Dominating cycles in graphs

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Tomoki Yamashita

Preface

This thesis is written on the subject "Dominating Cycles in Graphs" to be submitted for the degree in Doctor at Kobe University.

The basis of thesis is formed by papers written during these three years. Some of the paper have been accepted for publication in graph theory-oriented journals; for the others I hope this will happen in the future. The papers that together underlie this thesis are listed at the end of this preface.

After two introductory chapters, the reader will find five chapters. General terminology can be found in Chapter 2. The order chapters can be read independently from one another.

Roughly speaking, this thesis consists of three parts. In the first part (Chapters 3 and 4), I will present my work about the edge-dominating cycle in graphs with large connectivity. In this work, we show there is an analogy between hamiltonian cycles and edge-dominating cycles on a degree condition for graphs with large connectivity.

In the second part (Chapters 5 and 6), I will present my work about the vertex-dominating cycle in tough graphs and bipartite graphs. Part of this work is joint work with A. Saito. We make a comparative study of each degree condition for the existence of hamiltonian cycles, edge-dominating cycles and vertex-dominating cycles.

In the third part (Chapter 7), I will present my work about a cycle such that every vertex is within the prescribed distance from the cycle. This work is joint work with A. Saito. We give a generalization of several theroems with the sufficient condition involving connectivity and stable number, called Chvátal-Erdős condition.

Papers underlying on the thesis

[1] A. Saito, T. Yamashita, *A Note on Dominating Cycles in Tough Graphs*, Ars Combinatoria **69**, 2003, 3–8 (Chapter 5)

[2] A. Saito, T. Yamashita, *Cycles within Specified Distance from Each Vertex*, to appear in Discrete Mathematics. (Chapter 7)

[3] T. Yamashita, *Vertex-dominating cycles in* 2*-connected bipartite graphs*, submitted. (Chapter 6)

[4] T. Yamashita, *Dominating cycles in k-connected graphs*, submitted. (Chapter 3)

[5] T. Yamashita, *Degree Sum and Connectivity Conditions for Dominating Cycles*, submitted. (Chapter 4)

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Chapter 1

Introduction

A hamiltonian cycle is a cycle passing through every vertex. We say that a graph G is hamiltonian if G has a hamiltonian cycle. In contrast with Eulerian cycle, the question whether a given graph is hamiltonian seems to be much harder to answer. Up to now, no easily verifiable necessary and sufficient condition is known. This fact gave rise to a number of sufficient conditions. In particular, the following two sufficient conditions are well-known. One is concerned with the minimum degree sum of two nonadjacent vertices, and the other is concerned with the stability number and the connectivity of a graph.

Theorem 1.1 (Ore [16]) *Let G be a graph on $n \geq 3$ vertices. If the minimum degree sum of nonadjacent vertices is at least n, then G is hamiltonian.*

Theorem 1.2 (Chvátal&Erdős [7]) *Let $k \geq 2$ be a positive integer, and let G be a k-connected graph. If the stability number of G is at most k, then G is hamiltonian.*

In recent years, the study of dominating cycles has been in full flood. A dominating cycle is defined as a cycle C such that every edge is incident with some vertex of C. However, in some papers, a cycle C is also called a dominating cycle if every vertex is adjacent to some vertex of C. In this thesis, we call the former an edge-dominating cycle and the latter a vertex-dominating cycle. Each of dominating cycles is a generalization of hamiltonian cycles. There may be sufficient conditions for a graph to have a dominating cycle which correspond to that for hamiltonicity. In this thesis, I will discuss sufficient conditions for each of dominating cycles.

In Chapter 3, we study a degree condition for a connected graph. Bondy gave a degree sum condition for a connected graph to be hamiltonian. This result is a common generalization of Ore's Theorem (1.1) and Chvátal–Erdős Theorem (1.2).

Theorem 1.3 (Bondy [4]) *Let $k \geq 2$ be a positive integer, and let G be a k-connected graph on n vertices. If the minimum degree sum of a stable set of order $k+1$ is greater than $(n-1)(k+1)/2$ then G is hamiltonian.*

We extend Theorem 1.3 as follows:

Theorem 1.4 *Let G be a k-connected graph ($k \geq 2$) on n vertices, and let S be any stable set of order $k+1$. If the maximum degree sum of a subset of S of order 2 is at least n, then G is hamiltonian.*

On the other hand, Bondy and Fan showed the result for a vertex-dominating cycle.

Theorem 1.5 (Bondy&Fan [5]) *Let $k \geq 2$ be a positive integer, and let G be a k-connected graph on n vertices. If the minimum degree sum of a 3-stable set of order $k+1$ is at least $n-2k$, then G has a vertex-dominating cycle.*

We consider a degree sum condition for a k-connected graph to have an edge-dominating cycle. The degree sum condition for a 2-connected graph was already known. The result can be derived from the following theorem due to Bondy. A graph is called k-path-connected if any two vertices are connected by a path of length at least k.

Theorem 1.6 (Bondy [4]) *Let $k \geq 2$ be a positive integer, and let G be a k-connected graph on n vertices, and let C be a longest cycle of G. If the minimum degree sum of a stable set of order $k+1$ is at least $n + k(k-1)$, then $G - C$ contains no $(k-1)$-path-connected subgraph.*

The case $k = 2$ of Theorem 1.6 implies the following:

Theorem 1.7 (Bondy [4]) *Let G be a 2-connected graph on n vertices. If the minimum degree sum of a stable set of order 3 is at least $n+2$, then any longest cycle is an edge-dominating cycle.*

On the other hand, Fraisse showed the following theorem for the existence of a D_k-cycle.

Theorem 1.8 (Fraisse [10]) *Let G be a k-connected graph on n vertices, where $n \geq 3$. If the minimum degree sum of a stable set of order $k+1$ is at least $n + k(k-1)$, then G has a D_k-cycle.*

Neither Theorem 1.6 nor Theorem 1.8 implies a degree sum condition for a k-connected graph ($k \geq 3$) to have a dominating cycle. We show the following result for an edge-dominating cycle.

Theorem 1.9 *Let G be a k-connected graph ($k \geq 2$) on n vertices, and let S be any stable set of order $k+1$. If the maximum degree sum of a subset of S of order 3 is at least $n+2$, then there exists a longest cycle which is an edge-dominating cycle.*

In Chapter 4, we focus the degree sum and connectivity condition. Motivated by the observation in Chapter 3, we can find there is an analogy between hamiltonian cycles and edge-dominating cycles on a degree condition for a graph with large connectivity. Bauer, Broersma, Li and Veldman showed the following result concerning the degree sum and connectivity, which is an extension of Ore's Theorem (1.1).

Theorem 1.10 (Bauer et al. [2]) *Let G be a 2-connected graph on n vertices. If the minimum degree sum of a stable set of order 3 is at least $n + \kappa$, then G is hamiltonian.*

Motivated by the observation above, we give an extension of Theorem 1.7.

Theorem 1.11 *Let G be a 3-connected graph on n vertices. If the minimum degree sum of a stable set of order 4 is at least $n + \kappa + 3$, then G contains a longest cycle which is an edge-dominating cycle.*

By observing Theorems 1.8, 1.10 and 1.11, we propose the following conjecture. which has been verified for $k = 2$ (Theorem 1.10) and $k = 3$ (Theorem 1.11).

Conjecture 1.12 *Let G be a k-connected graph on n vertices, where $n \geq 3$. If the minimum degree sum of a stable set of order $k + 1$ is at least $n + \kappa + k(k - 2)$, then G has a longest cycle of which is a D_{k-1}-cycle.*

In Chapter 5, we consider a degree condition for a tough graph. Jung gave a degree sum condition for a tough graph to be hamiltonian.

Theorem 1.13 (Jung [12]) *Let G be a 1-tough graph on $n \geq 11$ vertices. If the minimum degree sum of nonadjacent vertices is at least $n - 4$, then G is hamiltonian.*

Bauer, Veldman, Morgana and Schmeichel gave a degree sum condition for edge-dominating cycles.

Theorem 1.14 (Bauer et al. [3]) *Let G be a 1-tough graph on n vertices. If the minimum degree sum of a stable set of order 3 is at least n, then G has an edge-dominating cycle.*

We show the following result for a vertex-dominating cycle.

Theorem 1.15 *Let $k \geq 2$ be a positive integer, and let G be a k-connected graph on n vertices such that the toughness of G is greater than $k/(k+1)$. If the minimum degree sum of a 4-stable set of order $k+1$ is at least $n-2k-2$, then G has a vertex-dominating cycle.*

In Chapter 6, we shall study a degree condition for a bipartite graph to have a vertex-dominating cycle. Moon and Moser gave a degree sum condition for a hamiltonian cycle.

Theorem 1.16 (Moon&Moser [13]) *Let G be a balanced bipartite graph of order $2n$ with partite sets X and Y ($|X| = |Y| = n$). If the minimum degree sum of nonadjacent vertices in different partite sets is at least $n+1$, then G is hamiltonian.*

Ash and Jackson gave a minimum degree condition for an edge-dominating cycle.

Theorem 1.17 (Ash&Jackson [1]) *Let G be a 2-connected bipartite graph with partite sets X and Y, and let $\max(|X|, |Y|) = n$. If the minimum degree of vertices is at least $(n-1)/3$, then there exists a longest cycle which is an edge-dominating cycle.*

We give a minimum degree condition for a bipartite graph to have a vertex-dominating cycles.

Theorem 1.18 *Let G be a 2-connected bipartite graph with partite sets X and Y, and let $\max(|X|,|Y|) = n$. If the minimum degree of vertices is at least $(n+1)/3$, then G has a vertex-dominating cycle.*

In Chapter 7, we consider a cycle such that every vertex is within the prescribed distance from the cycle. Let G be a graph and let f be a non-negative integer-valued function defined on $V(G)$. A cycle C is called an f-dominating cycle if $d_G(v,C) \le f(v)$ holds for each $v \in V(G)$, where $d_G(v,C)$ denotes the distance between v and C. A set S is called an f-stable set if $d_G(u,v) \ge f(u) + f(v)$ holds for each pair of distinct vertices u, v in S, and denote by $\alpha_f(G)$ the order of a largest f-stable set in G. We prove the following theorem.

Theorem 1.19 *Let G be a 2-connected graph, and let f be a non-negative integer-valued function defined on $V(G)$. If $\alpha_{f+1}(G) \le \kappa(G)$, then G has an f-dominating cycle.*

By taking an appropriate function as f, we can deduce a number of known results (Chvátal–Erdős Theorem 1.2 etc.) from this theorem. Furthermore, we consider a vertex-dominating cycle passing through prescribed vertices, and give a further extension as follows. For a set of vertices $X \subset V(G)$ and a positive integer d, we define

$\hat{\alpha}_{G,d}(X)$ by

$$\hat{\alpha}_{G,d}(X) = \max\{|S| \colon S \subset X, S \text{ is a } d\text{-stable in } G\}.$$

Note that $\hat{\alpha}_{G,2}(X) = \hat{\alpha}_2(G[X]) = \alpha(X)$, but for $d \ge 3$, $\hat{\alpha}_{G,d}(X)$ and $\hat{\alpha}_d(G[X])$ may take different values.

Corollary 1.20 *Let G be a k-connected graph ($k \ge 2$), and let X and Y be disjoint subsets of $V(G)$. If $\alpha(X) + \hat{\alpha}_{G,4}(Y) \le k$, then G has a cycle C which contains X and dominates Y.*

Chapter 2

Terminology, notation and preliminary

2.1 Graphs

A *graph* $G = (V, E)$ consists of a finite nonempty set V whose elements are called *vertices* and a set E of 2-element subsets of V whose elements are called *edges*. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$ respectively. We denote by $|X|$ the number of the elements of a finite set X, called the *order*order!of a set of X. The *order of a graph* is the number of vertices in the graph, and is written by $|G|$.

The edge $e = \{u, v\}$ is said to *join* the vertices u and v. If $e = \{u, v\}$ is an edge of graph G, u and v are called *adjacent*, while u and e are *incident*, as are v and e. It is convenient to henceforth denote an edge by uv or vu rather than by $\{u, v\}$.

A graph is *complete* if every two of its vertices are adjacent. We denote a complete graph of order n by K_n. A graph is *bipartite* if its vertex set can be partitioned into subsets X and Y that each edge joins a vertex of X and a vertex of Y. We denote a bipartite graph G with partition (X, Y)

by $G = (X \cup Y, E)$. A graph $G = (X \cup Y, E)$ is *balanced* if $|X| = |Y|$ and *complete* if $E(G) = \{uv \colon u \in X, v \in Y\}$ A complete bipartite graph $G = (X \cup Y, E)$ in which $|X| = n$ and $|Y| = m$ is denoted by $K_{n,m}$.

2.2 Subgraph and operations on graphs

A graph H is a *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. Particularly if $V(H) = V(G)$ then H is called a *spanning subgraph* of G. For $X \subset V(G)$, a graph $G[X]$ is an *induced subgraph* by X if $V(G[X]) = X$ and $E(G[X]) = \{uv \in E(G) \colon u, v \in X\}$.

Let G and H be two graphs. The *union* of G and H, denoted by $G \cup H$, is the graph with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$. If $V(G) \cap V(H) = \emptyset$, then the *join* of G and H, denoted by $G + H$, is the graph obtained from $G \cup H$ by joining each vertex in $V(G)$ to each vertex in $V(H)$, notation $G + H$.

If $X \subset V(G)$, we denote by $G - X$ a graph induced by $V(G) - X$. If $X \subset E(G)$, we denote $G' = (V, E \cup X)$ and $G'' = (V, E - X)$ by $G + X$ and $G - X$ respectively. For $v \in V(G)$ and $e \in E(G)$, we denote $G - \{v\}$ and $G - \{e\}$ simply by $G - v$ and $G - e$ respectively. Furthermore, if H is a subgraph of a graph G, the subgraph $G - V(H)$ is denoted simply by $G - H$.

2.3 Neighborhoods, degrees, stable sets

The *neighborhood* $N_G(x)$ of a vertex x in a graph G is the set of all vertices adjacent to x in G. For $X \subset V(G)$, $N_G(X)$ denote the set of vertices in $G - X$ which are adjacent to some vertex in X. Furthermore, for a subgraph H of G, $x \subset V(G)$ and $X \subset V(G) - V(H)$, we sometimes write $N_H(x)$ and $N_H(X)$ instead of $N_G(x) \cap V(H)$ and $N_G(X) \cap V(H)$, respectively.

The *degree of a vertex* x, denoted by $d_G(x)$, is the cardinality of the neighborhood of x. Similarly, we define the *degree of a set of vertices* X as $d_G(X) = |N_G(X)|$, and the *degree of a subgraph* H as $d_G(H) = |N_G(V(H))|$. A vertex of degree 1 is called an *endvertex*. The *minimum degree* of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$. The *maximum degree* is defined similarly and is denoted by $\Delta(G)$.

We call $X \subseteq V(G)$ a *stable set* in G if any two vertices x, $y \in X$ are not adjacent in G. A stable set is also called an independent set. The *stable number* of G is defined to be the cardinality of a maximum stable set in G, denoted by $\alpha(G)$.

For a graph G, let $\sigma_k(G)$ be the minimum degree sum of a stable set of k vertices:

$$\sigma_k(G) = \begin{cases} \min\{\sum_{x \in X} d_G(x) \colon X \text{ is a stable set of } G \text{ of order } k\} & \text{if } \alpha(G) \geq k \\ +\infty & \text{if } \alpha(G) < k. \end{cases}$$

If no ambiguity can arise, we omit the index G, hence we denote $N(x)$, $d(x)$, etc. We often simply write δ, α and σ_k instead of $\delta(G)$, $\alpha(G)$ and $\sigma_k(G)$, respectively.

For $S \subset V(G)$ with $S \neq \emptyset$, let $\Delta_k(S)$ denote the maximum degree sum of a subset of S of order k:

$$\Delta_k(S) = \max\{\sum_{x \in X} d_G(x) \colon X \text{ is a subset of } S \text{ of order } k\}$$

2.4 Walks, paths and cycles

A sequence of vertices $W = x_0 x_1 \dots x_l$ is called a *walk* of G if $x_i \in V(G)$, $x_l \in V(G)$ and $x_i x_{i+1} \in E(G)$ for $0 \leq i \leq l-1$. Let $W = x_0 x_1 \dots x_l$ be a walk in a graph G. Then l is called the *length* of W and denoted by $l(W)$. A walk W is said to *join* x_0 and x_l. The vertices x_0 and x_l are called the starting vertex and the terminal vertex of W, respectively. For i, j with $0 \le i \le j \le l$, we denote the subwalk $x_i x_{i+1} \dots x_j$ by $x_i \overrightarrow{W} x_j$ and its reverse $x_j x_{j-1} \dots x_i$ by $x_j \overleftarrow{W} x_i$. For X, $Y \subset V(G)$, if $x_0 \in X$ and $x_l \in Y$, then W is said to be a walk from X to Y. If $(W - \{x_0, x_l\}) \cap X = \emptyset$, we say that W is internally disjoint from X. For x, $y \in V(G)$, a walk from $\{x\}$ to $\{y\}$ is called an x-y walk. Let $P = x_0 x_1 \dots x_k$ and $Q = y_0 y_1 \dots y_l$ be walks in G. If $x_k = y_0$, the walk $x_0 x_1 \dots x_k y_1 y_2 \dots y_l$ is denoted by $x_0 P x_k Q y_l$. If P and Q have no common vertices, P and Q are called *disjoint*.

