Limit theorems for percolation

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平成１７年１月

神戸大学大学院自然科学研究科

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(パーコレーションに関する極限定理)

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Chapter 1

Introduction

The mathematical theory of percolation started in 1957, when Broadbent and Hammersley[10] studied the behaviour of fluids penetrating into porous media. Percolation theory contains many interesting problems, and at the same time it is very powerful for studying phase transitions of various models. In this paper, we investigate limit theorems for the number of percolation clusters in a finite region and the phase structure of the two-dimensional Widom-Rowlinson model. We review related existing results.

First let us explain the Bernoulli percolation model. Consider an infinite connected graph $G = (V, E)$. The following types of problems are mainly studied:

- Bond problem : Each bond $e \in E$ is independently declared to be open with probability p and closed with probability $1 - p$.
- Site problem : Each site $v \in V$ is independently declared to be open with probability p and closed with probability $1 - p$.

Let us concentrate to the bond problem for a while. We write P_p for the Bernoulli measure on $\{\text{open}, \text{closed}\}^E$. The expectation and the variance with respect to P_p are denoted by E_p and var_p, respectively. The open cluster containing $v \in V$, denoted by C_v, is the set of sites which can be reached from v via open bonds. There is a critical probability $p_c(G)$ such that for any $v \in V$,

$$P_p\{|C_v| = \infty\} \begin{cases} = 0 & \text{if } p < p_c(G), \\ > 0 & \text{if } p > p_c(G). \end{cases}$$

The central limit theorem (CLT) for some quantities related to percolation are studied by many authors (see e.g. [23] §11.6 and [17]). The CLT for martingale differences turns out to be useful at the critical point, as in [39]. For Bernoulli bond percolation problem on the d-dimensional hypercubic lattice $\mathbb{Z}^d$ ($d \geq 2$), Zhang[65] proved a CLT for the number of open clusters in a finite box for all $p \in (0,1)$, that is, including the case $p = p_c(\mathbb{Z}^d)$. His proof is based on McLeish's martingale CLT[43]. Using this method together with the ergodic theorem, Penrose[49] proved a general CLT which can be applied to several models.

In Chapter 2, by using the argument in [65], we study a CLT for percolation problems on regular trees and Sierpiński carpet lattices, where the ergodic theorem is not available. This result is contained in

> Nobuaki Sugimine and Masato Takei ; Remarks on central limit theorems for the number of percolation clusters, to appear in *Publications of the Research Institute for Mathematical Sciences*.

The Sierpiński carpet is a self-similar set. The construction is the following: We divide a unit square into 3×3 smaller squares and delete the center square. We repeat the same procedure for each of the remaining eight squares. Then we get 8^2 smaller squares, and so on. This construction can be generalized as follows: We divide a unit square into $L \times L$ smaller squares and delete the squares *not* corresponding to $T \subset \mathbf{T}_L \equiv \{0, \cdots, L-1\}^2$, and repeat the same procedure for smaller squares (the Sierpiński carpet corresponds to the case $L = 3$ and $T = \mathbf{T}_3 \setminus \{(1,1)\}$). The resulting self-similar set is denoted by K^T. We can construct an infinite graph $G_T (\subset \mathbb{Z}^2)$ corresponding to K^T, which is called a *Sierpiński carpet lattice* (see Figure 2.1 in section 2.1.3).

Theorem A. *Let $L \geq 3$. We assume that $T \subset \mathbf{T}_L$ satisfies the following conditions:*

- *K^T is connected,*
- *$(i,j) \in T \Longrightarrow (j,i) \in T$, $(i, L-1-j) \in T$,*
- *$\{(0,j), (1,j); 0 \leq j \leq L-1\} \subset T$.*

We consider Bernoulli bond percolation on G_T. Let $G_n^T = G_T \cap [0, L^n]^2$ (with abuse of notation). The number of bonds and sites of G_n^T are denoted by $||G_n^T||$ and $|G_n^T|$, respectively. Let K_n be the number of open clusters in G_n^T. Then, for all $p \in (0,1)$ the following statements hold:

(i) The limits

$$m(p) \equiv \lim_{n\to\infty} \frac{E_p K_n}{||G_n^T||} \text{ and } \hat{m}(p) \equiv \lim_{n\to\infty} \frac{E_p K_n}{|G_n^T|}$$

exist and with probability one

$$\lim_{n\to\infty} \frac{K_n}{||G_n^T||} = m(p) \text{ and } \lim_{n\to\infty} \frac{K_n}{|G_n^T|} = \hat{m}(p).$$

(ii) $\dfrac{K_n - E_p K_n}{\sqrt{\mathrm{var}_p K_n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$ *as* $n \to \infty$, *where* $\xrightarrow{\mathcal{L}}$ *denotes the convergence in law and* $\mathcal{N}(0,1)$ *denotes the standard normal distribution.*

(iii) For any $\varepsilon_1 \in (0, 1 - \hat{m}(p))$ *and* $\varepsilon_2 \in (0, \hat{m}(p))$, *there exists positive constants* $\sigma_1(\varepsilon_1, p)$ *and* $\sigma_2(\varepsilon_2, p)$ *such that*

$$\lim_{n\to\infty} \frac{-1}{|G_n^T|} \log P_p \left\{ \frac{K_n}{|G_n^T|} \geq \hat{m}(p) + \varepsilon_1 \right\} = \sigma_1(\varepsilon_1, p),$$
$$\lim_{n\to\infty} \frac{-1}{|G_n^T|} \log P_p \left\{ \frac{K_n}{|G_n^T|} \leq \hat{m}(p) - \varepsilon_2 \right\} = \sigma_2(\varepsilon_2, p).$$

Remark. The conditions of Theorem A imply that $p_c(G_T) < 1$ (see Lemma 2.1.3 below). Our proof can be applied to some cases where $p_c(G_T) = 1$. For example, we can treat (a variant of) the pre-Sierpiński gasket (i.e. G_T with $T = \{0,1\}^2 \setminus \{(1,1)\}$).

As for trees, we have the following

Theorem B. *We consider the Bernoulli (site or bond) percolation problem on* $(d+1)$*-trees with* $d \geq 2$. *Fix an arbitrary point as the origin. Let* $B(n)$ *be the sites whose distance from the origin is within* n. *For all* $p \in (0,1)$, *the central limit theorem holds for the number of open clusters in* $B(n)$.

Next we turn to explain the Widom-Rowlinson model. In 1970, Widom and Rowlinson[60] invented a model on $\mathbb{R}^d$. Lebowitz and Gallavotti[41] introduced its lattice version, which is somewhat similar to the Ising model. We compare these two models:

- The Ising model (one of lattice spin systems) :
  There is a $+$ spin or a $-$ spin on each site.
  A configuration where different $+$ spin and $-$ spin sit next to each other is discouraged.

- The lattice Widom-Rowlinson model (one of lattice gas models) :
  Each site is occupied by either a $+$ particle or a $-$ particle, or it is vacant.
  A configuration where a $+$ particle and a $-$ particle sit next to each other is prohibited.

Let us consider these models on a graph $G = (V, E)$. We say that $x, y \in V$ is adjacent if $\{x, y\} \in E$. Let Λ be a finite region. It is known that the probability that a configuration σ in Λ appears is proportional to $\exp\{-\beta H(\sigma)\}$, where $H(\sigma)$ is the corresponding energy to σ and $\beta > 0$ denotes the inverse temperature. This is called the *finite volume Gibbs distribution*. Energies for the above models are given as follows:

- The Ising model : For $\sigma \in \{-1, +1\}^\Lambda$, $h \in \mathbb{R}$,

$$H(\sigma) = -\sum_{x,y\in\Lambda;\text{adjacent}} \sigma(x)\sigma(y) - \sum_{x\in\Lambda} h\sigma(x).$$

- The Widom-Rowlinson model : for $\sigma \in \{-1, 0, +1\}^\Lambda$, $a, h \in \mathbb{R}$,

$$H(\sigma) = \sum_{x,y\in\Lambda;\text{adjacent}} U(\sigma(x)\sigma(y)) - \sum_{x\in\Lambda} \left\{a\sigma(x)^2 + h\sigma(x)\right\},$$

$$U(s) = \begin{cases} \infty & \text{if } s = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Without loss of generality, we may assume that $\beta = 1$ for the Widom-Rowlinson model by a simple change of variables $\beta a \to a$, $\beta h \to h$.

In order to study phase transitions, we consider infinite particle systems as an idealized version of 'sufficiently large systems'. Probability measures satisfying the Dobrushin-Lanford-Ruelle equation (see section 3.1), which are called *Gibbs measures*, describe equilibrium states of infinite particle systems. For example, the accumulation points of the finite volume Gibbs distribution as Λ tends to the whole lattice are Gibbs measures. The set of all Gibbs measures is non-empty, compact and convex. There may be two cases: One case is that there is a unique Gibbs measure, and the other case is that there are multiple Gibbs measures. Transition from unique Gibbs measure case to multiple Gibbs measures case is called the phase transition. At this transition point, expectations of several random variables exhibit singularities.

On the other hand, appearance of an infinite cluster in Bernoulli site percolation problem is a kind of phase transition, which we call the percolation transition. This problem can be rephrased as follows: Each spin is $+$ with probability p and $-$ with probability $1 - p$ with no interaction. At $p = p_c$, the character of typical configurations are dramatically changed:

	typical spin configurations
$p < p_c$	Each + cluster is finite with probability 1,
$p > p_c$	A + cluster with infinite sites exists with probability 1 (we say that + spins 'percolate').

Phase transitions in the two-dimensional (2D) Ising model is closely related to the change of the geometric character of typical configurations:

Ising model	typical spin configurations
β small (unique Gibbs measure)	Both + clusters and − clusters are small
β large (multiple Gibbs measures)	The majority of spins have the same direction. Each Gibbs measure is characterized by percolation of spins of same sign

Indeed, in the 2D Ising model, the critical temperature of the phase transition is equal to that of the percolation transition. This relation was proved by Coniglio, Nappi, Peruggi and Russo[14], on the basis of the works of Harris[30] and Miyamoto[44].

Russo[52] introduced the infinite cluster method for studying the phase structure of the 2D Ising model. Exploiting this method, Aizenman[1] and Higuchi[31] independently proved the absence of the non-translationally invariant Gibbs states. In 2000, Georgii and Higuchi[20] simplified the proofs of Aizenman and Higuchi. The structure of Gibbs states is described in terms of percolation, and possible extensions, for example to the 2D antiferromagnetic Ising model with arbitrary external field and to the 2D hard-core lattice gas model, are given. Although the 2D Widom-Rowlinson model is generally thought to be similar to the 2D Ising model, the proof in [20] does not work.

The main part of Chapter 3 is published in

> Yasunari Higuchi and Masato Takei ; Some results on the phase structure of two-dimensional Widom-Rowlinson model, *Osaka Journal of Mathematics* **41-2** 237-255, (2004).

We study the phase structure of the lattice Widom-Rowlinson model by using percolation method. To describe our results, we prepare some notations. For Bernoulli site percolation on the d-dimensional hypercubic lattice, we consider two critical probabilities: Let $p_c(d)$ be the critical probability when we connect pairs of distinct sites with the Euclidean distance 1 and $p_c^*(d)$ be the one when we connect the sites with the Euclidean distance not larger than $\sqrt{d}$. It is well-known that $p_c^*(d) < p_c(d)$ and $p_c(2) + p_c^*(2) = 1$. Let $\lambda = e^a$.

Theorem C. *Let $d \geq 2$. For the d-dimensional Widom-Rowlinson model, if $h = 0$ and $\lambda > 2^{2d-1} p_c(d)/(1-p_c(d))$, then there exist two different extremal Gibbs measures μ_λ^+ and μ_λ^- which are invariant under spatial shifts. If $\lambda > 2^{2d-1}(1 - p_c^*(d))/p_c^*(d)$, then the limiting Gibbs measure with a boundary condition in which particles of different kinds do not coexist is a convex combination of μ_λ^+ and μ_λ^-.*

Theorem D. *For the two-dimensional Widom-Rowlinson model, if $h = 0$ and $\lambda > 8(1 - p_c^*(2))/p_c^*(2) = 8p_c(2)/(1 - p_c(2))$, then for every Gibbs measure μ which is invariant under horizontal shifts or vertical shifts, we have $\mu = \alpha\mu_\lambda^+ + (1-\alpha)\mu_\lambda^-$ for some $\alpha \in [0, 1]$.*

Theorem E. *For each $\lambda > 0$, there exists $h_c = h_c(\lambda) \in [0, \infty)$ such that the set of Gibbs measures is a singleton if $|h| > h_c(\lambda)$. Especially, if there are multiple Gibbs measures with parameter $(\lambda, 0)$, then $h_c(\lambda) = 0$.*

Remark. It is shown up to now that the d-dimensional Widom-Rowlinson model admits no phase transition when $h = 0$ and $0 < \lambda < p_c(d)/(1-p_c(d))$.

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Chapter 2

Percolation

2.1 Bernoulli bond percolation

Let $G = (V, E)$ be an infinite connected graph. We fix an arbitrary point as the origin. Each bond $e \in E$ is independently declared to be open with probability p and closed with probability $1 - p$. We denote the Bernoulli measure on $\{\text{open}, \text{closed}\}^E$ by P_p. The expectation, the variance and the covariance relative to P_p are denoted by E_p, var_p and cov_p, respectively. The open cluster containing the origin is denoted by C. The percolation probability $\theta(p) = P_p(|C| = \infty)$ is an increasing function in p, where $|C|$ denotes the number of vertices in C. We define the critical probability $p_c(G) = p_c(G, \text{bond}) \equiv \sup\{p; \theta(p) = 0\}$. We introduce some typical examples of graphs below.

2.1.1 d-dimensional hypercubic lattice $\mathbb{Z}^d$

Let $d \geq 1$. We write $\mathbb{Z}$ for the set of all integers, and put

$$\mathbb{Z}^d = \{(x_1, \cdots, x_d); x_1, \cdots, x_d \in \mathbb{Z}\}.$$

We add edges between all pairs $x, y \in \mathbb{Z}^d$ with the Euclidean distance one. We also write $\mathbb{Z}^d$ for the resulting graph.

If $d = 1$, then no percolation occurs for $p < 1$. It is well-known that $p_c(\mathbb{Z}^2) = 1/2$ (see e.g. [23] or [33]). Generally, $p_c(\mathbb{Z}^{d+1}) < p_c(\mathbb{Z}^d)$ for any $d \geq 1$ ([24]). For $d \geq 2$, when $p > p_c(\mathbb{Z}^d)$ there exists the unique infinite cluster almost surely. Periodicity under spatial shifts plays an important role for proving these facts. So it is natural to ask when we can obtain similar results for more general periodic graphs. A systematic investigation of planar periodic percolation problems is found in [38].

2.1.2 $(d+1)$-regular trees $\mathbb{T}^d$

This is one of graphs which we are interested in. An infinite connected graph containing no cycles is called a tree. For an integer $d \geq 1$, let $\mathbb{T}^d = (V^d, E^d)$ be the tree in which each vertex has $(d+1)$ edges. We call this graph the $(d+1)$-*regular tree.* We fix an arbitrary point as the origin, denoted by $O \in V^d$. For the Bernoulli bond percolation problem on $\mathbb{T}^d$ with $d \geq 2$, It is well-known that $p_c(\mathbb{T}^d) = 1/d$ and there are infinitely many infinite clusters when $p \in (p_c(\mathbb{T}^d), 1)$. One of the main tools for analyzing the percolation problems on trees is the branching process argument (see e.g. [23] §10.1), and another is the mass-transport technique (see e.g. [27]).

2.1.3 Sierpiński carpet lattices G_T

We are also interested in a class of pre-fractal graphs, which are lacking of periodicity.

Generalized Sierpiński carpets. Let $L \geq 2$. For $0 \leq i_1, i_2 \leq L-1$, let $\Psi_{(i_1,i_2)} : \mathbb{R}^2 \to \mathbb{R}^2$ be an affine map which maps $[0,1]^2$ to $[i_1/L, (i_1+1)/L] \times [i_2/L, (i_2+1)/L]$, preserving the directions. For $T \subset \mathbf{T}_L \equiv \{0, \cdots, L-1\}^2$, there exists a unique compact set $K^T \subset [0,1]^2$ such that $K^T = \bigcup_{t \in T} \Psi_t(K^T)$. This is called a generalized Sierpiński carpet.

Sierpiński carpet lattices. Hereafter we assume that $(0,0) \in T$ and K^T is connected. Let

$$F_n^T = \bigcup_{t_1, \cdots, t_n \in T} \Psi_{t_1} \circ \cdots \circ \Psi_{t_n}([0,1]^2).$$

A sequence of graphs $G_n^T = (V_n^T, E_n^T)$ $(n = 1, 2, \cdots)$ are defined by

$$V_n^T = L^n F_n^T \cap \mathbb{Z}^2, E_n^T = \{\langle u, v \rangle; u, v \in V_n^T, |u - v| = 1\},$$

where $|\cdot|$ denotes the usual Euclidean norm. The graph $G_T = (V_T, E_T)$ which is defined by

$$V_T = \bigcup_{n=1}^{\infty} V_n^T, E_T = \bigcup_{n=1}^{\infty} E_n^T$$

is called the Sierpiński carpet lattice corresponding to K^T.

We consider the bond percolation on G_T. Shinoda[56] obtained a sufficient condition for T to satisfy $p_c(G_T) < 1$.

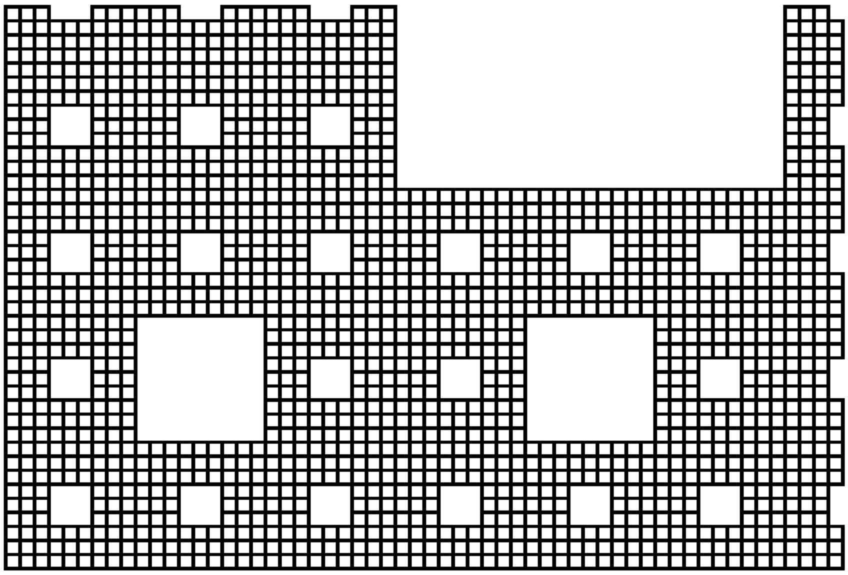

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Figure 2.1: The pre-Sierpiński carpet (G_T with $T = \mathbf{T}_3 \setminus \{(1,1)\}$).

Theorem 2.1.1 ([56]). *For $T \subset \mathbf{T}_L$, put*

$$T_u = \{i; (i, L-1) \in T\},\ T_d = \{i; (i,0) \in T\},$$
$$T_l = \{j; (0,j) \in T\},\ T_r = \{j; (L-1,j) \in T\}.$$

If the following conditions hold, then we have $p_c(G_T) < 1$:

- *For any $t \in T$, $T \setminus \{t\}$ is connected,*
- *$|T_u \cap T_d| \geq 2$ and $|T_l \cap T_r| \geq 2$.*

However, since G_T is in general not periodic, it is difficult to know further properties, e.g. whether the infinite cluster is unique or not.

Example 2.1.2. When $L = 3$ and $T = \mathbf{T}_3 \setminus \{(1,1)\}$, K^T is the Sierpiński carpet. We call corresponding G_T the pre-Sierpiński carpet (see Figure 2.1). By Theorem 2.1.1, we can see that $p_c(G_T) < 1$. Using the rescaling argument in [2], which can be applied to higher dimensional cases, Wu[63] proved the uniqueness of the infinite cluster when p is sufficiently close to 1.

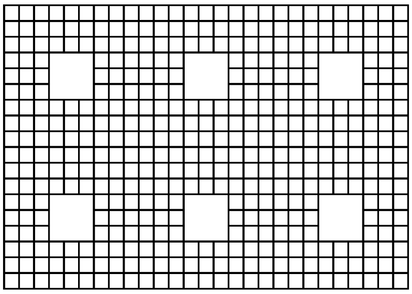

Figure 2.2: Sponge $G_{2,[2,3]}$.

From now on, we treat a class of planar Sierpiński carpet lattices, which was considered by Kumagai[40]. We consider the following conditions for $T \subset \mathbf{T}_L$:

$$K^T \text{ is connected.} \tag{2.1}$$
$$(i,j) \in T \Longrightarrow (j,i) \in T,\ (i, L-1-j) \in T. \tag{2.2}$$
$$\{(0,j); 0 \leq j \leq L-1\} \subset T. \tag{2.3}$$

We consider sponge percolation problems on G_T. Hereafter we sometimes omit T and we write G_n also for the graph congruent to the "original" $G_n = G_n^T$. Let $G_{n,[l,m]}$ be the rectangle which is defined by placing m G_n's horizontally and l G_n's vertically. We also consider a dual graph $G^*_{n,[l,m]}$ of $G_{n,[l,m]}$ (see Figures 2.2 and 2.3. The precise definition is found in [40]). We define the following crossing probabilities:

$$\begin{aligned} &Q_{n,[l,m]}(p) \\ &\quad = P_p(\text{there is an open crossing from the bottom to the top in } G_{n,[l,m]}), \\ &Q^*_{n,[l,m]}(p) \\ &\quad = P_p(\text{there is an } \textit{closed} \text{ crossing from the bottom to the top in } G^*_{n,[l,m]}). \end{aligned}$$

Note that $Q_{n,[l,m]}(p) + Q^*_{n,[m,l]}(p) = 1$. We define the following critical points:

$$p_s = \sup\{p; \limsup_{n\to\infty} Q_{n,[1,3L]}(p) = 0\},\ p_s^* = \inf\{p; \limsup_{n\to\infty} Q^*_{n,[1,3L]}(p) = 0\}.$$

Lemma 2.1.3 ([40]). *Let $L \geq 2$. Suppose that $T \subset \mathbf{T}_L$ satisfies (2.1), (2.2) and (2.3).*

(i) We have $0 < p_s \leq p_c \leq p_s^* < 1$.

(ii) When $p < p_s$, there exist $n_0 \in \mathbb{N}$, $\theta < 1$ and $C > 0$ such that

$$Q_{n_0+m,[1,3L]}(p) \le C\theta^{2^m} \text{ for all } m \ge 0.$$

(iii) (an RSW-type lemma) For $k \ge m \ge 2$,

$$\limsup_{n\to\infty} Q_{n,[m,m]}(p) = 1 \Longleftrightarrow \limsup_{n\to\infty} Q_{n,[k,m]}(p) = 1,$$

which also holds for dual crossing probabilities.

