1}{2}\Delta_{n+2,n+3}}
\end{aligned}
\tag{3.85}
$$

and γ_k $(k=1,\ldots,n+3)$ *are non-zero constants.*

We note that the Toda equation for T_{n+1} is equivalent to the equation given in [8].

Remark 3.4 *If non-zero constants* γ_k $(k=1,\dots,n+3)$ *do not contain the parameters* θ_j $(j=1,\dots,n+3)$*, the non-zero constants* γ_k $(k=1,\dots,n+3)$ *are determined with the ambiguity for at most finite number of*

$$e\left(\frac{\mathbb{Z}}{2\,(n+1)(n+2)}\right). \tag{3.86}$$

3.3 Hirota-Miwa equations

In the following, we denote

$$\tau_{k,l}=T_{k,l}(\tau_0),\quad \tau_{k,-l}=T_{k,-l}(\tau_0)\quad (k,l=1,\dots,n+3,\ k\neq l). \tag{3.87}$$

We first present the Hirota-Miwa equation for the following six τ-functions:

$$\tau_{n+2,n+3},\quad \tau_{n+1,n+2},\quad \tau_{n+2,-(n+1)},\quad \tau_{n+1,n+3},\quad \tau_{n+3,-(n+1)},\quad \tau_0. \tag{3.88}$$

The action of transformations $T_{n+1,n+2}$, $T_{n+3,-(n+1)}$ and $T_{n+2,n+3}$ on the Hamiltonians H_i $(i=1,\dots,n)$ is described as follows:

$$\begin{aligned}
T_{n+1,n+2}(H_i)&=H_i-\frac{\mathrm{tr}A_iR_{n+1,n+2}}{t_i(t_i-1)}+\frac{\Delta^i_{n+1}}{t_i}+\frac{\Delta^{-i}_{n+2}}{t_i-1}\\
&\quad+\sum_{j=1,j\neq i}^{n+2}\frac{\Delta_{n+1,n+2}}{t_i-t_j},\\
T_{n+3,-(n+1)}(H_i)&=H_i+\frac{1}{t_i}\left(d_i-b_i\,\frac{a_{n+1}}{b_{n+1}}\right)\\
&\quad+\frac{\Delta^i_{-(n+1)}}{t_i-1}+\sum_{j=1,j\neq i}^{n+2}\frac{\Delta_{n+3,-(n+1)}}{t_i-t_j},\\
T_{n+2,n+3}(H_i)&=H_i+\frac{1}{t_i-1}\left(a_i+b_i\,\frac{d_{n+2}}{b_{n+2}}\right)\\
&\quad+\frac{\Delta^i_{n+2}}{t_i-1}+\sum_{j=1,j\neq i}^{n+2}\frac{\Delta_{n+2,n+3}}{t_i-t_j}.
\end{aligned} \tag{3.89}$$

By using (3.89) and

$$\partial_{t_i}\log\tau_{k,l}=T_{k,l}(H_i)\quad (i=1,\dots,n,\ k,l=1,\dots,n+3,\ k\neq l), \tag{3.90}$$

we obtain

$$\frac{\tau_{n+1,n+2}\,\tau_{n+3,-(n+1)}}{\tau_0\,\tau_{n+2,n+3}} = \varphi_{n+1}^{n+2,n+3}\left(d_{n+1} - b_{n+1}\frac{d_{n+2}}{b_{n+2}}\right) \times \prod_{i=1}^{n} t_i^{\frac{1}{n+1}} \prod_{i=1}^{n+2}\prod_{j=1,j\neq i}^{n+2} (t_i - t_j)^{\frac{-1}{2\,(n+1)(n+2)}}, \tag{3.91}$$

where $\varphi_{n+1}^{n+2,n+3}$ is a non-zero constant. Assuming that $\varphi_{n+1}^{n+2,n+3}$ do not contain the parameters θ_j $(j = 1, \ldots, n+3)$, the Hirota-Miwa-equation

$$\tau_{n+1,n+2}\,\tau_{n+3,-(n+1)} - \tau_{n+1,n+3}\,\tau_{n+2,-(n+1)} = \varphi_{n+1}^{n+2,n+3}\,\theta_{n+1} \prod_{i=1}^{n} t_i^{\frac{1}{n+1}} \prod_{i=1}^{n+2}\prod_{j=1,j\neq i}^{n+2} (t_i - t_j)^{\frac{-1}{2\,(n+1)(n+2)}}\,\tau_0\,\tau_{n+2,n+3}. \tag{3.92}$$

is obtained by the action of the transformation r_{n+1} on the both sides of (3.91).

For the other indexies $i, j, k = 1, \ldots, n+3$ with i, j, k mutually distinct, Hirota-Miwa equations are obtained in a similar way.