A walk is called a *path* if its vertices are distinct. If $x, y \in V(G)$, then we denote by xPy a path from x to y. For $X, Y \subset V(G)$, a path xPy is called an X-Y path if $V(P) \cap X = \{x\}$ and $V(P) \cap Y = \{y\}$. We abbreviate an $\{x\}$-Y path by an x-Y path. For a subgraph H of G, a path xPy is called an H-path if $V(xPy) \cap V(H) = \{x, y\}$ and $E(H) \cap E(P) = \emptyset$.

A path is called a *cycle* if the endvertices are the same. Let C be a cycle of G and $u, v \in V(C)$. We denote C with a given orientation by $\overrightarrow{C}$. We denote by $u\overrightarrow{C}v$ a path from u to v on $\overrightarrow{C}$. The reverse sequence of $u\overrightarrow{C}v$ is denoted by $v\overleftarrow{C}u$. For $u \in V(C)$, we denote the h-th successor and the h-th predecessor of u on $\overrightarrow{C}$ by u^{+h} and u^{-h} respectively. We abbreviate u^{+1} and u^{-1} by u^+ and u^- respectively. For a cycle $\overrightarrow{C}$ and $X \subset V(C)$, we define $X^+ = \{x^+ \colon x \in X\}$ and $X^- = \{x^- \colon x \in X\}$.

2.5 Connectivity and toughness

A graph G is *connected* if any two vertices of G are joined by a path. A maximal connected subgraph is called a *component* of G. We denote the number of components of G by $\omega(G)$. $S \subset V(G)$ is a *cutset* in G if G is connected and $G - S$ is not connected. The cardinality of a minimum cutset in G is called the *connectivity* of G, denoted by $\kappa(G)$. A graph G is called *k-connected* if $k \le \kappa(G)$.

A connected graph G is defined to be *t-tough* if $|S| \geq t \cdot \omega(G-S)$ for every cutset S of $V(G)$. The *toughness* of G, denoted by $t(G)$, is the maximum value of t for which G is t-tough (taking $t(K_n) = \infty$ for all $n \geq 1$).

2.6 Distance and d-stable

For a graph G and $x, y \in V(G)$, the *distance* between vertices x and y, denoted by $d_G(x, y)$, is the length of a shortest path in G joining them. For a non-empty set of vertices S of G, we define the distance between x and S by $d_G(x, S) = \min\{d_G(x, s) \colon s \in S\}$. If H is a subgraph of G, we often write $d_G(x, H)$ instead of $d_G(x, V(H))$.

A set S of vertices in a graph G is said to be *d-stable* if the distance between each pair of distinct vertices in S is at least d. We define $\hat{\alpha}_k(G)$ by $\hat{\alpha}_k(G) = \max\big\{|S| : S \text{ is a } k\text{-stable set}\big\}$. Note that a 2-stable set is a stable set and $\hat{\alpha}_2(G) = \alpha(G)$.

2.7 Hamiltonian and dominating cycles

A cycle containing all vertices of the graph is called a *hamiltonian cycle* in a graph. We say that a graph G is *hamiltonian* if G has a hamiltonian cycle.

An *edge-dominating cycle* is defined as a cycle such that every edge in G is incident with a vertex in C, that is, $V(G - C)$ is a stable set. A cycle C of a graph G is said to be a *D_λ-cycle* if $|H| < \lambda$ for any component H of $G - C$. Note that a D_1-cycle is a hamiltonian cycle and a D_2-cycle is an edge-dominating cycle.

Let S and T be subsets of $V(G)$. Then S is said to *dominate* T if every vertex in T either belongs to S or has a neighbor in S. If S dominates $V(G)$, then S is called a *dominating set*. A cycle C is said to be a *vertex-dominating cycle* if every vertex in G is adjacent to a vertex in C, that is,

$V(C)$ is a dominating set in G. A cycle C in a graph G is said to be a d_λ-*cycle* if $d_G(x, C) < \lambda$ for each $x \in V(G)$. Note that a d_1-cycle is a hamiltonian cycle and a d_2-cycle is a vertex-dominating cycle.

Chapter 3

Edge-dominating cycles in connected graphs

3.1 Introduction

The following is a classical result due to Dirac in hamiltonian graph theory.

Theorem 3.1 (Dirac [8]) *Let G be a graph of order $n \geq 3$. If $\delta \geq n/2$, then G is hamiltonian.*

In 1960, Ore introduced a degree sum condition for a graph to be hamiltonian.

Theorem 3.2 (Ore [16]) *Let G be a graph of order $n \geq 3$. If $\sigma_2 \geq n$ then G is hamiltonian.*

For several sufficient conditions for hamiltonicity, it is observed that weaker conditions guarantee the existence of a hamiltonian cycle for graphs with large connectivity. In 1980, Bondy gave a degree sum condition for a connected graph.

Theorem 3.3 (Bondy [4]) *Let $k \geq 2$ be a positive integer, and let G be a k-connected graph on n vertices. If $\sigma_{k+1} > (k+1)(n-1)/2$, then G is hamiltonian.*

In Theorems 3.1–3.3, the degree sum conditions are sharp and cannot be relaxed. We consider the graph G_1 satisfying $kK_1 + (k+1)K_1 \subset G_1 \subset K_k + (k+1)K_1$. Then G_1 is a k-connected graph with $\delta(G_1) = (|V(G_1)|-1)/2$ and $\sigma_{k+1}(G_1) = (k+1)(|V(G_1)|-1)/2$, but G_1 is not hamiltonian.

For $S \subset V(G)$ with $S \neq \emptyset$, let $\Delta_k(S)$ denote the maximum degree sum of a subset of S of order k:

$$\Delta_k(S) = \max\{\sum_{x \in X} d_G(x) : X \subset S, |X| = k\}.$$

The first purpose of this paper is to show that Theorem 3.3 admits the following extension.

Theorem 3.4 *Let $k \geq 2$ be a positive integer, and let G be a k-connected graph on n vertices. If $\Delta_2(S) \geq n$ for any stable set S of order $k+1$, then G is hamiltonian.*

Since $\frac{\Delta_2(S)}{2} \geq \frac{\Delta_{k+1}(S)}{k+1}$ for $S \subset V(G)$ with $|S| = k+1$, it is easy to see that Theorem 3.4 implies Theorem 3.3. On the other hand, Theorem 3.3 does not imply Theorem 3.4. Let $G_2 = K_k + [(k-1)K_1 \cup \{K_1 + (K_m \cup K_{n-2k-m})\}]$, where $k \geq 2$ and $n \geq 2k+4$. Then G_2 is a k-connected hamiltonian graph with $\Delta_2(S) = n$ for any stable set S of order $k+1$, but $\sigma_{k+1}(G_2) \leq (k+1)(n-1)/2$.

Since an edge-dominating cycle is a generalization of a hamiltonian cycle, there may be sufficient conditions for a graph to have an edge-dominating cycle, which correspond to known sufficient conditions for hamiltonicity.

In 1971, Nash-Williams gave a minimum degree condition for an edge-dominating cycle.

Theorem 3.5 (Nash-Williams [14]) *Let G be a 2-connected graph on n vertices. If $\delta \geq (n+2)/3$, then any longest cycle of G is an edge-dominating cycle.*

The second purpose of this paper is to show that Theorem 3.5 also admits a similar relaxation under an additional assumption on connectivity. The degree sum condition for a 2-connected graph is already known. The result can be derived from the following theorem due to Bondy!%A graph is called *k-path-connected* if any two vertices are connected by a path of length at least k.

Theorem 3.6 (Bondy [4]) *Let $k \geq 2$ be a positive integer, and let G be a k-connected graph on n vertices, and let C be a longest cycle of G. If $\sigma_{k+1} \geq n+k(k-1)$, then $G-C$ contains no $(k-1)$-path-connected subgraph.*

The case $k=2$ of Theorem 3.6 implies the following (a generalization of Theorem 3.5):

Theorem 3.7 (Bondy [4]) *Let G be a 2-connected graph on n vertices. If $\sigma_3 \geq n+2$, then any longest cycle is an edge-dominating cycle.*

On the other hand, Fraisse showed the following theorem for the existence of a D_λ-cycle.

Theorem 3.8 (Fraisse [10]) *Let $k \geq 2$ be a positive integer, and let G be a k-connected graph on n vertices. If $\sigma_{k+1} \geq n + k(k-1)$, then G has a D_k-cycle.*

Unfortunately, neither Theorem 3.6 nor Theorem 4.9 implies a degree sum condition for a k-connected graph ($k \geq 3$) to have an edge-dominating cycle. So we show the following result.

Theorem 3.9 *Let $k \geq 2$ be a positive integer, and let G be a k-connected graph on n vertices. If $\Delta_3(S) \geq n+2$ for any stable set S of order $k+1$, then there exists a longest cycle which is an edge-dominating cycle.*

Theorem 3.9 is best possible in a sense. Consider the graph $G_3 = K_k + (k+1)K_2$, where $k \geq 2$ and $n \geq 3k$. Then G_3 is a k-connected graph with $\Delta_3(S) = |V(G_3)| + 1$ for any stable set S of order $k+1$, but any longest cycle of G_3 is not edge-dominating. On the other hand, Theorem 3.9 does not assert that any longest cycle is an edge-dominating cycle. It remains unresolved whether any longest cycle is an edge-dominating cycle under the hypothesis of Theorem 3.9.

3.2 Proof of Theorem 3.4

In each lemma below we assume that G is a nonhamiltonian graph on n vertices, C a longest cycle in G and H a component of $G - V(C)$. Since the first lemma is standard in hamiltonian graph theory, we do not prove them in this paper.

Lemma 3.10 *If uPv is a path with $|V(P)| > |V(C)|$, then $uv \notin E(G)$ and $d_G(u) + d_G(v) \leq n-1$.*

By Lemma 3.10, the following lemma is plain.

Lemma 3.11 *Let $u_0 \in V(H)$ and $u_1, u_2 \in N_C(H)^+$. The following statements hold:*

(i) *$N_C(H)^+ \cup \{u_0\}$ is a stable set.*

(ii) $d_G(u_0) + d_G(u_1) \leq n-1$.

(iii) $d_G(u_1) + d_G(u_2) \leq n-1$.

Proof of Theorem 3.4. Suppose that G is a k-connected nonhamiltonian graph on n vertices satisfying the hypothesis. Hence there exists a component H of $G - C$. Since G is k-connected, we have $|N_C(H)| \geq k$. Let $\{u_1, u_2, \ldots, u_k\} \subset N_C(H)^+$ and $u_0 \in V(H)$. By Lemma 3.11 (i), $\{u_0, u_1, u_2, \ldots, u_k\}$ is a stable set of order $k+1$. By Lemmas 3.11 (ii) and (iii), we have $d_G(u_i) + d_G(u_j) \leq n-1$ for any i, j with $0 \leq i < j \leq k$. This contradicts the hypothesis. □

3.3 Proof of Theorem 3.9

Let G be a graph on n vertices such that there exists no longest and edge-dominating cycle. Choose a cycle C such that

(i) $|V(C)|$ is as large as possible, and

(ii) $|E(G-C)|$ is as small as possible, subject to (i).

We give an orientation to C. Let H_0 be a component of $G-C$ with $|V(H_0)| \geq 2$.

Lemma 3.12 $d_G(u_1) + d_G(u_2) + d_G(u_3) \leq n+1$ *for* $u_1, u_2, u_3 \in N_C(H)^+$.

Proof. Since C is a longest cycle, we can easily obtain the following claim.

Claim 1 *For* $1 \leq i \neq j \leq 3$, *the following statements hold:*

(1) *For any* $u_i \in N_C(H_0)^+$, $u_i \notin N_C(H)$.

(2) *For any* $u_i, u_j \in N_C(H_0)^+$, *there exists no* C*-path joining* u_i *and* u_j.

(3) *For any* $u_i, u_j \in N_C(H_0)^+$ *and* $w \in u_i \overrightarrow{C} u_j^-$, *if there exists a* C*-path joining* u_i *and* w, *there exists no* C*-path joining* u_j *and* w^-.

Without loss of generality, we may assume that u_1, u_2 and u_3 appear in this order along $\overrightarrow{C}$. Let $x_i = u_i^-$. Note that $x_i \in N_C(H_0)$. Let $C_i := u_i \overrightarrow{C} x_{i+1}$ (indices are taken modulo 3) and $x_i' \in N_{H_0}(x_i)$.

Claim 2 *If $v^{-2} \in N_{C_i}(u_{i+1})$ and $v \in N_{C_i}(u_i)$, then $N_{G-C}(v^-) \neq \emptyset$.*

Proof. Suppose that $v \in N_{C_i}(u_i)$ and $v^{-2} \in N_{C_i}(u_{i+1})$. Let $x_i' P x_{i+1}'$ be a path in H_0. Then $C' = x_i \overleftarrow{C} u_{i+1} v^{-2} \overleftarrow{C} u_i v \overrightarrow{C} x_{i+1} x_{i+1}' P x_i' x_i$ is a cycle. By (i), $x_{i+1} P x_i$ contains only one vertex of H_0, since $V(C - C') = \{v^-\}$. Let $u \in N_{H_0}(x_i) \cap N_{H_0}(x_j)$. Therefore $|E(G-C')| - |E(G-C)| = |N_{G-C}(v^-)| - |N_{H_0}(u)|$. By $|V(H)| \geq 2$, we obtain $|N_{H_0}(u)| \geq 1$. By (ii), we have $|E(G - C')| \geq |E(G-C)|$. Hence $|N_{G-C}(v^-)| \geq 1$, that is, $N_{G-C}(v^-) \neq \emptyset$. □

By Claim 2, for each vertex $v \in N_{C_i}(u_i)$, we define v^* as folllows: if $v^{-2} \in N_{C_i}(u_{i+1})$, then $v^* = w$ for an arbitrary vertex $w \in N_{G-C}(v^-)$; otherwise $v^* = v^{-2}$. For $1 \leq i \leq 3$, we define $H_i := \{v^* \in V(G-C) \colon v \in N_{C_i}(u_i)\}$.