Theorem 2.1.4 ([40]). *Let $L \ge 2$. Suppose that $T \subset \mathbf{T}_L$ satisfies (2.1), (2.2) and (2.3). We assume that*

$$\sup\{p; \limsup_{n\to\infty} Q_{n,[3L,2]}(p) < 1\} = \sup\{p; \limsup_{n\to\infty} Q_{n,[3L,1]}(p) < 1\} (= p_s^*). \tag{2.4}$$

(It is noted in [40] that a sufficient condition for (2.4) is that T satisfies (2.1), (2.2) and $\{(0,j),(1,j); 0 \le j \le L-1\} \subset T$.) Then, the following statements hold.

(i) $p_s = p_c = p_s^*$.
(ii) $\theta(p_c) = 0$.
(iii) The uniqueness of the infinite cluster holds for $p > p_c$.

Shinoda[55] studied the correlation length in Bernoulli bond percolation on the pre-Sierpiński gasket and obtained some hyperscaling relations involving the Hausdorff dimension of the Sierpiński gasket.

For $d \ge 2$, we can define d-dimensional Sierpiński carpet lattices corresponding to $T \subset \mathbf{T}_L^d \equiv \{0, \cdots, L-1\}^d$.

Example 2.1.5. Let $d \ge 2$ and γ be a positive integer strictly less than d. Put

$$T(d,\gamma) = \{(t_1, \cdots, t_d) \in \mathbf{T}_3^d; \sum_{k=1}^{d} 1_{\{t_k=1\}} \le d - \gamma\}.$$

We call $G_{T(d,\gamma)}$ the d-dimensional γ-cavity Menger sponge lattice. Note that $G_{T(d,1)}$ is the d-dimensional Sierpiński carpet. Murai [47] studied an asymptotic behavior of $p_c(G_{T(d,\gamma)})$ as $d \to \infty$: Let $\varepsilon = \varepsilon(d) = o(1)$ with $\varepsilon d^{1/3} \to \infty$ as $d \to \infty$. For sufficiently large d,

$$\frac{1}{2d-1} \le p_c(G_{T(d,\gamma)}, \text{bond}) \le p_c(G_{T(d,\gamma)}, \text{site}) \le \frac{1+\varepsilon}{2(d-\gamma)}.$$

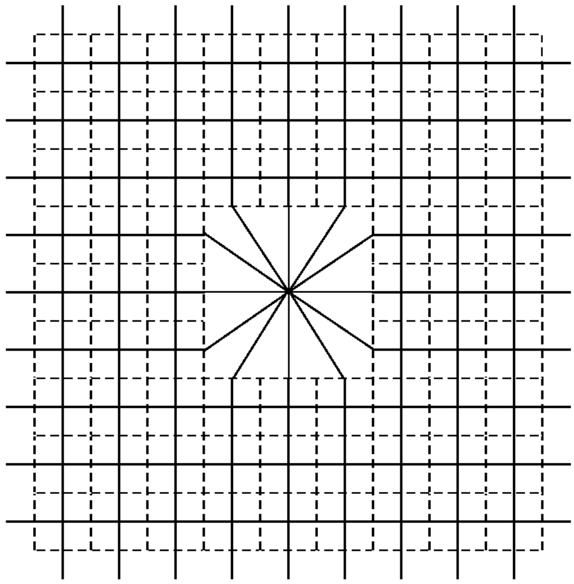

Figure 2.3: 'Dual sponge' $G^*_{2,[1,1]}$.

It is shown in [28] that $p_c(G, \text{bond}) < 1$ is a necessary and sufficient condition for the existence of the phase transition in the Ising model on G. Ising models on fractal-like lattices are studied in several references. Among them, we refer to [13], [37], [62] and [59].

For some classes of Sierpiński carpet lattices, it is shown in [57] that the critical probability of the oriented percolation equals 1 (for oriented percolation problems, we refer to [23] §12.8).

2.2 Bernoulli site percolation

In the site percolation problem, each site $v \in V$ is declared to be open or closed. Corresponding Bernoulli measure on $\{\text{open}, \text{closed}\}^V$ is denoted by $\hat{P}_p$. A sequence $p = (x_1, \cdots, x_k)$ of distinct points of V is a (finite) *path* from x_1 to x_k if $\{x_i, x_{i+1}\} \in E$ $(i = 1, \cdots, k-1)$. We similarly define an infinite path $p = (x_1, x_2, \cdots)$. A region $C \subset V$ is said to be *connected* if for any $x, y \in C$ there exists a path in C from x to y. A *cluster* in $S \subset V$ is a maximal connected component of S. A cluster which contains infinitely many points is called an *infinite cluster*. We use the notations

$\hat{E}_p$, $\widehat{\mathrm{var}}_p$, $\hat{C}, \hat{\theta}(p)$, $p_c(G, \mathrm{site})$ and so on.

Theorem 2.2.1 ([24]). *Let $G = (V, E)$ be an infinite connected graph with countably many edges and maximum vertex degree $\Delta < \infty$. The critical probabilities of G satisfy*

$$\frac{1}{\Delta - 1} \leq p_c(G, \mathrm{bond}) \leq p_c(G, \mathrm{site}) \leq 1 - (1 - p_c(G, \mathrm{bond}))^{\Delta - 1}.$$

2.3 The number of open clusters

Let us consider the Bernoulli bond percolation problem on $G = (V, E)$. We fix an arbitrary site as the origin. Let $\{B(n)\}$ be an increasing sequence of finite regions containing the origin. The number of open clusters in $B(n)$ is denoted by $K_n = K_n(\omega)$. We can see that

$$K_n = \sum_{x \in B(n)} \frac{1}{|C_n(x)|},$$

where $C_n(x) = \{y \in B(n); \text{there is an open path in } B(n) \text{ from } x \text{ to } y\}$. Indeed,

$$\begin{aligned} \sum_{x \in B(n)} \frac{1}{|C_n(x)|} &= \sum_{C:\text{a cluster in } B(n)} \sum_{x \in B(n): C_n(x) = C} \frac{1}{|C_n(x)|} \\ &= \sum_{C:\text{a cluster in } B(n)} |C| \times \frac{1}{|C|} = K_n. \end{aligned}$$

In the site problem, we write the number of open clusters in $B(n)$ for $\hat{K}_n$. In this case,

$$\hat{K}_n = \sum_{x \in B(n)} \frac{1}{|\hat{C}_n(x)|} 1_{\{|\hat{C}_n(x)| > 0\}}.$$

As an analogue of this, in the bond problem we also consider

$$\tilde{K}_n = \sum_{x \in B(n)} \frac{1}{|C_n(x)|} 1_{\{|C_n(x)| > 1\}},$$

In other words, when we regard isolated points as clusters, we denote the number of open clusters in $B(n)$ by K_n. Otherwise we denote it by $\tilde{K}_n$.

Example 2.3.1. Let us consider Bernoulli bond percolation on trees. We can see that

$$K_n = |B(n)| - ||\omega|_{B(n)}|| = ||B(n)|| + 1 - ||\omega|_{B(n)}||, \tag{2.5}$$

where $||\cdot||$ denotes the number of the bonds.

When G is a regular tree, we can obtain the exact form of $\kappa(p)$. For example, when $G = \mathbb{T}^2$,

$$\kappa(p) = \frac{1+2\varepsilon}{1-2\varepsilon}\sum_{n=1}^{\infty}\frac{1}{n^2}\binom{2n}{n-1}\frac{1}{4^n}(1-4\varepsilon^2)^n,$$

where $\varepsilon = 1/2 - p$ (see e.g. [23] §10.1). We can see that $\kappa'''(p)$ has a jump discontinuity at $p = p_c(\mathbb{T}^2) = 1/2$.

Example 2.3.2. We consider Bernoulli bond percolation on $\mathbb{Z}^d$ with $d \geq 2$. Assume that $0 < p < 1$. The number of open clusters in $B(n) = [-n, n]^d \cap \mathbb{Z}^d$ is denoted by $K_n = K_n(\omega)$. Put $\kappa(p) \equiv E_p(|C|^{-1})$. Note that $0 < \kappa(p) < 1$. The strong law of large numbers for K_n holds:

$$\frac{1}{|B(n)|}K_n \longrightarrow \kappa(p) \text{ a.s. and in } L^1.$$

It is the reason why $\kappa(p)$ is called *the number of open clusters per vertex* (see [23] §4.1 for details).

Sykes and Essam[58] derived 'exact' calculations of the critical probabilities of some planar graphs, under the assumption that $\kappa(p)$ has a unique singularity at the critical point. It is conjectured that $\kappa(p)$ is not thrice differentiable at $p = p_c(\mathbb{Z}^2)$. For the site percolation on the triangular lattice, Zhang[66] gave an affirmative answer to this conjecture.

It is widely believed that the phase transition for large d is qualitatively similar to that of a binary tree. Yang and Zhang[64] studied the differentiability of $\kappa(p)$ near the critical point for sufficiently large d.

2.4 CLT for martingale differences

We quote the central limit theorem for the martingale difference array, proved by McLeish[43].

Let $(\Omega, \mathcal{F}, P)$ be the basic probability space. We consider a family of random variables $\{X_{j,n}; j = 1, \cdots q_n\}$ with $EX_{j,n}^2 < \infty$ and put $S_n = \sum_{j=1}^{q_n} X_{j,n}$. Let $\{\mathcal{F}_{j,n}; j = 1, \cdots q_n\}$ be a family of sub σ-field of $\mathcal{F}$ with

$\mathcal{F}_{j-1,n} \subset \mathcal{F}_{j,n}$. We call $\{X_{j,n}\}$ a *martingale difference array* with respect to $\{\mathcal{F}_{j,n}\}$ if $X_{j,n}$ is $\mathcal{F}_{j,n}$-measurable and it holds that $E[X_{j,n}|\mathcal{F}_{j-1,n}] = 0$ a.s. for all j and n.

Theorem 2.4.1 ([43] (2.3)). *Let $\{X_{j,n}\}$ be a martingale difference array with the following conditions:*

$$\max_{1\leq k\leq q_n} |X_{k,n}| \text{ is bounded under } L^2\text{-norm, uniformly over } n, \tag{2.6}$$

$$\max_{1\leq k\leq q_n} |X_{k,n}| \xrightarrow{\text{p}} 0 \text{ as } n \to \infty, \tag{2.7}$$

$$\sum_{k=1}^{q_n} X_{k,n}^2 \xrightarrow{\text{p}} 1 \text{ as } n \to \infty, \tag{2.8}$$

where $\xrightarrow{\text{p}}$ denotes the convergence in probability. Then, it holds that $S_n \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$ as $n \to \infty$.

2.5 CLT for the number of open clusters on trees

The uniqueness of the infinite cluster plays an important role in [65], but we do not need the uniqueness for proving the following theorem, due to the geometric character of trees.

Theorem 2.5.1. *We consider the Bernoulli bond percolation problem on $\mathbb{T}^d$ $(d \geq 2)$. Let $B(n) = \{x; d(O,x) \leq n\}$, where $d(\cdot,\cdot)$ denotes the graph distance. For any $p \in (0,1)$, the central limit theorems for $\{K_n\}$ and $\{\tilde{K_n}\}$ hold.*

Some remarks are in order. We note that the CLT for $\{K_n\}$ follows from the CLT for i.i.d. sequences, using (2.5). We can obtain the CLT for the number of open clusters in Bernoulli site percolation by a similar proof as for $\tilde{K_n}$. Our proof is valid for trees for which we can verify (2.11) and (2.12), e.g. trees of bounded degree.

Our proof is based on the argument in [65]. We enumerate the elements of E^d as $e_1, e_2, \cdots$ according to the following rule:

Let $A(n) = B(n) \setminus B(n-1)$, where $B(0) = \emptyset$.

1) If $m < n$, then for any $e_i \in A(m)$ and $e_j \in A(n)$ we have $i < j$.

2) For any i, $\{e_1, e_2, \cdots, e_i\}$ is connected.

Let $q_n = ||B(n)||$. We write $\Omega = \{0,1\}^{E^d} \ni \omega = (\omega_1, \omega_2, \cdots)$. We define σ-fields $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \sigma[\omega_1, \cdots, \omega_k]$ $(k \geq 1)$. Let $f_n = K_n$ or $\tilde{K}_n$. Noting that f_n depends only on the first q_n coordinates, we can write

$$f_n - E_p f_n = \sum_{k=1}^{q_n} \{E_p[f_n|\mathcal{F}_k] - E_p[f_n|\mathcal{F}_{k-1}]\}.$$

Let $\Delta_{k,n} = E_p[f_n|\mathcal{F}_k] - E_p[f_n|\mathcal{F}_{k-1}]$ $(1 \leq k \leq q_n)$. These are martingale differences : $E_p[\Delta_{k,n}|\mathcal{F}_{k-1}] = 0$. This implies that $\mathrm{var}_p f_n = \sum_{k=1}^{q_n} E_p \Delta_{k,n}^2$. Thus we have

$$\frac{f_n - E_p f_n}{\sqrt{\mathrm{var}_p f_n}} = \sum_{k=1}^{q_n} X_{k,n}, \text{ where } X_{k,n} = \frac{\Delta_{k,n}}{\sqrt{\sum_{k=1}^{q_n} E_p \Delta_{k,n}^2}}.$$

Since $\Delta_{k,n}(\omega)$ is $\mathcal{F}_k$-measurable, we regard $\Delta_{k,n}(\omega)$ as a function of the first k coordinates of ω. We have

$$\begin{aligned}
&\Delta_{k,n}(\omega_1, \cdots, \omega_{k-1}, \alpha) \\
&= \sum_{\omega'_{k+1}, \cdots, \omega'_{q_n}=0,1} \delta_k f_n(\omega_1, \cdots, \omega_{k-1}, \alpha, \omega'_{k+1}, \cdots, \omega'_{q_n}) \\
&\quad \times P_p\{(\omega_{k+1}, \cdots, \omega_{q_n}) = (\omega'_{k+1}, \cdots, \omega'_{q_n})\},
\end{aligned}$$

where $\alpha \in \{0,1\}$ and

$$\begin{aligned}
&\delta_k f_n(\omega_1, \cdots, \omega_{k-1}, \alpha, \omega'_{k+1}, \cdots, \omega'_{q_n}) \\
&= p^{1-\alpha}(1-p)^{\alpha}\{f_n(\omega_1, \cdots, \omega_{k-1}, \alpha, \omega'_{k+1}, \cdots, \omega'_{q_n}) \\
&\quad - f_n(\omega_1, \cdots, \omega_{k-1}, 1-\alpha, \omega'_{k+1}, \cdots, \omega'_{q_n})\}.
\end{aligned} \tag{2.9}$$

We will check that $\{X_{k,n}\}$ satisfy the conditions (2.6)-(2.8) of Theorem 2.4.1. To verify the conditions (2.6) and (2.7), it is sufficient to check (2.10) and (2.11):

There exists $M > 0$ such that $|\Delta_{k,n}| \leq M$ for all n and k. (2.10)

There exists $\sigma = \sigma(p) > 0$ such that $\sum_{k=1}^{q_n} E_p \Delta_{k,n}^2 \geq \sigma q_n$ for all n. (2.11)

To prove (2.8), we have only to show that

$$\frac{1}{q_n} \sum_{k=1}^{q_n} (\Delta_{k,n}^2 - E_p \Delta_{k,n}^2) \xrightarrow{\mathrm{P}} 0 \text{ as } n \to \infty, \tag{2.12}$$

thanks to (2.11). This says that the weak law of large numbers for $\{\Delta_{k,n}^2\}$ implies the central limit theorem for $\{X_{k,n}\}$.

Proof of Theorem 2.5.1. We consider the case $f_n = \tilde{K}_n$ only. Since $\{\quad\}$ in RHS of (2.9) is the difference of $\tilde{K}_n$ caused by changing only the state of e_k, we can see that $|\delta_k \tilde{K}_n| \leq 1$ and $|\Delta_{k,n}| \leq 1$, which proves (2.10).

Next we prove (2.11) (our method is similar to the proof of (3.5) in [15]). Fix an integer $k \in \{1, 2, \cdots, q_n\}$. We denote the set of the indices of edges which contain at least one endpoint of e_k by $N(e_k)$. Let

$$D(k) = \{\omega = (\omega_1, \cdots, \omega_k); \omega_i = 1 \text{ for } i \in N(e_k) \cap \{1, \cdots, k-1\} \text{ and } \omega_k = 0\},$$
$$D'(k) = \{\omega' = (\omega'_{k+1}, \cdots, \omega'_{q_n}); \omega'_i = 1 \text{ for } i \in N(e_k) \cap \{k+1, \cdots, q_n\}\}.$$

Noting that $\mathbb{T}^d$ has no cycles and $\{e_1, \cdots, e_k\}$ is connected, we can see that

$$\delta_k \tilde{K}_n(\omega_1, \cdots, \omega_k, \omega'_{k+1}, \cdots, \omega'_{q_n}) \geq 0 \text{ if } \omega \in D(k)$$

and $\delta_k \tilde{K}_n(\omega_1, \cdots, \omega_k, \omega'_{k+1}, \cdots, \omega'_{q_n}) = p$ if $\omega \in D(k)$ and $\omega' \in D'(k)$. For any $\omega \in D(k)$,

$$\begin{aligned}\Delta_{k,n}(\omega_1, \cdots, \omega_k) &\geq \sum_{\omega' \in D'(k)} p P_p\{(\omega_{k+1}, \cdots, \omega_{q_n}) = (\omega'_{k+1}, \cdots, \omega'_{q_n})\} \\ &\geq p \cdot p^{2d}(1-p) = p^{2d+1}(1-p).\end{aligned}$$

Thus we have

$$\begin{aligned}E_p \Delta_{k,n}^2 &\geq E_p[\{\Delta_{k,n}(\omega_1, \cdots, \omega_k)\}^2; D(k)] \\ &\geq \{p^{2d+1}(1-p)\}^2 \cdot p^{2d} = p^{6d+2}(1-p)^2 \equiv \sigma(p).\end{aligned}$$

Finally we verify (2.12). We write $e = \langle x_1(e), x_2(e) \rangle$ when $d(O, x_1(e)) < d(O, x_2(e))$. For $e_1, e_2 \in E$, let $d(e_1, e_2) = \min_{i,j=1,2} d(x_i(e_1), x_j(e_2))$. Since there are no cycles on $\mathbb{T}^d$, we can see that $\delta_k \tilde{K}_n$ depends only on the state of $N(e_k)$. So $\Delta_{i,n}$ and $\Delta_{j,n}$ are independent if $d(e_i, e_j) > 1$. Now (2.12) easily follows from Chebyshev's inequality. □

2.6 LLN for the number of open clusters on Sierpiński carpet lattices

Theorem 2.6.1. *We consider Bernoulli bond percolation on a Sierpiński carpet lattice G_T. We assume that T satisfies (2.1), (2.2) and (2.3). Let*

$B(n) = G_n^T$ *(with abuse of notation). For all* $p \in [0,1]$, *the limit* $m(p) \equiv \lim_{n\to\infty} \frac{E_p K_n}{||B(n)||}$ *exists and*

$$\lim_{n\to\infty} \frac{K_n}{||B(n)||} = m(p) \ a.s.$$

For the proof, we prepare some notations. For $\underline{x} = (x_1, x_2) \in \mathbb{Z}_+^2 \equiv \{0, 1, 2, \cdots\}^2$, we define $G_m^T(\underline{x}) = (V_m^T(\underline{x}), E_m^T(\underline{x}))$ with

$$V_m^T(\underline{x}) = V_T \cap \{[x_1 L^m, (x_1+1)L^m] \times [x_2 L^m, (x_2+1)L^m]\},$$
$$E_m^T(\underline{x}) = \{\langle u, v\rangle \in E_T; u, v \in V_m^T(\underline{x})\}.$$

Let $m < n$. For $e \in E_T$, let $\underline{x}_m(e) = (x_1^m(e), x_2^m(e)) \in \mathbb{Z}_+^2$ be such that e belongs to $G_m^T(e) \equiv G_m^T(\underline{x}_m(e))$. For $\underline{x} = (x_1, x_2) \in \mathbb{Z}_+^2$, let $||\underline{x}||_1 = |x_1| + |x_2|$ and $||\underline{x}||_\infty = \max\{|x_1|, |x_2|\}$. When we regard G_n as a union of G_m's, we often index these G_m's by T^{n-m}. For fixed m, we identify an element of T^{n-m} as that of $\mathbb{Z}_+^2$ in obvious fashion.

For a region S of G_T, the border points of S is defined by the inner boundary sites when we regard S as a subset of $\mathbb{Z}^2$. For $m > 0$, $\operatorname{int} G_m$ denotes the graph obtained by deleting the border points of G_m and the edges connecting them. For $m > l > 0$, let

$$\partial_l G_m = \{G_l(\underline{x}) : \underline{x} \in T^{m-l}, G_l(\underline{x}) \text{ contains some of border points of } G_m\}$$

(see Figure 2.4). We often use the following facts.

Lemma 2.6.2. *We assume that T satisfies (2.1), (2.2) and (2.3).*
(i) For $m > l > 0$, $||G_m|| \geq |T|^{m-l} \times ||\operatorname{int} G_l||$.
(ii) $||\operatorname{int} G_m|| / ||G_m|| \to 1$ *as* $m \to \infty$.
(iii) $\sup_{l\geq 1} \frac{||\partial_l G_{l+j}||}{||G_{l+j}||} \to 0$ *as* $j \to \infty$.

Proof. (i) is obvious. By (2.3), we have $|T| \geq 4(L-1)$ and $||\operatorname{int} G_1|| \geq 4L$. Noting that $||\operatorname{int} G_m|| = ||G_m|| - 4L^m$ and $||G_m|| \geq \{4(L-1)\}^{m-1} \cdot 4L$, we can prove (ii). Using (i) and (ii), we have

$$\sup_{l\geq 1} \frac{||\partial_l G_{l+j}||}{||G_{l+j}||} \leq \frac{4(L^j - 1)}{\{4(L-1)\}^j} \sup_{l\geq 1} \frac{||G_l||}{||\operatorname{int} G_l||} \to 0 \text{ as } j \to \infty,$$

which proves (iii). □

Noting that $|\operatorname{int} G_l| = |G_l| - 4L^l$ and $|\operatorname{int} G_1| \geq 4(L-2)$, we obtain the following

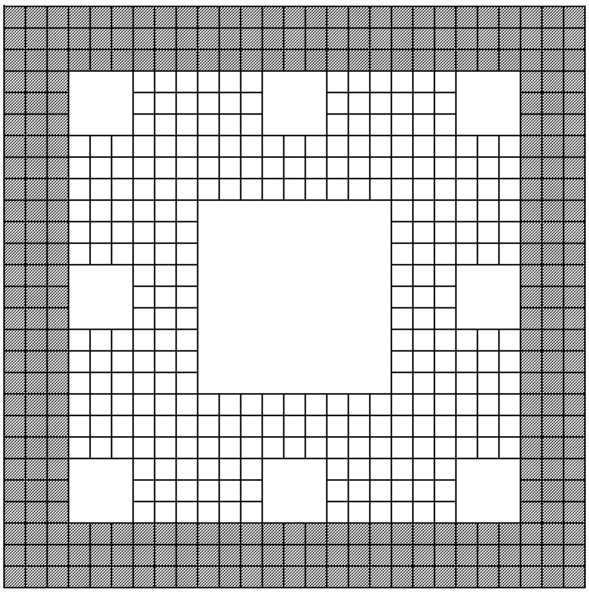

Figure 2.4: The shaded region indicates $\partial_1 G_3$.