Theorem 3.5 *For each $i, j, k = 1, \ldots, n+3$ with i, j, k mutually distinct, the following Hirota-Miwa equations are satisfied:*

$$F_k^{ij}\,\tau_0\,\tau_{i,j} = \tau_{i,k}\,\tau_{j,-k} - \tau_{j,k}\,\tau_{i,-k} \quad (i, j, k = 1, \ldots, n+3), \tag{3.93}$$

where

$$\begin{aligned}
F_k^{ij} &= \varphi_k^{ij}\,\theta_k \left(\frac{(t_i - t_j)}{(t_i - t_k)(t_j - t_k)}\right)^{\frac{1}{2}} \\
&\quad \times \prod_{l=1,l\neq k}^{n+2} (t_k - t_l)^{\frac{1}{n+1}} \prod_{l_1=1}^{n+2}\prod_{l_2=1,l_2\neq l_1}^{n+2} (t_{l_1} - t_{l_2})^{\frac{-1}{2\,(n+1)(n+2)}}, \\
F_j^{i,n+3} &= \varphi_j^{i,n+3}\,\theta_j \prod_{k=1,k\neq j}^{n+2} (t_j - t_k)^{\frac{1}{n+1}} \prod_{k=1}^{n+2}\prod_{l=1,l\neq k}^{n+2} (t_k - t_l)^{\frac{-1}{2\,(n+1)(n+2)}}, \\
F_{n+3}^{ij} &= \varphi_{n+3}^{ij}\,\theta_{n+3}\,(t_i - t_j)^{\frac{1}{2}} \prod_{k=1}^{n+2}\prod_{l=1,l\neq k}^{n+2} (t_k - t_l)^{\frac{-1}{2\,(n+1)(n+2)}}
\end{aligned} \tag{3.94}$$

and φ_k^{ij} is a non-zero constant which do not contain the parameters θ_j $(j = 1, \ldots, n+3)$.

Remark 3.6 *The non-zero constants* φ_k^{ij} $(i,j,k=1,\ldots,n+3,\ i,j,k$ *mutually distinct) are determined with the ambiguity for at most finite number of*

$$e\left(\frac{\mathbb{Z}}{2\,(n+1)(n+2)}\right). \tag{3.95}$$

3.4 Bilinear equations

In this subsection, we present the bilinear differential equations for the τ-functions τ_0 and $\tau_1=\tau_{n+1,n+2}$. We consider the polynomial functions in (a,b,c,d)

$$\widehat{H}_i=\sum_{j=1,j\neq i}^{n+2}\frac{t_i(t_i-1)}{t_i-t_j}\left(\mathrm{tr}A_iA_j-\frac{1}{2}\theta_i\,\theta_j\right)\quad(i=1,\ldots,n) \tag{3.96}$$

and

$$\widehat{H}_i^*=T_{n+1,n+2}(\widehat{H}_i)=\widehat{H}_i-\mathrm{tr}A_iR_{n+1,n+2}+\frac{\theta_i}{2}\quad(i=1,\ldots,n). \tag{3.97}$$

In the following, we denote $\widehat{R}=R_{n+1,n+2}$. For each $i=1,\ldots,n$, from

$$\partial_{t_i}(\widehat{R})=\frac{\widehat{R}A_i(\widehat{R}-I_2)}{t_i-1}-\frac{(\widehat{R}-I_2)A_i\widehat{R}}{t_i}, \tag{3.98}$$

it follows that

$$\begin{aligned}
\delta_i(\widehat{H}_i)&=\sum_{j=1,j\neq i}^{n+2}\frac{t_i(t_i-1)(t_i^2-2\,t_it_j+t_j)}{(t_i-t_j)^2}\left(\mathrm{tr}A_iA_j-\frac{1}{2}\theta_i\,\theta_j\right),\\
\delta_j(\widehat{H}_i)&=\frac{t_i(t_i-1)\,t_j(t_j-1)}{(t_i-t_j)^2}\left(\mathrm{tr}A_iA_j-\frac{1}{2}\theta_i\,\theta_j\right)\\
&\qquad\qquad(j=1,\ldots,n,\ j\neq i),\\
\delta_i(\widehat{H}_i-\widehat{H}_i^*)&=\mathrm{tr}A_i(\widehat{R}-I_2)\,A_i\widehat{R}-\sum_{j=1,j\neq i}^{n+2}\frac{t_i(t_i-1)}{t_i-t_j}\,\mathrm{tr}\,[A_i,A_j]\widehat{R},\\
\delta_j(\widehat{H}_i-\widehat{H}_i^*)&=t_j\,\mathrm{tr}A_i\widehat{R}A_j(\widehat{R}-I_2)-(t_j-1)\,\mathrm{tr}A_i(\widehat{R}-I_2)\,A_j\widehat{R}\\
&\quad-\frac{t_j(t_j-1)}{t_i-t_j}\,\mathrm{tr}\,[A_i,A_j]\widehat{R}\qquad(j=1,\ldots,n,\ j\neq i).
\end{aligned} \tag{3.99}$$