Claim 3 *If $v_1 \neq v_2$, then $v_1^* \neq v_2^*$.*

Proof. Suppose that $v_1 \neq v_2$ and $v_1^* = v_2^*$. Then we know that $v_1^* \in V(G-C)$. Let $v_1 \in V(C_i)$ and $v_2 \in V(C_j)$. If $i = j$, then, without loss of generality, we may assume that v_1 and v_2 appear in this order along $\overrightarrow{C_1}$, and let $x_h = x_2$. If $i \neq j$, then, without loss of generality, we may assume that $v_1 \in V(C_1)$ and $v_2 \in V(C_2)$, and let $x_h = x_3$. By (i), v_1 and v_2 are not consecutive vertices. Let $x_1' P x_h'$ be a path in H_0. Then $x_1 \overleftarrow{C} u_h v_2^- \overleftarrow{C} v_1^+ u_1 \overrightarrow{C} v_1 v_1^* v_2 \overrightarrow{C} x_h x_h' P x_1' x_1$ is a cycle longer than C, a contradiction. □

Let $N_{C_i}(u_i)^* := \{v^* \colon v \in N_{C_i}(u_i)\}$. By Claim 3, note that $|N_{C_i}(u_i)| = |N_{C_i}(u_i)^*|$. Clearly, we have $N_{C_i}(u_i)^* \cup N_{C_i}(u_{i+1}) \cup N_{C_i}(u_{i+2})^- \subset (C_i \cup H_i \cup \{u_{i+1}\})$. By Claim 1 (3), $N_{C_i}(u_i)^*$, $N_{C_i}(u_{i+1})$ and $N_{C_i}(u_{i+2})^-$ are pairwise disjoint. By Claim 3, we have $|N_{C_i}(u_i)^*| + |N_{C_i}(u_{i+1})| + |N_{C_i}(u_{i+2})^-| \leq$

$|V(C_i)| + |V(H_i)| + 1$. Thus, we obtain that

$$\begin{aligned} d_{C_i}(u_i) + d_{C_i}(u_{i+1}) + d_{C_i}(u_{i+2}) &= |N_{C_i}(u_i)| + |N_{C_i}(u_{i+1})| + |N_{C_i}(u_{i+2})| \\ &= |N_{C_i}(u_i)^*| + |N_{C_i}(u_{i+1})| + |N_{C_i}(u_{i+2})^-| \\ &\leq |V(C_i)| + |V(H_i)| + 1. \end{aligned}$$

Therefore the following inequality holds:

$$\begin{aligned} d_C(u_1) + d_C(u_2) + d_C(u_3) &\leq \sum_{h=1}^{3}(|V(C_h)| + |V(H_h)| + 1) \\ &\leq |V(C)| + \sum_{h=1}^{3}|V(H_h)| + 3. \end{aligned} \tag{3.1}$$

Claim 4 *For any $1 \leq i, j \leq 3$, $N_{H_i}(u_j) = \emptyset$.*

Proof. We consider only the case $i = 1$ and $j = 2$. The other cases can be proved similarly. Suppose that $w \in V(H_1)$ and $v \in N_C(w)$. Since $u_1 v^+ \in E(G)$, by Claim 1 (3), we obtain that $wu_2 \notin E(G)$. □

Let i, j be integers satisfying $0 \leq i \leq 3$, $1 \leq j \leq 3$. By Claim 1(1), we have $N_{H_0}(u_j) = \emptyset$. By Claim 1 (1) and Claim 3, we have $H_i \cap H_j = \emptyset$. By Claim 1 (2), we have $N_{G-C}(u_i) \cap N_{G-C}(u_j) = \emptyset$. Since $\bigcup_{h=0}^{3} V(H_h) \subset V(G - C)$, we obtain that

$$\begin{aligned} &d_{G-C}(u_1) + d_{G-C}(u_2) + d_{G-C}(u_3) \\ &\leq |V(G - C) - \bigcup_{h=0}^{3} V(H_h)| \\ &\leq |V(G - C)| - \sum_{h=0}^{3}|V(H_h)| \\ &\leq |V(G - C)| - \sum_{h=1}^{3}|V(H_h)| - 2. \end{aligned} \tag{3.2}$$

Thus, by the inequality (3.1) and (3.2), we obtain

$$d_G(u_1) + d_G(u_2) + d_G(u_3) \leq |V(G)| + |V(G - C)| + 1 = n + 1.$$

□

Lemma 3.13 *$d_G(u_0) + d_G(u_1) + d_G(u_2) \leq n + 1$ for any $u_0 \in V(H_0)$ and $u_1, u_2 \in N_C(H_0)^+$.*

Proof. By using the similar argument as in the proof of Lemma 1, we can prove this lemma. □

Proof of Theorem 3.9. Suppose that G is a k-connected graph on n vertices satisfying the hypothesis, and there exists no longest and edge-dominating cycle. Hence there exists a component H_0 of $G-C$ with $|V(H_0)| \geq 2$. Since G is k-connected, we have $|N_C(H_0)| \geq k$. Let $\{u_1, u_2, \ldots, u_k\} \subset N_C(H_0)^+$ and $u_0 \in V(H_0)$. By Lemma 3.11 (i), $\{u_0, u_1, u_2, \ldots, u_k\}$ is a stable set of order $k+1$. By Lemmas 3.12 and 3.13, for any h, i, j, $0 \leq h < i < j \leq k$, we have $d_G(u_h) + d_G(u_i) + d_G(u_j) \leq n + 1$. This contradicts the hypothesis.
□

3.4 Remarks

Remark 1.

In 1987, Bondy and Fan proved that a much weaker degree sum condition guarantees the existence of a vertex-dominating cycle.

Theorem 3.14 (Bondy&Fan [5]) *Let G be a k-connected graph ($k \geq 2$) on n vertices. If $\sum_{x \in S} d_G(x) \geq n - 2k$ for every 3-stable set S of order $k + 1$, then G has a vertex-dominating cycle.*

Under an additional assumption on connectivity, the relaxation of the degree condition for a dominating cycle is similar to that for a hamiltonian cycle but not that for a vertex-dominating cycle.

Remark 2.

In 1983, Veldman studied the relation between the edge degree and the existence of an edge-dominating cycle. The *edge degree* $d_G(e)$ of the edge $e = uv$ is defined as the number of neighbours of e, i.e., $|N(u) \cup N(v)| - 2$. Two edges are called *remote* if they are disjoint and there is no edge joining them. Veldman proved the following result.

Theorem 3.15 (Veldman [18]) *Let $k \geq 2$ be a positive integer, and let G be a k-connected graph on n vertices. If $\sum_{i=0}^{k} d_G(e_i) > k(n-k)/2$ for any $k+1$ mutually remote edges $e_0, e_1, \ldots, e_k$ of G, then G has an edge-dominating cycle.*

Wang and Zhang proved the following fact which was conjectured by Veldman.

Theorem 3.16 (Wang&Zhang [20]) *Let G be a k-connected graph on n vertices. If $\sum_{i=0}^{k} d_G(e_i) > (k+1)(n-2)/3$ for any $k+1$ mutually remote edges $e_0, e_1, \ldots, e_k$ of G, then G has an edge-dominating cycle.*

Theorem 3.16 also admits the following extension. For $F \subset E(G)$ with $S \neq \emptyset$, let $\Delta_k(F)$ denote the maximum value of the edge degree sums of the subset of S of order k:

$$\Delta_k(F) = \max\{\sum_{e \in X} d_G(e) : X \subset F, |X| = k\}.$$

Theorem 3.17 *Let G be a k-connected graph on n vertices. If $\Delta_3(F) \geq n-1$ for every set F of $k+1$ mutually remote edges, then G has an edge-dominating cycle.*

Theorem 3.17 can be proved by a similar approach of the proof of Theorem 3.9.

Chapter 4

Degree sum and connectivity for edge-dominating cycles

4.1 Introduction

In 1960, Ore gave a degree sum condition for the existence of a hamiltonian cycle.

Theorem 4.1 (Ore [16]) *Let G be a graph of order $n \geq 3$. If $\sigma_2 \geq n$, then G is hamiltonian.*

In 1989, Bauer, Broersma, Li and Veldman showed the following result concerning the degree sum and connectivity.

Theorem 4.2 (Bauer et al. [2]) *Let G be a 2-connected graph on n vertices. If $\sigma_3 \geq n + \kappa$, then G is hamiltonian.*

Theorem 4.2 is a generalization of the following result.

Theorem 4.3 (Haggkvist&Nicoghossian [11]) *Let G be a 2-connected graph on n vertices. If $\delta \geq (n + \kappa)/3$, then G is hamiltonian.*

Theorem 4.2 is best possible in a sense. Let $2 \leq \kappa$ and $\frac{n+\kappa-1}{3} \leq m \leq \frac{n-1}{2}$. We consider the graph $G_1 = K_m + mK_1 \cup K_{n-2m}$ by joining κ vertices of K_m and each vertex of K_{n-2m}. Then $\kappa(G_1) = \kappa$, $\sigma_3(G_1) = n + \kappa - 1$ and G_1 is not hamiltonian.

In 1980, Bondy showed the following theorem for an edge-dominating cycle.

Theorem 4.4 (Bondy [4]) *Let G be a 2-connected graph on n vertices. If $\sigma_3 \geq n+2$, then each longest cycle of G is an edge-dominating cycle.*

Motivated by the above observations, we extend Theorem 4.4 just as Theorem 4.1 is extended to Theorem 4.2. However, the extension of Theorem 4.4 was already established by Sun, Tian and Wei.

Theorem 4.5 (Sun et al. [17]) *Let G be a 3-connected graph on n vertices. If $\sigma_4 \geq n + 2\kappa$, then G contains a longest cycle which is an edge-dominating cycle.*

Theorem 4.5 is best possible for $\kappa = 3$, but is not best possible for $\kappa \geq 4$ by considering the following graph G_2. Let $3 \leq \kappa$ and $\frac{n+\kappa-2}{4} \leq m \leq \frac{n-2}{3}$. We consider the graph $G_2 = K_m + mK_2 \cup K_{n-3m}$ by joining κ vertices of K_m and each vertex of K_{n-3m}. Then $\kappa(G_2) = \kappa$, $\sigma_4(G_2) = n + \kappa + 2$ and any longest cycle of G_2 is not an edge-dominating cycle. So we prove the following:

Theorem 4.6 *Let G be a 3-connected graph on n vertices. If $\sigma_4 \geq n+\kappa+3$, then G contains a longest cycle which is an edge-dominating cycle.*

The following corollary can be derived easily.

Corollary 4.7 *Let G be a 3-connected graph on n vertices with $\sigma_4 \geq n + \kappa + 3$. If $\alpha \leq \delta$, then G is hamiltonian.*

Corollary 4.7 is a generalization of the following theorem.

Theorem 4.8 (Nikogosyan [15]) *Let G be a 3-connected graph on n vertices. If $\delta \geq \max\{(n+2\kappa)/4, \alpha\}$, then G is hamiltonian.*

Fraisse showed the following theorem for the existence of a D_k-cycle.

Theorem 4.9 (Fraisse [10]) *Let G be a k-connected graph on n vertices, where $n \geq 3$. If $\sigma_{k+1}(G) \geq n + k(k-1)$, then G contains a D_k-cycle.*

In [17], Sun et al. proposed the following question: Replacing the condition $\sigma_{k+1}(G) \geq n + k(k-1)$ of Theorem 4.9 by the stronger condition $\sigma_{k+1}(G) \geq n + (k-1)\kappa$, can we replace k of the conclusion of Theorem 4.9 by $k-1$? However, we propose the following conjecture.

Conjecture 4.10 *Let G be a k-connected graph on n vertices, where $n \geq 3$. If $\sigma_{k+1}(G) \geq n + \kappa + k(k-2)$, then G contains a longest cycle of which is a D_{k-1}-cycle.*

Conjecture 4.10 has been verified for $k = 2$ (Theorem 4.2) and $k = 3$ (Theorem 4.6), and, if it is true, is best possible in a sense. Let $k \leq \kappa$ and $\frac{n+\kappa-k+1}{k+1} \leq m \leq \frac{n-k+1}{k}$. We consider the graph $G_3 = K_m + mK_{k-1} \cup K_{n-km}$ by joining κ vertices of K_m and each vertex of K_{n-km}. Then $\kappa(G_3) = \kappa$, $\sigma_4(G_3) = n + \kappa + k(k-2)$ and any longest cycle of G_k is not a D_{k-1}-cycle.

4.2 Claims

Let G be a graph on n vertices satisfying the hypothesis of Theorem 4.6. Assume that there exists no longest and edge-dominating cycle in G. Choose a cycle C such that

(C1) C is a longest cycle of G, and

(C2) $|E(G-C)|$ is as small as possible, subject to (C1).

Since C is not an edge-dominating cycle, there exists a component H of $G - C$ such that $u, v \in V(H)$. Let $N_C(H) = \{x_1, x_2, \ldots, x_t\}$. We give an orientation to C. Let $u_i = x_i^+$ and $T = N_C(H)^+$. Furthermore, let S be any vertex cut set with $|S| = \kappa$, and $B_1, B_2, \ldots, B_p$ the components of $G - S$, where p is the number of components of $G - S$.

By (C1), we obtain the following claim.

Claim 1 (Sun et al. [17] Claim 2)

(1) *For any $u_i \in T$, $u_i \notin N_C(H)$.*

(2) *For any $u_i, u_j \in T$, there exists no C-path joining u_i and u_j.*

(3) *For any $u_i, u_j \in T$ and $w \in u_i \overrightarrow{C} x_j$, if there exists a C-path joining u_i and w, there exists no C-path joining u_j and w^-.*

By Claims 1 (1) and (2), the following facts are plain.

Fact 1 *$T \cup \{u\}$ is a stable set.*

Fact 2 $\sum_{u_i \in T} d_{G-C}(u_i) \leq n - |C| - |H|$.

Furthermore, by (C2), we obtain the following claim.

Claim 2 (Sun et al. [17] Claim 3)

(1) *For any $u_i \in T$, $u_i^+ \notin N_C(H)$.*

(2) *For any $u_i, u_j \in T$, there exists no C-path joining u_i and u_j^+.*

(3) *For any $u_i, u_j \in T$ and $w \in u_i^+ \overrightarrow{C} x_j$, if there exists a C-path joining u_i and w^+, there exists no C-path joining u_j and w^-.*

Claim 3 *Let $u_i, u_j, u_k \in T$ and $X \subseteq (N_C(u) \cap N_C(v)) - N_C(\{u_i, u_j, u_k\})$. Then $d_G(u_i) + d_G(u_j) + d_G(u_k) \leq n - |X| - |H| + 3$.*

Proof. Without loss of generality, we may assume that u_i, u_j and u_k appear in the consecutive order along $\overrightarrow{C}$. Let $C_1 = u_i\overrightarrow{C}x_j$, $C_2 = u_j\overrightarrow{C}x_k$ and $C_3 = u_k\overrightarrow{C}x_i$. First we show that for $h = 1, 2, 3$,

$$d_{C_h}(u_i) + d_{C_h}(u_j) + d_{C_h}(u_k) \leq |C_h| - |X \cap C_h| + 1.$$

We consider only the case $h = 1$. The case $h = 2, 3$ can be proved similarly. Since $X \subseteq (N_C(u) \cap N_C(v)) - N_C(\{u_i, u_j, u_k\})$, we have $w_1 \notin N_{C_1}(u_i)^{-2} \cup N_{C_1}(u_j) \cup N_{C_1}(u_k)^-$ for any $w_1 \in C_1 \cap X$. Thus $N_{C_1}(u_i)^{-2} \cup N_{C_1}(u_j) \cup N_{C_1}(u_k)^- \subseteq C_1 - (C_1 \cap X) \cup \{x_i\}$. By Claims 1 (3) and 2 (3), $N_{C_1}(u_i)^{-2}$, $N_{C_1}(u_j)$ and $N_{C_1}(u_k)^-$ are pairwise disjoint. Hence we have $d_{C_1}(u_i)+d_{C_1}(u_j)+d_{C_1}(u_k) \leq |C_1| - |C_1 \cap X| + 1$. Thus we obtain

$$d_C(u_i) + d_C(u_j) + d_C(u_k) \leq \sum_{h=1}^{3}(|C_h| - |X \cap C_h| + 1) = |C| - |X| + 3.$$

By Fact 2, we have $d_G(u_i) + d_G(u_j) + d_G(u_k) \leq n - |X| - |H| + 3$. □

Claim 4 $d_C(u) \geq \kappa + 1$.

Proof. Let $u_i, u_j, u_k \in T$. By Claim 3, we obtain $d_G(u_i)+d_G(u_j)+d_G(u_k) \leq n - |H| + 3$ letting $X = \emptyset$. By Fact 1 and the degree condition, we have $d_G(u) \geq \kappa + |H|$. From $d_H(u) = |H| - 1$, we have $d_C(u) \geq \kappa + 1$. □

Claim 5 $|T - S| \geq 3$.

Proof. By Claim 4, we have $|N_C(H)| \geq \kappa+1$. This implies $|N_C(H)^+ - S| + |N_C(H)^- - S| \geq |N_C(H)^+| + |N_C(H)^-| - |S| \geq 2(\kappa + 1) - \kappa = \kappa + 2 \geq 5$. Thus either $|N_C(H)^+ - S| \geq 3$ or $|N_C(H)^- - S| \geq 3$ holds. Without loss of generality, we may assume that $|T - S| = |N_C(H)^+ - S| \geq 3$. □

Let $|B_1 \cap T| \geq |B_2 \cap T| \geq \cdots \geq |B_p \cap T|$.

Claim 6 $T \subset B_1 \cup S$.

Proof. First, we show that $T \subset B_1 \cup B_2 \cup S$. By Claim 5, we may assume that $B_h \cap T \neq \emptyset$ for $h = 1, 2, 3$. By Claims 1 (1), (2) and 2 (2), we have

$$d_G(y_h) \leq |B_h| + |S| - |(B_i \cup S) \cap (H \cup T \cup T^+ - \{y_h^+\})|$$

for $y_h \in B_i \cap T$, $h = 1, 2, 3$. By Claim 4, $|S| + 1 \leq |T|$. Since $N_G(u) \subset T \cup H - \{u\}$, we have $d_G(u) \leq |T| + |H| - 1$. Thus we obtain

$$\begin{aligned} & d_G(y_1) + d_G(y_2) + d_G(y_3) \\ \leq \; & \sum_{h=1}^{3} |B_h| + 3|S| - \sum_{h=1}^{3} |(B_h \cup S) \cap (H \cup T \cup T^+ - \{y_h^+\})| \\ \leq \; & n + 2|S| - \sum_{h=4}^{p} |B_h| - \sum_{h=1}^{3} |(B_h \cup S) \cap (H \cup T \cup T^+ - \{y_h^+\})| \\ \leq \; & n + 2|S| - \big(|H| + |T| + |T^+| - 3\big) \leq n + \kappa - d_G(u) + 1. \end{aligned}$$

Hence $d_G(y_1) + d_G(y_2) + d_G(y_3) + d_G(u) \leq n + \kappa + 1$, a contradiction.

