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博　士　論　文

Spaces of initial conditions of the two dimensional Garnier system and its degenerate ones

(2次元Garnier系とその退化系の初期値空間)

平成17年1月

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Contents

Spaces of initial conditions of the two dimensional Garnier system and its degenerate ones

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0 Introduction

In this paper, we construct spaces of initial conditions $E_J(s)$, $s=(s_1,s_2)\in B_J$ for the two dimensional Garnier system $\mathcal{H}_J,\ J=11111$ and its degenerate systems $\mathcal{H}_J,\ J=1112,113,122,14,23,5$, which are completely integrable Hamiltonian systems of degree 2 of the form

$$dq_k=\sum_{i=1,2}\frac{\partial H_{Ji}}{\partial p_k}ds_i,\quad dp_k=-\sum_{i=1,2}\frac{\partial H_{Ji}}{\partial q_k}ds_i\quad k=1,2.$$

Note that the label J is a partition of 5. For every J, two Hamiltonians are certain polynomials of q_1,q_2,p_1,p_2 whose coefficients are rational functions of $s=(s_1,s_2)$ holomorphic in a domain $B_J\subset\mathbb{C}^2$. For example, the Hamiltonians $H_i=H_{11111i},\ i=1,2$ of the Garnier system $\mathcal{H}_{11111}$ are of the following form:

$$\begin{aligned}
s_1(s_1-1)H_1=&\Big\{q_1(q_1-1)(q_1-s_1)-\frac{s_1(s_1-1)}{s_1-s_2}q_1q_2\Big\}p_1^2\\
&+2q_1q_2\Big(q_1-\frac{s_1(s_2-1)}{s_2-s_1}\Big)p_1p_2+q_1q_2\Big(q_2-\frac{s_2(s_1-1)}{s_1-s_2}\Big)p_2^2\\
&-\Big\{(\alpha_0-1)q_1(q_1-1)+\alpha_1q_1(q_1-s_1)+\alpha_2(q_1-1)(q_1-s_1)\\
&\qquad+\alpha_3q_1\Big(q_1-\frac{s_1(s_2-1)}{s_2-s_1}\Big)-\alpha_2\frac{s_1(s_1-1)}{s_1-s_2}q_2\Big\}p_1\\
&+\Big\{(\alpha_\infty+2\nu)q_1q_2+\alpha_2\frac{s_1(s_2-1)}{s_2-s_1}q_2+\alpha_3\frac{s_2(s_1-1)}{s_1-s_2}q_1\Big\}p_2+\nu(\nu+\alpha_\infty)q_1,
\end{aligned}$$

$$\begin{aligned}
s_2(s_2-1)H_2=&\,q_1q_2\Big(q_1-\frac{s_1(s_2-1)}{s_2-s_1}\Big)p_1^2+2q_1q_2\Big(q_2-\frac{s_2(s_1-1)}{s_1-s_2}\Big)p_1p_2\\
&+\Big\{q_2(q_2-1)(q_2-s_2)-\frac{s_2(s_2-1)}{s_2-s_1}q_1q_2\Big\}p_2^2\\
&+\Big\{(\alpha_\infty+2\nu)q_1q_2+\alpha_2\frac{s_1(s_2-1)}{s_2-s_1}q_2+\alpha_3\frac{s_2(s_1-1)}{s_1-s_2}q_1\Big\}p_1\\
&-\Big\{(\alpha_0-1)q_2(q_2-1)+\alpha_1q_2(q_2-s_2)+\alpha_3(q_2-1)(q_2-s_2)\\
&\qquad+\alpha_2q_2\Big(q_2-\frac{s_2(s_1-1)}{s_1-s_2}\Big)-\alpha_3\frac{s_2(s_2-1)}{s_2-s_1}q_1\Big\}p_2+\nu(\nu+\alpha_\infty)q_2,
\end{aligned}$$

where $\alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_\infty)$ are complex parameters and

$$\nu=-\frac{1}{2}(\alpha_0+\alpha_1+\alpha_2+\alpha_3-1+\alpha_\infty).$$

In this case, we see

$$B_{11111} = \mathbb{C}^2 \setminus \cup\{s_1 s_2(s_1-1)(s_2-1)(s_1-s_2)=0\}.$$

The forms of the Hamiltonians for the other J are given in the next section. Here we explain the meaning of the label J, a partition of 5. Our systems are obtained as monodromy preserving deformation equations of the second order linear ordinary differential equations with regular or irregular singular points and apparent singular points. Let us assign 1 to a regular singular point and $r+1$ to an irregular singular point of Poincaré rank r. Then we can express by a sequence of positive integers the distribution of regular or irregular singular points with the data of Poincaré ranks of a linear differnatial equation. For example, $\mathcal{H}_{11111}$ (or $\mathcal{H}_{1112}$) is a monodromy preserving deformation system of a linear differenatial equation with five regular singular points (or with three regular singular points and an irregular singular point of Poincaré rank 1).

Each $\mathcal{H}_J$ defines a nonsingular foliation of the trivial fiber space $\mathbb{C}^4 \times B_J \ni (q,p,s)$, $q = (q_1,q_2)$, $p=(p_1,p_2)$, $s=(s_1,s_2)$, because $\partial H_{Ji}/\partial p_k$, $\partial H_{Ji}/\partial q_k$ $i,k=1,2$ are holomorphic on $\mathbb{C}^4 \times B_J$. However, since the differential system is nonlinear, its leaves or the solutions may not be prolonged along some curves in B_J, in other words, they may have movable singularities. Therefore it is preferabale to obtain a fiber space over B_J which contains $\mathbb{C}^4 \times B_J$ as fiber subspace so that every solution can be prolonged in the space along any curve in B_J. The most typical and well known such spaces are those for Painlevé systems. The purpose of this paper is to construct such fiber spaces E_J over B_J (namely to construct the fibers $E_J(s)$, $s \in B_J$ which are called the spaces of initial conditions) for the Garnier system $\mathcal{H}_{11111}$ and its degenerate systems $\mathcal{H}_J,\ J = 1112, 113, 122, 14, 23, 5$.

For every J, we first compactify the fiber $\mathbb{C}^4 \times s \ni (q,p) \times s$ suitably. As such a compact manifold we choose Hirzebruch manifold of dimension 4 $\overline{\Sigma}_\nu$ which is a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$. The manifold is covered by nine affine charts. Then we write the system $\mathcal{H}_J$ in the coordinates of all charts of the manifold. It can be seen that on certain three charts the differential systems are polynomial Hamiltonian systems, however on the other charts they are not Hamiltonian systems and have pole singularities on a divisor $D \times s$, $s \in B_J$. We next determine the so-called accessible singular points on $D \times s$. An accessible singular point is a point through which many holomorphic solution curves may pass. We can verify that the set of accessible singular points is a disjoint union of connected components $A_i(s)$ each of which is isomorphic to $\mathbb{P}^1$. We can assign to each component $A_i(s)$ an element of the partition J denoted by $n_i \in \mathbb{Z}_>$ with $\sum n_i = 5$, which implies the number of components is equal to the lenghth of J. We then make quadratic transformation $Q_{A_i(s)}$ along each $A_i(s)$. We see that our differential system has yet pole singularities on the excetional divisor $D_i^{(1)}(s) = Q_{A_i(s)}(A_i(s))$. Therefore we have to determine the accessible singular points and make quadratic transformation again. After repeating such quadratic transformations several times and auxiliary transformations, we can arrive at a holomorphic system, namely we can obtain coordinate systems which separates infinitely many solution curves of the original system $\mathcal{H}_J$ passing through any point on $A_i(s)$.

Let $\overline{E}_J(s)$ be the compact manifold obtained from $\overline{\Sigma}_\nu \times s$ by the composition of all the quadratic transformations and auxiliary transformations. Then we obtain $E_J(s)$ by removing the inaccessible singular points. The fiber space $E_J = \coprod_{s \in B_J} E_J(s)$ is sometimes called defining manifold for the system $\mathcal{H}_J$. The space is covered by finitely many charts which are isomorphic to $\mathbb{C}^4 \times B_J$. We notice that the original polynomial Hamiltonian system is extended to each chart as a polynomial Hamiltonian system.

We give here more remarks. The number of quadratic transformations along $A_i(s)$ is $2n_i$ where n_i is a positive integer assigned to it as above. The first n_i quadratic transformations are simultaneous replacement of every point on curves by $\mathbb{P}^2$ and the second n_i transformations are simultaneous replacement of every point on surfaces by $\mathbb{P}^1$. In the case where $n_i \geqslant 2$, we have to insert some simple change of variables after the n_i-th transformation and make certain change of variables after the last transformation by investigating carefully the fundamental 2-form $dq_1\wedge dp_1+dq_2\wedge dp_2$ in order to obtain good symplectic coordinate systems $(q^*,p^*)=(q_1^*,q_1^*,p_1^*,p_2^*)$, where we say that a coordinate system $(q^*,p^*)=(q_1^*,q_1^*,p_1^*,p_2^*)$ is symplectic if it satisfies

$$dq_1\wedge dp_1+dq_2\wedge dp_2=dq_1^*\wedge dp_1^*+dq_2^*\wedge dp_2^*.$$

Remark that the Hamiltonians $H_i,\ i=1,2$ in the coordinate system (q,p,s) are changed to $H_i(*),\ i=1,2$ in (q^*,p^*,s) determined by

$$\sum_{i=1,2}dq_i\wedge dp_i+\sum_{i=1,2}dH_i\wedge ds_i=\sum_{i=1,2}dq_i^*\wedge dp_i^*+\sum_{i=1,2}dH_i(*)\wedge ds_i.$$

The pull back of $A_i(s)$ is a $\mathbb{C}^2$-bundle over $\mathbb{P}^1$. We also notice that there are Bäcklund transformations which act on some parameters as permutations in the case where J has several elements of the same integer. Since the transformations also act as permutations of the corresponding components $A_i(s)$, a coordinate system for a component derives ones for the other. However some coordinate systmes thus obtained are not good, which means that the relation between the coordinate system and the original one is not of simple form, therefore we use the symmetries in the case where they produce good coordinate systems.

This paper is organized as follows. In Section 1, we give the explicit forms of the Hamiltonians of the two dimensional degenerate Garnier systems $\mathcal{H}_J,\ J\neq 11111$. In Section 2, we give symmetric group actions on systems $\mathcal{H}_J$ for some J. In Section 3, we explain a compactification of the original phase space $\mathbb{C}^4\ni(q_1,q_2,p_1,p_2)$. In Sections from 4 to 10, we consturuct spaces of initial conditions for two dimensional Garnier and all its degenerate systems. The results thus obtained are collected in the last section, Section 11, as theorems. Each theorem gives the description of the spaces of initial conditions for each $\mathcal{H}_J$.

1 Hamiltonians of the systems other than $\mathcal{H}_{11111}$

We give explicitly the forms of the Hamiltonians H_{J1} and H_{J2} (abbreviated as H_1 and H_2 respectively) of $\mathcal{H}_J$ $(J\neq 11111)$:

$\mathcal{H}_{1112}$:

$$\begin{aligned}s_1^2H_1=&\ q_1^2(q_1-s_1)p_1^2+2q_1^2q_2p_1p_2+q_1q_2(q_2-s_2)p_2^2\\&-\{(\alpha_0+\alpha_2-1)q_1^2+\alpha_1q_1(q_1-s_1)+\eta(q_1-s_1)+\eta s_1q_2\}p_1\\&-\{(\alpha_0+\alpha_1-1)q_1q_2+\alpha_2q_1(q_2-s_2)-\eta(s_2-1)q_2\}p_2+\nu(\nu+\alpha_\infty)q_1,\end{aligned}$$

$$\begin{aligned}s_2(s_2-1)H_2=&\ q_1^2q_2p_1^2+2q_1q_2(q_2-s_2)p_1p_2\\&+\Big\{q_2(q_2-1)(q_2-s_2)+\frac{s_2(s_2-1)}{s_1}q_1q_2\Big\}p_2^2\\&-\{(\alpha_0+\alpha_1-1)q_1q_2+\alpha_2q_1(q_2-s_2)-\eta(s_2-1)q_2\}p_1\\&-\Big\{(\alpha_0-1)q_2(q_2-1)+\alpha_1q_2(q_2-s_2)+\alpha_2(q_2-1)(q_2-s_2)\\&\qquad+\frac{s_2(s_2-1)}{s_1}(\alpha_2q_1+\eta q_2)\Big\}p_2+\nu(\nu+\alpha_\infty)q_2,\end{aligned}$$

$$\Big(\nu=-\frac{1}{2}(\alpha_0+\alpha_1+\alpha_2-1+\alpha_\infty)\Big),$$

$\mathcal{H}_{113}$:

$$H_1 = q_1^3p_1^2 + 2q_1^2\Big(q_2 + \frac{1}{s_2}\Big)p_1p_2 + q_1\Big\{q_2\Big(q_2 + \frac{1}{s_2}\Big) - \Big(\frac{s_1}{s_2^2} + \frac{1}{2}\Big)q_1\Big\}p_2^2$$
$$-\Big\{(\alpha_0 + \alpha_1 - 1)q_1^2 + \eta\Big(q_1 + \frac{q_2}{s_2}\Big)\Big\}p_1$$
$$-\Big\{(\alpha_0 + \alpha_1 - 1)q_1q_2 + \frac{\alpha_1}{s_2}q_1 - \eta\Big(\frac{s_1}{s_2^2} - \frac{1}{2}\Big)q_2 + \frac{\eta}{s_2}\Big\}p_2 + \nu(\nu + \alpha_\infty)q_1,$$

$$H_2 = q_1^2\Big(q_2 + \frac{1}{s_2}\Big)p_1^2 + 2q_1\Big\{q_2\Big(q_2 + \frac{1}{s_2}\Big) - \Big(\frac{s_1}{s_2^2} + \frac{1}{2}\Big)q_1\Big\}p_1p_2$$
$$+\Big\{q_2^2\Big(q_2 + \frac{1}{s_2}\Big) + \Big(\frac{s_1^2}{s_2^3} - \frac{s_2}{4}\Big)q_1^2 - \Big(\frac{s_1}{s_2^2} + \frac{3}{2}\Big)q_1q_2 - \frac{q_1}{s_2}\Big\}p_2^2$$
$$-\Big\{(\alpha_0 + \alpha_1 - 1)q_1q_2 + \frac{\alpha_1}{s_2}q_1 - \eta\Big(\frac{s_1}{s_2^2} - \frac{1}{2}\Big)q_2 + \frac{\eta}{s_2}\Big\}p_1$$
$$-\Big[(\alpha_0 + \alpha_1 - 1)q_2^2 - \Big\{\alpha_0 - 1 + \alpha_1\Big(\frac{s_1}{s_2^2} + \frac{1}{2}\Big)\Big\}q_1$$
$$+\Big\{\eta\Big(\frac{s_1^2}{s_2^3} - \frac{s_2}{4}\Big) + \frac{\alpha_1}{s_2}\Big\}q_2 - \eta\Big(\frac{s_1}{s_2^2} + \frac{1}{2}\Big)\Big]p_2 + \nu(\nu + \alpha_\infty)q_2,$$

$$\Big(\nu = -\frac{1}{2}(\alpha_0 + \alpha_1 - 1 + \alpha_\infty)\Big),$$

$\mathcal{H}_{122}$:

$$s_1^2H_1 = q_1^2(q_1 - s_1)p_1^2 + 2q_1^2q_2p_1p_2 + q_1q_2^2p_2^2$$
$$-\{(\alpha_0 - 1)q_1^2 + \alpha_1q_1(q_1 - s_1) + \eta_1(q_1 - s_1) + \eta_1s_1q_2\}p_1$$
$$-\{(\alpha_0 + \alpha_1 - 1)q_1q_2 + \eta_0s_2q_1 + \eta_1q_2\}p_2 + \nu(\nu + \alpha_\infty)q_1,$$

$$-s_2H_2 = q_1^2q_2p_1^2 + 2q_1q_2^2p_1p_2 + q_2^2(q_2 - 1)p_2^2$$
$$-\{(\alpha_0 + \alpha_1 - 1)q_1q_2 + \eta_0s_2q_1 + \eta_1q_2\}p_1$$
$$-\Big\{(\alpha_0 - 1)q_2(q_2 - 1) + \alpha_1q_2^2 + \frac{\eta_0s_2}{s_1}q_1 + \eta_0s_2(q_2 - 1)\Big\}p_2 + \nu(\nu + \alpha_\infty)q_2,$$

$$\Big(\nu = -\frac{1}{2}(\alpha_0 + \alpha_1 - 1 + \alpha_\infty)\Big),$$

$\mathcal{H}_{14}$:

$$H_1 = p_1^2 - 2s_2p_1p_2 - (q_2 + s_2q_1 + s_1 - \tfrac{1}{2}s_2^2)p_2^2 - \{q_1(q_1 + s_2) - q_2\}p_1,$$
$$-\{q_1q_2 + (s_1 - \tfrac{1}{2}s_2^2)q_1 + s_2q_2 + 1 - \alpha_0\}p_2 - \nu q_1,$$

$$H_2 = -s_2p_1^2 - 2(q_2 + s_2q_1 + s_1 - \tfrac{1}{2}s_2^2)p_1p_2$$
$$-\{s_2q_1^2 + q_1q_2 + (s_1 - \tfrac{1}{2}s_2^2)q_1 - s_2q_2 - s_2(s_1 - \tfrac{1}{2}s_2^2)\}p_2^2,$$
$$-\{q_1q_2 + (s_1 - \tfrac{1}{2}s_2^2)q_1 + s_2q_2 - \alpha_0 + 1\}p_1$$
$$-[q_2^2 - \{\alpha_0 - 1 + s_2(s_1 - \tfrac{1}{2}s_2^2)\}q_1 + (s_1 - \tfrac{1}{2}s_2^2)q_2]p_2 - \nu q_2,$$

$$\Big(\nu = -\alpha_\infty\Big),$$

$\mathcal{H}_{23}$:

$$H_1 = (q_1 - s_1)p_1^2 + 2q_2p_1p_2 - \tfrac{1}{2}\{q_1(q_1 - s_1) - q_2 + 2(\alpha_0 - 1)\}p_1 - \tfrac{1}{2}(q_1q_2 - 2\eta s_2)p_2 - \tfrac{1}{2}\nu q_1,$$

$$-s_2H_2 = q_2p_1^2 - q_2^2p_2^2 - \tfrac{1}{2}(q_1q_2 - 2\eta s_2)p_1 - \tfrac{1}{2}\{q_2^2 - 2\eta s_2(q_1 - s_1) - 2(\alpha_0 - 1)q_2\}p_2 - \tfrac{1}{2}\nu q_2,$$

$$\Big(\nu = -\alpha_\infty\Big),$$

$\mathcal{H}_5$:

$$H_1 = (q_2^2 - q_1 - s_1)p_1^2 + 2q_2p_1p_2 + p_2^2 + 2(q_1^2 - s_1^2 + s_2q_2)p_1 + 2(q_1q_2 + s_1q_2 + s_2)p_2 + 2\nu q_1,$$

$$H_2 = q_2p_1^2 + 2p_1p_2 + 2(q_1q_2 + s_1q_2 + s_2)p_1 + 2(q_2^2 - q_1 + s_1)p_2 + 2\nu q_2,$$

$$\Big(\nu = \alpha + \frac{1}{2}\Big).$$

Here q_1, q_2, p_1, p_2 and s_1, s_2 are complex variables and $\alpha_0, \alpha_1, ...$ are complex constans. We notice that Hamiltonians $H_1 = H_{J1}$ and $H_2 = H_{J2}$ are polynomials of $q = (q_1, q_2)$ and $p = (p_1, p_2)$ whose coefficients are rational functions of $s = (s_1, s_2)$ holomorphic in B_J where

$$B_{1112} = \mathbb{C}^2 \setminus \cup\{s_1s_2(s_2 - 1) = 0\}, \quad B_{113} = \mathbb{C}^2 \setminus \{s_2 = 0\},$$
$$B_{122} = \mathbb{C}^2 \setminus \{s_1 = 0\}, \quad B_{14} = B_{23} = B_5 = \mathbb{C}^2.$$

2 Symmetric group actions

The systems $\mathcal{H}_J$ for $J = 11111, 1112, 113$ and 122 admit symmetric group actions of degree $5, 3, 2$ and 2 respectively. The actions are realized by certain rational symplectic transformations each of which preserves the form of the system while it changes some parameters as a permutation.

As we make use of the transformations in order to avoid analogous calculations of blowing up, we give here the explicit forms of the generators of the transformation groups.

2.1 S_5 action on $\mathcal{H}_{11111}$

Let us consider the following symplectic transformations

$$\sigma_1 : \quad \begin{cases} q_1' = \dfrac{q_1}{s_1}, \quad q_2' = \dfrac{q_2}{s_2}, \\[2mm] p_1' = s_1p_1, \quad p_2' = s_2p_2, \\[2mm] s_1' = \dfrac{1}{s_1}, \quad s_2' = \dfrac{1}{s_2}, \end{cases}$$

$$\sigma_2 : \quad \begin{cases} q_1' = \dfrac{s_1}{s_1 - 1}\Big(1 - \dfrac{q_1}{s_1} - \dfrac{q_2}{s_2}\Big), \\[2mm] q_2' = \dfrac{(s_1 - s_2)q_2}{s_2(s_1 - 1)}, \\[2mm] p_1' = -(s_1 - 1)p_1, \quad p_2' = \dfrac{s_2(s_1 - 1)}{s_1 - s_2}\Big(p_2 - \dfrac{s_1}{s_2}p_1\Big), \\[2mm] s_1' = \dfrac{s_1}{s_1 - 1}, \quad s_2' = \dfrac{s_1 - s_2}{s_1 - 1}, \end{cases}$$

$$\sigma_3: \quad \begin{cases} q_1' = \dfrac{(s_2-s_1)q_1}{s_1(s_2-1)}, \quad q_2' = \dfrac{s_2}{s_2-1}\Big(1-\dfrac{q_1}{s_1}-\dfrac{q_2}{s_2}\Big), \\[2ex] p_1' = \dfrac{s_1(s_2-1)}{s_2-s_1}\Big(p_1-\dfrac{s_2}{s_1}p_2\Big), \quad p_2' = -(s_2-1)p_2, \\[2ex] s_1' = \dfrac{s_2-s_1}{s_2-1}, \quad s_2' = \dfrac{s_2}{s_2-1}, \end{cases}$$

$$\sigma_4: \quad \begin{cases} q_1' = \dfrac{q_1}{q_1+q_2-1}, \quad q_2' = \dfrac{q_2}{q_1+q_2-1}, \\[2ex] p_1' = (q_1+q_2-1)(p_1+\nu-q_1p_1-q_2p_2), \\[2ex] p_2' = (q_1+q_2-1)(p_2+\nu-q_1p_1-q_2p_2), \\[2ex] s_1' = \dfrac{s_1}{s_1-1}, \quad s_2' = \dfrac{s_2}{s_2-1}. \end{cases}$$

We can verify that each transformation $\sigma_m,\ 1 \leqslant m \leqslant 4$ changes the system $\mathcal{H}_{11111}$ in variables $q_1, q_2, p_1, p_2, s_1, s_2$ with parameters $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_\infty)$ into the same system in variables $q_1', q_2', p_1', p_2', s_1', s_2'$ with parameters $\alpha' = (\alpha_0', \alpha_1', \alpha_2', \alpha_3', \alpha_\infty')$ denoted by $\sigma_m(\alpha)$ where

$$\sigma_1(\alpha) = (\alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_\infty), \qquad \sigma_2(\alpha) = (\alpha_2, \alpha_1, \alpha_0, \alpha_3, \alpha_\infty),$$
$$\sigma_3(\alpha) = (\alpha_3, \alpha_1, \alpha_2, \alpha_0, \alpha_\infty), \qquad \sigma_4(\alpha) = (\alpha_0, \alpha_\infty, \alpha_2, \alpha_3, \alpha_1).$$

The transformation group generated by $\sigma_m,\ 1 \leqslant m \leqslant 4$ is algebraically isomorphic to the symmetric group S_5.

2.2 S_3 action on $\mathcal{H}_{1112}$

The symplectic transformations

$$\sigma_1 \quad \begin{cases} q_1' = -\dfrac{q_1}{s_2-1}, \quad q_2' = \dfrac{s_2}{s_2-1}\Big(1-\dfrac{q_1}{s_1}-\dfrac{q_2}{s_2}\Big), \\[2ex] p_1' = -(s_2-1)\Big(p_1-\dfrac{s_2}{s_1}p_2\Big), \quad p_2' = -(s_2-1)p_2, \\[2ex] s_1' = -\dfrac{s_1}{s_2-1}, \quad s_2' = \dfrac{s_2}{s_2-1}, \end{cases}$$

$$\sigma_2 \quad \begin{cases} q_1' = -\dfrac{q_1}{1-\frac{q_1}{s_1}-\frac{q_2}{s_2}}, \quad q_2' = \dfrac{s_2-1}{s_2}\dfrac{q_2}{1-\frac{q_1}{s_1}-\frac{q_2}{s_2}}, \\[2ex] p_1' = \Big(1-\dfrac{q_1}{s_1}-\dfrac{q_2}{s_2}\Big)\Big\{-\Big(1-\dfrac{q_1}{s_1}\Big)p_1+\dfrac{q_2}{s_1}p_2\Big\}, \\[2ex] p_2' = \Big(1-\dfrac{q_1}{s_1}-\dfrac{q_2}{s_2}\Big)\Big\{-\dfrac{q_1}{s_2-1}p_1+\Big(\dfrac{s_2-q_2}{s_2-1}\Big)p_2\Big\}, \\[2ex] s_1' = -s_1, \quad s_2' = -\dfrac{1}{s_2-1} \end{cases}$$

change the system $\mathcal{H}_{1112}$ in variables $q_1, q_2, p_1, p_2, s_1, s_2$ with parameters $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_\infty)$ into the systems in variables $q_1', q_2', p_1', p_2', s_1', s_2'$ with parameters $\alpha' = (\alpha_0', \alpha_1', \alpha_2', \alpha_\infty')$ denoted by $\sigma_m(\alpha)$ where

$$\sigma_1(\alpha) = (\alpha_2, \alpha_1, \alpha_0, \alpha_\infty), \qquad \sigma_2(\alpha) = (\alpha_\infty, \alpha_1, \alpha_2, \alpha_0).$$

2.3 S_2 action on $\mathcal{H}_{113}$

The transformation

$$\sigma \quad \begin{cases} q_1' = \dfrac{q_1}{(s_1 + \frac{1}{2}s_2^2)q_1 + s_2 q_2 + 1}, \\[2ex] q_2' = -\dfrac{s_2 q_1 + q_2}{(s_1 + \frac{1}{2}s_2^2)q_1 + s_2 q_2 + 1}, \\[2ex] p_1' = \left\{\left(s_1 + \dfrac{1}{2}s_2^2\right)q_1 + s_2 q_2 + 1\right\}\left[\left\{\left(s_1 - \dfrac{1}{2}s_2^2\right)q_1 + 1\right\}p_1 + \left\{\left(s_1 - \dfrac{1}{2}s_2^2\right)q_2 - s_2\right\}p_2\right], \\[2ex] p_2' = -\left\{\left(s_1 + \dfrac{1}{2}s_2^2\right)q_1 + s_2 q_2 + 1\right\}\{s_2 q_1 p_1 + (s_2 q_2 + 1)p_2\}, \\[2ex] s_1' = -s_1, \quad s_2' = s_2 \end{cases}$$

changes the system $\mathcal{H}_{113}$ in variables $q_1, q_2, p_1, p_2, s_1, s_2$ with parameters $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_\infty)$ into the system in variables $q_1', q_2', p_1', p_2', s_1', s_2'$ with parameters $\alpha' = (\alpha_0', \alpha_1', \alpha_2', \alpha_\infty')$ denoted by $\sigma(\alpha)$ where

$$\sigma_1(\alpha) = (\alpha_\infty, \alpha_1, \alpha_2, \alpha_0).$$

2.4 S_2 action on $\mathcal{H}_{122}$

The transformation

$$\sigma \quad \begin{cases} q_1' = -\dfrac{q_2}{s_2}, \quad q_2' = \dfrac{q_1}{s_1}, \\[2ex] p_1' = -s_2 p_2, \quad p_2' = s_1 p_1, \\[2ex] s_1' = -s_2, \quad s_2' = -\dfrac{1}{s_1} \end{cases}$$

changes the system $\mathcal{H}_{122}$ in variables $q_1, q_2, p_1, p_2, s_1, s_2$ with parameters $\alpha = (\alpha_0, \alpha_1, \alpha_\infty)$ into the system in variables $q_1', q_2', p_1', p_2', s_1', s_2'$ with parameters $\alpha' = (\alpha_0', \alpha_1', \alpha_\infty')$ denoted by $\sigma(\alpha)$ where

$$\sigma(\alpha) = (\alpha_1, \alpha_0, \alpha_\infty).$$

3 Compactification of the original phase spaces

We construct spaces of initial conditions $E_J(s)$, $s = (s_1, s_2) \in B_J$ for each system $\mathcal{H}_J$ by compactification of the complex spaces $\mathbb{C}^4$ and by successive quadratic transformations. We choose a compact complex manifold $\overline{\Sigma}_\nu$, ν being a complex constant depending on the system $\mathcal{H}_J$, as an compactification of the fibers $\mathbb{C}^4$.

The manifold $\overline{\Sigma}_\nu$ is a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$ defined as flows. Let $\xi := (\xi_0, \xi_1, \xi_2)$ be the homogeneous coordinates of $\mathbb{P}^2$, $U_i := \{(\xi_0/\xi_i, \xi_1/\xi_i, \xi_2/\xi_i) \mid \xi_i \neq 0\} \simeq \mathbb{C}^2$ be the i-th affine chart. Set $W_i := U_i \times \mathbb{P}^2 (i = 0, 1, 2)$ and let $\eta_i := {}^t(\eta_{i0}, \eta_{i1}, \eta_{i2})$ be the homogeneous coordinates of the second component $\mathbb{P}^2$ of W_i. Then we define $\overline{\Sigma}_\nu$ to be the quotient space of $\bigsqcup_{0\leq i\leq 2} W_i$ by the relations

$$\eta_i = g_{i0} \cdot \eta_0,$$

$$g_{10} = \begin{pmatrix} \xi_0^2 & 0 & 0 \\ -\nu\xi_0\xi_1 & -\xi_1^2 & -\xi_1\xi_2 \\ 0 & 0 & \xi_0\xi_1 \end{pmatrix}, \quad g_{20} = \begin{pmatrix} \xi_0^2 & 0 & 0 \\ 0 & \xi_0\xi_2 & 0 \\ -\nu\xi_0\xi_2 & -\xi_1\xi_2 & -\xi_2^2 \end{pmatrix}$$

up to multiplication of nonzero constant. Set

$$W_{ij} := \{(\xi_0/\xi_i, \xi_1/\xi_i, \xi_2/\xi_i, \eta_{i0}/\eta_{ij}, \eta_{i1}/\eta_{ij}, \eta_{i2}/\eta_{ij}) \mid \xi_i, \eta_{ij} \neq 0\} \simeq \mathbb{C}^4, \quad 0 \leqslant i, j \leqslant 2,$$

then we see that $\{W_{ij}\}_{0\leq i,j\leq 2}$ form an atlas consisting of affine charts of the manifold $\overline{\Sigma}_\nu$ and

$$W_i = \bigcup_{0\leqslant j\leqslant 2} W_{ij}.$$

We notice that $\overline{\Sigma}_\nu$ is isomorphic to $T^*\mathbb{P}^2 \sqcup (\mathbb{P}^2 \times \mathbb{P}^1)$ if $\nu = 0$ and to $\mathbb{P}^2 \times \mathbb{P}^2$ if $\nu \neq 0$.

Let us extend the original system $\mathcal{H}_J$ defined on $\mathbb{C}^4 \times B_J \ni (q_1, q_2, p_1, p_2, s_1, s_2)$ to that on $\overline{\Sigma}_\nu \times B_J$ assuming that $(q, p) = (q_1, q_2, p_1, p_2)$ is the coordinate system of W_{00} namely

$$q_1 = \frac{\xi_1}{\xi_0}, \quad q_2 = \frac{\xi_2}{\xi_0}, \quad p_1 = \frac{\eta_{01}}{\eta_{00}}, \quad p_2 = \frac{\eta_{02}}{\eta_{00}}.$$

Denote by $\mathcal{H}_J^{(0)}$ the extended system on $\overline{\Sigma}_\nu \times B_J$. Notice that the patching matrices $g_{i0}, i = 1, 2$ are given by

$$g_{10} = \begin{pmatrix} 1 & 0 & 0 \\ -\nu q_1 & -q_1^2 & -q_1q_2 \\ 0 & 0 & q_1 \end{pmatrix}, \quad g_{20} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q_2 & 0 \\ -\nu q_2 & -q_1q_2 & -q_2^2 \end{pmatrix}$$

up to nonzero constant multiplication.

We see that the transformations from the original chart W_{00} to the charts $W_{i0}, i = 1, 2$ are symplectic. In fact, by setting

$$q_1^1 = \frac{\xi_0}{\xi_1}, \quad q_2^1 = \frac{\xi_2}{\xi_1}, \quad p_1^1 = \frac{\eta_{11}}{\eta_{10}}, \quad p_2^1 = \frac{\eta_{12}}{\eta_{10}},$$
$$q_1^2 = \frac{\xi_1}{\xi_2}, \quad q_2^2 = \frac{\xi_0}{\xi_2}, \quad p_1^2 = \frac{\eta_{21}}{\eta_{20}}, \quad p_2^2 = \frac{\eta_{22}}{\eta_{20}},$$

we have

$$q_1 = \frac{1}{q_1^1}, \quad q_2 = \frac{q_2^1}{q_1^1}, \quad p_1 = -q_1^1(\nu + q_1^1p_1^1 + q_2^1p_2^1), \quad p_2 = q_1^1p_2^1, \tag{3.1}$$

$$q_1 = \frac{q_1^2}{q_2^2}, \quad q_2 = \frac{1}{q_2^2}, \quad p_1 = q_2^2p_1^2, \quad p_2 = -q_2^2(\nu + q_1^2p_1^2 + q_2^2p_2^2), \tag{3.2}$$

which yields

$$dp_1 \wedge dq_1 + dp_2 \wedge dq_2 = dp_1^1 \wedge dq_1^1 + dp_2^1 \wedge dq_2^1 = dp_1^2 \wedge dq_1^2 + dp_2^2 \wedge dq_2^2.$$

Therefore the original Hamiltonian system is written also as a Hamiltonian system on each $W_{i0} \ni (q_1^i, q_2^i, p_1^i, p_2^i),\ i = 1, 2$. Moreover, we can verify that the Hamiltonians become polynomials of the dependent variables $(q^i, p^i) = (q_1^i, q_2^i, p_1^i, p_2^i)$ whose coefficients are rational functions of $s = (s_1, s_2)$ holomorphic in B_J if the values of $\nu = \nu_J$ are chosen as

$$\nu_{11111} = -\frac{1}{2}(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 - 1 + \alpha_\infty), \quad \nu_{1112} = -\frac{1}{2}(\alpha_0 + \alpha_1 + \alpha_2 - 1 + \alpha_\infty),$$

$$\nu_{113} = \nu_{122} = -\frac{1}{2}(\alpha_0 + \alpha_1 - 1 + \alpha_\infty), \quad \nu_{14} = \nu_{23} = -\alpha_\infty, \quad \nu_5 = \alpha + \frac{1}{2}.$$

However, we see that the differential systems on $W_{ij} \times B_J,\ j \neq 0$ are not Hamiltonian systems and have pole singularities on $W_{ij} \backslash W_{i0}$.

Setting

$$W^0 = \bigcup_{i=0}^{2} W_{i0},$$

we state these facts as

Proposition 3.1. *For every J, the extended differential system $\mathcal{H}_J^{(0)}$ on $\overline{\Sigma}_{\nu_J} \times B_J$ is a polynomial type Hamiltonian system on $W^0 \times B_J$ with coefficients holomorphic in $B_J \ni s = (s_1, s_2)$ but it has pole singularities on $D \times B_J$ where*

$$D = \overline{\Sigma}_\nu \backslash W^0.$$

In the following sections, we will find suitable coordinate systems which express the families of solutions passing through D by successive quadratic transformations in all cases of $J = 11111, 1112, 113, 122, 14, 23, 5$.

4 Spaces of initial conditions for $\mathcal{H}_{11111}$

Remind that

$$\nu = -\frac{1}{2}(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 - 1 + \alpha_\infty)$$

in the present case. In this section, we often omitt the label 11111.

4.1 Accessible singularities on $D \times B$

Let us determine the set of accessible singular points of the system $\mathcal{H}^{(0)}$ on $\overline{\Sigma}_\nu \times B\,(B = B_{11111})$. Notice that the system is a holomorphic Hamiltonian system on $W^0 \times B$. By definition, an accessible singular point is a point through which (potentially infinitely many) solutions of the system $\mathcal{H}^{(0)}$ in $W^0 \times B$ passe holomorphically.

Let us investigate the form of the system on $W_{01} \times B (\subset W_0 \times B)$, for example. By setting $\xi_0 = \eta_{01} = 1$, we take $(\xi_1, \xi_2, \eta_{00}, \eta_{02})$ as the coordinates of W_{01}. By the use of them, the system is written as

$$
\begin{aligned}
e(s)\eta_{00}d\xi_1 &= \sum_{i=1,2} P_i(\xi_1,\xi_2,\eta_{00},\eta_{02},s)ds_i,\\
e(s)\eta_{00}d\xi_2 &= \sum_{i=1,2} Q_i(\xi_1,\xi_2,\eta_{00},\eta_{02},s)ds_i,\\
e(s)\eta_{00}d\eta_{00} &= \sum_{i=1,2} X_i(\xi_1,\xi_2,\eta_{00},\eta_{02},s)ds_i,\\
e(s)\eta_{00}d\eta_{02} &= \sum_{i=1,2} Y_i(\xi_1,\xi_2,\eta_{00},\eta_{02},s)ds_i,
\end{aligned}
$$

where $e(s) = s_1s_2(s_1-1)(s_2-1)(s_1-s_2)$ and $P_i, Q_i, X_i, Y_i \in \mathbb{C}[\xi_1,\xi_2,\eta_{00},\eta_{02},s_1,s_2]$ (polynomial ring) are given by

$$
\begin{aligned}
P_1 &= O(\eta_{00}) + 2s_2(s_2-1)\xi_1[\{(s_1-s_2)\xi_1 + s_1(s_2-1)\}\xi_2\eta_{02}\\
&\quad + (\xi_1-1)(\xi_1-s_1)(s_1-s_2) - s_1(s_1-1)\xi_2],\\
P_2 &= O(\eta_{00}) + 2s_1(s_1-1)\xi_1\xi_2[\{(s_1-s_2)\xi_2 - s_2(s_1-1)\}\eta_{02} + (s_1-s_2)\xi_1 + s_1(s_2-1)],\\
Q_1 &= O(\eta_{00}) + 2s_2(s_2-1)\xi_1\xi_2[\{(s_1-s_2)\xi_2 - s_2(s_1-1)\}\eta_{02} + (s_1-s_2)\xi_1 + s_1(s_2-1)],\\
Q_2 &= O(\eta_{00}) + 2s_1(s_1-1)\xi_2[\{(s_2-s_1)\xi_2 + s_2(s_1-1)\}\xi_1\\
&\quad + \{(\xi_2-1)(\xi_2-s_2)(s_2-s_1) - s_2(s_2-1)\xi_1\}\eta_{02}],\\
X_1 &= O(\eta_{00}),\\
X_2 &= O(\eta_{00}),\\
Y_1 &= O(\eta_{00}) + s_2(s_2-1)[\{(s_1-s_2)\xi_2 - s_2(s_1-1)\}\xi_2\eta_{02}^3 + \{2(s_1-s_2)\xi_1\xi_2 + s_2(s_1-1)\xi_1\\
&\quad + 2s_1(s_2-1)\xi_2\}\eta_{02}^2 + \{(s_1-s_2)\xi_1^2 - 2(s_1^2-s_2)\xi_1 - s_1(s_1-1)\xi_2 + s_1(s_1-s_2)\}\eta_{02}\\
&\quad + s_1(s_1-1)\xi_1],\\
Y_2 &= O(\eta_{00}) + s_1(s_1-1)[s_2(s_2-1)\xi_2\eta_{02}^3 - \{s_2(s_2-1)\xi_1 + (s_1-s_2)\xi_2^2 - 2(s_1-s_2^2)\xi_2\\
&\quad + s_2(s_1-s_2)\}\eta_{02}^2 - \{2\xi_1\xi_2(s_1-s_2) - 2\xi_1s_2(s_1-1)\\
&\quad - s_1(s_2-1)\xi_2\}\eta_{02} - (s_1-s_2)\xi_2^2 - s_1(s_2-1)\xi_1],
\end{aligned}
$$

where $O(\eta_{00})$ denotes a polynomial of $\xi_1,\xi_2,\eta_{00},\eta_{02},s_1,s_2$ with a factor η_{00}. Therefore, accessible singular points are the points satisfying the equations

$$\eta_{00} = 0, \quad P_i = Q_i = Y_i = 0, \quad (i = 1,2),$$

We see that the equations have the following three solutions

$$
\begin{gathered}
\eta_{00} = 0, \quad s_1s_2 - s_2\xi_1 - s_1\xi_2 = 0, \quad s_1 - s_2\eta_{02} = 0;\\
\eta_{00} = 0, \quad 1 - \xi_1 - \xi_2 = 0, \quad 1 - \eta_{02} = 0;\\
\eta_{00} = 0, \quad \xi_1 = 0, \quad \eta_{02} = 0.
\end{gathered}
$$

Observing the system $\mathcal{H}^{(0)}$ on all $W_{ij},\ j \neq 0$, we can verify

Proposition 4.1. *The set of accessible singular points of the system $\mathcal{H}^{(0)}_{11111}$ for each $s = (s_1, s_2) \in B_{11111}$ is a disjoint union of five connected components $A_0(s), A_1(s), A_2(s), A_3(s), A_\infty(s) \simeq \mathbb{P}^1$*

given by

$$\begin{aligned}
A_0(s) &= \{(\xi,\eta_0,s)\in W_0\times B|s_1s_2\xi_0-s_2\xi_1-s_1\xi_2=0,\eta_{00}=0,s_1\eta_{01}-s_2\eta_{02}=0\}\\
&\cup\{(\xi,\eta_1,s)\in W_1\times B|s_1s_2\xi_0-s_2\xi_1-s_1\xi_2=0,\eta_{10}=0,\eta_{11}+s_2\eta_{12}=0\}\\
&\cup\{(\xi,\eta_2,s)\in W_2\times B|s_1s_2\xi_0-s_2\xi_1-s_1\xi_2=0,\eta_{20}=0,s_1\eta_{21}+\eta_{22}=0\},\\
A_1(s) &= \{(\xi,\eta_0,s)\in W_0\times B|\xi_0-\xi_1-\xi_2=0,\eta_{00}=0,\eta_{01}-\eta_{02}=0\}\\
&\cup\{(\xi,\eta_1,s)\in W_1\times B|\xi_0-\xi_1-\xi_2=0,\eta_{10}=0,\eta_{11}+\eta_{12}=0\}\\
&\cup\{(\xi,\eta_2,s)\in W_2\times B|\xi_0-\xi_1-\xi_2=0,\eta_{20}=0,\eta_{21}+\eta_{22}=0\},\\
A_2(s) &= \{(\xi,\eta_0,s)\in W_0\times B|\xi_1=\eta_{00}=\eta_{02}=0\}\cup\{(\xi,\eta_2,s)\in W_2\times B|\xi_1=\eta_{20}=\eta_{22}=0\},\\
A_3(s) &= \{(\xi,\eta_0,s)\in W_0\times B|\xi_2=\eta_{00}=\eta_{01}=0\}\cup\{(\xi,\eta_1,s)\in W_1\times B|\xi_2=\eta_{10}=\eta_{11}=0\},\\
A_\infty(s) &= \{(\xi,\eta_1,s)\in W_1\times B|\xi_0=\eta_{10}=\eta_{12}=0\}\cup\{(\xi,\eta_2,s)\in W_2\times B|\xi_0=\eta_{20}=\eta_{21}=0\}.
\end{aligned}$$

We study here some actions of the Bäcklund transformations given in Section 2.

Every $\sigma \in S_5$ changes the system $\mathcal{H}(\alpha)$ in the variables (q,p,s) to the system $\mathcal{H}(\alpha')$ in the variables (q',p',s') with $\alpha'=\sigma(\alpha)$. Noting that

$$\nu'=\sigma(\nu)=-\frac{1}{2}(\alpha'_0+\alpha'_1+\alpha'_2+\alpha'_3-1+\alpha'_\infty)=\nu,$$

construct the manifold $\overline{\Sigma}'_\nu$ for the system $\mathcal{H}(\alpha')$ in the variables (q',p',s') in the same way as $\overline{\Sigma}_\nu$. It is covered by the affine charts $\{W'_{ij}\}_{0\leqslant i,j\leqslant 2}$ where

$$\begin{gathered}
q'_1=\frac{\xi'_1}{\xi'_0},\quad q'_2=\frac{\xi'_2}{\xi'_0},\quad p'_1=\frac{\eta'_{01}}{\eta'_{00}},\quad p'_2=\frac{\eta'_{02}}{\eta'_{00}},\\
W'_{ij}=\{(\xi'_0/\xi'_i,\xi'_1/\xi'_i,\xi'_2/\xi'_i,eta'_{i0}/\eta'_{ij},\eta'_{i1}/\eta'_{ij},\eta'_{i2}/\eta'_{ij})\mid \xi'_i,\eta'_{ij}\neq 0\}\simeq\mathbb{C}^4,\\
{}^t(\eta'_{i0},\eta'_{i1},\eta'_{i2})=g'_{i0}\cdot {}^t(\eta'_{00},\eta'_{01},\eta'_{02}),\quad i=1,2.
\end{gathered}$$

Here g'_{i0}, $i=1,2$ are matrices of the same form as g_{i0}, $i=1,2$, q_1,q_2 being replaced by q'_1,q'_2.