Corollary 2.6.3. *We assume that T satisfies (2.1), (2.2) and (2.3).*
(i) For $m > l > 0$, $|G_m| \geq |T|^{m-l} \times |\operatorname{int} G_l|$.
(ii) $|\operatorname{int} G_m|/|G_m| \to 1$ as $m \to \infty$.

Proof of Theorem 2.6.1. We generalize the definition of K_n. For a finite subgraph S of G_T, $K(S)$ denotes the number of open clusters in S. Note that $K(S_1 \cup S_2) \leq K(S_1) + K(S_2)$. Suppose that $n > m > 0$. Noting that $K_n \leq \sum_{\underline{x} \in T^{n-m}} K(G_m^T(\underline{x}))$ and $E_p K(G_m^T(\underline{x})) = E_p K_m$ for any $\underline{x} \in T^{n-m}$, we have $E_p K_n \leq |T|^{n-m} E_p K_m$. Dividing by $q_n \equiv ||B(n)||$ and using Lemma 2.6.2 (i), we can see that

$$\frac{E_p K_n}{q_n} \leq \frac{|T|^{n-m} E_p K_m}{q_n} \leq \frac{E_p K_m}{||\operatorname{int} B(m)||} = \frac{E_p K_m}{q_m} \frac{q_m}{||\operatorname{int} B(m)||}.$$

This implies the existence of the limit of $E_p K_n / q_n$.

Let $E_n = \{|(K_n - E_p K_n)/q_n| \geq |T|^{-n/4}\}$. Noting that $|\Delta_{k,n}^2| \leq 1$, we have

$$E_p \left(\frac{K_n - E_p K_n}{q_n} \right)^2 = \frac{\operatorname{var}_p K_n}{q_n^2} = \frac{1}{q_n^2} \sum_{k=1}^{q_n} E_p \Delta_{k,n}^2 \leq \frac{1}{q_n}.$$

It follows from Chebyshev's inequality and Lemma 2.6.2 (i) that

$$P(E_n) \le |T|^{n/2} \cdot \frac{1}{q_n} \le \frac{|T|^{n/2}}{4L|T|^{n-1}} = \frac{1}{4L|T|^{n/2-1}}.$$

By Borel-Cantelli's lemma, we can show the almost sure convergence of K_n/q_n. □

Noting that $|B(n)| \le 2q_n$, we can prove the following theorem in a similar manner.

Theorem 2.6.4. *We consider Bernoulli bond percolation on a Sierpiński carpet lattice G_T. We assume that T satisfies (2.1), (2.2) and (2.3). For all $p \in [0,1]$, the limit $\hat{m}(p) \equiv \lim_{n\to\infty} \frac{E_p K_n}{|B(n)|}$ exists and*

$$\lim_{n\to\infty} \frac{K_n}{|B(n)|} = \hat{m}(p) \ \ a.s.$$

2.7 CLT for the number of open clusters on Sierpiński carpet lattices

Theorem 2.7.1. *We consider Bernoulli bond percolation on a Sierpiński carpet lattice G_T. We assume that T satisfies (2.1), (2.2) and (2.3). Let $B(n) = G_n^T$. We can prove the CLT for $\{K_n\}$ for $p \in (0,1) \setminus [p_s, p_s^*]$. Moreover, if (2.4) is satisfied, then the CLT for $\{K_n\}$ holds for all $p \in (0,1)$.*

We remark that similar results can be proved for $\{\tilde{K}_n\}$.

Let $L \ge 2$. Fix $T \subset \mathbf{T}_L$ which satisfies (2.1), (2.2) and (2.3). In the same way as in section 2.5, we define $E_T = \{e_1, e_2, \cdots\}$, q_n, $\{\mathcal{F}_k\}$, $\Delta_{k,n}$ and $X_{k,n}$.

We can easily check the condition (2.10) for $f_n = K_n$ or $\tilde{K}_n$ as in section 2.5. While we can verify (2.11) for K_n by using the FKG inequality as in [65], we cannot apply this method to $\tilde{K}_n$. So we prove (2.11) for $\tilde{K}_n$ by the same argument as in section 2.5. Fix an integer $k \in \{1, 2, \cdots, q_n\}$. Let $\bar{N}(e_k) = \{i \in \{1, 2, \cdots, q_n\}; e_i \in E_n^T \setminus N(e_k) \text{ and } e_i \cap e_j \ne \emptyset \text{ for some } j \in N(e_k)\}$. Note that $||N(e_k)|| \le 7$ and $||\bar{N}(e_k)|| \le 16$. We modify the definitions of $D(k)$ and $D'(k)$:

$$D(k) = \left\{ \omega = (\omega_1, \cdots, \omega_k); \begin{array}{c} \omega_i = 1 \text{ for } i \in N(e_k) \cap \{1, \cdots, k-1\}, \omega_k = 0, \\ \omega_j = 0 \text{ for } j \in \bar{N}(e_k) \cap \{1, \cdots, k-1\} \end{array} \right\},$$

$$D'(k) = \left\{ \omega' = (\omega'_{k+1}, \cdots, \omega'_{q_n}); \begin{array}{c} \omega'_i = 1 \text{ for } i \in N(e_k) \cap \{k+1, \cdots, q_n\}, \\ \omega'_j = 0 \text{ for } j \in \bar{N}(e_k) \cap \{k+1, \cdots, q_n\} \end{array} \right\}.$$

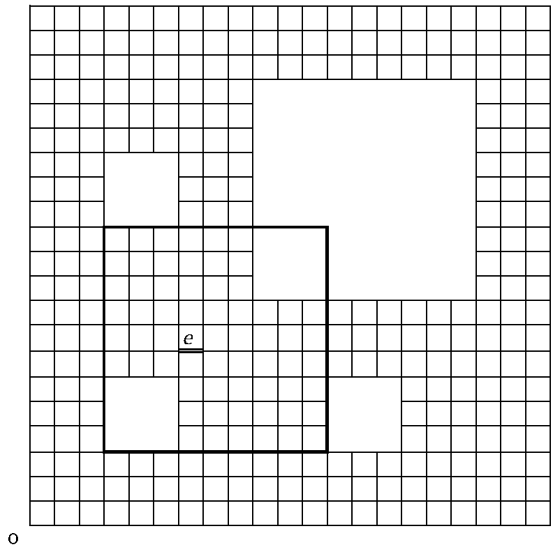

Figure 2.5: The thick-lined box is $\bar{G}^T_{1,\cdot}(e)$.

Now we can prove (2.11) for $\tilde{K}_n$ along the same line as in section 2.5.

The condition (2.12) will be checked in the following three subsections. We shall give a proof only for K_n. Almost the same proof works for $\tilde{K}_n$.

2.7.1 Subcritical regime : $p < p_s$

Let $n > m$. We define $\bar{G}^T_{m,n}(e) = (\bar{V}^T_{m,n}(e), \bar{E}^T_{m,n}(e))$ by

$$\bar{V}^T_{m,n}(e)$$
$$= V^T_n \cap \{[(x^m_1(e) - 1)L^m, (x^m_1(e) + 2)L^m] \times [(x^m_2(e) - 1)L^m, (x^m_2(e) + 2)L^m]\},$$
$$\bar{E}^T_{m,n}(e) = \{\langle u, v\rangle \in E^T_n; u, v \in \bar{V}^T_{m,n}(e)\}$$

(see Figure 2.5). The (inner) boundary of $\bar{G}^T_{m,n}(e)$ is defined by

$$\left\{u \in \bar{V}^T_{m,n}(e); \langle u, v\rangle \in E_T \text{ for some } v \in V^T_n \setminus \bar{V}^T_{m,n}(e)\right\}.$$

For $e \in E^T_n$, let

$$D(e, m, n) = \left\{\begin{array}{c}\text{each endpoints of edge } e \text{ belong to}\\ \text{different open clusters in } \bar{G}^T_{m,n}(e)\\ \text{which are connected to the boundary of } \bar{G}^T_{m,n}(e)\end{array}\right\}.$$

Noting that $P_p(D(e,m,n)) \le 4Q_{m,[1,3]}$, we can see the following by Lemma 2.1.3(ii).

Lemma 2.7.2. *If $p < p_s$, then for given $\varepsilon > 0$ we can take a sufficiently large m_0 such that $P_p(D(e,m,n)) \le \varepsilon$ for all $n > m \ge m_0$ and $e \in E_n^T$.*

Now we prove Theorem 2.7.1 for $p < p_s$. We shall verify (2.12). Fix $\varepsilon > 0$. We take sufficiently large m so that the statement of Lemma 2.7.2 holds. Let $n > m$. We compare $\Delta_{k,n}$ with $\Delta'_{k,n} = {}_m\Delta'_{k,n} \equiv \Delta_{k,n} 1_{D(e_k,m,n)^c}$. We have

$$\begin{aligned}
&\frac{1}{q_n}\left|\sum_{k=1}^{q_n}(\Delta_{k,n}^2 - E_p\Delta_{k,n}^2)\right| \\
&\le \frac{1}{q_n}\left|\sum_{k=1}^{q_n}\{\Delta_{k,n}^2 - (\Delta'_{k,n})^2\}\right| + \frac{1}{q_n}\left|\sum_{k=1}^{q_n}\{(\Delta'_{k,n})^2 - E_p(\Delta'_{k,n})^2\}\right| \\
&\quad + \frac{1}{q_n}\left|\sum_{k=1}^{q_n}\{E_p(\Delta'_{k,n})^2 - E_p\Delta_{k,n}^2\}\right| = S_{\mathrm{I}} + S_{\mathrm{II}} + S_{\mathrm{III}}.
\end{aligned}$$

Noting that $|\Delta_{k,n}| \le 1$ and $E_p|\Delta_{k,n} - \Delta'_{k,n}| \le P_p(D(e_k,m,n))$, we have

$$\begin{aligned}
E_p(S_{\mathrm{I}}) &\le \frac{1}{q_n}\sum_{k=1}^{q_n} E_p|\Delta_{k,n} + \Delta'_{k,n}||\Delta_{k,n} - \Delta'_{k,n}| \\
&\le \frac{1}{q_n} q_n \cdot 2 \cdot \varepsilon = 2\varepsilon.
\end{aligned}$$

Similarly, we have $E_p(S_{\mathrm{III}}) \le \dfrac{1}{q_n}\displaystyle\sum_{k=1}^{q_n} E_p|(\Delta'_{k,n})^2 - \Delta_{k,n}^2| \le 2\varepsilon$.

Next we show that $E_p(S_{\mathrm{II}}^2) \to 0$ as $n \to \infty$. To this end, we shall prove $\Delta'_{i,n}$ depends on the states of the edges in $\bar{G}_{m,n}^T(e_i)$, so that $\Delta'_{i,n}$ and $\Delta'_{j,n}$ are independent if $||\underline{x}_m(e_i) - \underline{x}_m(e_j)||_\infty > 3$. We split $D(e_i,m,n)^c$ into two disjoint events:

$$G(e_i,m,n) = \left\{\begin{array}{c}\text{in } \bar{G}_{m,n}^T(e_i)\text{, endpoints of } e_i \text{ is connected to each other} \\ \text{by an open path not traversing } e_i\end{array}\right\},$$

$$H(e_i,m,n) = \left\{\begin{array}{c}\text{each endpoints of edge } e_i \text{ belong to different} \\ \text{open clusters, but not both of these clusters are} \\ \text{connected to the boundary of } \bar{G}_{m,n}^T(e_i)\end{array}\right\}.$$

When $G(e_i, m, n)$ occurs, the number of open clusters are independent of the state of e_i. Hence $\delta_i K_n 1_{G(e_i,m,n)} = 0$. On the other hand, we can see that

$$\delta_i K_n 1_{H(e_i,m,n)} = \begin{cases} -(1-p) & \text{if } \omega_i = 1, \\ p & \text{if } \omega_i = 0. \end{cases}$$

Thus $\Delta'_{i,n}$ is measurable with respect to the states of edges in $\bar{G}^T_{m,n}(e_i)$. Now we have

$$\begin{aligned} E_p(S_{\text{II}}^2) &= \frac{1}{q_n^2} \sum_{i=1}^{q_n} \sum_{j=1}^{q_n} \text{cov}_p \left((\Delta'_{i,n})^2, (\Delta'_{j,n})^2 \right) \\ &\leq \frac{1}{q_n^2} \sum_{i=1}^{q_n} \sum_{j:||\underline{x}_m(e_j)-\underline{x}_m(e_i)||_\infty \leq 3} 4 \leq \frac{1}{q_n^2} q_n \cdot 49 \cdot 4 = \frac{196}{q_n}. \end{aligned}$$

Using Markov's and Chebyshev's inequalities, we can prove (2.12). This completes the proof.

2.7.2 Critical regime : $p \in [p_s, p_s^*]$

Once we prove the following lemma, we can obtain the CLT for $p \in [p_s, p_s^*]$ by the same argument as in the preceding subsection.

Lemma 2.7.3. *We assume that (2.4) holds. If $p \leq p_s^*$, then for given $\varepsilon > 0$ we can take a sufficiently large m_0 such that $P_p(D(e, m, n)) \leq \varepsilon$ for all $n > m \geq m_0$ and $e \in E_n^T$.*

Proof. Since $T = \mathbf{T}_2$ (i.e. $G_T = \mathbb{Z}^2$) is the only case that $T \subset \mathbf{T}_2$ satisfies (2.1), (2.2) and (2.3), we consider the case $L \geq 3$. Using the duality equation, (2.4) and Lemma 2.1.3(iii), we can prove that there exists a positive constant δ such that $Q^*_{n,[3L,1]} \geq \delta$ for all n. For $e \in E_T$ and $i \geq 1$, let $D_i(e)$ be the union of four G_i's (or $L^i \times L^i$ holes), which are connected to the corner closest to e among the corners of $G_i^T(e)$. Let $A_i(e)$ be G_{i-1}'s (or $L^{i-1} \times L^{i-1}$ holes) which contain the border points of $D_i(e)$. Note that for all i there is an dual closed circuit in $A_i(e)$ with probability $\geq \delta^4$. We take m_0 such that $(1 - \delta^4)^{m_0} \leq \varepsilon$. For $m > m_0$, we have

$$\begin{aligned} & P_p(D(e, m, n)) \\ & \leq P_p \left(\bigcap_{i=2,\cdots,m_0+1} \{\text{there is no dual closed circuit in } A_i(e)\} \right) \\ & \leq (1 - \delta^4)^{m_0} \leq \varepsilon. \end{aligned}$$

This completes the proof. ☐

2.7.3 Supercritical regime : $p > p_s^*$

First we note that by Lemma 2.6.2, for given $\eta > 0$ we can find j_0 such that $||\partial_l G_{l+j}||/||\operatorname{int} G_{l+j}|| \leq \eta$ for all $l \geq 1$ and $j \geq j_0$.

Lemma 2.7.4. *Suppose that $p > p_s^*$. For fixed j, we have*

$$\alpha(l) = \alpha(l,p) \equiv P_p(\text{there exists an open circuit in } \partial_l G_{l+j}) \to 1 \text{ as } l \to \infty.$$

Proof. By the FKG inequality, we have

$$\alpha(p) \geq [Q_{l,[L^j,1]}(p)]^4 \geq [\{Q_{l,[3,1]}(p)\}^{N_j}]^4, \text{ where } N_j = 2\lceil (L^j+1)/3\rceil - 1.$$

Since $\limsup_{l\to\infty} Q_{l,[3,1]}(p) = 1$ for $p > p_s^*$, we get the conclusion. ☐

Now we check (2.12). We fix an integer l. Suppose that $n > m > l + j_0$. We regard $B(n) = G_n^T$ as $\bigcup_{\underline{x}\in T^{n-m}} G_m^T(\underline{x})$. For $\underline{x} \in T^{n-m}$, let $1(\underline{x})$ be the indicator function of {there exists an open circuit in $\partial_l G_m(\underline{x})$}. Let $S_k = \Delta_{k,n}^2 - E_p\Delta_{k,n}^2$ and $\tilde{S}_k = S_k 1(\underline{x}_m(e_k))$. Note that $|\tilde{S}_k| \leq |S_k| \leq 1$ and $|E_p\tilde{S}_k| = |E_p(S_k - \tilde{S}_k)| = |E_pS_k(1 - 1(\underline{x}_m(e_k)))| \leq 1 - \alpha(l)$. For $i, k \in \{1, 2, \cdots, q_n\}$, we have

$$\begin{aligned} E_p[S_iS_k] &= E_p[\tilde{S}_i\tilde{S}_k] + E_p[S_i(S_k - \tilde{S}_k)] + E_p[\tilde{S}_k(S_i - \tilde{S}_i)] \\ &\leq E_p[\tilde{S}_i\tilde{S}_k] + 2(1 - \alpha(l)). \end{aligned}$$

Let $U(n) = \bigcup_{\underline{x}\in T^{n-m}} \partial_l G_m(\underline{x})$. Note that by the choice of m and Lemma 2.6.2 (i),

$$||U(n)|| \leq \sum_{\underline{x}\in T^{n-m}} ||\partial_l G_m(\underline{x})|| \leq \sum_{\underline{x}\in T^{n-m}} \eta||\operatorname{int} G_m(\underline{x})|| \leq \eta q_n.$$

Let $I_n = \{(i,k); 1 \leq i,k \leq q_n\}$ and $\hat{I}_n = \{(i,k); e_i, e_k \notin U(n) \text{ and } ||\underline{x}_m(e_i) - \underline{x}_m(e_k)||_1 \geq 2\}$. Note that $|I_n \setminus \hat{I}_n| \leq 2q_n||U(n)|| + q_n \cdot 5q_m \leq 2\eta q_n^2 + 5q_mq_n$. By the same argument as in section 2.7.1, we can see that if $(i,k) \in \hat{I}_n$, then $\tilde{S}_i$ and $\tilde{S}_k$ are independent and $E_p[\tilde{S}_i\tilde{S}_k] = E_p[\tilde{S}_i]E_p[\tilde{S}_k] \leq (1-\alpha(l))^2$. Thus we have

$$\frac{1}{q_n^2}\sum_{(i,k)\in I_n} E_p[S_iS_k] \le \frac{1}{q_n^2}\Big(|\hat{I}_n|\cdot(1-\alpha(l))^2 + |I_n\setminus\hat{I}_n|\cdot 1 + q_n^2\cdot 2(1-\alpha(l))\Big)$$
$$\le (1-\alpha(l))^2 + \left(2\eta + \frac{5q_m}{q_n}\right) + 2(1-\alpha(l)).$$

For fixed l and m we have

$$\limsup_{n\to\infty}\frac{1}{q_n^2}E_p\left\{\sum_{k=1}^{q_n}(\Delta_{k,n}^2 - E_p\Delta_{k,n}^2)\right\}^2 \le (1-\alpha(l))^2 + 2\eta + 2(1-\alpha(l)).$$

Letting $\eta \searrow 0$ and $l\to\infty$, we get the desired result by Lemma 2.7.4.

2.7.4 Some extensions

Our proof of the above CLT for $p\in(0,p_s)$ is based on the fact that $Q_{n,[1,3]}(p)\to 0$ as $n\to\infty$ (see section 2.7.1). Even if T does not satisfy (2.1), (2.2) or (2.3), we obtain the CLT for p when we can show that suitable analogues of $Q_{n,[1,3]}(p)\to 0$ as $n\to\infty$. We give some examples.

Example 2.7.5. When $L=2$ and $T=\mathbf{T}_2\setminus\{(1,1)\}$, K^T is the Sierpiński gasket. We call corresponding G_T (a variant of) the pre-Sierpiński gasket. Since K^T is a finitely-ramified fractal, it is easily checked that $p_c(G_T)=1$. Since T has a reflection symmetry and $Q_{n,[1,3]}(p)\to 0$ as $n\to\infty$, we can prove the CLT for $p\in(0,1)$ by using the argument in section 2.7.1.

Example 2.7.6. Let $L=2l+1$ with $l\ge 1$ and

$$T=\{(0,j),(L-1,j),(j,l);0\le j\le L-1\}.$$

Since T is anisotropic, we have to consider both left-right and top-bottom crossing probabilities. While K^T is an infinitely-ramified fractal, it is known that these crossing probabilities tend to zero as $n\to\infty$ (see [40] and [56]). Thus we can obtain the CLT for $p\in(0,1)$.

2.8 Large deviations for the number of open clusters on Sierpiński carpet lattices

We derive large deviation estimates for K_n, using a slight modification of the argument in [65].

Theorem 2.8.1 (Azuma-Hoeffding's inequality). *(cf. [3], [25] §12.2, [61] Exercise 14.2) Let* $\{M_k\}$ *be a martingale with* $M_0 = 0$ *and* $|M_k - M_{k-1}| \le c_k\,(k \ge 1)$ *a.s. For any* $x > 0$*, we have*

$$P\left\{\sup_{k\le n} M_k \ge x\right\} \le \exp\left(-\frac{x^2}{2\sum_{k=1}^n c_k^2}\right).$$

Proof. Fix $\theta \in \mathbb{R}$. Since $\{M_k\}$ is a martingale and $x \mapsto e^{\theta x}$ is convex, we can see that $\{e^{\theta M_k}\}$ is a submartingale. By Doob-Kolmogorov's inequality, we have

$$P\left\{\sup_{k\le n} M_k \ge x\right\} \le P\left\{\sup_{k\le n} e^{\theta M_k} \ge e^{\theta x}\right\} \le \frac{E[e^{\theta M_n}]}{e^{\theta x}}.$$

To estimate $E[e^{\theta M_n}]$, we use the following lemma.