By using (3.99), we obtain

$$
\begin{aligned}
&\sum_{j=1}^{n} \frac{2}{2\,t_i t_j - t_i - t_j} \left\{ \delta_j (\widehat{H}_i + \widehat{H}_i^*) + (\widehat{H}_i - \widehat{H}_i^*)(\widehat{H}_j - \widehat{H}_j^*) \right\} \\
&\quad = -\frac{\mathrm{tr} A_i (\widehat{R} - I_2)\, A_i \widehat{R}}{t_i(t_i-1)} + \frac{1}{t_i(t_i-1)} \left(\mathrm{tr} A_i \widehat{R} - \frac{\theta_i}{2} \right)^2 \\
&\qquad + \sum_{j=1, j\neq i}^{n} \frac{2}{2\,t_i t_j - t_i - t_j} \left\{ \left(\mathrm{tr} A_i \widehat{R} - \frac{\theta_i}{2} \right) \left(\mathrm{tr} A_j \widehat{R} - \frac{\theta_j}{2} \right) \right. \\
&\qquad\qquad + (t_j - 1)\, \mathrm{tr} A_i (\widehat{R} - I_2)\, A_j \widehat{R} \\
&\qquad\qquad \left. - t_j\, \mathrm{tr} A_i \widehat{R} A_j (\widehat{R} - I_2) \right\} \\
&\qquad + \sum_{j=1, j\neq i}^{n+2} \frac{1}{2\,t_i t_j - t_i - t_j} \left\{ (2\,t_j - 1)\, \mathrm{tr}\, [A_i, A_j] \widehat{R} - \mathrm{tr} A_i A_j + \frac{1}{2}\, \theta_i\, \theta_j \right\} \\
&\qquad + \sum_{j=1, j\neq i}^{n+2} \frac{2\,t_i - 1}{t_i - t_j} \left(\mathrm{tr} A_i A_j - \frac{1}{2}\, \theta_i\, \theta_j \right).
\end{aligned}
\tag{3.100}
$$

On the other hand, for each $i = 1, \ldots, n$, we obtain

$$
\begin{aligned}
\mathrm{tr} A_i (\widehat{R} - I_2)\, A_j \widehat{R} &= \left(\mathrm{tr} A_i \widehat{R} - \frac{\theta_i}{2} \right) \left(\mathrm{tr} A_j \widehat{R} - \frac{\theta_j}{2} \right) - \frac{1}{2}\, \mathrm{tr}\, [A_i, A_j] \widehat{R} \\
&\quad - \frac{1}{2}\, \mathrm{tr} A_i A_j + \frac{1}{4}\, \theta_i\, \theta_j \qquad (j = 1, \ldots, n,\ j \neq i), \\
\mathrm{tr} A_i \widehat{R}\, A_j (\widehat{R} - I_2) &= \left(\mathrm{tr} A_i \widehat{R} - \frac{\theta_i}{2} \right) \left(\mathrm{tr} A_j \widehat{R} - \frac{\theta_j}{2} \right) + \frac{1}{2}\, \mathrm{tr}\, [A_i, A_j] \widehat{R} \\
&\quad - \frac{1}{2}\, \mathrm{tr} A_i A_j + \frac{1}{4}\, \theta_i\, \theta_j \qquad (j = 1, \ldots, n,\ j \neq i), \\
\mathrm{tr} A_i (\widehat{R} - I_2)\, A_i \widehat{R} &= \left(\mathrm{tr} A_i \widehat{R} - \frac{\theta_i}{2} \right)^2 - \frac{\theta_i^2}{4\, t_i (t_i - 1)}
\end{aligned}
\tag{3.101}
$$

and

$$
\begin{aligned}
\mathrm{tr}[A_i, A_{n+1}] \widehat{R} + \mathrm{tr} A_i A_{n+1} &= \theta_{n+1}\, \mathrm{tr} A_i \widehat{R}, \\
\mathrm{tr}[A_i, A_{n+2}] \widehat{R} - \mathrm{tr} A_i A_{n+2} &= \theta_{n+2}\, \mathrm{tr} A_i (\widehat{R} - I_2)
\end{aligned}
\tag{3.102}
$$

by direct computations. From (3.100), (3.101) and (3.102), the following differential equations are obtained:

$$\sum_{j=1}^{n} \frac{2}{2\,t_i t_j - t_i - t_j} \left\{ \delta_j (\widehat{H}_i + \widehat{H}_i^*) + (\widehat{H}_i - \widehat{H}_i^*)(\widehat{H}_j - \widehat{H}_j^*) \right\}$$
$$= \left(\frac{\theta_{n+1}}{t_i} + \frac{\theta_{n+2}}{t_i - 1} \right) (\widehat{H}_i - \widehat{H}_i^*) + \frac{2\,t_i - 1}{t_i(t_i-1)}\, \widehat{H}_i + \frac{\theta_i^2}{4\,t_i(t_i-1)} \qquad (i = 1, \dots, n). \tag{3.103}$$

For each $i = 1, \dots, n$, substituting

$$\widehat{H}_i = \delta_i \log \tau_0 + \widehat{B}_i, \quad \widehat{H}_i^* = \delta_i \log \tau_1 + T_{n+1,n+2}(\widehat{B}_i), \tag{3.104}$$

where

$$\widehat{B}_i = \sum_{j=1, j\neq i}^{n+2} \frac{t_i(t_i-1)}{t_i - t_j} \left(\frac{\theta_i^2 + \theta_j^2}{2\,(n+1)} - \frac{\sum_{i=1}^{n+3} \theta_i^2}{2\,(n+1)(n+2)} \right), \tag{3.105}$$

into the equation (3.103), we obtain the bilinear differential equations for the τ-functions τ_0 and τ_1.