Then we can verify that the transformation σ defines a biholomorphic mapping from $\overline{\Sigma}_\nu$ to $\overline{\Sigma}'_\nu$, which is also denoted by σ. Moreover, let $A'_i(s')$, $i=0,1,2,3,\infty$ be the components of the accessible singular points of the system $\mathcal{H}^{(0)'}(\alpha')$ on $\overline{\Sigma}'_\nu$, then $\sigma(A_i(s))=A'_{\sigma(i)}(s')$ for every i as sets, namely not pointwise. For simplicity, we say that the Bäcklund transformation σ act on the componets $\{A_i(s)\}$ as a permutation. The following proposition holds.

Proposition 4.2. *The Bäcklund transformation group S_5 acts on the components of the accessible singular points of $\mathcal{H}_{11111}$ according to the following diagram*

	A_0	A_1	A_2	A_3	A_∞
σ_1	A_1	A_0	*	*	*
σ_2	A_2	*	A_0	*	*
σ_3	A_3	*	*	A_0	*
σ_4	*	A_∞	*	*	A_1

Here the diagram should be read as σ_1 transposes $A_0(s)$ and $A_1(s)$, more precisely $\sigma_1(A_0(s))=A'_1(s')$, $\sigma_1(A_1(s))=A'_0(s')$, $\sigma_1(A_i(s))=A'_i(s'), i=2,3,\infty$, and so on.

In the following subsections, we obtain coordinate systems corresponding to $A_i(s)$, $s=(s_1,s_2)\in B=B_{11111}$ which separate completely the solution curves passing through $A_i(s)$. The systems for

$A_2(s)$ and $A_3(s)$ are obtained by quadratic transformations, while the other systems are obtained from that for $A_3(s)$ by the use of Bäcklund transformations given in section 2. Notice that we can obtain coordinate systems for $A_3(s)$(or $A_2(s)$) from those for $A_2(s)$(or $A_3(s)$) and Bäcklund transformations. But the coordinate system obtained by the procedure is not good, namely the form of the relation between the coordinates and the original ones is not good. Therefore we perform the quadratic transformation to obtain good coordinate system for $A_2(s)$ and $A_3(s)$.

The quadratic transformation along a set A is denoted by Q_A, and the superscript (k) of a letter indicates that it is concerned with a k-th quadratic transformation.

4.2 Coordinate systems for $A_2(s)$

We make successively the quadratic transformations along $A_2(s) \cap W_0$, $A_2(s) \cap W_2$ and find coordinate systems for $A_2(s)$.

4.2.1 Coordinate system for $A_2(s) \cap W_0$

The first quadratic transformation along $A_2(s) \cap W_0$. Note that $A_2(s) \cap W_0 \subset W_{01}$ and

$$A_2(s) \cap W_0 = \{(\xi_1, \xi_2, \eta_{00}, \eta_{02}) \in W_{01} \simeq \mathbb{C}^4 \mid \xi_2 \in \mathbb{C}, \xi_1 = \eta_{00} = \eta_{02} = 0\}.$$

We replace every point $(\xi_1, \xi_2, \eta_{00}, \eta_{02}) = (0, \xi_2, 0, 0)$ with $\xi_2 \in \mathbb{C}$ by $\mathbb{P}^2$ simultaneously. Let $(\xi_2, x_{20}^{(1)}, y_{20}^{(1)}, z_{20}^{(1)}) \in \mathbb{C}^4$, $(\xi_2, x_{21}^{(1)}, y_{21}^{(1)}, z_{21}^{(1)}) \in \mathbb{C}^4$ and $(\xi_2, x_{22}^{(1)}, y_{22}^{(1)}, z_{22}^{(1)}) \in \mathbb{C}^4$ be coordinate systems of $V_{21}^{(1)}(s) = Q_{A_2(s)\cap W_0}(W_{01} \times s)$ defined by

$$\begin{aligned}
\xi_1 &= x_{20}^{(1)}, \quad \eta_{00} = x_{20}^{(1)} y_{20}^{(1)}, \quad \eta_{02} = x_{20}^{(1)} z_{20}^{(1)}, \\
\xi_1 &= x_{21}^{(1)} y_{21}^{(1)}, \quad \eta_{00} = y_{21}^{(1)}, \quad \eta_{02} = y_{21}^{(1)} z_{21}^{(1)}, \\
\xi_1 &= x_{22}^{(1)} z_{22}^{(1)}, \quad \eta_{00} = y_{22}^{(1)} z_{22}^{(1)}, \quad \eta_{02} = z_{22}^{(1)},
\end{aligned}$$

then the exceptional divisor $D_{21}^{(1)}(s) = Q_{A_2(s)\cap W_0}(A_2(s) \cap W_0)$ is given by

$$\{(\xi_2, x_{20}^{(1)}, y_{20}^{(1)}, z_{20}^{(1)}) \mid x_{20}^{(1)} = 0\} \cup \{(\xi_2, x_{21}^{(1)}, y_{21}^{(1)}, z_{21}^{(1)}) \mid y_{21}^{(1)} = 0\} \cup \{(\xi_2, x_{22}^{(1)}, y_{22}^{(1)}, z_{22}^{(1)}) \mid z_{22}^{(1)} = 0\}.$$

Let us write our system in the three coordinate systems near the exeptional divisor. In the first coordinate, it is written as

$$\frac{\partial \xi_2}{\partial s_1} = \frac{1}{s_1(s_1-1)(s_1-s_2)}\Big(\frac{O(x_{20}^{(1)})}{x_{20}^{(1)}} + \frac{2s_1(s_2-1)\xi_2}{y_{20}^{(1)}}\Big),$$

$$\frac{\partial \xi_2}{\partial s_2} = \frac{1}{s_2(s_2-1)(s_1-s_2)}\Big(\frac{O(x_{20}^{(1)})}{x_{20}^{(1)}} + \frac{P_1(\xi_2, z_{20}^{(1)})}{y_{20}^{(1)}}\Big),$$

$$\frac{\partial x_{20}^{(1)}}{\partial s_1} = \frac{1}{s_1(s_1-1)(s_1-s_2)}\Big(\frac{O(x_{20}^{(1)})}{x_{20}^{(1)}} + \frac{2s_1(\xi_2 - s_1^2 + s_1 - s_2)}{y_{20}^{(1)}}\Big),$$

$$\frac{\partial x_{20}^{(1)}}{\partial s_2} = \frac{1}{s_2(s_2-1)(s_1-s_2)}\Big(\frac{O(x_{20}^{(1)})}{x_{20}^{(1)}} + \frac{2s_1(s_2-1)\xi_2}{y_{20}^{(1)}}\Big),$$

$$\frac{\partial y_{20}^{(1)}}{\partial s_1}=\frac{O(x_{20}^{(1)})+P_2(\xi_2)(\alpha_2 y_{20}^{(1)}-1)}{s_1(s_1-1)(s_1-s_2)x_{20}^{(1)}},\quad \frac{\partial y_{20}^{(1)}}{\partial s_2}=\frac{O(x_{20}^{(1)})+s_1\xi_2(\alpha_2 y_{20}^{(1)}-1)}{s_2(s_1-s_2)x_{20}^{(1)}},$$

$$\frac{\partial z_{20}^{(1)}}{\partial s_1}=\frac{1}{s_1(s_1-1)(s_1-s_2)}\Big(\frac{O(x_{20}^{(1)})+P_3(\xi_2,z_{20}^{(1)})(\alpha_2-1/y_{20}^{(1)})}{x_{20}^{(1)}}+\frac{P_4(\xi_2,z_{20}^{(1)})}{y_{20}^{(1)}}\Big),$$

$$\frac{\partial z_{20}^{(1)}}{\partial s_2}=\frac{1}{s_2(s_2-1)(s_1-s_2)}\Big(\frac{O(x_{20}^{(1)})+P_5(\xi_2,z_{20}^{(1)})(\alpha_2-1/y_{20}^{(1)})}{x_{20}^{(1)}}+\frac{P_6(\xi_2,z_{20}^{(1)})}{y_{20}^{(1)}}\Big)$$

in a neighborhood of $D_{21}^{(1)}(s)=\{x_{20}^{(1)}=0\}$, in the second coordinate system, it is written as

$$\frac{\partial \xi_2}{\partial s_1}=\frac{O(y_{21}^{(1)})}{s_1(s_1-1)(s_1-s_2)y_{21}^{(1)}},\quad \frac{\partial \xi_2}{\partial s_2}=\frac{O(y_{21}^{(1)})}{s_2(s_2-1)(s_1-s_2)y_{21}^{(1)}},$$

$$\frac{\partial y_{21}^{(1)}}{\partial s_1}=\frac{O(y_{21}^{(1)})}{s_1(s_1-1)(s_1-s_2)y_{21}^{(1)}},\quad \frac{\partial y_{21}^{(1)}}{\partial s_2}=\frac{O(y_{21}^{(1)})}{s_2(s_2-1)(s_1-s_2)y_{21}^{(1)}},$$

$$\frac{\partial x_{21}^{(1)}}{\partial s_1}=\frac{O(y_{21}^{(1)})+\{(s_1-1)\xi_2-s_1+s_2\}(\alpha_2-x_{21}^{(1)})}{s_1(s_1-1)(s_1-s_2)y_{21}^{(1)}},$$

$$\frac{\partial x_{21}^{(1)}}{\partial s_2}=\frac{O(y_{21}^{(1)})+s_1(s_2-1)\xi_2(x_{21}^{(1)}-\alpha_2)}{s_2(s_2-1)(s_1-s_2)y_{21}^{(1)}},$$

$$\frac{\partial z_{21}^{(1)}}{\partial s_1}=\frac{O(y_{21}^{(1)})+s_1(s_1-1)(x_{21}^{(1)}-\alpha_2)}{s_1(s_1-1)(s_1-s_2)y_{21}^{(1)}},\quad \frac{\partial z_{21}^{(1)}}{\partial s_2}=\frac{O(y_{21}^{(1)})+s_1(s_2-1)(x_{21}^{(1)}-\alpha_2)}{s_2(s_2-1)(s_1-s_2)y_{21}^{(1)}}$$

in a neighborhood of $D_{21}^{(1)}(s)=\{y_{21}^{(1)}=0\}$, and in the third coordinate system it is written as

$$\frac{\partial \xi_2}{\partial s_1}=\frac{1}{s_1(s_1-1)(s_1-s_2)}\Big(\frac{O(z_{22}^{(1)})}{z_{22}^{(1)}}+\frac{2s_1(s_2-1)\xi_2 x_{22}^{(1)}}{y_{22}^{(1)}}\Big),$$

$$\frac{\partial \xi_2}{\partial s_2}=\frac{1}{s_2(s_2-1)(s_1-s_2)}\Big(\frac{O(z_{22}^{(1)})}{z_{22}^{(1)}}+\frac{P_7(\xi_2,x_{22}^{(1)})}{y_{22}^{(1)}}\Big),$$

$$\frac{\partial z_{22}^{(1)}}{\partial s_1}=\frac{1}{s_1(s_1-1)}\Big(\frac{O(z_{22}^{(1)})}{(s_1-s_2)z_{22}^{(1)}}+\frac{s_1(s_1-1)(x_{22}^{(1)}-\xi_2)-s_1s_2}{y_{22}^{(1)}}\Big),$$

$$\frac{\partial z_{22}^{(1)}}{\partial s_2}=\frac{1}{s_2(s_2-1)(s_1-s_2)}\Big(\frac{O(z_{22}^{(1)})}{z_{22}^{(1)}}+\frac{s_1(s_2-1)(s_1-s_2)(\xi_2-x_{22}^{(1)})}{y_{22}^{(1)}}\Big),$$

$$\frac{\partial x_{22}^{(1)}}{\partial s_1}=\frac{1}{s_1(s_1-1)(s_1-s_2)}\Big(\frac{O(z_{22}^{(1)})+P_8(\xi_2,x_{22}^{(1)})(x_{22}^{(1)}/y_{22}^{(1)}-\alpha_2)}{z_{22}^{(1)}}\Big),$$

$$\frac{\partial x_{22}^{(1)}}{\partial s_2}=\frac{1}{s_2(s_2-1)(s_1-s_2)}\Big(\frac{O(z_{22}^{(1)})+P_9(\xi_2,x_{22}^{(1)})(x_{22}^{(1)}/y_{22}^{(1)}-\alpha_2)}{z_{22}^{(1)}}\Big),$$

$$\frac{\partial y_{22}^{(1)}}{\partial s_1}=\frac{O(z_{22}^{(1)})+s_1(s_1-1)(\alpha_2 y_{22}^{(1)}-x_{22}^{(1)})}{s_1(s_1-1)(s_1-s_2)z_{22}^{(1)}},\quad \frac{\partial y_{22}^{(1)}}{\partial s_2}=\frac{O(z_{22}^{(1)})+s_1(s_2-1)(x_{22}^{(1)}-\alpha_2 y_{22}^{(1)})}{s_2(s_2-1)(s_1-s_2)z_{22}^{(1)}}$$

in a neighborhood of $D_{21}^{(1)}(s)=\{z_{22}^{(1)}=0\}$. Here $P_i(*)$ denotes a polynomial of $*$. In the following, $P_i(*)$ always denotes a polynomial of some variables $*$. Investigating carefully these systems in a neighborhood of $D_{21}^{(1)}(s)$ in the same way of deriving Proposition 4.1, we can verity that the set of accessible singular points $A_{21}^{(1)}(s)$ is given by

$$A_{21}^{(1)}(s)=\{(\xi_2,x_{21}^{(1)},y_{21}^{(1)},z_{21}^{(1)})=(\xi_2,\alpha_2,0,z_{21}^{(1)})\}\subset D_{21}^{(1)}(s),$$

which means that $A_{21}^{(1)}(s)$ is included in a coordinate neighborhood of the coordinates $(\xi_2,x_{21}^{(1)},y_{21}^{(1)},z_{21}^{(1)})$.

The second quadratic transformation along $A_{21}^{(1)}(s)$. We next replace the points $(\xi_2,x_{21}^{(1)},y_{21}^{(1)},z_{21}^{(1)})=(\xi_2,\alpha_2,0,z_{21}^{(1)})$ with $(\xi_2,z_{21}^{(1)})\in\mathbb{C}^2$ by $\mathbb{P}^1$ simultaneously. Let $(\xi_2,z_{21}^{(1)},x_{20}^{(2)},y_{20}^{(2)})\in\mathbb{C}^4$ and $(\xi_2,z_{21}^{(1)},x_{21}^{(2)},y_{21}^{(2)})\in\mathbb{C}^4$ be coordinate systems of $V_{21}^{(2)}(s)=Q_{A_{21}^{(1)}(s)}(V_{21}^{(1)}(s))$ defined by

$$x_{21}^{(1)}=\alpha_2+x_{20}^{(2)},\quad y_{21}^{(1)}=x_{20}^{(2)}y_{20}^{(2)},$$
$$x_{21}^{(1)}=\alpha_2+x_{21}^{(2)}y_{21}^{(2)},\quad y_{21}^{(1)}=y_{21}^{(2)},$$

then the exceptional divisor $D_{21}^{(2)}(s)=Q_{A_{21}^{(1)}(s)}(A_{21}^{(1)}(s))$ is given by

$$\{(\xi_2,z_{21}^{(1)},x_{20}^{(2)},y_{20}^{(2)})\mid x_{20}^{(2)}=0\}\cup\{(\xi_2,z_{21}^{(1)},x_{21}^{(2)},y_{22}^{(2)})\mid y_{21}^{(2)}=0\}.$$

We can verify that our system is written in the second coordinate system as

$$s_1s_2(s_1-1)(s_2-1)(s_1-s_2)d\xi_2-\sum_{i=1,2}P_{2i}(\xi_2,z_{21}^{(1)},x_{21}^{(2)},y_{21}^{(2)},s)ds_i=0,$$
$$s_1s_2(s_1-1)(s_2-1)(s_1-s_2)dz_{21}^{(1)}-\sum_{i=1,2}Q_{2i}(\xi_2,z_{21}^{(1)},x_{21}^{(2)},y_{21}^{(2)},s)ds_i=0,$$
$$s_1s_2(s_1-1)(s_2-1)(s_1-s_2)dx_{21}^{(2)}-\sum_{i=1,2}X_{2i}(\xi_2,z_{21}^{(1)},x_{21}^{(2)},y_{21}^{(2)},s)ds_i=0,$$
$$s_1s_2(s_1-1)(s_2-1)(s_1-s_2)dy_{21}^{(2)}-\sum_{i=1,2}Y_{2i}(\xi_2,z_{21}^{(1)},x_{21}^{(2)},y_{21}^{(2)},s)ds_i=0.$$

Here $P_{2i},Q_{2i},X_{2i},Y_{2i}$ in the second differential system are certain polynomials of $\xi_2,z_{21}^{(1)},x_{21}^{(2)},y_{21}^{(2)}$ and s. This means that the foliation has no singular points in $(\xi_2,z_{21}^{(1)},x_{21}^{(2)},y_{21}^{(2)},s)$-space $\mathbb{C}^4\times B$ and every leaf in the space is transversal with fibers. On the other hand, we can verify that the points $(\xi_2, z_{21}^{(1)}, x_{20}^{(2)}, y_{20}^{(2)}) = (\xi_2, z_{21}^{(1)}, 0, 0)$ are inaccessible singular points, because our system is written as

$$\frac{\partial \xi_2}{\partial s_1} = \frac{O(x_{20}^{(2)})}{s_1(s_1-1)(s_1-s_2)x_{20}^{(2)}}, \quad \frac{\partial \xi_2}{\partial s_2} = \frac{O(x_{20}^{(2)})}{s_2(s_2-1)(s_1-s_2)x_{20}^{(2)}},$$

$$\frac{\partial z_{21}^{(1)}}{\partial s_1} = \frac{1}{s_1(s_1-1)(s_1-s_2)}\Big(\frac{O(x_{20}^{(2)})}{x_{20}^{(2)}} + \frac{s_1(s_1-1)}{y_{20}^{(2)}}\Big),$$

$$\frac{\partial z_{21}^{(1)}}{\partial s_2} = \frac{1}{s_2(s_2-1)(s_1-s_2)}\Big(\frac{O(x_{20}^{(2)})}{x_{20}^{(2)}} - \frac{s_1(s_2-1)}{y_{20}^{(2)}}\Big)$$

$$\frac{\partial x_{20}^{(2)}}{\partial s_1} = \frac{1}{s_1(s_1-1)(s_1-s_2)}\Big(\frac{O(x_{20}^{(2)})}{x_{20}^{(2)}} + \frac{s_1\{(s_1-1)\xi_2 - s_1 + s_2\}}{y_{20}^{(2)}}\Big),$$

$$\frac{\partial x_{20}^{(2)}}{\partial s_2} = \frac{1}{s_2(s_1-s_2)}\Big(\frac{O(x_{20}^{(2)})}{x_{20}^{(2)}} + \frac{s_1\xi_2}{y_{20}^{(2)}}\Big),$$

$$\frac{\partial y_{20}^{(2)}}{\partial s_1} = \frac{O(x_{20}^{(2)})}{s_1(s_1-1)(s_1-s_2)x_{20}^{(2)}}, \quad \frac{\partial y_{20}^{(2)}}{\partial s_2} = \frac{O(x_{20}^{(2)})}{s_2(s_2-1)(s_1-s_2)x_{20}^{(2)}}$$

in a neighborhood of $(\xi_2, z_{21}^{(1)}, x_{20}^{(2)}, y_{20}^{(2)}) = (\xi_2, z_{21}^{(1)}, 0, 0)$.

Thus we have obtained a coordinate system $(\xi_2, z_{21}^{(1)}, x_{21}^{(2)}, y_{21}^{(2)}) \in \mathbb{C}^4$ which separates the solutions passing through $A_2(s) \cap W_0 = A_2(s) \cap W_{01}$. It is related to the coordinate system $(\xi_1, \xi_2, \eta_{00}, \eta_{02}) \in W_{01}$ by

$$\xi_1 = y_{21}^{(2)}(\alpha_2 + x_{21}^{(2)} y_{21}^{(2)}), \quad \xi_2 = \xi_2, \quad \eta_{00} = y_{21}^{(2)}, \quad \eta_{02} = y_{21}^{(2)} z_{21}^{(1)}$$

and then

$$q_1 = y_{21}^{(2)}(\alpha_2 + x_{21}^{(2)} y_{21}^{(2)}), \quad q_2 = \xi_2, \quad p_1 = \frac{1}{y_{21}^{(2)}}, \quad p_2 = z_{21}^{(1)}.$$

If we set

$$q_1^{21} = -x_{21}^{(2)}, \quad q_2^{21} = \xi_2, \quad p_1^{21} = y_{21}^{(2)}, \quad p_2^{21} = z_{21}^{(1)},$$

then we have

$$q_1 = p_2^{21}(\alpha_2 - q_1^{21} p_1^{21}), \quad q_2 = q_2^{21}, \quad p_1 = \frac{1}{p_1^{21}}, \quad p_2 = p_2^{21} \tag{4.1}$$

and

$$dq_1 \wedge dp_1 + dq_2 \wedge dp_1 = dq_1^{21} \wedge dp_1^{21} + dq_2^{21} \wedge dp_2^{21}.$$

We should notice that the transformation from (q, p) to (q^{21}, p^{21}) is symplectic and then our system $\mathcal{H}_{11111}$ is also written as an Hamiltonian system in the variables (q^{21}, p^{21}). In our teminology, (q^{21}, p^{21}) is a symplectic coordinate system. We can verify that the Hamiltonians in (q^{21}, p^{21}) are polynomials of the variables whose coefficients are rational functions of $s = (s_1, s_2)$ holomorphic in B.

4.2.2 Coordinate system for $A_2(s) \cap W_2$

The first quadratic transformation along $A_2(s) \cap W_2$. Note that $A_2(s) \cap W_2 \subset W_{21}$ and

$$A_2(s) \cap W_2 = \{(\xi_0, \xi_1, \eta_{20}, \eta_{22}) \in W_{21} \simeq \mathbb{C}^4 \mid \xi_0 \in \mathbb{C}, \xi_1 = \eta_{20} = \eta_{22} = 0\}.$$

We replace every point $(\xi_0, \xi_1, \eta_{20}, \eta_{22}) = (\xi_0, 0, 0, 0)$ with $\xi_0 \in \mathbb{C}$ by $\mathbb{P}^2$ simultaneously. Let $(\xi_0, X_{20}^{(1)}, Y_{20}^{(1)}, Z_{20}^{(1)}) \in \mathbb{C}^4$, $(\xi_0, X_{21}^{(1)}, Y_{21}^{(1)}, Z_{21}^{(1)}) \in \mathbb{C}^4$ and $(\xi_0, X_{22}^{(1)}, Y_{22}^{(1)}, Z_{22}^{(1)}) \in \mathbb{C}^4$ be coordinate systems of $V_{22}^{(1)}(s) = Q_{A_2(s) \cap W_2}(W_{21} \times s)$ defined by

$$\begin{aligned}
&\xi_1 = X_{20}^{(1)}, \quad \eta_{20} = X_{20}^{(1)} Y_{20}^{(1)}, \quad \eta_{22} = X_{20}^{(1)} Z_{20}^{(1)}, \\
&\xi_1 = X_{21}^{(1)} Y_{21}^{(1)}, \quad \eta_{20} = Y_{21}^{(1)}, \quad \eta_{22} = Y_{21}^{(1)} Z_{21}^{(1)}, \\
&\xi_1 = X_{22}^{(1)} Z_{22}^{(1)}, \quad \eta_{20} = Y_{22}^{(1)} Z_{22}^{(1)}, \quad \eta_{22} = Z_{22}^{(1)},
\end{aligned}$$

then the exceptional divisor $D_{22}^{(1)}(s) = Q_{A_2(s) \cap W_2}(A_2(s) \cap W_2)$ is given by

$$\{X_{20}^{(1)} = 0\} \cup \{Y_{21}^{(1)} = 0\} \cup \{Z_{22}^{(1)} = 0\}.$$

We can verity that the set of accessible singular points $A_{22}^{(1)}(s)$ is given by

$$A_{22}^{(1)}(s) = \{(\xi_1, X_{21}^{(1)}, Y_{21}^{(1)}, Z_{21}^{(1)}) = (\xi_1, \alpha_2, 0, Z_{21}^{(1)})\} \subset D_{22}^{(1)}(s).$$

The second quadratic transformation along $A_{22}^{(1)}(s)$. We next replace the points $(\xi_1, X_{21}^{(1)}, Y_{21}^{(1)}, Z_{21}^{(1)}) = (\xi_1, \alpha_2, 0, Z_{21}^{(1)})$ with $(\xi_1, Z_{21}^{(1)}) \in \mathbb{C}^2$ by $\mathbb{P}^1$ simultaneously. Let $(\xi_1, Z_{21}^{(1)}, X_{20}^{(2)}, Y_{20}^{(2)}) \in \mathbb{C}^4$ and $(\xi_1, Z_{21}^{(1)}, X_{21}^{(2)}, Y_{21}^{(2)}) \in \mathbb{C}^4$ be coordinate systems of $V_{22}^{(2)}(s) = Q_{A_{22}^{(1)}(s)}(V_{22}^{(1)}(s))$ defined by

$$\begin{aligned}
&X_{21}^{(1)} = \alpha_2 + X_{20}^{(2)}, \quad Y_{21}^{(1)} = X_{20}^{(2)} Y_{20}^{(2)}, \\
&X_{21}^{(1)} = \alpha_2 + X_{21}^{(2)} Y_{21}^{(2)}, \quad Y_{21}^{(1)} = Y_{21}^{(2)},
\end{aligned}$$

then the exceptional divisor $D_{22}^{(2)}(s) = Q_{A_{22}^{(1)}(s)}(A_{22}^{(1)}(s))$ is given by

$$\{X_{20}^{(2)} = 0\} \cup \{Y_{21}^{(2)} = 0\}.$$

We see that, in the $(\xi_0, Z_{21}^{(1)}, X_{21}^{(2)}, Y_{21}^{(2)}, s)$-space $\mathbb{C}^4 \times B$, the Pfaffin system has no singular points and every leaf is transversal with fibers, moreover, the points $(\xi_0, Z_{21}^{(1)}, X_{20}^{(2)}, Y_{20}^{(2)}) = (\xi_0, Z_{21}^{(1)}, 0, 0)$ are inaccessible singular points.

Thus we have obtained a coordinate system $(\xi_1, Z_{21}^{(1)}, X_{21}^{(2)}, Y_{21}^{(2)}) \in \mathbb{C}^4$ which separates the solutions passing through $A_2(s) \cap W_2 = A_2(s) \cap W_{21}$. We see that the system is related to $(\xi_0, \xi_1, \eta_{20}, \eta_{22}) \in W_{21}$ by

$$\xi_0 = \xi_0, \quad \xi_1 = Y_{21}^{(2)}(\alpha_2 + X_{21}^{(2)} Y_{21}^{(2)}), \quad \eta_{20} = Y_{21}^{(2)}, \quad \eta_{22} = Y_{21}^{(2)} Z_{21}^{(1)}$$

and then we have

$$q_1 = \frac{Y_{21}^{(2)}}{\xi_0}(\alpha_2 + X_{21}^{(2)} Y_{21}^{(2)}), \quad q_2 = \frac{1}{\xi_0}, \quad p_1 = \frac{\xi_0}{Y_{21}^{(2)}}, \quad p_2 = -\xi_0(\nu + \alpha_2 + \xi_0 Z_{21}^{(1)} + X_{21}^{(2)} Y_{21}^{(2)}),$$

which is equivalent to

$$q_1^2 = Y_{21}^{(2)}(\alpha_2 + X_{21}^{(2)} Y_{21}^{(2)}), \quad q_2^2 = \xi_0, \quad p_1^2 = \frac{1}{Y_{21}^{(2)}}, \quad p_2^2 = Z_{21}^{(1)}$$

by means of (3.2). If we set

$$q_1^{22} = -X_{21}^{(2)}, \quad q_2^{22} = \xi_0, \quad p_1^{22} = Y_{21}^{(2)}, \quad p_2^{22} = Z_{21}^{(1)},$$

then we have

$$q_1^2 = p_2^{22}(\alpha_2 - q_1^{22}p_1^{22}), \quad q_2^2 = q_2^{22}, \quad p_1^2 = \frac{1}{p_1^{22}}, \quad p_2^2 = p_2^{22} \tag{4.2}$$

and

$$dq_1 \wedge dp_1 + dq_2 \wedge dp_1 = dq_1^2 \wedge dp_1^2 + dq_2^2 \wedge dp_1^2 = dq_1^{22} \wedge dp_1^{22} + dq_2^{22} \wedge dp_2^{22}.$$

Thus we have obtained a symplectic coordinate system (q^{22}, p^{22}) for $A_2(s) \cap W_2$. We can verify that the Hamiltonians with respect to the coordinates are polynomials of these variables.

4.3 Coordinate systems for $A_3(s)$

We make successively the quadratic transformations along $A_3(s) \cap W_0$, $A_3(s) \cap W_1$ and find coordinate systems for $A_3(s)$.

4.3.1 Coordinate system for $A_3(s) \cap W_0$

The first quadratic transformation along $A_3(s) \cap W_0$. Note that $A_3(s) \cap W_0 \subset W_{02}$ and

$$A_3(s) \cap W_0 = \{(\xi_1, \xi_2, \eta_{00}, \eta_{01}) \in W_{02} \simeq \mathbb{C}^4 \mid \xi_1 \in \mathbb{C}, \xi_2 = \eta_{00} = \eta_{01} = 0\}.$$

We replace every point $(\xi_1, \xi_2, \eta_{00}, \eta_{01}) = (\xi_1, 0, 0, 0)$ with $\xi_1 \in \mathbb{C}$ by $\mathbb{P}^2$ simultaneously. Let $(\xi_1, x_{30}^{(1)}, y_{30}^{(1)}, z_{30}^{(1)}) \in \mathbb{C}^4$, $(\xi_1, x_{31}^{(1)}, y_{31}^{(1)}, z_{31}^{(1)}) \in \mathbb{C}^4$ and $(\xi_1, x_{32}^{(1)}, y_{32}^{(1)}, z_{32}^{(1)}) \in \mathbb{C}^4$ be coordinate systems of $V_{31}^{(1)}(s) = Q_{A_3(s) \cap W_0}(W_{02} \times s)$ defined by

$$\begin{aligned}
\xi_2 &= x_{30}^{(1)}, \quad \eta_{00} = x_{30}^{(1)} y_{30}^{(1)}, \quad \eta_{01} = x_{30}^{(1)} z_{30}^{(1)}, \\
\xi_2 &= x_{31}^{(1)} y_{31}^{(1)}, \quad \eta_{00} = y_{31}^{(1)}, \quad \eta_{01} = y_{31}^{(1)} z_{31}^{(1)}, \\
\xi_2 &= x_{32}^{(1)} z_{32}^{(1)}, \quad \eta_{00} = y_{32}^{(1)} z_{32}^{(1)}, \quad \eta_{01} = z_{32}^{(1)},
\end{aligned}$$

then the exceptional divisor $D_{31}^{(1)}(s) = Q_{A_3(s) \cap W_0}(A_3(s) \cap W_0)$ is given by

$$\{x_{30}^{(1)} = 0\} \cup \{y_{31}^{(1)} = 0\} \cup \{z_{32}^{(1)} = 0\}.$$

We can verity that the set of accessible singular points $A_{31}^{(1)}$ is given by

$$A_{31}^{(1)}(s) = \{(\xi_1, x_{31}^{(1)}, y_{31}^{(1)}, z_{31}^{(1)}) = (\xi_1, \alpha_3, 0, z_{31}^{(1)})\} \subset D_{31}^{(1)}(s).$$

The second quadratic transformation along $A_{31}^{(1)}(s)$. We next replace the points $(\xi_1, x_{31}^{(1)}, y_{31}^{(1)}, z_{31}^{(1)}) = (\xi_1, \alpha_3, 0, z_{31}^{(1)})$ with $(\xi_1, z_{31}^{(1)}) \in \mathbb{C}^2$ by $\mathbb{P}^1$ simultaneously. Let $(\xi_1, z_{31}^{(1)}, x_{30}^{(2)}, y_{30}^{(2)}) \in \mathbb{C}^4$ and $(\xi_1, z_{31}^{(1)}, x_{31}^{(2)}, y_{31}^{(2)}) \in \mathbb{C}^4$ be coordinate systems of $V_{31}^{(2)}(s) = Q_{A_{31}^{(1)}(s)}(V_{31}^{(1)}(s))$ defined by

$$\begin{aligned}
x_{31}^{(1)} &= \alpha_3 + x_{30}^{(2)}, \quad y_{31}^{(1)} = x_{30}^{(2)} y_{30}^{(2)}, \\
x_{31}^{(1)} &= \alpha_3 + x_{31}^{(2)} y_{31}^{(2)}, \quad y_{31}^{(1)} = y_{31}^{(2)},
\end{aligned}$$

then the exceptional divisor $D_{31}^{(2)}(s) = Q_{A_{31}^{(1)}(s)}(A_{31}^{(1)}(s))$ is given by

$$\{x_{30}^{(2)} = 0\} \cup \{y_{31}^{(2)} = 0\}.$$

We can verify that our system has no singular points, every leaf is transversal with fibers in $(\xi_1, z_{31}^{(1)}, x_{31}^{(2)}, y_{31}^{(2)}, s)$-space $\mathbb{C}^4 \times B$. On the other hand, the points $(\xi_1, z_{31}^{(1)}, x_{30}^{(2)}, y_{30}^{(2)}) = (\xi_1, z_{31}^{(1)}, 0, 0)$ are inaccessible singular points,

Thus we have obtained a coordinate system $(\xi_1, z_{31}^{(1)}, x_{31}^{(2)}, y_{31}^{(2)}) \in \mathbb{C}^4$ which separates the solutions passing through $A_3(s) \cap W_0 = A_3(s) \cap W_{02}$. It is related to the coordinate system $(\xi_1, \xi_2, \eta_{00}, \eta_{01}) \in W_{02}$ by

$$\xi_1 = \xi_1, \quad \xi_2 = y_{31}^{(2)}(\alpha_3 + x_{31}^{(2)} y_{31}^{(2)}), \quad \eta_{00} = y_{31}^{(2)}, \quad \eta_{01} = y_{31}^{(2)} z_{31}^{(1)}$$

and then

$$q_1 = \xi_1, \quad q_2 = y_{31}^{(2)}(\alpha_3 + x_{31}^{(2)} y_{31}^{(2)}), \quad p_1 = z_{31}^{(1)}, \quad p_2 = \frac{1}{y_{31}^{(2)}}.$$

If we set

$$q_1^{31} = \xi_1, \quad q_2^{31} = -x_{31}^{(2)}, \quad p_1^{31} = z_{31}^{(1)}, \quad p_2^{31} = y_{31}^{(2)},$$

then we have

$$q_1 = q_1^{31}, \quad q_2 = p_2^{31}(\alpha_3 - q_2^{31} p_2^{31}), \quad p_1 = p_1^{31}, \quad p_2 = \frac{1}{p_2^{31}}. \tag{4.3}$$

Thus we have obtained a symplectic coordinate system $(q^{31}, p^{31}) \in \mathbb{C}^4$ for $A_3(s) \cap W_0$.

4.3.2 Coordinate system for $A_3(s) \cap W_1$

The first quadratic transformation along $A_3(s) \cap W_1$. Note that $A_3(s) \cap W_1 \subset W_{12}$ and

$$A_3(s) \cap W_1 = \{(\xi_0, \xi_2, \eta_{10}, \eta_{11}) \in W_{12} \simeq \mathbb{C}^4 \mid \xi_0 \in \mathbb{C}, \xi_2 = \eta_{10} = \eta_{11} = 0\}.$$

We replace every point $(\xi_0, \xi_2, \eta_{10}, \eta_{11}) = (\xi_0, 0, 0, 0)$ with $\xi_1 \in \mathbb{C}$ by $\mathbb{P}^2$ simultaneously. Let $(\xi_0, X_{30}^{(1)}, Y_{30}^{(1)}, Z_{30}^{(1)}) \in \mathbb{C}^4$, $(\xi_0, X_{31}^{(1)}, Y_{31}^{(1)}, Z_{31}^{(1)}) \in \mathbb{C}^4$ and $(\xi_0, X_{32}^{(1)}, Y_{32}^{(1)}, Z_{32}^{(1)}) \in \mathbb{C}^4$ be coordinate systems of $V_{32}^{(1)}(s) = Q_{A_3(s) \cap W_1}(W_{12} \times s)$ defined by

$$\begin{aligned}
\xi_2 &= X_{30}^{(1)}, \quad \eta_{10} = X_{30}^{(1)} Y_{30}^{(1)}, \quad \eta_{11} = X_{30}^{(1)} Z_{30}^{(1)}, \\
\xi_2 &= X_{31}^{(1)} Y_{31}^{(1)}, \quad \eta_{10} = Y_{31}^{(1)}, \quad \eta_{11} = Y_{31}^{(1)} Z_{31}^{(1)}, \\
\xi_2 &= X_{32}^{(1)} Z_{32}^{(1)}, \quad \eta_{10} = Y_{32}^{(1)} Z_{32}^{(1)}, \quad \eta_{11} = Z_{32}^{(1)},
\end{aligned}$$

then the exceptional divisor $D_{32}^{(1)}(s) = Q_{A_3(s) \cap W_1}(A_3(s) \cap W_1)$ is given by

$$\{X_{30}^{(1)} = 0\} \cup \{Y_{31}^{(1)} = 0\} \cup \{Z_{32}^{(1)} = 0\},$$

the set of accessible singular points $A_{32}^{(1)}$ is given by

$$A_{32}^{(1)}(s) = \{(\xi_0, X_{31}^{(1)}, Y_{31}^{(1)}, Z_{31}^{(1)}) = (\xi_0, \alpha_3, 0, Z_{31}^{(1)})\} \subset D_{32}^{(1)}(s).$$

The second quadratic transformation along $A_{32}^{(1)}(s)$. We next replace the points $(\xi_0, X_{31}^{(1)}, Y_{31}^{(1)}, Z_{31}^{(1)}) = (\xi_0, \alpha_3, 0, Z_{31}^{(1)})$ with $(\xi_0, Z_{31}^{(1)}) \in \mathbb{C}^2$ by $\mathbb{P}^1$ simultaneously. Let $(\xi_0, Z_{31}^{(1)}, X_{30}^{(2)}, Y_{30}^{(2)}) \in \mathbb{C}^4$ and $(\xi_0, Z_{31}^{(1)}, X_{31}^{(2)}, Y_{31}^{(2)}) \in \mathbb{C}^4$ be coordinate systems of $V_{32}^{(2)}(s) = Q_{A_{32}^{(1)}(s)}(V_{32}^{(1)}(s))$ defined by

$$X_{31}^{(1)} = \alpha_3 + X_{30}^{(2)}, \quad Y_{31}^{(1)} = X_{30}^{(2)} Y_{30}^{(2)},$$
$$X_{31}^{(1)} = \alpha_3 + X_{31}^{(2)} Y_{31}^{(2)}, \quad Y_{31}^{(1)} = Y_{31}^{(2)}$$

then the exceptional divisor $D_{32}^{(2)}(s) = Q_{A_{32}^{(1)}(s)}(A_{32}^{(1)}(s))$ is given by

$$\{X_{30}^{(2)} = 0\} \cup \{Y_{31}^{(2)} = 0\}.$$

We see that, in the $(\xi_0, Z_{31}^{(1)}, X_{31}^{(2)}, Y_{31}^{(2)}, s)$-space $\mathbb{C}^4 \times B$, the Pfaffian system has no singular points, every leaf is transversal with fibers, moreover, the points $(\xi_0, Z_{31}^{(1)}, X_{30}^{(2)}, Y_{30}^{(2)}) = (\xi_0, Z_{31}^{(1)}, 0, 0)$ are inaccessible singular points.

Thus we have obtained a coordinate system $(\xi_0, Z_{31}^{(1)}, X_{31}^{(2)}, Y_{31}^{(2)}) \in \mathbb{C}^4$ which separates the solutions passing through $A_3(s) \cap W_1 = A_3(s) \cap W_{12}$. It is related to the coordinate system $(\xi_0, \xi_2, \eta_{10}, \eta_{11}) \in W_{02}$ by

$$\xi_0 = \xi_0, \quad \xi_2 = Y_{31}^{(2)}(\alpha_3 + X_{31}^{(2)} Y_{31}^{(2)}), \quad \eta_{10} = Y_{31}^{(2)}, \quad \eta_{11} = Y_{31}^{(2)} Z_{31}^{(1)}$$

and then we have

$$q_1 = \frac{1}{\xi_0}, \quad q_2 = \frac{Y_{31}^{(2)}}{\xi_0}(\alpha_3 + X_{31}^{(2)} Y_{31}^{(2)}), \quad p_1 = -\xi_0(\nu + \alpha_3 + \xi_0 Z_{31}^{(1)} + X_{31}^{(2)} Y_{31}^{(2)}), \quad p_2 = \frac{\xi_0}{Y_{31}^{(2)}},$$

which is equivalent to

$$q_1^1 = \xi_0, \quad q_2^1 = Y_{31}^{(2)}(\alpha_3 + X_{31}^{(2)} Y_{31}^{(2)}), \quad p_1^1 = Z_{31}^{(1)}, \quad p_2^1 = \frac{1}{Y_{31}^{(2)}}$$

by means of (3.1). If we set

$$q_1^{32} = \xi_1, \quad q_2^{32} = -X_{31}^{(2)}, \quad p_1^{32} = Z_{31}^{(1)}, \quad p_2^{32} = Y_{31}^{(2)},$$

then we have

$$q_1^1 = q_1^{32}, \quad q_2^1 = p_2^{32}(\alpha_3 - q_2^{32} p_2^{32}), \quad p_1^1 = p_1^{32}, \quad p_2^1 = \frac{1}{p_2^{32}}. \tag{4.4}$$

The system $(q^{32}, p^{32}) \in \mathbb{C}^4$ is a symplectic coordinate system for $A_3(s) \cap W_1$.

4.4 Coordinate systems for $A_0(s)$

We obtain the coordinate systems for $A_0(s)$ from those for $A_3(s)$ by the use of Bäcklund transformations given in Section 2. Let $\mathcal{H}(\alpha')$ be the transform of $\mathcal{H}(\alpha)$ by σ_3 and $\overline{\Sigma}_\nu$ be the Hirzebruch manifold for $\mathcal{H}(\alpha')$.

4.4.1 Coordinate system for $A_0(s) \cap W_0$

We derive a coordinate system for $A_0(s) \cap W_0$ from that for $A_3(s) \cap W_0$ and the Bäcklund transformation σ_3 given in Section 2. We can verify $\sigma_3(A_0(s) \cap W_{02}) = A_3'(s') \cap W_{02}'$. As we have shown, there exists a coordinate system $({q'}_1^{31}, {q'}_2^{31}, {p'}_1^{31}, {p'}_2^{31})$ for $A_3'(s') \cap W_{02}'$ with

$$q_1' = {q'}_1^{31}, \quad q_2' = {p'}_2^{31}(\alpha_3' - {q'}_2^{31}{p'}_2^{31}), \quad p_1' = {p'}_1^{31}, \quad p_2' = \frac{1}{{p'}_2^{31}}.$$

Observing the relations between (q_1, q_2, p_1, p_2) and $({q'}_1^{31}, {q'}_2^{31}, {p'}_1^{31}, {p'}_2^{31})$ we take $(q_1^{01}, q_2^{01}, p_1^{01}, p_2^{01})$ as a coordinate system for $A_0(s) \cap W_0$ where

$${q'}_1^{31} = \frac{s_1(s_2-1)q_1^{01}}{s_2 - s_1}, \quad {q'}_2^{31} = -(s_2-1)q_2^{01}, \quad {p'}_1^{31} = \frac{(s_2-s_1)p_1^{01}}{s_1(s_2-1)}, \quad {p'}_2^{31} = -\frac{p_2^{01}}{s_2-1}.$$

We note that

$$q_1 = q_1^{01}, \quad q_2 = p_2^{01}(\alpha_0 - q_2^{01}p_2^{01}) - \frac{s_2 q_1^{01}}{s_1} + s_2, \quad p_1 = \frac{s_2}{s_1 p_2^{01}} + p_1^{01}, \quad p_2 = \frac{1}{p_2^{01}}. \tag{4.5}$$

Thus we have obtained a symplectic coordinate system (q^{01}, p^{01}) for $A_0(s) \cap W_0$.

4.4.2 Coordinate system for $A_0(s) \cap W_1$

A coordinate system for $A_0(s) \cap W_1$ is obtained from that for $A_3(s) \cap W_1$ and the Bäcklund transformation σ_3. We see $\sigma_3(A_0(s) \cap W_{12}) = A_3'(s') \cap W_{12}'$. There exists a coordinate system $({q'}_1^{32}, {q'}_2^{32}, {p'}_1^{32}, {p'}_2^{32})$ for $A_3'(s') \cap W_{12}'$ with

$${q'}_1^{1} = {q'}_1^{32}, \quad {q'}_2^{1} = {p'}_2^{32}(\alpha_3' - {q'}_2^{32}{p'}_2^{32}), \quad {p'}_1^{1} = {p'}_1^{32}, \quad {p'}_2^{1} = \frac{1}{{p'}_2^{32}}.$$

Observing the relations between $(q_1^1, q_2^1, p_1^1, p_2^1)$ and $({q'}_1^{32}, {q'}_2^{32}, {p'}_1^{32}, {p'}_2^{32})$ we take $(q_1^{02}, q_2^{02}, p_1^{02}, p_2^{02})$ as a coordinate system for $A_0(s) \cap W_1$ where

$${q'}_1^{32} = \frac{s_1(s_2-1)q_1^{02}}{s_2 - s_1}, \quad {q'}_2^{32} = -(s_2-1)q_2^{02}, \quad {p'}_1^{32} = \frac{(s_2-s_1)p_1^{02}}{s_1(s_2-1)}, \quad {p'}_2^{32} = -\frac{p_2^{02}}{s_2-1}.$$

We note that

$$q_1^1 = q_1^{02}, \quad q_2^1 = p_2^{02}(\alpha_0 - q_2^{02}p_2^{02}) + s_2 q_1^{02} - \frac{s_2}{s_1}, \quad p_1^1 = -\frac{s_2}{p_2^{02}} + p_1^{02}, \quad p_2^1 = \frac{1}{p_2^{02}}. \tag{4.6}$$

The system $(q^{02}, p^{02}) \in \mathbb{C}^4$ is a symplectic coordinate system for $A_0(s) \cap W_1$.