Lemma 2.8.2. *Let* Y *be a random variable with* $E[Y] = 0$ *and* $|Y| \le c$ *a.s. It holds that*

$$E[e^{\theta Y}] \le \cosh(\theta c) \le \exp\left(\frac{1}{2}(\theta c)^2\right).$$

Proof of Lemma. Since $x \mapsto e^{\theta x}$ is convex, for $y \in [-c, c]$

$$\begin{aligned} e^{\theta y} &\le \frac{c-y}{2c}e^{-\theta c} + \frac{c+y}{2c}e^{\theta c} = \frac{1}{2}\left(e^{\theta c} + e^{-\theta c}\right) + \frac{y}{2c}\left(e^{\theta c} - e^{-\theta c}\right) \\ &= \cosh(\theta c) + \frac{y}{c}\sinh(\theta c). \end{aligned}$$

Thus we have

$$\begin{aligned} E[e^{\theta Y}] &\le E\left[\cosh(\theta c) + \frac{Y}{c}\sinh(\theta c)\right] \\ &= \cosh(\theta c) + \frac{\sinh(\theta c)}{c}E[Y] = \cosh(\theta c). \end{aligned}$$

The second inequality follows from

$$\begin{aligned} \cosh x &= \frac{1}{2}\left(e^x + e^{-x}\right) = \frac{1}{2}\sum_{k=0}^{\infty}\left\{\frac{x^k}{k!} + \frac{(-x)^k}{k!}\right\} \\ &= \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!} = \sum_{k=0}^{\infty}\frac{(x^2)^k}{2k\cdot(2k-1)\cdots(k+1)\cdot k!} \\ &\le \sum_{k=0}^{\infty}\frac{(x^2)^k}{2^k\cdot k!} = \exp\left(\frac{1}{2}x^2\right). \end{aligned}$$

□

By this lemma, we can see that $E[e^{\theta(M_k-M_{k-1})}|\mathcal{F}_{k-1}] \le e^{\theta^2 c_k^2/2}$ a.s. for $k = 1, 2, \cdots$. We have

$$\begin{aligned} E[e^{\theta M_n}] = E[e^{\theta M_{n-1}} e^{\theta(M_n - M_{n-1})}] &= E[e^{\theta M_{n-1}} E[e^{\theta(M_n-M_{n-1})}|\mathcal{F}_{n-1}]] \\ &\le E[e^{\theta M_{n-1}}] e^{\theta^2 c_n^2/2} \\ &\le \cdots \le \exp\left(\frac{\theta^2}{2}\sum_{k=1}^n c_k^2\right). \end{aligned}$$

This estimate gives that

$$\begin{aligned} P\left\{\sup_{k\le n} M_k \ge x\right\} &\le \exp\left(\frac{\theta^2}{2}\sum_{k=1}^n c_k^2 - \theta x\right) \\ &= \exp\left\{\frac{\sum_{k=1}^n c_k^2}{2}\left(\theta - \frac{x}{\sum_{k=1}^n c_k^2}\right)^2 - \frac{x^2}{2\sum_{k=1}^n c_k^2}\right\}. \end{aligned}$$

Letting $\theta = x/(\sum_{k=1}^n c_k^2)$, we can get the best estimate. □

Theorem 2.8.3. *(cf. [65] Theorem 3) We consider Bernoulli bond percolation on G_T, where T satisfies (2.1), (2.2) and (2.3). For all $p \in (0,1)$, there exists a positive constant C_1 such that*

$$P_p\left\{\frac{|K_n - E_p K_n|}{\sqrt{q_n}} \ge x\right\} \le 2e^{-x^2}.$$

Proof. Let $Y_k = E_p[K_n|\mathcal{F}_k] - E_p K_n$. Noting that $\{Y_k\}$ is a martingale with

$$Y_0 = E_p[K_n|\mathcal{F}_0] - E_p K_n = 0,\ |Y_k - Y_{k-1}| = |\Delta_{k,n}| \le 1 \text{ a.s.},$$

it follows from Azuma-Hoeffding's inequality that

$$\begin{aligned} &P_p\{Y_{q_n} = K_n - E_p K_n \ge x\sqrt{q_n}\} \\ &\le P_p\left\{\sup_{k\le q_n} Y_k \ge x\sqrt{q_n}\right\} \le \exp\left\{-\frac{(x\sqrt{q_n})^2}{\sum_{k=1}^{q_n} 1^2}\right\} = \exp\left(-x^2\right). \end{aligned}$$

A similar argument for $\{-Y_k\}$ gives that

$$P_p\{-Y_{q_n} = -(K_n - E_p K_n) \ge x\sqrt{q_n}\} \le \exp\left(-x^2\right).$$

□

Theorem 2.8.4 (Large deviations for K_n). *(cf. [65] Theorem 2) We consider Bernoulli bond percolation on a Sierpiński carpet lattice G_T. We assume that T satisfies (2.1), (2.2) and (2.3). Let $p \in (0,1)$.*

(i) There exists a positive constant $\sigma_1(\varepsilon, p)$ such that for all $\varepsilon \in (0, 1 - \hat{m}(p))$,

$$\lim_{n\to\infty} \frac{-1}{|B(n)|} \log P_p\left\{ \frac{K_n}{|B(n)|} \geq \hat{m}(p) + \varepsilon \right\} = \sigma_1(\varepsilon, p).$$

(ii) There exists a positive constant $\sigma_2(\varepsilon, p)$ such that for all $\varepsilon \in (0, \hat{m}(p))$,

$$\lim_{n\to\infty} \frac{-1}{|B(n)|} \log P_p\left\{ \frac{K_n}{|B(n)|} \leq \hat{m}(p) - \varepsilon \right\} = \sigma_2(\varepsilon, p).$$

Proof. We prove (ii) first. Let S_1 and S_2 be subgraphs of G_T such that S_1 and S_2 are edge-disjoint. Since $K(S_1)$ and $K(S_2)$ are mutually independent, we have

$$\begin{aligned}
&P_p\{K(S_1 \cup S_2) \leq |S_1 \cup S_2|(\hat{m}(p) - \varepsilon)\} \\
&\geq P_p\{K(S_1) + K(S_2) \leq |S_1| + |S_2|(\hat{m}(p) - \varepsilon)\} \\
&\geq P_p\{K(S_1) \leq |S_1|(\hat{m}(p) - \varepsilon),\ K(S_2) \leq |S_2|(\hat{m}(p) - \varepsilon)\} \\
&= P_p\{K(S_1) \leq |S_1|(\hat{m}(p) - \varepsilon)\} P_p\{K(S_2) \leq |S_2|(\hat{m}(p) - \varepsilon)\}.
\end{aligned}$$

Let $n > m$. We can see that

$$P_p\{K_n \leq |B(n)|(\hat{m}(p) - \varepsilon)\} \geq P_p\{K_m \leq |B(m)|(\hat{m}(p) - \varepsilon)\}^{|T|^{n-m}}.$$

Noting that if all bonds in S are open, then $K(S) = 1$ and $1 \leq |S|(\hat{m}(p) - \varepsilon)$ for $\varepsilon < \hat{m}(p)$ and sufficiently large $|S|$, we have $P_p\{K(S) \leq |S|(\hat{m}(p) - \varepsilon)\} \geq p^{||S||}$. We can see that

$$\begin{aligned}
&\frac{-\log P_p\{K_n \leq |B(n)|(\hat{m}(p) - \varepsilon)\}}{|B(n)|} \\
&\qquad \leq \frac{-|T|^{n-m} \log P_p\{K_m \leq |B(m)|(\hat{m}(p) - \varepsilon)\}}{|B(n)|} \\
&\qquad \leq \frac{-\log P_p\{K_m \leq |B(m)|(\hat{m}(p) - \varepsilon)\}}{|\operatorname{int} B(m)|},
\end{aligned}$$

where we used Corollary 2.6.3(i). By Corollary 2.6.3(ii), we have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{-\log P_p\{K_n \leq |B(n)|(\hat{m}(p) - \varepsilon)\}}{|B(n)|} \\
&\qquad \leq \liminf_{m\to\infty} \frac{-\log P_p\{K_m \leq |B(m)|(\hat{m}(p) - \varepsilon)\}}{|B(m)|}.
\end{aligned}$$

So we set

$$\sigma_2(\varepsilon,p) \equiv \lim_{n\to\infty} \frac{-1}{|B(n)|} \log P_p\{K_n \le |B(n)|(\hat{m}(p)-\varepsilon)\} < +\infty.$$

By the definition of $\hat{m}(p)$, we have

$$(\hat{m}(p)-\varepsilon/2)|B(n)| \le E_pK_n \le (\hat{m}(p)+\varepsilon/2)|B(n)|$$

for sufficiently large n. By Theorem 2.8.3, we have

$$\begin{aligned}
&P_p\{K_n \le |B(n)|(\hat{m}(p)-\varepsilon)\}\\
&\le P_p\left\{|K_n - E_pK_n| \ge \frac{\varepsilon}{2}|B(n)|\right\}\\
&\le 2\exp\left\{-\left(\varepsilon|B(n)|/(2\sqrt{q_n})\right)^2\right\} \le 2\exp\left(-\varepsilon^2|B(n)|/16\right),
\end{aligned}$$

where we used $q_n \le 4|B(n)|$. This implies that $\sigma_2(\varepsilon,p) > 0$.

Next we prove (i). Let S_1 and S_2 be subgraphs of G_T such that S_1 and S_2 are edge-disjoint. Put $V(S_1) = \{$all edges in ∂S_1 are closed$\}$, where ∂S_1 denotes the edge boundary of S_1. Noting that $K(S_1\cup S_2) = K(S_1)+K(S_2)$ on $V(S_1)$, we have

$$\begin{aligned}
&P_p\{K(S_1\cup S_2) \ge |S_1\cup S_2|(\hat{m}(p)+\varepsilon)\}\\
&\ge P_p\{K(S_1\cup S_2) \ge |S_1\cup S_2|(\hat{m}(p)+\varepsilon), V(S_1)\}\\
&\ge P_p\{K(S_1) \ge |S_1|(\hat{m}(p)+\varepsilon), K(S_2) \ge |S_2|(\hat{m}(p)+\varepsilon), V(S_1)\}\\
&\ge P_p\{K(S_1) \ge |S_1|(\hat{m}(p)+\varepsilon)\}P_p\{K(S_2) \ge |S_2|(\hat{m}(p)+\varepsilon)\}(1-p)^{||\partial S_1||}.
\end{aligned}$$

In the last inequality, we used the FKG inequality since $V(S_1)$ and $\{K(S_i) \ge |S_i|(\hat{m}(p)+\varepsilon)\}$ $(i=1,2)$ are decreasing events.

Let $n > m$. We can see that

$$\begin{aligned}
&P_p\{K_n \ge |B(n)|(\hat{m}(p)+\varepsilon)\}\\
&\ge P_p\{K_m \ge |B(m)|(\hat{m}(p)+\varepsilon)\}^{|T|^{n-m}} \cdot \{(1-p)^{||\partial B(m)||}\}^{|T|^{n-m}}.
\end{aligned}$$

Noting that if all bonds in S are closed, then $K(S) = |S|$ and $|S| \ge |S|(\hat{m}(p)+\varepsilon)$ for $\varepsilon < 1-\hat{m}(p)$, we have $P_p\{K(S) \ge |S|(\hat{m}(p)+\varepsilon)\} \ge (1-p)^{||S||}$. By Corollary 2.6.3, we can see that

$$\begin{aligned}
&\frac{-\log P_p\{K_n \ge |B(n)|(\hat{m}(p)+\varepsilon)\}}{|B(n)|}\\
&\le \frac{-|T|^{n-m}\log P_p\{K_m \ge |B(m)|(\hat{m}(p)+\varepsilon)\}}{|B(n)|} + \frac{-|T|^{n-m}||\partial B(m)||\log(1-p)}{|B(n)|}\\
&\le \frac{-\log P_p\{K_m \ge |B(m)|(\hat{m}(p)+\varepsilon)\}}{|\operatorname{int} B(m)|} + \frac{-4|\partial B(m)|\log(1-p)}{|\operatorname{int} B(m)|},
\end{aligned}$$

and

$$\limsup_{n\to\infty} \frac{-\log P_p\{K_n \geq |B(n)|(\hat{m}(p)+\varepsilon)\}}{|B(n)|}$$
$$\leq \liminf_{m\to\infty} \frac{-\log P_p\{K_m \geq |B(m)|(\hat{m}(p)+\varepsilon)\}}{|B(m)|}.$$

So we set

$$\sigma_1(\varepsilon, p) \equiv \lim_{n\to\infty} \frac{-1}{|B(n)|} \log P_p\{K_n \geq |B(n)|(\hat{m}(p)+\varepsilon)\} < +\infty.$$

As before, for sufficiently large n, it follows from Theorem 2.8.3 that

$$\begin{aligned}
&P_p\{K_n \geq |B(n)|(\hat{m}(p)+\varepsilon)\} \\
&\leq P_p\left\{|K_n - E_pK_n| \geq \frac{\varepsilon}{2}|B(n)|\right\} \\
&\leq 2\exp\left\{-\left(\varepsilon|B(n)|/(2\sqrt{q_n})\right)^2\right\} \leq 2\exp\left(-\varepsilon^2|B(n)|/16\right).
\end{aligned}$$

This implies that $\sigma_1(\varepsilon, p) > 0$. $\square$

Chapter 3

Widom-Rowlinson models

3.1 Definition of the model

We define the Widom-Rowlinson model on an infinite connected graph $G = (V, E)$. We write $x \sim y$ if $x, y \in V$ are adjacent, namely $\{x, y\} \in E$. Let $\Omega = \{-1, 0, +1\}^V$ be the configuration space with product topology. The Borel σ-algebra of Ω is denoted by $\mathcal{F}$. For $\Lambda \subset V$, we consider $\Omega_\Lambda = \{-1, 0, +1\}^\Lambda$ and its Borel σ-algebra $\mathcal{F}_\Lambda$. A configuration $\omega \in \Omega_\Lambda$ is said to be *feasible* if $\omega(x)\omega(y) \neq -1$ for all adjacent $x, y \in \Lambda$. We write $\Lambda \subset\subset V$ if Λ is a finite subset of V. For $\Lambda \subset\subset V$ and a feasible boundary condition $\omega \in \Omega$, the *finite volume Gibbs distribution* $\mu^\omega_{\Lambda,\lambda,h}$ is defined by

$$\mu^\omega_{\Lambda,\lambda,h}(\sigma) = \frac{1}{Z^\omega_{\Lambda,\lambda,h}} 1_{\{\sigma*\omega:\text{feasible}\}} \prod_{x\in\Lambda} \lambda^{\sigma(x)^2} e^{h\sigma(x)} \quad (\sigma \in \Omega_\Lambda).$$

Here $\lambda > 0$ is a parameter called *activity*, and $h \in \mathbb{R}$ is a parameter which plays a similar role as the external field in the Ising model. The normalizing constant $Z^\omega_{\Lambda,\lambda,h}$ is called the *partition function*. The configuration $\sigma * \omega \in \Omega$ is defined by

$$\sigma * \omega(x) = \begin{cases} \sigma(x) & \text{if } x \in \Lambda, \\ \omega(x) & \text{if } x \in \Lambda^c. \end{cases}$$

A probability measure μ on $(\Omega, \mathcal{F})$ which satisfies the *DLR equation*

$$\mu\left(\cdot|\mathcal{F}_{\Lambda^c}\right)(\omega) = \mu^\omega_{\Lambda,\lambda,h}(\cdot) \quad \mu\text{-a.a.}\omega \quad (\Lambda \subset\subset V)$$

is said to be a *Gibbs measure with parameter* (λ, h). The set of all Gibbs measures with parameter (λ, h) is denoted by $\mathcal{G}(\lambda, h)$. It is well-known that

$\mathcal{G}(\lambda, h)$ is a non-empty compact convex set. We write $\mathcal{G}_{\text{ex}}(\lambda, h)$ for the set of all extremal Gibbs measures.

3.2 Basic properties of Gibbs measures

Let us consider the Widom-Rowlinson model on a graph $G = (V, E)$. For $\omega, \omega' \in \Omega$ and $\Lambda \subset V$, we write $\omega = \omega'$ on [off] Λ if $\omega(x) = \omega'(x)$ for all $x \in \Lambda$ [$x \in \Lambda^c$]. Let $\partial\Lambda$ and $\partial^-\Lambda$ be outer and inner boundaries of Λ, respectively:

$$\partial\Lambda = \{y \notin \Lambda; y \sim x \text{ for some } x \in \Lambda\},$$
$$\partial^-\Lambda = \{x \in \Lambda; y \sim x \text{ for some } y \notin \Lambda\}.$$

A *cylinder function* is a function which is $\mathcal{F}_\Delta$-measurable for some $\Delta \subset\subset V$. For a cylinder function f, supp f denotes the smallest Δ such that f is $\mathcal{F}_\Delta$-measurable, i.e.

$$\text{supp } f = \bigcap \{\Delta \subset\subset V; f \text{ is } \mathcal{F}_\Delta\text{-measurable}\}.$$

An event E is called a *cylinder event* if its indicator function 1_E is a cylinder function.

3.2.1 Strong Markov property

By definition, $\mu^\omega_{\Lambda,\lambda,h}$ enjoys the Markov property, namely $\mu^\omega_{\Lambda,\lambda,h}(\sigma)$ depends only on the values of ω on $\partial\Lambda$. Moreover we can state the strong Markov property as follows. Let $\mu \in \mathcal{G}(\lambda, h)$. We say that a random subset Γ of V is *determined from outside* if $\{\Gamma = \Lambda\} \in \mathcal{F}_{\Lambda^c}$ for any $\Lambda \subset\subset V$. We consider a σ-algebra

$$\mathcal{F}_{\Gamma^c} = \{A \in \mathcal{F}; A \cap \{\Gamma = \Lambda\} \in \mathcal{F}_{\Lambda^c} \text{ for any } \Lambda \subset\subset V\}.$$

Lemma 3.2.1 (Strong Markov property). *Each Gibbs measure μ enjoys the strong Markov property: If Γ is finite μ-a.s. and determined from outside, then*

$$\mu\left(\cdot|\mathcal{F}_{\Gamma^c}\right)(\omega) = \mu^\omega_{\Gamma(\omega),\lambda,h}(\cdot) \quad \mu\text{-}a.a.\omega.$$

Remark 3.2.2. Let A be a cylinder event. If $\Gamma(\omega) = \emptyset$, then we set $\mu^\omega_{\Gamma(\omega)}(A) = 1_A(\omega)$. If $\Gamma(\omega)$ contains infinitely many points, then we set $\mu^\omega_{\Gamma(\omega)}(A) = \mu(A)$.

The proof is elementary and we omit it.

3.2.2 Stochastic domination

First we state the Holley-FKG inequality for rather general settings.

Let Λ be a finite set and S be a finite subset of $\mathbb{R}$. We set $\tilde{\Omega}_\Lambda = S^\Lambda$. For $\sigma, \sigma' \in \tilde{\Omega}_\Lambda$, we write $\sigma \le \sigma'$ if $\sigma(x) \le \sigma'(x)$ for all $x \in \Lambda$. We say $\sigma \sim \sigma'$ if there exists $x \in \Lambda$ such that $\sigma(x) \neq \sigma'(x)$ and $\sigma = \sigma'$ off x. Let μ, μ' be probability measures on $\tilde{\Omega}_\Lambda$. We write $\mu \le \mu'$ if $\mu(f) \le \mu'(f)$ for any increasing function f on $\tilde{\Omega}_\Lambda$. For a probability measure μ on $\tilde{\Omega}_\Lambda$, we define $\tilde{\Omega}^\mu_\Lambda = \{\sigma \in \tilde{\Omega}_\Lambda; \mu(\sigma) > 0\}$. We say that μ is *nice* if there exists $M = M(\mu) \in \tilde{\Omega}^\mu_\Lambda$ such that $\sigma \le M$ for all $\sigma \in \tilde{\Omega}^\mu_\Lambda$. We can define the connectedness of a subset of $\tilde{\Omega}_\Lambda$ with respect to the relation $\sim$. We call μ *irreducible* if $\tilde{\Omega}^\mu_\Lambda$ is connected in this sense.

Theorem 3.2.3.

(i) (Holley's inequality) Let μ, μ' be nice and irreducible probability measures. In addition we assume that $M(\mu) \le M(\mu')$. If for any $x \in \Lambda$, $a \in S$, $\eta, \eta' \in \tilde{\Omega}_{\Lambda \setminus \{x\}}$ such that $\eta \le \eta'$, $\mu(\sigma = \eta$ off $x) > 0$ and $\mu'(\sigma = \eta'$ off $x) > 0$,

$$\mu(\sigma(x) \ge a | \sigma = \eta \text{ off } x) \le \mu'(\sigma(x) \ge a | \sigma = \eta' \text{ off } x)$$

holds, then $\mu \le \mu'$.

(ii) (the FKG inequality) Let μ be a nice and irreducible probability measure on $\tilde{\Omega}_\Lambda$. If for any $x \in \Lambda$, $a \in S$, $\eta, \eta' \in \tilde{\Omega}_{\Lambda \setminus \{x\}}$ such that $\eta \le \eta'$, $\mu(\sigma = \eta$ off $x) > 0$ and $\mu(\sigma = \eta'$ off $x) > 0$,

$$\mu(\sigma(x) \ge a | \sigma = \eta \text{ off } x) \le \mu(\sigma(x) \ge a | \sigma = \eta' \text{ off } x)$$

is satisfied, then μ has positive correlations, i.e. $\mu(fg) \ge \mu(f)\mu(g)$ holds for increasing functions f, g on $\tilde{\Omega}_\Lambda$.

The proof of this theorem is obtained by a slight modification of the argument in [19] §4.2.

Now we return to the Widom-Rowlinson model. For $\omega, \omega' \in \Omega$, we write $\omega \le \omega'$ if $\omega(x) \le \omega'(x)$ for all $x \in V$, regarding $\{-1, 0, +1\} \subset \mathbb{R}$. Let μ and ν be probability measures on $(\Omega, \mathcal{F})$. We say $\mu \le \nu$ if $\mu(f) \le \nu(f)$ for any increasing cylinder function f on Ω. The finite Gibbs distribution in $\Lambda \subset\subset V$ with the boundary condition $\omega \equiv +1$ (resp. 0, -1) is denoted by $\mu^+_{\Lambda,\lambda,h}$ (resp. $\mu^0_{\Lambda,\lambda,h}$, $\mu^-_{\Lambda,\lambda,h}$).

Lemma 3.2.4. *The finite Gibbs distributions have following properties:*

(i) The FKG inequality holds for $\mu^\omega_{\Lambda,\lambda,h}$.

(ii) $\mu^\omega_{\Lambda,\lambda,h} \le \mu^{\omega'}_{\Lambda,\lambda,h}$ if $\omega \le \omega'$.

(iii) $\mu^\omega_{\Lambda,\lambda,h} \le \mu^\omega_{\Lambda,\lambda,h'}$ if $h \le h'$.

(iv) If $\Lambda \subset \Delta$, then $\mu^+_{\Lambda,\lambda,h} \ge \mu^+_{\Delta,\lambda,h}$ and $\mu^-_{\Lambda,\lambda,h} \le \mu^-_{\Delta,\lambda,h}$.

Proof. We can see that the set of feasible configurations is connected, i.e. $\mu^{\omega}_{\Lambda,\lambda,h}$ is irreducible. It is clear that both $\mu^{\omega}_{\Lambda,\lambda,h}$ and $\mu^{\omega'}_{\Lambda,\lambda,h}$ are nice. Indeed,

$$M(\mu^{\omega}_{\Lambda,\lambda,h}) = \begin{cases} 0 & \text{on } \{x \in \partial^{-}\Lambda; \omega(y) = -1 \text{ for some } y \in \partial\Lambda \text{ with } y \sim x\}, \\ +1 & \text{otherwise,} \end{cases}$$

and $M(\mu^{\omega'}_{\Lambda,\lambda,h})$ is similar. We note that $M(\mu^{\omega}_{\Lambda,\lambda,h}) \leq M(\mu^{\omega'}_{\Lambda,\lambda,h})$ because $\omega \leq \omega'$.