Theorem 3.7 *The τ-functions τ_0 and τ_1 satisfy the following bilinear equations.*

$$\sum_{j=1}^{n} \frac{2}{2\,t_i t_j - t_i - t_j} \left\{ \mathcal{D}_i^* \mathcal{D}_j^*\, \tau_0 \cdot \tau_1 + M_j^i\, \mathcal{D}_j^*\, \tau_0 \cdot \tau_1 \right\} + M^{i,0}\, \mathcal{D}_i^*\, \tau_0 \cdot \tau_1$$
$$- \frac{2\,t_i - 1}{t_i(t_i-1)}\, \delta_i\, \tau_0 \cdot \tau_1 + M^{i,1}\, \tau_0 \cdot \tau_1 = 0 \qquad (i = 1, \dots, n), \tag{3.106}$$

where

$$\begin{aligned}
M^i_j &= \frac{t_j(t_j-1)}{2\,(n+1)}\left(\frac{2\,\theta_{n+1}+1}{t_j}+\frac{2\,\theta_{n+2}+1}{t_j-1}\right)\\
&\quad+\frac{\theta_{n+1}+\theta_{n+2}+1}{(n+1)(n+2)}\sum_{k=1,k\neq j}^{n+2}\frac{t_j(t_j-1)}{t_j-t_k},\\
M^{i,0} &= \sum_{j=1}^{n}\frac{2\,M^i_j}{2\,t_it_j-t_i-t_j}-\frac{\theta_{n+1}}{t_i}-\frac{\theta_{n+2}}{t_i-1},\\
M^{i,1} &= \sum_{j=1}^{n}\frac{2\,M^i_iM^i_j}{2\,t_it_j-t_i-t_j}+\left(\frac{2\,\theta_{n+1}+1}{t_i}+\frac{2\,\theta_{n+2}+1}{t_i-1}\right)M^i_i \qquad (3.107)\\
&\quad+\sum_{j=1,j\neq i}^{n+2}\frac{2}{2\,t_it_j-t_i-t_j}\left(\frac{\theta_i^2+\theta_j^2}{2\,(n+1)}-\frac{\sum_{i=1}^{n+3}\theta_i^2}{2\,(n+1)(n+2)}\right)\\
&\quad+\frac{\theta_{n+1}+\theta_{n+2}+1}{(n+1)(n+2)}\sum_{j=1,j\neq i}^{n+2}\frac{2\,t_i(t_i-1)(2\,t_j-1)}{(t_i-t_j)(2\,t_it_j-t_i-t_j)}\\
&\quad-\frac{t_i(t_i-1)}{2\,(n+1)}\left(\frac{2\,\theta_{n+1}+1}{t_i^2}+\frac{2\,\theta_{n+2}+1}{(t_i-1)^2}\right)-\frac{\theta_i^2}{4\,t_i(t_i-1)}
\end{aligned}$$

and $\mathcal{D}^*_i$ *stands for the Hirota derivative with respect to the derivation* δ_i*.*

4 Garnier system

Following [1], we define the rational functions in a_j, b_j, c_j, d_j and t_i by

$$
\begin{aligned}
q_j &= t_j \frac{b_j}{\mathcal{X}} && (j = 1, \ldots, n), \\
p_j &= \frac{\mathcal{X}}{t_j} \left(\frac{a_j}{b_j} + (t_j - 1) \frac{a_{n+1}}{b_{n+1}} - t_j \frac{a_{n+2}}{b_{n+2}} \right) && (j = 1, \ldots, n), \\
x_i &= \frac{t_i}{t_i - 1} && (i = 1, \ldots, n),
\end{aligned}
\tag{4.108}
$$

where $\mathcal{X} = \sum_{j=1}^{n+2} t_j \, b_j$. Then we can describe the total differential equations for q_j and p_j $(j = 1, \ldots, n)$.