Next, we show that $T \subset B_1 \cup S$. Suppose that $B_h \cap T \neq \emptyset$ for $h = 1, 2$. By Claim 5, suppose that $y_1, y_2 \in B_1 \cap T$ and $y_3 \in B_2 \cap T$. Without loss of generality, we may assume that y_1, y_2 and y_3 appear in the consecutive order along $\overrightarrow{C}$. Let $C_1 = y_1 \overrightarrow{C} y_2^-$, $C_2 = y_2 \overrightarrow{C} y_3^-$, $C_3 = y_3 \overrightarrow{C} y_1^-$, $W_h = B_2 \cap C_h$ and $T_h = T \cap C_h$. By Claims 1 and 2, we obtain the following statement.

(i) $N_{C_1}(y_1)^-$ and $N_{C_1}(y_2)$ are disjoint, and $N_{C_1}(y_1)^- \cup N_{C_1}(y_2) \subseteq (C_1 - (W_1 \cup (T_1 - \{y_1\})) \cup C_1'$, where $C_1' = \{w \in W_1 \colon w^+ \in N_{C_1}(y_1)\}$.

(ii) $N_{C_2}(y_2)^-$ and $N_{C_2}(y_1)$ are disjoint, and $N_{C_2}(y_2)^- \cup N_{C_2}(y_1) \subseteq (C_2 - (W_2 \cup (T_2 - \{y_2\})) \cup C_2'$, where $C_2' = \{w \in W_2 \colon w^+ \in N_{C_2}(y_2)\}$.

(iii) $N_{C_3}(y_1)^+$ and $N_{C_3}(y_2)$ are disjoint, and $N_{C_3}(y_1)^+ \cup N_{C_3}(y_2) \subseteq (C_3 \cup \{y_1\} - (W_3 \cup T_3^+)) \cup C_3'$, where $C_3' = \{w \in W_3 \colon w^- \in N_{C_3}(y_1)\}$.

By (i)-(iii), the following inequality holds:

$$\begin{aligned} d_C(y_1)+d_C(y_2) &\leq |C\cup\{y_1\}|-|B_2\cap C|-|T_1\cup T_2\cup T_3^+-\{y_1,y_2\}| \\ &\qquad +|B_2\cap(T_1\cup T_2\cup T_3^+)|+\sum_{h=1}^{3}|C'_h| \\ &\leq |C|-|B_2\cap C|-|T|+3 \\ &\qquad +|B_2\cap(T_1\cup T_2\cup T_3^+)|+\sum_{h=1}^{3}|C'_h|. \end{aligned}$$

By Claim 1 (3), we have $N_C(y_3)\cap C'_h=\emptyset$ for $h=1,2,3$. By the definition of C'_h and Claims 1 (2) and 2 (2), we have $\bigcup_{h=1}^{3} C'_h \subseteq B_2\cap C-(T_1\cup T_2\cup {T_3}^+)$. This implies

$$d_C(y_3)\leq |B_2\cap C|+|S\cap C|-|B_2\cap(T_1\cup T_2\cup T_3^+)|-\sum_{h=1}^{3}|C'_h|.$$

Thus we obtain

$$d_C(y_1)+d_C(y_2)+d_C(y_3)\leq |C|+|S\cap C|-|T|+3\leq |C|+\kappa-|T|+3.$$

By Fact 2, we have

$$d_G(y_1)+d_G(y_2)+d_G(y_3)\leq n+\kappa-|T|-|H|+3\leq n+\kappa-(d_G(u)+1)+3.$$

Therefore $d_G(y_1)+d_G(y_2)+d_G(y_3)+d_G(u)\leq n+\kappa+2$, a contradiction. □

Claim 7 $V(H)\cap B_2=\emptyset$.

Proof. Suppose that $u\in V(H)\cap B_2$. By Claim 6, we have $N_C(u)^+\subseteq B_1\cup S$. Hence $(N_C(u)^+\cap B_1)^-\cup(N_C(u)^+\cap S)\subseteq S$. By Claim 4, we have $\kappa+1\leq |N_C(u)^+|\leq |S|=\kappa$, a contradiction. □

4.3 Proof of Theorem 4.6

Suppose that Theorem 4.6 fails to hold, then none of the longest cycles in G is an edge-dominating cycle. Let C and H be chosen as in Section 2. Let $N_C(H) = \{x_1, x_2, \dots, x_t\}$. We give an orientation to C. Let $u_i = x_i^+$ and $T = N_C(H)^+$.

Let S be any vertex cut set with $|S| = \kappa$, and $B_1, B_2, \dots, B_p$ the components of $G - S$, where $p \geq 2$ is the number of components of $G - S$. By Claims 5 and 6, we may assume that $u_i, u_j \in B_1 \cap T$ and $\bigcup_{h=2}^{p} B_h \cap T = \emptyset$. Let $C_1 = u_i \overrightarrow{C} x_j$, $C_2 = u_j \overrightarrow{C} x_i$, $W_h = B_2 \cap C_h$, $T_h = T \cap C_h$, $D_1 = \{w \in W_1 \colon w^+ \in N_{C_1}(u_i)\}$ and $D_2 = \{w \in W_2 \colon w^+ \in N_{C_2}(u_j)\}$. By Claims 1 and 2, we obtain

(i) $N_{C_1}(u_i)^-$, $N_{C_1}(u_j)$ are disjoint and $N_{C_1}(u_i)^- \cup N_{C_1}(u_j) \subseteq C_1 - (W_1 \cup T_1 - \{u_i\}) \cup D_1$.

(ii) $N_{C_2}(u_j)^-$, $N_{C_2}(u_i)$ are disjoint and $N_{C_2}(u_j)^- \cup N_{C_2}(u_i) \subseteq C_2 - (W_2 \cup T_2 - \{u_j\}) \cup D_2$.

By (i) and (ii), we have $d_C(u_i) + d_C(u_j) \leq |C| - |B_2 \cap C| - |T| + 2 + \sum_{h=1}^{2} |D_h|$. On the other hand, by Fact 2 and $u_i, u_j \in B_1$, we have $d_{G-C}(u_i) + d_{G-C}(u_j) \leq n - |C| - |H| - |B_2 \cap (G - C)|$. Thus we obtain

$$\begin{aligned} d_G(u_i) + d_G(u_j) &\leq n - |B_2| - |T| - |H| + \sum_{h=1}^{2} |D_h| + 2 \\ &\leq n - |B_2| - (d_G(u) + 1) + \sum_{h=1}^{2} |D_h| + 2, \end{aligned}$$

and so

$$d_G(u_i) + d_G(u_j) + d_G(u) \leq n - |B_2| + \sum_{h=1}^{2} |D_h| + 1. \tag{4.1}$$

We consider two cases.

Case 1. $B_2 - (N_G(u) \cap N_G(v)) \neq \emptyset$.

We divide into three subcases.

Case 1.1. $D_h = \emptyset$ for any $h = 1, 2$.

By the inequality (4.1), we have $d_G(u_i) + d_G(u_j) + d_G(u) \leq n - |B_2| + 1$. Without loss of generality, we may assume $y \in B_2 - N_G(u)$. By Fact 1 and $y \in B_2$, $\{u_i, u_j, u, y\}$ is a stable set. Since $d_G(y) \leq |B_2| + |S| - 1$, we have $d_G(u_i) + d_G(u_j) + d_G(u) + d_G(y) \leq n + |S| \leq n + \kappa$, a contradiction. $\square$

Case 1.2. $D_h \neq \emptyset$ for any $h = 1, 2$.

Without loss of generality, we may assume that $|D_1| \geq |D_2|$. For each vertex $x \in V(C)$, we define x^* as follows: if $N_{G-C}(x) \neq \emptyset$ then $x^* = w$ for an arbitrary vertex $w \in N_{G-C}(x)$; otherwise $x^* = x^-$. We define $D_h^* := \{x^* \colon x \in D_h\}$. Let $y_2 \in D_2$. By Claim 1 (1), we have $y_2 \notin N_G(u)$. Hence $\{u_i, u_j, u, y_2\}$ is a stable set.

Claim 8 $N_C(y_2) \subset B_2 \cup S - (D_1 \cup D_1^*)$.

We shall show $(D_1 \cup D_1^*) \cap N_C(y_2) = \emptyset$. Let $x_1 P_H x_2$ be a path in H. First, we show that $D_1 \cap N_G(y_2) = \emptyset$. If $y_1 \in D_1 \cap N_G(y_2)$, then $x_1 \overleftarrow{C} y_2^+ u_j \overrightarrow{C} y_2 y_1 \overleftarrow{C} u_i y_1^+ \overrightarrow{C} x_2 P_H x_1$ is longer than C, contradicting (C1). Next, we show that $D_1^* \cap N_G(y_2) = \emptyset$. Suppose that $y_1 \in D_1^* \cap N_G(y_2)$. If $y_1 \in V(G - C)$, then the cycle $x_1 \overleftarrow{C} y_2^+ u_j \overrightarrow{C} y_2 y_1 \overleftarrow{C} u_i y_1^+ \overrightarrow{C} x_2 P_H x_1$ is longer than C, a contradiction. If $y_1 \in V(C)$, then $C' = x_1 \overleftarrow{C} y_2^+ u_j \overrightarrow{C} y_2 y_1 \overleftarrow{C} u_i y_1^{+2} \overrightarrow{C} x_2 P_H x_1$ is a cycle with $|V(C')| = |V(C)|$. Since $V(C) - V(C') = \{y_1^+\}$ and $N_{G-C}(y_1^+) = \emptyset$, we have $|E(G - C')| < |E(G - C)|$. This contradicts (C2). Thus we have $(D_1 \cup D_1^*) \cap N_C(y_2) = \emptyset$. $\square$

By the definition of D_h^*, we have $D_h \cap D_h^* = \emptyset$ and $|D_h| = |D_h^*|$. Furthermore, since $D_h \subset B_2$, we get $D_h^* \subset B_2 \cup S$. Thus, by Claim 8, we have

$$\begin{aligned} d_G(y_2) &\leq |B_2| + |S| - (|D_1 \cup D_1^*|) \\ &\leq |B_2| + |S| - 2|D_1| \\ &\leq |B_2| + |S| - \sum_{h=1}^{2} |D_h|. \end{aligned}$$

Hence, by the inequality (4.1), we obtain $d_G(u_i) + d_G(u_j) + d_G(u) + d_G(y_2) \leq n + |S| + 1 \leq n + \kappa + 1$, a contradiction.

Case 1.3. $D_h \neq \emptyset$ and $D_{3-h} = \emptyset$ for $h = 1, 2$.

By symmetry, we may assume that $D_1 \neq \emptyset$ and $D_2 = \emptyset$. We define

$$\tilde{D}_1 = \{w \in V(C - B_2) \colon y \in D_1, V(w^+ \overrightarrow{C} y) \subseteq B_2\}.$$

Note that $|D_1| = |\tilde{D}_1|$, since $(T \cup D_1^+) \cap B_2 = \emptyset$. Since $\tilde{D}_1^+ \subseteq B_2$, we have $\tilde{D}_1 \cap N_C(u_i)^- = \emptyset$. This implies $(\tilde{D}_1 - N_C(u_j)) \cap (N_{C_1}(u_i)^- \cup N_{C_1}(u_j)) = \emptyset$. Thus, since $N_C(u_j) \cap T = \emptyset$, we obtain

$$\begin{aligned} d_C(u_i) + d_C(u_j) &\leq |C| - |B_2| - |T| + 2 + |D_1| - |\tilde{D}_1 - N_C(u_j)| + |\tilde{D}_1 \cap T| \\ &= |C| - |B_2| - |T| + 2 + |\tilde{D}_1 \cap N_C(u_j)| + |\tilde{D}_1 \cap T| \\ &= |C| - |B_2| - |T| + 2 + |\tilde{D}_1 \cap (N_C(u_j) \cup T)|. \end{aligned}$$

Let $T' = \{u_h \in T \colon N_{G-C}(u_h) = \emptyset,\ 1 \leq h \leq t\}$. By Claims 1 (1) (2), we have $N_{G-C}(T - T') \cap (V(H) \cup N_{G-C}(u_i) \cup N_{G-C}(u_j)) = \emptyset$. Thus, by Fact 2, we have

$$d_{G-C}(u_i) + d_{G-C}(u_j) \leq n - |C| - |H| - |N_{G-C}(\tilde{D}_1 \cap (T - T'))|.$$

Let $y_1, y_2 \in T - T'$. By Claim 1 (2), $N_{G-C}(y_1) \cap N_{G-C}(y_2) = \emptyset$. Hence we have $|N_{G-C}(T-T')| \geq |T-T'|$. Thus we obtain

$$\begin{aligned} & d_G(u_i)+d_G(u_j)+d_G(u) \\ \leq\ & n-|B_2|+1+|\tilde{D}_1 \cap (N_C(u_j)\cup T)| - |N_{G-C}(\tilde{D}_1 \cap (T-T'))| \\ \leq\ & n-|B_2|+1+|\tilde{D}_1 \cap (N_C(u_j)\cup T)| - |\tilde{D}_1 \cap (T-T')| \\ =\ & n-|B_2|+1+|\tilde{D}_1 \cap (N_C(u_j)\cup T')|. \end{aligned} \tag{4.2}$$

Suppose that $y \in B_2 - N_G(u)$. Since $d_G(y) \leq |B_2|+|S|-1$, we have $d_G(u_i)+d_G(u_j)+d_G(u)+d_G(y) \leq n+\kappa+|\tilde{D}_1 \cap (N_C(u_j)\cup T')|$. By Fact 1 and $y \in B_2$, $\{u_i, u_j, u, y\}$ is a stable set. Thus, by the degree condition, we have $|\tilde{D}_1 \cap (N_C(u_j)\cup T')| \geq 3$, that is, $\tilde{D}_1 \cap (N_C(u_j)\cup T') \neq \emptyset$. Choose $z \in \tilde{D}_1 \cap (N_C(u_j)\cup T')$ such that $|u_i \overrightarrow{C} z|$ is as large as possible.

Claim 9 $\{u_i, u_j, u, z^+\}$ *is a stable set.*

Proof. Suppose that $u \in N_G(z^+)$. Then $z^{+2} \in T$. By Claim 6, we have $z^{+2} \in B_1 \cup S$. This contradicts the definition of $\tilde{D}_1$. Hence we have $u \notin N_G(z^+)$. On the other hand, since $u_i, u_j \in B_1$ and $z^+ \in B_2$, we obtain $u_i, u_j \notin N_G(z^+)$. Thus, by Fact 1, $\{u_i, u_j, u, z^+\}$ is a stable set. □

Claim 10 $N_C(z^+) \subset (B_2 - \{z^+\}) \cup S - (D_1 \cap u_i \overrightarrow{C} z)$.

Proof. Suppose that $y \in N_C(z^+) \cap (D_1 \cap u_i \overrightarrow{C} z)$. By the definition of D_1, we have $y^+ \in N_C(u_i)$. If $z \in N_C(u_j)$, then the cycle $x_i \overleftarrow{C} u_j z \overleftarrow{C} y^+ u_i \overrightarrow{C} z^+ \overrightarrow{C} x_j P x_i$ is longer than C, contradicting (C1). If $z \in T'$, then $C' = x_i \overleftarrow{C} z^+ y \overleftarrow{C} u_i y^+ \overrightarrow{C} z^-$ $P x_i$ is a cycle with $|V(C')| = |V(C)|$ and $|E(G-C')| < |E(G-C)|$, since $N_{G-C}(z) = \emptyset$. This contradicts (C2). Thus we have $N_C(z^+) \cap (D_1 \cap u_i \overrightarrow{C} z) \neq \emptyset$. □

By the choice of z, we obtain $|D_1 \cap u_i \overrightarrow{C} z| \geq |\tilde{D}_1 \cap (N_C(u_j)\cup T')| - 1$. Thus, by Claim 10, we have $d_G(z^+) \leq |B_2| - 1 + |S| - |D_1 \cap u_i \overrightarrow{C} z| \leq$

$|B_2| + |S| - |\tilde{D}_1 \cap (N_C(u_j) \cup T')|$. Hence, by the inequality (4.2), we obtain

$$d_G(u_i) + d_G(u_j) + d_G(u) + d_G(z^+) \le n + |S| + 1 \le n + \kappa + 1,$$

a contradiction. This completes the proof of Case 1.

Case 2. $B_2 - (N_G(u) \cap N_G(v) = \emptyset$.