4.5 Coordinate systems for $A_1(s)$

We obtain the coordinate systems for $A_1(s)$ from those for $A_3(s)$ by the use of Bäcklund transformations. Let $\mathcal{H}(\alpha')$ be the transform of $\mathcal{H}(\alpha)$ by $\sigma_3\sigma_1$ and $\overline{\Sigma}_\nu$ be the Hirzebruch manifold for $\mathcal{H}(\alpha')$.

4.5.1 Coordinate system for $A_1(s) \cap W_0$

We derive a coordinate system for $A_1(s) \cap W_0$ from that for $A_3(s) \cap W_0$ and the Bäcklund transformation $\sigma_3\sigma_1$. We can verify $\sigma_3\sigma_1(A_1(s) \cap W_{02}) = A_3'(s') \cap W_{02}'$. Observing the relations between (q_1, q_2, p_1, p_2) and $({q'}_1^{31}, {q'}_2^{31}, {p'}_1^{31}, {p'}_2^{31})$ we take $(q_1^{11}, q_2^{11}, p_1^{11}, p_2^{11})$ as a coordinate system for $A_0(s) \cap W_0$ where

$${q'}_1^{31} = \frac{(s_2-s_1)q_1^{11}}{s_1(s_2-1)}, \quad {q'}_2^{31} = (s_2-1)q_2^{11}, \quad {p'}_1^{31} = \frac{s_1(s_2-1)p_1^{11}}{(s_2-s_1)}, \quad {p'}_2^{31} = \frac{p_2^{11}}{s_2-1}.$$

We note that

$$q_1 = q_1^{11}, \quad q_2 = p_2^{11}(\alpha_1 - q_2^{11}p_2^{11}) - q_1^{11} + 1, \quad p_1 = \frac{1}{p_2^{11}} + p_1^{11}, \quad p_2 = \frac{1}{p_2^{11}}. \tag{4.7}$$

Thus we have obtained a symplectic coordinate system $(q^{11}, p^{11}) \in \mathbb{C}^4$ for $A_1(s) \cap W_0$.

4.5.2 Coordinate system for $A_1(s) \cap W_1$

A coordinate system for $A_1(s) \cap W_1$ is obtained from that for $A_3(s) \cap W_1$ and the Bäcklund transformation $\sigma_3\sigma_1$. We see $\sigma_3\sigma_1(A_1(s) \cap W_{12}) = A_3'(s') \cap W_{12}'$. Observing the relations between $(q_1^1, q_2^1, p_1^1, p_2^1)$ and $({q'}_1^{32}, {q'}_2^{32}, {p'}_1^{32}, {p'}_2^{32})$ we take $(q_1^{12}, q_2^{12}, p_1^{12}, p_2^{12})$ as a coordinate system for $A_0(s) \cap W_1$ where

$${q'}_1^{32} = \frac{(s_2 - s_1)q_1^{12}}{s_1(s_2 - 1)}, \quad {q'}_2^{32} = (s_2 - 1)q_2^{12}, \quad {p'}_1^{32} = \frac{s_1(s_2 - 1)p_1^{12}}{(s_2 - s_1)}, \quad {p'}_2^{32} = \frac{p_2^{12}}{s_2 - 1}.$$

We note that

$$q_1 = q_1^{12}, \quad q_2 = p_2^{12}(\alpha_1 - q_2^{12}p_2^{12}) + q_1^{12} - 1, \quad p_1 = -\frac{1}{p_2^{12}} + p_1^{12}, \quad p_2 = \frac{1}{p_2^{12}}. \tag{4.8}$$

The system $(q^{12}, p^{12}) \in \mathbb{C}^4$ is a symplectic coordinate system for $A_1(s) \cap W_1$.

4.6 Coordinate systems for $A_\infty(s)$

We obtain the coordinate systems for $A_\infty(s)$ from those for $A_3(s)$ by the use of Bäcklund transformations. Let $\mathcal{H}(\alpha')$ be the transform of $\mathcal{H}(\alpha)$ by $\tau = (\sigma_4\sigma_3)\sigma_1(\sigma_4\sigma_3)^{-1}$ and $\overline{\Sigma}_\nu$ be the Hirzebruch manifold for $\mathcal{H}(\alpha')$.

4.6.1 Coordinate system for $A_\infty(s) \cap W_1$

We derive a coordinate system for $A_\infty(s) \cap W_1$ from that for $A_3(s) \cap W_1$ and the Bäcklund transformation $\tau = (\sigma_4\sigma_3)\sigma_1(\sigma_4\sigma_3)^{-1}$. We can verify $\tau(A_\infty(s) \cap W_{11}) = A_3'(s') \cap W_{12}'$. Observing the relations between $(q_1^1, q_2^1, p_1^1, p_2^1)$ and $({q'}_1^{32}, {q'}_2^{32}, {p'}_1^{32}, {p'}_2^{32})$ we take $(q_1^{\infty 1}, q_2^{\infty 1}, p_1^{\infty 1}, p_2^{\infty 1})$ as a coordinate system for $A_\infty(s) \cap W_1$ where

$${q'}_1^{32} = -q_1^{\infty 1}, \quad {q'}_2^{32} = q_2^{\infty 1}, \quad {p'}_1^{32} = -p_1^{\infty 1}, \quad {p'}_2^{32} = p_2^{\infty 1},$$

We note that

$$q_1^1 = p_1^{\infty 1}(\alpha_\infty - q_1^{\infty 1}p_1^{\infty 1}), \quad q_2^1 = q_2^{\infty 1}, \quad p_1^1 = \frac{1}{p_1^{\infty 1}}, \quad p_2^1 = p_2^{\infty 1}. \tag{4.9}$$

Thus we have obtained a symplectic coordinate system $(q^{\infty 1}, p^{\infty 1}) \in \mathbb{C}^4$ for $A_\infty(s) \cap W_1$.

4.6.2 Coordinate system for $A_\infty(s) \cap W_2$

A coordinate system for $A_\infty(s) \cap W_2$ is obtained from that for $A_3(s) \cap W_0$ and the Bäcklund transformation $\tau = (\sigma_4\sigma_3)\sigma_1(\sigma_4\sigma_3)^{-1}$. We see $\tau(A_\infty(s) \cap W_{22}) = A_3'(s') \cap W_{02}'$. Observing the relations between $(q_1.q_2, p_1, p_2)$ and $({q'}_1^{31}, {q'}_2^{31}, {p'}_1^{31}, {p'}_2^{31})$ we take $(q_1^{\infty 2}, q_2^{\infty 2}, p_1^{\infty 2}, p_2^{\infty 2})$ as a coordinate system for $A_\infty(s) \cap W_2$ where

$${q'}_1^{31} = -q_1^{\infty 2}, \quad {q'}_2^{31} = q_2^{\infty 2}, \quad {p'}_1^{31} = -p_1^{\infty 2}, \quad {p'}_2^{31} = p_2^{\infty 2},$$

We note that

$$q_1^2 = q_1^{\infty 2}, \quad q_2^2 = p_2^{\infty 2}(\alpha_\infty - q_2^{\infty 2} p_2^{\infty 2}), \quad p_1^2 = p_1^{\infty 2}, \quad p_2^2 = \frac{1}{p_2^{\infty 2}}. \tag{4.10}$$

The system $(q^{\infty 2}, p^{\infty 2}) \in \mathbb{C}^4$ is a symplectic coordinate system for $A_\infty(s) \cap W_2$.

Thus we have obtained ten symplectic coordinate systems $(q_1^*, q_2^*, p_1^*, p_2^*)$ each of which separates solution curves passing through the accessible singular points (see (4.1)-(4.10)). We notice that the Hamiltonians of each coordinate system are also polynomials whose coefficients are rational functions of s holomorphic in B.

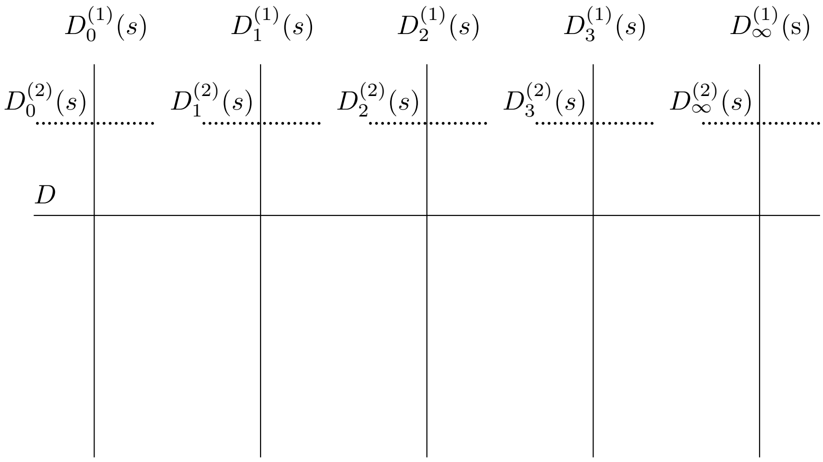

Figure 1. J=11111

5 Spaces of initial conditions for $\mathcal{H}_{1112}$

In the present case,

$$\nu = -\frac{1}{2}(\alpha_0 + \alpha_1 + \alpha_2 - 1 + \alpha_\infty).$$

In this section, we omitt the label 1112.

5.1 Accessible singularities on $D \times B$

Observing the system $\mathcal{H}^{(0)}$ on all W_{ij} $j \neq 0$, in the same way of deriving Propotison 4.1, we can obtain

Proposition 5.1. *The set of accessible singular points of the system $\mathcal{H}_{1112}^{(0)}$ for each $s = (s_1, s_2) \in B_{1112}$ is a disjoint union of four connected components $A_0(s), A_1(s), A_2(s), A_\infty(s) \simeq \mathbb{P}^1$ given by*

$$\begin{aligned} A_0(s) = &\{(\xi, \eta_0, s) \in W_0 \times B | s_1 s_2 \xi_0 - s_2 \xi_1 - s_1 \xi_2 = 0, \eta_{00} = 0, s_1 \eta_{01} - s_2 \eta_{02} = 0\} \\ &\cup \{(\xi, \eta_1, s) \in W_1 \times B | s_1 s_2 \xi_0 - s_2 \xi_1 - s_1 \xi_2 = 0, \eta_{10} = 0, \eta_{11} + s_2 \eta_{12} = 0\} \\ &\cup \{(\xi, \eta_2, s) \in W_2 \times B | s_1 s_2 \xi_0 - s_2 \xi_1 - s_1 \xi_2 = 0, \eta_{20} = 0, s_1 \eta_{21} + \eta_{22} = 0\}, \end{aligned}$$

$$A_1(s) = \{(\xi, \eta_0, s) \in W_0 \times B | \xi_1 = \eta_{00} = \eta_{02} = 0\} \cup \{(\xi, \eta_2, s) \in W_2 \times B | \xi_1 = \eta_{20} = \eta_{22} = 0\},$$
$$A_2(s) = \{(\xi, \eta_0, s) \in W_0 \times B | \xi_2 = \eta_{00} = \eta_{01} = 0\} \cup \{(\xi, \eta_1, s) \in W_1 \times B | \xi_2 = \eta_{10} = \eta_{11} = 0\},$$
$$A_\infty(s) = \{(\xi, \eta_1, s) \in W_1 \times B | \xi_0 = \eta_{10} = \eta_{12} = 0\} \cup \{(\xi, \eta_2, s) \in W_2 \times B | \xi_0 = \eta_{20} = \eta_{21} = 0\}.$$

Moreover, we can verify

Proposition 5.2. *The Bäcklund transformation group S_3 acts on the components of the accessible singular points of $\mathcal{H}_{1112}$ according to the following diagram*

	A_0	A_2	A_∞
σ_1	A_2	A_0	$*$
σ_2	A_∞	$*$	A_0

where $A_1(s)$ is invariant under the action of this group.

In the following subsections, we obtain coordinate systems for $A_i(s)$. The systems for $A_1(s)$ and $A_2(s)$ are obtained by quadratic transformations, while the systems for $A_0(s)$ and $A_\infty(s)$ are obtained from that for $A_2(s)$ by the use of Bäcklund transformations.

5.2 Coordinate systems for $A_2(s)$

We make successively the quadratic transformations along $A_2(s) \cap W_0$, $A_2(s) \cap W_1$ and find coordinate systems for $A_2(s)$.

5.2.1 Coordinate system for $A_2(s) \cap W_0$

The first quadratic transformation along $A_2(s) \cap W_0$. Let

$$\xi_2 = x_{20}^{(1)}, \quad \eta_{00} = x_{20}^{(1)} y_{20}^{(1)}, \quad \eta_{01} = x_{20}^{(1)} z_{20}^{(1)},$$
$$\xi_2 = x_{21}^{(1)} y_{21}^{(1)}, \quad \eta_{00} = y_{21}^{(1)}, \quad \eta_{01} = y_{21}^{(1)} z_{21}^{(1)},$$
$$\xi_2 = x_{22}^{(1)} z_{22}^{(1)}, \quad \eta_{00} = y_{22}^{(1)} z_{22}^{(1)}, \quad \eta_{01} = z_{22}^{(1)},$$

then

$$D_{21}^{(1)}(s) = Q_{A_2(s)\cap W_0}(A_2(s) \cap W_0) = \{x_{20}^{(1)} = 0\} \cup \{y_{21}^{(1)} = 0\} \cup \{z_{22}^{(1)} = 0\},$$

the set of accessible singular points $A_{21}^{(1)}$ is given by

$$A_{21}^{(1)}(s) = \{(\xi_1, x_{21}^{(1)}, y_{21}^{(1)}, z_{21}^{(1)}) = (\xi_1, \alpha_2, 0, z_{21}^{(1)})\} \subset D_{21}^{(1)}(s).$$

The second quadratic transformation along $A_{21}^{(1)}(s)$. Let

$$x_{21}^{(1)} = \alpha_2 + x_{20}^{(2)}, \quad y_{21}^{(1)} = x_{20}^{(2)} y_{20}^{(2)},$$
$$x_{21}^{(1)} = \alpha_2 + x_{21}^{(2)} y_{21}^{(2)}, \quad y_{21}^{(1)} = y_{21}^{(2)},$$

then

$$D_{21}^{(2)}(s) = Q_{A_{21}^{(1)}(s)}(A_{21}^{(1)}(s)) = \{x_{20}^{(2)} = 0\} \cup \{y_{21}^{(2)} = 0\}.$$

We can verify that our system has no singular points, every leaf is transversal with fibers in $(\xi_1, z_{21}^{(1)}, x_{21}^{(2)}, y_{21}^{(2)}, s)$-space $\mathbb{C}^4 \times B$. On the other hand, the points $(\xi_1, z_{21}^{(1)}, x_{20}^{(2)}, y_{20}^{(2)}) = (\xi_1, z_{21}^{(1)}, 0, 0)$ are inaccessible singular points.

Thus we have obtained a coordinate system $(\xi_1, z_{21}^{(1)}, x_{21}^{(2)}, y_{21}^{(2)}) \in \mathbb{C}^4$ which separates the solutions passing through $A_2(s) \cap W_0 = A_2(s) \cap W_{02}$. If we set

$$q_1^{21} = \xi_1, \quad q_2^{21} = -x_{21}^{(2)}, \quad p_1^{21} = z_{21}^{(1)}, \quad p_2^{21} = y_{21}^{(2)},$$

then we have

$$q_1 = q_1^{21}, \quad q_2 = p_2^{21}(\alpha_3 - q_2^{21} p_2^{21}), \quad p_1 = p_1^{21}, \quad p_2 = \frac{1}{p_2^{21}}. \tag{5.1}$$

Thus we have obtained a symplectic coordinate system $(q^{21}, p^{21}) \in \mathbb{C}^4$ for $A_2(s) \cap W_0$.

5.2.2 Coordinate system for $A_2(s) \cap W_1$

The first quadratic transformation along $A_2(s) \cap W_1$. Let

$$\begin{aligned} \xi_2 &= X_{20}^{(1)}, \quad \eta_{10} = X_{20}^{(1)} Y_{20}^{(1)}, \quad \eta_{11} = X_{20}^{(1)} Z_{20}^{(1)}, \\ \xi_2 &= X_{21}^{(1)} Y_{21}^{(1)}, \quad \eta_{10} = Y_{21}^{(1)}, \quad \eta_{11} = Y_{21}^{(1)} Z_{21}^{(1)}, \\ \xi_2 &= X_{22}^{(1)} Z_{22}^{(1)}, \quad \eta_{10} = Y_{22}^{(1)} Z_{22}^{(1)}, \quad \eta_{11} = Z_{22}^{(1)}, \end{aligned}$$

then

$$D_{22}^{(1)}(s) = Q_{A_2(s) \cap W_1}(A_2(s) \cap W_1) = \{X_{20}^{(1)} = 0\} \cup \{Y_{21}^{(1)} = 0\} \cup \{Z_{22}^{(1)} = 0\},$$

the set of accessible singular points $A_{22}^{(1)}$ is given by

$$A_{22}^{(1)}(s) = \{(\xi_0, X_{21}^{(1)}, Y_{21}^{(1)}, Z_{21}^{(1)}) = (\xi_0, \alpha_2, 0, Z_{21}^{(1)})\} \subset D_{22}^{(1)}(s).$$

The second quadratic transformation along$A_{22}^{(1)}(s)$. Let

$$\begin{aligned} X_{21}^{(1)} &= \alpha_2 + X_{20}^{(2)}, \quad Y_{21}^{(1)} = X_{20}^{(2)} Y_{20}^{(2)}, \\ X_{21}^{(1)} &= \alpha_2 + X_{21}^{(2)} Y_{21}^{(2)}, \quad Y_{21}^{(1)} = Y_{21}^{(2)}, \end{aligned}$$

then

$$D_{22}^{(2)}(s) = Q_{A_{22}^{(1)}(s)}(A_{22}^{(1)}(s)) = \{X_{20}^{(2)} = 0\} \cup \{Y_{21}^{(2)} = 0\}.$$

We see that, in the $(\xi_0, Z_{21}^{(1)}, X_{21}^{(2)}, Y_{21}^{(2)}, s)$-space $\mathbb{C}^4 \times B$, the Pfaffin system has no singular points, every leaf is transversal with fibers, moreover, the points $(\xi_0, Z_{21}^{(1)}, X_{20}^{(2)}, Y_{20}^{(2)}) = (\xi_0, Z_{21}^{(1)}, 0, 0)$ are inaccessible singular points.

Thus we have obtained a coordinate system $(\xi_0, Z_{21}^{(1)}, X_{21}^{(2)}, Y_{21}^{(2)}) \in \mathbb{C}^4$ which separates the solutions passing through $A_2(s) \cap W_1 = A_2(s) \cap W_{12}$. If we set

$$q_1^{22} = \xi_1, \quad q_2^{22} = -X_{21}^{(2)}, \quad p_1^{22} = Z_{21}^{(1)}, \quad p_2^{22} = Y_{21}^{(2)},$$

then we have

$$q_1^1 = q_1^{22}, \quad q_2^1 = p_2^{22}(\alpha_2 - q_2^{22} p_2^{22}), \quad p_1^1 = p_1^{22}, \quad p_2^1 = \frac{1}{p_2^{22}}. \tag{5.2}$$

The system $(q^{22}, p^{22}) \in \mathbb{C}^4$ is a symplectic coordinate system for $A_2(s) \cap W_1$.

5.3 Coordinate systems for $A_0(s)$

We obtain the systems for $A_0(s)$ from those for $A_2(s)$ and σ_1.

5.3.1 Coordinate system for $A_0(s) \cap W_0$

We derive a coordinate system for $A_0(s) \cap W_0$ from that for $A_2(s) \cap W_0$ and σ_1. We can verify $\sigma_1(A_0(s) \cap W_{02}) = A_2'(s') \cap W_{02}'$. Observing the relations between $(q_1.q_2, p_1, p_2)$ and the coordinate system $({q'}_1^{21}, {q'}_2^{21}, {p'}_1^{21}, {p'}_2^{21})$ for $A_2'(s')$, we take $(q_1^{01}, q_2^{01}, p_1^{01}, p_2^{01})$ as a coordinate system for $A_0(s) \cap W_0$ where

$$ {q'}_1^{21} = -\frac{q_1^{01}}{s_2 - 1}, \quad {q'}_2^{21} = -(s_2 - 1) q_2^{01}, \quad {p'}_1^{21} = -(s_2 - 1) p_1^{01}, \quad {p'}_2^{21} = -\frac{p_2^{01}}{s_2 - 1}. $$

We note that

$$ q_1 = q_1^{01}, \quad q_2 = p_2^{01}(\alpha_0 - q_2^{01} p_2^{01}) - \frac{s_2 q_1^{01}}{s_1} + s_2, \quad p_1 = \frac{s_2}{s_1 p_2^{01}} + p_1^{01}, \quad p_2 = \frac{1}{p_2^{01}}. \tag{5.3} $$

Thus we have obtained a symplectic coordinate system $(q^{01}, p^{01}) \in \mathbb{C}^4$ for $A_0(s) \cap W_0$.

5.3.2 Coordinate system for $A_0(s) \cap W_1$

A coordinate system for $A_0(s) \cap W_1$ is obtained from that for $A_2(s) \cap W_1$ and σ_1. We see $\sigma_1(A_0(s) \cap W_{12}) = A_2'(s') \cap W_{12}'$. Observing the relations between $(q_1^1.q_2^1, p_1^1, p_2^1)$ and $({q'}_1^{22}, {q'}_2^{22}, {p'}_1^{22}, {p'}_2^{22})$ we take $(q_1^{02}, q_2^{02}, p_1^{02}, p_2^{02})$ as a coordinate system for $A_0(s) \cap W_1$ where

$$ {q'}_1^{22} = -\frac{q_1^{02}}{s_2 - 1}, \quad {q'}_2^{22} = -(s_2 - 1) q_2^{02}, \quad {p'}_1^{22} = -(s_2 - 1) p_1^{02}, \quad {p'}_2^{22} = -\frac{p_2^{02}}{s_2 - 1}. $$

We note that

$$ q_1^1 = q_1^{02}, \quad q_2^1 = p_2^{02}(\alpha_0 - q_2^{01} p_2^{01}) + s_2 q_1^{02} - \frac{s_2}{s_1}, \quad p_1^1 = -\frac{s_2}{p_2^{01}} + p_1^{01}, \quad p_2^1 = \frac{1}{p_2^{01}}. \tag{5.4} $$

The system $(q^{02}, p^{02}) \in \mathbb{C}^4$ is a symplectic coordinate system for $A_0(s) \cap W_1$.

5.4 Coordinate systems for $A_\infty(s)$

We obtain the systems for $A_\infty(s)$ from those for $A_2(s)$ and $\tau = (\sigma_2)\sigma_1(\sigma_2)^{-1}$.

5.4.1 Coordinate system for $A_\infty(s) \cap W_1$

We derive a coordinate system for $A_\infty(s) \cap W_1$ from that for $A_2(s) \cap W_1$ and $\tau = (\sigma_2)\sigma_1(\sigma_2)^{-1}$. We can verify $\tau(A_\infty(s) \cap W_{11}) = A_3'(s') \cap W_{12}'$. Observing the relations between $(q_1^1.q_2^1, p_1^1, p_2^1)$ and $({q'}_1^{22}, {q'}_2^{22}, {p'}_1^{22}, {p'}_2^{22})$ we take $(q_1^{\infty 1}, q_2^{\infty 1}, p_1^{\infty 1}, p_2^{\infty 1})$ as a coordinate system for $A_\infty(s) \cap W_1$ where

$$ {q'}_1^{22} = -q_1^{\infty 1}, \quad {q'}_2^{22} = q_2^{\infty 1}, \quad {p'}_1^{22} = -p_1^{\infty 1}, \quad {p'}_2^{22} = p_2^{\infty 1}, $$

We note that

$$ q_1^1 = p_1^{\infty 1}(\alpha_\infty - q_1^{\infty 1} p_1^{\infty 1}), \quad q_2^1 = q_2^{\infty 1}, \quad p_1^1 = \frac{1}{p_1^{\infty 1}}, \quad p_2^1 = p_2^{\infty 1}. \tag{5.5} $$

Thus we have obtained a symplectic coordinate system $(q^{\infty 1}, p^{\infty 1}) \in \mathbb{C}^4$ for $A_\infty(s) \cap W_1$.

5.4.2 Coordinate system for $A_\infty(s)\cap W_2$

A coordinate system for $A_\infty(s)\cap W_2$ is obtained from that for $A_2(s)\cap W_0$ and $\tau=(\sigma_2)\sigma_1(\sigma_2)^{-1}$. We see $\tau(A_\infty(s)\cap W_{22}) = A'_3(s')\cap W'_{02}$. Observing the relations between $(q_1.q_2,p_1,p_2)$ and $({q'}^{21}_1,{q'}^{21}_2,{p'}^{21}_1,{p'}^{21}_2)$ we take $(q^{\infty2}_1,q^{\infty2}_2,p^{\infty2}_1,p^{\infty2}_2)$ as a coordinate system for $A_\infty(s)\cap W_2$ where

$$ {q'}^{21}_1=-q^{\infty2}_1,\quad {q'}^{21}_2=q^{\infty2}_2,\quad {p'}^{21}_1=-p^{\infty2}_1,\quad {p'}^{21}_2=p^{\infty2}_2. $$

We note that

$$ q^2_1=q^{\infty2}_1,\quad q^2_2=p^{\infty2}_2(\alpha_\infty-q^{\infty2}_2p^{\infty2}_2),\quad p^2_1=p^{\infty2}_1,\quad p^2_2=\frac{1}{p^{\infty2}_2}. \tag{5.6} $$

The system $(q^{\infty2},p^{\infty2})\in\mathbb{C}^4$ is a symplectic coordinate system for $A_\infty(s)\cap W_2$.

5.5 Coordinate systems for $A_1(s)$

In this subsection, we obtain coordinate systems for $A_1(s)$ by making quadratic transformations four times. In order to get good symplectic coordinates, we insert a simple change of variables after the second quadratic transformation and make a suitable change of variables after the last quadratic transformation. The last procedure is very important because it not only produces symplectic cordinates but also resolves a kind of singularity $\xi_2=1$ in the present case.

5.5.1 Coordinate system for $A_1(s)\cap W_0$

The first quadratic transformation along $A_1(s)\cap W_0$. Note that $A_1(s)\cap W_0\subset W_{01}$ and

$$ A_1(s)\cap W_0=\{(\xi_1,\xi_2,\eta_{00},\eta_{02})\in W_{01}\simeq\mathbb{C}^4\mid \xi_2\in\mathbb{C},\xi_1=\eta_{00}=\eta_{02}=0\}. $$

We replace every point $(\xi_1,\xi_2,\eta_{00},\eta_{02})=(0,\xi_2,0,0)$ with $\xi_2\in\mathbb{C}$ by $\mathbb{P}^2$ simultaneously. Let $(\xi_2,x^{(1)}_{10},y^{(1)}_{10},z^{(1)}_{10})\in\mathbb{C}^4$, $(\xi_2,x^{(1)}_{11},y^{(1)}_{11},z^{(1)}_{11})\in\mathbb{C}^4$ and $(\xi_2,x^{(1)}_{12},y^{(1)}_{12},z^{(1)}_{12})\in\mathbb{C}^4$ be coordinate systems of $V^{(1)}_{11}(s)=Q_{A_1(s)\cap W_0}(W_{01}\times s)$ defined by

$$ \begin{aligned} &\xi_1=x^{(1)}_{10},\quad \eta_{00}=x^{(1)}_{10}y^{(1)}_{10},\quad \eta_{02}=x^{(1)}_{10}z^{(1)}_{10},\\ &\xi_1=x^{(1)}_{11}y^{(1)}_{11},\quad \eta_{00}=y^{(1)}_{11},\quad \eta_{02}=y^{(1)}_{11}z^{(1)}_{11},\\ &\xi_1=x^{(1)}_{12}z^{(1)}_{12},\quad \eta_{00}=y^{(1)}_{12}z^{(1)}_{12},\quad \eta_{02}=z^{(1)}_{12}, \end{aligned} $$

then the exceptional divisor $D^{(1)}_{11}(s)=Q_{A_1(s)\cap W_0}(A_1(s)\cap W_0)$ is given by

$$ \{x^{(1)}_{10}=0\}\cup\{y^{(1)}_{11}=0\}\cup\{z^{(1)}_{12}=0\}. $$

Let us write our system in the three coordinate systems near the exeptional divisor. In the first coordinate, it is written as

$$ \frac{\partial\xi_2}{\partial s_1}=\frac{O(x^{(1)}_{10})}{s_1^2x^{(1)}_{10}},\quad \frac{\partial\xi_2}{\partial s_2}=\frac{1}{s_1s_2(s_2-1)}\Big(\frac{O(x^{(1)}_{10})}{x^{(1)}_{10}}+\frac{P_1(\xi_2,z^{(1)}_{10})}{y^{(1)}_{10}}\Big), $$

$$ \frac{\partial x^{(1)}_{10}}{\partial s_1}=\frac{O(x^{(1)}_{10})}{s_1^2x^{(1)}_{10}},\quad \frac{\partial x^{(1)}_{10}}{\partial s_2}=\frac{O(x^{(1)}_{10})}{s_2(s_2-1)x^{(1)}_{10}}, $$

$$\frac{\partial y_{10}^{(1)}}{\partial s_1}=\frac{O(x_{10}^{(1)})+\eta s_1 y_{10}^{(1)}(\xi_2-1)}{s_1^2 x_{10}^{(1)}}, \quad \frac{\partial y_{10}^{(1)}}{\partial s_2}=\frac{O(x_{10}^{(1)})-\eta\xi_2 y_{10}^{(1)}}{s_1 s_2 x_{10}^{(1)}},$$

$$\frac{\partial z_{10}^{(1)}}{\partial s_1}=\frac{O(x_{10}^{(1)})+\eta s_1\{1+(\xi_2-1)z_{10}^{(1)}\}}{s_1^2 x_{10}^{(1)}},$$

$$\frac{\partial z_{10}^{(1)}}{\partial s_2}=\frac{1}{s_1 s_2(s_2-1)}\Big(\frac{O(x_{10}^{(1)})+P_2(\xi_2,z_{10}^{(1)})}{x_{10}^{(1)}}+\frac{P_3(\xi_2,z_{10}^{(1)})}{y_{10}^{(1)}}\Big)$$

in a neighborhood of $D_{11}^{(1)}(s)=\{x_{10}^{(1)}=0\}$, in the second coordinate system, it is written as

$$\frac{\partial \xi_2}{\partial s_1}=\frac{O(y_{11}^{(1)})}{s_1^2 y_{11}^{(1)}}, \quad \frac{\partial \xi_2}{\partial s_2}=\frac{O(y_{11}^{(1)})}{s_1 s_2(s_2-1)y_{11}^{(1)}},$$

$$\frac{\partial y_{11}^{(1)}}{\partial s_1}=\frac{O(y_{11}^{(1)})}{s_1^2 y_{11}^{(1)}}, \quad \frac{\partial y_{11}^{(1)}}{\partial s_2}=\frac{O(y_{11}^{(1)})}{s_1 s_2(s_2-1)},$$

$$\frac{\partial x_{11}^{(1)}}{\partial s_1}=\frac{O(y_{11}^{(1)})-\eta s_1(\xi_2-1)}{s_1^2 y_{11}^{(1)}}, \quad \frac{\partial x_{11}^{(1)}}{\partial s_2}=\frac{O(y_{11}^{(1)})+\eta\xi_2}{s_2 y_{11}^{(1)}},$$

$$\frac{\partial z_{11}^{(1)}}{\partial s_1}=\frac{O(y_{11}^{(1)})+\eta s_1}{s_1^2 y_{11}^{(1)}}, \quad \frac{\partial z_{11}^{(1)}}{\partial s_2}=\frac{O(y_{11}^{(1)})-\eta s_1(s_2-1)}{s_1 s_2(s_2-1)y_{11}^{(1)}}$$

in a neighborhood of $D_{11}^{(1)}(s)=\{y_{11}^{(1)}=0\}$, and in the third coordinate system it is written as

$$\frac{\partial \xi_2}{\partial s_1}=\frac{O(z_{12}^{(1)})}{s_1^2 z_{12}^{(1)}}, \quad \frac{\partial \xi_2}{\partial s_2}=\frac{1}{s_1 s_2(s_2-1)}\Big(\frac{O(z_{12}^{(1)})}{z_{12}^{(1)}}+\frac{P_1(\xi_2,x_{12}^{(1)})}{y_{12}^{(1)}}\Big),$$

$$\frac{\partial z_{12}^{(1)}}{\partial s_1}=\frac{O(z_{12}^{(1)})}{s_1^2 z_{12}^{(1)}}, \quad \frac{\partial z_{12}^{(1)}}{\partial s_2}=\frac{O(z_{12}^{(1)})}{s_1 s_2(s_2-1)z_{12}^{(1)}},$$

$$\frac{\partial x_{12}^{(1)}}{\partial s_1}=\frac{O(z_{12}^{(1)})-\eta s_1(\xi_2-1+x_{12}^{(1)})}{s_1^2 z_{12}^{(1)}},$$

$$\frac{\partial x_{12}^{(1)}}{\partial s_2}=\frac{1}{s_1 s_2(s_2-1)}\Big(\frac{O(z_{12}^{(1)})+P_2(\xi_2,x_{12}^{(1)})}{z_{12}^{(1)}}+\frac{P_3(\xi_2,x_{12}^{(1)})}{y_{12}^{(1)}}\Big),$$

$$\frac{\partial y_{12}^{(1)}}{\partial s_1}=\frac{O(z_{12}^{(1)})-\eta s_1 y_{12}^{(1)}}{s_1^2 z_{12}^{(1)}}, \quad \frac{\partial y_{12}^{(1)}}{\partial s_2}=\frac{O(z_{12}^{(1)})-s_1(s_2-1)y_{12}^{(1)}}{s_1 s_2(s_2-1)z_{12}^{(1)}}$$

in a neighborhood of $D_{11}^{(1)}(s)=\{z_{12}^{(1)}=0\}$. Investigating carefully these systems in a neighborhood of $D_{11}^{(1)}(s)$, we can verity that the set of accessible singular points $A_{11}^{(1)}(s)$ is given by

$$A_{11}^{(1)}(s)=\{(\xi_2,x_{10}^{(1)},y_{10}^{(1)},z_{10}^{(1)})=(\xi_2,0,0,-1/(\xi_2-1)\}\subset D_{11}^{(1)}(s).$$

The second quadratic transformation along $A_{11}^{(1)}(s)$. We next replace the points $(\xi_2, x_{10}^{(1)}, y_{10}^{(1)}, z_{10}^{(1)}) = (\xi_2, 0, 0, -1/(\xi_2-1))$ with $\xi_2 \in \mathbb{C} \setminus \{\xi_2 = 1\}$ by $\mathbb{P}^2$ simultaneously. Note that $\xi_2 = 1$ is excluded. Let $(\xi_2, x_{10}^{(2)}, y_{10}^{(2)}, z_{10}^{(2)}) \in \mathbb{C}^4$, $(\xi_2, x_{11}^{(2)}, y_{11}^{(2)}, z_{11}^{(2)}) \in \mathbb{C}^4$ and $(\xi_2, x_{12}^{(2)}, y_{12}^{(2)}, z_{12}^{(2)}) \in \mathbb{C}^4$ be coordinate systems of $V_{11}^{(2)}(s) = Q_{A_{11}^{(1)}(s)}(V_{11}^{(1)}(s))$ defined by

$$
\begin{aligned}
x_{10}^{(1)} &= x_{10}^{(2)}, \quad y_{10}^{(1)} = x_{10}^{(2)} y_{10}^{(2)}, \quad z_{10}^{(1)} = -1/(\xi_2-1) + x_{10}^{(2)} z_{10}^{(2)}, \\
x_{10}^{(1)} &= x_{11}^{(2)} y_{11}^{(2)}, \quad y_{10}^{(1)} = y_{11}^{(2)}, \quad z_{10}^{(1)} = -1/(\xi_2-1) + y_{11}^{(2)} z_{11}^{(2)}, \\
x_{10}^{(1)} &= x_{12}^{(2)} z_{12}^{(2)}, \quad y_{10}^{(1)} = y_{12}^{(2)} z_{12}^{(2)}, \quad z_{10}^{(1)} = -1/(\xi_2-1) + z_{12}^{(2)},
\end{aligned}
$$

then the exceptional divisor $D_{11}^{(2)}(s) = Q_{A_{11}^{(1)}(s)}(A_{11}^{(1)}(s))$ is given by

$$\{x_{10}^{(2)} = 0\} \cup \{y_{11}^{(2)} = 0\} \cup \{z_{12}^{(2)} = 0\}.$$

Let us write our system in the three coordinate systems near the exeptional divisor. In the first coordinate, it is written as

$$\frac{\partial \xi_2}{\partial s_1} = \frac{1}{s_1^2(\xi_2-1)}\Big(\frac{O(x_{10}^{(1)})}{x_{10}^{(1)}} + \frac{2(s_2-1)\xi_2}{y_{10}^{(2)}}\Big), \quad \frac{\partial \xi_2}{\partial s_2} = \frac{1}{s_1 s_2 (s_2-1)(\xi_2-1)}\Big(\frac{O(x_{10}^{(1)})}{x_{10}^{(1)}} + \frac{P_1(\xi_2, z_{10}^{(2)})}{y_{10}^{(2)}}\Big),$$

$$\frac{\partial x_{10}^{(2)}}{\partial s_1} = \frac{1}{s_1^2(\xi_2-1)}\Big(\frac{O(x_{10}^{(1)})}{x_{10}^{(1)}} - \frac{2s_1(\xi_2-1)}{y_{10}^{(2)}}\Big),$$

$$\frac{\partial x_{10}^{(2)}}{\partial s_2} = \frac{1}{s_1 s_2 (s_2-1)(\xi_2-1)}\Big(\frac{O(x_{10}^{(1)})}{x_{10}^{(1)}} + \frac{2\xi_2(s_2-1)}{y_{10}^{(2)}}\Big),$$

$$\frac{\partial y_{10}^{(2)}}{\partial s_1} = \frac{O(x_{10}^{(2)}) + 2s_1(\xi_2-1)^2\{1+\eta(\xi_2-1)y_{10}^{(1)}\}}{s_1^2(\xi_2-1)^2 x_{10}^{(2)}},$$

$$\frac{\partial y_{10}^{(2)}}{\partial s_2} = \frac{O(x_{10}^{(2)}) - 2s_1(s_2-1)(\xi_2-1)\xi_2\{1+\eta(\xi_2-1)y_{10}^{(1)}\}}{s_1 s_2 (s_2-1)(\xi_2-1)^2 x_{10}^{(2)}},$$

$$\frac{\partial z_{10}^{(2)}}{\partial s_1} = \frac{1}{s_1^2(\xi_2-1)^3}\Big(\frac{O(x_{10}^{(2)})}{x_{10}^{(2)}} + \frac{P_2(\xi_2, z_{10}^{(2)})}{x_{10}^{(2)}} + \frac{P_3(\xi_2, z_{10}^{(2)})}{x_{10}^{(2)} y_{10}^{(2)}}\Big),$$

$$\frac{\partial z_{10}^{(1)}}{\partial s_2} = \frac{O(x_{10}^{(2)})}{s_1 s_2 (s_2-1)(\xi_2-1)^3 (x_{10}^{(2)})^2}$$

in a neighborhood of $D_{11}^{(2)}(s) = \{x_{10}^{(2)} = 0\}$, in the second coordinate system, it is written as

$$\frac{\partial \xi_2}{\partial s_1} = \frac{O(y_{11}^{(2)})}{s_1^2(\xi_2-1)y_{11}^{(2)}}, \quad \frac{\partial \xi_2}{\partial s_2} = \frac{O(y_{11}^{(2)}) - 2s_1\xi_2(\xi_2-s_2)}{s_1 s_2 (s_2-1)(\xi_2-1) y_{11}^{(2)}},$$

$$\frac{\partial y_{11}^{(2)}}{\partial s_1} = \frac{1}{s_1^2(\xi_2-1)^2}\Big(\frac{O(y_{11}^{(2)})}{y_{11}^{(2)}} - \frac{\eta s_1(\xi_2-1)^3}{x_{11}^{(2)}}\Big),$$

$$\frac{\partial y_{11}^{(2)}}{\partial s_2} = \frac{1}{s_1 s_2(\xi_2-1)^2}\Big(\frac{O(y_{11}^{(2)})}{y_{11}^{(2)}} - \frac{\eta s_1 \xi_2 (xi_2-1)^2}{x_{11}^{(2)}}\Big),$$

$$\frac{\partial x_{11}^{(2)}}{\partial s_1} = \frac{O(y_{11}^{(2)}) + P_1(\xi_2, x_{11}^{(2)})\{\eta(\xi_2-1)+x_{11}^{(2)}\}}{s_1^2(\xi_2-1)^2 y_{11}^{(2)}},$$

$$\frac{\partial x_{11}^{(2)}}{\partial s_2} = \frac{O(y_{11}^{(2)}) + P_2(\xi_2, x_{11}^{(2)})\{\eta(\xi_2-1)-x_{11}^{(2)}\}}{s_1 s_2(s_2-1)(1-\xi_2)^2 y_{11}^{(2)}},$$

$$\frac{\partial z_{11}^{(2)}}{\partial s_1} = \frac{O(y_{11}^{(2)})}{s_1^2(\xi_2-1)^3 y_{11}^{(2)}}, \quad \frac{\partial z_{11}^{(2)}}{\partial s_2} = \frac{O(y_{11}^{(2)})}{s_1 s_2(s_2-1)(\xi_2-1)^3 x_{11}^{(2)}(y_{11}^{(2)})^2}$$

in a neighborhood of $D_{11}^{(2)}(s) = \{y_{11}^{(2)} = 0\}$, and in the third coordinate system, it is written as

$$\frac{\partial \xi_2}{\partial s_1} = \frac{1}{s_1^2(\xi_2-1)}\Big(\frac{O(z_{12}^{(2)})}{z_{12}^{(2)}} + \frac{2(s_2-1)\xi_2 x_{12}^{(2)}}{y_{12}^{(2)}}\Big),$$

$$\frac{\partial \xi_2}{\partial s_2} = \frac{1}{s_1 s_2(s_2-1)(\xi_2-1)}\Big(\frac{O(z_{12}^{(2)})}{z_{12}^{(2)}} + \frac{P_1(\xi_2, x_{12}^{(2)})}{y_{12}^{(2)}}\Big),$$

$$\frac{\partial z_{12}^{(2)}}{\partial s_1} = \frac{1}{s_1^2(\xi_2-1)^3}\Big(\frac{O(z_{12}^{(2)})}{z_{12}^{(2)}} + \frac{P_2(\xi_2, x_{12}^{(2)})}{x_{12}^{(2)}} + \frac{P_3(\xi_2, x_{12}^{(2)})}{y_{12}^{(2)}}\Big),$$

$$\frac{\partial z_{12}^{(2)}}{\partial s_2} = \frac{1}{s_1 s_2(s_2-1)(\xi_2-1)^3}\Big(\frac{O(z_{12}^{(2)})}{z_{12}^{(2)}} + \frac{P_4(\xi_2)}{x_{12}^{(2)} z_{12}^{(2)}} + \frac{P_5(\xi_2, z_{12}^{(2)})}{y_{12}^{(2)} z_{12}^{(2)}}\Big),$$

$$\frac{\partial x_{12}^{(2)}}{\partial s_1} = \frac{1}{s_1^2(\xi_2-1)^3}\Big(\frac{O(z_{12}^{(2)})}{z_{12}^{(2)}} + \frac{P_6(\xi_2, z_{12}^{(2)})}{y_{12}^{(2)} z_{12}^{(2)}}\Big),$$

$$\frac{\partial x_{12}^{(2)}}{\partial s_2} = \frac{1}{s_1 s_2(s_2-1)(\xi_2-1)^3}\Big(\frac{O(z_{12}^{(2)})}{z_{12}^{(2)}} + \frac{P_8(\xi_2, z_{12}^{(2)}, x_{12}^{(2)})}{y_{12}^{(2)}(z_{12}^{(2)})^2}\Big),$$

$$\frac{\partial y_{12}^{(2)}}{\partial s_1} = \frac{P_9(\xi_2, x_{12}^{(2)}, y_{12}^{(2)})}{s_1^2(\xi_2-1)^3 z_{12}^{(2)}}, \quad \frac{\partial y_{12}^{(2)}}{\partial s_2} = \frac{P_{10}(\xi_2, x_{12}^{(2)}, y_{12}^{(2)})}{s_1 s_2(s_2-1)(\xi_2-1)^3 x_{12}^{(2)} z_{12}^{(2)}}$$

in a neighborhood of $D_{11}^{(2)}(s) = \{z_{12}^{(2)} = 0\}$. Investigating carefully these systems in a neighborhood of $D_{11}^{(2)}(s)$, we can verity that the set of accessible singular points $A_{11}^{(2)}(s)$ is given by

$$A_{11}^{(2)}(s) = \{(\xi_2, x_{10}^{(2)}, y_{10}^{(2)}, z_{10}^{(2)}) = (\xi_2, 0, -1/(\eta(\xi_2-1)), z_{10}^{(2)})\} \subset D_{11}^{(2)}(s).$$

The third quadratic transformation along $A_{11}^{(2)}(s)$. Here we insert a change of variables

$$\xi_2 = \xi_2, \quad x_{10}^{(2)} = x_{10}^{(2)}, \quad y_{10}^{(2)} = 1/v_{10}^{(2)}, \quad z_{10}^{(2)} = z_{10}^{(2)},$$

namely, a change of local coordinates of a neighborhood of the set $A_{11}^{(2)}(s)$. The change of variables is necessary for obtaing a symplectic coordinate system.