Proposition 4.1 *If* a_j, b_j, c_j, d_j $(j = 1, \ldots, n+2)$ *and* t_i $(i = 1, \ldots, n)$ *satisfy the Schlesinger system. Then* q_j, p_j, x_i $(i, j = 1, \ldots, n)$ *defined by* (4.108) *satisfy the Garnier system*

$$
dq_j = \sum_{i=1}^{n} \{K_i, q_j\} \, dx_i, \quad dp_j = \sum_{i=1}^{n} \{K_i, p_j\} \, dx_i, \tag{4.109}
$$

with Hamiltonians

$$
K_i = T_{n+3,-(n+1)}(H_i) \quad (i = 1, \ldots, n), \tag{4.110}
$$

where $\{\,,\,\}$ *stands for the Poisson bracket*

$$
\{\varphi, \psi\} = \sum_{j=1}^{n} \left(\frac{\partial \varphi}{\partial p_j} \frac{\partial \psi}{\partial q_j} - \frac{\partial \varphi}{\partial q_j} \frac{\partial \psi}{\partial p_j} \right). \tag{4.111}
$$

We remark that the Hamiltonians K_i $(i = 1, \ldots, n)$ are equivalent to (1.9) denoting the parameters by

$$
\kappa_0 = \theta_{n+1}, \quad \kappa_1 = \theta_{n+2}, \quad \kappa_\infty = \theta_{n+3} + 1. \tag{4.112}
$$

In this section, we show that the Garnier system has affine Weyl group symmetry of type $W(B_{n+3}^{(1)})$. We also show that the τ-functions for the Garnier system, formulated on the root lattice $Q(C_{n+3})$, satisfy the relations of three types, Toda equations, Hirota-Miwa equations and bilinear differential equations.

4.1 Affine Weyl group symmetries

The transformations σ_k, r_l and T_μ given in Section 2 can be lifted to the birational canonical transformations of variables q_j, p_j, x_i $(i, j = 1, \ldots, n)$, which is already known in [7], [8]. In this subsection, we formurate the action of those transformations as the realization of affine Weyl group.

Here we denote the parameter by

$$\begin{aligned} &\varepsilon_1 = \theta_{n+1}, \quad \varepsilon_2 = \theta_{n+2}, \quad \varepsilon_3 = \theta_{n+3} + 1, \\ &\varepsilon_j = \theta_{j-3} \qquad (j = 4, \ldots, n+3). \end{aligned} \tag{4.113}$$

Then the group of symmetries for the Garnier system is generated by the transformations s_k $(k = 0, \ldots, n+3)$, which act on the parameters ε_j $(j = 1, \ldots, n+3)$ as follows:

$$\begin{aligned} &s_0(\varepsilon_1) = 1 - \varepsilon_2, \qquad s_0(\varepsilon_2) = 1 - \varepsilon_1, \quad s_0(\varepsilon_j) = \varepsilon_j \quad (j \neq 1, 2), \\ &s_{n+3}(\varepsilon_{n+3}) = -\varepsilon_{n+3}, \quad s_{n+3}(\varepsilon_j) = \varepsilon_j \quad (j \neq n+3), \\ &s_k(\varepsilon_j) = \varepsilon_{\sigma_k(j)} \qquad (k \neq 0, n+3). \end{aligned} \tag{4.114}$$

We describe the action of the transformations s_k on the variables q_j, p_j, x_i $(i, j = 1, \ldots, n)$. For $k = 0$,

$$s_0(q_j) = \frac{p_j(q_j p_j - \varepsilon_{j+3})}{Q_1(Q_1 + \varepsilon_3)}, \quad s_0(q_j p_j) = \varepsilon_{j+3} - q_j p_j, \quad s_0(x_i) = \frac{1}{x_i}, \tag{4.115}$$

where

$$Q_1 = \sum_{l=1}^{n} q_l p_l + \frac{1}{2}\left(1 - \sum_{l=1}^{n+3} \varepsilon_l\right). \tag{4.116}$$

For $k = 1$,

$$s_1(q_j) = \frac{q_j}{x_j}, \quad s_1(p_j) = x_j\, p_j, \quad s_1(x_i) = \frac{1}{x_i}. \tag{4.117}$$

For $k = 2$,

$$s_2(q_j) = \frac{q_j}{Q_2}, \quad s_2(p_j) = (p_j - Q_1)\, Q_2, \quad s_2(x_i) = \frac{x_i}{x_i - 1}, \tag{4.118}$$

where

$$Q_2 = \sum_{j=1}^{n} q_j - 1. \tag{4.119}$$

For $k = 3$,

$$\begin{aligned} &s_3(q_1) = \frac{1}{q_1}, \qquad s_3(q_j) = -\frac{q_j}{q_1} \quad (j \neq 1), \\ &s_3(p_1) = -q_1 Q_1, \quad s_3(p_j) = -q_1 p_j \quad (j \neq 1), \\ &s_3(x_1) = \frac{1}{x_1}, \qquad s_n(x_i) = \frac{x_i}{x_1} \quad (i \neq 1), \end{aligned} \tag{4.120}$$

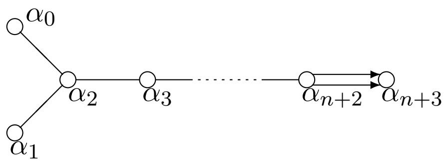

Figure 1: Dynkin diagram of type $B^{(1)}_{n+3}$

For $k = 4, \dots, n+2$,

$$s_k(q_j) = q_{\sigma_{k-3}(j)}, \quad p_k(q_j) = p_{\sigma_{k-3}(j)}, \quad s_k(x_j) = x_{\sigma_{k-3}(j)} \tag{4.121}$$

For $k = n+3$,

$$\begin{aligned} &s_{n+3}(p_n) = p_n - \frac{\varepsilon_{n+3}}{q_n}, \quad s_{n+3}(p_j) = p_j \quad (j \neq n), \\ &s_{n+3}(q_j) = q_j, \qquad\qquad s_{n+3}(x_i) = x_i. \end{aligned} \tag{4.122}$$

Then we obtain the following theorem.