By Claims 5 and 6, $|B_1 \cap T| \ge 3$. Let $u_i, u_j, u_k \in B_1 \cap T$. By Claim 7, $B_2 \subset (N_C(u) \cap N_C(v)) - N_C(\{u_i, u_j, u_k\})$. Hence we obtain $d_G(u_i) + d_G(u_j) + d_G(u_k) \le n - |B_2| - |H| + 3$ letting $X = B_2$ in Claim 3. Let $y \in B_2$. Since $d_G(y) \le |B_2| + |S| - 1$, we have $d_G(u_i) + d_G(u_j) + d_G(u_k) + d_G(y) \le n + \kappa - |H| + 2 \le n + \kappa$, a contradiction. This completes the proof of Case 2 and the proof of Theorem 4.6. □

Chapter 5

Vertex-dominating cycles in tough graphs

5.1 Introduction

There are a number of sufficient conditions for a graph to have a hamiltonian cycle. Since a vertex-dominating cycle is a generalization of a hamiltonian cycle, there may be sufficient conditions for a graph to have a vertex-dominating cycle, which correspond to known sufficient conditions for hamiltonicity. One such example was obtained by Bondy and Fan [5]. Bondy [4] proved the following theorem concerning degree sums and hamiltonicity.

Theorem 5.1 (Bondy [4]) *Let k be a positive integer, and let G be a k-connected graph. If $\sum_{x \in S} d_G(x) > \frac{1}{2}(p-1)(k+1)$ for every stable set S in G of order $k+1$, then G has a hamiltonian cycle.*

In 1987, Bondy and Fan proved that a much weaker degree sum condition guarantees the existence of a vertex-dominating cycle.

Theorem 5.2 (Bondy&Fan [5]) *Let $k \geq 2$ and let G be a k-connected graph of order p. If $\sum_{x \in S} d_G(x) \geq p - 2k$ for every 3-stable set S of G of* *order $k+1$, then G has a vertex-dominating cycle.*

In Theorem 5.1 and Theorem 5.2, the degree sum conditions are sharp and cannot be relaxed. For example, for Theorem 5.1, $G = K_k + (k+1)K_1$ satisfies $\sum_{x \in S} d_G(x) = \frac{1}{2}(|V(G)|-1)(k+1)$ for the unique stable set S of order $k+1$ (which is the largest stable set), but G is not hamiltonian. For Theorem 5.2, Bondy and Fan gave the following example. Let $H = K_k + (k+1)K_l$, where $l \geq k$. Then for each K_l, we introduce a new vertex and join each new vertex and every vertex in the corresponding K_l by an edge. Let G be the resulting graph. Then $|V(G)| = k + (k+1)(l+1)$ and G is k-connected. Let $S = V(G) - V(H)$. Since each vertex in S has neighbors only in the corresponding K_l, S is a 3-stable set (actually, S is a 4-stable set). Since each vertex in S has degree l, $\sum_{x \in S} d_G(x) = (k+1)l = |V(G)| - 2k - 1$. However, G has no vertex-dominating cycle.

Note that both sharpness examples in the previous paragraph have toughness $k/(k+1)$. Actually, for several sufficient conditions for hamiltonicity, it is observed that weaker conditions guarantee the existence of a hamiltonian cycle for graphs with large toughness. For example, the case $k = 1$ of Theorem 5.1, known as Ore's theorem, admits a weaker degree sum condition for 1-tough graphs.

Theorem 5.3 (Ore [16]) *Every graph G of order $p \geq 3$ which satisfies $d_G(x) + d_G(y) \geq p$ for every pair of distinct nonadjacent vertices x and y has a hamiltonian cycle.*

Theorem 5.4 (Jung [12]) *Every 1-tough graph of order $p \geq 11$ which satisfies $d_G(x) + d_G(y) \geq p - 4$ for every pair of distinct nonadjacent vertices x and y has a hamiltonian cycle.*

Like a hamiltonian cycle, some sufficient conditions for the existence of an edge-dominating cycle can be relaxed if we put a further assumption on toughness. One such example is the following.

Theorem 5.5 (Bondy [4]) *Let G be a 2-connected graph of order p. If $\sum_{x \in S} d_G(x) \geq p+2$ for every stable subset S of $V(G)$ of order 3, then any longest cycle of G is an edge-dominating cycle.*

Theorem 5.6 (Bauer et al. [3]) *Let G be a 1-tough graph of order p. If $\sum_{x \in S} d_G(x) \geq p$ for every stable subset S of $V(G)$ of order 3, then any longest cycle of G is an edge-dominating cycle.*

The purpose of this paper is to show that Theorem 5.2 also admits a similar relaxation under an additional assumption on toughness.

Theorem 5.7 *Let $k \geq 2$, and let G be a k-connected graph of order p with $t(G) > k/(k+1)$. If $\sum_{x \in S} d_G(x) \geq p - 2k - 2$ for every 4-stable subset S of $V(G)$ of order $k+1$, then G has a vertex-dominating cycle.*

The example of sharpness for Theorem 5.2 shows that the toughness $k/(k+1)$ in Theorem 5.7 is sharp. On the other hand, we know the sharpness of degree condition only for $k = 2$. Let $m \geq 3$ and take three complete graphs H_1, H_2 and H_3 of the same order m. Take three distinct vertices x_i, y_i and u_i in each H_i ($1 \leq i \leq 3$). Let K_3 be a complete graph of order three with $V(K_3) = \{v_1, v_2, v_3\}$. Let v_0 be a new vertex. Then construct G by

$$V(G) = V(H_1) \cup V(H_2) \cup V(H_3) \cup V(K_3) \cup \{v_0\}, \quad \text{and}$$
$$E(G) = E(H_1) \cup E(H_2) \cup E(H_3) \cup E(K_3) \cup \{v_0y_1, v_0y_2, v_0y_3, v_1x_1, v_2x_2, v_3x_3\}$$

Then G is a 2-connected graph with $t(G) = 1 > \frac{2}{3}$ and $\sum_{i=1}^{3} d_G(u_i) \geq |G| - 7$ for the 4-stable set S of order three, but G has no vertex-dominating cycle. This example shows that even under a stronger condition $t(G) \geq 1$, we cannot relax the condition on the degree sum. For $k \geq 3$, we do not know whether the degree bound $p - 2k - 2$ is best possible.

5.2 Proof of Theorem 1

To prove Theorem 5.7, we use two known results. Chvátal-Erdős [7] showed that for $k \geq 2$, a k-connected graph G with independence number at most k and order at least three is hamiltonian. Broersma [6] proved a Chvátal-Erdős type theorem for the existence of a vertex-dominating cycle.

Theorem 5.8 (Broersma [6]) *Let G be a k-connected graph ($k \geq 2$). If $\hat{\alpha}_4(G) \leq k$, then G has a vertex-dominating cycle.*

We also use a degree sum condition for a balanced bipartite graph to be hamiltonian.

Theorem 5.9 (Moon&Moser [13]) *Let G be a balanced bipartite graph of order $2n$ with partite sets X and Y ($|X| = |Y| = n$). If $d_G(x) + d_G(y) \geq n+1$ for each $x \in X$ and $y \in Y$ with $xy \notin E(G)$, then G has a hamiltonian cycle.*

Proof of Theorem 5.7. Assume that G has no vertex-dominating cycle. By Theorem 5.8, we have $\hat{\alpha}_4(G) > k$. Let $S = \{x_0, x_1, \ldots, x_k\}$ be a 4-stable set in $V(G)$, and let $B_i = \{x_i\} \cup N_G(x_i)$ ($0 \leq i \leq k$). For each i, j with $0 \leq i < j \leq k$, we have $B_i \cap B_j = \emptyset$ and $e_G(B_i, B_j) = 0$ since $d_G(x_i, x_j) \geq 4$, where $e_G(X, Y)$ is the number of edges which have one endvertex in X and the other in Y. Let $R = V(G) - \bigcup_{i=0}^{k} B_i$. Then $G - R$ is disconnected, and the components of $G - R$ are $B_0, \ldots, B_k$. Since $t(G) > k/(k+1)$, $|R| \geq k+1$. On the other hand, since $N_G(S) \cap (R \cup \{x_0, \ldots, x_k\}) = \emptyset$, $p - 2k - 2 \leq \sum_{i=0}^{k} d_G(x_i) = |N_G(S)| = p - |R| - (k+1)$, which implies $|R| \leq k+1$. Therefore, we have $|R| = k+1$. Let $R = \{y_0, y_1, \ldots, y_k\}$.

Now contract each B_i into a single vertex b_i ($0 \leq i \leq k$) and remove all the edges in $G[R]$. Let H be the resulting graph. Then H is a balanced bipartite graph with partite sets $\{b_0, \ldots, b_k\}$ and R. Since G is k-connected, $|N_G(B_i) \cap R| \geq k$. This implies $d_H(b_i) \geq k$. If $d_H(y_i) \leq 1$ for some i,

$0 \le i \le k$, then $N_G(y_i) \subset R \cup B_j$ for some j with $0 \le j \le k$. Without loss of generality, we may assume $j = 0$. Let $R' = R - \{y_i\}$. Then $G - R'$ has $k+1$ components $B_0 \cup \{y_i\}$, $B_1, \dots, B_k$. Since $|R'| = k$, this contradicts the assumption that $t(G) > k/(k+1)$. Thus, we have $d_H(y_i) \ge 2$ for each y_i $(0 \le i \le k)$.

Now since we have $d_H(b_i) + d_H(y_j) \ge k+2$ for each i, j with $1 \le i, j \le k$, H has a hamiltonian cycle C' by Theorem 5.9. Without loss of generality, we may assume $C' = y_0 b_0 y_1 b_1 \dots y_k b_k y_0$. For each i, $0 \le i \le k$, and each $y_i b_i$ and $b_i y_{i+1}$, there exist vertices $z_i^{(1)}$, $z_i^{(2)} \in B_i$ with $y_i z_i^{(1)}$, $z_i^{(2)} y_{i+1} \in E(G)$ (indices modulo k). If $z_i^{(1)} \ne z_i^{(2)}$, let $P_i = y_i z_i^{(1)} x_i z_i^{(2)} y_{i+1}$. If $z_i^{(1)} = z_i^{(2)}$, let $P_i = y_i z_i^{(1)} y_{i+1}$. Then $C = y_0 P_0 y_1 P_1 \dots y_k P_k y_0$ is a cycle in G.

We claim that C is a vertex-dominating cycle of G. Assume, to the contrary, that there exists a vertex v in G which is not dominated by C. Since $R \subset V(C)$ and $N_G(x_i) \cap V(C) \ne \emptyset$ for each i, $0 \le i \le k$, every vertex in $R \cup \{x_0, \dots x_k\}$ is dominated by C. Therefore, $v \in N_G(x_{i_0})$ for some i_0, $0 \le i_0 \le k$. Since v is not dominated by C, $x_{i_0} \notin V(C)$ and hence $z_{i_0}^{(1)} = z_{i_0}^{(2)}$. Since $R \cup \{z_{i_0}^{(1)}\} \subset V(C)$, $N_G(v) \cap (R \cup \{z_{i_0}^{(1)}\}) = \emptyset$. This implies $N_G(v) \subset B_{i_0} - \{z_{i_0}^{(1)}\}$ and, if $i \ne i_0$, $d_G(v, x_i) \ge 4$. Let $S' = (S - \{x_{i_0}\}) \cup \{v\}$. Then S' is a 4-stable set of G of order $k+1$ and since $N_G(S') \cap (R \cup S' \cup \{z_{i_0}^{(1)}\}) = \emptyset$, we have $\sum_{x \in S'} d_G(x) \le p - |R| - (k+1) - 1 = p - 2k - 3$. This contradicts the assumption, and hence G has a vertex-dominating cycle C of G. $\square$

Chapter 6

Vertex-dominating cycles in 2-connected bipartite graphs

6.1 Introduction

In 1953, Dirac studied that the minimum degree and hamiltonian cycles.

Theorem 6.1 (Dirac [8]) *Let G be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then G is hamiltonian.*

In 1960, Ore showed the following theorem, which is a generalization Theorem 6.1.

Theorem 6.2 (Ore [16]) *Let G be a graph on $n \geq 3$ vertices. If $d_G(x) + d_G(y) \geq n$ for any nonadjacent vertices x and y, then G is hamiltonian.*

In 1963, Moon and Moser gave a degree sum condition for a bipartite graph to be hamiltonian.

Theorem 6.3 (Moon&Moser [13]) *Let G be a bipartite graph with partite sets V_1 and V_2, where $|V_1| = |V_2| = n$. If $d_G(x) + d_G(y) \geq n + 1$ for each pair of nonadjacent vertices $x \in V_1$ and $y \in V_2$, then G is hamiltonian.*

In 1971, Nash-Williams showed the following result for an edge-dominating cycle.

Theorem 6.4 (Nash-Williams [14]) *Let G be a 2-connected graph on n vertices. If $\delta(G) \geq (n-2)/3$, then any longest cycle of G is an edge-dominating cycle.*

In 1984, Ash and Jackson showed the following theorem for a bipartite graph to have an edge-dominating cycle.

Theorem 6.5 (Ash&Jackson [1]) *Let G be a 2-connected bipartite graph with partite sets V_1 and V_2, where $\max(|V_1|, |V_2|) = n$. If $\delta(G) \geq (n-1)/3$, then there exists a longest cycle which is an edge-dominating cycle.*

In 1987, Bondy and Fan [5] proved the following theorem for a vertex-dominating cycle.

Theorem 6.6 (Bondy&Fan [5]) *Let $k \geq 2$ and let G be a k-connected graph on n vertices. If $\sum_{x \in S} d_G(x) \geq n - 2k$ for every 3-stable set S of G of order $k+1$, then G has a vertex-dominating cycle.*

Theorem 6.6 implies the following corollary.

Corollary 6.7 *Let G be a 2-connected graph on n vertices. If $\delta(G) \geq (n-4)/3$, then G has a vertex-dominating cycle.*

In this paper, we give a minimum degree condition for a bipartite graph to have a vertex-dominating cycle.

Theorem 6.8 *Let G be a 2-connected bipartite graph with partite sets V_1 and V_2, where $\max(|V_1|, |V_2|) = n$. If $\delta(G) \geq (n+1)/3$, then G has a vertex-dominating cycle.*

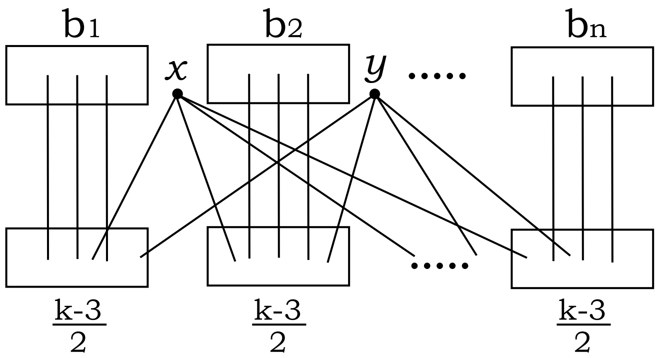


Figure 6.1: $\mathcal{F}_k$

In Theorem 6.8, the degree condition is sharp in the following sense. By adding new two vertices v_1 and v_2 to the graph $K_{l,m} \cup K_{l,n} \cup K_{l,n}$, where $(l, m, n) = (k+1, k, k), (k+1, k+1, k), (k+2, k, k), (k+2, k+1, k), (k+2, k, k+1), (k+2, k+2, k)$, and joining both v_1 and v_2 with every vertex of three partite sets of order l, we obtain a 2-connected bipartite graph G. Let V_1 and V_2 be partite sets of G. It is easy to see that $\delta(G) = \max(|V_1|, |V_2|)/3$. However, since any cycle does not contain all vertices of at least one of three complete bipartite subgraphs $K_{l,m}$, $K_{l,n}$ and $K_{l,n}$, G has no vertex-dominating cycle.

6.2 Proof of Theorem 6.8

We define the following sets $\mathcal{F}_k$ and $\mathcal{H}_k$ of graphs for each odd integer $k \geq 5$. Let $n, b_1, b_2, \ldots, b_n$ be integers with $n \geq 3$ and $b_i \geq (k+1)/2$ $(1 \leq i \leq n)$. Let $\bigcup_{i=1}^{n} K_{(k-3)/2, b_i}$ denote the vertex-disjoint union of $K_{(k-3)/2, b_i}$ for all $i \in \{1, 2, \ldots, n\}$. Then the graph $F_{k, b_1, \ldots, b_n}$ is obtained from $\bigcup_{i=1}^{n} K_{(k-3)/2, b_i}$ by adding two new vertices x and y, and joining both x and y with every vertex of $\bigcup_{i=1}^{n} K_{(k-3)/2, b_i}$ whose degree in $\bigcup_{i=1}^{n} K_{(k-3)/2, b_i}$ is $(k-3)/2$. Let $\mathcal{F}_k$ be the set of all such graphs (Figure 6.1).