We next replace the points $(\xi_2, z_{10}^{(2)}, x_{10}^{(2)}, v_{10}^{(2)}) = (\xi_2, z_{10}^{(2)}, 0, -\eta(\xi_2-1))$ with $(\xi_2, z_{10}^{(2)}) \in \mathbb{C}^2 \setminus \{\xi_2 = 1\}$ by $\mathbb{P}^1$ simultaneously. Let $(\xi_2, z_{10}^{(2)}, x_{10}^{(3)}, y_{10}^{(3)}) \in \mathbb{C}^4$ and $(\xi_2, z_{10}^{(2)}, x_{11}^{(3)}, y_{11}^{(3)}) \in \mathbb{C}^4$ be coordinate systems of $V_{11}^{(3)}(s) = Q_{A_{11}^{(2)}(s)}(V_{11}^{(2)}(s))$ defined by

$$x_{10}^{(2)} = x_{10}^{(3)}, \quad v_{10}^{(2)} = -\eta(\xi_2-1) + x_{10}^{(3)} y_{10}^{(3)},$$
$$x_{10}^{(2)} = x_{11}^{(3)} y_{11}^{(3)}, \quad v_{10}^{(2)} = -\eta(\xi_2-1) + y_{11}^{(3)},$$

then the exceptional divisor $D_{11}^{(3)}(s) = Q_{A_{11}^{(2)}(s)}(A_{11}^{(2)}(s))$ is given by

$$\{x_{10}^{(3)} = 0\} \cup \{y_{11}^{(3)} = 0\}.$$

Let us write our system in the two coordinate systems near the exeptional divisor. In the first coordinate, it is written as

$$\frac{\partial \xi_2}{\partial s_1} = \frac{O(x_{10}^{(1)})}{s_1^2(\xi_2-1)x_{10}^{(1)}}, \quad \frac{\partial \xi_2}{\partial s_2} = \frac{O(x_{10}^{(1)})}{s_1 s_2(s_2-1)(\xi_2-1)x_{10}^{(1)}},$$

$$\frac{\partial z_{10}^{(2)}}{\partial s_1} = \frac{P_1(\xi_2, z_{10}^{(2)}, x_{10}^{(3)}, y_{10}^{(3)})}{s_1^2(\xi_2-1)^3\{x_{10}^{(3)} y_{10}^{(3)} - \eta(\xi_2-1)\}},$$

$$\frac{\partial z_{10}^{(2)}}{\partial s_2} = \frac{P_2(\xi_2, z_{10}^{(2)}, x_{10}^{(3)}, y_{10}^{(3)})}{s_1 s_2(s_2-1)(\xi_2-1)^3 x_{10}^{(3)}\{x_{10}^{(3)} y_{10}^{(3)} - \eta(\xi_2-1)\}},$$

$$\frac{\partial x_{10}^{(3)}}{\partial s_1} = \frac{O(x_{10}^{(1)})}{s_1^2(\xi_2-1)x_{10}^{(3)}}, \quad \frac{\partial x_{10}^{(3)}}{\partial s_2} = \frac{O(x_{10}^{(2)})}{s_2(s_2-1)(\xi_2-1)x_{10}^{(2)}},$$

$$\frac{\partial y_{10}^{(3)}}{\partial s_1} = \frac{O(x_{10}^{(3)}) + (\alpha_1 - y_{10}^{(3)})P_3(\xi_2)}{s_1^2(\xi_2-1)^2 x_{10}^{(3)}}, \quad \frac{\partial y_{10}^{(3)}}{\partial s_2} = \frac{O(x_{10}^{(3)}) + (\alpha_1 - y_{10}^{(3)})P_4(\xi_2)}{s_1 s_2(\xi_2-1)^2 x_{10}^{(3)}}$$

in a neighborhood of $D_{11}^{(3)}(s) = \{x_{10}^{(3)} = 0\}$, in the second coordinate system, it is written as

$$\frac{\partial \xi_2}{\partial s_1} = \frac{O(y_{11}^{(3)})}{s_1^2(\xi_2-1)y_{11}^{(3)}}, \quad \frac{\partial \xi_2}{\partial s_2} = \frac{O(y_{11}^{(3)})}{s_1 s_2(s_2-1)(\xi_2-1)y_{11}^{(3)}},$$

$$\frac{\partial z_{10}^{(2)}}{\partial s_1} = \frac{O(y_{11}^{(3)})}{s_1^2(\xi_2-1)^3\{y_{11}^{(3)} - \eta(\xi_2-1)\}x_{11}^{(3)}},$$

$$\frac{\partial z_{10}^{(2)}}{\partial s_2}=\frac{O(y_{11}^{(3)})+P_1(\xi_2,z_{10}^{(2)},x_{11}^{(3)})}{s_1s_2(s_2-1)(\xi_2-1)^3\{y_{11}^{(3)}-\eta(\xi_2-1)\}y_{11}^{(3)}(x_{11}^{(3)})^2},$$

$$\frac{\partial y_{11}^{(3)}}{\partial s_1}=\frac{1}{s_1^2(\xi_2-1)^2}\Big(\frac{O(y_{11}^{(3)})}{y_{11}^{(3)}}+\frac{P_2(\xi_2,x_{11}^{(3)})}{x_{11}^{(3)}}\Big),$$

$$\frac{\partial y_{11}^{(3)}}{\partial s_2}=\frac{1}{s_1s_2(s_2-1)(\xi_2-1)^2}\Big(\frac{O(y_{11}^{(3)})}{y_{11}^{(3)}}+\frac{P_3(\xi_2,x_{11}^{(3)})}{x_{11}^{(3)}}\Big),$$

$$\frac{\partial x_{11}^{(3)}}{\partial s_1}=\frac{O(y_{11}^{(3)})-\eta s_1(\xi_2-1)^3(\alpha_1x_{11}^{(3)}-1)}{s_1^2(\xi_2-1)^2y_{11}^{(3)}},$$

$$\frac{\partial x_{11}^{(3)}}{\partial s_2}=\frac{O(y_{11}^{(3)})+\eta\xi_2s_1(\xi_2-1)^2(\alpha_1x_{11}^{(3)}-1)}{s_1s_2(\xi_2-1)^2y_{11}^{(3)}}$$

in a neighborhood of $D_{11}^{(3)}(s)=\{y_{11}^{(3)}=0\}$. Investigating carefully these systems in a neighborhood of $D_{11}^{(3)}(s)$, we can verity that the set of accessible singular points $A_{11}^{(3)}(s)$ is given by

$$A_{11}^{(3)}(s)=\{(\xi_2,z_{10}^{(2)},x_{10}^{(3)},y_{10}^{(3)})=(\xi_2,z_{10}^{(2)},0,\alpha_1)\}\subset D_{11}^{(3)}(s).$$

The fourth quadratic transformation along $A_{11}^{(3)}(s)$. We next replace the points $(\xi_2,z_{10}^{(2)},x_{10}^{(3)},y_{10}^{(3)})=(\xi_2,z_{10}^{(2)},0,\alpha_1)$ with $(\xi_2,z_{10}^{(2)})\in\mathbb{C}^2\backslash\{\xi_2=1\}$ by $\mathbb{P}^1$ simultaneously. Let $(\xi_2,z_{10}^{(2)},x_{10}^{(4)},y_{10}^{(4)})\in\mathbb{C}^4$ and $(\xi_2,z_{10}^{(2)},x_{11}^{(4)},y_{11}^{(4)})\in\mathbb{C}^4$ be coordinate systems of $V_{11}^{(4)}(s)=Q_{A_{11}^{(3)}(s)}(V_{11}^{(3)}(s))$ defined by

$$\begin{aligned}x_{10}^{(3)}&=x_{10}^{(4)},\quad y_{10}^{(3)}=\alpha_1+x_{10}^{(4)}y_{10}^{(4)},\\x_{10}^{(3)}&=x_{11}^{(4)}y_{11}^{(4)},\quad y_{10}^{(3)}=\alpha_1+y_{11}^{(4)},\end{aligned}$$

then the exceptional divisor is given by

$$D_1^{(4)}(s)=Q_{A_{10}^{(3)}(s)}(A_{10}^{(3)}(s))=\{x_{10}^{(4)}=0\}\cup\{y_{11}^{(4)}=0\}.$$

We can verify that the Pfaffian system is written as

$$\begin{aligned}&s_1^2s_2(s_2-1)(\xi_2-1)d\xi_2-\sum_{i=1,2}P'_{1i}(\xi_2,z_{10}^{(2)},x_{10}^{(4)},y_{10}^{(4)},s)ds_i=0,\\&s_1^2s_2(s_2-1)(\xi_2-1)^3\{\eta(\xi_2-1)+O(x_{10}^{(4)})\}dz_{10}^{(2)}-\sum_{i=1,2}Q'_{1i}(\xi_2,z_{10}^{(2)},x_{10}^{(4)},y_{10}^{(4)},s)ds_i=0,\\&s_1^2s_2(s_2-1)(\xi_2-1)dx_{10}^{(4)}-\sum_{i=1,2}X'_{1i}(\xi_2,z_{10}^{(2)},x_{10}^{(4)},y_{10}^{(4)},s)ds_i=0,\\&s_1^2s_2(s_2-1)(\xi_2-1)^2dy_{10}^{(4)}-\sum_{i=1,2}Y'_{1i}(\xi_2,z_{10}^{(2)},x_{10}^{(4)},y_{10}^{(4)},s)ds_i=0\end{aligned}$$

in the coordinates $(\xi_2, z_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)})$ and s where $P'_{1i}, Q'_{1i}, X'_{1i}, Y'_{1i}$ are certain polynomials of $\xi_2, z_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}$ and s. Therefore the differential system in the cordinates $(\xi_2, z_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)})$ is holomorphic in a neighborhood of $\{x_{10}^{(4)} = 0\}$ except for $\xi_2 = 1$. On the other hand, we can verify that

$$\frac{\partial \xi_2}{\partial s_1} = \frac{P_1(\xi_2, z_{10}^{(2)}, x_{11}^{(4)}, y_{11}^{(4)})}{s_1^2(\xi_2 - 1)}, \quad \frac{\partial \xi_2}{\partial s_2} = \frac{P_2(\xi_2, z_{10}^{(2)}, x_{11}^{(4)}, y_{11}^{(4)})}{s_1 s_2 (s_2 - 1)(\xi_2 - 1)},$$

$$\frac{\partial z_{10}^{(2)}}{\partial s_1} = \frac{P_3(\xi_2, z_{10}^{(2)}, x_{11}^{(4)}, y_{11}^{(4)})}{s_1^2(\xi_2 - 1)^3\{x_{11}^{(4)} y_{11}^{(4)}(\alpha_1 + y_{11}^{(4)}) - \eta(\xi_2 - 1)\}},$$

$$\frac{\partial z_{10}^{(2)}}{\partial s_2} = \frac{O(y_{11}^{(4)}) + O(x_{11}^{(4)}) + 4\eta s_1\{(\xi_2 - 1)^2 + s_2 \xi_2 (2 - \xi_2)\}}{s_1 s_2 (s_2 - 1)(\xi_2 - 1)^3\{x_{11}^{(4)} y_{11}^{(4)}(\alpha_1 + y_{11}^{(4)}) - \eta(\xi_2 - 1)\} x_{11}^{(4)}},$$

$$\frac{\partial y_{11}^{(4)}}{\partial s_1} = \frac{O(y_{11}^{(4)}) + \eta s_1 (\xi_2 - 1)^2}{s_1^2 (\xi_2 - 1)^2 x_{11}^{(4)}}, \quad \frac{\partial y_{11}^{(4)}}{\partial s_2} = \frac{O(y_{11}^{(4)}) + \eta s_1 \xi_2 (\xi_2 - 1)^2}{s_1 s_2 (s_2 - 1)(\xi_2 - 1)^2 x_{11}^{(4)}},$$

$$\frac{\partial x_{11}^{(4)}}{\partial s_1} = \frac{P_4(\xi_2, z_{10}^{(2)}, x_{11}^{(4)}, y_{11}^{(4)})}{s_1^2 (\xi_2 - 1)^2}, \quad \frac{\partial x_{11}^{(4)}}{\partial s_2} = \frac{P_5(\xi_2, z_{10}^{(2)}, x_{11}^{(4)}, y_{11}^{(4)})}{s_1 s_2 (s_2 - 1)(\xi_2 - 1)^2}$$

in a neighborhood of $(\xi_2, z_{10}^{(2)}, x_{11}^{(4)}, y_{11}^{(4)}) = (\xi_2, z_{10}^{(2)}, 0, 0)$ with $\xi_2 \neq 1$, which shows that the points $(\xi_2, z_{10}^{(2)}, x_{11}^{(4)}, y_{11}^{(4)}) = (\xi_2, z_{10}^{(2)}, 0, 0)$ are inaccessible with $\xi_2 \neq 1$.

Thus we have obtained a coordinate system $(\xi_2, z_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}) \in \mathbb{C}^4$ which separates the solutions passing through $A_1(s) \cap W_0 = A_1(s) \cap W_{01}$ with $\xi_2 \neq 1$. It is related to the coordinate system $(\xi_1, \xi_2, \eta_{00}, \eta_{02}) \in W_{01}$ by

$$\xi_1 = x_{10}^{(4)}, \quad \xi_2 = \xi_2, \quad \eta_{00} = \frac{(x_{10}^{(4)})^2}{x_{10}^{(4)}(\alpha_1 + x_{10}^{(4)} y_{10}^{(4)}) - \eta(\xi_2 - 1)}, \quad \eta_{02} = x_{10}^{(4)}\left(x_{10}^{(4)} z_{10}^{(2)} - \frac{1}{\xi_2 - 1}\right)$$

and then

$$q_1 = x_{10}^{(4)}, \quad q_2 = \xi_2, \quad p_1 = -\frac{\eta(\xi_2 - 1)}{(x_{10}^{(4)})^2} + \frac{\alpha_1}{x_{10}^{(4)}} + y_{10}^{(4)},$$

$$p_2 = \frac{\eta}{x_{10}^{(4)}} - \eta(\xi_2 - 1) z_{10}^{(2)} - \frac{\alpha_1}{\xi_2 - 1} + \left(\alpha_1 z_{10}^{(2)} - \frac{y_{10}^{(4)}}{\xi_2 - 1}\right) x_{10}^{(4)} + z_{10}^{(2)} y_{10}^{(4)} (x_{10}^{(4)})^2.$$

Here we notice that this coordinate system is not symplectic and the form of the differential system is very complicated. Therefore we proceed to finding another good coordinate system. For this purpose, let us calculate the 2-form $dq_1 \wedge dp_1 + dq_2 \wedge dp_2$ in the coordinates $(\xi_2, z_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)})$:

$$\begin{aligned} & dq_1 \wedge dp_1 + dq_2 \wedge dp_2 \\ = \ & dx_{10}^{(4)} \wedge dy_{10}^{(4)} + \{-\eta(\xi_2 - 1) + x_{10}^{(4)}(\alpha_1 + x_{10}^{(4)} y_{10}^{(4)})\} d\xi_2 \wedge dz_{10}^{(2)} \\ & + \left(\alpha_1 z_{10}^{(2)} + 2 z_{10}^{(2)} x_{10}^{(4)} y_{10}^{(4)} - \frac{y_{10}^{(4)}}{\xi_2 - 1}\right) d\xi_2 \wedge dx_{10}^{(4)} \\ & + \left(z_{10}^{(2)} (x_{10}^{(4)})^2 - \frac{x_{10}^{(4)}}{\xi_2 - 1}\right) d\xi_2 \wedge dy_{10}^{(4)} \end{aligned}$$

$$
\begin{aligned}
&= dx_{10}^{(4)} \wedge dy_{10}^{(4)} \\
&\quad + d\xi_2 \wedge d\Big\{\{-\eta(\xi_2-1) + x_{10}^{(4)}(\alpha_1 + x_{10}^{(4)}y_{10}^{(4)})\}z_{10}^{(2)} - \frac{x_{10}^{(4)}y_{10}^{(4)}}{\xi_2-1}\Big\} \\
&= dx_{10}^{(4)} \wedge dy_{10}^{(4)} \\
&\quad + d\xi_2 \wedge d\Big\{\{-\eta(\xi_2-1) + x_{10}^{(4)}(\alpha_1 + x_{10}^{(4)}y_{10}^{(4)})\}z_{10}^{(2)} - \frac{x_{10}^{(4)}y_{10}^{(4)}}{\xi_2-1} - \frac{\alpha_1}{\xi_2-1}\Big\}.
\end{aligned}
$$

Therefore, setting

$$
w_{10}^{(2)} = \{-\eta(\xi_2-1) + x_{10}^{(4)}(\alpha_1 + x_{10}^{(4)}y_{10}^{(4)})\}z_{10}^{(2)} - \frac{x_{10}^{(4)}y_{10}^{(4)}}{\xi_2-1} - \frac{\alpha_1}{\xi_2-1},
$$

we have symplectic coordinates $(\xi_2, w_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)})$. Futhermore, in this coordinate system, we can verify that the our system is written as

$$
\begin{aligned}
s_1^2 s_2(s_2-1)d\xi_2 - \sum_{i=1,2} P_{1i}(\xi_2, w_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}, s)ds_i = 0,\\
s_1^2 s_2(s_2-1)dw_{10}^{(2)} - \sum_{i=1,2} Q_{1i}(\xi_2, w_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}, s)ds_i = 0,\\
s_1^2 s_2(s_2-1)dx_{10}^{(4)} - \sum_{i=1,2} X_{1i}(\xi_2, w_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}, s)ds_i = 0,\\
s_1^2 s_2(s_2-1)dy_{10}^{(4)} - \sum_{i=1,2} Y_{1i}(\xi_2, w_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}, s)ds_i = 0,
\end{aligned}
$$

where $P_{1i}, Q_{1i}, X_{1i}, Y_{1i}$ are certain polynomials of $\xi_2, w_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}$ and s. This means that the foliation has no singular points in $(\xi_2, w_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}, s)$-space $\mathbb{C}^4 \times B$ and every leaf in the space is transversal with fibers. Remark that the system has no singularity on $\xi_2 = 1$. We write this affine symplectic coordinate system as $(q_1^{11}, q_2^{11}, p_1^{11}, p_2^{11})$, namely

$$
q_1^{11} = x_{10}^{(4)}, \quad q_2^{11} = \xi_2, \quad p_1^{11} = y_{10}^{(4)}, \quad p_2^{11} = w_{10}^{(2)}.
$$

The relation between it and the original coordinate system is given by

$$
q_1 = q_1^{11}, \quad q_2 = q_2^{11}, \quad p_1 = -\frac{\eta(q_2^{11}-1)}{(q_1^{11})^2} + \frac{\alpha_1}{q_1^{11}} + p_1^{11}, \quad p_2 = \frac{\eta}{q_1^{11}} + p_2^{11}. \tag{5.7}
$$

5.5.2 Coordinate system for $A_1(s) \cap W_2$

The first quadratic transformation along $A_1(s) \cap W_2$. Note that $A_1(s) \cap W_2 \subset W_{21}$ and

$$
A_1(s) \cap W_2 = \{(\xi_0, \xi_1, \eta_{20}, \eta_{22}) \in W_{21} \simeq \mathbb{C}^4 \mid \xi_1 \in \mathbb{C}, \xi_0 = \eta_{20} = \eta_{22} = 0\}.
$$

We replace every point $(\xi_0, \xi_1, \eta_{20}, \eta_{22}) = (\xi_0, 0, 0, 0)$ with $\xi_0 \in \mathbb{C}$ by $\mathbb{P}^2$ simultaneously. Let $(\xi_0, X_{10}^{(1)}, Y_{10}^{(1)}, Z_{10}^{(1)}) \in \mathbb{C}^4$, $(\xi_0, X_{11}^{(1)}, Y_{11}^{(1)}, Z_{11}^{(1)}) \in \mathbb{C}^4$ and $(\xi_0, X_{12}^{(1)}, Y_{12}^{(1)}, Z_{12}^{(1)}) \in \mathbb{C}^4$ be coordinate systems of $V_{12}^{(1)}(s) = Q_{A_1(s)\cap W_2}(W_{21} \times s)$ defined by

$$
\begin{aligned}
\xi_1 = X_{10}^{(1)}, \quad \eta_{20} = X_{10}^{(1)}Y_{10}^{(1)}, \quad \eta_{22} = X_{10}^{(1)}Z_{10}^{(1)},\\
\xi_1 = X_{11}^{(1)}Y_{11}^{(1)}, \quad \eta_{20} = Y_{11}^{(1)}, \quad \eta_{22} = Y_{11}^{(1)}Z_{11}^{(1)},\\
\xi_1 = X_{12}^{(1)}Z_{12}^{(1)}, \quad \eta_{20} = Y_{12}^{(1)}Z_{12}^{(1)}, \quad \eta_{22} = Z_{12}^{(1)},
\end{aligned}
$$

then the exceptional divisor $D_{12}^{(1)}(s)=Q_{A_1(s)\cap W_2}(A_1(s)\cap W_2)$ is given by

$$\{X_{10}^{(1)}=0\}\cup\{Y_{11}^{(1)}=0\}\cup\{Z_{12}^{(1)}=0\},$$

the set of accessible singular points $A_{12}^{(1)}(s)$ is given by

$$A_{12}^{(1)}(s)=\{(\xi_0,X_{10}^{(1)},Y_{10}^{(1)},Z_{10}^{(1)})=(\xi_0,0,0,1/(\xi_0-1)\}\subset D_{12}^{(1)}(s).$$

The second quadratic transformation along $A_{12}^{(1)}(s)$. We next replace the points $(\xi_0,X_{10}^{(1)},Y_{10}^{(1)},Z_{10}^{(1)})=(\xi_0,0,0,1/(\xi_0-1))$ with $\xi_0\in\mathbb{C}\setminus\{\xi_0=1\}$ by $\mathbb{P}^2$ simultaneously. Let $(X_{10}^{(2)},Y_{10}^{(2)},Z_{10}^{(2)})\in\mathbb{C}^3$, $(X_{11}^{(2)},Y_{11}^{(2)},Z_{11}^{(2)})\in\mathbb{C}^3$ and $(X_{12}^{(2)},Y_{12}^{(2)},Z_{12}^{(2)})\in\mathbb{C}^3$ be coordinate systems of $V_{12}^{(2)}(s)=Q_{A_{12}^{(1)}(s)}(V_{12}^{(1)}(s))$ defined by

$$\begin{aligned}
X_{10}^{(1)}&=X_{10}^{(2)},\quad Y_{10}^{(1)}=X_{10}^{(2)}Y_{10}^{(2)},\quad Z_{10}^{(1)}=1/(\xi_0-1)+X_{10}^{(2)}Z_{10}^{(2)},\\
X_{10}^{(1)}&=X_{11}^{(2)}Y_{11}^{(2)},\quad Y_{10}^{(1)}=Y_{11}^{(2)},\quad Z_{10}^{(1)}=1/(\xi_0-1)+Y_{11}^{(2)}Z_{11}^{(2)},\\
X_{10}^{(1)}&=X_{12}^{(2)}Z_{12}^{(2)},\quad Y_{10}^{(1)}=Y_{12}^{(2)}Z_{12}^{(2)},\quad Z_{10}^{(1)}=1/(\xi_0-1)+Z_{12}^{(2)},
\end{aligned}$$

then the exceptional divisor $D_{12}^{(2)}(s)=Q_{A_{12}^{(1)}(s)}(A_{12}^{(1)}(s))$ is given by

$$\{X_{10}^{(2)}=0\}\cup\{Y_{11}^{(2)}=0\}\cup\{Z_{12}^{(2)}=0\},$$

the set of accessible singular points $A_{12}^{(2)}(s)$ is given by

$$A_{12}^{(2)}(s)=\{(\xi_0,X_{10}^{(2)},Y_{10}^{(2)},Z_{10}^{(2)})=(\xi_0,0,1/(\eta(\xi_0-1),Z_{10}^{(2)})\}\subset D_{12}^{(2)}(s).$$

The third quadratic transformation along $A_{12}^{(2)}(s)$. Here we insert a change of variables

$$\xi_0=\xi_0,\quad X_{10}^{(2)}=X_{10}^{(2)},\quad Y_{10}^{(2)}=1/V_{10}^{(2)},\quad Z_{10}^{(1)}=Z_{10}^{(1)}$$

namely, a change of local coordinates of a neighborhood of the set $A_{12}^{(2)}(s)$. The change of variables is necessary for obtaing a symplectic coordinate system.

We next replace the points $(\xi_0,Z_{10}^{(2)},X_{10}^{(2)},V_{10}^{(2)})=(\xi_0,Z_{10}^{(2)},0,\eta(\xi_0-1))$ with $(\xi_0,Z_{10}^{(2)})\in\mathbb{C}^2\setminus\{\xi_0=1\}$ by $\mathbb{P}^1$ simultaneously. Let $(\xi_0,Z_{10}^{(2)},X_{10}^{(3)},Y_{10}^{(3)})\in\mathbb{C}^4$ and $(\xi_0,Z_{10}^{(2)},X_{11}^{(3)},Y_{11}^{(3)})\in\mathbb{C}^4$ be coordinate systems of $V_{12}^{(3)}(s)=Q_{A_{12}^{(2)}(s)}(V_{12}^{(2)}(s))$ defined by

$$\begin{aligned}
X_{10}^{(2)}&=X_{10}^{(3)},\quad V_{10}^{(2)}=\eta(\xi_0-1)+X_{10}^{(3)}Y_{10}^{(3)},\\
X_{10}^{(2)}&=X_{11}^{(3)}Y_{11}^{(3)},\quad V_{10}^{(2)}=\eta(\xi_0-1)+Y_{11}^{(3)},
\end{aligned}$$

then the exceptional divisor $D_{12}^{(3)}(s)=Q_{A_{12}^{(2)}(s)}(A_{12}^{(2)}(s))$ is given by

$$\{X_{10}^{(3)}=0\}\cup\{Y_{11}^{(3)}=0\},$$

the set of accessible singular points $A_{12}^{(3)}(s)$ is given by

$$A_{12}^{(3)}(s)=\{(\xi_0,Z_{10}^{(2)},X_{10}^{(3)},Y_{10}^{(3)})=(\xi_0,Z_{10}^{(2)},0,\alpha_1)\}\subset D_{12}^{(3)}(s).$$

The fourth quadratic transformation along $A^{(3)}_{12}(s)$. We next replace the points $(\xi_0, Z^{(2)}_{10}, X^{(3)}_{10}, Y^{(3)}_{10}) = (\xi_0, Z^{(2)}_{10}, 0, \alpha_1)$ with $(\xi_0, Z^{(2)}_{10}) \in \mathbb{C}^2 \setminus \{\xi_0 = 1\}$ by $\mathbb{P}^1$ simultaneously. Let $(\xi_2, Z^{(2)}_{10}, X^{(4)}_{10}, Y^{(4)}_{10}) \in \mathbb{C}^4$ and $(\xi_2, Z^{(2)}_{10}, X^{(4)}_{11}, Y^{(4)}_{11}) \in \mathbb{C}^4$ be coordinate systems of $V^{(4)}_{12}(s) = Q_{A^{(3)}_{12}(s)}(V^{(3)}_{12}(s))$ defined by

$$X^{(3)}_{10} = X^{(4)}_{10}, \quad Y^{(3)}_{10} = \alpha_1 + X^{(4)}_{10} Y^{(4)}_{10},$$
$$X^{(3)}_{10} = X^{(4)}_{11} Y^{(4)}_{11}, \quad Y^{(3)}_{10} = \alpha_1 + Y^{(4)}_{11},$$

then the exceptional divisor $D^{(4)}_{12}(s) = Q_{A^{(3)}_{12}(s)}(A^{(3)}_{12}(s))$ is given by

$$\{X^{(4)}_{10} = 0\} \cup \{Y^{(4)}_{11} = 0\}.$$

We see that, in the $(\xi_0, Z^{(2)}_{10}, X^{(4)}_{10}, Y^{(4)}_{10}, s)$-space $\mathbb{C}^4 \times B$, the differential system is holomorphic in a neighborhood of $\{X^{(4)}_{10} = 0\}$ except for $\xi_0 = 1$, moreover, the points $(\xi_0, Z^{(2)}_{10}, X^{(4)}_{11}, Y^{(4)}_{11}) = (\xi_0, Z^{(2)}_{10}, 0, 0)$ are inaccessible with $\xi_0 \neq 1$.

Thus we have obtained a coordinate system $(\xi_0, Z^{(2)}_{10}, X^{(4)}_{10}, Y^{(4)}_{10}) \in \mathbb{C}^4$ which is separates the solutions passing through $A_1(s) \cap W_2 = A_1(s) \cap W_{21}$ with $\xi_0 \neq 1$. It is related to the coordinate system $(\xi_0, \xi_1, \eta_{20}, \eta_{22}) \in W_{21}$ by

$$\xi_0 = \xi_0, \quad \xi_1 = X^{(4)}_{10}, \quad \eta_{20} = \frac{(X^{(4)}_{10})^2}{X^{(4)}_{10}(\alpha_1 + X^{(4)}_{10} Y^{(4)}_{10}) + \eta(\xi_0 - 1)}, \quad \eta_{22} = X^{(4)}_{10}\Big(X^{(4)}_{10} Z^{(2)}_{10} - \frac{1}{\xi_0 - 1}\Big)$$

and then

$$q^2_1 = X^{(4)}_{10}, \quad q^2_2 = \xi_0, \quad q^2_1 = \frac{\eta(\xi_0 - 1)}{(x^{(4)}_{10})^2} + \frac{\alpha_1}{X^{(4)}_{10}} + Y^{(4)}_{10},$$
$$q^2_2 = -\frac{\eta}{X^{(4)}_{10}} + \eta(\xi_0 - 1) Z^{(2)}_{10} - \frac{\alpha_1}{\xi_0 - 1} + \Big(\alpha_1 Z^{(2)}_{10} - \frac{Y^{(4)}_{10}}{\xi_0 - 1}\Big) X^{(4)}_{10} + Z^{(2)}_{10} Y^{(4)}_{10} (X^{(4)}_{10})^2.$$

Here we notice that this coordinate system is not symplectic and the form of the differential system is very complicated. Therefore we proceed to finding another good coordinate system. For this purpose, let us calculate the 2-form $dq^2_1 \wedge dp^2_1 + dq^2_2 \wedge dp^2_2$ $(= dq_1 \wedge dp_1 + dq_2 \wedge dp_2)$ in the coordinates $(\xi_2, Z^{(2)}_{10}, X^{(4)}_{10}, Y^{(4)}_{10})$:

$$\begin{aligned}
& dq^2_1 \wedge dp^2_1 + dq^2_2 \wedge dp^2_2 \\
= \;& dX^{(4)}_{10} \wedge dY^{(4)}_{10} + \{\eta(\xi_0 - 1) + X^{(4)}_{10}(\alpha_1 + X^{(4)}_{10} Y^{(4)}_{10})\} d\xi_0 \wedge dZ^{(2)}_{10} \\
& + \Big(\alpha_1 Z^{(2)}_{10} + 2 Z^{(2)}_{10} X^{(4)}_{10} Y^{(4)}_{10} - \frac{Y^{(4)}_{10}}{\xi_0 - 1}\Big) d\xi_0 \wedge dX^{(4)}_{10} \\
& + \Big(Z^{(2)}_{10} (X^{(4)}_{10})^2 - \frac{X^{(4)}_{10}}{\xi_0 - 1}\Big) d\xi_0 \wedge dY^{(4)}_{10} \\
= \;& dX^{(4)}_{10} \wedge dY^{(4)}_{10} \\
& + d\xi_0 \wedge d\Big\{\{\eta(\xi_0 - 1) + X^{(4)}_{10}(\alpha_1 + X^{(4)}_{10} Y^{(4)}_{10})\} Z^{(2)}_{10} - \frac{X^{(4)}_{10} Y^{(4)}_{10}}{\xi_0 - 1}\Big\} \\
= \;& dX^{(4)}_{10} \wedge dY^{(4)}_{10} \\
& + d\xi_0 \wedge d\Big\{\{\eta(\xi_0 - 1) + X^{(4)}_{10}(\alpha_1 + X^{(4)}_{10} Y^{(4)}_{10})\} Z^{(2)}_{10} - \frac{X^{(4)}_{10} Y^{(4)}_{10}}{\xi_0 - 1} - \frac{\alpha_1}{\xi_0 - 1}\Big\}.
\end{aligned}$$

Therefore, setting

$$W_{10}^{(2)} = \{\eta(\xi_0 - 1) + X_{10}^{(4)}(\alpha_1 + X_{10}^{(4)}Y_{10}^{(4)})\}Z_{10}^{(2)} - \frac{X_{10}^{(4)}Y_{10}^{(4)}}{\xi_0 - 1} - \frac{\alpha_1}{\xi_0 - 1},$$

we have symplectic coordinates $(\xi_0, W_{10}^{(2)}, X_{10}^{(4)}, Y_{10}^{(4)})$. Writing

$$q_1^{12} = X_{10}^{(4)}, \quad q_2^{12} = \xi_0, \quad p_1^{12} = Y_{10}^{(4)}, \quad p_2^{12} = W_{10}^{(2)},$$

we have

$$q_1^2 = q_1^{12}, \quad q_2^2 = q_2^{12}, \quad q_1^2 = \frac{\eta(q_2^{12} - 1)}{(q_1^{12})^2} + \frac{\alpha_1}{q_1^{12}} + p_1^{12}, \quad q_2^2 = -\frac{\eta}{q_1^{12}} + p_2^{12}. \tag{5.8}$$

The system $(q^{12}, p^{12}) \in \mathbb{C}^4$ separates solution curves passing through $A_1 \cap W_2$ and the Hamiltonians have no singularity on $\xi_0 = 1$.

Thus we have obtained eight symplectic coordinate systems $(q_1^*, q_2^*, p_1^*, p_2^*)$ each of which separates solution curves passing through the accessible singular points (see (5.1)-(5.8)). We notice that the Hamiltonians of each coordinate system are also polynomials whose coefficients are rational functions of s holomorphic in B.

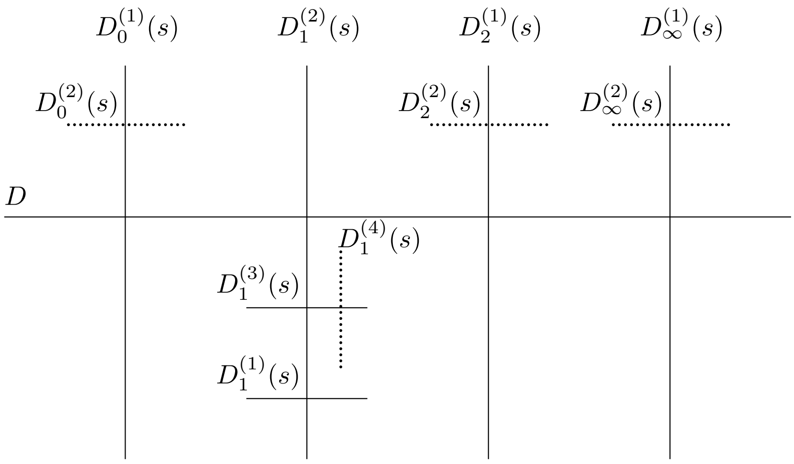

Figure 2. J=1112

6 Spaces of initial conditions for $\mathcal{H}_{113}$

In the present case,

$$\nu = -\frac{1}{2}(\alpha_0 + \alpha_1 - 1 + \alpha_\infty).$$

In this section, we omitt the label 113.

6.1 Accessible singularities on $D \times B$

Observing the system $\mathcal{H}^{(0)}$ on all W_{ij} $j \neq 0$, we can obtain

Proposition 6.1. *The set of accessible singular points of the system $\mathcal{H}_{113}^{(0)}$ for each $s = (s_1, s_2) \in B_{113}$ is a disjoint union of three connected components $A_0(s), A_1(s), A_\infty(s) \simeq \mathbb{P}^1$ given by*

$$\begin{aligned}
A_0(s) &= \{(\xi, \eta_0, s) \in W_0 \times B | 2\xi_0 + (2s_1 + s_2^2)\xi_1 + 2s_2\xi_2 = 0, \eta_{00} = 0, 2s_2\eta_{01} - (2s_1 + s_2^2)\eta_{02} = 0\} \\
&\cup \{(\xi, \eta_1, s) \in W_1 \times B | 2\xi_0 + (2s_1 + s_2^2)\xi_1 + 2s_2\xi_2 = 0, \eta_{10} = 0, s_2\eta_{11} - \eta_{12} = 0\} \\
&\cup \{(\xi, \eta_2, s) \in W_2 \times B | 2\xi_0 + (2s_1 + s_2^2)\xi_1 + 2s_2\xi_2 = 0, \eta_{20} = 0, 2\eta_{21} - (2s_1 + s_2^2)\eta_{22} = 0\}, \\
A_1(s) &= \{(\xi, \eta_0, s) \in W_0 \times B | \xi_1 = \eta_{00} = \eta_{02} = 0\} \cup \{(\xi, \eta_2, s) \in W_2 \times B | \xi_1 = \eta_{20} = \eta_{22} = 0\}, \\
A_\infty(s) &= \{(\xi, \eta_1, s) \in W_1 \times B | \xi_0 = \eta_{10} = \eta_{12} = 0\} \cup \{(\xi, \eta_2, s) \in W_2 \times B | \xi_0 = \eta_{20} = \eta_{21} = 0\}.
\end{aligned}$$

Moreover, we can verify

Proposition 6.2. *The Bäcklund transformation group S_2 acts on the components of the accessible singular points of $\mathcal{H}_{113}$ according to the following diagram*

	A_0	A_∞
σ	A_∞	A_0

where $A_1(s)$ is invariant under the action of this group.

In the following subsections, we obtain coordinate system corresponding to $A_i(s)$ which separate completely the solution curves passing through $A_i(s)$. The systems for all $A_i(s)$ are obtained by quadratic transformations. Notice that we can obtain the coordinate systems for $A_0(s)$(or $A_\infty(s)$) from those for $A_\infty(s)$(or $A_0(s)$) and Bäcklund transformation σ. But the coordinate system obtained by the procedure is not good. Therefore we perform the quadratic transformation to obtain good coordinate system for $A_0(s)$ and $A_\infty(s)$.

6.2 Coordinate systems for $A_0(s)$

We make successively the quadratic transformations along $A_0(s) \cap W_0$, $A_0(s) \cap W_1$ and find coordinate systems for $A_0(s)$. Note that although $A_0(s)$ is expressed by the three coordinate systems W_0, W_1 and W_2, it can be done by two of them. In this paper, we choose W_0 and W_1.

6.2.1 Coordinate system for $A_0(s) \cap W_0$

We choose the coordinate system $W_{02} \subset W_0$. By setting $\xi_0 = \eta_{01} = 1$, we take $(\xi_1, \xi_2, \eta_{00}, \eta_{02})$ as the coordinates of W_{02}.

The first quadratic transformation along $A_0(s) \cap W_{02}$. Let

$$\begin{aligned}
&\xi_2 = -(s_1/s_2 + s_2/2)\xi_1 - 1/s_2 + x_{00}^{(1)}, \quad \eta_{00} = \xi_1 x_{00}^{(1)} y_{00}^{(1)}, \quad \eta_{01} = (2s_1 + s_2^2)/2s_2 + x_{00}^{(1)} z_{00}^{(1)}, \\
&\xi_2 = -(s_1/s_2 + s_2/2)\xi_1 - 1/s_2 + x_{01}^{(1)} y_{01}^{(1)}, \quad \eta_{00} = y_{01}^{(1)}, \quad \eta_{01} = (2s_1 + s_2^2)/2s_2 + y_{01}^{(1)} z_{01}^{(1)}, \\
&\xi_2 = -(s_1/s_2 + s_2/2)\xi_1 - 1/s_2 + x_{02}^{(1)} z_{02}^{(1)}, \quad \eta_{00} = y_{02}^{(1)} z_{02}^{(1)}, \quad \eta_{01} = (2s_1 + s_2^2)/2s_2 + z_{02}^{(1)},
\end{aligned}$$

then

$$D_{01}^{(1)}(s) = Q_{A_0(s) \cap W_{02}}(A_0(s) \cap (W_{02} \times B)) = \{x_{00}^{(1)} = 0\} \cup \{y_{01}^{(1)} = 0\} \cup \{z_{02}^{(1)} = 0\},$$

the set of accessible singular points is given by

$$A_{01}^{(1)}(s) = \{(\xi_2, x_{01}^{(1)}, y_{01}^{(1)}, z_{01}^{(1)}) = (\xi_2, \alpha_0, 0, z_{01}^{(1)})\} \subset D_{01}^{(1)}(s).$$

The second quadratic transformation with $A^{(1)}_{01}(s)$. Let

$$x^{(1)}_{01} = \alpha_0 + x^{(2)}_{00}, \quad y^{(1)}_{01} = x^{(2)}_{00} y^{(2)}_{00},$$
$$x^{(1)}_{01} = \alpha_0 + x^{(2)}_{01} y^{(2)}_{01}, \quad y^{(1)}_{01} = y^{(2)}_{01},$$

then

$$D^{(2)}_{01}(s) = Q_{A^{(1)}_{01}(s)}(A^{(1)}_{01}(s)) = \{x^{(2)}_{00} = 0\} \cup \{y^{(2)}_{01} = 0\}.$$

We can verify that our system has no singular points and every leaf is transversal with fibers in $(\xi_1, z^{(1)}_{01}, x^{(2)}_{01}, y^{(2)}_{01}, s)$-space $\mathbb{C}^4 \times B$. On the other hand, the points $(\xi_1, z^{(1)}_{01}, x^{(2)}_{00}, y^{(2)}_{00}) = (\xi_1, z^{(1)}_{01}, 0, 0)$ are inaccessible singular points.

Thus we have obtained a coordinate system $(\xi_1, z^{(1)}_{01}, x^{(2)}_{01}, y^{(2)}_{01}) \in \mathbb{C}^4$ which separates the solutions passing through $A_0(s) \cap W_{02}$. If we set

$$q^{01}_1 = \xi_1, \quad q^{01}_2 = -x^{(2)}_{01}, \quad p^{01}_1 = z^{(1)}_{01}, \quad p^{01}_2 = y^{(2)}_{01},$$

then we have

$$(6.1)\quad q_1 = q^{01}_1, \quad q_2 = p^{01}_2(\alpha_0 - q^{01}_2 p^{01}_2) - \Big(\frac{2s_1 + s_2^2}{2s_2}\Big) q^{01}_1 - \frac{1}{s_2}, \quad p_1 = \frac{2s_1 + s_2^2}{2s_2 p^{01}_2} + p^{01}_1, \quad p_2 = \frac{1}{p^{01}_2}.$$

Thus we have obtained a symplectic coordinate system $(q^{01}, p^{01}) \in \mathbb{C}^4$ for $A_0(s) \cap W_{02}$.

6.2.2 Coordinate system for $A_0(s) \cap W_1$

We choose the coordinate system $W_{12} \subset W_1$. By setting $\xi_1 = \eta_{12} = 1$, we take $(\xi_0, \xi_2, \eta_{10}, \eta_{11})$ as the coordinates of W_{12}.

The first quadratic transformation along $A_0(s) \cap W_{12}$. Let

$$\xi_2 = -\xi_0/s_2 - (2s_1 + s_2^2)/2s_2 + X^{(1)}_{00}, \quad \eta_{20} = X^{(1)}_{00} Y^{(1)}_{00}, \quad \eta_{21} = 1/s_2 + X^{(1)}_{00} Z^{(1)}_{00},$$
$$\xi_2 = -\xi_0/s_2 - (2s_1 + s_2^2)/2s_2 + X^{(1)}_{01} Y^{(1)}_{01}, \quad \eta_{20} = Y^{(1)}_{01}, \quad \eta_{21} = 1/s_2 + Y^{(1)}_{01} Z^{(1)}_{01},$$
$$\xi_2 = -\xi_0/s_2 - (2s_1 + s_2^2)/2s_2 + X^{(1)}_{02} Z^{(1)}_{02}, \quad \eta_{20} = Y^{(1)}_{02} Z^{(1)}_{02}, \quad \eta_{21} = 1/s_2 + Z^{(1)}_{02},$$

then

$$D^{(1)}_{02}(s) = Q_{A_0(s) \cap W_{12}}(A_0(s) \cap (W_{12} \times B)) = \{X^{(1)}_{00} = 0\} \cup \{Y^{(1)}_{01} = 0\} \cup \{Z^{(1)}_{02} = 0\},$$

the set of accessible singular points is given by

$$A^{(1)}_{02}(s) = \{(\xi_0, X^{(1)}_{01}, Y^{(1)}_{01}, Z^{(1)}_{01}) = (\xi_0, \alpha_0, 0, Z^{(1)}_{01})\} \subset D^{(1)}_{02}(s).$$

The second quadratic transformation along $A^{(1)}_{02}(s)$. Let

$$X^{(1)}_{01} = \alpha_0 + X^{(2)}_{00}, \quad Y^{(1)}_{01} = X^{(2)}_{00} Y^{(2)}_{00},$$
$$X^{(1)}_{01} = \alpha_0 + X^{(2)}_{01} Y^{(2)}_{01}, \quad Y^{(1)}_{01} = Y^{(2)}_{01},$$

then

$$D^{(2)}_{02}(s) = Q_{A^{(1)}_{02}(s)}(A^{(1)}_{02}(s)) = \{X^{(2)}_{00} = 0\} \cup \{Y^{(2)}_{01} = 0\}.$$

We see that, in the $(\xi_0, Z_{01}^{(1)}, X_{01}^{(2)}, Y_{01}^{(2)}, s)$-space $\mathbb{C}^4 \times B$, the Pfaffin system has no singular points, every leaf is transversal with fibers, moreover, the points $(\xi_0, Z_{01}^{(1)}, X_{00}^{(2)}, Y_{00}^{(2)}) = (\xi_0, Z_{01}^{(1)}, 0, 0)$ are inaccessible singular points.

Thus we have obtained a coordinate system $(\xi_1, Z_{01}^{(1)}, X_{01}^{(2)}, Y_{01}^{(2)}) \in \mathbb{C}^4$ which separates the solutions passing through $A_0(s) \cap W_{12}$. If we set

$$q_1^{02} = \xi_0, \quad q_2^{02} = -X_{01}^{(2)}, \quad p_1^{02} = Z_{01}^{(1)}, \quad p_2^{02} = Y_{01}^{(2)},$$

then we have

$$q_1^1 = q_1^{02}, \quad q_2^1 = p_2^{02}(\alpha_0 - p_2^{02} q_2^{02}) - \frac{q_1^{02}}{s_2} - \frac{2s_1 + s_2^2}{2s_2}, \quad p_1^1 = \frac{1}{s_2 p_2^{02}} + p_1^{02}, \quad p_2^1 = \frac{1}{p_2^{02}}. \tag{6.2}$$

The system $(q^{02}, p^{02}) \in \mathbb{C}^4$ is a symplectic coordinate system for $A_0(s) \cap W_{12}$.