Theorem 4.2 *The birational canonical transformations s_k ($k = 0, \dots, n+3$) satisfy the fundamental relations for the generators of $W(B^{(1)}_{n+3})$*

$$\begin{aligned} &s_k^2 = 1 && (k = 0, \dots, n+3), \\ &(s_k s_l)^2 = 1 && (k, l \neq 0, 1, 2,\ |k-l| > 1), \\ &(s_k s_{k+1})^3 = 1 && (k = 1, \dots, n+1), \\ &(s_0 s_1)^2 = 1, && (s_0 s_2)^3 = 1, \qquad (s_{n+2} s_{n+3})^4 = 1. \end{aligned} \tag{4.123}$$

The simple affine roots of $B^{(1)}_{n+3}$ is given by

$$\begin{aligned} &\alpha_0 = 1 - \varepsilon_1 - \varepsilon_2, \\ &\alpha_j = \varepsilon_j - \varepsilon_{j+1}, \qquad (j = 1, \dots, n+2) \\ &\alpha_{n+3} = \varepsilon_{n+3} \end{aligned} \tag{4.124}$$

and the action of the transformations s_k on α_j ($j = 0, 1, \dots, n+3$) is described as follows. For $k = 0$,

$$s_0(\alpha_0) = -\alpha_0, \quad s_0(\alpha_2) = \alpha_0 + \alpha_2, \quad s_0(\alpha_j) = \alpha_j \quad (j \neq 0, 2). \tag{4.125}$$

For $k = 1$,

$$s_1(\alpha_1) = -\alpha_1, \quad s_1(\alpha_2) = \alpha_1 + \alpha_2, \quad s_1(\alpha_j) = \alpha_j \quad (j \neq 0, 1). \tag{4.126}$$

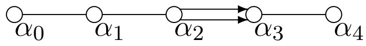

Figure 2: Dynkin diagram of type $F_4^{(1)}$

For $k=2$,

$$\begin{aligned} s_2(\alpha_2) &= -\alpha_2, \\ s_2(\alpha_j) &= \alpha_j + \alpha_2 \quad (j=0,1,3), \\ s_2(\alpha_j) &= \alpha_j \qquad\quad\ (j \neq 0,1,2,3). \end{aligned} \tag{4.127}$$

For $k=3,\dots,n+2$,

$$\begin{aligned} &s_k(\alpha_k) = -\alpha_k, \quad s_k(\alpha_{k+1}) = \alpha_{k+1} + \alpha_k, \quad s_k(\alpha_{k-1}) = \alpha_{k-1} + \alpha_k, \\ &s_k(\alpha_j) = \alpha_j \qquad (j \neq k, k+1, k-1). \end{aligned} \tag{4.128}$$

For $k=n+3$,

$$\begin{aligned} &s_{n+3}(\alpha_{n+3}) = -\alpha_{n+3}, \quad s_{n+3}(\alpha_{n+2}) = \alpha_{n+2} + 2\,\alpha_{n+3}, \\ &s_{n+3}(\alpha_j) = \alpha_j \qquad\qquad (j \neq n+2, n+3). \end{aligned} \tag{4.129}$$

We note that the group generated by the transformations s_k $(k=1,\dots,n+2)$ is isomorphic to the symmetric group $\mathcal{S}_{n+3}$ ([1]). We also remark that the group generated by s_k $(k=1,\dots,n+3)$ is isomorphic to affine Weyl group $W(B_{n+3})$. (In the case $n=1$, this is remarked in [5].)

Remark 4.3 *In the only case $n=1$ (P_{VI}), there is the following birational canonical transformation:*

$$\begin{aligned} &\widehat{s}_0(q) = q - \frac{\varepsilon_4}{p}, \quad \widehat{s}_0(p) = p, \quad \widehat{s}_0(t) = t, \\ &\widehat{s}_0(\varepsilon_j) = \varepsilon_j + \frac{1}{2}(1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \qquad (j=1,\dots,4). \end{aligned} \tag{4.130}$$