To define $\mathcal{H}_k$, let m, $c_1, \ldots, c_m$ be integers at least $(k+1)/2$. The

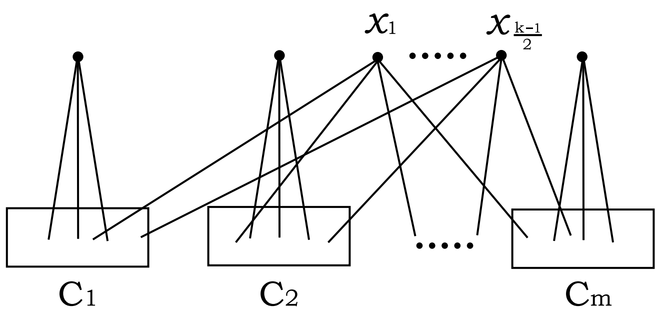


Figure 6.2: $\mathcal{H}_k$

graph $H_{k,c_1,\dots,c_m}$ is obtained from $\bigcup_{i=1}^{m} K_{1,c_i}$ by adding $(k-1)/2$ new vertices $x_1, \dots, x_{(k-1)/2}$, and joining each x_i with every endvertex of $\bigcup_{i=1}^{m} K_{1,c_i}$. Let $\mathcal{H}_k$ be the set of all such graphs (Figure 6.2.).

To prove Theorem 6.8, we use the following result due to Wang.

Theorem 6.9 (Wang [19]) *Let k be an integer with $k \geq 2$. Let G be a 2-connected bipartite graph with partite sets V_1 and V_2, where $\min(|V_1|, |V_2|) = a$. If $d_G(x) + d_G(y) \geq k+1$ for every pair of non-adjacent vertices x and y, then G contains a cycle of length at least $\min(2a, 2k)$, unless $a \geq k \geq 5$, k is odd and $G \in \mathcal{F}_k \cup \mathcal{H}_k$.*

Proof of Theorem 6.8.

Let $|V_1| = n_1$, $|V_2| = n_2$ and $n_1 \leq n_2$. Let G be a graph satisfying the hypothesis, and let C a longest cycle in G.

Claim 1 $|V(C)| \geq \min\left(2n_1, \frac{2}{3}(2n_2-1)\right)$.

Proof. First, suppose that $G \in \mathcal{F}_k$. Since $\delta(G) = \frac{1}{2}(k+1)$ and $n \geq 3$, we have

$$\frac{1}{3}(n_2+1) = \frac{1}{3}\left(\sum_{i=1}^{n} b_i + 1\right) \geq \frac{1}{3}\left(\frac{n(k+1)}{2} + 1\right) = \frac{n}{3}\left(\delta + \frac{1}{n}\right) > \delta.$$

This contradicts the degree condition. Hence $G \notin \mathcal{F}_k$. Next, suppose that $G \in \mathcal{H}_k$. Since $\delta(G) = \frac{1}{2}(k+1)$ and $m \geq 3$, we have

$$\frac{1}{3}(n_2+1) = \frac{1}{3}\left(\sum_{i=1}^{m} c_i + 1\right) \geq \frac{1}{3}\left(\frac{m(k+1)}{2} + 1\right) = \frac{m}{3}\left(\delta + \frac{1}{m}\right) > \delta,$$

contradiction. Therefore $G \notin \mathcal{H}_k$.

Thus, since $\delta(G) \geq \frac{1}{3}(n_2+1)$, we have $d_G(x) + d_G(y) \geq \frac{2}{3}(n_2+1) = \frac{1}{3}(2n_2-1)+1$. By Theorem 6.9, we obtain $|V(C)| \geq \min\left(2n_1, \frac{2}{3}(2n_2-1)\right)$, since $G \notin \mathcal{F}_k \cup \mathcal{H}_k$. $\square$

If $\min\left(2n_1, \frac{2}{3}(2n_2-1)\right) = 2n_1$, then $V_1 \subset V(C)$. This implies $N_C(v_2) \neq \emptyset$ for any $v_2 \in V_2 - V(C)$. Hence C is a vertex-dominating cycle. Thus we may assume $\min\left(2n_1, \frac{2}{3}(2n_2-1)\right) = \frac{2}{3}(2n_2-1)$. Therefore we obtain $|V_1 - V(C)| \leq |V_2 - V(C)| \leq n_2 - \frac{2}{3}(2n_2-1) \leq \frac{1}{3}(n_2+1)$. If $|V_2 - V(C)| < \frac{1}{3}(n_2+1)$, then we have $N_C(v_i) \neq \emptyset$ for any $v_i \in V_i - V(C)$ for $i = 1, 2$, since $\delta(G) \geq \frac{1}{3}(n_2+1)$. Hence C is a vertex-dominating cycle. Thus we may assume that

$$|V_1 - V(C)| \leq |V_2 - V(C)| = \frac{1}{3}(n_2+1) \text{ and } |V(C)| = \frac{2}{3}(2n_2-1). \quad (6.1)$$

Note that $\frac{1}{3}(n_2+1)$ is an integer. For $i = 1, 2$, we define as follows:

$$\begin{aligned} X_i &:= \{x_i \in V_i - V(C) \colon N_C(x_i) \neq \emptyset, N_{G-C}(x_i) \neq \emptyset\}, \\ Y_i &:= \{y_i \in V_i - V(C) \colon N_{G-C}(y_i) = \emptyset\} \text{ and} \\ Z_i &:= \{z_i \in V_i - V(C) \colon N_C(z_i) = \emptyset\}. \end{aligned}$$

Claim 2 $|N_C(X_2)| \geq |Y_1|$.

Proof. By the definition of X_i, Y_i and Z_i and (6.1), we have $|N_C(x_2)| \geq \delta(G) - (|X_1| + |Z_1|) \geq \frac{1}{3}(n_2+1) - \left(\frac{1}{3}(n_2+1) - |Y_1|\right) = |Y_1|$ for any $x_2 \in X_2$. $\square$

Claim 3 *If $Z_i \neq \emptyset$, then $N_G(z_i) = V_{3-i} - V(C)$ for any $z_i \in Z_i$ and $Y_{3-i} = \emptyset$. In particular, if $Z_2 \neq \emptyset$ then $|V_1| = |V_2|$.*

Proof. Suppose that $Z_i \neq \emptyset$, say $z_i \in Z_i$. By the degree condition and the definition of Z_i, we have $\frac{1}{3}(n_2+1) \leq d_G(z_i) \leq |V_{3-i} - V(C)|$. Thus, by (6.1), we obtain $d_G(z_i) = |V_{3-i} - V(C)|$, that is, $N_G(z_i) = V_{3-i} - V(C)$. Furthermore, if $Z_2 \neq \emptyset$ then we have $|V_1| = |V_2|$. $\square$

If $Z_1 = \emptyset$ and $Z_2 = \emptyset$, then C is a hamiltonian cycle, that is, a vertex-dominating cycle. Therefore by Claim 3, we may assume, without loss of generality,

$$Z_1 \neq \emptyset \quad \textit{and} \quad Y_2 = \emptyset \tag{6.2}$$

Now, we choose two vertices x_a and x_b as follows: if $X_2 = \emptyset$, choose $x_a, x_b \in X_1$ such that a and b as close as possible on C; otherwise, choose $x_a \in X_1 \cup X_2$ and $x_b \in X_2$ such that a and b as close as possible on C, where $a \in N_C(x_a)$, $b \in N_C(x_b)$ and $a \neq b$.

We give an orientation on C such that $|a^+ \overrightarrow{C} b^-| \leq |b^+ \overrightarrow{C} a^-|$. By the choice of x_a and x_b, we obtain

$$|a^+ \overrightarrow{C} b^-| \leq \frac{1}{2}|C| - 1 = \frac{1}{3}(2n_2 - 1) - 1 = 2\left(\frac{1}{3}(n_2+1) - 1\right). \tag{6.3}$$

Claim 4 *There exists a path $x_a P_0 x_b$ in $G - C$. In particular, if $x_a \neq x_b$, then there exists a path $x_a P_0 x_b$ in $G - C$ which dominates Z_1 and Z_2.*

Proof. First, suppose that $x_a = x_b$, then x_a is a path of order one in $G - C$. Next, suppose $x_a, x_b \in X_i$ ($x_a \neq x_b$) for $i = 1, 2$. If $x_a, x_b \in X_1$, then, by the choice of x_a and x_b, note that $X_2 = \emptyset$, that is, $Z_2 \neq \emptyset$. By Claim 3, we have $x_a, x_b \in N_G(z_{3-i})$ for any $z_{3-i} \in Z_{3-i}$. Hence $x_a z_{3-i} x_b$ is a path in $G - C$ which dominates Z_1 and Z_2. Finally, suppose $x_a \in X_1$ and $x_b \in X_2$. If $x_a x_b \in E(G)$, then $x_a x_b$ is a path in $G - C$ which dominates Z_1 and Z_2, by Claim 3. If $x_a x_b \notin E(G)$, then there exists a vertex $v_2 \in N_{G-C}(x_a)$.

Since $v_2, x_b \in N_{G-C}(z_1)$ for any $z_1 \in Z_1$, $x_a v_2 z_1 x_b$ is a path in $G - C$ which dominates Z_1 and Z_2. $\square$

By Claim 4, we obtain $C_0 = x_a a \overleftarrow{C} b x_b P_0 x_a$ is a cycle. Let $U_i = V(a^+ \overrightarrow{C} b^-) \cap V_i$ and $U'_i = V(b^+ \overrightarrow{C} a^-) \cap V_i$.

Claim 5 *X_1, X_2, Y_1 and U_1 are dominated by $a \overleftarrow{C} b$.*

Proof. By the choice of x_a and x_b, we have $N_G(x_i) \cap V(a \overleftarrow{C} b) \neq \emptyset$ for any $x_i \in X_i$. By (6.3), we obtain $|U_2| \leq \frac{1}{2}|a^+ \overrightarrow{C} b^-| = \frac{1}{3}(n_2 + 1) - 1$. Hence $N_G(y_1) \cap V(a \overleftarrow{C} b) \neq \emptyset$ for any $y_1 \in Y_1$. Moreover, $N_G(u_1) \cap V(a \overleftarrow{C} b) \neq \emptyset$ for any $u_1 \in U_1$, since $Y_2 = \emptyset$. $\square$

Case 1. $|a^+ \overrightarrow{C} b^-|$ is even.

Note that $x_a \in X_1$ and $x_b \in X_2$. By Claim 4, $x_a P_0 x_b$ dominates Z_1 and Z_2. If C_0 dominates U_2, then C_0 is a vertex-dominating cycle, by Claim 5. Thus, suppose that C_0 does not dominate U_2, that is, there exists $u_2 \in U_2$ such that $N_G(u_2) \subset U_1 \cup Y_1$. From $u_2 \neq a^+, b^-$, we have $|a^+ \overrightarrow{C} b^-| \neq 2$, and so

$$|a^+ \overrightarrow{C} b^-| \geq 4. \tag{6.4}$$

Since $\frac{1}{3}(n_2 + 1) \leq d_G(u_2) \leq |U_1| + |Y_1| = \frac{1}{2}|a^+ \overrightarrow{C} b^-| + |Y_1|$, we obtain

$$n_2 \leq \frac{3}{2}|a^+ \overrightarrow{C} b^-| + 3|Y_1| - 1. \tag{6.5}$$

By (6.3) and (6.5), we have $|Y_1| \geq 1$. We first show that $|Y_1| = 1$. Suppose that $|Y_1| \geq 2$. Since $C - N_C(\{x_a\} \cup X_2)$ consists of at least $|N_C(X_2)| + 1$ paths of order at least $|a^+ \overrightarrow{C} b^-|$, we have $|C| \geq (|N_C(X_2)| + 1)(|a^+ \overrightarrow{C} b^-| + 1)$.

On the other hand, by Claim 2, (6.4) and (6.5), we obtain

$$\begin{aligned}
& (|N_C(X_1 \cup X_2)| + 1)(|a^+ \overrightarrow{C} b^-| + 1) - |C| \\
= \ & (|Y_1| + 1)(|a^+ \overrightarrow{C} b^-| + 1) - \frac{2}{3}(2n_2 - 1) \\
\geq \ & (|Y_1| + 1)(|a^+ \overrightarrow{C} b^-| + 1) - \frac{2}{3}\left\{2\left(\frac{3}{2}|a^+ \overrightarrow{C} b^-| + 3|Y_1| - 1\right) - 1\right\} \\
= \ & (|Y_1| + 1)(|a^+ \overrightarrow{C} b^-| + 1) - (2|a^+ \overrightarrow{C} b^-| + 4|Y_1| - 2) \\
= \ & (|Y_1| - 1)(|a^+ \overrightarrow{C} b^-| - 3) > 0.
\end{aligned}$$

Thus we get a contradiction. Therefore we have $|Y_1| = 1$.

By (6.3) and (6.5), note that $|a^+ \overrightarrow{C} b^-| = |b^+ \overrightarrow{C} a^-| = 2\left(\frac{1}{3}(n_2 + 1) - 1\right)$. By the choice of x_a and x_b, we have $N_C(x) \subset \{a, b\}$ for any $x \in X_1 \cup X_2$. This implies that X_1 and X_2 are dominated by $\{a, b\}$.

On the other hand, since $Y_2 = \emptyset$, $N_G(u'_1) \cap V(a \overrightarrow{C} b) \neq \emptyset$ for any $u'_1 \in U'_1$, that is, $a \overrightarrow{C} b$ dominates U'_1. Thus, if $a \overrightarrow{C} b$ dominates U'_2, then $x_a a \overrightarrow{C} b x_b P_0 x_a$ is a vertex-dominating cycle. Thus, suppose that $a \overrightarrow{C} b$ does not dominate U'_2, that is, there exists $u'_2 \in U'_2$ such that $N_G(u'_2) \subset U'_1 \cup Y_1$.

Since $|U_1| = \frac{1}{3}(n_2 + 1) - 1$, we have $y_1 \in N_G(u_2)$, where $y_1 \in Y_1$. Similarly, $y_1 \in N_G(u'_2)$. By the existence of the C-path $u_2 y_1 u'_2$, there exists a C-path $w P_1 w'$ joining $V(a^+ \overrightarrow{C} u_2)$ and $V(b^+ \overrightarrow{C} u'_2)$. Choose P_1 such that $V(a^+ \overrightarrow{C} w) \cup V(b^+ \overrightarrow{C} w')$ is inclusion-minimal. Then $C_1 = x_a a \overleftarrow{C} w' P_1 w \overrightarrow{C} b x_b P_0 x_a$ is a cycle.

By the choice of P_1, $w \overrightarrow{C} b \cup w' \overrightarrow{C} a$ dominates Y_1. By $u_2 \neq a^+$, we have $|a^+ \overrightarrow{C} w^-| \leq |a^+ \overrightarrow{C} u_2| \leq 2\left(\frac{1}{3}(n_2 + 1) - 1\right) - 2$. By the choice of P_1 and the fact that $|Y_1| = 1$ and $Y_2 = \emptyset$, we have $N(v_a) \cap (w \overrightarrow{C} b \cup w' \overrightarrow{C} a) \neq \emptyset$ for any $v_a \in a^+ \overrightarrow{C} w^-$, that is, $w \overrightarrow{C} b \cup w' \overrightarrow{C} a$ dominates $V(a^+ \overrightarrow{C} w^-)$. Similarly, $w \overrightarrow{C} b \cup w' \overrightarrow{C} a$ dominates $V(b^+ \overrightarrow{C} w'^-)$. Hence C_1 is a vertex-dominating cycle.

This completes the proof of Case 1.

Case 2. $|a^+\overrightarrow{C}b^-|$ is odd.

Note that $x_a \in X_i$ and $x_b \in X_i$ for $i = 1, 2$.

Case 2.1. $Z_2 = \emptyset$.

If $X_2 = \emptyset$, then $V(G) - V(C) = \emptyset$. Hence C is a hamiltonian cycle, that is, a vertex-dominating cycle. Thus, we may assume $X_2 \neq \emptyset$. By the choice x_a and x_b, we have $x_a, x_b \in X_2$. By Claim 3, C_0 dominates Z_1. If C_0 dominates U_2, then C_0 is a vertex-dominating cycle, by Claim 5. Thus, suppose that C_0 does not dominate U_2, that is, there exists $u_2 \in U_2$ such that $N_G(u_2) \subset U_1 \cup Y_1$. From $u_2 \neq a^+, b^-$, we have $|a^+\overrightarrow{C}b^-| \neq 3$, and so

$$|a^+\overrightarrow{C}b^-| \geq 5. \tag{6.6}$$

Since $|a^+\overrightarrow{C}b^-|$ is odd and $a^+, b^- \in V_2$, we have $|U_1| = \frac{1}{2}(|a^+\overrightarrow{C}b^-| - 1)$. Hence we obtain $\frac{1}{3}(n_2 + 1) \leq d_G(u_2) \leq \frac{1}{2}(|a^+\overrightarrow{C}b^-| - 1) + |Y_1|$, and so

$$n_2 \leq \frac{3}{2}(|a^+\overrightarrow{C}b^-| - 1) + 3|Y_1| - 1. \tag{6.7}$$

By (6.3) and (6.7), we have $|Y_1| \geq 2$. Now we show that $|Y_1| = 2$ or $|Y_1| = 3$. Assume that $|Y_1| \geq 4$. Since $C - N_C(X_1 \cup X_2)$ has at least $|N_C(X_1 \cup X_2)|$ paths of order at least $|a^+\overrightarrow{C}b^-|$, we have $|C| \geq |N_C(X_1 \cup X_2)|(|a^+\overrightarrow{C}b^-| + 1)$. On the other hand, by Claim 2, (6.6) and (6.7), we obtain

$$\begin{aligned}
&|N_C(X_1 \cup X_2)|(|a^+\overrightarrow{C}b^-| + 1) - |C| \\
\geq\ & |Y_1|(|a^+\overrightarrow{C}b^-| + 1) - \frac{2}{3}\left\{2\left(\frac{3}{2}(|a^+\overrightarrow{C}b^-| - 1) + 3|Y_1| - 1\right) - 1\right\} \\
=\ & |Y_1|(|a^+\overrightarrow{C}b^-| + 1) - (2|a^+\overrightarrow{C}b^-| + 4|Y_1| - 4) \\
=\ & (|Y_1| - 2)(|a^+\overrightarrow{C}b^-| - 3) - 2 > 0,
\end{aligned}$$

a contradiction. Therefore we have $|Y_1| = 2$ or $|Y_1| = 3$.