6.3 Coordinate systems for $A_\infty(s)$

We make successively the quadratic transformations along $A_\infty(s) \cap W_1$, $A_\infty(s) \cap W_2$ and find coordinate systems for $A_\infty(s)$.

6.3.1 Coordinate system for $A_\infty(s) \cap W_1$

The first quadratic transformation along $A_\infty(s) \cap W_1$. Let

$$\begin{aligned}
\xi_0 &= x_{\infty 0}^{(1)}, \quad \eta_{10} = x_{\infty 0}^{(1)} y_{\infty 0}^{(1)}, \quad \eta_{12} = x_{\infty 0}^{(1)} z_{\infty 0}^{(1)}, \\
\xi_0 &= x_{\infty 1}^{(1)} y_{\infty 1}^{(1)}, \quad \eta_{10} = y_{\infty 1}^{(1)}, \quad \eta_{12} = y_{\infty 1}^{(1)} z_{\infty 1}^{(1)}, \\
\xi_0 &= x_{\infty 2}^{(1)} z_{\infty 2}^{(1)}, \quad \eta_{10} = y_{\infty 2}^{(1)} z_{\infty 2}^{(1)}, \quad \eta_{12} = z_{\infty 2}^{(1)},
\end{aligned}$$

then

$$D_{\infty 1}^{(1)}(s) = Q_{A_\infty(s) \cap W_1}(A_\infty(s) \cap W_1) = \{x_{\infty 0}^{(1)} = 0\} \cup \{y_{\infty 1}^{(1)} = 0\} \cup \{z_{\infty 2}^{(1)} = 0\},$$

the set of accessible singular points is given by

$$A_{\infty 1}^{(1)}(s) = \{(\xi_2, x_{\infty 1}^{(1)}, y_{\infty 1}^{(1)}, z_{\infty 1}^{(1)}) = (\xi_2, \alpha_\infty, 0, z_{\infty 1}^{(1)})\} \subset D_{\infty 1}^{(1)}(s).$$

The second quadratic transformation along $A_{\infty 1}^{(1)}(s)$. Let

$$\begin{aligned}
x_{\infty 1}^{(1)} &= \alpha_\infty + x_{\infty 0}^{(2)}, \quad y_{\infty 1}^{(1)} = x_{\infty 0}^{(2)} y_{\infty 0}^{(2)}, \\
x_{\infty 1}^{(1)} &= \alpha_\infty + x_{\infty 1}^{(2)} y_{\infty 1}^{(2)}, \quad y_{\infty 1}^{(1)} = y_{\infty 1}^{(2)},
\end{aligned}$$

then

$$D_{\infty 1}^{(2)}(s) = Q_{A_{\infty 1}^{(1)}(s)}(A_{\infty 1}^{(1)}(s)) = \{x_{\infty 0}^{(2)} = 0\} \cup \{y_{\infty 1}^{(2)} = 0\},$$

We can verify that our system has no singularity, every leaf is transversal with the fibers in $(\xi_2, z_{\infty 1}^{(1)}, x_{\infty 1}^{(2)}, y_{\infty 1}^{(2)})$-space $\mathbb{C}^4 \times B$. On the other hand, the points $(\xi_2, z_{\infty 1}^{(1)}, x_{\infty 0}^{(2)}, y_{\infty 0}^{(2)}) = (\xi_2, z_{\infty 1}^{(1)}, 0, 0)$ are inaccessible singular points.

Thus we have obtained a coordinate system $(\xi_2, z_{\infty 1}^{(1)}, x_{\infty 1}^{(2)}, y_{\infty 1}^{(2)}) \in \mathbb{C}^4$ which separates the solutions passing through $A_\infty(s) \cap W_1 = A_\infty(s) \cap W_{11}$. If we set

$$q_1^{\infty 1} = -x_{\infty 1}^{(2)}, \quad q_2^{\infty 1} = \xi_2, \quad p_1^{\infty 1} = y_{\infty 1}^{(2)}, \quad p_2^{\infty 1} = z_{\infty 1}^{(1)},$$

then we have

$$q_1^1 = p_1^{\infty 1}(\alpha_\infty - q_1^{\infty 1} p_1^{\infty 1}), \quad q_2^1 = q_2^{\infty 1}, \quad p_1^1 = \frac{1}{p_1^{\infty 1}}, \quad p_2^1 = p_2^{\infty 1}. \tag{6.3}$$

Thus we have obtained a symplectic coordinate system $(q^{\infty 1}, p^{\infty 1}) \in \mathbb{C}^4$ for $A_\infty(s) \cap W_1$.

6.3.2 Coordinate system for $A_\infty(s) \cap W_2$

The first quadratic transformation along $A_\infty(s) \cap W_2$. Let

$$\begin{aligned}
\xi_0 &= X_{\infty 0}^{(1)}, \quad \eta_{20} = X_{\infty 0}^{(1)} Y_{\infty 0}^{(1)}, \quad \eta_{21} = X_{\infty 0}^{(1)} Z_{\infty 0}^{(1)}, \\
\xi_0 &= X_{\infty 1}^{(1)} Y_{\infty 1}^{(1)}, \quad \eta_{20} = Y_{\infty 1}^{(1)}, \quad \eta_{21} = Y_{\infty 1}^{(1)} Z_{\infty 1}^{(1)}, \\
\xi_0 &= X_{\infty 2}^{(1)} Z_{\infty 2}^{(1)}, \quad \eta_{20} = Y_{\infty 2}^{(1)} Z_{\infty 2}^{(1)}, \quad \eta_{21} = Z_{\infty 2}^{(1)},
\end{aligned}$$

then

$$D_{\infty 2}^{(1)}(s) = Q_{A_\infty(s) \cap W_2}(A_\infty(s) \cap W_2) = \{X_{\infty 0}^{(1)} = 0\} \cup \{Y_{\infty 1}^{(1)} = 0\} \cup \{Z_{\infty 2}^{(1)} = 0\},$$

the set of accessible singular points is given by

$$A_{\infty 2}^{(1)}(s) = \{(\xi_1, X_{\infty 1}^{(1)}, Y_{\infty 1}^{(1)}, Z_{\infty 1}^{(1)}) = (\xi_1, \alpha_\infty, 0, Z_{\infty 1}^{(1)})\} \subset D_{\infty 2}^{(1)}(s).$$

The second quadratic transformation along $A_{\infty 2}^{(1)}(s)$. Let

$$\begin{aligned}
X_{\infty 1}^{(1)} &= \alpha_\infty + X_{\infty 0}^{(2)}, \quad Y_{\infty 1}^{(1)} = X_{\infty 0}^{(2)} Y_{\infty 0}^{(2)}, \\
X_{\infty 1}^{(1)} &= \alpha_\infty + X_{\infty 1}^{(2)} Y_{\infty 1}^{(2)}, \quad Y_{\infty 1}^{(1)} = Y_{\infty 1}^{(2)},
\end{aligned}$$

then

$$D_{\infty 2}^{(2)}(s) = Q_{A_{\infty 2}^{(1)}(s)}(A_{\infty 2}^{(1)}(s)) = \{X_{\infty 0}^{(2)} = 0\} \cup \{Y_{\infty 1}^{(2)} = 0\}.$$

We see that, in the $(\xi_1, Z_{\infty 1}^{(1)}, X_{\infty 1}^{(2)}, Y_{\infty 1}^{(2)})$-space $\mathbb{C}^4 \times B$, the Pfaffian system has no singularity, every leaf is transversal with the fibers, moreover, the points $(\xi_1, Z_{\infty 1}^{(1)}, X_{\infty 0}^{(2)}, Y_{\infty 0}^{(2)}) = (\xi_1, Z_{\infty 1}^{(1)}, 0, 0)$ are inaccessible singular points.

Thus we have obtained a coordinate system $(\xi_1, Z_{\infty 1}^{(1)}, X_{\infty 1}^{(2)}, Y_{\infty 1}^{(2)}) \in \mathbb{C}^4$ which separates the solutions passing through $A_\infty(s) \cap W_2 = A_\infty(s) \cap W_{22}$. If we set

$$q_1^{\infty 2} = \xi_1, \quad q_2^{\infty 2} = -X_{\infty 1}^{(2)}, \quad p_1^{\infty 2} = Z_{\infty 1}^{(1)}, \quad p_2^{\infty 2} = Y_{\infty 1}^{(2)},$$

then we have

$$q_1^2 = q_1^{\infty 2}, \quad q_2^2 = p_2^{\infty 2}(\alpha_\infty - q_2^{\infty 2} p_2^{\infty 2}), \quad p_1^2 = p_1^{\infty 2}, \quad p_2^2 = \frac{1}{p_2^{\infty 2}}. \tag{6.4}$$

The system $(q^{\infty 2}, p^{\infty 2}) \in \mathbb{C}^4$ is a symplectic coordinate system for $A_\infty(s) \cap W_2$.

6.4 Coordinate systems for $A_1(s)$

We obtain coordinate systems for $A_1(s)$ by making quadratic transformations six times along $A_1(s) \cap W_0$, $A_1(s) \cap W_2$.

6.4.1 Coordinate system for $A_1(s) \cap W_0$

The first quadratic transformation along $A_1(s) \cap W_0$. Let

$$\xi_1 = x_{10}^{(1)}, \quad \eta_{00} = x_{10}^{(1)} y_{10}^{(1)}, \quad \eta_{02} = x_{10}^{(1)} z_{10}^{(1)},$$
$$\xi_1 = x_{11}^{(1)} y_{11}^{(1)}, \quad \eta_{00} = y_{11}^{(1)}, \quad \eta_{02} = y_{11}^{(1)} z_{11}^{(1)},$$
$$\xi_1 = x_{12}^{(1)} z_{12}^{(1)}, \quad \eta_{00} = y_{12}^{(1)} z_{12}^{(1)}, \quad \eta_{02} = z_{12}^{(1)},$$

then

$$D_{11}^{(1)}(s) = Q_{A_1(s)\cap W_0}(A_1(s) \cap W_0) = \{x_{10}^{(1)} = 0\} \cup \{y_{11}^{(1)} = 0\} \cup \{z_{12}^{(1)} = 0\},$$

the set of accessible singular points is given by

$$A_{11}^{(1)}(s) = \{(\xi_2, x_{10}^{(1)}, y_{10}^{(1)}, z_{10}^{(1)}) = (\xi_2, 0, 0, -1/\xi_2)\} \subset D_{11}^{(1)}(s).$$

The second quadratic transformation along $A_{11}^{(1)}(s)$. Note that $\xi_2 = 0$ is excluded. Let

$$x_{10}^{(1)} = x_{10}^{(2)}, \quad y_{10}^{(1)} = x_{10}^{(2)} y_{10}^{(2)}, \quad z_{10}^{(1)} = -1/\xi_2 + x_{10}^{(2)} z_{10}^{(2)},$$
$$x_{10}^{(1)} = x_{11}^{(2)} y_{11}^{(2)}, \quad y_{10}^{(1)} = y_{11}^{(2)}, \quad z_{10}^{(1)} = -1/\xi_2 + y_{11}^{(2)} z_{11}^{(2)},$$
$$x_{10}^{(1)} = x_{12}^{(1)} z_{12}^{(2)}, \quad y_{10}^{(1)} = x_{12}^{(2)} y_{12}^{(2)}, \quad z_{10}^{(1)} = -1/\xi_2 + z_{12}^{(2)},$$

then

$$D_{11}^{(2)}(s) = Q_{A_{11}^{(1)}(s)}(A_{11}^{(1)}(s)) = \{x_{10}^{(2)} = 0\} \cup \{y_{11}^{(2)} = 0\} \cup \{z_{12}^{(2)} = 0\},$$

the set of accessible singular points is given by

$$A_{11}^{(2)}(s) = \{(\xi_2, x_{10}^{(2)}, y_{10}^{(2)}, z_{10}^{(2)}) = (\xi_2, 0, 0, -1/\xi_2^3)\} \subset D_{11}^{(2)}(s).$$

The third quadratic transformation along $A_{11}^{(2)}(s)$. Let

$$x_{10}^{(2)} = x_{10}^{(3)}, \quad y_{10}^{(2)} = x_{10}^{(3)} y_{10}^{(3)}, \quad z_{10}^{(2)} = -1/\xi_2^3 + x_{10}^{(3)} z_{10}^{(3)},$$
$$x_{10}^{(2)} = x_{11}^{(3)} y_{11}^{(3)}, \quad y_{10}^{(2)} = y_{11}^{(3)}, \quad z_{10}^{(2)} = -1/\xi_2^3 + y_{11}^{(3)} z_{11}^{(3)},$$
$$x_{10}^{(2)} = x_{12}^{(3)} z_{12}^{(3)}, \quad y_{10}^{(2)} = y_{12}^{(3)} z_{12}^{(3)}, \quad z_{10}^{(2)} = -1/\xi_2^3 + z_{12}^{(3)},$$

then

$$D_{11}^{(3)}(s) = Q_{A_1^{(2)}(s)}(A_1^{(2)}(s)) = \{x_{10}^{(3)} = 0\} \cup \{y_{11}^{(3)} = 0\} \cup \{z_{12}^{(3)} = 0\},$$

the set of accessible singular points is given by

$$A_{11}^{(3)}(s) = \{(\xi_2, x_{10}^{(3)}, y_{10}^{(3)}, z_{10}^{(3)}) = (\xi_2, 0, -1/(\eta\xi_2^2), z_{10}^{(3)})\} \subset D_{11}^{(3)}(s).$$

The fourth quadratic transformation along $A_{11}^{(3)}(s)$. Here we insert a change of variables

$$\xi_2 = \xi_2, \quad x_{10}^{(3)} = x_{10}^{(3)}, \quad y_{10}^{(3)} = 1/v_{10}^{(3)}, \quad z_{10}^{(3)} = z_{10}^{(3)}.$$

Let

$$x_{10}^{(3)} = x_{10}^{(4)}, \quad v_{10}^{(3)} = -\eta\xi_2^2 + x_{10}^{(4)} y_{10}^{(4)},$$
$$x_{10}^{(3)} = x_{11}^{(4)} y_{11}^{(4)}, \quad v_{10}^{(3)} = -\eta\xi_2^2 + y_{11}^{(4)},$$

then

$$D_{11}^{(4)}(s) = Q_{A_1^{(3)}(s)}(A_1^{(3)}(s)) = \{x_{10}^{(4)} = 0\} \cup \{y_{11}^{(4)} = 0\},$$

the set of accessible singular points is given by

$$A_{11}^{(4)}(s) = \{(\xi_2, z_{10}^{(3)}, x_{10}^{(4)}, y_{10}^{(4)}) = (\xi_2, z_{10}^{(3)}, 0, \eta)\} \subset D_{11}^{(4)}(s).$$

The fifth quadratic transformation with $A_{11}^{(4)}(s)$. Let

$$x_{10}^{(4)} = x_{10}^{(5)}, \quad y_{10}^{(4)} = \eta + x_{10}^{(5)} y_{10}^{(5)},$$
$$x_{10}^{(4)} = x_{11}^{(5)} y_{11}^{(5)}, \quad y_{10}^{(4)} = \eta + y_{11}^{(5)},$$

then

$$D_{11}^{(5)}(s) = Q_{A_1^{(4)}(s)}(A_1^{(4)}(s)) = \{x_{10}^{(5)} = 0\} \cup \{y_{11}^{(5)} = 0\},$$

the set of accessible singular points is given by

$$A_{11}^{(5)}(s) = \{(\xi_2, z_{10}^{(3)}, x_{10}^{(5)}, y_{10}^{(5)}) = (\xi_2, z_{10}^{(3)}, 0, \alpha_1)\} \subset D_{11}^{(5)}(s).$$

The sixth quadratic transformation with $A_{11}^{(5)}(s)$. Let

$$x_{10}^{(5)} = x_{10}^{(6)}, \quad y_{10}^{(5)} = \alpha_1 + x_{10}^{(6)} y_{10}^{(6)},$$
$$x_{10}^{(5)} = x_{11}^{(6)} y_{11}^{(6)}, \quad y_{10}^{(5)} = \alpha_1 + y_{11}^{(6)},$$

then

$$D_{11}^{(6)}(s) = Q_{A_{11}^{(5)}(s)}(A_{11}^{(5)}(s)) = \{x_{10}^{(6)} = 0\} \cup \{y_{11}^{(6)} = 0\}.$$

We can verify that the differential system in the cordinates $(\xi_2, z_{10}^{(3)}, x_{10}^{(6)}, y_{10}^{(6)})$ is holomorphic in a neighborhood of $\{x_{10}^{(6)} = 0\}$ except for $\xi_0 = 0$ and the points $(\xi_2, z_{10}^{(3)}, x_{11}^{(6)}, y_{11}^{(6)}) = (\xi_2, z_{10}^{(3)}, 0, 0)$ are inaccessible with $\xi_0 \neq 0$.

Thus we have obtained a coordinate system $(\xi_2, z_{10}^{(3)}, x_{10}^{(6)}, y_{10}^{(6)}) \in \mathbb{C}^4$ which is separates the solutions passing through $A_1(s) \cap W_0 = A_1(s) \cap W_{01}$ with $\xi_2 \neq 0$. It is related to the original coordinate system (q_1, q_2, p_1, p_2) by

$$q_1 = x_{10}^{(6)}, \quad q_2 = \xi_2, \quad p_1 = -\frac{\eta \xi_2^2}{(x_{10}^{(6)})^3} + \frac{\eta}{(x_{10}^{(6)})^2} + \frac{\alpha_1}{x_{10}^{(6)}} + y_{10}^{(6)},$$

$$\begin{aligned} p_2 = {} & \frac{\eta \xi_2}{(x_{10}^{(6)})^2} - \eta \xi_2^2 z_{10}^{(3)} - \frac{\alpha_1}{\xi_2} - \frac{\eta}{\xi_2^3} + \Big(\eta z_{10}^{(3)} - \frac{y_{10}^{(6)}}{\xi_2} - \frac{\alpha_1}{\xi_2^3}\Big) x_{10}^{(6)} \\ & + \Big(\alpha_1 z_{10}^{(3)} - \frac{y_{10}^{(6)}}{\xi_2^3}\Big)(x_{10}^{(6)})^2 + z_{10}^{(3)} y_{10}^{(6)} (x_{10}^{(6)})^3. \end{aligned}$$

Now we calculate the 2-form $dq_1 \wedge dp_1 + dq_2 \wedge dp_2$ in the coordinates $(\xi_2, z_{10}^{(3)}, x_{10}^{(6)}, y_{10}^{(6)})$:

$$\begin{aligned} & dq_1 \wedge dp_1 + dq_2 \wedge dp_2 \\ = {} & dx_{10}^{(6)} \wedge dy_{10}^{(6)} \\ & + \{y_{10}^{(6)} (x_{10}^{(6)})^3 + \alpha_1 (x_{10}^{(6)})^2 + \eta x_{10}^{(6)} - \eta \xi_2^2\} d\xi_2 \wedge dz_{10}^{(3)} \\ & + \Big\{3 z_{10}^{(3)} y_{10}^{(6)} (x_{10}^{(6)})^2 + 2\Big(\alpha_1 z_{10}^{(3)} - \frac{y_{10}^{(6)}}{\xi_2^3}\Big) x_{10}^{(6)} + \eta z_{10}^{(3)} - \frac{y_{10}^{(6)}}{\xi_2} - \frac{\alpha_1}{\xi_2^3}\Big\} d\xi_2 \wedge dx_{\infty 0}^{(6)} \\ & + \Big(z_{10}^{(3)} (x_{10}^{(6)})^3 - \frac{(x_{10}^{(6)})^2}{\xi_2^3} - \frac{x_{10}^{(6)}}{\xi_2}\Big) d\xi_2 \wedge dy_{10}^{(6)} \\ = {} & dx_{10}^{(6)} \wedge dy_{10}^{(6)} \\ & + d\xi_2 \wedge d\Big[\{y_{10}^{(6)} (x_{10}^{(6)})^3 + \alpha_1 (x_{10}^{(6)})^2 + \eta x_{10}^{(6)} - \eta \xi_2^2\} z_{10}^{(3)} - \Big\{\frac{(x_{10}^{(6)})^2}{\xi_2^3} + \frac{x_{10}^{(6)}}{\xi_2}\Big\} y_{10}^{(6)} - \frac{\alpha_1}{\xi_2^3} x_{10}^{(6)}\Big] \end{aligned}$$

$$
\begin{aligned}
= \quad & dx_{10}^{(6)} \wedge dy_{10}^{(6)} \\
& + d\xi_2 \wedge d\Big[\{y_{10}^{(6)}(x_{10}^{(6)})^3 + \alpha_1(x_{10}^{(6)})^2 + \eta x_{10}^{(6)} - \eta\xi_2^2\}z_{10}^{(3)} \\
& \qquad - \Big\{\frac{(x_{10}^{(6)})^2}{\xi_2^3} + \frac{x_{10}^{(6)}}{\xi_2}\Big\}y_{10}^{(6)} - \frac{\alpha_1}{\xi_2^3}x_{10}^{(6)} - \frac{\alpha_1}{\xi_2} - \frac{\eta}{\xi_2^3}\Big].
\end{aligned}
$$

Therefore, setting

$$
w_{10}^{(3)} = \{y_{10}^{(6)}(x_{10}^{(6)})^3 + \alpha_1(x_{10}^{(6)})^2 + \eta x_{10}^{(6)} - \eta\xi_2^2\}z_{10}^{(3)} - \Big\{\frac{(x_{10}^{(6)})^2}{\xi_2^3} + \frac{x_{10}^{(6)}}{\xi_2}\Big\}y_{10}^{(6)} - \frac{\alpha_1}{\xi_2^3}x_{10}^{(6)} - \frac{\alpha_1}{\xi_2} - \frac{\eta}{\xi_2^3},
$$

we have symplectic coordinates $(\xi_2, w_{10}^{(3)}, x_{10}^{(6)}, y_{10}^{(6)})$. Writing

$$
q_1^{11} = x_{10}^{(6)}, \quad q_2^{11} = \xi_2, \quad p_1^{11} = y_{10}^{(6)}, \quad p_2^{11} = w_{10}^{(3)},
$$

we have

$$
q_1 = q_1^{11}, \quad q_2 = q_2^{11}, \quad p_1 = -\frac{\eta(q_2^{11})^2}{(q_1^{\infty 1})^3} + \frac{\eta}{(q_1^{11})^2} + \frac{\alpha_1}{q_1^{11}} + p_1^{11}, \quad p_2 = \frac{\eta q_2^{11}}{(q_1^{11})^2} + p_2^{11}. \tag{6.5}
$$

Thus we have obtained a symplectic coordinate system $(q^{11}, p^{11}) \in \mathbb{C}^4$ in which the Hamiltonians have no singularity on $\xi_2 = 0$.

6.4.2 Coordinate system for $A_1(s) \cap W_2$

The first quadratic transformation along $A_1 \cap W_2$. Let

$$
\begin{aligned}
\xi_1 &= X_{10}^{(1)}, \quad \eta_{20} = X_{10}^{(1)}Y_{10}^{(1)}, \quad \eta_{22} = X_{10}^{(1)}Z_{10}^{(1)}, \\
\xi_1 &= X_{11}^{(1)}Y_{11}^{(1)}, \quad \eta_{20} = Y_{11}^{(1)}, \quad \eta_{22} = Y_{11}^{(1)}Z_{11}^{(1)}, \\
\xi_1 &= X_{12}^{(1)}Z_{12}^{(1)}, \quad \eta_{20} = Y_{12}^{(1)}Z_{12}^{(1)}, \quad \eta_{22} = Z_{12}^{(1)},
\end{aligned}
$$

then

$$
D_{12}^{(1)}(s) = Q_{A_1(s)\cap W_2}(A_1(s) \cap W_2) = \{X_{10}^{(1)} = 0\} \cup \{Y_{11}^{(1)} = 0\} \cup \{Z_{12}^{(1)} = 0\},
$$

the set of accessible singular points is given by

$$
A_{12}^{(1)}(s) = \{(\xi_0, X_{10}^{(1)}, Y_{10}^{(1)}, Z_{10}^{(1)}) = (\xi_0, 0, 0, 0)\} \subset D_{12}^{(1)}(s).
$$

The second quadratic transformation along $A_{12}^{(1)}(s)$. Let

$$
\begin{aligned}
X_{10}^{(1)} &= X_{10}^{(2)}, \quad Y_{10}^{(1)} = X_{10}^{(2)}Y_{10}^{(2)}, \quad Z_{10}^{(1)} = X_{10}^{(2)}Z_{10}^{(2)}, \\
X_{10}^{(1)} &= X_{11}^{(2)}Y_{11}^{(2)}, \quad Y_{10}^{(1)} = Y_{11}^{(2)}, \quad Z_{10}^{(1)} = Y_{11}^{(2)}Z_{11}^{(2)}, \\
X_{10}^{(1)} &= X_{12}^{(1)}Z_{12}^{(2)}, \quad Y_{10}^{(1)} = X_{12}^{(2)}Y_{12}^{(2)}, \quad Z_{10}^{(1)} = Z_{12}^{(2)},
\end{aligned}
$$

then

$$
D_{12}^{(2)}(s) = Q_{A_{12}^{(1)}(s)}(A_{12}^{(1)}(s)) = \{X_{10}^{(2)} = 0\} \cup \{Y_{11}^{(2)} = 0\} \cup \{Z_{12}^{(2)} = 0\},
$$

the set of accessible singular points is given by

$$
A_{12}^{(2)}(s) = \{(\xi_0, X_{10}^{(2)}, Y_{10}^{(2)}, Z_{10}^{(2)}) = (\xi_0, 0, 0, 1)\} \subset D_{12}^{(2)}(s).
$$

The third quadratic transformation along $A_{12}^{(2)}(s)$. Let

$$X_{10}^{(2)} = X_{10}^{(3)}, \quad Y_{10}^{(2)} = X_{10}^{(3)} Y_{10}^{(3)}, \quad Z_{10}^{(2)} = 1 + X_{10}^{(3)} Z_{10}^{(3)},$$
$$X_{10}^{(2)} = X_{11}^{(3)} Y_{11}^{(3)}, \quad Y_{10}^{(2)} = Y_{11}^{(3)}, \quad Z_{10}^{(2)} = 1 + Y_{11}^{(3)} Z_{11}^{(3)},$$
$$X_{10}^{(2)} = X_{12}^{(3)} Z_{12}^{(3)}, \quad Y_{10}^{(2)} = Y_{12}^{(3)} Z_{12}^{(3)}, \quad Z_{10}^{(2)} = 1 + Z_{12}^{(3)},$$

then

$$D_{12}^{(3)}(s) = Q_{A_{12}^{(2)}(s)}(A_{12}^{(2)}(s)) = \{X_{10}^{(3)} = 0\} \cup \{Y_{11}^{(3)} = 0\} \cup \{Z_{12}^{(3)} = 0\},$$

the set of accessible singular points is given by

$$A_{12}^{(3)}(s) = \{(\xi_0, X_{10}^{(3)}, Y_{10}^{(3)}, Z_{10}^{(3)}) = (\xi_0, 0, -1/\eta, Z_{10}^{(3)})\} \subset D_{12}^{(3)}(s).$$

The fourth quadratic transformation along $A_{12}^{(3)}(s)$. We insert here the transformations

$$\xi_0 = \xi_0, \quad X_{10}^{(3)} = X_{10}^{(3)}, \quad Y_{10}^{(3)} = 1/V_{10}^{(3)}, \quad Z_{10}^{(3)} = Z_{10}^{(3)}.$$

Let

$$X_{10}^{(3)} = X_{10}^{(4)}, \quad V_{10}^{(3)} = -\eta + X_{10}^{(4)} Y_{10}^{(4)},$$
$$X_{10}^{(3)} = X_{11}^{(4)} Y_{11}^{(4)}, \quad V_{10}^{(3)} = -\eta + Y_{11}^{(4)},$$

then

$$D_{12}^{(4)}(s) = Q_{A_{12}^{(3)}(s)}(A_{12}^{(3)}(s)) = \{X_{10}^{(4)} = 0\} \cup \{Y_{11}^{(4)} = 0\},$$

the set of accessible singular points is given by

$$A_{12}^{(4)}(s) = \{(\xi_0, Z_{10}^{(3)}, X_{10}^{(4)}, Y_{10}^{(4)}) = (\xi_0, Z_{10}^{(3)}, 0, \eta\xi_0)\} \subset D_{12}^{(4)}(s).$$

The fifth quadratic transformation with $A_{12}^{(4)}(s)$. Let

$$X_{10}^{(4)} = X_{10}^{(5)}, \quad Y_{10}^{(4)} = \eta\xi_0 + X_{10}^{(5)} Y_{10}^{(5)},$$
$$X_{10}^{(4)} = X_{11}^{(5)} Y_{11}^{(5)}, \quad Y_{10}^{(4)} = \eta\xi_0 + Y_{11}^{(5)},$$

then

$$D_{12}^{(5)}(s) = Q_{A_{12}^{(4)}(s)}(A_{12}^{(4)}(s)) = \{X_{10}^{(5)} = 0\} \cup \{Y_{11}^{(5)} = 0\},$$

the set of accessible singular points is given by

$$A_{12}^{(5)}(s) = \{(\xi_0, Z_{10}^{(3)}, X_{10}^{(5)}, Y_{10}^{(5)}) = (\xi_0, Z_{10}^{(3)}, 0, \alpha_1)\} \subset D_{12}^{(5)}(s).$$

The sixth quadratic transformation with $A_{12}^{(5)}(s)$. Let

$$X_{10}^{(5)} = X_{10}^{(6)}, \quad Y_{10}^{(5)} = \alpha_1 + X_{10}^{(6)} Y_{10}^{(6)},$$
$$X_{10}^{(5)} = X_{11}^{(6)} Y_{11}^{(6)}, \quad Y_{10}^{(5)} = \alpha_1 + Y_{11}^{(6)},$$

then

$$D_{12}^{(6)}(s) = Q_{A_{12}^{(5)}(s)}(A_{12}^{(5)}(s)) = \{X_{10}^{(6)} = 0\} \cup \{Y_{11}^{(6)} = 0\}.$$

We see that, in the $(\xi_0, Z_{10}^{(3)}, X_{10}^{(6)}, Y_{10}^{(6)}, s)$-space $\mathbb{C}^4 \times B$, the differential system is holomorphic in a neighborhood of $\{X_{10}^{(6)} = 0\}$, moreover, the points $(\xi_0, Z_{10}^{(3)}, X_{11}^{(6)}, Y_{11}^{(6)}) = (\xi_0, Z_{10}^{(3)}, 0, 0)$ are inaccessible.

Thus we have obtained a coordinate system $(\xi_0, Z_{10}^{(3)}, X_{10}^{(6)}, Y_{10}^{(6)}) \in \mathbb{C}^4$ which is related to the coordinate system $(q_1^2, q_2^2, p_1^2, p_2^2)$ as

$$q_1^2 = X_{10}^{(6)}, \quad q_2^2 = \xi_0, \quad p_1^2 = -\frac{\eta}{(X_{10}^{(8)})^3} + \frac{\eta\xi_0}{(X_{10}^{(6)})^2} + \frac{\alpha_1}{X_{10}^{(6)}} + Y_{10}^{(6)},$$
$$p_2^2 = -\frac{\eta}{X_{10}^{(6)}} + \eta\xi_0 - \eta Z_{10}^{(3)} + (\eta\xi_0 Z_{10}^{(3)} + \alpha_1)X_{10}^{(6)} + (\alpha_1 Z_{10}^{(3)} + Y_{10}^{(6)})(X_{10}^{(6)})^2 + Z_{10}^{(3)} Y_{10}^{(6)} (X_{10}^{(6)})^3.$$

Now we calculate the 2-form $dq_1^2 \wedge dp_1^2 + dq_2^2 \wedge dp_2^2$ in the coordinates $(\xi_0, Z_{10}^{(3)}, X_{10}^{(6)}, Y_{10}^{(6)})$:

$$\begin{aligned}
&dq_1^2 \wedge dp_1^2 + dq_2^2 \wedge dp_2^2 \\
&= dX_{10}^{(6)} \wedge dY_{10}^{(6)} \\
&\quad + \{Y_{10}^{(6)}(X_{10}^{(6)})^3 + \alpha_1 (X_{10}^{(6)})^2 + \eta\xi_0 X_{10}^{(6)} - \eta\} d\xi_0 \wedge dZ_{10}^{(3)} \\
&\quad + \{3Z_{10}^{(3)} Y_{10}^{(6)} (X_{10}^{(6)})^2 + 2(\alpha_1 Z_{10}^{(3)} + Y_{10}^{(6)}) X_{10}^{(6)} + \eta\xi_0 Z_{10}^{(3)} + \alpha_1\} d\xi_0 \wedge dX_{10}^{(6)} \\
&\quad + (Z_{10}^{(3)} (X_{10}^{(6)})^3 + (X_{10}^{(6)})^2) d\xi_2 \wedge dY_{10}^{(6)} \\
&= dX_{10}^{(6)} \wedge dY_{10}^{(6)} \\
&\quad + d\xi_0 \wedge d[\{Y_{10}^{(6)}(X_{10}^{(6)})^3 + \alpha_1 (X_{10}^{(6)})^2 + \eta\xi_0 X_{10}^{(6)} - \eta\} Z_{10}^{(3)} + (X_{10}^{(6)})^2 Y_{10}^{(6)} + \alpha_1 X_{10}^{(6)}] \\
&= dX_{10}^{(6)} \wedge dY_{10}^{(6)} \\
&\quad + d\xi_0 \wedge d[\{Y_{10}^{(6)}(X_{10}^{(6)})^3 + \alpha_1 (X_{10}^{(6)})^2 + \eta\xi_0 X_{10}^{(6)} - \eta\} Z_{10}^{(3)} + (X_{10}^{(6)})^2 Y_{10}^{(6)} + \alpha_1 X_{10}^{(6)} + \eta\xi_0].
\end{aligned}$$

Therefore, setting

$$W_{10}^{(3)} = \{Y_{10}^{(6)}(X_{10}^{(6)})^3 + \alpha_1 (X_{10}^{(6)})^2 + \eta\xi_0 X_{10}^{(6)} - \eta\} Z_{10}^{(3)} + (X_{10}^{(6)})^2 Y_{10}^{(6)} + \alpha_1 X_{10}^{(6)} + \eta\xi_0,$$

we have symplectic coordinates $(\xi_0, W_{10}^{(3)}, X_{10}^{(6)}, Y_{10}^{(6)})$.

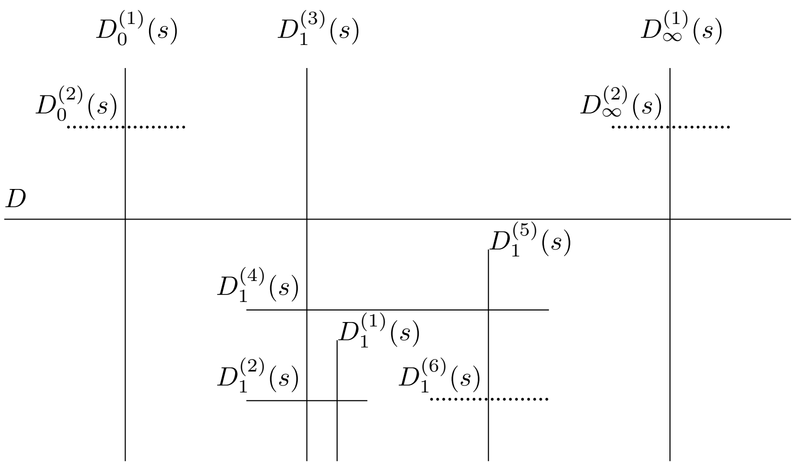

Figure 3. J=113

Writing

$$q_1^{12} = X_{10}^{(6)}, \quad q_2^{12} = \xi_0, \quad p_1^{12} = Y_{10}^{(6)}, \quad p_2^{12} = W_{10}^{(3)},$$

we have

$$q_1^2 = q_1^{12}, \quad q_2^2 = q_2^{12}, \quad p_1^2 = -\frac{\eta}{(q_1^{12})^3} + \frac{\eta q_2^{12}}{(q_1^{12})^2} + \frac{\alpha_1}{q_1^{12}} + p_1^{12}, \quad p_2^2 = -\frac{\eta}{q_1^{12}} + p_2^{12}. \tag{6.6}$$

The system $(q^{12}, p^{12}) \in \mathbb{C}^4$ separates solution curves passing through $A_1(s) \cap W_2$.

Thus we have obtained six symplectic coordinate systems $(q_1^*, q_2^*, p_1^*, p_2^*)$ each of which separates solution curves passing through the accessible singular points (see (6.1)-(6.6)). We notice that the Hamiltonians of each coordinate system are also polynomials whose coefficients are rational functions of s holomorphic in B.

7 Spaces of initial conditions for $\mathcal{H}_{122}$

In the present case,

$$\nu = -\frac{1}{2}(\alpha_0 + \alpha_1 + \alpha_2 - 1 + \alpha_\infty)$$

In this section, we omitt the label 122.

7.1 Accessible singularities on $D \times B$

Observing the system $\mathcal{H}^{(0)}$ on all W_{ij} $j \neq 0$, we can obtain

Proposition 7.1. *The set of accessible singular points of the system $\mathcal{H}_{122}^{(0)}$ for each $s = (s_1, s_2) \in B_{122}$ is a disjoint union of three connected components $A_0(s), A_1(s), A_\infty(s) \simeq \mathbb{P}^1$ given by*

$$\begin{aligned} A_0(s) &= \{(\xi, \eta_0, s) \in W_0 \times B | \xi_2 = \eta_{00} = \eta_{01} = 0\} \cup \{(\xi, \eta_1, s) \in W_1 \times B | \xi_2 = \eta_{10} = \eta_{11} = 0\}, \\ A_1(s) &= \{(\xi, \eta_0, s) \in W_0 \times B | \xi_1 = \eta_{00} = \eta_{02} = 0\} \cup \{(\xi, \eta_2, s) \in W_2 \times B | \xi_1 = \eta_{20} = \eta_{22} = 0\}, \\ A_\infty(s) &= \{(\xi, \eta_1, s) \in W_1 \times B | \xi_0 = \eta_{10} = \eta_{12} = 0\} \cup \{(\xi, \eta_2, s) \in W_2 \times B | \xi_0 = \eta_{20} = \eta_{21} = 0\}. \end{aligned}$$

Moreover, we can verify

Proposition 7.2. *The Bäcklund transformation group S_2 acts on the components of the accessible singular points of $\mathcal{H}_{122}$ according to the following diagram*

	A_0	A_1
σ	A_1	A_0

where $A_\infty(s)$ is invariant under the action of this group.

In the following subsections, we obtain coordinate systems corresponding to $A_i(s)$ which separate completely the solution curves passing through $A_i(s)$. The systems for $A_1(s)$ and $A_\infty(s)$ are obtained by quadratic transformations, while the systems for $A_0(s)$ are obtained from that for $A_1(s)$ by the use of Bäcklund transformations.

7.2 Coordinate systems for $A_\infty(s)$

We make successively the quadratic transformations along $A_\infty(s)\cap W_1$, $A_\infty(s)\cap W_2$ and find coordinate systems for $A_\infty(s)$.

7.2.1 Coordinate system for $A_\infty(s)\cap W_1$

The first quadratic transformation along $A_\infty \cap W_1$. Let

$$\xi_0 = x_{\infty 0}^{(1)}, \quad \eta_{10} = x_{\infty 0}^{(1)} y_{\infty 0}^{(1)}, \quad \eta_{12} = x_{\infty 0}^{(1)} z_{\infty 0}^{(1)},$$
$$\xi_0 = x_{\infty 1}^{(1)} y_{\infty 1}^{(1)}, \quad \eta_{10} = y_{\infty 1}^{(1)}, \quad \eta_{12} = y_{\infty 1}^{(1)} z_{\infty 1}^{(1)},$$
$$\xi_0 = x_{\infty 2}^{(1)} z_{\infty 2}^{(1)}, \quad \eta_{10} = y_{\infty 2}^{(1)} z_{\infty 2}^{(1)}, \quad \eta_{12} = z_{\infty 2}^{(1)},$$

then

$$D_{\infty 1}^{(1)}(s) = Q_{A_{\infty 1}\cap W_1}(A_\infty \cap W_1) = \{x_{\infty 0}^{(1)} = 0\} \cup \{y_{\infty 1}^{(1)} = 0\} \cup \{z_{\infty 2}^{(1)} = 0\},$$

the set of accessible singular points is given by

$$A_{\infty 1}^{(1)}(s) = \{(\xi_2, x_{\infty 1}^{(1)}, y_{\infty 1}^{(1)}, z_{\infty 1}^{(1)}) = (\xi_2, \alpha_\infty, 0, z_{\infty 1}^{(1)})\} \subset D_{\infty 1}^{(1)}(s).$$

The second quadratic transformation with $A_{\infty 1}^{(1)}(s)$. Let

$$x_{\infty 1}^{(1)} = \alpha_\infty + x_{\infty 0}^{(2)}, \quad y_{\infty 1}^{(1)} = x_{\infty 0}^{(2)} y_{\infty 0}^{(2)},$$
$$x_{\infty 1}^{(1)} = \alpha_\infty + x_{\infty 1}^{(2)} y_{\infty 1}^{(2)}, \quad y_{\infty 1}^{(1)} = y_{\infty 1}^{(2)},$$

then

$$D_{\infty 1}^{(2)}(s) = Q_{A_{\infty 1}^{(1)}(s)}(A_{\infty 1}^{(1)}(s)) = \{x_{\infty 0}^{(2)} = 0\} \cup \{y_{\infty 1}^{(2)} = 0\}.$$

We can verify that our system has no singularity and every leaf is transversal with the fibers in $(\xi_2, z_{\infty 1}^{(1)}, x_{\infty 1}^{(2)}, y_{\infty 1}^{(2)})$-space $\mathbb{C}^4 \times B$. On the other hand, the points $(\xi_2, z_{\infty 1}^{(1)}, x_{\infty 0}^{(2)}, y_{\infty 0}^{(2)}) = (\xi_2, z_{\infty 1}^{(1)}, 0, 0)$ are inaccessible singular points.

Thus we have obtained a coordinate system $(\xi_2, z_{\infty 1}^{(1)}, x_{\infty 1}^{(2)}, y_{\infty 1}^{(2)}) \in \mathbb{C}^4$ which separates the solutions passing through $A_\infty(s)\cap W_1 = A_\infty(s) \cap W_{11}$. If we set

$$q_1^{\infty 1} = -x_{\infty 1}^{(2)}, \quad q_2^{\infty 1} = \xi_2, \quad p_1^{\infty 1} = y_{\infty 1}^{(2)}, \quad p_2^{\infty 1} = z_{\infty 1}^{(1)},$$

then we have

$$q_1^1 = p_1^{\infty 1}(\alpha_\infty - q_1^{\infty 1} p_1^{\infty 1}), \quad q_2^1 = q_2^{\infty 1}, \quad p_1^1 = \frac{1}{p_1^{\infty 1}}, \quad p_2^1 = p_2^{\infty 1} \tag{7.1}$$

Thus we have obtained a symplectic coordinate system $(q^{\infty 1}, p^{\infty 1}) \in \mathbb{C}^4$ for $A_\infty(s)\cap W_1$.

7.2.2 Coordinate system for $A_\infty(s)\cap W_2$

The first quadratic transformation along $A_\infty(s) \cap W_2$. Let

$$\xi_0 = X_{\infty 0}^{(1)}, \quad \eta_{20} = X_{\infty 0}^{(1)} Y_{\infty 0}^{(1)}, \quad \eta_{21} = X_{\infty 0}^{(1)} Z_{\infty 0}^{(1)},$$
$$\xi_0 = X_{\infty 1}^{(1)} Y_{\infty 1}^{(1)}, \quad \eta_{20} = Y_{\infty 1}^{(1)}, \quad \eta_{21} = Y_{\infty 1}^{(1)} Z_{\infty 1}^{(1)},$$
$$\xi_0 = X_{\infty 2}^{(1)} Z_{\infty 2}^{(1)}, \quad \eta_{20} = Y_{\infty 2}^{(1)} Z_{\infty 2}^{(1)}, \quad \eta_{21} = Z_{\infty 2}^{(1)},$$

then

$$D^{(1)}_{\infty 2}(s) = Q_{A_\infty \cap W_2}(A_\infty \cap W_2) = \{X^{(1)}_{\infty 0} = 0\} \cup \{Y^{(1)}_{\infty 1} = 0\} \cup \{Z^{(1)}_{\infty 2} = 0\},$$

the set of accessible singular points is given by

$$A^{(1)}_{\infty 2}(s) = \{(\xi_1, X^{(1)}_{\infty 1}, Y^{(1)}_{\infty 1}, Z^{(1)}_{\infty 1}) = (\xi_1, \alpha_\infty, 0, Z^{(1)}_{\infty 1})\} \subset D^{(1)}_{\infty 2}(s).$$

The second quadratic transformation with $A^{(1)}_{\infty 2}(s)$. Let

$$X^{(1)}_{\infty 1} = \alpha_\infty + X^{(2)}_{\infty 0}, \quad Y^{(1)}_{\infty 1} = X^{(2)}_{\infty 0} Y^{(2)}_{\infty 0},$$
$$X^{(1)}_{\infty 1} = \alpha_\infty + X^{(2)}_{\infty 1} Y^{(2)}_{\infty 1}, \quad Y^{(1)}_{\infty 1} = Y^{(2)}_{\infty 1},$$

then

$$D^{(2)}_{\infty 2}(s) = Q_{A^{(1)}_{\infty 2}(s)}(A^{(1)}_{\infty 2}(s)) = \{X^{(2)}_{\infty 0} = 0\} \cup \{Y^{(2)}_{\infty 1} = 0\}.$$

We see that, in the $(\xi_1, Z^{(1)}_{\infty 1}, X^{(2)}_{\infty 1}, Y^{(2)}_{\infty 1})$-space $\mathbb{C}^4 \times B$, the Pfaffian system has no singularity, every leaf is transversal with the fibers, moreover, the points $(\xi_1, Z^{(1)}_{\infty 1}, X^{(2)}_{\infty 0}, Y^{(2)}_{\infty 0}) = (\xi_1, Z^{(1)}_{\infty 1}, 0, 0)$ are inaccessible singular points.