Fix any $x \in \Lambda$. For $\eta \in \tilde{\Omega}_{\Lambda\setminus\{x\}}$ such that $\eta * \omega$ is feasible, we can easily see that $\mu^{\omega}_{\Lambda,\lambda,h}(\sigma(x) = +1|\sigma = \eta \text{ off } x)$ is equal to

$$\begin{cases} 0 & \text{if } \eta * \omega(y) = -1 \text{ for some } y \sim x, \\ \frac{\lambda e^h}{\lambda e^h + 1 + \lambda e^{-h}} & \text{if } \eta * \omega(y) = 0 \text{ for all } y \sim x, \\ \frac{\lambda e^h}{\lambda e^h + 1} & \text{otherwise.} \end{cases}$$

It turns out that this conditional probability is increasing in ω, η and h. Similarly, we can see that $\mu^{\omega}_{\Lambda,\lambda,h}(\sigma(x) \geq 0|\sigma = \eta \text{ off } x)$ is increasing in ω, η and h (but not in λ!). Hence (i)-(iii) follows from Theorem 3.2.3. (iv) is proved by a standard application of (i). □

Remark 3.2.5. Since the above conditional probability is not increasing in λ, the monotonicity of phase transition depends on the underlying graph (see section 3.4.1 below).

3.2.3 Extremal Gibbs measures

Let $\mu^{+}_{\lambda,h}$ and $\mu^{-}_{\lambda,h}$ be the limiting Gibbs measures of $\mu^{+}_{\Lambda,\lambda,h}$ and $\mu^{-}_{\Lambda,\lambda,h}$ as $\Lambda \nearrow V$. These exist by virtue of Lemma 3.2.4(iv). It is well-known that limiting Gibbs measures satisfy the DLR equation. Both $\mu^{+}_{\lambda,h}$ and $\mu^{-}_{\lambda,h}$ are invariant under any graph automorphism of G. It follows from Lemma 3.2.4(ii) that

$$\mu^{-}_{\lambda,h} \leq \mu \leq \mu^{+}_{\lambda,h}$$

for any $\mu \in \mathcal{G}(\lambda, h)$. From this, it is easy to see that $\mu^{+}_{\lambda,h}, \mu^{-}_{\lambda,h} \in \mathcal{G}_{\text{ex}}(\lambda, h)$. Let $\mathcal{T} = \bigcap_{\Lambda \subset\subset V} \mathcal{F}_{\Lambda^c}$, which is called the *tail* σ*-algebra*. The following lemma is well-known.

Lemma 3.2.6. *Following conditions (i)-(iii) are equivalent.*

(i) $\mu \in \mathcal{G}_{\text{ex}}(\lambda, h)$

(ii) μ *is tail-trivial, which means that* $\mu(A) = 0$ *or* 1 *for any* $A \in \mathcal{T}$.

(iii) $\lim_{\Lambda \nearrow V} \mu^{\omega}_{\Lambda,\lambda,h} = \mu$ *for* μ*-a.a.*ω.

From this lemma, we can find that every extremal Gibbs measure satisfies the FKG inequality. It is also well-known that any Gibbs measure is uniquely represented as a convex combination of extremal Gibbs measures. For details, we refer to [18] and [46].

The following criterion of the uniqueness of Gibbs measure is useful (see [19] Theorem 4.17).

Proposition 3.2.7. *Following conditions (i)-(iii) are equivalent.*

(i) $\mathcal{G}(\lambda, h)$ *is a singleton.*

(ii) $\mu^{+}_{\lambda,h} = \mu^{-}_{\lambda,h}$

(iii) For all $x \in V$, $\mu^{+}_{\lambda,h}(\sigma(x)) = \mu^{-}_{\lambda,h}(\sigma(x))$.

3.2.4 Differentiability of the pressure and uniqueness of Gibbs measures

In this section, we consider the Widom-Rowlinson model on $\mathbb{Z}^d$ with $d \geq 2$. We review the relation between the differentiability of the pressure and the uniqueness of Gibbs measures, which will be used in section 3.6.

We set

$$p(\Lambda, \lambda, h, \omega) = \frac{1}{|\Lambda|} \log Z^{\omega}_{\Lambda,\lambda,h}.$$

Differentiating twice by h, we can see that $p(\Lambda, \lambda, h, \omega)$ is a convex function of h.

Lemma 3.2.8. *Let* Λ_n *be a box in* $\mathbb{Z}^d$ *with centre at the origin and side length* $2n+1$*. The limit*

$$P(\lambda, h) = \lim_{n \to \infty} p(\Lambda_n, \lambda, h, \omega)$$

exists and is independent of ω*. It is also a convex function of* h*, therefore it is differentiable except at most countably many* h*'s. We call* $P(\lambda, h)$ *the pressure.*

Proof. By a standard subadditive argument, we can show that $p(\Lambda_n, \lambda, h, 0)$ converges as $n \to \infty$. We write $P(\lambda, h)$ for the limit. For any feasible boundary condition ω, we can see that $Z^{0}_{\Lambda_n,\lambda,h} \geq Z^{\omega}_{\Lambda_n,\lambda,h} \geq Z^{0}_{\Lambda_{n-1},\lambda,h}$ for all $n \geq 3$, which implies that $p(\Lambda_n, \lambda, h, \omega) \to P(\lambda, h)$ as $n \to \infty$. □

The following result is well-known.

Theorem 3.2.9 ([11]). $|\mathcal{G}(\lambda, h)| = 1$ *if and only if* $P(\lambda, x)$ *is differentiable at* $x = h$*.*

Together with the preceding lemma, for each $\lambda > 0$, except at most countably many h's, there is a unique Gibbs measure for (λ, h).

Zeros of the partition function are studied in [51] and [16].

3.3 Percolation and phase transition in the Widom-Rowlinson model

We consider the Widom-Rowlinson model on a graph $G = (V, E)$.

3.3.1 Percolation in the Widom-Rowlinson model

For $\omega \in \Omega$, we set

$$\begin{aligned} S^{+}(\omega) &= \{x \in V; \omega(x) = +1\}, \\ S^{0}(\omega) &= \{x \in V; \omega(x) = 0\}, \\ S^{-}(\omega) &= \{x \in V; \omega(x) = -1\}, \\ S^{0+}(\omega) &= \{x \in V; \omega(x) \geq 0\}, \\ S^{0-}(\omega) &= \{x \in V; \omega(x) \leq 0\}. \end{aligned}$$

A path in $S^{+}(\omega)$ is called a (+)path in ω. In the analogous way, we define a (+)circuit and a (+)cluster. We say that $x, y \in V$ are (+)connected in ω if there is a (+)path from x to y in ω. The event that x and y are (+)connected is denoted by $\{x \stackrel{+}{\longleftrightarrow} y\}$. For $C \subset V$, we write $\{x \stackrel{+}{\longleftrightarrow} C\}$ for the event that x and some point in C are (+)connected. Let E^{+} be the event that there exists an infinite (+)cluster. The event that x belongs to an infinite (+)cluster is denoted by $\{x \stackrel{+}{\longleftrightarrow} \infty\}$. Let $I^{+} = I^{+}(\omega) = \{x \in V; x \stackrel{+}{\longleftrightarrow} \infty \text{ in } \omega\}$, which is equal to the union of all infinite (+)clusters in ω. There correspond in obvious fashion analogous notions for $S^{0}(\omega), S^{-}(\omega), S^{0+}(\omega)$ and $S^{0-}(\omega)$ as well.

3.3.2 Site random-cluster representation

In this subsection we assume that $h = 0$ and omit h. The *site random-cluster representation* of Widom-Rowlinson model is used in several papers; e.g. [9], [19], [28] and [29]. Here we introduce the site random-cluster representation of Gibbs distribution with an arbitrary boundary condition.

Fix $\Lambda \subset\subset V$. For $\xi \in \{0, 1\}^{\Lambda}$, let

$$\tilde{S}^{1}(\xi) = \{x \in \Lambda; \xi(x) = 1\} \ , \ \tilde{S}^{0}(\xi) = \{x \in \Lambda; \xi(x) = 0\}.$$

A path in $\tilde{S}^1(\xi)$ is called a (1)path in ξ. Analogously, we define a (1)circuit and a (1)cluster. We say that $x, y \in V$ are (1)connected if there is a (1)path from x to y in ξ. The event that x and y are (1)connected is denoted by $\{x \overset{1}{\longleftrightarrow} y\}$. For $C \subset V$, $\{x \overset{1}{\longleftrightarrow} C\}$ denotes the event that x and some point in C are (1)connected.

Let $\omega \in \Omega$ be a feasible boundary condition. We set

$$W_\Lambda^+(\omega) = \{x \in \partial\Lambda; \omega(x) = +1\} \ , \ W_\Lambda^-(\omega) = \{x \in \partial\Lambda; \omega(x) = -1\}.$$

For $\xi \in \{0,1\}^\Lambda$, let

$$1_{D(\omega,\xi)} = \begin{cases} 1 & \text{if there is no (1)path connecting } W_\Lambda^+(\omega) \text{ and } W_\Lambda^-(\omega) \text{ in } \xi, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\lambda > 0$. The *site random-cluster distribution* $R_{\Lambda,\lambda}^\omega$ is a probability measure on $\{0,1\}^\Lambda$ which is defined by

$$R_{\Lambda,\lambda}^\omega(\xi) = \frac{1}{\tilde{Z}_{\Lambda,\lambda}^\omega} 1_{D(\omega,\xi)} \prod_{x\in\Lambda} \lambda^{\xi(x)} \cdot 2^{k(\xi,\omega,\Lambda)} \quad (\xi \in \{0,1\}^\Lambda),$$

where $k(\xi, \omega, \Lambda)$ is the number of (1)clusters in ξ which touch neither $W_\Lambda^+(\omega)$ nor $W_\Lambda^-(\omega)$, and $\tilde{Z}_{\Lambda,\lambda}^\omega$ is a normalizing constant.

Lemma 3.3.1 (Site random-cluster representation).

The finite volume Gibbs distribution $\mu_{\Lambda,\lambda}^\omega$ is related to the site random-cluster distribution $R_{\Lambda,\lambda}^\omega$ as follows.

(i) First we pick $Y \in \{0,1\}^\Lambda$ according to $R_{\Lambda,\lambda}^\omega$. For $x \in \Lambda$ with $Y(x) = 0$, we set $X(x) = 0$. For each (1)cluster C of Y, we assign $+1$ or -1 to all the sites of this cluster as follows. If C is connected to $W_\Lambda^+(\omega)$, then we set $X \equiv +1$ on C. If C is connected to $W_\Lambda^-(\omega)$, then we set $X \equiv -1$ on C. Otherwise we toss a fair coin to determine the sign. Then, the distribution of $X \in \Omega_\Lambda$ is $\mu_{\Lambda,\lambda}^\omega$.

(ii) We choose $X \in \Omega_\Lambda$ according to $\mu_{\Lambda,\lambda}^\omega$ and set $Y(x) = X(x)^2$ for each $x \in \Lambda$. Then, the distribution of $Y \in \{0,1\}^\Lambda$ is $R_{\Lambda,\lambda}^\omega$.

The proof is straightforward and we omit it. Note that the distribution of $\tilde{S}^0(\sigma^2) = S^0(\sigma)$ with respect to $\mu_{\lambda,\Lambda}^\omega$ is equal to the distribution of $\tilde{S}^0(\xi)$ with respect to $R_{\lambda,\Lambda}^\omega$. For example, we have $\mu_{\Lambda,\lambda}^\omega(x \overset{0}{\longleftrightarrow} y) = R_{\Lambda,\lambda}^\omega(x \overset{0}{\longleftrightarrow} y)$ for any $x, y \in \Lambda$.

Using the site random-cluster representation, we can prove the following characterization of the phase transition of Widom-Rowlinson model in terms of percolation.

Proposition 3.3.2. *(cf. [19] Theorem 4.17 and 8.13, [29] §3) Following (i)-(v) are equivalent.*

(i) $\mathcal{G}(\lambda)$ *is a singleton.*

(ii) $\mu_\lambda^+ = \mu_\lambda^-$

(iii) $\mu_\lambda^+(\sigma(x) = +1) = \mu_\lambda^-(\sigma(x) = +1)$ *for all* $x \in V$.

(iv) $\lim_{\Lambda \nearrow V} R_{\Lambda,\lambda}^+(x \overset{1}{\longleftrightarrow} \partial\Lambda) = 0$ *for all* $x \in V$.

(v) $\mu_\lambda^+(x \overset{+}{\longleftrightarrow} \infty) = 0$ *for all* $x \in V$.

Remark 3.3.3. An analogue of the equivalence between (i) and (v) holds for the 2D Ising model, but fails for the Ising model on $\mathbb{Z}^d$ with $d \geq 3$.

For any $x \in \Lambda$, we shall calculate the conditional probability $R_{\Lambda,\lambda}^\omega(\xi(x) = 1|\xi = \eta \text{ off } x)$, where $\eta \in \{0,1\}^{\Lambda\setminus\{x\}}$ satisfies $R_{\Lambda,\lambda}^\omega(\xi = \eta \text{ off } x) > 0$. For $s \in \{0,1\}$, we define $\eta_{x,s} \in \{0,1\}^\Lambda$ by

$$\eta_{x,s}(y) = \begin{cases} \eta(y) & \text{if } y \neq x, \\ s & \text{if } y = x. \end{cases}$$

Then we have

$$R_{\Lambda,\lambda}^\omega(\xi(x) = 1|\xi = \eta \text{ off } x) = \frac{R_{\Lambda,\lambda}^\omega(\xi = \eta_{x,1})}{R_{\Lambda,\lambda}^\omega(\xi = \eta_{x,1}) + R_{\Lambda,\lambda}^\omega(\xi = \eta_{x,0})}.$$

From $R_{\Lambda,\lambda}^\omega(\xi = \eta \text{ off } x) > 0$, it follows that

$$1_{D(\omega,\eta_{x,0})} = 1.$$

Thus we have

$$\frac{R_{\Lambda,\lambda}^\omega(\xi = \eta_{x,1})}{R_{\Lambda,\lambda}^\omega(\xi = \eta_{x,0})} = \lambda \cdot 1_{D(\omega,\eta_{x,1})} \cdot 2^{k(\eta_{x,1},\omega,\Lambda) - k(\eta_{x,0},\omega,\Lambda)}.$$

The values of $1_{D(\omega,\eta_{x,1})}$ and $k(\eta_{x,1},\omega,\Lambda) - k(\eta_{x,0},\omega,\Lambda)$ are closely related to the number of (1)clusters in η each of which contains a site adjacent to x. The number of such (1)clusters is denoted by N, and the number of ones which touch neither $W_\Lambda^+(\omega)$ nor $W_\Lambda^-(\omega)$ is denoted by n. It is clear that $0 \leq n \leq N \leq 4$. We define $\kappa(\eta,\omega,x,\Lambda)$ as follows: If there are two disjoint (1)clusters containing sites adjacent to x, of which one touches $W_\Lambda^+(\omega)$ and another touches $W_\Lambda^-(\omega)$, then we set $\kappa(\eta,\omega,x,\Lambda) = -\infty$. Otherwise, we set

$$\kappa(\eta,\omega,x,\Lambda) = \begin{cases} 1 - n & \text{if } N = n, \\ -n & \text{if } N > n. \end{cases}$$

Noting that $R^{\omega}_{\Lambda,\lambda}(\xi = \eta_{x,1})/R^{\omega}_{\Lambda,\lambda}(\xi = \eta_{x,0}) = \lambda \cdot 2^{\kappa(\eta,\omega,x,\Lambda)}$, we have

$$R^{\omega}_{\Lambda,\lambda}(\xi(x) = 1 | \xi = \eta \text{ off } x) = \frac{\lambda \cdot 2^{\kappa(\eta,\omega,x,\Lambda)}}{\lambda \cdot 2^{\kappa(\eta,\omega,x,\Lambda)} + 1}.$$

Remark 3.3.4. By the definition of κ, it turns out that $R^{\omega}_{\Lambda,\lambda}$ does not satisfy the conditions of Theorem 3.2.3.

3.3.3 Uniqueness region

The disagreement percolation method (see [19] §7.1) gives the following estimate.

Proposition 3.3.5. *(cf. [19] Example 7.5) If $\lambda(e^h + e^{-h}) < p_c(\mathbb{Z}^d, \text{site})$, then $\mathcal{G}(\lambda, h)$ is a singleton. In particular, this holds if $\lambda < p_c(\mathbb{Z}^d, \text{site})/2(1 - p_c(\mathbb{Z}^d, \text{site}))$ and $|h|$ is sufficiently small.*

We quote another uniqueness result. It has already been mentioned in [32] that the proof given below is also valid for $\mathbb{Z}^d$ with $d > 2$. In fact, this proof works for an arbitrary infinite locally finite connected graph (see e.g. [29]).

Theorem 3.3.6 ([32]). *For the Widom-Rowlinson model on $\mathbb{Z}^2$, $|\mathcal{G}(\lambda, 0)| = 1$ if $\lambda < p_c(\mathbb{Z}^2, \text{site})/(1 - p_c(\mathbb{Z}^2, \text{site}))$.*

Proof. The mapping $\varphi : \Omega = \{-1, 0, +1\}^{\ \ 2} \to \{0, 1\}^{\ \ 2}$ is defined by

$$(\varphi(\sigma))(x) = \begin{cases} 1 & \text{if } \sigma(x) \in \{+1, 0\}, \\ 0 & \text{if } \sigma(x) = -1 \end{cases} \quad (\sigma \in \Omega,\ x \in \mathbb{Z}^2).$$

Let $\Lambda \subset\subset \mathbb{Z}^2$. We abbreviate $\varphi|_{\Omega_\Lambda}$ to φ and $p_c(\mathbb{Z}^2, \text{site})$ to p_c.

For a feasible boundary condition $\omega \in \Omega$, we consider the probability measure $\mu^{\omega}_{\Lambda,\lambda} \circ \varphi^{-1}$ on $\{0, 1\}^{\Lambda}$. For $x \in \Lambda$ and $\eta \in \{0, 1\}^{\Lambda \setminus \{x\}}$, we can see that

$$\mu^{\omega}_{\Lambda,\lambda} \circ \varphi^{-1}(\xi(x) = 1 | \xi = \eta \text{ off } x) = \begin{cases} \frac{1}{\lambda+1} & \text{if } \eta * (\varphi(\omega)) = 0 \text{ for all } y \sim x, \\ c_1 & \text{if } \eta * (\varphi(\omega)) = 1 \text{ for all } y \sim x, \\ c_2 & \text{otherwise}, \end{cases}$$

where the constants c_1, c_2 satisfy that

$$\frac{\lambda + 1}{2\lambda + 1} < c_1 < 1, \ \frac{1}{\lambda + 1} < c_2 < 1.$$

Noting that $1/(\lambda+1) < (\lambda+1)/(2\lambda+1)$ for all $\lambda > 0$, it follows from Holley's inequality that

$$P_{\frac{1}{\lambda+1}} \leq \mu^{\omega}_{\Lambda,\lambda} \circ \varphi^{-1} \leq P_{c_3},$$

where $c_3 = c_1 \vee c_2$. Note that $(\lambda+1)/(2\lambda+1) < c_3 < 1$.

Assume that $1 - 1/(\lambda+1) < p_c$ i.e. $\lambda < p_c/(1-p_c)$. For $x \in \mathbb{Z}^2$, we have

$$P_{\frac{1}{\lambda+1}}(x \overset{0}{\longleftrightarrow} \infty) = 0.$$

For $\varepsilon > 0$, we can choose a sufficiently large $\Lambda \subset\subset \mathbb{Z}^2$ so that

$$P_{\frac{1}{\lambda+1}}(x \overset{0}{\longleftrightarrow} \partial\Lambda) < \varepsilon.$$

We fix such a Λ. For a finite subset Δ of $\mathbb{Z}^2$ containing Λ, we have

$$\mu^{\omega}_{\Delta,\lambda}(x \overset{-}{\longleftrightarrow} \partial\Lambda) = \mu^{\omega}_{\Delta,\lambda} \circ \varphi^{-1}(x \overset{0}{\longleftrightarrow} \partial\Lambda) \leq P_{\frac{1}{\lambda+1}}(x \overset{0}{\longleftrightarrow} \partial\Lambda) < \varepsilon.$$

For any $\mu \in \mathcal{G}(\lambda, 0)$, we can see that

$$\mu(x \overset{-}{\longleftrightarrow} \partial\Lambda) = \int \mu^{\omega}_{\Delta,\lambda}(x \overset{-}{\longleftrightarrow} \partial\Lambda)\mu(d\omega) < \varepsilon.$$

Letting $\Lambda \nearrow \mathbb{Z}^2$ and $\varepsilon \searrow 0$, $\mu(x \overset{-}{\longleftrightarrow} \infty) = 0$ for all x, which implies that $\mu(E^-) = 0$.

Next we define the mapping $\psi : \Omega = \{-1, 0, +1\}^{\mathbb{Z}^2} \to \{0, 1\}^{\mathbb{Z}^2}$ by

$$(\psi(\sigma))(x) = \begin{cases} 1 & \text{if } \sigma(x) \in \{-1, 0\}, \\ 0 & \text{if } \sigma(x) = +1 \end{cases} \quad (\sigma \in \Omega,\ x \in \mathbb{Z}^2).$$

In a similar way, we can see that

$$P_{\frac{1}{\lambda+1}} \leq \mu^{\omega}_{\Lambda,\lambda} \circ \psi^{-1} \leq P_{c_3}.$$

Using this, we can prove that $\mu(E^+) = 0$ if $\lambda < p_c/(1-p_c)$.

When $\lambda < p_c/(1-p_c)$, $\mu(E^+) = \mu(E^-) = 0$ for all $\mu \in \mathcal{G}(\lambda, 0)$. For $\varepsilon > 0$, we can choose a sufficiently large box Δ containing the origin such that with μ-probability $> 1-\varepsilon$ there is a subset Γ of Δ containing the origin with $\omega \equiv 0$ on $\partial\Gamma$. Let $\Gamma(\omega)$ be the maximal one in Δ among such Γ's. We have

$$\begin{aligned} \mu(\sigma(0)) &= \mu(\sigma(0) \cdot 1_{\{\Gamma(\omega)\neq\emptyset\}}) + \mu(\sigma(0) \cdot 1_{\{\Gamma(\omega)=\emptyset\}}) \\ &= \mu(\mu^{0}_{\Gamma(\omega),\lambda}(\sigma(0)) \cdot 1_{\{\Gamma(\omega)\neq\emptyset\}}) + \mu(\sigma(0) \cdot 1_{\{\Gamma(\omega)=\emptyset\}}). \end{aligned}$$

The second equality follows from the strong Markov property. Noting that

$$\mu^0_{\Gamma(\omega),\lambda}(\sigma(0)) = \mu^0_{\Gamma(\omega),\lambda}(T\sigma(0)) = -\mu^0_{\Gamma(\omega),\lambda}(\sigma(0)),$$

we have $\mu^0_{\Gamma(\omega),\lambda}(\sigma(0)) = 0$. This implies that $-\varepsilon < \mu(\sigma(0)) < \varepsilon$. Letting $\varepsilon \searrow 0$, we have $\mu(\sigma(0)) = 0$. Since $\mu^+_\lambda(\sigma(0)) = \mu^-_\lambda(\sigma(0))$, we have the desired result by Proposition 3.2.7. □

Remark 3.3.7. In [32], Russo's comparison lemma[54] is used instead of Holley's inequality.