Suppose that $|Y_1| = 2$. By (6.3), we have $|a^+\overrightarrow{C}b^-| \leq 2\left(\frac{1}{3}(n_2+1)-1\right) - 1$, since $|a^+\overrightarrow{C}b^-|$ is odd. Thus, by (6.7), we obtain $|a^+\overrightarrow{C}b^-| = 2\left(\frac{1}{3}(n_2+1)-1\right) - 1$, $|b^+\overrightarrow{C}a^-| = 2\left(\frac{1}{3}(n_2+1)-1\right)+1$ and $Y_1 \subset N_G(u_2)$. By the choice of x_a and x_b, we have $X_1 \neq \emptyset$ and X_2 is dominated by $\{a, b\}$. Since $|U_2| = \frac{1}{3}(n_2+1)-1$ and $a, b \in V_1$, there exists $y_1 \in Y_1$ such that $N(y_1) \cap V(b^+\overrightarrow{C}a^-) \neq \emptyset$, say $u_2' \in N(y_1) \cap V(b^+\overrightarrow{C}a^-)$. By (6.6) and $|b^+\overrightarrow{C}a^-| = |a^+\overrightarrow{C}b^-| + 2$, either $|b^+\overrightarrow{C}u_2'^-| \leq |a^+\overrightarrow{C}b^-|$ or $|u_2'^-\overrightarrow{C}a^-| \leq |a^+\overrightarrow{C}b^-|$ holds. Without loss of generality, we may assume $|b^+\overrightarrow{C}u_2'^-| \leq |a^+\overrightarrow{C}b^-|$. By the existence of the C-path $u_2y_1u_2'$, there exists a C-path wP_2w' joining $V(a^+\overrightarrow{C}u_2)$ and $V(b^+\overrightarrow{C}u_2')$. Choose P_2 such that (i) $|V(b^+\overrightarrow{C}w')|$ is as small as possible, and (ii) subject to (i), $|V(a^+\overrightarrow{C}w)|$ is as small as possible. Then $C_2 = x_a a\overleftarrow{C}w'P_2w\overrightarrow{C}bx_bP_0x_a$ is a cycle.

First, we show that $V(a^+\overrightarrow{C}w^-)$ is dominated by $w\overrightarrow{C}b \cup w'\overrightarrow{C}a$. From $u_2 \neq b^-$, note that $|a^+\overrightarrow{C}w^-| \leq |a^+\overrightarrow{C}u_2^-| \leq |a^+\overrightarrow{C}b^-| - 2 \leq 2\left(\frac{1}{3}(n_2+1)-1\right) - 3$. By the choice of P_2 and the fact that $|Y_1| = 2$ and $Y_2 = \emptyset$, we have $N(v_a) \cap V(w\overrightarrow{C}b \cup w'\overrightarrow{C}a) \neq \emptyset$ for any $v_a \in V(a^+\overrightarrow{C}w^-)$. This implies that $V(a^+\overrightarrow{C}w^-)$ is dominated by $w\overrightarrow{C}b \cup w'\overrightarrow{C}a$. Next, we show that $V(b^+\overrightarrow{C}w')$ is dominated by $w\overrightarrow{C}b \cup w'\overrightarrow{C}a$. By the choice of P_2 and $Y_1 \subset N_G(u_2)$, we have $N(v_b) \cap (Y_1 \cup V(a^+\overrightarrow{C}w^-)) = \emptyset$ for any $v_b \in V(b^+\overrightarrow{C}w'^-)$. Since $|b^+\overrightarrow{C}u_2'| \leq |b^+\overrightarrow{C}w'| \leq |a^+\overrightarrow{C}b^-|$, we have $N(v_b) \cap V(w\overrightarrow{C}b \cup w'\overrightarrow{C}a) \neq \emptyset$ for any $v_b \in V(b^+\overrightarrow{C}w'^-)$. This implies that $V(b^+\overrightarrow{C}w')$ is dominated by $w\overrightarrow{C}b \cup w'\overrightarrow{C}a$. Hence C_2 is a vertex-dominating cycle.

Suppose that $|Y_1| = 3$. By Claim 2, $C - N_C(X_2)$ has at least three paths of order at least 5. Therefore $|a^+\overrightarrow{C}b^-| \leq \frac{1}{2}(|C|-6) - 1 \leq 2\left(\frac{1}{3}(n_2+1)-1\right) - 3$. Thus, by (6.7), we have $|a^+\overrightarrow{C}b^-| = 2\left(\frac{1}{3}(n_2+1)-1\right) - 3$, $|b^+\overrightarrow{C}a^-| = 2\left(\frac{1}{3}(n_2+1)-1\right)+3$ and $Y_1 \subset N_G(u_2)$. Since $|b^+\overrightarrow{C}a^-| = 2\left(\frac{1}{3}(n_2+1)-1\right)+3$, by (6.6) and the choice of x_a and x_b, we have $|N_C(X_1 \cup X_2)| \leq 3$. If $|N_C(X_1 \cup X_2)| = 3$, then let $c \in N_C(X_1 \cup X_2) - \{a, b\}$. Note that $X_1 \cup X_2$ is dominated by $\{a, b, c\}$. Since $|U_2| = \frac{1}{3}(n_2+1) - 2$ and $a, b \in V_1$, there exists

$y_1 \in Y_1$ such that $N(y_1) \cap V(b^+ \overrightarrow{C} a^-) \neq \emptyset$, say $u_2' \in N(y_1) \cap V(b^+ \overrightarrow{C} a^-)$.

We show that $|b^+ \overrightarrow{C} u_2'| \leq |a^+ \overrightarrow{C} b^-|$ and $c \notin V(b^+ \overrightarrow{C} u_2')$. Suppose that $|N_C(X_1 \cup X_2)| = 2$. By (6.6) and $|b^+ \overrightarrow{C} a^-| = |a^+ \overrightarrow{C} b^-| + 6$, either $|b^+ \overrightarrow{C} {u_2'}^-| \leq |a^+ \overrightarrow{C} b^-|$ or $|{u_2'}^+ \overrightarrow{C} a^-| \leq |a^+ \overrightarrow{C} b^-|$ holds. Without loss of generality, we may assume $|b^+ \overrightarrow{C} {u_2'}^-| \leq |a^+ \overrightarrow{C} b^-|$. If $|N_C(X_1 \cup X_2)| = 3$, then either $c \notin V(b^+ \overrightarrow{C} {u_2'}^-)$ or $c \notin V({u_2'}^+ \overrightarrow{C} a^-)$. Without loss of generality, we may assume that $c \notin V(b^+ \overrightarrow{C} u_2')$. By (6.6) and $|b^+ \overrightarrow{C} a^-| = |a^+ \overrightarrow{C} b^-| + 6$, we have $|b^+ \overrightarrow{C} u_2'| \leq |a^+ \overrightarrow{C} b^-|$.

Thus, using the same argument as in the case of $|Y_1| = 2$, we get a vertex-dominating cycle. This completes the proof of Case 2.1. □

Case 2.2. $Z_2 \neq \emptyset$.

By Claim 3, note that $Y_1 = \emptyset$. Since $|U_1| \leq \frac{1}{3}(n_2 + 1) - 1$, we have $N(u_2) \cap V(b \overrightarrow{C} a) \neq \emptyset$ for any $u_2 \in U_2$, that is, C_0 dominates U_2. Since $Z_1 \neq \emptyset$ and $Z_2 \neq \emptyset$, we may assume $x_a = x_b$; otherwise C_0 is a vertex-dominating cycle, by Claim 4.

By the 2-connectivity of G and the choice of x_a and x_b, there exists $x_d \in X_1 \cup X_2$ such that $x_d \neq x_a$ and $N_C(x_d) \cap V(b^+ \overrightarrow{C} a^-) \neq \emptyset$. Choose x_d such that a^- as close to d on C as possible, where $d \in N_C(x_d) \cap V(a^- \overleftarrow{C} b^+)$. Note that $a \overrightarrow{C} d$ dominates X_1 and X_2. By Claim 4, there exists a path $x_a P_4 x_d$ in $G - C$ which dominates Z_1 and Z_2. Since $|a \overrightarrow{C} b| \geq 3$, we have $|d^+ \overrightarrow{C} a^-| \leq \frac{1}{2}(|C| - 4) \leq 2\left(\frac{1}{3}(n_2 + 1) - 1\right) - 1$. Define $U_i'' = V(d^+ \overrightarrow{C} a^-) \cap V_i$. Since $|U_1''|, |U_2''| \leq \frac{1}{3}(n_2 + 1) - 1$ and $Y_1 = Y_2 = \emptyset$, $a \overrightarrow{C} d$ dominates U_1'' and U_2''. Hence $x_a \overrightarrow{C} d x_d P_4 x_a$ is a vertex-dominating cycle. This completes the proof of Case 2.2 and the proof of Theorem 6.8. □

Chapter 7

Cycles within specified distance from each vertex

7.1 Introduction

Let f be a non-negative integer-valued function defined on $V(G)$. Then a cycle C is called an *f-dominating cycle* if $d_G(x, C) \leq f(x)$ for every $x \in V(G)$. By taking an appropriate function as f, we can give a unified view to many cycle-related problems. If f is a constant function taking the value 0 (resp. 1), then an f-dominating cycle is a hamiltonian cycle (resp. a dominating cycle) of G, respectively. Furthermore, given a prescribed set $S \subset V(G)$, if we set

$$f(v) = \begin{cases} 0 & \text{if } v \in S \\ \operatorname{diam}(G) & \text{if } v \in V(G) - S, \end{cases}$$

where $\operatorname{diam}(G)$ is the diameter of G, then an f-dominating cycle is a cycle passing through every vertex in S.

The purpose of this chapter is to give a Chvátal-Erdős type theorem for a graph to have an f-dominating cycle. Then we show that this theorem is a generalization of a number of known results.

A set of vertices $S \subset V(G)$ is said to be an *f-stable set* if $d_G(u, v) \geq f(u) + f(v)$ holds for each pair of distinct vertices u, $v \in S$. If we take a constant function taking the value one as f, an f-stable set is an ordinary stable set. The f-stability number, denoted by $\alpha_f(G)$, is defined by $\alpha_f(G) = \max\{|S| \colon S \text{ is an } f\text{-stable set}\}$. For an integer constant c, we define the function $f + c$ by $(f + c)(v) = f(v) + c$ $(v \in V(G))$.

We prove the following theorem.

Theorem 7.1 *Let G be a k-connected graph ($k \geq 2$) and let f be a non-negative integer-valued function defined on $V(G)$. If $\alpha_{f+1}(G) \leq k$, then G has an f-dominating cycle.*

Before we prove Theorem 7.1, we give its application. By setting $f(v) = 0$ for each $v \in V(G)$, we immediately obtain Chvátal-Erdős' theorem. Let $\alpha(G)$ be the independence number of a graph G.

Theorem 7.2 (Chvátal&Erdős [7]) *A k-connected graph ($k \geq 2$) G with $\alpha(G) \leq k$ is hamiltonian.*

Bondy and Fan [5] gave a degree sum condition for a graph to have a dominating cycle. Later Broersma [6] proved the following theorem and showed that it extends the result of Bondy and Fan.

Theorem 7.3 (Broersma [6]) *Let G be a k-connected graph ($k \geq 2$) and let λ be a positive integer. If $\hat{\alpha}_{2\lambda}(G) \leq k$, then G has a d_λ-cycle.*

It is easy to see that if f is a constant function taking the value $\lambda - 1$, then Theorem 7.1 coincides with Theorem 7.3.

Theorem 7.1 also generalizes previous results on cycles passing through specified vertices. The following is a classical result by Dirac [8].

Theorem 7.4 (Dirac [8]) *Every set of k vertices in a k-connected graph ($k \geq 2$) lies in a cycle.*

Theorem 7.4 was later extended by Fournier [9]. For a set of vertices $X \subset V(G)$, we define the independence number of X, denoted by $\alpha(X)$, by $\alpha(X) = \alpha(G[X])$, where $G[X]$ is the subgraph of G induced by X.

Theorem 7.5 (Fournier [9]) *Let G be a k-connected graph ($k \geq 2$), and let $X \subset V(G)$. If $\alpha(X) \leq k$, then G has a cycle which contains X.*

In order to demonstrate the usefulness of the Theorem 7.1, we give a further extension as a corollary. For a set of vertices $X \subset V(G)$ and a positive integer d, we define $\hat{\alpha}_{G,d}(X)$ by

$$\hat{\alpha}_{G,d}(X) = \max\{|S| \colon S \subset X, S \text{ is a } d\text{-stable in } G\}.$$

Note that $\hat{\alpha}_{G,2}(X) = \hat{\alpha}_2(G[X]) = \alpha(X)$, but for $d \geq 3$, $\hat{\alpha}_{G,d}(X)$ and $\hat{\alpha}_d(G[X])$ may take different values.

Corollary 7.6 *Let G be a k-connected graph ($k \geq 2$), and let X and Y be disjoint subsets of $V(G)$. If $\alpha(X) + \hat{\alpha}_{G,4}(Y) \leq k$, then G has a cycle C which contains X and dominates Y.*

Corollary 7.6 coincides with Theorem 7.5 if $Y = \emptyset$.

Proof of Corollary 7.6. Define an integer-valued function f on $V(G)$ by

$$f(v) = \begin{cases} 0 & \text{if } v \in X \\ 1 & \text{if } v \in Y \\ \text{diam}(G) & \text{if } v \in V(G) - (X \cup Y). \end{cases}$$

Then an f-dominating cycle is a required cycle. Let S be a maximum $(f+1)$-stable set, i.e. an $(f+1)$-stable set of cardinality $\alpha_{f+1}(G)$. We claim $|S| \leq k$.

First, suppose $S \not\subset X \cup Y$. If $|S| \geq 2$, then we can take a pair of distinct vertices u, v such that $u \in S - (X \cup Y)$ and $v \in S$. Then $d_G(u, v) \geq f(u) + f(v) + 2 \geq \text{diam}(G) + 2$, a contradiction. Therefore, we have $|S| \leq 1 < k$.

Next, suppose $S \subset X \cup Y$. Then by the definition of f, $S \cap X$ is a stable set contained in X and $Y \cap S$ is a 4-stable set of G contained in Y. Thus, $|S \cap X| \leq \alpha(X)$ and $|S \cap Y| \leq \hat{\alpha}_{G,4}(Y)$. Therefore, $|S| = |S \cap X| + |S \cap Y| \leq \alpha(X) + \hat{\alpha}_{G,4}(Y) \leq k$. Now in either case, we have $\alpha_{f+1}(G) = |S| \leq k$, and G has an f-dominating cycle by Theorem 7.1. $\square$

7.2 Lemmas

Let $f\colon V(G) \to \boldsymbol{N}$. Then for a vertex $x \in V(G)$ and $X \subset V(G)$, we define $N_f(x)$ and $D_f(X)$ by

$$N_f(x) = \{v \in V(G)\colon d_G(x, v) \leq f(x)\}, \text{ and}$$
$$D_f(X) = \{v \in V(G)\colon d_G(v, X) \leq f(v)\}.$$

Using this notation, we can say that an f-dominating cycle is a cycle C satisfying $D_f(C) = V(G)$.

Although the following lemma is a trivial observation on $D_f(X)$, it is frequently used in the subsequent arguments.

Lemma 7.7 *Let G be a graph and let $f\colon V(G) \to \boldsymbol{N}$. Furthermore, let X, $Y \subset V(G)$.*

(1) *If $X \subset Y$, then $D_f(X) \subset D_f(Y)$.*

(2) $D_f(X \cup Y) = D_f(X) \cup D_f(Y)$

(3) $D_f(X) - D_f(Y) \subset D_f(X - Y)$

The next lemma is a simple but useful observation on a walk.