Thus we have obtained a coordinate system $(\xi_1, Z^{(1)}_{\infty 1}, X^{(2)}_{\infty 1}, Y^{(2)}_{\infty 1}) \in \mathbb{C}^4$ which separates the solutions passing through $A_\infty(s) \cap W_2 = A_\infty(s) \cap W_{22}$. If we set

$$q^{\infty 2}_1 = \xi_1, \quad q^{\infty 2}_2 = -X^{(2)}_{\infty 1}, \quad p^{\infty 2}_1 = X^{(2)}_{\infty 1}, \quad p^{\infty 2}_2 = Y^{(1)}_{\infty 1},$$

then we have

$$q^2_1 = q^{\infty 2}_1, \quad q^2_2 = p^{\infty 2}_2(\alpha_\infty - q^{\infty 2}_2 p^{\infty 2}_2), \quad p^2_1 = p^{\infty 2}_1, \quad p^2_2 = \frac{1}{p^{\infty 2}_2} \tag{7.2}$$

The system $(q^{\infty 2}, p^{\infty 2}) \in \mathbb{C}^4$ is a symplectic coordinate system for $A_\infty(s) \cap W_2$.

7.3 Coordinate systems for $A_1(s)$

We obtain coordinate systems for $A_1(s)$ by making quadratic transformations four times along $A_1(s) \cap W_0$, $A_1(s) \cap W_2$.

7.3.1 Coordinate system for $A_1(s) \cap W_0$

The first quadratic transformation along $A_1(s) \cap W_0$. Let

$$\xi_1 = x^{(1)}_{10}, \quad \eta_{00} = x^{(1)}_{10} y^{(1)}_{10}, \quad \eta_{02} = x^{(1)}_{10} z^{(1)}_{10},$$
$$\xi_1 = x^{(1)}_{11} y^{(1)}_{11}, \quad \eta_{00} = y^{(1)}_{11}, \quad \eta_{02} = y^{(1)}_{11} z^{(1)}_{11},$$
$$\xi_1 = x^{(1)}_{12} z^{(1)}_{12}, \quad \eta_{00} = y^{(1)}_{12} z^{(1)}_{12}, \quad \eta_{02} = z^{(1)}_{12}$$

then

$$D^{(1)}_{11}(s) = Q_{A_1(s) \cap W_0}(A_1(s) \cap W_0) = \{x^{(1)}_{10} = 0\} \cup \{y^{(1)}_{11} = 0\} \cup \{z^{(1)}_{12} = 0\},$$

the set of accessible singular points is given by

$$A^{(1)}_{11}(s) = \{(\xi_2, x^{(1)}_{10}, y^{(1)}_{10}, z^{(1)}_{10}) = (\xi_2, 0, 0, -1/(\xi_2 - 1)\} \subset D^{(1)}_{11}(s).$$

The second quadratic transformation along $A_{11}^{(1)}(s)$. Note that $\xi_2 = 1$ is excluded. Let

$$x_{10}^{(1)} = x_{10}^{(2)}, \quad y_{10}^{(1)} = x_{10}^{(2)} y_{10}^{(2)}, \quad z_{10}^{(1)} = -1/(\xi_2 - 1) + x_{10}^{(2)} z_{10}^{(2)},$$
$$x_{10}^{(1)} = x_{11}^{(2)} y_{11}^{(2)}, \quad y_{10}^{(1)} = y_{11}^{(2)}, \quad z_{10}^{(1)} = -1/(\xi_2 - 1) + y_{11}^{(2)} z_{11}^{(2)},$$
$$x_{10}^{(1)} = x_{12}^{(2)} z_{12}^{(2)}, \quad y_{10}^{(1)} = y_{12}^{(2)} z_{12}^{(2)}, \quad z_{10}^{(1)} = -1/(\xi_2 - 1) + z_{12}^{(2)}$$

then

$$D_{11}^{(2)}(s) = Q_{A_{11}^{(1)}(s)}(A_{11}^{(1)}(s)) = \{x_{10}^{(2)} = 0\} \cup \{y_{11}^{(2)} = 0\} \cup \{z_{12}^{(2)} = 0\},$$

the set of accessible singular points is given by

$$A_{11}^{(2)}(s) = \{(\xi_2, x_{10}^{(1)}, y_{10}^{(1)}, z_{10}^{(1)}) = (\xi_2, 0, -1/(\eta_1(\xi_2 - 1)), z_{10}^{(2)})\} \subset D_{11}^{(2)}(s).$$

The third quadratic transformation along $A_{11}^{(2)}(s)$. Here we insert a change of variables

$$\xi_2 = \xi_2, \quad x_{10}^{(2)} = x_{10}^{(2)}, \quad y_{10}^{(2)} = 1/v_{10}^{(2)}, \quad z_{10}^{(2)} = z_{10}^{(2)}.$$

Let

$$x_{10}^{(2)} = x_{10}^{(3)}, \quad v_{10}^{(2)} = -\eta_1(\xi_2 - 1) + x_{10}^{(3)} y_{10}^{(3)},$$
$$x_{10}^{(2)} = x_{11}^{(3)} y_{11}^{(3)}, \quad v_{10}^{(2)} = -\eta_1(\xi_2 - 1) + y_{11}^{(3)}$$

then

$$D_{11}^{(3)}(s) = Q_{A_{11}^{(2)}(s)}(A_{11}^{(2)}(s)) = \{x_{10}^{(3)} = 0\} \cup \{y_{11}^{(3)} = 0\},$$

that the set of accessible singular points is given by

$$A_{11}^{(3)}(s) = \{(\xi_2, z_{10}^{(2)}, x_{10}^{(3)}, y_{10}^{(3)}) = (\xi_2, z_{10}^{(2)}, 0, \alpha_1)\} \subset D_{11}^{(3)}(s).$$

The fourth quadratic transformation along $A_{11}^{(3)}(s)$. Let

$$x_{10}^{(3)} = x_{10}^{(4)}, \quad y_{10}^{(3)} = \alpha_1 + x_{10}^{(4)} y_{10}^{(4)},$$
$$x_{10}^{(3)} = x_{11}^{(4)} y_{11}^{(4)}, \quad y_{10}^{(3)} = \alpha_1 + y_{11}^{(4)}$$

then the exceptional divisor is given by

$$D_{11}^{(4)}(s) = Q_{A_{10}^{(3)}(s)}(A_{10}^{(3)}(s)) = \{x_{10}^{(4)} = 0\} \cup \{y_{11}^{(4)} = 0\}.$$

We can verify that the differential system in the cordinates $(\xi_2, z_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)})$ is holomorphic in a neighborhood of $\{x_{10}^{(6)} = 0\}$ except for $\xi_2 = 1$ and the points $(\xi_2, z_{10}^{(2)}, x_{11}^{(4)}, y_{11}^{(4)}) = (\xi_2, z_{10}^{(2)}, 0, 0)$ are inaccessible with $\xi_2 \neq 1$.

Thus we have obtained a coordinate system $(\xi_2, z_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}) \in \mathbb{C}^4$ which is separates the solutions passing through $A_1(s) \cap W_0 = A_1(s) \cap W_{01}$ with $\xi_2 \neq 1$. It is related to the original coordinate system (q_1, q_2, p_1, p_2) by

$$q_1 = x_{10}^{(4)}, \quad q_2 = \xi_2, \quad p_1 = -\frac{\eta_1(\xi_2 - 1)}{(x_{10}^{(4)})^2} + \frac{\alpha_1}{x_{10}^{(4)}} + y_{10}^{(4)},$$

$$p_2 = \frac{\eta_1}{x_{10}^{(4)}} - \eta_1(\xi_2 - 1) z_{10}^{(2)} - \frac{\alpha_1}{\xi_2 - 1} + \Big(\alpha_1 z_{10}^{(2)} - \frac{y_{10}^{(4)}}{\xi_2 - 1}\Big) x_{10}^{(4)} + z_{10}^{(2)} y_{10}^{(4)} (x_{10}^{(4)})^2.$$

Now we calculate the 2-form $dq_1 \wedge dp_1 + dq_2 \wedge dp_2$ in the coordinates $(\xi_2, z_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)})$:

$$\begin{aligned}
&dq_1 \wedge dp_1 + dq_2 \wedge dp_2 \\
&= dx_{10}^{(4)} \wedge dy_{10}^{(4)} + \{-\eta_1(\xi_2 - 1) + x_{10}^{(4)}(\alpha_1 + x_{10}^{(4)} y_{10}^{(4)})\} d\xi_2 \wedge dz_{10}^{(2)} \\
&\quad + \left(\alpha_1 z_{10}^{(2)} + 2 z_{10}^{(2)} x_{10}^{(4)} y_{10}^{(4)} - \frac{y_{10}^{(4)}}{\xi_2 - 1}\right) d\xi_2 \wedge dx_{10}^{(4)} \\
&\quad + \left(z_{10}^{(2)} (x_{10}^{(4)})^2 - \frac{x_{10}^{(4)}}{\xi_2 - 1}\right) d\xi_2 \wedge dy_{10}^{(4)} \\
&= dx_{10}^{(4)} \wedge dy_{10}^{(4)} \\
&\quad + d\xi_2 \wedge d\left\{\{-\eta(\xi_2 - 1) + x_{10}^{(4)}(\alpha_1 + x_{10}^{(4)} y_{10}^{(4)})\} z_{10}^{(2)} - \frac{x_{10}^{(4)} y_{10}^{(4)}}{\xi_2 - 1}\right\} \\
&= dx_{10}^{(4)} \wedge dy_{10}^{(4)} \\
&\quad + d\xi_2 \wedge d\left\{\{-\eta(\xi_2 - 1) + x_{10}^{(4)}(\alpha_1 + x_{10}^{(4)} y_{10}^{(4)})\} z_{10}^{(2)} - \frac{x_{10}^{(4)} y_{10}^{(4)}}{\xi_2 - 1} - \frac{\alpha_1}{\xi_2 - 1}\right\}.
\end{aligned}$$

Therefore, setting

$$w_{10}^{(2)} = \{-\eta(\xi_2 - 1) + x_{10}^{(4)}(\alpha_1 + x_{10}^{(4)} y_{10}^{(4)})\} z_{10}^{(2)} - \frac{x_{10}^{(4)} y_{10}^{(4)}}{\xi_2 - 1} - \frac{\alpha_1}{\xi_2 - 1},$$

we have symplectic coordinates $(\xi_2, w_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)})$. Writing

$$q_1^{11} = x_{10}^{(4)}, \quad q_2^{11} = \xi_2, \quad p_1^{11} = y_{10}^{(4)}, \quad p_2^{11} = w_{10}^{(2)},$$

we have

$$q_1 = q_1^{11}, \quad q_2 = q_2^{11}, \quad p_1 = -\frac{\eta_1(q_2^{11} - 1)}{(q_1^{11})^2} + \frac{\alpha_1}{q_1^{11}} + p_1^{11}, \quad p_2 = \frac{\eta}{q_1^{11}} + p_2^{11}. \tag{7.3}$$

Thus we have obtained a symplectic coordinate system $(q^{11}, p^{11}) \in \mathbb{C}^4$ in which Hamiltonians have no singularity on $\xi_2 = 1$.

7.3.2 Coordinate system for $A_1(s) \cap W_2$

The first quadratic transformation along $A_1(s) \cap W_2$. Let

$$\begin{aligned}
\xi_1 &= X_{10}^{(1)}, \quad \eta_{20} = X_{10}^{(1)} Y_{10}^{(1)}, \quad \eta_{22} = X_{10}^{(1)} Z_{10}^{(1)}, \\
\xi_1 &= X_{11}^{(1)} Y_{11}^{(1)}, \quad \eta_{20} = Y_{11}^{(1)}, \quad \eta_{22} = Y_{11}^{(1)} Z_{11}^{(1)}, \\
\xi_1 &= X_{12}^{(1)} Z_{12}^{(1)}, \quad \eta_{20} = Y_{12}^{(1)} Z_{12}^{(1)}, \quad \eta_{22} = Z_{12}^{(1)},
\end{aligned}$$

then

$$D_{12}^{(1)}(s) = Q_{A_1(s) \cap W_2}(A_1(s) \cap W_2) = \{X_{10}^{(1)} = 0\} \cup \{Y_{11}^{(1)} = 0\} \cup \{Z_{12}^{(1)} = 0\},$$

the set of accessible singular points is given by

$$A_{12}^{(1)}(s) = \{(\xi_0, X_{10}^{(1)}, Y_{10}^{(1)}, Z_{10}^{(1)}) = (\xi_0, 0, 0, 1/(\xi_0 - 1)\} \subset D_{12}^{(1)}(s).$$

The second quadratic transformation along $A_{12}^{(1)}(s)$. Let

$$X_{10}^{(1)} = X_{10}^{(2)}, \quad Y_{10}^{(1)} = X_{10}^{(2)} Y_{10}^{(2)}, \quad Z_{10}^{(1)} = 1/(\xi_0 - 1) + X_{10}^{(2)} Z_{10}^{(2)},$$
$$X_{10}^{(1)} = X_{11}^{(2)} Y_{11}^{(2)}, \quad Y_{10}^{(1)} = Y_{11}^{(2)}, \quad Z_{10}^{(1)} = 1/(\xi_0 - 1) + Y_{11}^{(2)} Z_{11}^{(2)},$$
$$X_{10}^{(1)} = X_{12}^{(2)} Z_{12}^{(2)}, \quad Y_{10}^{(1)} = Y_{12}^{(2)} Z_{12}^{(2)}, \quad Z_{10}^{(1)} = 1/(\xi_0 - 1) + Z_{12}^{(2)},$$

then

$$D_{12}^{(2)}(s) = Q_{A_{12}^{(1)}(s)}(A_{12}^{(1)}(s)) = \{X_{10}^{(2)} = 0\} \cup \{Y_{11}^{(2)} = 0\} \cup \{Z_{12}^{(2)} = 0\},$$

the set of accessible singular points is given by

$$A_{12}^{(2)}(s) = \{(\xi_0, X_{10}^{(2)}, Y_{10}^{(2)}, Z_{10}^{(2)}) = (\xi_0, 0, 1/(\eta_1(\xi_0 - 1), Z_{10}^{(2)})\} \subset D_{12}^{(2)}(s).$$

The third quadratic transformation along $A_{12}^{(2)}(s)$. We insert here the transformations

$$\xi_0 = \xi_0, \quad X_{10}^{(2)} = X_{10}^{(2)}, \quad Y_{10}^{(2)} = 1/V_{10}^{(2)}, \quad Z_{10}^{(1)} = Z_{10}^{(1)}.$$

Let

$$X_{10}^{(2)} = X_{10}^{(3)}, \quad V_{10}^{(2)} = \eta_1(\xi_0 - 1) + X_{10}^{(3)} Y_{10}^{(3)},$$
$$X_{10}^{(2)} = X_{11}^{(3)} Y_{11}^{(3)}, \quad V_{10}^{(2)} = \eta_1(\xi_0 - 1) + Y_{11}^{(3)},$$

then

$$D_{12}^{(3)}(s) = Q_{A_{12}^{(2)}(s)}(A_{12}^{(2)}(s)) = \{X_{10}^{(3)} = 0\} \cup \{Y_{11}^{(3)} = 0\},$$

the set of accessible singular points is given by

$$A_{12}^{(3)}(s) = \{(\xi_0, Z_{10}^{(2)}, X_{10}^{(3)}, Y_{10}^{(3)}) = (\xi_0, Z_{10}^{(2)}, 0, \alpha_1)\} \subset D_{12}^{(3)}(s).$$

The fourth quadratic transformation along $A_{12}^{(3)}(s)$. Let

$$X_{10}^{(3)} = X_{10}^{(4)}, \quad Y_{10}^{(3)} = \alpha_1 + X_{10}^{(4)} Y_{10}^{(4)},$$
$$X_{10}^{(3)} = X_{11}^{(4)} Y_{11}^{(4)}, \quad Y_{10}^{(3)} = \alpha_1 + Y_{11}^{(4)},$$

then

$$D_{12}^{(4)}(s) = Q_{A_{12}^{(3)}(s)}(A_{12}^{(3)}(s)) = \{X_{10}^{(4)} = 0\} \cup \{Y_{11}^{(4)} = 0\}.$$

We see that, in the $(\xi_0, Z_{10}^{(2)}, X_{10}^{(4)}, Y_{10}^{(4)}, s)$-space $\mathbb{C}^4 \times B$, the differential system is holomorphic in a neighborhood of $\{X_{10}^{(4)} = 0\}$ except for $\xi_0 = 1$, moreover, the points $(\xi_0, Z_{10}^{(2)}, X_{11}^{(4)}, Y_{11}^{(4)}) = (\xi_0, Z_{10}^{(2)}, 0, 0)$ are inaccessible.

Thus we have obtained a coordinate system $(\xi_0, Z_{10}^{(2)}, X_{10}^{(4)}, Y_{10}^{(4)}) \in \mathbb{C}^4 \setminus \{\xi_0 = 1\}$ which is related to the coordinate system $(q_1^2, q_2^2, p_1^2, p_2^2)$ as

$$q_1^2 = X_{10}^{(4)}, \quad q_2^2 = \xi_0, \quad q_1^2 = \frac{\eta(\xi_0 - 1)}{(X_{10}^{(4)})^2} + \frac{\alpha_1}{X_{10}^{(4)}} + Y_{10}^{(4)},$$

$$q_2^2 = -\frac{\eta_1}{X_{10}^{(4)}} + \eta_1(\xi_0 - 1) Z_{10}^{(2)} - \frac{\alpha_1}{\xi_0 - 1} + \Big(\alpha_1 Z_{10}^{(2)} - \frac{Y_{10}^{(4)}}{\xi_0 - 1}\Big) X_{10}^{(4)} + Z_{10}^{(2)} Y_{10}^{(4)} (X_{10}^{(4)})^2.$$

Now we calculate the 2-form $dq_1^2 \wedge dp_1^2 + dq_2^2 \wedge dp_2^2$ in the coordinates $(\xi_0, Z_{10}^{(2)}, X_{10}^{(4)}, Y_{10}^{(4)})$:

$$\begin{aligned}
& dq_1^2 \wedge dp_1^2 + dq_2^2 \wedge dp_2^2 \\
= \; & dX_{10}^{(4)} \wedge dY_{10}^{(4)} + \{\eta_1(\xi_0 - 1) + X_{10}^{(4)}(\alpha_1 + X_{10}^{(4)}Y_{10}^{(4)})\} d\xi_0 \wedge dZ_{10}^{(2)} \\
& + \Big(\alpha_1 Z_{10}^{(2)} + 2Z_{10}^{(2)}X_{10}^{(4)}Y_{10}^{(4)} - \frac{Y_{10}^{(4)}}{\xi_0 - 1}\Big) d\xi_0 \wedge dX_{10}^{(4)} \\
& + \Big(Z_{10}^{(2)}(X_{10}^{(4)})^2 - \frac{X_{10}^{(4)}}{\xi_0 - 1}\Big) d\xi_0 \wedge dY_{10}^{(4)} \\
= \; & dX_{10}^{(4)} \wedge dY_{10}^{(4)} \\
& + d\xi_0 \wedge d\Big\{\{\eta_1(\xi_0 - 1) + X_{10}^{(4)}(\alpha_1 + X_{10}^{(4)}Y_{10}^{(4)})\}Z_{10}^{(2)} - \frac{X_{10}^{(4)}Y_{10}^{(4)}}{\xi_0 - 1}\Big\} \\
= \; & dX_{10}^{(4)} \wedge dY_{10}^{(4)} \\
& + d\xi_0 \wedge d\Big\{\{\eta_1(\xi_0 - 1) + X_{10}^{(4)}(\alpha_1 + X_{10}^{(4)}Y_{10}^{(4)})\}Z_{10}^{(2)} - \frac{X_{10}^{(4)}Y_{10}^{(4)}}{\xi_0 - 1} - \frac{\alpha_1}{\xi_0 - 1}\Big\}.
\end{aligned}$$

Therefore, setting

$$W_{10}^{(2)} = \{\eta_1(\xi_0 - 1) + X_{10}^{(4)}(\alpha_1 + X_{10}^{(4)}Y_{10}^{(4)})\}Z_{10}^{(2)} - \frac{X_{10}^{(4)}Y_{10}^{(4)}}{\xi_0 - 1} - \frac{\alpha_1}{\xi_0 - 1},$$

we have symplectic coordinates $(\xi_0, W_{10}^{(2)}, X_{10}^{(4)}, Y_{10}^{(4)})$. Writing

$$q_1^{12} = X_{10}^{(4)}, \quad q_2^{12} = \xi_0, \quad p_1^{12} = Y_{10}^{(4)}, \quad p_2^{12} = W_{10}^{(2)},$$

we have

$$q_1^2 = q_1^{12}, \quad q_2^2 = q_2^{12}, \quad q_1^2 = \frac{\eta_1(q_2^{12} - 1)}{(q_1^{12})^2} + \frac{\alpha_1}{q_1^{12}} + p_1^{12}, \quad q_2^2 = -\frac{\eta_1}{q_1^{12}} + p_2^{12}. \tag{7.4}$$

The system $(q^{12}, p^{12}) \in \mathbb{C}^4$ separates solution curves passing through $A_1(s) \cap W_2$ and the Hamiltonians have no singularity on $\xi_0 = 1$.

7.4 Coordinate systems for $A_0(s)$

We obtain the systems for $A_0(s)$ from those for $A_1(s)$ and σ.

7.4.1 Coordinate system for $A_0(s) \cap W_0$

We derive a coordinate system for $A_0(s) \cap W_0$ from that for $A_1(s) \cap W_0$ and σ. We can verify $\sigma(A_0(s) \cap W_{02}) = A_1'(s') \cap W_{01}'$. Observing the relations between $(q_1.q_2, p_1, p_2)$ and the coordinate system $({q'}_1^{11}, {q'}_2^{11}, {p'}_1^{11}, {p'}_2^{11})$ for $A_1'(s')$, we take $(q_1^{01}, q_2^{01}, p_1^{01}, p_2^{01})$ as a coordinate system for $A_0(s) \cap W_0$ where

$${q'}_1^{11} = -\frac{q_2^{01}}{s_2}, \quad {q'}_2^{11} = \frac{q_1^{01}}{s_1}, \quad {p'}_1^{11} = -s_2 p_2^{01}, \quad {p'}_2^{11} = s_1 p_1^{01}.$$

We note that

$$q_1 = q_1^{01}, \quad q_2 = q_2^{01}, \quad p_1 = \frac{\eta_0 s_2}{s_1 q_2^{01}} + p_1^{01}, \quad p_2 = -\frac{\eta_0 s_2(q_1^{01} - s_1)}{s_1 (q_2^{01})^2} + \frac{\alpha_0}{q_2^{01}} + p_2^{01}. \tag{7.5}$$

Thus we have a symplectic coordinate system $(q^{01}, p^{01}) \in \mathbb{C}^4$ for $A_0(s) \cap W_0$.

7.4.2 Coordinate system for $A_0(s) \cap W_1$

A coordinate system for $A_0(s) \cap W_1$ is obtained from that for $A_1(s) \cap W_2$ and σ. We see $\sigma(A_0(s) \cap W_{12}) = A_1'(s') \cap W_{21}'$. Observing the relations between $(q_1.q_2, p_1, p_2)$ and $({q'}_1^{12}, {q'}_2^{12}, {p'}_1^{12}, {p'}_2^{12})$ we take $(q_1^{02}, q_2^{02}, p_1^{02}, p_2^{02})$ as a coordinate system for $A_0(s) \cap W_1$ where

$$ {q'}_1^{12} = -\frac{q_2^{02}}{s_2}, \quad {q'}_2^{12} = \frac{q_1^{02}}{s_1}, \quad {p'}_1^{12} = -s_2 p_2^{02}, \quad {p'}_2^{12} = s_1 p_1^{02}. $$

We note that

$$ q_1^1 = q_1^{02}, \quad q_2^1 = q_2^{02}, \quad p_1^1 = -\frac{\eta_0 s_2}{q_2^{01}} + p_1^{02}, \quad p_2^1 = \frac{\eta_0 s_2 (s_1 q_1^{01} - 1)}{s_1 (q_2^{01})^2} + \frac{\alpha_0}{q_2^{02}} + p_2^{02}. \tag{7.6} $$

The system $(q^{02}, p^{02}) \in \mathbb{C}^4$ is a symplectic coordinate system for $A_0(s) \cap W_1$.

Thus we have obtained six symplectic coordinate systems $(q_1^*.q_2^*, p_1^*, p_2^*)$ each of which separates solution curves passing through the accessible singular points (see (7.1)-(7.6)). We notice that the Hamiltonians of each coordinate system are also polynomials whose coefficients are rational functions of s holomorphic in B.

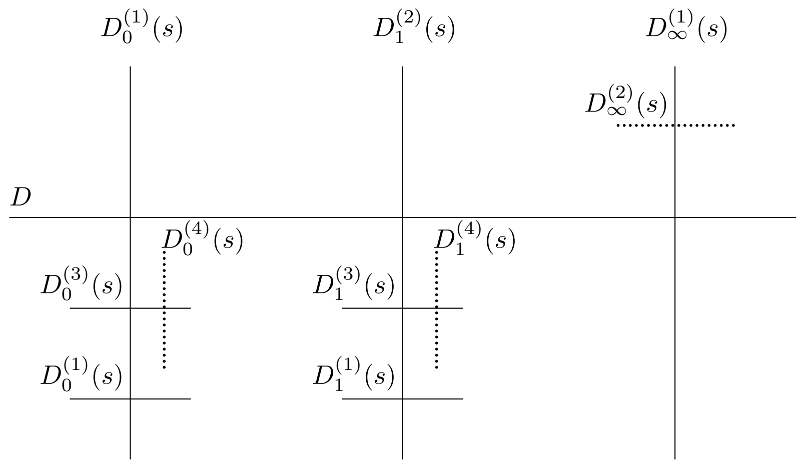

Figure 4. J=122

8 Spaces of initial conditions for $\mathcal{H}_{14}$

In the present case,

$$ \nu = -\alpha_\infty $$

In this section, we omitt the label 14.

8.1 Accessible singularities on $D \times B$

Observing the system $\mathcal{H}^{(0)}$ on all W_{ij} $j \neq 0$, we can obtain

Proposition 8.1. *The set of accessible singular points of the system* $\mathcal{H}_{14}^{(0)}$ *for each* $s = (s_1, s_2) \in B_{14}$ *is a disjoint union of two connected components* $A_0(s), A_\infty(s) \simeq \mathbb{P}^1$

$$\begin{aligned} A_0(s) = &\{(\xi, \eta_0, s) \in W_0 \times B | (s_1 + s_2^2/2)\xi_0 + s_2\xi_1 + \xi_2 = 0, \eta_{00} = 0, \eta_{01} - s_2\eta_{02} = 0\} \\ &\cup \{(\xi, \eta_1, s) \in W_1 \times B | (s_1 + s_2^2/2)\xi_0 + s_2\xi_1 + \xi_2 = 0, \eta_{10} = 0, \eta_{11} - (s_1 + s_2^2/2)\eta_{12} = 0\} \\ &\cup \{(\xi, \eta_2, s) \in W_2 \times B | (s_1 + s_2^2/2)\xi_0 + s_2\xi_1 + \xi_2 = 0, \eta_{20} = 0, (s_1 + s_2^2/2)\eta_{21} - s_2\eta_{22} = 0\} \\ A_\infty(s) = &\{(\xi, \eta_1, s) \in W_1 \times B | \xi_0 = \eta_{10} = \eta_{12} = 0\} \cup \{(\xi, \eta_2, s) \in W_2 \times B | \xi_0 = \eta_{20} = \eta_{21} = 0\}. \end{aligned}$$

In the following subsections, we obtain coordinate systems corresponding to $A_i(s)$ which separate completely the solution curves passing through $A_i(s)$. The systems for $A_0(s)$ and $A_\infty(s)$ are obtained by quadratic transformations.

8.2 Coordinate systems for $A_0(s)$

We make successively the quadratic transformations along $A_0(s) \cap W_0$, $A_0(s) \cap W_1$ and find coordinate systems for $A_0(s)$. Note that although $A_0(s)$ is expressed by the three coordinate systems W_0, W_1 and W_2, it can be done by two of them. In this paper, we choose W_0 and W_1.

8.2.1 Coordinate system for $A_0(s) \cap W_0$

We choose the coordinate system $W_{02} \subset W_0$. By setting $\xi_0 = \eta_{02} = 1$, we take $(\xi_1, \xi_2, \eta_{00}, \eta_{02})$ as the coordinates of W_{02}.

The first quadratic transformation along $A_0(s) \cap W_0$. Let

$$\begin{aligned} \xi_2 &= -s_2\xi_1 - (s_1 + s_2^2/2) + x_{00}^{(1)}, \quad \eta_{00} = x_{00}^{(1)} y_{00}^{(1)}, \quad \eta_{01} = s_2 + x_{00}^{(1)} z_{00}^{(1)}, \\ \xi_2 &= -s_2\xi_1 - (s_1 + s_2^2/2) + x_{01}^{(1)} y_{01}^{(1)}, \quad \eta_{00} = y_{01}^{(1)}, \quad \eta_{01} = s_2 + y_{01}^{(1)} z_{01}^{(1)}, \\ \xi_2 &= -s_2\xi_1 - (s_1 + s_2^2/2) + x_{02}^{(1)} z_{02}^{(1)}, \quad \eta_{00} = y_{02}^{(1)} z_{02}^{(1)}, \quad \eta_{01} = s_2 + z_{02}^{(1)}, \end{aligned}$$

then

$$D_{01}^{(1)}(s) = Q_{A_0(s) \cap W_0}(A_0(s) \cap (W_0 \times B)) = \{x_{00}^{(1)} = 0\} \cup \{y_{01}^{(1)} = 0\} \cup \{z_{02}^{(1)} = 0\},$$

the set of accessible singular points is given by

$$A_{01}^{(1)}(s) = \{(\xi_1, x_{01}^{(1)}, y_{01}^{(1)}, z_{01}^{(1)}) = (\xi_1, \alpha_0, 0, z_{01}^{(1)})\} \subset D_{01}^{(1)}(s).$$

The second quadratic transformation with $A_{01}^{(1)}(s)$. Let

$$\begin{aligned} x_{01}^{(1)} &= \alpha_0 + x_{00}^{(2)}, \quad y_{01}^{(1)} = x_{00}^{(2)} y_{00}^{(2)}, \\ x_{01}^{(1)} &= \alpha_0 + x_{01}^{(2)} y_{01}^{(2)}, \quad y_{01}^{(1)} = y_{01}^{(2)}, \end{aligned}$$

then

$$D_{01}^{(2)}(s) = Q_{A_{01}^{(1)}(s)}(A_{01}^{(1)}(s)) = \{x_{00}^{(2)} = 0\} \cup \{y_{01}^{(2)} = 0\}.$$

We can verify that our system has no singular points and every leaf is transversal with fibers in $(\xi_1, z_{01}^{(1)}, x_{01}^{(2)}, y_{01}^{(2)}, s)$-space $\mathbb{C}^4 \times B$. On the other hand, the point $(\xi_1, z_{01}^{(1)}, x_{00}^{(2)}, y_{00}^{(2)}) = (\xi_1, z_{01}^{(1)}, 0, 0)$ are inaccessible singular points.

Thus we have obtained a coordinate system $(\xi_1, z_{01}^{(1)}, x_{01}^{(2)}, y_{01}^{(2)}) \in \mathbb{C}^4$ which separates the solutions passing through $A_0(s) \cap W_{02}$. If we set

$$q_1^{01} = \xi_1, \quad q_2^{01} = -x_{01}^{(2)}, \quad p_1^{01} = z_{01}^{(1)}, \quad p_2^{01} = y_{01}^{(2)},$$

then we have

$$(8.1) \qquad q_1 = q_1^{01}, \quad q_2 = p_2^{01}(\alpha_0 - q_2^{01}p_2^{01}) - s_2 q_1^{01} - \frac{2s_1 + s_2^2}{2}, \quad p_1 = \frac{s_2}{p_2^{01}} + p_1^{01}, \quad p_2 = \frac{1}{p_2^{01}}.$$

Thus we have a simplectic coordinate system $(q^{01}, p^{01}) \in \mathbb{C}^4$ for $A_0(s) \cap W_0$.

8.2.2 Coordinate system for $A_0(s) \cap W_1$

We choose the coordinate system $W_{12} \subset W_1$. By setting $\xi_1 = \eta_{12} = 1$, we take $(\xi_1, \xi_2, \eta_{00}, \eta_{02})$ as the coordinates of W_{02}.

The first quadratic transformation along $A_0(s) \cap W_2$. Let

$$\begin{aligned}
\xi_2 &= -(s_1 + s_2^2/2)\xi_0 - s_2 + X_{00}^{(1)}, \quad \eta_{10} = X_{00}^{(1)}Y_{00}^{(1)}, \quad \eta_{11} = s_1 + s_2^2/2 + X_{00}^{(1)}Z_{00}^{(1)}, \\
\xi_2 &= -(s_1 + s_2^2/2)\xi_0 - s_2 + X_{01}^{(1)}Y_{01}^{(1)}, \quad \eta_{10} = Y_{01}^{(1)}, \quad \eta_{11} = s_1 + s_2^2/2 + Y_{01}^{(1)}Z_{01}^{(1)}, \\
\xi_2 &= -(s_1 + s_2^2/2)\xi_0 - s_2 + X_{02}^{(1)}Z_{02}^{(1)}, \quad \eta_{10} = Y_{02}^{(1)}Z_{02}^{(1)}, \quad \eta_{11} = s_1 + s_2^2/2 + Z_{02}^{(1)},
\end{aligned}$$

then

$$D_{02}^{(1)}(s) = Q_{A_0(s)\cap W_1}(A_0(s) \cap (W_1 \times B)) = \{X_{00}^{(1)} = 0\} \cup \{Y_{01}^{(1)} = 0\} \cup \{Z_{02}^{(1)} = 0\},$$

the set of accessible singular points is given by

$$A_{02}^{(1)}(s) = \{(\xi_0, X_{01}^{(1)}, Y_{01}^{(1)}, Z_{01}^{(1)}) = (\xi_0, \alpha_0, 0, Z_{01}^{(1)})\} \subset D_{02}^{(1)}(s).$$

The second quadratic transformation along $A_{02}^{(1)}(s)$. Let

$$\begin{aligned}
X_{01}^{(1)} &= \alpha_0 + X_{00}^{(2)}, \quad Y_{01}^{(1)} = X_{00}^{(2)}Y_{00}^{(2)}, \\
X_{01}^{(1)} &= \alpha_0 + X_{01}^{(2)}Y_{01}^{(2)}, \quad Y_{01}^{(1)} = Y_{01}^{(2)},
\end{aligned}$$

then

$$D_{02}^{(2)}(s) = Q_{A_{02}^{(1)}(s)}(A_{02}^{(1)}(s)) = \{X_{00}^{(2)} = 0\} \cup \{Y_{01}^{(2)} = 0\}.$$

We see that, in the $(\xi_0, Z_{01}^{(1)}, X_{01}^{(2)}, Y_{01}^{(2)}, s)$-space $\mathbb{C}^4 \times B$, the Pfaffin system has no singular points, every leaf is transversal with fibers, moreover, the points $(\xi_0, Z_{01}^{(1)}, X_{00}^{(2)}, Y_{00}^{(2)}) = (\xi_0, Z_{01}^{(1)}, 0, 0)$ are inaccessible singular points.

Thus we have obtained a coordinate system $(\xi_0, Z_{01}^{(1)}, X_{01}^{(2)}, Y_{01}^{(2)}) \in \mathbb{C}^4$ which separates the solutions passing through $A_0(s) \cap W_{12}$. If we set

$$q_1^{02} = \xi_0, \quad q_2^{02} = -X_{01}^{(2)}, \quad p_1^{02} = Z_{01}^{(1)}, \quad p_2^{02} = Y_{01}^{(2)},$$

then we have

$$(8.2) \quad q_1^1 = q_1^{02}, \quad q_2^1 = p_2^{02}(\alpha_0 - q_2^{02}p_2^{02}) - \Big(\frac{2s_1 + s_2^2}{2}\Big)q_1^{02} - s_2, \quad p_1^1 = \frac{2s_1 + s_2^2}{2p_2^{02}} + p_1^{02}, \quad p_2^1 = \frac{1}{p_2^{02}}.$$

The system $(q^{02}, p^{02}) \in \mathbb{C}^4$ is a symplectic coordinate system for $A_0(s) \cap W_1$.

8.3 Coordinate systems for $A_\infty(s)$

We obtain coordinate systems for $A_\infty(s)$ by making quadratic transformations eight times along $A_\infty(s) \cap W_1,\ A_\infty(s) \cap W_2$.

8.3.1 Coordinate system for $A_\infty(s) \cap W_1$

The first quadratic transformation along $A_\infty(s) \cap W_1$. Let

$$\begin{aligned}
\xi_0 = x^{(1)}_{\infty 0}, \quad \eta_{10} = x^{(1)}_{\infty 0} y^{(1)}_{\infty 0}, \quad \eta_{12} = x^{(1)}_{\infty 0} z^{(1)}_{\infty 0},\\
\xi_0 = x^{(1)}_{\infty 1} y^{(1)}_{\infty 1}, \quad \eta_{10} = y^{(1)}_{\infty 1}, \quad \eta_{12} = y^{(1)}_{\infty 1} z^{(1)}_{\infty 1},\\
\xi_0 = x^{(1)}_{\infty 2} z^{(1)}_{\infty 2}, \quad \eta_{10} = y^{(1)}_{\infty 2} z^{(1)}_{\infty 2}, \quad \eta_{12} = z^{(1)}_{\infty 2},
\end{aligned}$$

then

$$D^{(1)}_{\infty 1}(s) = Q_{A_\infty(s) \cap W_1}(A_\infty(s) \cap W_1) = \{x^{(1)}_{\infty 0} = 0\} \cup \{y^{(1)}_{\infty 1} = 0\} \cup \{z^{(1)}_{\infty 2} = 0\},$$

the set of accessible singular points is given by

$$A^{(1)}_{\infty 1}(s) = \{(\xi_2, x^{(1)}_{\infty 0}, y^{(1)}_{\infty 0}, z^{(1)}_{\infty 0}) = (\xi_2, 0, 0, 0)\} \subset D^{(1)}_{\infty 1}(s).$$

The second quadratic transformation along $A^{(1)}_{\infty 1}(s)$. Let

$$\begin{aligned}
x^{(1)}_{\infty 0} = x^{(2)}_{\infty 0}, \quad y^{(1)}_{\infty 0} = x^{(2)}_{\infty 0} y^{(2)}_{\infty 0}, \quad z^{(1)}_{\infty 0} = x^{(2)}_{\infty 0} z^{(2)}_{\infty 0},\\
x^{(1)}_{\infty 0} = x^{(2)}_{\infty 1} y^{(2)}_{\infty 1}, \quad y^{(1)}_{\infty 0} = y^{(2)}_{\infty 1}, \quad z^{(1)}_{\infty 0} = y^{(2)}_{\infty 1} z^{(2)}_{\infty 1},\\
x^{(1)}_{\infty 0} = x^{(1)}_{\infty 2} z^{(2)}_{\infty 2}, \quad y^{(1)}_{\infty 0} = x^{(2)}_{\infty 2} y^{(2)}_{\infty 2}, \quad z^{(1)}_{\infty 0} = z^{(2)}_{\infty 2},
\end{aligned}$$

then

$$D^{(2)}_{\infty 1}(s) = Q_{A^{(1)}_{\infty 1}(s)}(A^{(1)}_{\infty 1}(s)) = \{x^{(2)}_{\infty 0} = 0\} \cup \{y^{(2)}_{\infty 1} = 0\} \cup \{z^{(2)}_{\infty 2} = 0\},$$

the set of accessible singular points is given by

$$A^{(2)}_{\infty 1}(s) = \{(\xi_2, x^{(2)}_{\infty 0}, y^{(2)}_{\infty 0}, z^{(2)}_{\infty 0}) = (\xi_2, 0, 0, 1)\} \subset D^{(2)}_{\infty 1}(s).$$

The third quadratic transformation along $A^{(2)}_{\infty 1}(s)$. Let

$$\begin{aligned}
x^{(2)}_{\infty 0} = x^{(3)}_{\infty 0}, \quad y^{(2)}_{\infty 0} = x^{(3)}_{\infty 0} y^{(3)}_{\infty 0}, \quad z^{(2)}_{\infty 0} = 1 + x^{(3)}_{\infty 0} z^{(3)}_{\infty 0},\\
x^{(2)}_{\infty 0} = x^{(3)}_{\infty 1} y^{(3)}_{\infty 1}, \quad y^{(2)}_{\infty 0} = y^{(3)}_{\infty 1}, \quad z^{(2)}_{\infty 0} = 1 + y^{(3)}_{\infty 1} z^{(3)}_{\infty 1},\\
x^{(2)}_{\infty 0} = x^{(3)}_{\infty 2} z^{(3)}_{\infty 2}, \quad y^{(2)}_{\infty 0} = y^{(3)}_{\infty 2} z^{(3)}_{\infty 2}, \quad z^{(2)}_{\infty 0} = 1 + z^{(3)}_{\infty 2},
\end{aligned}$$

then

$$D^{(3)}_{\infty 1}(s) = Q_{A^{(2)}_{\infty 1}(s)}(A^{(2)}_{\infty 1}(s)) = \{x^{(3)}_{\infty 0} = 0\} \cup \{y^{(3)}_{\infty 1} = 0\} \cup \{z^{(3)}_{\infty 2} = 0\},$$

the set of accessible singular points is given by

$$A^{(3)}_{\infty 1}(s) = \{(\xi_2, x^{(3)}_{\infty 0}, y^{(3)}_{\infty 0}, z^{(3)}_{\infty 0}) = (\xi_2, 0, 0, 2\xi_2)\} \subset D^{(3)}_{\infty 1}(s).$$

The fourth quadratic transformation along $A^{(3)}_{\infty 1}(s)$. Let

$$\begin{aligned}
x^{(3)}_{\infty 0} = x^{(4)}_{\infty 0}, \quad y^{(3)}_{\infty 0} = x^{(4)}_{\infty 0} y^{(4)}_{\infty 0}, \quad z^{(3)}_{\infty 0} = 2\xi_2 + x^{(4)}_{\infty 0} z^{(4)}_{\infty 0},\\
x^{(3)}_{\infty 0} = x^{(4)}_{\infty 1} y^{(4)}_{\infty 1}, \quad y^{(3)}_{\infty 0} = y^{(4)}_{\infty 1}, \quad z^{(3)}_{\infty 0} = 2\xi_2 + y^{(4)}_{\infty 1} z^{(4)}_{\infty 1},\\
x^{(3)}_{\infty 0} = x^{(4)}_{\infty 2} z^{(4)}_{\infty 2}, \quad y^{(3)}_{\infty 0} = y^{(4)}_{\infty 2} z^{(4)}_{\infty 2}, \quad z^{(3)}_{\infty 0} = 2\xi_2 + z^{(4)}_{\infty 2},
\end{aligned}$$

then

$$D^{(4)}_{\infty 1}(s) = Q_{A^{(3)}_{\infty 1}(s)}(A^{(3)}_{\infty 1}(s)) = \{x^{(4)}_{\infty 0} = 0\} \cup \{y^{(4)}_{\infty 1} = 0\} \cup \{z^{(4)}_{\infty 2} = 0\},$$

the set of accessible singular points is given by

$$A^{(4)}_{\infty 1}(s) = \{(\xi_2, x^{(4)}_{\infty 0}, y^{(4)}_{\infty 0}, z^{(4)}_{\infty 0}) = (\xi_2, 0, -1, z^{(4)}_{\infty 0})\} \subset D^{(4)}_{\infty 1}(s).$$

The fifth quadratic transformation along $A^{(4)}_{\infty 1}(s)$. Here we insert a change of variables

$$\xi_2 = \xi_2, \quad x^{(4)}_{\infty 0} = x^{(4)}_{\infty 0}, \quad y^{(4)}_{\infty 0} = 1/v^{(4)}_{\infty 0}, \quad z^{(4)}_{\infty 0} = z^{(4)}_{\infty 0},$$

Let

$$\begin{aligned} x^{(4)}_{\infty 0} &= x^{(5)}_{\infty 0}, \quad v^{(4)}_{\infty 0} = -1 + x^{(5)}_{\infty 0} y^{(5)}_{\infty 0}, \\ x^{(4)}_{\infty 0} &= x^{(5)}_{\infty 1} y^{(5)}_{\infty 1}, \quad v^{(4)}_{\infty 0} = -1 + y^{(5)}_{\infty 1}, \end{aligned}$$

then

$$D^{(5)}_{\infty 1}(s) = Q_{A^{(4)}_{\infty 1}(s)}(A^{(4)}_{\infty 1}(s)) = \{x^{(5)}_{\infty 0} = 0\} \cup \{y^{(5)}_{\infty 1} = 0\},$$

the set of accessible singular points is given by

$$A^{(5)}_{\infty 1}(s) = \{(\xi_2, z^{(4)}_{\infty 0}, x^{(5)}_{\infty 0}, y^{(5)}_{\infty 0}) = (\xi_2, z^{(4)}_{\infty 0}, 0, 2\xi_2)\} \subset D^{(5)}_{\infty 1}(s).$$

The sixth quadratic transformation along $A^{(5)}_{\infty 1}(s)$. Let

$$\begin{aligned} x^{(5)}_{\infty 0} &= x^{(6)}_{\infty 0}, \quad y^{(5)}_{\infty 0} = 2\xi_2 + x^{(6)}_{\infty 0} y^{(6)}_{\infty 0}, \\ x^{(5)}_{\infty 0} &= x^{(6)}_{\infty 1} y^{(6)}_{\infty 1}, \quad y^{(5)}_{\infty 0} = 2\xi_2 + y^{(6)}_{\infty 1}, \end{aligned}$$

then

$$D^{(6)}_{\infty 1}(s) = Q_{A^{(5)}_{\infty 1}(s)}(A^{(5)}_{\infty 1}(s)) = \{x^{(6)}_{\infty 0} = 0\} \cup \{y^{(6)}_{\infty 1} = 0\},$$

the set of accessible singular points is given by

$$A^{(6)}_{\infty 1}(s) = \{(\xi_2, z^{(4)}_{\infty 0}, x^{(6)}_{\infty 0}, y^{(6)}_{\infty 0}) = (\xi_2, z^{(4)}_{\infty 0}, 0, 0)\} \subset D^{(6)}_{\infty 1}(s).$$

The seventh quadratic transformation along $A^{(6)}_{\infty 1}(s)$. Let

$$\begin{aligned} x^{(6)}_{\infty 0} &= x^{(7)}_{\infty 0}, \quad y^{(6)}_{\infty 0} = x^{(7)}_{\infty 0} y^{(7)}_{\infty 0}, \\ x^{(6)}_{\infty 0} &= x^{(7)}_{\infty 1} y^{(7)}_{\infty 1}, \quad y^{(6)}_{\infty 0} = y^{(7)}_{\infty 1} \end{aligned}$$

then

$$D^{(7)}_{\infty 1}(s) = Q_{A^{(6)}_{\infty 1}(s)}(A^{(6)}_{\infty 1}(s)) = \{x^{(7)}_{\infty 0} = 0\} \cup \{y^{(7)}_{\infty 1} = 0\}.$$

the set of accessible singular points is given by

$$A^{(7)}_{\infty 1}(s) = \{(\xi_2, z^{(4)}_{\infty 0}, x^{(7)}_{\infty 0}, y^{(7)}_{\infty 0}) = (\xi_2, z^{(4)}_{\infty 0}, 0, 1 - \alpha_0 + 2\alpha_\infty)\} \subset D^{(7)}_{\infty 1}(s).$$

The eighth quadratic transformation along $A^{(7)}_{\infty 1}(s)$. Let

$$\begin{aligned} x^{(7)}_{\infty 0} &= x^{(8)}_{\infty 0}, \quad y^{(7)}_{\infty 0} = 1 - \alpha_0 + 2\alpha_\infty + x^{(8)}_{\infty 0} y^{(8)}_{\infty 0}, \\ x^{(7)}_{\infty 0} &= x^{(8)}_{\infty 1} y^{(8)}_{\infty 1}, \quad y^{(7)}_{\infty 0} = 1 - \alpha_0 + 2\alpha_\infty + y^{(8)}_{\infty 1}, \end{aligned}$$

then

$$D^{(8)}_{\infty 1}(s) = Q_{A^{(7)}_{\infty 1}(s)}(A^{(7)}_{\infty 1}(s)) = \{x^{(8)}_{\infty 0} = 0\} \cup \{y^{(8)}_{\infty 1} = 0\},$$

We can verify that the differential system in the cordinates $(\xi_2, z_{\infty 0}^{(4)}, x_{\infty 0}^{(8)}, y_{\infty 0}^{(8)})$ is holomorphic in a neighborhood of $\{x_{\infty 0}^{(8)} = 0\}$ and the points $(\xi_2, z_{\infty 0}^{(4)}, x_{\infty 1}^{(8)}, y_{\infty 1}^{(8)}) = (\xi_2, z_{\infty 0}^{(4)}, 0, 0)$ are inaccessible.