3.3.4 Phase coexistence region

Let us consider the Widom-Rowlinson model on $G = (V, E)$ with $h = 0$. In this subsection, we assume that G is of bounded degree. We denote the maximum degree of vertices in G by $\Delta(G)$. Let P_p denote the Bernoulli probability measure on $\{0,1\}^V$ with density p. We consider the site random-cluster model on G also. For $x \in \Lambda$ and $\eta \in \{0,1\}^{\Lambda\setminus\{x\}}$ such that $R^\omega_{\Lambda,\lambda}(\xi = \eta \text{ off } x) > 0$, we can see that $-\infty \leq \kappa(\eta, \omega, x, \Lambda) \leq 1$. Holley's inequality implies that $R^\omega_{\Lambda,\lambda} \leq P_{2\lambda/(2\lambda+1)}$. Moreover, if $\omega \in \Omega$ satisfies $\omega \geq 0$ or $\omega \leq 0$ on $\partial\Lambda$, then $1 - \Delta(G) \leq \kappa(\eta, \omega, x, \Lambda) \leq 1$. Thus we obtain the following lemma.

Lemma 3.3.8. *If a feasible boundary condition $\omega \in \Omega$ satisfies $\omega \geq 0$ or $\omega \leq 0$ on $\partial\Lambda$, then we have*

$$P_{\lambda/(\lambda+2^{\Delta(G)-1})} \leq R^\omega_{\Lambda,\lambda} \leq P_{2\lambda/(2\lambda+1)}.$$

Theorem 3.3.9. *(cf. [32], [28]) We consider the Widom-Rowlinson model on an infinite connected graph $G = (V, E)$ of bounded degree with $p_c(G, \text{site}) < 1$. If $\lambda > 2^{\Delta(G)-1} p_c(G, \text{site})/(1 - p_c(G, \text{site}))$, then $\mu^+_\lambda \neq \mu^-_\lambda$.*

Proof. When $\lambda/(\lambda + 2^{\Delta(G)-1}) > p_c(G, \text{site})$ (i.e. $\lambda > 2^{\Delta(G)-1} p_c(G, \text{site})/(1 - p_c(G, \text{site}))$), $P_{\lambda/(\lambda+2^{\Delta(G)-1})}(0 \stackrel{1}{\longleftrightarrow} \infty) = \theta > 0$. For $\Lambda \subset\subset V$, it follows from Lemmata 3.3.1 and 3.3.8 that

$$\begin{aligned} R^+_{\Lambda,\lambda}(0 \stackrel{1}{\longleftrightarrow} \partial\Lambda) &\geq P_{\lambda/(\lambda+2^{\Delta(G)-1})}(0 \stackrel{1}{\longleftrightarrow} \partial\Lambda) \\ &\geq P_{\lambda/(\lambda+2^{\Delta(G)-1})}(0 \stackrel{1}{\longleftrightarrow} \infty) = \theta > 0. \end{aligned}$$

Thus we have $\liminf_{\Lambda \nearrow V} R^+_{\Lambda,\lambda}(0 \stackrel{1}{\longleftrightarrow} \partial\Lambda) \geq \theta > 0$. It follows from this and Proposition 3.3.2 that $\mu^+_\lambda \neq \mu^-_\lambda$. □

3.3.5 Widom-Rowlinson model and Bernoulli site percolation

Theorem 3.3.10 ([28]). *We consider the following classes of graphs:*

$$\begin{aligned}
\mathcal{G} &= \{\textit{infinite locally finite connected graphs}\},\\
\mathcal{G}^b &= \{\textit{infinite locally finite connected graphs with bounded degree}\},\\
\mathcal{G}_{\mathrm{SP}} &= \{G \in \mathcal{G}; p_c(G, \mathrm{site}) < 1\},\\
\mathcal{G}_{\mathrm{WR}} &= \{G \in \mathcal{G}; \textit{the Widom-Rowlinson model on } G \textit{ exhibits phase transition}\},\\
\mathcal{G}^b_{\mathrm{SP}} &= \mathcal{G}^b \cap \mathcal{G}_{\mathrm{SP}},\ \mathcal{G}^b_{\mathrm{WR}} = \mathcal{G}^b \cap \mathcal{G}_{\mathrm{WR}}.
\end{aligned}$$

Then, it holds that $\mathcal{G}_{\mathrm{SP}} \supset \mathcal{G}_{\mathrm{WR}}$, $\mathcal{G}_{\mathrm{SP}} \setminus \mathcal{G}_{\mathrm{WR}} \neq \emptyset$ *and* $\mathcal{G}^b_{\mathrm{SP}} = \mathcal{G}^b_{\mathrm{WR}}$.

Proof. If $G \in \mathcal{G} \setminus \mathcal{G}_{\mathrm{SP}}$, then the proof of Theorem 3.3.6 shows that $G \in \mathcal{G} \setminus \mathcal{G}_{\mathrm{WR}}$. This proves that $\mathcal{G}_{\mathrm{SP}} \supset \mathcal{G}_{\mathrm{WR}}$. An example of the graph in $\mathcal{G}_{\mathrm{SP}} \setminus \mathcal{G}_{\mathrm{WR}}$ is found in [28]. It follows from Theorem 3.3.9 that $\mathcal{G}^b_{\mathrm{SP}} \subset \mathcal{G}^b_{\mathrm{WR}}$. This completes the proof. □

3.4 Phase structure

In the preceding section, we showed that phase transition occurs when $h = 0$ and λ is sufficiently large. In this section, we quote some examples where the fine properties of the phase structure are studied. Here we assume that $h = 0$ and omit 'h'.

For the Widom-Rowlinson model on a graph G, the phase transition is said to be *monotonic* if there exists $\lambda_c = \lambda_c(G) \in (0, \infty)$ such that

$$\begin{cases} |\mathcal{G}(\lambda)| = 1 & \text{if } \lambda < \lambda_c,\\ |\mathcal{G}(\lambda)| > 1 & \text{if } \lambda > \lambda_c. \end{cases}$$

For the Ising models, we can show the monotonicity of the phase transition by virtue of Griffiths' inequalities. However, the monotonicity for the Widom-Rowlinson model depends on G.

3.4.1 The Widom-Rowlinson model on trees

As for $(d+1)$-regular tree $\mathbb{T}^d$ with $d \geq 2$,

Theorem 3.4.1 ([9](1998)). *The Widom-Rowlinson model on* $\mathbb{T}^d$ $(d \geq 2)$ *is monotonic, and*

$$\lambda_c(\mathbb{T}^d) = \frac{1}{d-1}\left(\frac{d+1}{d}\right)^d.$$

Using this theorem, a class of graphs on which the Widom-Rowlinson model is not monotonic can be constructed. To each site in $\mathbb{T}^d$, we add $n \in \mathbb{N}$ edges which terminate in a single site. We write $\mathbb{T}^d_n$ for the resulting tree.

Theorem 3.4.2 ([9]). *For the Widom-Rowlinson model on $\mathbb{T}^{40}_7$, there are three thresholds $\lambda_1 \approx 0.2179, \lambda_2 \approx 0.4013, \lambda_3 \approx 4.5519$ such that*

$$\begin{cases} |\mathcal{G}(\lambda)| = 1 & \text{if } \lambda \in (0, \lambda_1] \cup [\lambda_2, \lambda_3] \\ |\mathcal{G}(\lambda)| > 1 & \text{if } \lambda \in (\lambda_1, \lambda_2) \cup (\lambda_3, \infty). \end{cases}$$

On the other hand, Häggström[29] constructed a class of d-dimensional periodic lattices on which the Widom-Rowlinson model is monotone.

3.4.2 The one-dimensional Widom-Rowlinson model

Miyamoto[45] treated the spatially-inhomogeneous Widom-Rowlinson model on $\mathbb{T}^1 = \mathbb{Z}^1$.

Theorem 3.4.3 ([45]). *We consider the one-dimensional Widom-Rowlinson models with the finite volume Gibbs distribution*

$$q^{\omega}_{\Lambda,\{\lambda_x\}}(\sigma) = \frac{1}{Z_{\Lambda,\{\lambda_x\}}} 1_{\{\sigma * \omega:\text{feasible}\}} \prod_{x\in\Lambda} (\lambda_x)^{\sigma(x)^2},$$

where $\{\lambda_x\} = \{\lambda_x > 0; x \in \mathbb{Z}^1\}$. For $(\tau', \tau) \in \{-1, 0, +1\}^2$, we define a boundary condition $\omega(\tau', \tau)$ by

$$\omega(\tau', \tau)(x) = \tau' \ (x < 0), = 0 \ (x = 0), = \tau \ (x > 0).$$

For any $\{\lambda_x\}$ and (τ', τ), limiting Gibbs measures

$$Q_{(\tau',\tau)} \equiv \lim_{\substack{n\to-\infty \\ m\to+\infty}} q^{\omega(\tau',\tau)}_{[n,m],\{\lambda_x\}}$$

exist. Put

$$\mathcal{M}_{+\infty}(\{\lambda_x\}) = \begin{cases} \{-1, +1\} & \text{if } \sum_{x=0}^{+\infty} \lambda_x^{-1} < +\infty, \\ \{0\} & \text{if } \sum_{x=0}^{+\infty} \lambda_x^{-1} = +\infty, \end{cases}$$

$$\mathcal{M}_{-\infty}(\{\lambda_x\}) = \begin{cases} \{-1, +1\} & \text{if } \sum_{x=-\infty}^{0} \lambda_x^{-1} < +\infty, \\ \{0\} & \text{if } \sum_{x=-\infty}^{0} \lambda_x^{-1} = +\infty, \end{cases}$$

$$\mathcal{M}(\{\lambda_x\}) = \mathcal{M}_{-\infty}(\{\lambda_x\}) \times \mathcal{M}_{+\infty}(\{\lambda_x\}).$$

The set of all extremal Gibbs states $\mathcal{G}_{\text{ex}}(\{\lambda_x\})$ is isomorphic to $\mathcal{M}(\{\lambda_x\})$. The mapping $\mathcal{M}(\{\lambda_x\}) \ni (\tau', \tau) \mapsto Q_{\tau',\tau} \in \mathcal{G}_{\text{ex}}(\{\lambda_x\})$ is the isomorphism.

3.4.3 The Widom-Rowlinson model on $\mathbb{Z}^d$ $(d \geq 2)$

It is widely believed that if $d = 2$ and $h = 0$, then each Gibbs measure is described by a mixture of two translation-invariant extremal Gibbs measures, as in the 2D Ising model.

Conjecture 3.4.4. Consider the Widom-Rowlinson model on $\mathbb{Z}^2$. If $h = 0$, then there exists a critical value $\lambda_c \in (0, \infty)$ such that

$$\begin{cases} |\mathcal{G}(\lambda)| = 1 & \text{if } \lambda < \lambda_c, \\ \mathcal{G}_{\text{ex}}(\lambda) = \{\mu_\lambda^+, \mu_\lambda^-\} & \text{if } \lambda > \lambda_c. \end{cases}$$

We give a partial answer to this conjecture later.

For $d = 2$, the translation-invariance of the limiting Gibbs measure with so-called Dobrushin boundary (see section 3.7.2 below) is analyzed by Bricmont, Lebowitz and Pfister[5]. Higuchi, Murai and Wang[36] studied a Dobrushin-Hryniv type limit theorem for the fluctuation of the phase separation line, i.e. the central limit theorem for the fluctuation of the phase separation line from the Wulff profile.

As for $d \geq 3$, Bricmont *et al.*[6, 7] showed that the Dobrushin boundary condition gives non-translation-invariant extremal Gibbs measures for the Widom-Rowlinson model on $\mathbb{Z}^d$, as in the Ising model. See [22] and [12] for a similar phenomenon for the random-cluster models, including Bernoulli bond percolation.

Recently, the structure of the set of the translation-invariant Gibbs measures for the Ising model on $\mathbb{Z}^d$ have been determined. We refer to [4].

It is conjectured in [19] §3.5 that the Widom-Rowlinson model on $\mathbb{Z}^d$ with asymmetric activities (i.e. $h \neq 0$) admits no phase transition.

3.5 The infinite cluster method

In sections 3.5-3.7, we study the phase structure of the two-dimensional lattice Widom-Rowlinson model.

Russo[52] created the infinite cluster method for determining the phase structure of the 2D Ising model. As in [20], we state his key results in the form of lemmata. In addition, we study the uniqueness of the infinite cluster under periodic Gibbs measures in section 3.5.6.

3.5.1 Some notations for percolation on $\mathbb{Z}^2$

We write $x \sim y$ if $x, y \in \mathbb{Z}^2$ are adjacent, namely $|x_1 - y_1| + |x_2 - y_2| = 1$. We say that x and y are $(*)$adjacent and write $x \overset{*}{\sim} y$ if $\max\{|x_1 - y_1|, |x_2 - y_2|\} =$

1.

We say p is a path in $S \subset \mathbb{Z}^2$ if $p \subset S$. A path p is called *circuit* if $x_k \sim x_1$.

A sequence $p = (x_1, \cdots, x_k)$ of distinct points of $\mathbb{Z}^2$ is a $(*)$*path* from x_1 to x_k if $x_i \overset{*}{\sim} x_{i+1}$ $(i = 1, \cdots, k-1)$. In the similar manner, we define a $(*)$circuit, a $(*)$cluster and $(*)$connectedness.

A $(*)$path (resp. $(*)$circuit, $(*)$cluster) in $S^+(\omega)$ is called a $(+*)$path (resp. a $(+*)$circuit, a $(+*)$cluster). We call a prefix such as '$+*$' the *type* of this path. Note that $E^+ \subset E^{+*} \subset E^{0+*}$ and so on. Similarly, for site random-cluster models, we define $(1*)$connectedness and so on.

We abbreviate $p_c(\mathbb{Z}^2, \text{site})$ to p_c.

3.5.2 Transformations of Ω

We consider the following transformations of Ω.

(i) The *translations* $\theta_s, s \in \mathbb{Z}^2$: which are defined by

$$(\theta_s \omega)(x) = \omega(x - s) \quad (x \in \mathbb{Z}^2)$$

for $\omega \in \Omega$. Particularly, let $\theta_{\text{hor}} = \theta_{(1,0)}$ and $\theta_{\text{vert}} = \theta_{(0,1)}$. The collection $(\theta_s)_{s \in \ ^2}$ is a group. For $a, b \in \mathbb{N}$, let $\mathbb{Z}^2(a,b) = \{(ak, bl) \in \mathbb{Z}^2; k, l \in \mathbb{Z}\}$. We say that $\mu \in \mathcal{G}(\lambda, h)$ is $((a,b)$-$)$*periodic* if it is invariant under the subgroup $(\theta_s)_{s \in \ ^2(a,b)}$. In particular, it is called *translation-invariant* if this holds for $(a, b) = (1, 1)$. We say that μ is *horizontally periodic* if it is invariant under $\theta_{(a,0)}$ for some $a \in \mathbb{N}$. Similarly, we define vertical periodicity.

(ii) The *spin-flip transformation* : For $\omega \in \Omega$, $T\omega \in \Omega$ is defined by

$$(T\omega)(x) = -\omega(x) \ (x \in \mathbb{Z}^2).$$

(iii) The *reflections* : For $k \in \mathbb{Z}$, let

$$R_{k,\text{hor}} : \mathbb{Z}^2 \ni x = (x_1, x_2) \mapsto (x_1, 2k - x_2) \in \mathbb{Z}^2,$$
$$R_{k,\text{vert}} : \mathbb{Z}^2 \ni x = (x_1, x_2) \mapsto (2k - x_1, x_2) \in \mathbb{Z}^2.$$

Let R be a reflection, i.e. $R = R_{k,\text{hor}}$ or $R_{k,\text{vert}}$ for some $k \in \mathbb{Z}$. We define $R : \Omega \to \Omega$ by

$$(R\omega)(x) = \omega(Rx) \ (\omega \in \Omega, x \in \mathbb{Z}^2).$$

3.5.3 Characterization of Gibbs measures by percolation

By the strong Markov property, the following lemma is easily obtained.

Lemma 3.5.1. *(cf. [20] Lemma 2.1) Let $\mu \in \mathcal{G}(\lambda, h)$. If $\mu(E^{0+}) = 0$, then $\mu = \mu^-_{\lambda,h}$.*

We need a variant of this lemma.

Proposition 3.5.2. *Let $\mu \in \mathcal{G}(\lambda, h)$. If $\mu(E^{0*}) = 0$, then μ is a convex combination of $\mu^+_{\lambda,h}$ and $\mu^-_{\lambda,h}$.*

Proof. Fix $\Lambda \subset\subset \mathbb{Z}^2$. By assumption, Λ is surrounded by either a $(+)$circuit or a $(-)$circuit μ-a.s. In such a case, we will say that Λ is surrounded by a $(+/-)$circuit. For any $\varepsilon > 0$, we can choose a large finite set $\Delta \supset \Lambda$ such that

$$\mu\left(\Lambda \text{ is surrounded by a } (+/-)\text{circuit in } \Delta\right) > 1 - \varepsilon.$$

For each circuit C surrounding Λ in Δ, we consider the events

$$\begin{aligned} &A^+_{\Lambda,\Delta,C} \\ &= \{C \text{ is the maximal } (+/-)\text{circuit surrounding } \Lambda \text{ in } \Delta \text{ and its type is } +\}, \\ &A^-_{\Lambda,\Delta,C} \\ &= \{C \text{ is the maximal } (+/-)\text{circuit surrounding } \Lambda \text{ in } \Delta \text{ and its type is } -\}, \end{aligned}$$

and

$$A^+_{\Lambda,\Delta} = \bigcup_C A^+_{\Lambda,\Delta,C},\ A^-_{\Lambda,\Delta} = \bigcup_C A^-_{\Lambda,\Delta,C},\ A^{+/-}_{\Lambda,\Delta} = A^+_{\Lambda,\Delta} \cup A^-_{\Lambda,\Delta},$$

where the union runs over all the circuit surrounding Λ in Δ. Clearly,

$$\mu\left(A^+_{\Lambda,\Delta}\right) + \mu\left(A^-_{\Lambda,\Delta}\right) = \mu\left(A^{+/-}_{\Lambda,\Delta}\right) > 1 - \varepsilon.$$

Let f be a nonnegative increasing function such that $\operatorname{supp} f \subset \Lambda$. We have

$$\begin{aligned} \mu(f) &= \mu\left(f \cdot 1_{A^+_{\Lambda,\Delta}}\right) + \mu\left(f \cdot 1_{A^-_{\Lambda,\Delta}}\right) + \mu\left(f \cdot 1_{(A^{+/-}_{\Lambda,\Delta})^c}\right) \\ &= \sum_C \left\{\mu\left(f \cdot 1_{A^+_{\Lambda,\Delta,C}}\right) + \mu\left(f \cdot 1_{A^-_{\Lambda,\Delta,C}}\right)\right\} + \mu\left(f \cdot 1_{(A^{+/-}_{\Lambda,\Delta})^c}\right). \end{aligned}$$

The Markov property of μ implies that

$$\mu(f) = \sum_C \left\{ \mu\left(\mu^+_{\mathrm{int}(C),\lambda,h}(f) \cdot 1_{A^+_{\Lambda,\Delta,C}}\right) + \mu\left(\mu^-_{\mathrm{int}(C),\lambda,h}(f) \cdot 1_{A^-_{\Lambda,\Delta,C}}\right)\right\}$$
$$+ \mu\left(f \cdot 1_{(A^{+/-}_{\Lambda,\Delta})^c}\right),$$

where $\mathrm{int}(C)$ is the bounded $(*)$connected component of $\mathbb{Z}^2 \setminus C$. For any circuit C surrounding Λ, we note that

$$\mu^+_{\lambda,h}(f) \le \mu^+_{\mathrm{int}(C),\lambda,h}(f) \le \mu^+_{\Lambda,\lambda,h}(f) \ , \ \mu^-_{\Lambda,\lambda,h}(f) \le \mu^-_{\mathrm{int}(C),\lambda,h}(f) \le \mu^-_{\lambda,h}(f).$$

So we have

$$\mu(f) \le \sum_C \left\{ \mu^+_{\Lambda,\lambda,h}(f)\mu\left(A^+_{\Lambda,\Delta,C}\right) + \mu^-_{\lambda,h}(f)\mu\left(A^-_{\Lambda,\Delta,C}\right)\right\} + \varepsilon||f||_\infty$$
$$= \mu^+_{\Lambda,\lambda,h}(f)\mu\left(A^+_{\Lambda,\Delta}\right) + \mu^-_{\lambda,h}(f)\mu\left(A^-_{\Lambda,\Delta}\right) + \varepsilon||f||_\infty.$$

Similarly,

$$\mu(f) \ge \mu^+_{\lambda,h}(f)\mu\left(A^+_{\Lambda,\Delta}\right) + \mu^-_{\Lambda,\lambda,h}(f)\mu\left(A^-_{\Lambda,\Delta}\right) - \varepsilon||f||_\infty.$$

Take a sequence $\Delta \nearrow \mathbb{Z}^2$. Note that $A^{+/-}_{\Lambda,\Delta}$ is increasing in Δ. Since finite subsets of $\mathbb{Z}^2$ are countably many and $\mu\left(A^+_{\Lambda,\Delta}\right) \in [0,1]$, by a diagonal-sequence argument we can choose a subsequence of Δ such that $\mu\left(A^+_{\Lambda,\Delta}\right)$ converges for all $\Lambda \subset\subset \mathbb{Z}^2$. We write α_Λ for this limit. By letting $\Delta \nearrow \mathbb{Z}^2$ along this subsequence and $\varepsilon \searrow 0$, we have $\mu(A^+_{\Lambda,\Delta}) \to \alpha_\Lambda$, $\mu(A^-_{\Lambda,\Delta}) \to 1 - \alpha_\Lambda$, and

$$\alpha_\Lambda \mu^+_{\lambda,h}(f) + (1-\alpha_\Lambda)\mu^-_{\Lambda,\lambda,h}(f) \le \mu(f) \le \alpha_\Lambda \mu^+_{\Lambda,\lambda,h}(f) + (1-\alpha_\Lambda)\mu^-_{\lambda,h}(f).$$

Next we take an increasing sequence $\Lambda \nearrow \mathbb{Z}^2$. As $\alpha_\Lambda \in [0,1]$, we can choose a suitable subsequence of Λ such that α_Λ converges to some $\alpha \in [0,1]$. By letting $\Lambda \nearrow \mathbb{Z}^2$ along this subsequence, we have

$$\mu(f) = \alpha\mu^+_{\lambda,h}(f) + (1-\alpha)\mu^-_{\lambda,h}(f)$$

for any nonnegative increasing f. Because both $\mu^+_{\lambda,h}$ and $\mu^-_{\lambda,h}$ are extremal in $\mathcal{G}(\lambda,h)$, the extremal decomposition theorem implies that α is unique and independent of the choice of subsequences. This completes the proof. □

Using the above proposition, we can determine the limiting Gibbs measure with free boundary condition, when activity is large.