The transformation s_0 is generated by a composition of $\widehat{s}_0$ and $s_1,\dots,s_4$, but $\widehat{s}_0$ cannot be generated by a composition of s_0, $s_1,\dots,s_4$. It follows that the group of symmetries for the Garnier system in 1-variable contains affine Weyl group $W(B_4^{(1)})$. Actually, it is known that P_{VI} has affine Weyl group symmetry of type $W(F_4^{(1)})$. The simple affine roots of $F_4^{(1)}$ is given by

$$\begin{aligned} &\alpha_0 = \varepsilon_1 - \varepsilon_2, \quad \alpha_1 = \varepsilon_2 - \varepsilon_3, \quad \alpha_2 = \varepsilon_3 - \varepsilon_4, \\ &\alpha_3 = \varepsilon_4, \qquad\quad \alpha_4 = \frac{1}{2}(1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4), \end{aligned} \tag{4.131}$$

and $\widehat{s}_0,\ s_1,\ \ldots, s_4$ *act on* α_j $(j=0,1,\ldots,4)$ *as follows:*

$$
\begin{array}{llll}
\widehat{s}_0(\alpha_4)=-\alpha_4, & \widehat{s}_0(\alpha_3)=\alpha_3+\alpha_4, & \widehat{s}_0(\alpha_j)=\alpha_j & (j\neq 3,4),\\
s_1(\alpha_0)=-\alpha_0, & s_1(\alpha_1)=\alpha_1+\alpha_0, & s_1(\alpha_j)=\alpha_j & (j\neq 0,1),\\
s_2(\alpha_1)=-\alpha_1, & s_2(\alpha_i)=\alpha_i+\alpha_1, & s_2(\alpha_j)=\alpha_j & (i=0,2,\ j=3,4),\\
s_3(\alpha_2)=-\alpha_2, & s_3(\alpha_i)=\alpha_i+\alpha_2, & s_3(\alpha_j)=\alpha_j & (i=1,3,\ j=1,4),\\
s_4(\alpha_3)=-\alpha_3, & s_4(\alpha_2)=\alpha_2+2\,\alpha_3, & s_4(\alpha_4)=\alpha_4+\alpha_3, & \\
s_4(\alpha_j)=\alpha_j & (j=1,2). & &
\end{array}
\tag{4.132}
$$

4.2 τ-Functions

For each solution of the Garnier system, we introduce the τ-functions $\bar{\tau}_\mu$ $(\nu\in\mathbb{Z}^{n+3},\ \nu_1+\ldots+\nu_{n+3}\in 2\,\mathbb{Z})$ satisfying the Pfaffian systems

$$d\log\bar{\tau}_\mu=\sum_{i=1}^{n} T_\mu(K_i)\,dt_i. \tag{4.133}$$

From the definition of Hamiltonians K_i, we can identify $\bar{\tau}_0$ with the τ-function for the Schlesinger system as follows:

$$\bar{\tau}_0=\tau_{n+3,-(n+1)}. \tag{4.134}$$

Therefore we can apply the properties of τ-functions for the Schlesinger system to the Garnier system. For each $\mu\in\mathbb{Z}^{n+3}$ with $\mu_1+\ldots+\mu_{n+3}\in 2\,\mathbb{Z}$, the action of the birational canonical transformations s_k on $\bar{\tau}_\mu$ is defined by

$$s_k(\bar{\tau}_\mu)=\bar{\tau}_{s_k(\mu)},\quad (k=0,1,\ldots,n+3), \tag{4.135}$$

where

$$
\begin{aligned}
&s_0(\mu)=(1-\mu_2,1-\mu_1,\mu_3,\ldots,\mu_{n+3}),\\
&s_k(\mu)=(\mu_{(k,k+1)1},\ldots,\mu_{(k,k+1)(n+3)})\qquad (k\neq 0,n+3),\\
&s_{n+3}(\mu)=(\mu_1,\ldots,\mu_{n+2},-\mu_{n+3}),
\end{aligned}
\tag{4.136}
$$

and $(k,k+1)$ stands for the adjacent transpositions. Then we obtain the relations which is satisfied by $\bar{\tau}_\mu$ formurated on the root lattice $Q(C_{n+3})$.

Theorem 4.4 *The τ-functions $\bar{\tau}_\mu$ $(\nu\in\mathbb{Z}^{n+3},\ \nu_1+\ldots+\nu_{n+3}\in 2\,\mathbb{Z})$ can be formulated on the root lattice $Q(C_{n+3})$ and satisfy the Toda equations, the Hirota-Miwa equations and the bilinear differential equations given in Section 3.*

In the last, we present the following proposition.

Proposition 4.5 *For the τ-functions*

$$\bar{\tau}_{1,-2} = \bar{\tau}_{\mathbf{e}_1-\mathbf{e}_2}, \quad \bar{\tau}_{1,3} = \bar{\tau}_{\mathbf{e}_1+\mathbf{e}_3}, \quad \bar{\tau}_{1,-3} = \bar{\tau}_{\mathbf{e}_1-\mathbf{e}_3} \tag{4.137}$$

and $\bar{\tau}_0$, the following relations are satisfied:

$$\begin{aligned} q_j &= -\frac{1}{\varepsilon_3}\, x_i(x_i-1)\,\partial_{x_i}\log\frac{\bar{\tau}_{1,3}}{\bar{\tau}_{1,-3}} + 2\,\bar{S}_j \quad (j=1,\ldots,n), \\ q_jp_j &= -x_j\,\partial_{x_j}\log\frac{\bar{\tau}_{1,-2}}{\bar{\tau}_0} - \frac{x_j\,\bar{\Delta}^{j+3}_{-2}}{x_j-1} + \frac{\bar{\Delta}^{j+3}_{-1}}{x_j-1} - \frac{(\varepsilon_1-\varepsilon_2)\,\bar{S}_j}{x_j-1} \\ &\qquad (j=1,\ldots,n), \end{aligned} \tag{4.138}$$

where

$$\bar{S}_j = \sum_{i=1,i\neq j}^{n+2} \frac{x_j(x_i-1)}{(n+1)(n+2)(x_i-x_j)}, \quad \bar{\Delta}^j_{-k} = -\frac{\varepsilon_j}{2} + \frac{1-2\,\varepsilon_k}{2\,(n+1)}. \tag{4.139}$$

Proof By using (4.108), (4.113) and (4.134), we can rewrite the relations (4.138) into

$$\begin{aligned} q_j &= \frac{t_j}{\theta_{n+3}+1}\,\partial_{t_j}\log\frac{\tau_{2\mathbf{e}_{n+3}}}{\tau_0} - \sum_{j=1,j\neq i}^{n+2}\frac{2\,t_j}{(n+1)(n+2)(t_i-t_j)}, \\ q_jp_j &= t_j(t_j-1)\,\partial_{t_j}\log\frac{\tau_{n+3,-(n+2)}}{\tau_{n+3,-(n+1)}} - t_j\,\Delta^j_{-(n+2)} + (t_j-1)\,\Delta^j_{-(n+1)} \\ &\quad - \frac{\theta_{n+1}-\theta_{n+2}}{(n+1)(n+2)}\sum_{j=1,j\neq i}^{n+2}\frac{t_i(t_i-1)}{t_i-t_j}, \end{aligned} \tag{4.140}$$

where

$$\Delta^j_{-k} = -\frac{\theta_j}{2} + \frac{1-2\,\theta_k}{2\,(n+1)}. \tag{4.141}$$

Therefore we show the relations (4.140) in the following.

The action of Schlesinger transformation $T_{2\mathbf{e}_{n+3}}$, which act on the parameters θ_j $(j=1,\ldots,n+3)$ as follows:

$$T_{2\mathbf{e}_{n+3}}(\theta_{n+3}) = \theta_{n+3}+2, \quad T_{2\mathbf{e}_{n+3}}(\theta_j) = \theta_j \quad (j\neq n+3), \tag{4.142}$$

on the Hamiltonians H_i $(i=1,\ldots,n)$ is described as follows:

$$T_{2\mathbf{e}_{n+3}}(H_i) = H_i + (\theta_{n+3}+1)\,\frac{b_i}{\mathcal{X}} + \sum_{j=1,j\neq i}^{n+2}\frac{2\,(\theta_{n+3}+1)}{(n+1)(n+2)(t_i-t_j)}. \tag{4.143}$$

From (4.143) and

$$\partial_{t_i} \log \tau_{2\mathbf{e}_{n+3}} = T_{2\mathbf{e}_{n+3}}(H_i) \quad (i = 1, \ldots, n), \tag{4.144}$$

the first relation of (4.140) is obtained.

The action of Schlesinger transformation $T_{n+3,-(n+1)}$ and $T_{n+3,-(n+2)}$ on the Hamiltonians H_i ($i = 1, \ldots, n$) is described as follows:

$$\begin{aligned} T_{n+3,-(n+1)}(H_i) &= H_i + \frac{1}{t_i}\left(a_i - b_i\,\frac{a_{n+1}}{b_{n+1}}\right) \\ &\qquad + \frac{\Delta^i_{-(n+1)}}{t_i} + \sum_{j=1, j\neq i}^{n+2} \frac{1-\theta_{n+1}+\theta_{n+3}}{(n+1)(n+2)(t_i - t_j)}, \\ T_{n+3,-(n+2)}(H_i) &= H_i + \frac{1}{t_i - 1}\left(a_i - b_i\,\frac{a_{n+2}}{b_{n+2}}\right) \\ &\qquad + \frac{\Delta^i_{-(n+2)}}{t_i - 1} + \sum_{j=1, j\neq i}^{n+2} \frac{1-\theta_{n+2}+\theta_{n+3}}{(n+1)(n+2)(t_i - t_j)}. \end{aligned} \tag{4.145}$$

From (4.145) and

$$\begin{aligned} \partial_{t_i} \log \tau_{n+3,-(n+1)} &= T_{n+3,-(n+1)}(H_i) \quad (i = 1, \ldots, n), \\ \partial_{t_i} \log \tau_{n+3,-(n+2)} &= T_{n+3,-(n+2)}(H_i) \quad (i = 1, \ldots, n), \end{aligned} \tag{4.146}$$

the second relation of (4.140) is obtained. □

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