Thus we have obtained a coordinate system $(\xi_2, z_{\infty 0}^{(4)}, x_{\infty 0}^{(8)}, y_{\infty 0}^{(8)}) \in \mathbb{C}^4$ which is separates the solutions passing through $A_\infty(s) \cap W_1 = A_\infty(s) \cap W_{11}$. It is related to the coordinate system $(q_1^1, q_2^1, p_1^1, p_2^1)$ by

$$\begin{aligned}
& q_1^1 = x_{\infty 0}^{(8)}, \quad q_2^1 = \xi_2, \quad p_1^1 = -\frac{1}{(x_{\infty 0}^{(8)})^4} + \frac{2\xi_2}{(x_{\infty 0}^{(8)})^3} + \frac{1-\alpha_0+2\alpha_\infty}{x_{\infty 0}^{(8)}} + y_{\infty 0}^{(8)}, \\
& p_2^1 = 4\xi_2^2 - z_{\infty 0}^{(4)} + (1-\alpha_0+2\alpha_\infty + 2\xi_2 z_{\infty 0}^{(4)}) x_{\infty 0}^{(8)} \\
& \qquad + \{2(1-\alpha_0+2\alpha_\infty)\xi_2 + y_{\infty 0}^{(8)}\}(x_{\infty 0}^{(8)})^2 \\
& \qquad + \{(1-\alpha_0+2\alpha_\infty) z_{\infty 0}^{(4)} + 2\xi_2 y_{\infty 0}^{(8)}\}(x_{\infty 0}^{(8)})^3 + z_{\infty 0}^{(4)} y_{\infty 0}^{(8)} (x_{\infty 0}^{(10)})^4.
\end{aligned}$$

Now we calculate the 2-form $dq_1^1 \wedge dp_1^1 + dq_2^1 \wedge dp_2^1$ in the coordinates $(\xi_2, z_{\infty 0}^{(4)}, x_{\infty 0}^{(8)}, y_{\infty 0}^{(8)})$:

$$\begin{aligned}
& dq_1^1 \wedge dp_1^1 + dq_2^1 \wedge dp_2^1 \\
= \; & dx_{\infty 0}^{(8)} \wedge dy_{\infty 0}^{(8)} \\
& + \{y_{\infty 0}^{(8)}(x_{\infty 0}^{(8)})^4 + (1-\alpha_0+2\alpha_\infty)(x_{\infty 0}^{(8)})^3 + 2\xi_2 x_{\infty 0}^{(8)} - 1\} d\xi_2 \wedge dz_{\infty 0}^{(4)} \\
& + [4 z_{\infty 0}^{(4)} y_{\infty 0}^{(8)} (x_{\infty 0}^{(8)})^3 + 3\{(1-\alpha_0+2\alpha_\infty) z_{\infty 0}^{(4)} + 2\xi_2 y_{\infty 0}^{(8)}\}(x_{\infty 0}^{(8)})^2 \\
& \qquad + 2\{2(1-\alpha_0+2\alpha_\infty)\xi_2 + y_{\infty 0}^{(8)}\} x_{\infty 0}^{(8)} + 2\xi_2 z_{\infty 0}^{(4)} + 1 - \alpha_0 + 2\alpha_\infty] d\xi_2 \wedge dx_{\infty 0}^{(8)} \\
& + \{z_{\infty 0}^{(4)}(x_{\infty 0}^{(8)})^4 + 2\xi_2 (x_{\infty 0}^{(8)})^3 + (x_{\infty 0}^{(8)})^2\} d\xi_2 \wedge dy_{\infty 0}^{(8)} \\
= \; & dx_{\infty 0}^{(8)} \wedge dy_{\infty 0}^{(8)} \\
& + d\xi_2 \wedge d[\{y_{\infty 0}^{(8)}(x_{\infty 0}^{(8)})^4 + (1-\alpha_0+2\alpha_\infty)(x_{\infty 0}^{(8)})^3 + 2\xi_2 x_{\infty 0}^{(8)} - 1\} z_{\infty 0}^{(4)} \\
& \qquad + \{2\xi_2 (x_{\infty 0}^{(8)})^3 + (x_{\infty 0}^{(8)})^2\} y_{\infty 0}^{(8)} + 2(1-\alpha_0+2\alpha_\infty)\xi_2 (x_{\infty 0}^{(8)})^2 \\
& \qquad + (1-\alpha_0+2\alpha_\infty) x_{\infty 0}^{(8)}] \\
= \; & dx_{\infty 0}^{(8)} \wedge dy_{\infty 0}^{(8)} \\
& + d\xi_2 \wedge d[\{y_{\infty 0}^{(8)}(x_{\infty 0}^{(8)})^4 + (1-\alpha_0+2\alpha_\infty)(x_{\infty 0}^{(8)})^3 + 2\xi_2 x_{\infty 0}^{(8)} - 1\} z_{\infty 0}^{(4)} \\
& \qquad + \{2\xi_2 (x_{\infty 0}^{(8)})^3 + (x_{\infty 0}^{(8)})^2\} y_{\infty 0}^{(8)} + 2(1-\alpha_0+2\alpha_\infty)\xi_2 (x_{\infty 0}^{(8)})^2 \\
& \qquad + (1-\alpha_0+2\alpha_\infty) x_{\infty 0}^{(8)} + 4\xi_2^2].
\end{aligned}$$

Therefore, setting

$$\begin{aligned}
w_{\infty 0}^{(4)} = \; & \{y_{\infty 0}^{(8)}(x_{\infty 0}^{(8)})^4 + (1-\alpha_0+2\alpha_\infty)(x_{\infty 0}^{(8)})^3 + 2\xi_2 x_{\infty 0}^{(8)} - 1\} z_{\infty 0}^{(4)} \\
& + \{2\xi_2 (x_{\infty 0}^{(8)})^3 + (x_{\infty 0}^{(8)})^2\} y_{\infty 0}^{(8)} + 2(1-\alpha_0+2\alpha_\infty)\xi_2 (x_{\infty 0}^{(8)})^2 \\
& + (1-\alpha_0+2\alpha_\infty) x_{\infty 0}^{(8)} + 4\xi_2^2,
\end{aligned}$$

we have symplectic coordinates $(\xi_2, w_{\infty 0}^{(4)}, x_{\infty 0}^{(8)}, y_{\infty 0}^{(8)})$. Writing

$$q_1^{\infty 1} = x_{\infty 0}^{(8)}, \quad q_2^{\infty 1} = \xi_2, \quad p_1^{\infty 1} = y_{\infty 0}^{(8)}, \quad p_2^{\infty 1} = w_{\infty 0}^{(4)}.$$

we have

$$\begin{aligned}
& q_1^1 = q_1^{\infty 1}, \quad q_2^1 = q_2^{\infty 1}, \\
& p_1^1 = -\frac{1}{(q_1^{\infty 1})^4} + \frac{2\xi_2}{(q_1^{\infty 1})^3} + \frac{1-\alpha_0+2\alpha_\infty}{q_1^{\infty 1}} + p_1^{\infty 1}, \\
& p_2^1 = -\frac{1}{(q_1^{\infty 1})^2} + p_2^{\infty 1}.
\end{aligned} \tag{8.3}$$

Thus we have a symplectic coordinate system $(q^{\infty 1}, p^{\infty 1}) \in \mathbb{C}^4$ which separates solution curves passing through $A_\infty(s) \cap W_1$.

8.3.2 Coordinate system for $A_\infty(s) \cap W_2$

The first quadratic transformation along $A_\infty(s) \cap W_2$. Let

$$\begin{aligned}
&\xi_0 = X^{(1)}_{\infty 0}, \quad \eta_{20} = X^{(1)}_{\infty 0} Y^{(1)}_{\infty 0}, \quad \eta_{21} = X^{(1)}_{\infty 0} Z^{(1)}_{\infty 0},\\
&\xi_0 = X^{(1)}_{\infty 1} Y^{(1)}_{\infty 1}, \quad \eta_{20} = Y^{(1)}_{\infty 1}, \quad \eta_{21} = Y^{(1)}_{\infty 1} Z^{(1)}_{\infty 1},\\
&\xi_0 = X^{(1)}_{\infty 2} Z^{(1)}_{\infty 2}, \quad \eta_{20} = Y^{(1)}_{\infty 2} Z^{(1)}_{\infty 2}, \quad \eta_{21} = Z^{(1)}_{\infty 2},
\end{aligned}$$

then

$$D^{(1)}_{\infty 2}(s) = Q_{A_\infty(s)\cap W_2}(A_\infty(s) \cap W_2) = \{X^{(1)}_{\infty 0} = 0\} \cup \{Y^{(1)}_{\infty 1} = 0\} \cup \{Z^{(1)}_{\infty 2} = 0\},$$

the set of accessible singular points is given by

$$A^{(1)}_{\infty 2}(s) = \{(\xi_1, X^{(1)}_{\infty 0}, Y^{(1)}_{\infty 0}, Z^{(1)}_{\infty 0}) = (\xi_1, 0, 0, -1/\xi_1)\} \subset D^{(1)}_{\infty 2}(s).$$

The second quadratic transformation along $A^{(1)}_{\infty 2}(s)$. Note that $\xi_1 = 0$ is excluded. Let

$$\begin{aligned}
&X^{(1)}_{\infty 0} = X^{(2)}_{\infty 0}, \quad Y^{(1)}_{\infty 0} = X^{(2)}_{\infty 0} Y^{(2)}_{\infty 0}, \quad Z^{(1)}_{\infty 0} = -1/\xi_1 + X^{(2)}_{\infty 0} Z^{(2)}_{\infty 0},\\
&X^{(1)}_{\infty 0} = X^{(2)}_{\infty 1} Y^{(2)}_{\infty 1}, \quad Y^{(1)}_{\infty 0} = Y^{(2)}_{\infty 1}, \quad Z^{(1)}_{\infty 0} = -1/\xi_1 + Y^{(2)}_{\infty 1} Z^{(2)}_{\infty 1},\\
&X^{(1)}_{\infty 0} = X^{(1)}_{\infty 2} Z^{(2)}_{\infty 2}, \quad Y^{(1)}_{\infty 0} = Y^{(2)}_{\infty 2} Z^{(2)}_{\infty 2}, \quad Z^{(1)}_{\infty 0} = -1/\xi_1 + Z^{(2)}_{\infty 2},
\end{aligned}$$

then

$$D^{(2)}_{\infty 2}(s) = Q_{A^{(1)}_{\infty 2}(s)}(A^{(1)}_{\infty 2}(s)) = \{X^{(2)}_{\infty 0} = 0\} \cup \{Y^{(2)}_{\infty 1} = 0\} \cup \{Z^{(2)}_{\infty 2} = 0\},$$

the set of accessible singular points is given by

$$A^{(2)}_{\infty 2}(s) = \{(\xi_1, X^{(2)}_{\infty 0}, Y^{(2)}_{\infty 0}, Z^{(2)}_{\infty 0}) = (\xi_1, 0, 0, -1/\xi_1^3)\} \subset D^{(2)}_{\infty 2}(s).$$

The third quadratic transformation along $A^{(2)}_{\infty 2}(s)$. Let

$$\begin{aligned}
&X^{(2)}_{\infty 0} = X^{(3)}_{\infty 0}, \quad Y^{(2)}_{\infty 0} = X^{(3)}_{\infty 0} Y^{(3)}_{\infty 0}, \quad Z^{(2)}_{\infty 0} = -1/\xi_1^3 + X^{(3)}_{\infty 0} Z^{(3)}_{\infty 0},\\
&X^{(2)}_{\infty 0} = X^{(3)}_{\infty 1} Y^{(3)}_{\infty 1}, \quad Y^{(2)}_{\infty 0} = Y^{(3)}_{\infty 1}, \quad Z^{(2)}_{\infty 0} = -1/\xi_1^3 + Y^{(3)}_{\infty 1} Z^{(3)}_{\infty 1},\\
&X^{(2)}_{\infty 0} = X^{(3)}_{\infty 2} Z^{(3)}_{\infty 2}, \quad Y^{(2)}_{\infty 0} = Y^{(3)}_{\infty 2} Z^{(3)}_{\infty 2}, \quad Z^{(2)}_{\infty 0} = -1/\xi_1^3 + Z^{(3)}_{\infty 2},
\end{aligned}$$

then

$$D^{(3)}_{\infty 2}(s) = Q_{A^{(2)}_{\infty 2}(s)}(A^{(2)}_{\infty 2}(s)) = \{X^{(3)}_{\infty 0} = 0\} \cup \{Y^{(3)}_{\infty 1} = 0\} \cup \{Z^{(3)}_{\infty 2} = 0\},$$

the set of accessible singular points is given by

$$A^{(3)}_{\infty 2}(s) = \{(\xi_1, X^{(3)}_{\infty 0}, Y^{(3)}_{\infty 0}, Z^{(3)}_{\infty 0}) = (\xi_1, 0, 0, -2/\xi_1^5)\} \subset D^{(3)}_{\infty 2}(s).$$

The fourth quadratic transformation along $A^{(3)}_{\infty 2}(s)$. Let

$$\begin{aligned}
&X^{(3)}_{\infty 0} = X^{(4)}_{\infty 0}, \quad Y^{(3)}_{\infty 0} = X^{(4)}_{\infty 0} Y^{(4)}_{\infty 0}, \quad Z^{(3)}_{\infty 0} = -2/\xi_1^5 + X^{(4)}_{\infty 0} Z^{(4)}_{\infty 0},\\
&X^{(3)}_{\infty 0} = X^{(4)}_{\infty 1} Y^{(4)}_{\infty 1}, \quad Y^{(3)}_{\infty 0} = Y^{(4)}_{\infty 1}, \quad Z^{(3)}_{\infty 0} = -2/\xi_1^5 + Y^{(4)}_{\infty 1} Z^{(4)}_{\infty 1},\\
&X^{(3)}_{\infty 0} = X^{(4)}_{\infty 2} Z^{(4)}_{\infty 2}, \quad Y^{(3)}_{\infty 0} = Y^{(4)}_{\infty 2} Z^{(4)}_{\infty 2}, \quad Z^{(3)}_{\infty 0} = -2/\xi_1^5 + Z^{(4)}_{\infty 2},
\end{aligned}$$

then

$$D^{(4)}_{\infty 2}(s) = Q_{A^{(3)}_{\infty 2}(s)}(A^{(3)}_{\infty 2}(s)) = \{X^{(4)}_{\infty 0} = 0\} \cup \{Y^{(4)}_{\infty 1} = 0\} \cup \{Z^{(4)}_{\infty 2} = 0\},$$

the set of accessible singular points is given by

$$A^{(4)}_{\infty 2}(s) = \{(\xi_1, X^{(4)}_{\infty 0}, Y^{(4)}_{\infty 0}, Z^{(4)}_{\infty 0}) = (\xi_1, 0, -1/\xi_1^3, Z^{(4)}_{\infty 0})\} \subset D^{(4)}_{\infty 2}(s).$$

The fifth quadratic transformation along $A^{(4)}_{\infty 2}(s)$. We insert here the transformations

$$\xi_1 = \xi_1, \quad X^{(4)}_{\infty 0} = X^{(4)}_{\infty 0}, \quad Y^{(4)}_{\infty 0} = 1/V^{(4)}_{\infty 0}, \quad Z^{(4)}_{\infty 0} = Z^{(4)}_{\infty 0}.$$

Let

$$X^{(4)}_{\infty 0} = X^{(5)}_{\infty 0}, \quad V^{(4)}_{\infty 0} = -\xi_1^3 + X^{(5)}_{\infty 0} Y^{(5)}_{\infty 0},$$
$$X^{(4)}_{\infty 0} = X^{(5)}_{\infty 1} Y^{(5)}_{\infty 1}, \quad V^{(4)}_{\infty 0} = -\xi_1^3 + Y^{(5)}_{\infty 1},$$

then

$$D^{(5)}_{\infty 2}(s) = Q_{A^{(4)}_{\infty 2}(s)}(A^{(4)}_{\infty 2}(s)) = \{X^{(5)}_{\infty 0} = 0\} \cup \{Y^{(5)}_{\infty 1} = 0\},$$

the set of accessible singular points is given by

$$A^{(5)}_{\infty 2}(s) = \{(\xi_1, Z^{(4)}_{\infty 0}, X^{(5)}_{\infty 0}, Y^{(5)}_{\infty 0}) = (\xi_1, Z^{(4)}_{\infty 0}, 0, 2\xi_1)\} \subset D^{(5)}_{\infty 2}(s).$$

The sixth quadratic transformation along $A^{(5)}_{\infty 2}(s)$. Let

$$X^{(5)}_{\infty 0} = X^{(6)}_{\infty 0}, \quad Y^{(5)}_{\infty 0} = 2\xi_1 + X^{(6)}_{\infty 0} Y^{(6)}_{\infty 0},$$
$$X^{(5)}_{\infty 0} = X^{(6)}_{\infty 1} Y^{(6)}_{\infty 1}, \quad Y^{(5)}_{\infty 0} = 2\xi_1 + Y^{(6)}_{\infty 1},$$

then

$$D^{(6)}_{\infty 2}(s) = Q_{A^{(5)}_{\infty 2}(s)}(A^{(5)}_{\infty 2}(s)) = \{X^{(6)}_{\infty 0} = 0\} \cup \{Y^{(6)}_{\infty 1} = 0\},$$

the set of accessible singular points is given by

$$A^{(6)}_{\infty 2}(s) = \{(\xi_1, Z^{(4)}_{\infty 0}, X^{(6)}_{\infty 0}, Y^{(6)}_{\infty 0}) = (\xi_1, Z^{(4)}_{\infty 0}, 0, 0)\} \subset D^{(6)}_{\infty 2}(s).$$

The seventh quadratic transformation along $A^{(6)}_{\infty 2}(s)$. Let

$$X^{(6)}_{\infty 0} = X^{(7)}_{\infty 0}, \quad Y^{(6)}_{\infty 0} = X^{(7)}_{\infty 0} Y^{(7)}_{\infty 0},$$
$$X^{(6)}_{\infty 0} = X^{(7)}_{\infty 1} Y^{(7)}_{\infty 1}, \quad Y^{(6)}_{\infty 0} = Y^{(7)}_{\infty 1},$$

then

$$D^{(7)}_{\infty 2}(s) = Q_{A^{(6)}_{\infty 2}(s)}(A^{(6)}_{\infty 2}(s)) = \{X^{(7)}_{\infty 0} = 0\} \cup \{Y^{(7)}_{\infty 1} = 0\},$$

the set of accessible singular points is given by

$$A^{(7)}_{\infty 2}(s) = \{(\xi_1, Z^{(4)}_{\infty 0}, X^{(7)}_{\infty 0}, Y^{(7)}_{\infty 0}) = (\xi_1, Z^{(4)}_{\infty 0}, 0, 1 - \alpha_0 + 2\alpha_\infty)\} \subset D^{(7)}_{\infty 2}(s).$$

The eighth quadratic transformation along $A^{(7)}_{\infty 2}(s)$. Let

$$X^{(7)}_{\infty 0} = X^{(8)}_{\infty 0}, \quad Y^{(7)}_{\infty 0} = 1 - \alpha_0 + 2\alpha_\infty + X^{(8)}_{\infty 0} Y^{(8)}_{\infty 0},$$
$$X^{(7)}_{\infty 0} = X^{(8)}_{\infty 1} Y^{(8)}_{\infty 1}, \quad Y^{(7)}_{\infty 0} = 1 - \alpha_0 + 2\alpha_\infty + Y^{(8)}_{\infty 1},$$

then

$$D^{(8)}_{\infty 2}(s) = Q_{A^{(7)}_{\infty 2}(s)}(A^{(7)}_{\infty 2}(s)) = \{X^{(8)}_{\infty 0} = 0\} \cup \{Y^{(8)}_{\infty 1} = 0\}.$$

We see that, in the $(\xi_1, Z^{(4)}_{\infty 0}, X^{(8)}_{\infty 0}, Y^{(8)}_{\infty 0}, s)$-space $\mathbb{C}^4 \times B$, the differential system is holomorphic in a neighborhood of $\{X^{(8)}_{\infty 0} = 0\}$ except for $\xi_1 = 0$, moreover, the points $(\xi_1, Z^{(4)}_{\infty 0}, X^{(8)}_{\infty 1}, Y^{(8)}_{\infty 1}) = (\xi_0, Z^{(4)}_{\infty 0}, 0, 0)$ are inaccessible with $\xi_1 \neq 0$.

Thus we have obtained a coordinate system $(\xi_1, Z^{(4)}_{\infty 0}, X^{(8)}_{\infty 0}, y^{(8)}_{\infty 0}) \in \mathbb{C}^4$ which is related to the coordinate system $(q_1^2, q_2^2, p_1^2, p_2^2)$ as

$$
\begin{aligned}
&q_1^2 = \xi_1, \quad q_2^2 = X^{(8)}_{\infty 0},\\
&p_1^1 = \frac{\xi_1^2}{(X^{(10)}_{\infty 0})^3} - \frac{1}{(X^{(10)}_{\infty 0})^2} - 2\xi_1^3 Z^{(4)}_{\infty 0} - \frac{1-\alpha_0+2\alpha_\infty}{\xi_1} - \frac{4}{\xi_1^4}\\
&\qquad + \Big(2\xi_1 Z^{(4)}_{\infty 0} - \frac{Y^{(10)}_{\infty 0}}{\xi_1} - \frac{1-\alpha_0+2\alpha_\infty}{\xi_1^3}\Big) X^{(10)}_{\infty 0} - \Big(\frac{Y^{(8)}_{\infty 0}}{\xi_1^3} + \frac{2(1-\alpha_0+2\alpha_\infty)}{\xi_1^5}\Big)(X^{(10)}_{\infty 0})^2\\
&\qquad - \Big\{(1-\alpha_0+2\alpha_\infty) Z^{(4)}_{\infty 0} + \frac{2Y^{(8)}_{\infty 0}}{\xi_2^5}\Big\}(X^{(8)}_{\infty 0})^3 + Z^{(4)}_{\infty 0} Y^{(8)}_{\infty 0} (X^{(8)}_{\infty 0})^4,\\
&p_1^2 = -\frac{1}{(X^{(8)}_{10})^4} + \frac{2\xi_1}{(X^{(8)}_{10})^3} + \frac{1-\alpha_0+2\alpha_\infty}{X^{(8)}_{\infty 0}} + Y^{(8)}_{\infty 0},
\end{aligned}
$$

We calculate the 2-form $dq_1^2 \wedge dp_1^2 + dq_2^2 \wedge dp_2^2$ in the coordinates $(\xi_1, Z^{(4)}_{\infty 0}, X^{(8)}_{\infty 0}, Y^{(8)}_{\infty 0})$:

$$
\begin{aligned}
&dq_1^2 \wedge dp_1^2 + dq_2^2 \wedge dp_2^2\\
&\quad = dX^{(8)}_{\infty 0} \wedge dY^{(8)}_{\infty 0}\\
&\qquad + \{Y^{(8)}_{\infty 0}(X^{(8)}_{\infty 0})^4 + (1-\alpha_0+2\alpha_\infty)(X^{(8)}_{\infty 0})^3 + 2\xi_1 X^{(8)}_{\infty 0} - \xi_1^3\} d\xi_1 \wedge dZ^{(4)}_{\infty 0}\\
&\qquad + \Big[4Z^{(4)}_{\infty 0} Y^{(8)}_{\infty 0} (X^{(8)}_{\infty 0})^3 + 3\Big\{(1-\alpha_0+2\alpha_\infty) Z^{(4)}_{\infty 0} - \frac{2Y^{(8)}_{\infty 0}}{\xi_1^5}\Big\}(X^{(8)}_{\infty 0})^2\\
&\qquad\qquad - 2\Big\{\frac{Y^{(8)}_{\infty 0}}{\xi_1^3} + \frac{2(1-\alpha_0+2\alpha_\infty)}{\xi_1^5}\Big\} X^{(8)}_{\infty 0} + 2\xi_2 Z^{(4)}_{\infty 0} - \frac{Y^{(8)}_{\infty 0}}{\xi_1^3} - \frac{1-\alpha_0+2\alpha_\infty}{\xi_1^5}\Big] d\xi_1 \wedge dX^{(8)}_{\infty 0}\\
&\qquad + \Big\{Z^{(4)}_{\infty 0}(X^{(8)}_{\infty 0})^4 - \frac{2(X^{(8)}_{\infty 0})^3}{\xi_1^5} - \frac{(X^{(8)}_{\infty 0})^2}{\xi_1^3} - \frac{X^{(8)}_{\infty 0}}{\xi_1^5}\Big\} d\xi_1 \wedge dY^{(8)}_{\infty 0}\\
&\quad = dX^{(8)}_{\infty 0} \wedge dY^{(8)}_{\infty 0}\\
&\qquad + d\xi_2 \wedge d\Big[\{Y^{(8)}_{\infty 0}(X^{(8)}_{\infty 0})^4 + (1-\alpha_0+2\alpha_\infty)(X^{(8)}_{\infty 0})^3 + 2\xi_1 X^{(8)}_{\infty 0} - \xi_1^3\} Z^{(4)}_{\infty 0}\\
&\qquad\qquad - \Big\{\frac{2(X^{(8)}_{\infty 0})^3}{\xi_1^5} + \frac{(X^{(8)}_{\infty 0})^2}{\xi_1^3} + \frac{X^{(8)}_{\infty 0}}{\xi_1^5}\Big\} Y^{(8)}_{\infty 0} - \frac{2(1-\alpha_0+2\alpha_\infty)}{\xi_1^5}(X^{(8)}_{\infty 0})^2\\
&\qquad\qquad - \frac{1-\alpha_0+2\alpha_\infty}{\xi_1^5} X^{(8)}_{\infty 0}\Big]\\
&\quad = dx^{(8)}_{\infty 0} \wedge dy^{(8)}_{\infty 0}\\
&\qquad + d\xi_2 \wedge d\Big[\{Y^{(8)}_{\infty 0}(X^{(8)}_{\infty 0})^4 + (1-\alpha_0+2\alpha_\infty)(X^{(8)}_{\infty 0})^3 + 2\xi_1 X^{(8)}_{\infty 0} - \xi_1^3\} Z^{(4)}_{\infty 0}\\
&\qquad\qquad - \Big\{\frac{2(X^{(8)}_{\infty 0})^3}{\xi_1^5} + \frac{(X^{(8)}_{\infty 0})^2}{\xi_1^3} + \frac{X^{(8)}_{\infty 0}}{\xi_1^5}\Big\} Y^{(8)}_{\infty 0} - \frac{2(1-\alpha_0+2\alpha_\infty)}{\xi_1^5}(X^{(8)}_{\infty 0})^2\\
&\qquad\qquad - \frac{1-\alpha_0+2\alpha_\infty}{\xi_1^5} X^{(8)}_{\infty 0} - \frac{1-\alpha_0+2\alpha_\infty}{\xi_1} - \frac{4}{\xi_1^4}\Big].
\end{aligned}
$$

Therefore, setting

$$\begin{aligned}W_{\infty 0}^{(4)} =& \{Y_{\infty 0}^{(8)}(X_{\infty 0}^{(8)})^4 + (1-\alpha_0+2\alpha_\infty)(X_{\infty 0}^{(8)})^3 + 2\xi_1 X_{\infty 0}^{(8)} - \xi_1^3\} Z_{\infty 0}^{(4)} \\ & -\left\{\frac{2(X_{\infty 0}^{(8)})^3}{\xi_1^5} + \frac{(X_{\infty 0}^{(8)})^2}{\xi_1^3} + \frac{X_{\infty 0}^{(8)}}{\xi_1^5}\right\} Y_{\infty 0}^{(8)} - \frac{2(1-\alpha_0+2\alpha_\infty)}{\xi_1^5}(X_{\infty 0}^{(8)})^2 \\ & -\frac{1-\alpha_0+2\alpha_\infty}{\xi_1^5} X_{\infty 0}^{(8)} - \frac{1-\alpha_0+2\alpha_\infty}{\xi_1} - \frac{4}{\xi_1^4},\end{aligned}$$

we have symplectic coordinates $(\xi_1, W_{\infty 0}^{(4)}, X_{\infty 0}^{(8)}, Y_{\infty 0}^{(8)})$. Writing

$$q_1^{\infty 2} = \xi_1, \quad q_2^{\infty 2} = X_{\infty 0}^{(8)}, \quad p_1^{\infty 2} = W_{\infty 0}^{(4)}, \quad p_2^{\infty 2} = Y_{\infty 0}^{(8)}.$$

we have

$$\begin{aligned}& q_1^2 = q_1^{\infty 2}, \quad q_2^2 = q_2^{\infty 2}, \\ (8.4) \qquad & p_1^2 = \frac{(q_1^{\infty 2})^2}{(q_2^{\infty 2})^2} - \frac{1}{(q_2^{\infty 2})^2} + p_1^{\infty 2}, \\ & p_2^2 = -\frac{(q_1^{\infty 2})^3}{(q_2^{\infty 2})^4} + \frac{2q_1^{\infty 2}}{(q_2^{\infty 2})^3} + \frac{1-\alpha_0+2\alpha_\infty}{q_2^{\infty 2}} + p_2^{\infty 2}\end{aligned}$$

The system $(q^{\infty 2}, p^{\infty 2}) \in \mathbb{C}^4$ separates solution curves passing through $A_\infty(s) \cap W_2$ and the Hamiltonians have no singularity on $\xi_1 = 0$.

Thus we have obtained four symplectic coordinate systems $(q_1^*, q_2^*, p_1^*, p_2^*)$ each of which separates solution curves passing through the accessible singular points (see (8.1)-(8.4)). We notice that the Hamiltonians of each coordinate system are also polynomials whose coefficients are rational functions of s holomorphic in B.

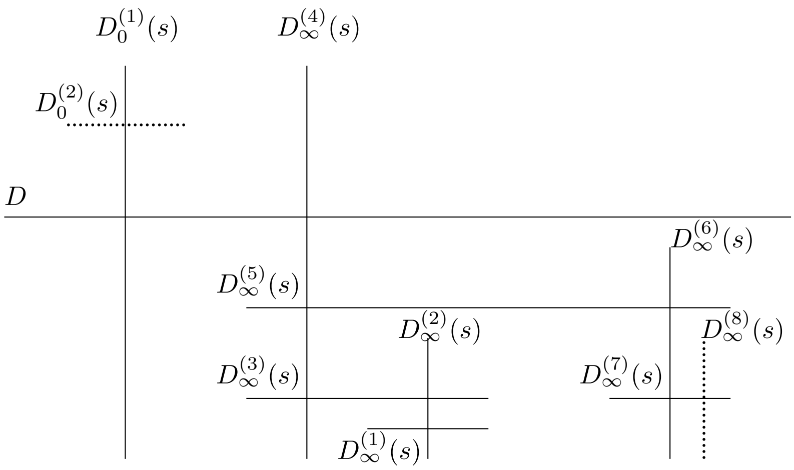

Figure 5. J=14

9 Spaces of initial conditions for $\mathcal{H}_{23}$

In the present case,

$$\nu = -\alpha_\infty$$

In this section, we omitt the label 23.

9.1 Accessible singularities on $D \times B$

Observing the system $\mathcal{H}^{(0)}$ on all W_{ij} $j \neq 0$, we can obtain

Proposition 9.1. *The set of accessible singular points of the system $\mathcal{H}_{23}^{(0)}$ for each $s = (s_1, s_2) \in B_{23}$ is a disjoint union of 2 connected components $A_0(s), A_\infty(s) \simeq \mathbb{P}^1$*

$$A_0(s) = \{(\xi, \eta_0, s) \in W_0 \times B | \xi_2 = \eta_{00} = \eta_{01} = 0\} \cup \{(\xi, \eta_1, s) \in W_1 \times B | \xi_2 = \eta_{10} = \eta_{11} = 0\},$$
$$A_\infty(s) = \{(\xi, \eta_1, s) \in W_1 \times B | \xi_0 = \eta_{10} = \eta_{12} = 0\} \cup \{(\xi, \eta_2, s) \in W_2 \times B | \xi_0 = \eta_{20} = \eta_{21} = 0\}.$$

In the following subsections, we obtain coordinate systems corresponding to $A_i(s)$ which separate completely the solution curves passing through $A_i(s)$. The systems for $A_0(s)$ and $A_\infty(s)$ are obtained by quadratic transformations.

9.2 Coordinate systems for $A_0(s)$

We obtain coordinate systems for $A_0(s)$ by making quadratic transformations four times along $A_0(s) \cap W_0,\ A_0(s) \cap W_1$.

9.2.1 Coordinate system for $A_0(s) \cap W_0$

The first quadratic transformation along $A_0(s) \cap W_0$. Let

$$\xi_2 = x_{00}^{(1)}, \quad \eta_{00} = x_{00}^{(1)} y_{00}^{(1)}, \quad \eta_{02} = x_{00}^{(1)} z_{00}^{(1)},$$
$$\xi_2 = x_{01}^{(1)} y_{01}^{(1)}, \quad \eta_{00} = y_{01}^{(1)}, \quad \eta_{02} = y_{01}^{(1)} z_{01}^{(1)},$$
$$\xi_2 = x_{02}^{(1)} z_{02}^{(1)}, \quad \eta_{00} = y_{02}^{(1)} z_{02}^{(1)}, \quad \eta_{02} = z_{02}^{(1)},$$

then

$$D_{01}^{(1)}(s) = Q_{A_0(s) \cap W_0}(A_0(s) \cap W_0) = \{x_{00}^{(1)} = 0\} \cup \{y_{01}^{(1)} = 0\} \cup \{z_{02}^{(1)} = 0\},$$

the set of accessible singular points is given by

$$A_{01}^{(1)}(s) = \{(\xi_1, x_{00}^{(1)}, y_{00}^{(1)}, z_{00}^{(1)}, s) = (\xi_1, 0, 0, -1/(\xi_1 - s_1), s)\} \subset D_{01}^{(1)}(s).$$

The second quadratic transformation with $A_{01}^{(1)}(s)$. Note that $\xi_1 = s_1$ is excluded. Let

$$x_{00}^{(1)} = x_{00}^{(2)}, \quad y_{00}^{(1)} = x_{00}^{(2)} y_{00}^{(2)}, \quad z_{00}^{(1)} = -1/(\xi_1 - s_1) + x_{00}^{(2)} z_{00}^{(2)},$$
$$x_{00}^{(1)} = x_{01}^{(2)} y_{01}^{(2)}, \quad y_{00}^{(1)} = y_{01}^{(2)}, \quad z_{00}^{(1)} = -1/(\xi_1 - s_1) + y_{01}^{(2)} z_{01}^{(2)},$$
$$x_{00}^{(1)} = x_{02}^{(2)} z_{02}^{(2)}, \quad y_{00}^{(1)} = y_{02}^{(2)} z_{02}^{(2)}, \quad z_{00}^{(1)} = -1/(\xi_1 - s_1) + z_{02}^{(2)},$$

then

$$D_{01}^{(2)}(s) = Q_{A_{01}(s)}(A_{01}(s)) = \{x_{00}^{(2)} = 0\} \cup \{y_{01}^{(2)} = 0\} \cup \{z_{02}^{(2)} = 0\},$$

the set of accessible singular points is given by

$$A_{01}^{(2)}(s) = \{(\xi_1, x_{00}^{(2)}, y_{00}^{(2)}, z_{00}^{(2)}) = (\xi_1, 0, 1/(\eta s_2(\xi_1 - s_1)), z_{00}^{(2)})\} \subset D_{01}^{(2)}(s).$$

The third quadratic transformation with $A_{01}^{(2)}(s)$. Here we insert a change of variables

$$\xi_1 = \xi_1, \quad x_{00}^{(2)} = x_{00}^{(2)}, \quad y_{00}^{(2)} = 1/v_{00}^{(2)}, \quad z_{00}^{(2)} = z_{00}^{(2)},$$

Let

$$x_{00}^{(2)} = x_{00}^{(3)}, \quad v_{00}^{(2)} = \eta s_2(\xi_1 - s_1) + x_{00}^{(3)} y_{00}^{(3)},$$
$$x_{00}^{(2)} = x_{01}^{(3)} y_{01}^{(3)}, \quad v_{00}^{(2)} = \eta s_2(\xi_1 - s_1) + y_{01}^{(3)},$$

then

$$D_{01}^{(3)}(s) = Q_{A_{01}^{(2)}(s)}(A_{01}^{(2)}(s)) = \{x_{00}^{(3)} = 0\} \cup \{y_{01}^{(3)} = 0\},$$

the set of accessible singular points is given by

$$A_{01}^{(3)}(s) = \{(\xi_1, z_{00}^{(2)}, x_{00}^{(3)}, y_{00}^{(3)}) = (\xi_1, z_{00}^{(2)}, 0, \alpha_0)\} \subset D_{01}^{(3)}(s).$$

The fourth quadratic transformation with $A_{01}^{(3)}(s)$. Let

$$x_{00}^{(3)} = x_{00}^{(4)}, \quad y_{00}^{(3)} = \alpha_0 + x_{00}^{(4)} y_{00}^{(4)},$$
$$x_{00}^{(3)} = x_{01}^{(4)} y_{01}^{(4)}, \quad y_{00}^{(3)} = \alpha_0 + y_{01}^{(4)},$$

then

$$D_{01}^{(4)}(s) = Q_{A_{01}^{(3)}(s)}(A_{01}^{(3)}(s)) = \{x_{00}^{(4)} = 0\} \cup \{y_{01}^{(4)} = 0\}.$$

We can verify that the differential system in the cordinates $(\xi_1, z_{00}^{(2)}, x_{00}^{(4)}, y_{00}^{(4)})$ is holomorphic in a neighborhood of $\{x_{00}^{(4)} = 0\}$ except for $\xi_1 = s_1$ and the points $(\xi_1, z_{00}^{(2)}, x_{01}^{(4)}, y_{01}^{(4)}) = (\xi_1, z_{00}^{(2)}, 0, 0)$ are inaccessible with $\xi_1 \neq s_1$.

Thus we have obtained a coordinate system $(\xi_2, z_{00}^{(2)}, x_{00}^{(4)}, y_{00}^{(4)}) \in \mathbb{C}^4$ which is separates the solutions passing through $A_0(s) \cap W_0 = A_0(s) \cap W_{01}$ with $\xi_1 = s_1$. It is related to the original coordinate system (q_1, q_2, p_1, p_2) by

$$\begin{aligned}
q_1 &= \xi_1, \quad q_2 = x_{00}^{(4)}, \\
p_1 &= -\frac{\eta s_2}{x_{00}^{(4)}} + \eta s_2(\xi_1 - s_1) z_{00}^{(2)} - \frac{\alpha_0}{\xi_1 - s_1} + \left(\alpha_0 z_{00}^{(2)} - \frac{y_{00}^{(4)}}{\xi_1 - s_1}\right) x_{00}^{(4)} + z_{00}^{(2)} y_{00}^{(4)} (x_{00}^{(4)})^2, \\
p_2 &= \frac{\eta s_2(\xi_1 - s_1)}{(x_{00}^{(4)})^2} + \frac{\alpha_0}{x_{00}^{(4)}} + y_{00}^{(4)}.
\end{aligned}$$

Now we calculate the 2-form $dq_1 \wedge dp_1 + dq_2 \wedge dp_2$ in the coordinates $(\xi_1, z_{00}^{(2)}, x_{00}^{(4)}, y_{00}^{(4)})$:

$$\begin{aligned}
&dq_1 \wedge dp_1 + dq_2 \wedge dp_2 \\
&\quad = \quad dx_{00}^{(4)} \wedge dy_{00}^{(4)} \\
&\qquad + \{y_{00}^{(2)} (x_{00}^{(4)})^2 + \alpha_0 x_{00}^{(4)} + \eta s_2(\xi_1 - s_1)\} d\xi_1 \wedge dz_{00}^{(2)} \\
&\qquad + \left(2 z_{00}^{(2)} y_{00}^{(4)} x_{00}^{(4)} + \alpha_0 z_{00}^{(2)} - \frac{y_{00}^{(4)}}{\xi_1 - s_1}\right) d\xi_1 \wedge dx_{00}^{(4)} \\
&\qquad + \left(z_{00}^{(2)} (x_{00}^{(4)})^2 - \frac{x_{00}^{(4)}}{\xi_1 - s_1}\right) d\xi_1 \wedge dy_{00}^{(4)}
\end{aligned}$$

$$\begin{aligned}
&= dx_{00}^{(4)} \wedge dy_{00}^{(4)} \\
&\quad + d\xi_1 \wedge d\Big[\{y_{00}^{(2)}(x_{00}^{(4)})^2 + \alpha_0 x_{00}^{(4)} + \eta s_2(\xi_1 - s_1)\} z_{00}^{(2)} - \frac{x_{00}^{(4)}}{\xi_1 - s_1} y_{00}^{(4)}\Big] \\
&= dx_{\infty 0}^{(8)} \wedge dy_{\infty 0}^{(8)} \\
&\quad + d\xi_2 \wedge d\Big[\{y_{00}^{(2)}(x_{00}^{(4)})^2 + \alpha_0 x_{00}^{(4)} + \eta s_2(\xi_1 - s_1)\} z_{00}^{(2)} - \frac{x_{00}^{(4)}}{\xi_1 - s_1} y_{00}^{(4)} - \frac{\alpha_1}{\xi_1 - s_1}\Big].
\end{aligned}$$

Therefore, setting

$$w_{00}^{(2)} = \{y_{00}^{(2)}(x_{00}^{(4)})^2 + \alpha_0 x_{00}^{(4)} + \eta s_2(\xi_1 - s_1)\} z_{00}^{(2)} - \frac{x_{00}^{(4)}}{\xi_1 - s_1} y_{00}^{(4)} - \frac{\alpha_1}{\xi_1 - s_1},$$

we have symplectic coordinates $(\xi_1, w_{00}^{(2)}, x_{00}^{(4)}, y_{00}^{(4)})$. Writing

$$q_1^{01} = \xi_1, \quad q_2^{01} = x_{00}^{(4)}, \quad p_1^{01} = w_{00}^{(2)}, \quad p_2^{01} = y_{00}^{(4)}.$$

we have

$$q_1 = q_1^{01}, \quad q_2 = q_2^{01}, \quad p_1 = -\frac{\eta s_2}{q_2^{01}} + p_1^{01}, \quad p_2 = \frac{\eta s_2(q_1^{01} - s_2)}{(q_2^{01})^2} + \frac{\alpha_0}{q_2^{01}} + p_2^{11}. \tag{9.1}$$

Thus we have obtained a simplectic coordinate system $(q^{01}, p^{01}) \in \mathbb{C}^4$ in which the Hamiltonians have no singularity on $\xi_1 = s_1$.