Theorem 3.5.3. *Assume that* $\lambda > 8p_c/(1-p_c)$ *and* $h = 0$. *Then we have*

$$\mu^0_{\lambda,0} = \lim_{\Lambda \nearrow \mathbb{Z}^2} \mu^0_{\Lambda,\lambda,0} = \frac{1}{2}\left(\mu^+_{\lambda,0} + \mu^-_{\lambda,0}\right).$$

Proof. Take a sequence $\Lambda \nearrow \mathbb{Z}^2$. By taking a suitable subsequence $\{\Lambda_n\}$, $\mu^0_{\Lambda_n,\lambda,0}$ converges to a probability measure on Ω, say $\mu^0_{\lambda,0}$, as $n \to \infty$.

We shall prove $\mu^0_{\lambda,0}(E^{0*}) = 0$ when $\lambda > 8p_c/(1-p_c)$. Let p^*_c be the critical probability of infinite $(*)$cluster of Bernoulli site percolation on $\mathbb{Z}^2$. It is well-known that $p_c + p^*_c = 1$ ([53]). Now, as $1 - \lambda/(\lambda+8) < p^*_c$, there is no infinite $(0*)$cluster $P_{\frac{\lambda}{\lambda+8}}$-a.s. Fix $x \in \mathbb{Z}^2$. For any $\varepsilon > 0$, we can choose a large N so that $P_{\frac{\lambda}{\lambda+8}}(x \stackrel{0*}{\longleftrightarrow} \partial\Lambda_n) < \varepsilon$ for all $n \geq N$. By Lemmata 3.3.1 and 3.3.8, for $m > n \geq N$ we have

$$\mu^0_{\Lambda_m,\lambda,0}(x \stackrel{0*}{\longleftrightarrow} \partial\Lambda_n) = R^0_{\Lambda_m,\lambda}(x \stackrel{0*}{\longleftrightarrow} \partial\Lambda_n) \leq P_{\frac{\lambda}{\lambda+8}}(x \stackrel{0*}{\longleftrightarrow} \partial\Lambda_n) < \varepsilon.$$

By letting $m \to \infty$, $n \to \infty$ and $\varepsilon \searrow 0$, we have $\mu^0_{\lambda,0}(x \stackrel{0*}{\longleftrightarrow} \infty) = 0$ for all $x \in \mathbb{Z}^2$. Thus $\mu^0_{\lambda,0}(E^{0*}) = 0$.

By Proposition 3.5.2, $\mu^0_{\lambda,0} = \alpha\mu^+_{\lambda,0} + (1-\alpha)\mu^-_{\lambda,0}$ for some coefficient $\alpha \in [0,1]$. We note that $\mu^0_{\Lambda_n,\lambda,0}(A) = \mu^0_{\Lambda_n,\lambda,0} \circ T(A)$ for each n and any $A \in \mathcal{F}_{\Lambda_n}$. By letting $n \to \infty$, we have $\mu^0_{\lambda,0}(A) = \mu^0_{\lambda,0} \circ T(A)$. This implies that $\alpha = 1/2$. We can conclude that $\mu^0_{\Lambda,\lambda,0}$ converges to $(\mu^+_{\lambda,0} + \mu^-_{\lambda,0})/2$, independent of the choice of the subsequence of $\Lambda \nearrow \mathbb{Z}^2$. □

We remark that the above proofs are valid for any dimension. Let $p^*_c(d)$ be the critical probability of Bernoulli site percolation on $\mathbb{Z}^d$ when we connect the distinct sites with the Euclidean distance not larger than $\sqrt{d}$. We can extend the above proofs to obtain the following

Corollary 3.5.4. *We consider the* d*-dimensional Widom-Rowlinson model with* $d \geq 2$. *If* $\lambda > 2^{2d-1}(1 - p^*_c(d))/p^*_c(d)$ *and* $h = 0$, *then every limit point of* $\mu^\omega_{\Lambda,\lambda,0}$ *with* $\omega \geq 0$ *or* $\omega \leq 0$ *is a mixture of* $\mu^+_{\lambda,0}$ *and* $\mu^-_{\lambda,0}$.

3.5.4 Flip-reflection domination

If $h = 0$, then the interaction is invariant under the *flip-reflection transformation* $R \circ T$, where R is any reflection. This implies that $\{\omega; \omega \text{ is feasible}\} = \{\omega; R \circ T(\omega) \text{ is feasible}\}$. Thus we can obtain the following lemma.

Lemma 3.5.5 (Flip-reflection domination). *(cf. [20] Lemma 2.3)*
Let $\mu \in \mathcal{G}(\lambda, 0)$ and R be any reflection. If μ-a.a.ω any $\Lambda \subset\subset \mathbb{Z}^2$ is surrounded by a $(*)$circuit which is R-invariant and on which $\omega \geq R \circ T(\omega)$, then we have $\mu \geq \mu \circ R \circ T$.

3.5.5 Percolation in half-planes

A *half-plane* is the set of the form $\pi = \{x = (x_1, x_2) \in \mathbb{Z}^2; x_i \geq (\leq) n\}$ for some $n \in \mathbb{Z}$ and $i \in \{1, 2\}$. The line $l = \{x = (x_1, x_2) \in \mathbb{Z}^2; x_i = n\}$ is called the *boundary line* of this half-plane. Let

$$\pi_{\mathrm{up},n} = \{x \in \mathbb{Z}^2; x_2 \geq n\}\ ,\ \pi_{\mathrm{down},n} = \{x \in \mathbb{Z}^2; x_2 \leq n\}.$$

We simply write π_{up}, π_{down} if $n = 0$. In the analogous way, $\pi_{\mathrm{left},n}$, $\pi_{\mathrm{right},n}$, π_{left} and π_{right} are defined.

A path $p = (x_1, \cdots, x_k)$ is called a *half-circuit* of the half-plane π with boundary line l if $p \subset \pi$ and $p \cap l = \{x_1, x_k\}$. For a half plane π, let E_π^+ be the event that there exists an infinite $(+)$cluster in π. The union of infinite $(+)$clusters in π is denoted by $I_\pi^+ = I_\pi^+(\omega) = \{x \in \pi; x \overset{+}{\longleftrightarrow} \infty \text{ in } \omega|_\pi\}$. When $\pi = \pi_{\mathrm{up}}$, we write E_{up}^+ or I_{up}^+ for short. Analogous notations will be used for infinite clusters of other types.

Lemma 3.5.6 (Shift lemma). *(cf. [20] Lemma 3.4) Let π and $\tilde{\pi}$ be half-planes. Assume that π is a translate of $\tilde{\pi}$. Then $E_\pi^+ = E_{\tilde{\pi}}^+$ μ-a.s. for every $\mu \in \mathcal{G}(\lambda, h)$. This also holds for infinite clusters of any other types.*

This lemma is proved by using so-called 'random Borel-Cantelli' argument(see [20]).

3.5.6 Percolation under periodic Gibbs measures

Proposition 3.5.7. *If $\mu \in \mathcal{G}(\lambda, h)$ is ((a, b)-)periodic, then there is at most one infinite cluster of each type μ-a.s.*

Proof. By the ergodic decomposition theorem([18] Chap.14), we can assume that μ is $(\theta_s)_{s\in\ {}^2(a,b)}$-ergodic. We want to apply the Burton-Keane uniqueness theorem, but its proof requires the finite energy property to connect different clusters with positive probability. In spite of lack of the finite energy property in our case, this is still possible in a similar manner as noted in [26] and [20] for the hard-core lattice gas model. The $(\theta_s)_{s\in\ {}^2(a,b)}$-ergodicity is sufficient to show that in a finite box there exist encounter points whose number has the same order as the volume of the box. Thus we can show the uniqueness of the infinite cluster. ☐

By virtue of this proposition, we can establish the non-coexistence of infinite clusters of different kinds by using Zhang's argument.

Proposition 3.5.8 (Zhang's argument). *The following statements hold.*

(i) (cf. [19] Theorem 5.18) If $\mu \in \mathcal{G}(\lambda, h)$ is a periodic and rotation-invariant probability measure with positive correlations, then we have $\mu(E^+ \cap E^{0-*}) = 0$.

(ii) (cf. [20] Lemma 3.1) If $\mu \in \mathcal{G}(\lambda, 0)$ has positive correlations and is flip-reflection invariant (i.e. $\mu = \mu \circ R \circ T$ for any reflection R), then we have $\mu(E^+ \cap E^-) = 0$.

Corollary 3.5.9. *If $\mu^+_{\lambda,h} \neq \mu^-_{\lambda,h}$, then $\mu^+_{\lambda,h}(E^{0-*}) = \mu^-_{\lambda,h}(E^{0+*}) = 0$.*

Proof. If $\mu^+_{\lambda,h} \neq \mu^-_{\lambda,h}$, then we have $\mu^+_{\lambda,h}(E^+) = \mu^-_{\lambda,h}(E^-) = 1$ by Proposition 3.3.2. We get the conclusion from Proposition 3.5.8(i). □

3.6 Number of phases in 2D : asymmetric case

Van den Berg and Steif conjectured that the hard-core lattice gas model on $\mathbb{Z}^d$ with parity-dependent activities has no phase transition, and Häggström proved it in the 2D case (see [19] §3.4 and [26]). In [19] §3.5, it is conjectured that the Widom-Rowlinson model on $\mathbb{Z}^d$ with asymmetric activities (i.e. $h \neq 0$) admits no phase transition. We expect that Häggström's method can be also adapted to the asymmetric Widom-Rowlinson model on $\mathbb{Z}^2$. In this section, we shall give a partial answer.

Theorem 3.6.1. *For each $\lambda > 0$, there exists $h_c = h_c(\lambda) \in [0, \infty)$ such that*

$$|h| > h_c \Longrightarrow |\mathcal{G}(\lambda, h)| = 1.$$

Especially, $h_c(\lambda) = 0$ when $|\mathcal{G}(\lambda, 0)| > 1$.

Let us prove this theorem in several steps. We assume that $h > 0$. The case $h < 0$ is treated analogously.

Lemma 3.6.2. *If $h > 0$, then $\mu^+_{\lambda,0} \leq \mu^-_{\lambda,h}$.*

Proof. Since phase transition occurs at most countably many h's as noted in section 3.2.4, there exists some $h' \in (0, h)$ with $\mu^+_{\lambda,h'} = \mu^-_{\lambda,h'}$. We can see that $\mu^+_{\lambda,0} \leq \mu^+_{\lambda,h'} = \mu^-_{\lambda,h'} \leq \mu^-_{\lambda,h}$. □

Proposition 3.6.3. *Let $\lambda > 0$. If $\mu^+_{\lambda,0} \neq \mu^-_{\lambda,0}$, then we have $|\mathcal{G}(\lambda, h)| = 1$ for all $h > 0$.*

Proof. We can show that $\mu^+_{\lambda,0}(E^{0-}) = 0$ if $\mu^+_{\lambda,0} \neq \mu^-_{\lambda,0}$ (see Corollary 3.5.9). So we have $\mu^-_{\lambda,h}(E^{0-}) \leq \mu^+_{\lambda,0}(E^{0-}) = 0$. By Lemma 3.5.1, we can see that $\mu^-_{\lambda,h} = \mu^+_{\lambda,h}$. Proposition 3.2.7 gives the result. □

Next, we fix $\lambda > 0$ such that $\mu^+_{\lambda,0} = \mu^-_{\lambda,0}$. For the unique Gibbs measure $\mu_0 \in \mathcal{G}(\lambda, 0)$, we can show that $\mu_0(E^+ \cup E^-) = 0$ (see Proposition 3.3.2). Therefore, for arbitrary $\mu \in \mathcal{G}(\lambda, h)$ we have $\mu(E^-) \leq \mu^-_{\lambda,h}(E^-) \leq \mu_0(E^-) = 0$. We define

$$h^+_c = h^+_c(\lambda) = \inf\{h \geq 0; \mu^+_{\lambda,h}(E^+) = 1\},$$
$$h^-_c = h^-_c(\lambda) = \inf\{h \geq 0; \mu^-_{\lambda,h}(E^+) = 1\}.$$

Because $\mu^+_{\lambda,h}(E^+) \geq \mu^-_{\lambda,h}(E^+)$, we have $h^+_c \leq h^-_c$. When $h^+_c < h^-_c$, $\mu^+_{\lambda,h}(E^+) = 1$ and $\mu^-_{\lambda,h}(E^+) = 0$ for all $h \in (h^+_c, h^-_c)$. This implies $\mu^+_{\lambda,h} \neq \mu^-_{\lambda,h}$ for uncountable h's, which is impossible. We can conclude $h^+_c = h^-_c$, say h_c. By a standard Peierls argument, we can prove that h_c is finite.

Proposition 3.6.4. *If* $\mu^+_{\lambda,0} = \mu^-_{\lambda,0}$, *then* $|\mathcal{G}(\lambda, h)| = 1$ *for* $h > h_c(\lambda)$.

Proof. When $h > h_c$, we have $\mu^-_{\lambda,h}(E^+) = 1$. It follows from Proposition 3.5.8(i) that $\mu^-_{\lambda,h}(E^{0-*}) = 0$. Lemma 3.5.1 again shows that $\mu^-_{\lambda,h} = \mu^+_{\lambda,h}$. □

Remark 3.6.5. For λ with $\mu^+_{\lambda,0} = \mu^-_{\lambda,0}$ and $h_c(\lambda) > 0$, any finite region of $\mathbb{Z}^2$ is surrounded by a $(0*)$circuit when $0 \leq |h| < h_c$. This implies the uniqueness of Gibbs state when $h = 0$, while we cannot deduce the uniqueness of Gibbs state when $h \neq 0$ by lack of symmetry.

In the 2D Ising model, the coexistence of $\infty(*)$clusters occurs when $\beta < \beta_c$ (see [34, 35]). An analogous problem remains open for the 2D Widom-Rowlinson model.

3.7 Number of phases in 2D : symmetric and large activity case

Now we turn to the symmetric case (i.e. $h = 0$). In this section we assume that $h = 0$ and omit 'h'. By Theorems 3.3.6 and 3.3.9, Gibbs measures are unique when $\lambda < p_c/(1 - p_c)$ and multiple when $\lambda > 8p_c/(1 - p_c)$, where $p_c = p_c(\mathbb{Z}^2, \text{site})$. Although our result is restricted to the large activity case, we can describe the structure of a class of Gibbs measures in which all translationally invariant ones are contained.

We begin with the following proposition.

Proposition 3.7.1. *Suppose that $\lambda > 8p_c/(1-p_c)$. If $\mu \in \mathcal{G}(\lambda)$ satisfies that $\mu(E^{0*}) > 0$, then $\mu(E^+ \cap E^-) > 0$.*

Proof. Without loss of generality, we can assume that $\mu \in \mathcal{G}_{\text{ex}}(\lambda)$ and $\mu(E^{0*}) = 1$. We shall show that $\mu(E^+ \cap E^-) = 1$.

Suppose that $\mu(E^+) = 0$, which implies that any finite set Λ of $\mathbb{Z}^2$ is surrounded by a $(0-*)$circuit μ-a.s. On the other hand, since $\lambda > 8p_c/(1-p_c)$, for $x \in \mathbb{Z}^2$ and $\varepsilon > 0$, we can choose a large $\Lambda \subset\subset \mathbb{Z}^2$ containing x such that $P_{\frac{\lambda}{\lambda+8}}(x \overset{0*}{\longleftrightarrow} \partial\Lambda) < \varepsilon$. As Λ is surrounded by a $(0-*)$circuit μ-a.s., we can choose a large $\Delta \subset\subset \mathbb{Z}^2$ such that with μ-probability$> 1-\varepsilon$ there is such a $(0-*)$circuit in Δ. Let Γ be the region surrounded by the maximal $(0-*)$circuit in Δ if it exists. Otherwise we set $\Gamma = \emptyset$. Because Γ is determined from outside, we can show by using the strong Markov property of μ that

$$\begin{aligned}&\mu(x \overset{0*}{\longleftrightarrow} \partial\Lambda)\\ &= \mu(\mu^{\omega}_{\lambda,\Gamma(\omega)}(x \overset{0*}{\longleftrightarrow} \partial\Lambda)1_{\{\Gamma(\omega)\neq\emptyset\}}) + \mu(\{x \overset{0*}{\longleftrightarrow} \partial\Lambda\} \cap \{\Gamma(\omega) = \emptyset\}).\end{aligned}$$

By Lemmata 3.3.1 and 3.3.8, we have

$$\mu^{\omega}_{\lambda,\Gamma(\omega)}(x \overset{0*}{\longleftrightarrow} \partial\Lambda) = R^{\omega}_{\lambda,\Gamma(\omega)}(x \overset{0*}{\longleftrightarrow} \partial\Lambda) \leq P_{\frac{\lambda}{\lambda+8}}(x \overset{0*}{\longleftrightarrow} \partial\Lambda) < \varepsilon.$$

Thus we have $\mu(x \overset{0*}{\longleftrightarrow} \partial\Lambda) < \varepsilon + \varepsilon = 2\varepsilon$. By letting $\Delta \nearrow \mathbb{Z}^2, \varepsilon \searrow 0$ and $\Lambda \nearrow \mathbb{Z}^2$, we can see that $\mu(x \overset{0*}{\longleftrightarrow} \infty) = 0$. Since x is arbitrary, we can conclude $\mu(E^{0*}) = 0$, which is a contradiction. Thus we have $\mu(E^+) = 1$.

In the same way, we can show that $\mu(E^-) = 1$. □

3.7.1 Periodic phases in two dimensions

When λ is large, we can get the complete description of periodic Gibbs measures.

Theorem 3.7.2. *If $\lambda > 8p_c/(1-p_c)$, then any periodic $\mu \in \mathcal{G}(\lambda)$ is a mixture of μ^+_λ and μ^-_λ.*

Before proving this, we prepare a lemma. We say $(\pi, \tilde{\pi})$ is a pair of *conjugate half-planes* if half-planes $\pi, \tilde{\pi}$ share only a common boundary line. An associated pair of infinite clusters $(I^{0+*}_{\pi}, I^{0+*}_{\tilde{\pi}})$ or $(I^{0-*}_{\pi}, I^{0-*}_{\tilde{\pi}})$ is called a *butterfly*. In particular, a butterfly in $(\pi_{\text{left}}, \pi_{\text{right}})$ is called a *horizontal butterfly*. A *vertical butterfly* is the one in $(\pi_{\text{up}}, \pi_{\text{down}})$.

Lemma 3.7.3 (Butterfly lemma). *(cf. [20] Lemma 3.1) Suppose that* $\lambda > 8p_c/(1-p_c)$ *and* $\mu \in \mathcal{G}(\lambda)$*. If* $\mu(E^{0*}) > 0$*, then there exists at least one butterfly with positive probability.*

Proof. By the extremal decomposition theorem, there exists $Q \in \mathcal{G}_{\text{ex}}(\lambda)$ such that $Q(E^{0*}) = 1$. By Proposition 3.7.1, $Q(E^+ \cap E^-) = 1$. If Q-a.s. there is no butterfly, then it turns out that Q is flip-reflection invariant. Because this is impossible by Proposition 3.5.8(ii), we can see that there exists at least one butterfly Q-a.s. This gives the result. $\square$

We can prove Theorem 3.7.2 by using Proposition 3.5.2 and the following proposition.

Proposition 3.7.4. *If* $\lambda > 8p_c/(1-p_c)$*, then* $\mu(E^{0*}) = 0$ *for any periodic* $\mu \in \mathcal{G}(\lambda)$*.*

Proof. By the ergodic decomposition theorem, it is sufficient to show that $\mu(E^{0*}) = 0$ for ergodic μ. So we assume that μ is ergodic.

Suppose that $\mu(E^{0*}) = 1$. By Proposition 3.7.1, we have $\mu(E^+ \cap E^-) > 0$. By butterfly lemma, we can assume that there is a vertical $(0+*)$ butterfly with positive probability. We can find a large square $\Lambda \subset\subset \mathbb{Z}^2$ such that with positive probability Λ intersects $I^{0+*}_{\text{up}}, I^{0+*}_{\text{down}}$ and I^-. Without loss of generality, we can assume that I^- leaves on the right between I^{0+*}_{up} and I^{0+*}_{down} with positive probability. For $k \in \mathbb{Z}$, let $A_k = \{(k,0) \in I^{0+*}_{\text{up}} \cap I^{0+*}_{\text{down}}, (k+1,0) \in I^-\}$ and A_∞ be the event that A_k occurs for infinitely many $k \in \mathbb{Z}$. By changing the configuration in Λ suitably, we have $\mu(A_0) > 0$. Poincaré's recurrence theorem([18] Lemma(18.15)) shows that $\mu(A_\infty) = 1$. But on A_∞ there exist infinitely many infinite $(-)$clusters. This contradicts Proposition 3.5.7. Consequently $\mu(E^{0*}) = 0$. $\square$

3.7.2 1-periodic phases in two dimensions

Let $\mu \in \mathcal{G}(\lambda)$. We say that an infinite cluster in a half-plane has the *line touching property* if the cluster touches the boundary line of the half-plane infinitely many times μ-a.s.

We define $\pm \in \Omega$ by

$$\pm(x) = \begin{cases} +1 & \text{if } x_2 > 0, \\ 0 & \text{if } x_2 = 0, \\ -1 & \text{if } x_2 < 0. \end{cases}$$

It follows from Lemma 3.2.4(iv) that $\mu^{\pm}_{\text{up}} = \lim_{\Lambda_{\text{up}} \nearrow \pi_{\text{up}}} \mu^{\pm}_{\Lambda_{\text{up}}}$ exists and is θ_{hor}-invariant.

Lemma 3.7.5. *(cf. [20] Lemma 4.2) $\mu^{\pm}_{\text{up}}(E^{0+*}_{\text{up}}) = 0$ when $\lambda > 8p_c/(1-p_c)$.*

This lemma is proved by using Theorem 3.7.2 and flip-reflection domination. Now we are ready to derive the line touching property of infinite clusters of several types. But note that the same argument as in the Ising model do not give the line touching property of the infinite clusters of types $+$, $+*$, 0, $0*$, $-$ and $-*$.

Lemma 3.7.6 (Line touching lemma). *(cf. [20] Lemma 4.1) Let $\lambda > 8p_c/(1-p_c)$ and $\mu \in \mathcal{G}(\lambda)$. The infinite $(0+)$cluster in any half-plane π have the line touching property μ-a.s. if it exists. The same holds for infinite clusters of type $0+*$ or $0-$ or $0-*$.*

Corollary 3.7.7. *Suppose $\lambda > 8p_c/(1-p_c)$ and $\mu \in \mathcal{G}(\lambda)$. In an arbitrary half plane π, there exists at most one infinite $(+)$cluster μ-a.s. The same holds for infinite clusters of types $+*$ or $-$ or $-*$.*

Lemma 3.7.8 (Orthogonal butterflies). *(cf. [20] Lemma 4.3) Let $\lambda > 8p_c/(1-p_c)$ and $\mu \in \mathcal{G}(\lambda)$. If $\mu(E^{0*}) > 0$, then there exist both horizontal butterflies and vertical butterflies with positive μ-probability.*

Proof. We can see that $\mu(E^+ \cap E^-) > 0$ by Proposition 3.7.1. By the extremal decomposition theorem, $Q(E^+ \cap E^-) = 1$ for some $Q \in \mathcal{G}_{\text{ex}}(\lambda)$. By butterfly lemma, there exist at least one butterfly Q-a.s.