Lemma 7.8 *Let G be a graph and let A, $B \subset V(G)$. If G has a walk W from A to B, Then G has a path P from A to B such that $V(P) \subset V(W)$ and that P is internally disjoint from $A \cup B$.*

7.3 Proof of Theorem 7.1

In this section, we prove Theorem 7.1.

Proof of Theorem 7.1. Assume G has no f-dominating cycle. Then for each cycle C in G, $D_f(C) \neq V(G)$, and hence there exists a vertex y_0 in $V(G) - D_f(C)$. Let H be the component of $G - V(C)$ with $y_0 \in V(H)$. Now choose such C and y_0 so that

(C1) $|D_f(C)|$ is as large as possible, and

(C2) $|V(H)|$ is as small as possible, subject to (C1).

The proof is divided into claims.

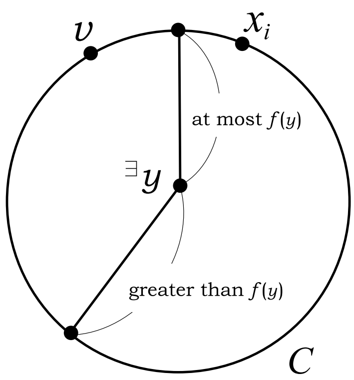
Claim 1 *There exists no cycle C' with $D_f(C) \subset D_f(C')$, $V(H) \cap V(C') \neq \emptyset$ and $N_G(H) \cap \big(V(C) - V(C')\big) = \emptyset$. In particular, G has no cycle C' with $V(C) \subset V(C')$ and $V(H) \cap V(C') \neq \emptyset$.*

Proof. Assume there exists a cycle C' with $D_f(C) \subset D_f(C')$, $V(H) \cap V(C') \neq \emptyset$ and $N_G(H) \cap \big(V(C) - V(C')\big) = \emptyset$. Then $D_f(C) = D_f(C')$ by the maximality of $D_f(C)$, and hence $y_0 \in V(H) - D_f(C')$. Since $N_G(H) \cap \big(V(C) - V(C')\big) = \emptyset$, $G - V(C')$ has a component H' with $y_0 \in V(H')$ and $V(H') \subsetneq V(H)$. This contradicts the minimality of H. □

Let $N_C(H) = \{x_1, x_2, \ldots, x_m\}$. We may assume $x_1, x_2, \ldots, x_m$ appear in the consecutive order along C. We consider $x_{m+1} = x_1$. Let $I_i = x_i \overrightarrow{C} x_{i+1}$ $(1 \leq i \leq m)$. We call each I_i $(1 \leq i \leq m)$ an *interval*. Let $x'_i \in N_H(x_i)$ and let $P_{i,j}$ be an $x'_i x'_j$-path in H $(1 \leq i, j \leq m)$. Possibly, $x'_i = x'_j$ and $l(P_{i,j}) = 0$ for some i and j $(i \neq j)$.

Claim 2 $x_{i+1} \neq x_i^+$

Proof. Assume $x_{i+1} = x_i^+$. Let $C' = x_i^+ \overrightarrow{C} x_i x'_i \overrightarrow{P}_{i,i+1} x'_{i+1} x_i^+$. Then $V(C) \subset V(C')$ and $V(C') \cap V(H) \neq \emptyset$. This contradicts Claim 1. □



If v is an essential vertex, there exists a vertex y such that $d(y, x_i^+ \overrightarrow{C} v) \leq f(y)$ and $d(y, v \overrightarrow{C} x_i) > f(y)$.

Figure 7.1: Essential vertex

Since G is k-connected, we have $m \geq 2k > k$ by Claim 2. A vertex $v \in I_i - \{x_i\}$ $(1 \leq i \leq m)$ is called an *essential* vertex if $D_f(x_i^+ \overrightarrow{C} v) \not\subset D_f(v \overrightarrow{C} x_i)$ (Figure 7.1). If v is not an essential vertex, it is called a *non-essential* vertex.

Claim 3 *For each i $(1 \leq i \leq m)$, x_i^+ is a non-essential vertex of I_i.*

Proof. Since $x_i^+ \overrightarrow{C} x_i^+ = \{x_i^+\} \subset x_i^+ \overrightarrow{C} x_i$, we have $D_f(x_i^+ \overrightarrow{C} x_i^+) \subset D_f(x_i^+ \overrightarrow{C} x_i)$ by Lemma 7.7 (1). □

Claim 4 *If v is a non-essential vertex of I_i, then each $u \in x_i^+ \overrightarrow{C} v^-$ is a non-essential vertex of I_i.*

Proof. Since $x_i^+ \overrightarrow{C} u \subset x_i^+ \overrightarrow{C} v$ and $v \overrightarrow{C} x_i \subset u \overrightarrow{C} x_i$, $D_f(x_i^+ \overrightarrow{C} u) \subset D_f(x_i^+ \overrightarrow{C} v)$ and $D_f(v \overrightarrow{C} x_i) \subset D_f(u \overrightarrow{C} x_i)$ by Lemma 7.7 (1). Since v is a non-essential vertex, $D_f(x_i^+ \overrightarrow{C} v) \subset D_f(v \overrightarrow{C} x_i)$. Thus, we obtain $D_f(x_i^+ \overrightarrow{C} u) \subset D_f(u \overrightarrow{C} x_i)$.
□

Claim 5 *For each i $(1 \leq i \leq m)$, x_{i+1} is an essential vertex of I_i.*

Proof. Assume x_{i+1} is a non-essential vertex. Then we have $D_f(x_i^+\overrightarrow{C}x_{i+1}) \subset D_f(x_{i+1}\overrightarrow{C}x_i)$. Let $C' = x_{i+1}\overrightarrow{C}x_i x_i'\overrightarrow{P}_{i,i+1}x_{i+1}'x_{i+1}$. Then $V(C) - V(C') = x_i^+\overrightarrow{C}x_{i+1}^-$ and $x_{i+1}\overrightarrow{C}x_i \subset V(C')$. Therefore, by Lemma 7.7,

$$D_f(C) - D_f(C') \subset D_f(x_i^+\overrightarrow{C}x_{i+1}^-) \subset D_f(x_i^+\overrightarrow{C}x_{i+1}) \subset D_f(x_{i+1}\overrightarrow{C}x_i) \subset D_f(C').$$

This implies $D_f(C) \subset D_f(C')$. Furthermore, $V(H) \cap V(C') \neq \emptyset$ and $N_G(H) \cap \big(V(C) - V(C')\big) = \emptyset$. However, this contradicts Claim 1. $\square$

By Claims 3, 4 and 5, each interval I_i has a unique vertex s_i such that s_i is a non-essential vertex of I_i but s_i^+ is an essential vertex of I_i. We call this s_i the *border* vertex of I_i.

Claim 6 *For each i, j with $1 \leq i < j \leq m$, there does not exist a walk from $x_i^+\overrightarrow{C}s_i$ to $x_j^+\overrightarrow{C}s_j$ which is internally disjoint from C. In particular, there does not exist an edge which joins $x_i^+\overrightarrow{C}s_i$ and $x_j^+\overrightarrow{C}s_j$*

Proof. Assume there exists a walk from $x_i^+\overrightarrow{C}s_i$ to $x_j^+\overrightarrow{C}s_j$ which is internally disjoint from C. Then by Lemma 7.8, there exists a path Q from $x_i^+\overrightarrow{C}s_i$ to $x_j^+\overrightarrow{C}s_j$ which is internally disjoint from C. Let the starting vertex and the terminal vertex of Q be u_i and u_j, respectively. Take such Q so that $x_i^+\overrightarrow{C}u_i \cup x_j^+\overrightarrow{C}u_j$ is inclusion-minimal. By the definition of an interval, $V(Q) \cap V(H) = \emptyset$. Let $C' = u_j\overrightarrow{C}x_i x_i'\overrightarrow{P}_{i,j}x_j'x_j\overleftarrow{C}u_i\overrightarrow{Q}u_j$. Then $V(C') \cap V(H) \neq \emptyset$ and $N_G(H) \cap \big(V(C) - V(C')\big) = N_G(H) \cap \big((x_i^+\overrightarrow{C}u_i^-) \cup (x_j^+\overrightarrow{C}u_j^-)\big) = \emptyset$. By Claim 1, $D_f(C) \not\subset D_f(C')$. Let $v \in D_f(C) - D_f(C')$. By Lemma 7.7, $D_f(C) - D_f(C') \subset D_f\big((x_i^+\overrightarrow{C}u_i^-) \cup (x_j\overrightarrow{C}u_j^-)\big) = D_f(x_i^+\overrightarrow{C}u_i^-) \cup D_f(x_j^+\overrightarrow{C}u_j^-)$. By symmetry, we may assume $v \in D_f(x_i^+\overrightarrow{C}u_i^-)$. Since $u_i \in x_i^+\overrightarrow{C}s_i$, u_i is a non-essential vertex by Claim 4, and hence $D_f(x_i^+\overrightarrow{C}u_i) \subset D_f(u_i\overrightarrow{C}x_i)$. This implies $v \in D_f(u_i\overrightarrow{C}x_i) - D_f(C') \subset D_f(u_i\overrightarrow{C}x_i - C') = D_f(x_j^+\overrightarrow{C}u_j^-)$. Now, since $v \in D_f(x_i^+\overrightarrow{C}u_i^-) \cap D_f(x_j^+\overrightarrow{C}u_j^-)$, there exist a vv_i-path R_i and vv_j-path R_j for some $v_i \in x_i^+\overrightarrow{C}u_i^-$ and $v_j \in x_j^+\overrightarrow{C}u_j^-$ such that $l(R_i) \leq f(v)$ and $l(R_j) \leq f(v)$ (Figure 7.2).

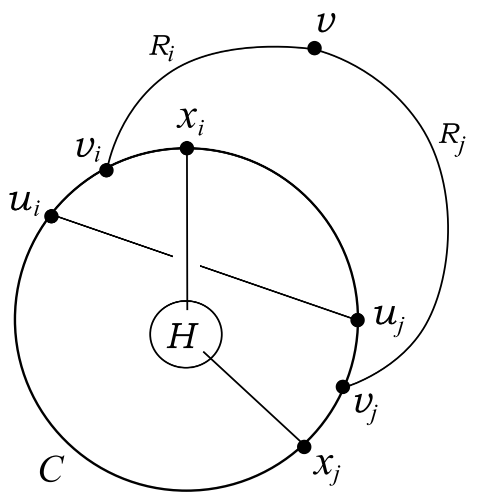


Figure 7.2: R_i and R_j

If $V(R_i) \cap u_j \overrightarrow{C} x_i \neq \emptyset$, then since $l(R_i) \leq f(v)$, we have $d_G(v, u_j \overrightarrow{C} x_i) \leq f(v)$. This implies $v \in D_f(u_j \overrightarrow{C} x_i) \subset D_f(C')$, a contradiction. Therefore, we have $V(R_i) \cap u_j \overrightarrow{C} x_i = \emptyset$. Similarly, we have $V(R_i) \cap u_i \overrightarrow{C} x_j = \emptyset$ and $V(R_j) \cap (u_j \overrightarrow{C} x_i \cup u_i \overrightarrow{C} x_j) = \emptyset$.

Let $W' = v_i \overleftarrow{R}_i v \overrightarrow{R}_j$. Then W' is a walk from $x_i^+ \overrightarrow{C} u_i^-$ to $x_j^+ \overrightarrow{C} u_j^-$, which is internally disjoint from $u_j \overrightarrow{C} x_i \cup u_i \overrightarrow{C} x_j$. By Lemma 7.8, there exists a path Q' from $x_i^+ \overrightarrow{C} u_i^-$ to $x_j \overrightarrow{C} u_j^-$ such that $V(Q') \subset V(W')$ and Q' is internally disjoint from C. This contradicts the minimality of $x_i^+ \overrightarrow{C} u_i \cup x_j^+ \overrightarrow{C} u_j$, and the claim follows. $\square$

Claim 7 *For each i, $1 \leq i \leq m$, there exists a vertex $y_i \in D_f(x_i^+ \overrightarrow{C} s_i)$ such that $N_f(y_i) \cap V(C) \subset x_i^+ \overrightarrow{C} s_i$.*

Proof. Since s_i is the border vertex, we have $D_f(x_i^+ \overrightarrow{C} s_i) \subset D_f(s_i \overrightarrow{C} x_i)$ but $D_f(x_i^+ \overrightarrow{C} s_i^+) \not\subset D_f(s_i^+ \overrightarrow{C} x_i)$. Take $y_i \in D_f(x_i^+ \overrightarrow{C} s_i^+) - D_f(s_i^+ \overrightarrow{C} x_i)$. Since $y_i \notin D_f(s_i^+ \overrightarrow{C} x_i)$, $N_f(y_i) \cap s_i^+ \overrightarrow{C} x_i = \emptyset$ and hence $N_f(y_i) \cap V(C) \subset x_i^+ \overrightarrow{C} s_i$. Since $s_i^+ \in D_f(s_i^+ \overrightarrow{C} x_i)$, if $y_i \in D_f(\{s_i^+\})$, then $y_i \in D_f(s_i^+ \overrightarrow{C} x_i)$ by Lemma

7.7 (1), a contradiction. Thus, we have $y_i \in D_f(x_i^+ \overrightarrow{C} s_i^+) - D_f(\{s_i^+\}) \subset D_f(x_i^+ \overrightarrow{C} s_i)$. $\square$

Claim 8 *For each i, j with $1 \leq i < j \leq m$, $d_G(y_i, y_j) \geq f(y_i) + f(y_j) + 2$.*

Proof. Let Q_i be a shortest path from $\{y_i\}$ to $V(C)$, and let Q_j be a shortest path from $\{y_j\}$ to $V(C)$. Then both Q_i and Q_j are internally disjoint from C. Let v_i and v_j be the terminal vertices of Q_i and Q_j, respectively. Then by Claim 7, $v_i \in x_i^+ \overrightarrow{C} s_i$ and $v_j \in x_j^+ \overrightarrow{C} s_j$.

Let R be a shortest $y_i y_j$-path in G, and let $W = v_i \overleftarrow{Q}_i \, y_i \overrightarrow{R} \, y_j \overrightarrow{Q}_j \, v_j$. Then W is a walk from $x_i^+ \overrightarrow{C} s_i$ to $x_j^+ \overrightarrow{C} s_j$. By Lemma 7.8, there exists a path Q' from $x_i^+ \overrightarrow{C} s_i$ to $x_j^+ \overrightarrow{C} s_j$ such that $V(Q') \subset V(W)$ and that Q' is internally disjoint from $x_i^+ \overrightarrow{C} s_i \cup x_j^+ \overrightarrow{C} s_j$. If $V(R) \cap (s_i^+ \overrightarrow{C} x_j \cup s_j^+ \overrightarrow{C} x_i) = \emptyset$, then $V(W) \cap (s_i^+ \overrightarrow{C} x_j \cup s_j^+ \overrightarrow{C} x_i) = \emptyset$ and $V(Q') \cap (s_i^+ \overrightarrow{C} x_j \cup s_j^+ \overrightarrow{C} x_i) = \emptyset$. Thus, Q' is internally disjoint from C, which contradicts Claim 6. Thus, we have $V(R) \cap (s_i^+ \overrightarrow{C} x_j \cup s_j^+ \overrightarrow{C} x_i) \neq \emptyset$. Let $v \in V(R) \cap (s_i^+ \overrightarrow{C} x_j \cup s_j^+ \overrightarrow{C} x_i)$. Then by Claim 7, $l(y_i \overrightarrow{R} v) \geq f(y_i) + 1$ and $l(v \overrightarrow{R} y_j) \geq f(y_j) + 1$. Therefore, we obtain $d_G(y_i, y_j) = l(R) \geq f(y_i) + f(y_j) + 2$. $\square$

Claim 9 *For each i, $1 \leq i \leq l$, $d_G(y_0, y_i) \geq f(y_0) + f(y_i) + 2$.*

Proof. Let R be a shortest $y_0 y_i$-path in G. Then since $y_0 \in V(H)$, by the definition of an interval, $\{x_1, \ldots, x_m\} \cap V(R) \neq \emptyset$. Let $x_j \in V(R)$. Since $y_0 \notin D_f(C)$, $l(y_0 \overrightarrow{R} x_j) \geq f(y_0) + 1$. By Claim 5, $s_i \notin \{x_1, \ldots, x_m\}$, and hence by Claim 7 $l(x_j \overrightarrow{R} y_i) \geq f(y_i) + 1$. Therefore, we obtain $d_G(y_0, y_i) = l(R) \geq f(y_0) + f(y_i) + 2$. $\square$

Let $Y = \{y_0, y_1, \ldots, y_m\}$. Then by Claims 8 and 9, Y is an $(f+1)$-stable set of order $m + 1 > k$ and hence $\alpha_{f+1}(G) > k$. This contradicts the assumption, and the theorem follows. $\square$

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