Assume that there is a vertical $(0+*)$butterfly but no horizontal butterfly, for example. In this case, $Q = Q \circ R_{k,\text{vert}} \circ T$ for any $k \in \mathbb{Z}$. Therefore Q is horizontally periodic. Fix $n \in \mathbb{N}$. By shift lemma, we have $Q(E^{0+*}_{\text{up},n} \cap E^{0+*}_{\text{down},-n}) = 1$. For $k \in \mathbb{Z}$, we set

$$A^n_k = \left\{ \omega \in \Omega; \begin{array}{c} (k,n) \in I^{0+*}_{\text{up},n}, (k,-n) \in I^{0+*}_{\text{down},-n}, \\ \omega(k,l) = 0 \text{ for } -(n-1) \leq l \leq n-1 \end{array} \right\}$$

and $A^n_\infty = \{A^n_k \text{ occurs for infinitely many } k \in \mathbb{Z}\}$. We can easily see that $Q(A^n_0) > 0$. Poincaré's recurrence theorem and tail-triviality of Q imply that $Q(A^n_\infty) = 1$ for all n. Thus we have $Q\left(\bigcap_{n=1}^\infty A^n_\infty\right) = 1$. If for some n there is an infinite $(-)$cluster in $\pi_{\text{up},n}$, Poincaré's recurrence theorem again shows that infinitely many infinite $(-)$clusters appear, which contradicts Corollary 3.7.7. Hence for any n there is a unique infinite $(0+*)$cluster in

$\pi_{\text{up},n}$. Similarly, the infinite $(0+*)$cluster in $\pi_{\text{down},-n}$ is also unique. We can find that any finite region in $\mathbb{Z}^2$ is surrounded by a $(0+*)$circuit in $\omega \in \bigcap_{n=1}^{\infty} A_{\infty}^n$, which contradicts $Q(E^+ \cap E^-) = 1$.

Consequently, both vertical butterflies and horizontal butterflies exist Q-a.s., which implies that this occurs with positive μ-probability. $\square$

Theorem 3.7.9. *If $\lambda > 8p_c/(1-p_c)$ and $\mu \in \mathcal{G}(\lambda)$ is either horizontally periodic or vertically periodic, then*

$$\mu = \alpha\mu_\lambda^+ + (1-\alpha)\mu_\lambda^-$$

with some $\alpha \in [0,1]$.

Proof. By the ergodic decomposition theorem, we can assume that μ is horizontally ergodic and satisfies $\mu(E^{0*}) = 1$. Because at least one vertical butterfly must exist, as in the proof of Lemma 3.7.8, we can show that $\mu(E^+ \cap E^-) = 0$. This is a contradiction, which implies that $\mu(E^{0*}) = 0$. Together with Proposition 3.5.2, we can find that μ is a mixture of μ_λ^+ and μ_λ^-. $\square$

3.8 Miscellanea

3.8.1 Point-to-half-circuit lemma and Pinning lemma

In this and next subsections, we assume that $d = 2$ and $h = 0$, and omit 'h'. We prove two lemmata, which have their origin in Russo[52].

Let $\theta^* = P_{\frac{\lambda}{\lambda+8}}(0 \overset{1*}{\longleftrightarrow} \infty)$. If $\lambda/(\lambda+8) > p_c^*$ (i.e. $\lambda > 8(1-p_c)/p_c$), then $\theta^* > 0$. For a half-circuit σ in a half-plane π with its boundary line l, let $\operatorname{int}\sigma$ be the unique subset of $\mathbb{Z}^2$ such that it is invariant under the reflection with respect to l and it satisfies that $\pi \cap \partial(\operatorname{int}\sigma) = \sigma$.

Lemma 3.8.1 (Point-to-half-circuit lemma). *(cf. [20] Lemma 2.3) Let π be a half-plane with its boundary line l and R be the reflection with respect to l. For $x \in l$, let $\sigma = \sigma(x)$ be a $(*)$half-circuit in π with $\operatorname{int}\sigma \ni x$. For a feasible boundary condition $\omega \in \Omega$ with $\omega \equiv +1$ on σ, we have*

$$\mu_{\Lambda,\lambda}^{\omega}\{x \text{ is } (0+*)\text{connected to } \sigma \text{ in } \operatorname{int}\sigma\} \geq \frac{\theta^*}{4}.$$

Proof. Put $\Lambda = \operatorname{int}\sigma$. Since the above event is increasing, we can assume that for $x \in \partial\Lambda \setminus \sigma$,

$$\omega(x) = \begin{cases} 0 & \text{if } x \text{ is adjacent to some point on } \sigma, \\ -1 & \text{otherwise.} \end{cases}$$

Noting that $\omega \geq R \circ T(\omega)$ on $\partial\Lambda$, we have the flip-reflection domination $\mu^{\omega}_{\Lambda,\lambda} \geq \mu^{\omega}_{\Lambda,\lambda} \circ R \circ T$.

Let us consider the following events:

$$B_{x,\sigma} = \{\text{there exists a } (0+*)\text{path in } \Lambda \text{ from } x \text{ to } \sigma\},$$
$$C_{x,\sigma} = \left\{ \begin{array}{c} x \text{ is surrounded by a } (0+*) \text{ circuit in } \Lambda \cup \sigma \\ \text{which is } (0+*)\text{connected to } \sigma \end{array} \right\}.$$

Put $D_{x,\sigma} = B_{x,\sigma} \cup C_{x,\sigma}$. We can easily see that $\mu^{\omega}_{\Lambda,\lambda}(D_{x,\sigma} \cup R \circ T(D_{x,\sigma})) = 1$ and $\mu^{\omega}_{\Lambda,\lambda}(D_{x,\sigma}) \geq 1/2$. On the other hand,

$$\begin{aligned} \frac{\mu^{\omega}_{\Lambda,\lambda}(B_{x,\sigma})}{\mu^{\omega}_{\Lambda,\lambda}(D_{x,\sigma})} &= \frac{\mu^{\omega}_{\Lambda,\lambda}(B_{x,\sigma} \cap C_{x,\sigma}) + \mu^{\omega}_{\Lambda,\lambda}(B_{x,\sigma} \cap C^c_{x,\sigma})}{\mu^{\omega}_{\Lambda,\lambda}(C_{x,\sigma}) + \mu^{\omega}_{\Lambda,\lambda}(B_{x,\sigma} \cap C^c_{x,\sigma})} \\ &\geq \frac{\mu^{\omega}_{\Lambda,\lambda}(B_{x,\sigma} \cap C_{x,\sigma})}{\mu^{\omega}_{\Lambda,\lambda}(C_{x,\sigma})}, \end{aligned}$$

where we used the fact that $y = (b+x)/(a+x)$ is increasing in x when $b < a$. From this, we can see that

$$\mu^{\omega}_{\Lambda,\lambda}(B_{x,\sigma}) \geq \mu^{\omega}_{\Lambda,\lambda}(B_{x,\sigma}|C_{x,\sigma})\mu^{\omega}_{\Lambda,\lambda}(D_{x,\sigma}) \geq \mu^{\omega}_{\Lambda,\lambda}(B_{x,\sigma}|C_{x,\sigma})/2.$$

When $C_{x,\sigma}$ occurs, there exist a $(0+*)$circuit in Λ which surrounds x and is $(0+*)$connected to σ. Let Γ be the region surrounded by the maximal $(0+*)$circuit which has the property mentioned above. By the strong Markov property, we have

$$\mu^{\omega}_{\Lambda,\lambda}(B_{x,\sigma}|C_{x,\sigma}) \geq \mu^{\omega}_{\Lambda,\lambda}(\mu^{\omega'}_{\Gamma,\lambda}(x \stackrel{+*}{\longleftrightarrow} \partial\Gamma)|C_{x,\sigma}).$$

Since $\omega' \geq 0$ on $\partial\Gamma$, $\mu^{\omega'}_{\Gamma,\lambda}(x \stackrel{+*}{\longleftrightarrow} \partial\Gamma) \geq \mu^{0}_{\Gamma,\lambda}(x \stackrel{+*}{\longleftrightarrow} \partial\Gamma)$ by the Markov property and the stochastic monotonicity. Using the site random-cluster representation, we can see that

$$\begin{aligned} \mu^{0}_{\Gamma,\lambda}(x \stackrel{+*}{\longleftrightarrow} \partial\Gamma) &= R^{0}_{\Gamma,\lambda}(x \stackrel{1*}{\longleftrightarrow} \partial\Gamma)/2 \\ &\geq P_{\frac{\lambda}{\lambda+8}}(x \stackrel{1*}{\longleftrightarrow} \partial\Gamma)/2 \geq P_{\frac{\lambda}{\lambda+8}}(x \stackrel{1*}{\longleftrightarrow} \infty)/2 = \theta^*/2. \end{aligned}$$

Thus we have $\mu^{\omega}_{\Lambda,\lambda}(B_{x,\sigma}) \geq \theta^*/4$. □

If we can prove the line touching property of infinite $(+*)$clusters in half-planes, then the next lemma will be useful. Let $l_{\text{right}} = \{x = (x_1, 0) \in \mathbb{Z}^2; x_1 \geq 0\}$ and $l_{\text{left}} = \{x = (x_1, 0) \in \mathbb{Z}^2; x_1 \leq 0\}$.

Lemma 3.8.2 (Pinning lemma). *(cf. [20] Lemma 5.2) Suppose that* $\lambda > 8(1-p_c)/p_c$, $\mu \in \mathcal{G}(\lambda)$ *and that* μ*-a.s. there exists an infinite* $(+*)$*cluster* I^{+*}_{up} *in* π_{up}*, which intersects* l_{right} *infinitely many times. Then, for all* n *and* $x \in l_{\mathrm{right}}$ *located far enough to the right, we have*

$$\mu\{x \text{ is } (0+*)\text{connected to } I^{+*}_{\mathrm{up}} \text{ in } (\Lambda_n \cup l_{\mathrm{left}})^c\} \geq \theta^*/8 (>0).$$

Similar statements are valid for π_{down}, π_{left} *and* π_{right}.

Proof. By the assumption, there is an infinite connected component in $I^{+*}_{\mathrm{up}} \setminus \Lambda_n$ which contains infinitely many points on l_{right}. If we choose $x \in l_{\mathrm{right}}$ located far enough to the right, then with probability $\geq 3/4$ at least one such point can be found left from x, and another such point can be found right from x. This implies that x is surrounded by a $(+*)$half-circuit σ in π_{up}, which is a part of the infinite connected component in $I^{+*}_{\mathrm{up}} \setminus \Lambda_n$. Note that $\Lambda_n \cap \operatorname{int} \sigma = \emptyset$.

We can choose a large box Δ containing x so that with probability $\geq 1/2$ we can find a $(+*)$half-circuit, like σ above, in Δ. The maximal one in Δ is again denoted by σ. Put $\Gamma(\omega) = \operatorname{int} \sigma$. By the strong Markov property and the point-to-half-circuit lemma, we have

$$\begin{aligned}
&\mu\{x \text{ is } (0+*)\text{connected to } I^{+*}_{\mathrm{up}} \text{ in } (\Lambda_n \cup l_{\mathrm{left}})^c\} \\
&\geq \mu(\{\Gamma \neq \emptyset\} \cap \{x \text{ is } (0+*)\text{connected to } \sigma \text{ in } \Gamma\}) \\
&= \mu(\mu^{\omega}_{\Gamma(\omega),\lambda}(x \text{ is } (0+*)\text{connected to } \sigma \text{ in } \Gamma) \cdot 1_{\{\Gamma \neq \emptyset\}}) \\
&\geq \frac{\theta^*}{4} \cdot \mu\{\Gamma \neq \emptyset\} \geq \frac{\theta^*}{8}.
\end{aligned}$$

□

3.8.2 Infinite $(0*)$ clusters in half-planes

We remark that the line touching property of the infinite $(0*)$clusters in half-planes cannot be obtained by the argument in [20] Lemma 4.1. Here we give a possible statement about it. It seems that the next lemma suggests the existence of the unique infinite $(0*)$'contour'.

Lemma 3.8.3. *Let* $\lambda > 8p_c/(1-p_c)$ *and* $\mu \in \mathcal{G}(\lambda)$. *For a half-plane* π *with* $\mu(E^{0*}_{\pi}) = 1$, *each infinite* $(0*)$*cluster in* π *intersects the boundary line of* π *at least once.*

Proof. For definiteness, we assume that $\pi = \pi_{\mathrm{up}}$. Put

$$G_x^{0*} = \left\{ \begin{array}{l} x \text{ belongs to an infinite } (0*)\text{cluster in } \pi_{\mathrm{up}} \\ \text{which does not intersect } l_{\mathrm{hor}} \end{array} \right\}.$$

Fix $k \geq 1$ and let

$$G_{x,k}^{0*} = \left\{ \begin{array}{l} x \text{ belongs to a } (0*)\text{cluster in } \pi_{\mathrm{up}} \\ \text{whose size} \geq k \text{ and which does not intersect } l_{\mathrm{hor}} \end{array} \right\}.$$

Take a box $\Delta \subset \pi_{\mathrm{up}}$ containing x and so large that there exists no path of length k from x to Δ^c. In $\omega \in G_{x,k}^{0*}$, there exists a $(+/-)$path in Δ which separates x from l_{hor}. For this ω, let $\Gamma(\omega)$ be a subset of $\Delta \setminus l_{\mathrm{hor}}$ containing x such that $\partial\Gamma(\omega) \setminus (\partial\Delta \cap \pi_{\mathrm{up}})$ is a $(+/-)$path and it is maximal in Δ. We put

$$E_{x,k}^{0*} = \{x \text{ belongs to a } (0*)\text{cluster in } \pi_{\mathrm{up}} \text{ whose size} \geq k\}.$$

For a path p in Δ, we consider the following events:

$$\begin{aligned} &A_{\Delta,p}^{+} \\ &= \{\Gamma \neq \emptyset\} \cap \{\partial\Gamma(\omega) \setminus (\partial\Delta \cap \pi_{\mathrm{up}}) = p \text{ is a } (+/-)\text{path and its type is } (+)\} \\ &A_{\Delta,p}^{-} \\ &= \{\Gamma \neq \emptyset\} \cap \{\partial\Gamma(\omega) \setminus (\partial\Delta \cap \pi_{\mathrm{up}}) = p \text{ is a } (+/-)\text{path and its type is } (-)\}. \end{aligned}$$

Note that these events are $\mathcal{F}_\Delta$-measurable. We can see that

$$\mu(G_{x,k}^{0*}) \leq \mu(\{\Gamma \neq \emptyset\} \cap E_{x,k}^{0*}) = \sum_p \left\{ \mu(E_{x,k}^{0*} \cap A_{\Delta,p}^{+}) + \mu(E_{x,k}^{0*} \cap A_{\Delta,p}^{-}) \right\},$$

where p in the summation runs over paths in Δ. Let $\mp = T\pm$. We have

$$\begin{aligned} \mu(E_{x,k}^{0*} \cap A_{\Delta,p}^{+}) &= \mu(\mu_{\Gamma(\omega),\lambda}(E_{x,k}^{0*}) 1_{A_{\Delta,p}^{+}}) \\ &\leq \mu(\mu_{\Gamma(\omega),\lambda}(E_{x,k}^{0-*}) 1_{A_{\Delta,p}^{+}}) \leq \mu_{\Delta,\lambda}^{\mp}(E_{x,k}^{0-*}) \mu(A_{\Delta,p}^{+}), \end{aligned}$$

where we used the strong Markov property, the inclusion of events and the stochastic monotonicity. Similarly,

$$\begin{aligned} \mu(E_{x,k}^{0*} \cap A_{\Delta,p}^{-}) &= \mu(\mu_{\Gamma(\omega),\lambda}(E_{x,k}^{0*}) 1_{A_{\Delta,p}^{-}}) \\ &\leq \mu(\mu_{\Gamma(\omega),\lambda}(E_{x,k}^{0+*}) 1_{A_{\Delta,p}^{-}}) \leq \mu_{\Delta,\lambda}^{\pm}(E_{x,k}^{0+*}) \mu(A_{\Delta,p}^{-}). \end{aligned}$$

Thus we have

$$\begin{aligned}
\mu(G^{0*}_{x,k}) &\le \sum_p \left\{\mu^{\mp}_{\Delta}(E^{0-*}_{x,k})\mu(A^{+}_{\Delta,p}) + \mu^{\pm}_{\Delta}(E^{0+*}_{x,k})\mu(A^{-}_{\Delta,p})\right\} \\
&= \mu^{\mp}_{\Delta}(E^{0-*}_{x,k})\left(\sum_p \mu(A^{+}_{\Delta,p})\right) + \mu^{\pm}_{\Delta}(E^{0+*}_{x,k})\left(\sum_p \mu(A^{-}_{\Delta,p})\right) \\
&= \mu^{\mp}_{\Delta}(E^{0-*}_{x,k})\mu\left(\bigcup_p A^{+}_{\Delta,p}\right) + \mu^{\pm}_{\Delta}(E^{0+*}_{x,k})\mu\left(\bigcup_p A^{-}_{\Delta,p}\right) \\
&\le \mu^{\mp}_{\Delta}(E^{0-*}_{x,k}) + \mu^{\pm}_{\Delta}(E^{0+*}_{x,k}).
\end{aligned}$$

Letting $\Delta \nearrow \pi_{\text{up}}$, we have $\mu(G^{0*}_{x,k}) \le \mu^{\mp}_{\text{up}}(E^{0-*}_{x,k}) + \mu^{\pm}_{\text{up}}(E^{0+*}_{x,k})$. Next letting $k \to \infty$, we have $\mu(G^{0*}_{x}) \le \mu^{\mp}_{\text{up}}(E^{0-*}_{\text{up}}) + \mu^{\pm}_{\text{up}}(E^{0+*}_{\text{up}}) = 0$, by Proposition 3.7.5. □

Remark 3.8.4. For example, if the type of the left $\infty(+/-)$cluster is $+$ and the type of the right $\infty(+/-)$cluster is $-$, then we cannot connect them in Δ.

3.8.3 The multitype Widom-Rowlinson model

Runnels and Lebowitz[50] introduced the multitype Widom-Rowlinson models with $q(\ge 2)$ species and showed the existence of intermediate phases for large q. More recently, Georgii and Zagrebnov[21] gave a new proof using percolation and chessboard estimate (see [18] Chapter 17), which can be easily generalized to a wider class of interactions.

We extend the definition of the Widom-Rowlinson model on $G = (V, E)$ to $q \ge 2$ case in a natural way: Let $\Omega = \{0, 1, \cdots, q\}^V$. For $a \in \{1, \cdots, q\}$, we regard a as a site occupied by a particle with color a, while 0 stands for a vacant site. For $\Lambda \subset V$, let $\Omega_\Lambda = \{0, 1, \cdots, q\}^\Lambda$. A configuration $\omega \in \Omega_\Lambda$ is said to be feasible if $\omega(x)\omega(y) = 0$ or $\omega(x) = \omega(y) \ne 0$ for all adjacent $x, y \in \Lambda$. For a feasible boundary condition $\omega \in \Omega$ and activity $\lambda > 0$, the finite volume Gibbs distribution in $\Lambda \subset\subset V$ is defined by

$$\mu^{\omega}_{\Lambda,\lambda}(\sigma) = \frac{1}{Z^{\omega}_{\Lambda,\lambda}} 1_{\{\sigma*\omega:\text{feasible}\}} \prod_{x\in\Lambda} \lambda^{1_{\{\sigma(x)\ne 0\}}} \quad (\sigma \in \Omega_\Lambda).$$

The set of Gibbs states is denoted by $\mathcal{G}(\lambda, q)$.

Let us consider the multitype Widom-Rowlinson model on $\mathbb{Z}^2$. For $\omega \in \Omega$ and $S \subset \mathbb{Z}^2$, we call S an *occupied sea* if S is an infinite cluster in

$\{x \in \mathbb{Z}^2; \omega(x) \neq 0\}$ and each $\Lambda \subset\subset \mathbb{Z}^2$ is surrounded by a circuit in S. We define an even occupied $(*)$sea S by an infinite $(*)$cluster in $\{x = (x_1, x_2) \in \mathbb{Z}^2; x_1 + x_2$ is even and $\omega(x) \neq 0\}$ and each finite region is surrounded by a $(*)$circuit in S. Similarly, we define an odd empty $(*)$sea.

Theorem 3.8.5 ([21]). *If q exceeds some q_0 (for example, we can take $q_0 = 2 \cdot 10^{85}$), then there exist $\lambda_c(q) \in (q/5, 5q)$ and $\varepsilon(q) \in (0, 1/3)$ with $\varepsilon(q) \to 0$ as $q \to \infty$ such that the following statements hold.*

(i) (colored phases) When $\lambda > \lambda_c(q)$, there are q different translation-invariant extremal Gibbs measures $\mu_a(a \in \{1, \cdots q\}) \in \mathcal{G}(\lambda, q)$. With μ_a-probability one, there is an occupied sea with color a, to which the origin belongs with μ_a-probability $\geq 1 - \varepsilon(q)$.

(ii) (staggered phases) When $q_0/q \leq \lambda < \lambda_c(q)$, there exist two different extremal Gibbs states μ_{even} and $\mu_{odd} \in \mathcal{G}(\lambda, q)$. These are invariant under even translations and μ_{odd} is an one-step translation of μ_{even}. There are an even occupied $(*)$sea and an odd empty $(*)$sea μ_{even}-a.s. With μ_{even}-probability $\geq 1 - \varepsilon(q)$, $(0, 0)$ belongs to the even occupied $(*)$sea and $(1, 0)$ belongs to the odd empty $(*)$sea. Each occupied cluster is finite μ_{even}-a.s. and its color is independent uniformly distributed.

(iii) (first-order phase transition) At $\lambda = \lambda_c$, $q + 2$ different Gibbs states μ_{even}, μ_{odd}, $\mu_a(a \in \{1, \cdots q\}) \in \mathcal{G}(\lambda, q)$ coexist.

A problem for this model is determining the minimum q such that staggered phases appear. In [42], it is conjectured that $q = 4$ for $\mathbb{Z}^2$. In [48], it is mentioned that second-order phase transition occurs when $q = 2$ and this can be proved by using Theorem 3.2.9. Unfortunately, details are not clear.

We mentioned the non-monotonic behaviour of the Widom-Rowlinson model and staggered phases for large q. These are related to lack of monotonicity in site random-cluster models. The site random-cluster model corresponding to the multitype Widom-Rowlinson model is

$$R^0_{\Lambda,\lambda}(\xi) = \frac{1}{\tilde{Z}^0_{\Lambda,\lambda}} \prod_{x \in \Lambda} \lambda^{\xi(x)} \cdot q^{k(\xi,\Lambda)} \quad (\xi \in \{0, 1\}^\Lambda),$$

where $k(\xi, \Lambda)$ denotes the number of (1)cluster in Λ on ξ. For large q, many (1)clusters appear with high probability. Because of hard-core interactions, staggered phases can appear. If activity is large, then colored phases appear.

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