9.2.2 Coordinate system for $A_0(s) \cap W_1$

The first quadratic transformation along $A_0(s) \cap W_1$. Let

$$\begin{aligned}
\xi_1 = X_{00}^{(1)}, \quad \eta_{10} = X_{00}^{(1)} Y_{00}^{(1)}, \quad \eta_{11} = X_{00}^{(1)} Z_{00}^{(1)}, \\
\xi_1 = X_{01}^{(1)} Y_{01}^{(1)}, \quad \eta_{10} = Y_{01}^{(1)}, \quad \eta_{11} = Y_{01}^{(1)} Z_{01}^{(1)}, \\
\xi_0 = X_{02}^{(1)} Z_{02}^{(1)}, \quad \eta_{10} = Y_{02}^{(1)} Z_{02}^{(1)}, \quad \eta_{11} = Z_{02}^{(1)},
\end{aligned}$$

then

$$D_{02}^{(1)}(s) = Q_{A_0(s)\cap W_1}(A_0(s) \cap W_1) = \{X_{00}^{(1)} = 0\} \cup \{Y_{01}^{(1)} = 0\} \cup \{Z_{02}^{(1)} = 0\},$$

the set of accessible singular points is given by

$$A_{02}^{(1)}(s) = \{(\xi_0, X_{00}^{(1)}, Y_{00}^{(1)}, Z_{00}^{(1)}) = (\xi_0, 0, 0, -s_1/(s_1\xi_0 - 1))\} \subset D_{02}^{(1)}(s).$$

The second quadratic transformation with $A_{02}^{(1)}(s)$. Let

$$\begin{aligned}
X_{00}^{(1)} = X_{00}^{(2)}, \quad Y_{00}^{(1)} = X_{00}^{(2)} Y_{00}^{(2)}, \quad Z_{00}^{(1)} = -s_1/(s_1\xi_0 - 1) + X_{00}^{(2)} Z_{00}^{(2)}, \\
X_{00}^{(1)} = X_{01}^{(2)} Y_{01}^{(2)}, \quad Y_{00}^{(1)} = Y_{01}^{(2)}, \quad Z_{00}^{(1)} = -s_1/(s_1\xi_0 - 1) + Y_{01}^{(2)} Z_{01}^{(2)}, \\
X_{00}^{(1)} = X_{02}^{(2)} Z_{02}^{(2)}, \quad Y_{00}^{(1)} = Y_{02}^{(2)} Z_{02}^{(2)}, \quad Z_{00}^{(1)} = -s_1/(s_1\xi_0 - 1) + Z_{02}^{(2)},
\end{aligned}$$

then

$$D_{02}^{(2)}(s) = Q_{A_{02}(s)}(A_{02}(s)) = \{X_{00}^{(2)} = 0\} \cup \{Y_{01}^{(2)} = 0\} \cup \{Z_{02}^{(2)} = 0\},$$

the set of accessible singular points is given by

$$A_{02}^{(2)}(s) = \{(\xi_0, X_{00}^{(2)}, Y_{00}^{(2)}, Z_{00}^{(2)}) = (\xi_0, 0, -1/(\eta s_2(s_1\xi_0 - 1)), Z_{00}^{(2)})\} \subset D_{02}^{(2)}(s).$$

The third quadratic transformation with $A_{02}^{(2)}(s)$. We insert here the transformations

$$\xi_1 = \xi_1, \quad X_{00}^{(2)} = X_{00}^{(2)}, \quad Y_{00}^{(2)} = 1/V_{00}^{(2)}, \quad Z_{00}^{(2)} = Z_{00}^{(2)}.$$

Let

$$X_{00}^{(2)} = X_{00}^{(3)}, \quad V_{00}^{(2)} = -\eta s_2(s_1\xi_0 - 1) + X_{00}^{(3)} y_{00}^{(3)},$$
$$X_{00}^{(2)} = X_{01}^{(3)} Y_{01}^{(3)}, \quad V_{00}^{(2)} = -\eta s_2(s_1\xi_0 - 1) + Y_{01}^{(3)},$$

then

$$D_{02}^{(3)}(s) = Q_{A_{02}^{(2)}(s)}(A_{02}^{(2)}(s)) = \{X_{00}^{(3)} = 0\} \cup \{Y_{01}^{(3)} = 0\},$$

the set of accessible singular points is given by

$$A_{02}^{(3)}(s) = \{(\xi_0, Z_{00}^{(2)}, X_{00}^{(3)}, Y_{00}^{(3)}) = (\xi_0, Z_{00}^{(2)}, 0, \alpha_0)\} \subset D_{02}^{(3)}(s).$$

The fourth quadratic transformation with $A_{02}^{(3)}(s)$. Let

$$X_{00}^{(3)} = X_{00}^{(4)}, \quad Y_{00}^{(3)} = \alpha_0 + X_{00}^{(4)} Y_{00}^{(4)},$$
$$X_{00}^{(3)} = X_{01}^{(4)} Y_{01}^{(4)}, \quad Y_{00}^{(3)} = \alpha_0 + Y_{01}^{(4)},$$

then

$$D_{02}^{(4)}(s) = Q_{A_{02}^{(3)}(s)}(A_{02}^{(3)}(s)) = \{X_{00}^{(4)} = 0\} \cup \{Y_{01}^{(4)} = 0\}$$

We see that, in the $(\xi_0, Z_{00}^{(2)}, X_{00}^{(4)}, Y_{00}^{(4)}, s)$-space $\mathbb{C}^4 \times B$, the differential system is holomorphic in a neighborhood of $\{X_{10}^{(6)} = 0\}$ except for $\xi_0 = 1/s_1$, moreover, the points $(\xi_0, Z_{00}^{(2)}, X_{00}^{(4)}, Y_{00}^{(4)}) = (\xi_0, Z_{10}^{(3)}, 0, 0)$ are inaccessible with $\xi_0 = 1/s_1$.

Thus we have obtained a coordinate system $(\xi_0, Z_{00}^{(2)}, X_{00}^{(4)}, Y_{00}^{(4)}) \in \mathbb{C}^4$ which is related to the coordinate system $(q_1^1, q_2^1, p_1^1, p_2^1)$ as

$$q_1^1 = \xi_0, \quad q_2^1 = X_{00}^{(4)},$$
$$p_1^1 = -\frac{\eta s_1 s_2}{X_{00}^{(4)}} - \eta s_2(s_1\xi_0 - 1)Z_{00}^{(2)} - \frac{s_1\alpha_0}{s_1\xi_0 - 1} + \Big(\alpha_0 Z_{00}^{(2)} - \frac{s_1 Y_{00}^{(4)}}{s_1\xi_0 - 1}\Big)X_{00}^{(4)} + Z_{00}^{(2)} Y_{00}^{(4)} (X_{00}^{(4)})^2,$$
$$p_2^1 = -\frac{\eta s_2(s_1\xi_1 - 1)}{(X_{00}^{(4)})^2} + \frac{\alpha_0}{X_{00}^{(4)}} + Y_{00}^{(4)}.$$

Now we calculate the 2-form $dq_1^1 \wedge dp_1^1 + dq_2^1 \wedge dp_2^1$ in the coordinates $(\xi_0, Z_{00}^{(2)}, X_{00}^{(4)}, Y_{00}^{(4)})$:

$$\begin{aligned}
&dq_1^1 \wedge dp_1^1 + dq_2^1 \wedge dp_2^1\\
&= \quad dX_{00}^{(4)} \wedge dY_{00}^{(4)}\\
&\qquad + \{Y_{00}^{(2)}(X_{00}^{(4)})^2 + \alpha_0 X_{00}^{(4)} - \eta s_2(s_1\xi_0 - 1)\} d\xi_0 \wedge dZ_{00}^{(2)}\\
&\qquad + \Big(2Z_{00}^{(2)} Y_{00}^{(4)} X_{00}^{(4)} + \alpha_0 Z_{00}^{(2)} - \frac{s_1 Y_{00}^{(4)}}{s_1\xi_0 - 1}\Big) d\xi_0 \wedge dX_{00}^{(4)}\\
&\qquad + \Big(Z_{00}^{(2)}(X_{00}^{(4)})^2 - \frac{s_1 X_{00}^{(4)}}{s_1\xi_0 - 1}\Big) d\xi_0 \wedge dY_{00}^{(4)}\\
&= \quad dX_{00}^{(4)} \wedge dY_{00}^{(4)}\\
&\qquad + d\xi_1 \wedge d\Big[\{Y_{00}^{(2)}(X_{00}^{(4)})^2 + \alpha_0 X_{00}^{(4)} - \eta s_2(s_1\xi_0 - 1)\} Z_{00}^{(2)} - \frac{s_1 X_{00}^{(4)}}{s_1\xi_0 - 1} Y_{00}^{(4)}\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \; dx^{(8)}_{\infty 0} \wedge dy^{(8)}_{\infty 0} \\
&\quad + d\xi_2 \wedge d\Big[\{Y^{(2)}_{00}(X^{(4)}_{00})^2 + \alpha_0 X^{(4)}_{00} - \eta s_2(s_1\xi_0 - 1)\}Z^{(2)}_{00} - \frac{s_1 X^{(4)}_{00}}{s_1\xi_0 - 1}Y^{(4)}_{00} - \frac{s_1\alpha_0}{s_1\xi_0 - 1}\Big].
\end{aligned}
$$

Therefore, setting

$$
W^{(2)}_{00} = \{Y^{(2)}_{00}(X^{(4)}_{00})^2 + \alpha_0 X^{(4)}_{00} - \eta s_2(s_1\xi_0 - 1)\}Z^{(2)}_{00} - \frac{s_1 X^{(4)}_{00}}{s_1\xi_0 - 1}Y^{(4)}_{00} - \frac{s_1\alpha_0}{s_1\xi_0 - 1},
$$

we have symplectic coordinates $(\xi_0, W^{(2)}_{00}, X^{(4)}_{00}, Y^{(4)}_{00})$. Writing

$$
q^{02}_1 = \xi_0, \quad q^{02}_2 = X^{(4)}_{00}, \quad p^{02}_1 = W^{(2)}_{00}, \quad p^{02}_2 = Y^{(4)}_{00}.
$$

then we have

$$
q_1 = q^{02}_1, \quad q_2 = q^{02}_2, \quad p_1 = \frac{\eta s_1 s_2}{q^{02}_2} + p^{02}_1, \quad p_2 = -\frac{\eta s_2(s_1 q^{02}_1 - 1)}{(q^{02}_2)^2} + \frac{\alpha_0}{q^{02}_2} + p^{02}_2. \tag{9.2}
$$

The system $(q^{02}, p^{02}) \in \mathbb{C}^4$ separates solution curves passing through $A_0(s) \cap W_1$ and the Hamiltonians have no singularity on $\xi_0 = 1/s_1$.

9.3 Coordinate systems for $A_\infty(s)$

We obtain coordinate systems for $A_\infty(s)$ by making quadratic transformations six times along $A_\infty(s) \cap W_1$, $A_\infty(s) \cap W_2$.

9.3.1 Coordinate system for $A_\infty(s) \cap W_1$

The first quadratic transformation along $A_\infty(s) \cap W_1$. Let

$$
\begin{aligned}
\xi_0 &= x^{(1)}_{\infty 0}, \quad \eta_{10} = x^{(1)}_{\infty 0} y^{(1)}_{\infty 0}, \quad \eta_{12} = x^{(1)}_{\infty 0} z^{(1)}_{\infty 0}, \\
\xi_0 &= x^{(1)}_{\infty 1} y^{(1)}_{\infty 1}, \quad \eta_{10} = y^{(1)}_{\infty 1}, \quad \eta_{12} = y^{(1)}_{\infty 1} z^{(1)}_{\infty 1}, \\
\xi_0 &= x^{(1)}_{\infty 2} z^{(1)}_{\infty 2}, \quad \eta_{10} = y^{(1)}_{\infty 2} z^{(1)}_{\infty 2}, \quad \eta_{12} = z^{(1)}_{\infty 2},
\end{aligned}
$$

then

$$
D^{(1)}_{\infty 1}(s) = Q_{A_\infty(s)\cap W_1}(A_\infty(s) \cap W_1) = \{x^{(1)}_{\infty 0} = 0\} \cup \{y^{(1)}_{\infty 1} = 0\} \cup \{z^{(1)}_{\infty 2} = 0\},
$$

the set of accessible singular points is given by

$$
A^{(1)}_{\infty 1}(s) = \{(\xi_2, x^{(1)}_{\infty 0}, y^{(1)}_{\infty 0}, z^{(1)}_{\infty 0}, s) = (\xi_2, 0, 0, 0, s)\} \subset D^{(1)}_{\infty}(s).
$$

The second quadratic transformation along $A^{(1)}_{\infty 1}(s)$. Let

$$
\begin{aligned}
x^{(1)}_{\infty 0} &= x^{(2)}_{\infty 0}, \quad y^{(1)}_{\infty 0} = x^{(2)}_{\infty 0} y^{(2)}_{\infty 0}, \quad z^{(1)}_{\infty 0} = x^{(2)}_{\infty 0} z^{(2)}_{\infty 0}, \\
x^{(1)}_{\infty 0} &= x^{(2)}_{\infty 1} y^{(2)}_{\infty 1}, \quad y^{(1)}_{\infty 0} = y^{(2)}_{\infty 1}, \quad z^{(1)}_{\infty 0} = y^{(2)}_{\infty 1} z^{(2)}_{\infty 1}, \\
x^{(1)}_{\infty 0} &= x^{(1)}_{\infty 2} z^{(2)}_{\infty 2}, \quad y^{(1)}_{\infty 0} = x^{(2)}_{\infty 2} y^{(2)}_{\infty 2}, \quad z^{(1)}_{\infty 0} = z^{(2)}_{\infty 2},
\end{aligned}
$$

then

$$
D^{(2)}_{\infty 1}(s) = Q_{A^{(1)}_{\infty 1}(s)}(A^{(1)}_{\infty 1}(s)) = \{x^{(2)}_{\infty 0} = 0\} \cup \{y^{(2)}_{\infty 1} = 0\} \cup \{z^{(2)}_{\infty 2} = 0\},
$$

the set of accessible singular points is given by

$$
A^{(2)}_{\infty 1}(s) = \{(\xi_2, x^{(2)}_{\infty 0}, y^{(2)}_{\infty 0}, z^{(2)}_{\infty 0}, s) = (\xi_2, 0, 0, 1, s)\} \subset D^{(2)}_{\infty 1}(s).
$$

The third quadratic transformation along $A^{(2)}_{\infty 1}(s)$. Let

$$\begin{aligned} x^{(2)}_{\infty 0} &= x^{(3)}_{\infty 0}, \quad y^{(2)}_{\infty 0} = x^{(3)}_{\infty 0} y^{(3)}_{\infty 0}, \quad z^{(2)}_{\infty 0} = 1 + x^{(3)}_{\infty 0} z^{(3)}_{\infty 0}, \\ x^{(2)}_{\infty 0} &= x^{(3)}_{\infty 1} y^{(3)}_{\infty 1}, \quad y^{(2)}_{\infty 0} = y^{(3)}_{\infty 1}, \quad z^{(2)}_{\infty 0} = 1 + y^{(3)}_{\infty 1} z^{(3)}_{\infty 1}, \\ x^{(2)}_{\infty 0} &= x^{(3)}_{\infty 2} z^{(3)}_{\infty 2}, \quad y^{(2)}_{\infty 0} = y^{(3)}_{\infty 2} z^{(3)}_{\infty 2}, \quad z^{(2)}_{\infty 0} = 1 + z^{(3)}_{\infty 2}, \end{aligned}$$

then

$$D^{(3)}_{\infty 1}(s) = Q_{A^{(2)}_{\infty}(s)}(A^{(2)}_{\infty}(s)) = \{x^{(3)}_{\infty 0} = 0\} \cup \{y^{(3)}_{\infty 1} = 0\} \cup \{z^{(3)}_{\infty 2} = 0\},$$

the set of accessible singular points is given by

$$A^{(3)}_{\infty 1}(s) = \{(\xi_2, x^{(3)}_{\infty 0}, y^{(3)}_{\infty 0}, z^{(3)}_{\infty 0}) = (\xi_2, 0, 0, -2)\} \subset D^{(3)}_{\infty 1}(s).$$

The fourth quadratic transformation along $A^{(3)}_{\infty 1}(s)$ Here we insert a change of variables

$$\xi_2 = \xi_2, \quad x^{(3)}_{\infty 0} = x^{(3)}_{\infty 0}, \quad y^{(3)}_{\infty 0} = 1/v^{(3)}_{\infty 0}, \quad z^{(3)}_{\infty 0} = z^{(3)}_{\infty 0}.$$

Let

$$\begin{aligned} x^{(3)}_{\infty 0} &= x^{(4)}_{\infty 0}, \quad v^{(3)}_{\infty 0} = -1/2 + x^{(4)}_{\infty 0} y^{(4)}_{\infty 0}, \\ x^{(3)}_{\infty 0} &= x^{(4)}_{\infty 1} y^{(4)}_{\infty 1}, \quad v^{(3)}_{\infty 0} = -1/2 + y^{(4)}_{\infty 1}, \end{aligned}$$

then

$$D^{(4)}_{\infty 1}(s) = Q_{A^{(3)}_{\infty 1}(s)}(A^{(3)}_{\infty 1}(s)) = \{x^{(4)}_{\infty 0} = 0\} \cup \{y^{(4)}_{\infty 1} = 0\},$$

the set of accessible singular points is given by

$$A^{(4)}_{\infty 1}(s) = \{(\xi_2, z^{(3)}_{\infty 0}, x^{(4)}_{\infty 0}, y^{(4)}_{\infty 0}) = (\xi_2, z^{(3)}_{\infty 0}, 0, \xi_2/2)\} \subset D^{(3)}_{\infty 1}(s).$$

The fifth quadratic transformation along $A^{(4)}_{\infty 1}(s)$. Let

$$\begin{aligned} x^{(4)}_{\infty 0} &= x^{(5)}_{\infty 0}, \quad v^{(4)}_{\infty 0} = \xi_2/2 + x^{(5)}_{\infty 0} y^{(5)}_{\infty 0}, \\ x^{(4)}_{\infty 0} &= x^{(5)}_{\infty 1} y^{(5)}_{\infty 1}, \quad v^{(4)}_{\infty 0} = \xi_2/2 + y^{(5)}_{\infty 1}, \end{aligned}$$

then

$$D^{(5)}_{\infty 1}(s) = Q_{A^{(4)}_{\infty 1}(s)}(A^{(4)}_{\infty 1}(s)) = \{x^{(5)}_{\infty 0} = 0\} \cup \{y^{(5)}_{\infty 1} = 0\},$$

the set of accessible singular points is given by

$$A^{(5)}_{\infty 1}(s) = \{(\xi_2, w^{(3)}_{\infty 0}, x^{(5)}_{\infty 0}, y^{(5)}_{\infty 0}) = (\xi_2, w^{(3)}_{\infty 0}, 0, 1 - \alpha_0 + 2\alpha_\infty)\} \subset D^{(5)}_{\infty 1}(s).$$

The sixth quadratic transformation along $A^{(5)}_{\infty 1}(s)$. Let

$$\begin{aligned} x^{(5)}_{\infty 0} &= x^{(6)}_{\infty 0}, \quad y^{(5)}_{\infty 0} = 1 - \alpha_0 + 2\alpha_\infty + x^{(6)}_{\infty 0} y^{(6)}_{\infty 0}, \\ x^{(5)}_{\infty 0} &= x^{(6)}_{\infty 1} y^{(6)}_{\infty 1}, \quad y^{(5)}_{\infty 0} = 1 - \alpha_0 + 2\alpha_\infty + y^{(6)}_{\infty 1}, \end{aligned}$$

then

$$D^{(6)}_{\infty 1}(s) = Q_{A^{(5)}_{\infty 1}(s)}(A^{(5)}_{\infty 1}(s)) = \{x^{(6)}_{\infty 0} = 0\} \cup \{y^{(6)}_{\infty 1} = 0\},$$

We can verify that the differential system in the cordinates $(\xi_2, z^{(3)}_{\infty 0}, x^{(6)}_{\infty 0}, y^{(6)}_{\infty 0})$ is holomorphic in a neighborhood of $\{x^{(6)}_{\infty 0} = 0\}$ and the points $(\xi_2, z^{(3)}_{\infty 0}, x^{(6)}_{\infty 1}, y^{(6)}_{\infty 1}) = (\xi_2, z^{(3)}_{\infty 0}, 0, 0)$ are inaccessible.

Thus we have obtained a coordinate system $(\xi_2, z^{(3)}_{\infty 0}, x^{(6)}_{\infty 0}, y^{(6)}_{\infty 0}) \in \mathbb{C}^4$ which is is separates the solutions passing through $A_\infty(s) \cap W_1 = A_\infty(s) \cap W_{11}$. It is related to the coordinate system $(q^1_1, q^1_2, p^1_1, p^1_2)$ by

$$q^1_1 = x^{(6)}_{\infty 0}, \quad q^1_2 = \xi_2, \quad p^1_1 = y^{(6)}_{\infty 0} + \frac{1-\alpha_0+2\alpha_\infty}{x^{(6)}_{\infty 0}} + \frac{\xi_2}{2(x^{(6)}_{\infty 0})^2} - \frac{1}{2(x^{(6)}_{\infty 0})^3},$$

$$\begin{aligned} p^1_2 = &-\frac{1}{2x^{(6)}_{\infty 0}} + \frac{\xi_2}{2} - \frac{z^{(3)}_{\infty 0}}{2} + \Big(\frac{\xi_2 z^{(3)}_{\infty 0}}{2} + 1 - \alpha_0 + 2\alpha_\infty\Big) x^{(6)}_{\infty 0} \\ &+ \{(1-\alpha_0+2\alpha_\infty) z^{(3)}_{\infty 0} + y^{(6)}_{\infty 0}\}(x^{(6)}_{\infty 0})^2 + z^{(3)}_{10} y^{(6)}_{\infty 0} (x^{(6)}_{\infty 0})^3. \end{aligned}$$

Now we calculate the 2-form $dq^1_1 \wedge dp^1_1 + dq^1_2 \wedge dp^1_2$ in the coordinates $(\xi_2, z^{(3)}_{\infty 0}, x^{(6)}_{\infty 0}, y^{(6)}_{\infty 0})$:

$$\begin{aligned} &dq^1_1 \wedge dp^1_1 + dq^1_2 \wedge dp^1_2 \\ &= dx^{(6)}_{\infty 0} \wedge dy^{(6)}_{\infty 0} \\ &\quad + \Big\{ y^{(6)}_{\infty 0} (x^{(6)}_{\infty 0})^3 + (1-\alpha_0+2\alpha_\infty)(x^{(6)}_{\infty 0})^2 + \frac{\xi_2 x^{(6)}_{\infty 0}}{2} - \frac{1}{2} \Big\} d\xi_2 \wedge dz^{(3)}_{\infty 0} \\ &\quad + \Big\{ 3 z^{(3)}_{\infty 0} y^{(6)}_{\infty 0} (x^{(6)}_{\infty 0})^2 + 2\{(1-\alpha_0+\alpha_\infty) z^{(3)}_{\infty 0} + y^{(6)}_{\infty 0}\} x^{(6)}_{\infty 0} \\ &\qquad + 1 - \alpha_0 + 2\alpha_\infty + \frac{\xi_2 z^{(3)}_{\infty 0}}{2} \Big\} d\xi_2 \wedge dx^{(6)}_{\infty 0} \\ &\quad + \{ z^{(3)}_{\infty 0} (x^{(6)}_{\infty 0})^3 + (x^{(6)}_{\infty 0})^2 \} d\xi_2 \wedge dy^{(6)}_{\infty 0} \\ &= dx^{(6)}_{\infty 0} \wedge dy^{(6)}_{\infty 0} \\ &\quad + d\xi_2 \wedge d\Big[\Big\{ y^{(6)}_{\infty 0} (x^{(6)}_{\infty 0})^3 + (1-\alpha_0+2\alpha_\infty)(x^{(6)}_{\infty 0})^2 + \frac{\xi_2 x^{(6)}_{\infty 0}}{2} - \frac{1}{2} \Big\} z^{(3)}_{\infty 0} \\ &\qquad + (x^{(6)}_{\infty 0})^2 y^{(6)}_{\infty 0} + (1-\alpha_0+2\alpha_\infty) x^{(6)}_{\infty 0} \Big] \\ &= dx^{(8)}_{\infty 0} \wedge dy^{(8)}_{\infty 0} \\ &\quad + d\xi_2 \wedge d\Big[\Big\{ y^{(6)}_{\infty 0} (x^{(6)}_{\infty 0})^3 + (1-\alpha_0+2\alpha_\infty)(x^{(6)}_{\infty 0})^2 + \frac{\xi_2 x^{(6)}_{\infty 0}}{2} - \frac{1}{2} \Big\} z^{(3)}_{\infty 0} \\ &\qquad + (x^{(6)}_{\infty 0})^2 y^{(6)}_{\infty 0} + (1-\alpha_0+2\alpha_\infty) x^{(6)}_{\infty 0} + \frac{\xi_2}{2} \Big]. \end{aligned}$$

Therefore, setting

$$\begin{aligned} w^{(3)}_{\infty 0} = &\Big\{ y^{(6)}_{\infty 0} (x^{(6)}_{\infty 0})^3 + (1-\alpha_0+2\alpha_\infty)(x^{(6)}_{\infty 0})^2 + \frac{\xi_2 x^{(6)}_{\infty 0}}{2} - \frac{1}{2} \Big\} z^{(3)}_{\infty 0} \\ &+ (x^{(6)}_{\infty 0})^2 y^{(6)}_{\infty 0} + (1-\alpha_0+2\alpha_\infty) x^{(6)}_{\infty 0} + \frac{\xi_2}{2}, \end{aligned}$$

we have a symplectic coordinates $(\xi_2, w^{(3)}_{\infty 0}, x^{(6)}_{\infty 0}, y^{(6)}_{\infty 0})$. Writing

$$q^{\infty 1}_1 = x^{(6)}_{\infty 0}, \quad q^{\infty 1}_2 = \xi_2, \quad p^{\infty 1}_1 = y^{(6)}_{\infty 0}, \quad p^{\infty 1}_2 = w^{(3)}_{\infty 0}.$$

we have

$$
(9.3) \qquad \begin{aligned}
&q_1^1 = q_1^{\infty 1}, \quad q_2^1 = q_2^{\infty 1},\\
&p_1^1 = -\frac{1}{(q_1^{\infty 1})^3} + \frac{q_2^{\infty 1}}{2(q_1^{\infty 1})^2} + \frac{1-\alpha_0+2\alpha_\infty}{q_1^{\infty 1}} + p_1^{\infty 1},\\
&p_2^1 = -\frac{1}{2q_1^{\infty 1}} + p_2^{\infty 1}.
\end{aligned}
$$

Thus we have obtained a symplectic coordinate system $(q^{\infty 1}, p^{\infty 1}) \in \mathbb{C}^4$ which separates solution curves passing through $A_\infty(s) \cap W_1$.

9.3.2 Coordinate system for $A_\infty(s) \cap W_2$

The first quadratic transformation along $A_\infty(s) \cap W_2$. Let

$$
\begin{aligned}
&\xi_0 = X_{\infty 0}^{(1)}, \quad \eta_{20} = X_{\infty 0}^{(1)} Y_{\infty 0}^{(1)}, \quad \eta_{21} = X_{\infty 0}^{(1)} Z_{\infty 0}^{(1)},\\
&\xi_0 = X_{\infty 1}^{(1)} Y_{\infty 1}^{(1)}, \quad \eta_{20} = Y_{\infty 1}^{(1)}, \quad \eta_{21} = Y_{\infty 1}^{(1)} Z_{\infty 1}^{(1)},\\
&\xi_0 = X_{\infty 2}^{(1)} Z_{\infty 2}^{(1)}, \quad \eta_{20} = Y_{\infty 2}^{(1)} Z_{\infty 2}^{(1)}, \quad \eta_{21} = Z_{\infty 2}^{(1)},
\end{aligned}
$$

then

$$
D_{\infty 2}^{(1)}(s) = Q_{A_\infty(s)\cap W_2}(A_\infty(s) \cap W_2) = \{X_{\infty 0}^{(1)} = 0\} \cup \{Y_{\infty 1}^{(1)} = 0\} \cup \{Z_{\infty 2}^{(1)} = 0\},
$$

the set of accessible singular points is given by

$$
A_{\infty 2}^{(1)}(s) = \{(\xi_1, X_{\infty 0}^{(1)}, Y_{\infty 0}^{(1)}, Z_{\infty 0}^{(1)}) = (\xi_1, 0, 0, -1/\xi_1)\} \subset D_{\infty 2}^{(1)}(s).
$$

The second quadratic transformation along $A_{\infty 2}^{(1)}(s)$. Note that $\xi_1 = 0$ is excluded. Let

$$
\begin{aligned}
&X_{\infty 0}^{(1)} = X_{\infty 0}^{(2)}, \quad Y_{\infty 0}^{(1)} = X_{\infty 0}^{(2)} Y_{\infty 0}^{(2)}, \quad Z_{\infty 0}^{(1)} = -1/\xi_1 + X_{\infty 0}^{(2)} Z_{\infty 0}^{(2)},\\
&X_{\infty 0}^{(1)} = X_{\infty 1}^{(2)} Y_{\infty 1}^{(2)}, \quad Y_{\infty 0}^{(1)} = Y_{\infty 1}^{(2)}, \quad Z_{\infty 0}^{(1)} = -1/\xi_1 + Y_{\infty 1}^{(2)} Z_{\infty 1}^{(2)},\\
&X_{\infty 0}^{(1)} = X_{\infty 2}^{(1)} Z_{\infty 2}^{(2)}, \quad Y_{\infty 0}^{(1)} = X_{\infty 2}^{(2)} Y_{\infty 2}^{(2)}, \quad Z_{\infty 0}^{(1)} = -1/\xi_1 + Z_{\infty 2}^{(2)},
\end{aligned}
$$

then

$$
D_{\infty 2}^{(2)}(s) = Q_{A_{\infty 2}^{(1)}(s)}(A_{\infty 2}^{(1)}(s)) = \{x_{\infty 0}^{(2)} = 0\} \cup \{y_{\infty 1}^{(2)} = 0\} \cup \{z_{\infty 2}^{(2)} = 0\},
$$

the set of accessible singular points is given by

$$
A_{\infty 2}^{(2)}(s) = \{(\xi_2, X_{\infty 0}^{(2)}, Y_{\infty 0}^{(2)}, Z_{\infty 0}^{(2)}) = (\xi_2, 0, 0, -1/\xi_1^3)\} \subset D_{\infty 2}^{(2)}(s).
$$

The third quadratic transformation along $A_{\infty 2}^{(2)}(s)$. Let

$$
\begin{aligned}
&X_{\infty 0}^{(2)} = X_{\infty 0}^{(3)}, \quad Y_{\infty 0}^{(2)} = X_{\infty 0}^{(3)} Y_{\infty 0}^{(3)}, \quad Z_{\infty 0}^{(2)} = -1/\xi_1^3 + X_{\infty 0}^{(3)} Z_{\infty 0}^{(3)},\\
&X_{\infty 0}^{(2)} = X_{\infty 1}^{(3)} Y_{\infty 1}^{(3)}, \quad Y_{\infty 0}^{(2)} = Y_{\infty 1}^{(3)}, \quad Z_{\infty 0}^{(2)} = -1/\xi_1^3 + Y_{\infty 1}^{(3)} Z_{\infty 1}^{(3)},\\
&X_{\infty 0}^{(2)} = X_{\infty 2}^{(3)} Z_{\infty 2}^{(3)}, \quad Y_{\infty 0}^{(2)} = Y_{\infty 2}^{(3)} Z_{\infty 2}^{(3)}, \quad Z_{\infty 0}^{(2)} = -1/\xi_1^3 + Z_{\infty 2}^{(3)},
\end{aligned}
$$

then

$$
D_{\infty 2}^{(3)}(s) = Q_{A_{\infty 2}^{(2)}(s)}(A_{\infty 2}^{(2)}(s)) = \{X_{\infty 0}^{(3)} = 0\} \cup \{Y_{\infty 1}^{(3)} = 0\} \cup \{Z_{\infty 2}^{(3)} = 0\},
$$

the set of accessible singular points is given by

$$
A_{\infty 2}^{(3)}(s) = \{(\xi_1, X_{\infty 0}^{(3)}, Y_{\infty 0}^{(3)}, Z_{\infty 0}^{(3)}) = (\xi_1, 0, 0, -2/\xi_1^2)\} \subset D_{\infty 2}^{(3)}(s).
$$

The fourth quadratic transformation along $A^{(3)}_{\infty 2}(s)$. We insert here the transformations

$$\xi_1=\xi_1,\quad X^{(3)}_{\infty 0}=X^{(3)}_{\infty 0},\quad Y^{(3)}_{\infty 0}=1/V^{(3)}_{\infty 0},\quad Z^{(3)}_{\infty 0}=Z^{(3)}_{\infty 0},$$

Let

$$X^{(3)}_{\infty 0}=X^{(4)}_{\infty 0},\quad V^{(3)}_{\infty 0}=-\xi_1^2/2+X^{(4)}_{\infty 0}Y^{(4)}_{\infty 0},$$
$$X^{(3)}_{\infty 0}=X^{(4)}_{\infty 1}Y^{(4)}_{\infty 1},\quad V^{(3)}_{\infty 0}=-\xi_1^2/2+Y^{(4)}_{\infty 1},$$

then

$$D^{(4)}_{\infty 2}(s)=Q_{A^{(3)}_{\infty 2}(s)}(A^{(3)}_{\infty 2}(s))=\{X^{(4)}_{\infty 0}=0\}\cup\{Y^{(4)}_{\infty 1}=0\},$$

the set of accessible singular points is given by

$$A^{(4)}_{\infty 2}(s)=\{(\xi_1,Z^{(3)}_{\infty 0},X^{(4)}_{\infty 0},Y^{(4)}_{\infty 0})=(\xi_1,Z^{(3)}_{\infty 0},0,1/2)\}\subset D^{(3)}_{\infty 2}(s).$$

The fifth quadratic transformation along $A^{(4)}_{\infty 1}(s)$. Let

$$X^{(4)}_{\infty 0}=X^{(5)}_{\infty 0},\quad Y^{(4)}_{\infty 0}=1/2+X^{(5)}_{\infty 0}Y^{(5)}_{\infty 0},$$
$$X^{(4)}_{\infty 0}=X^{(5)}_{\infty 1}Y^{(5)}_{\infty 1},\quad Y^{(4)}_{\infty 0}=1/2+Y^{(5)}_{\infty 1},$$

then

$$D^{(5)}_{\infty 2}(s)=Q_{A^{(4)}_{\infty 2}(s)}(A^{(4)}_{\infty 2}(s))=\{X^{(5)}_{\infty 0}=0\}\cup\{Y^{(5)}_{\infty 1}=0\},$$

the set of accessible singular points is given by

$$A^{(5)}_{\infty 2}(s)=\{(\xi_1,Z^{(3)}_{\infty 0},X^{(5)}_{\infty 0},Y^{(5)}_{\infty 0})=(\xi_1,Z^{(3)}_{\infty 0},0,1-\alpha_0+2\alpha_\infty)\}\subset D^{(5)}_{\infty 2}(s).$$

The sixth quadratic transformation along $A^{(5)}_{\infty 2}(s)$. Let

$$X^{(5)}_{\infty 0}=X^{(6)}_{\infty 0},\quad Y^{(5)}_{\infty 0}=1-\alpha_0+2\alpha_\infty+X^{(6)}_{\infty 0}Y^{(6)}_{\infty 0},$$
$$X^{(5)}_{\infty 0}=X^{(6)}_{\infty 1}Y^{(6)}_{\infty 1},\quad Y^{(5)}_{\infty 0}=1-\alpha_0+2\alpha_\infty+Y^{(6)}_{\infty 1},$$

then

$$D^{(6)}_{\infty 2}(s)=Q_{A^{(5)}_{\infty 2}(s)}(A^{(5)}_{\infty 2}(s))=\{X^{(6)}_{\infty 0}=0\}\cup\{Y^{(6)}_{\infty 1}=0\},$$

We see that, in the $(\xi_1,Z^{(3)}_{\infty 0},X^{(6)}_{\infty 0},Y^{(6)}_{\infty 0},s)$-space $\mathbb{C}^4\times B$, the differential system is holomorphic in a neighborhood of $\{X^{(5)}_{\infty 0}=0\}$ except for $\xi_1=0$. moreover, the points $(\xi_1,Z^{(3)}_{\infty 0},X^{(6)}_{\infty 1},Y^{(6)}_{\infty 1})=(\xi_0,Z^{(3)}_{\infty 0},0,0)$ are inaccessible with $\xi_1\neq 0$.

Thus we have obtained a coordinate system $(\xi_1,Z^{(3)}_{\infty 0},X^{(6)}_{\infty 0},Y^{(6)}_{\infty 0})\in\mathbb{C}^4$ which is related to the coordinate system $(q_1^2,q_2^2,p_1^2,p_2^2)$ as

$$\begin{aligned}
&q_1^2=\xi_1,\quad q_2^2=X^{(6)}_{\infty 0},\\
&p_1^2=\frac{\xi_1}{2(X^{(6)}_{\infty 0})^2}-\frac{\xi_1^2Z^{(3)}_{\infty 0}}{2}-\frac{1-\alpha_0+2\alpha_\infty}{\xi_1}-\frac{1}{2\xi_1^3}+\Big(\frac{z^{(3)}_{\infty 0}}{2}-\frac{Y^{(6)}_{\infty 0}}{\xi_1}-\frac{1-\alpha_0+2\alpha_\infty}{\xi_1^3}\Big)X^{(6)}_{\infty 0},\\
&\quad+\Big\{(1-\alpha_0+2\alpha_\infty)Z^{(3)}_{\infty 0}-\frac{Y^{(6)}_{\infty 0}}{\xi_1^3}\Big\}(X^{(6)}_{\infty 0})^2+Z^{(3)}_{10}Y^{(6)}_{\infty 0}(X^{(6)}_{\infty 0})^3\\
&p_2^2=-\frac{\xi_1^2}{2(X^{(6)}_{\infty 0})^3}+\frac{1}{2(X^{(6)}_{\infty 0})^2}+\frac{1-\alpha_0+2\alpha_\infty}{X^{(6)}_{\infty 0}}+Y^{(6)}_{\infty 0}
\end{aligned}$$

Now we calculate the 2-form $dq_1^2 \wedge dp_1^2 + dq_2^2 \wedge dp_2^2$ in the coordinates $(\xi_1, Z_{\infty 0}^{(3)}, X_{\infty 0}^{(6)}, Y_{\infty 0}^{(6)})$:

$$
\begin{aligned}
&dq_1^2 \wedge dp_1^2 + dq_2^2 \wedge dp_2^2 \\
&\quad = dX_{\infty 0}^{(6)} \wedge dY_{\infty 0}^{(6)} \\
&\qquad + \Big\{ Y_{\infty 0}^{(6)} (X_{\infty 0}^{(6)})^3 + (1-\alpha_0+2\alpha_\infty)(X_{\infty 0}^{(6)})^2 + \frac{X_{\infty 0}^{(6)}}{2} - \frac{\xi_1^2}{2} \Big\} d\xi_1 \wedge dZ_{\infty 0}^{(3)} \\
&\qquad + \Big[3Z_{\infty 0}^{(3)} Y_{\infty 0}^{(6)} (X_{\infty 0}^{(6)})^2 + 2\Big\{ (1-\alpha_0+\alpha_\infty) Z_{\infty 0}^{(3)} - \frac{Y_{\infty 0}^{(6)}}{\xi_1^3} \Big\} X_{\infty 0}^{(6)} \\
&\qquad\qquad + \frac{Z_{\infty 0}^{(3)}}{2} - \frac{Y_{\infty 0}^{(6)}}{\xi_1} + \frac{1-\alpha_0+2\alpha_\infty}{\xi_1^3} \Big] d\xi_1 \wedge dX_{\infty 0}^{(6)} \\
&\qquad + \Big\{ Z_{\infty 0}^{(3)} (X_{\infty 0}^{(6)})^3 - \frac{(X_{\infty 0}^{(6)})^2}{\xi_1^3} - \frac{X_{\infty 0}^{(6)}}{\xi_1} \Big\} d\xi_1 \wedge dY_{\infty 0}^{(6)} \\
&\quad = dX_{\infty 0}^{(6)} \wedge dY_{\infty 0}^{(6)} \\
&\qquad + d\xi_2 \wedge d\Big[\Big\{ Y_{\infty 0}^{(6)} (X_{\infty 0}^{(6)})^3 + (1-\alpha_0+2\alpha_\infty)(X_{\infty 0}^{(6)})^2 + \frac{X_{\infty 0}^{(6)}}{2} - \frac{\xi_1^2}{2} \Big\} Z_{\infty 0}^{(3)} \\
&\qquad\qquad - \Big\{ \frac{(X_{\infty 0}^{(6)})^2}{\xi_1^3} + \frac{X_{\infty 0}^{(6)}}{\xi_1} \Big\} Y_{\infty 0}^{(3)} + \frac{1-\alpha_0+2\alpha_\infty}{\xi_1^3} X_{\infty 0}^{(6)} \Big] \\
&\quad = dX_{\infty 0}^{(6)} \wedge dY_{\infty 0}^{(6)} \\
&\qquad + d\xi_2 \wedge d\Big[\Big\{ Y_{\infty 0}^{(6)} (X_{\infty 0}^{(6)})^3 + (1-\alpha_0+2\alpha_\infty)(X_{\infty 0}^{(6)})^2 + \frac{X_{\infty 0}^{(6)}}{2} - \frac{\xi_1^2}{2} \Big\} Z_{\infty 0}^{(3)} \\
&\qquad\qquad - \Big\{ \frac{(X_{\infty 0}^{(6)})^2}{\xi_1^3} + \frac{X_{\infty 0}^{(6)}}{\xi_1} \Big\} Y_{\infty 0}^{(6)} + \frac{1-\alpha_0+2\alpha_\infty}{\xi_1^3} X_{\infty 0}^{(6)} - \frac{1-\alpha_0+2\alpha_\infty}{\xi_1} - \frac{1}{2\xi_1^3} \Big].
\end{aligned}
$$

Therefore, setting

$$
\begin{aligned}
W_{\infty 0}^{(3)} = &\Big\{ Y_{\infty 0}^{(6)} (X_{\infty 0}^{(6)})^3 + (1-\alpha_0+2\alpha_\infty)(X_{\infty 0}^{(6)})^2 + \frac{X_{\infty 0}^{(6)}}{2} - \frac{\xi_1^2}{2} \Big\} Z_{\infty 0}^{(3)} \\
&- \Big\{ \frac{(X_{\infty 0}^{(6)})^2}{\xi_1^3} + \frac{X_{\infty 0}^{(6)}}{\xi_1} \Big\} Y_{\infty 0}^{(3)} + \frac{1-\alpha_0+2\alpha_\infty}{\xi_1^3} X_{\infty 0}^{(6)} - \frac{1-\alpha_0+2\alpha_\infty}{\xi_1} - \frac{1}{2\xi_1^3},
\end{aligned}
$$

we have symplectic coordinates $(\xi_1, W_{\infty 0}^{(3)}, X_{\infty 0}^{(6)}, Y_{\infty 0}^{(6)})$. Writing

$$
q_1^{\infty 2} = \xi_1, \quad q_2^{\infty 2} = X_{\infty 0}^{(6)}, \quad p_1^{\infty 2} = W_{\infty 0}^{(3)}, \quad p_2^{\infty 2} = Y_{\infty 0}^{(6)},
$$

we have

$$
\begin{aligned}
&q_1^2 = q_1^{\infty 2}, \quad q_2^2 = q_2^{\infty 2}, \\
&p_1^2 = \frac{q_1^{\infty 2}}{2(q_2^{\infty 2})^2} + p_1^{\infty 2}, \\
&p_2^2 = -\frac{(q_1^{\infty 2})^2}{2(q_2^{\infty 2})^3} + \frac{1}{2(q_2^{\infty 2})^2} + \frac{1-\alpha_0+2\alpha_\infty}{q_2^{\infty 2}} + p_2^{\infty 1}.
\end{aligned}
\tag{9.4}
$$

The system $(q^{\infty 2}, p^{\infty 2}) \in \mathbb{C}^4$ separates solution curves passing through $A_\infty(s) \cap W_2$ and the Hamiltonians have no singularity on $\xi_1 = 0$.

Thus we have obtained four symplectic coordinate systems $(q_1^*, q_2^*, p_1^*, p_2^*)$ each of which separates solution curves passing through the accessible singular points (see (9.1)-(9.4)). We notice that the Hamiltonians of each coordinate system are also polynomials whose coefficients are rational functions of s holomorphic in B.

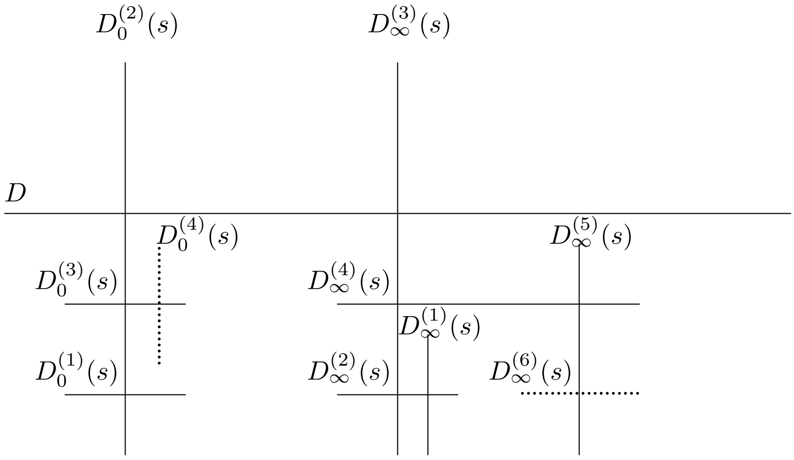

Figure 6. J=23

10 Spaces of initial conditions for $\mathcal{H}_5$

In the present case,

$$\nu = \alpha + \frac{1}{2}$$

In this section, we omitt the label 5.

10.1 Accessible singularities on $D \times B$

Observing the system $\mathcal{H}^{(0)}$ on all W_{ij} $j \neq 0$, we can obtain

Proposition 10.1. *The set of accessible singular points of the system $\mathcal{H}_5^{(0)}$ for each $s = (s_1, s_2) \in B_5$ is a disjoint union of 1 connected components $A_\infty(s) \simeq \mathbb{P}^1$ given by*

$$A_\infty(s) = \{(\xi, \eta_1, s) \in W_1 \times B | \xi_0 = \eta_{10} = \eta_{12} = 0\} \cup \{(\xi, \eta_2, s) \in W_2 \times B | \xi_0 = \eta_{20} = \eta_{21} = 0\}.$$

In the following subsections, we obtain coordinate systems corresponding to $A_\infty(s)$ which separate completely the solution curves passing through $A_\infty(s)$. The systems for $A_\infty(s)$ are obtained by quadratic transformations.

10.2 Coordinate systems for $A_\infty(s)$

We obtain coordinate systems for $A_\infty(s)$ by making quadratic transformations ten times along $A_\infty(s) \cap W_1,\ A_\infty(s) \cap W_2$.

10.2.1 Coordinate system for $A_\infty(s) \cap W_1$

The first quadratic transformation along $A_\infty(s) \cap W_1$. Let

$$\xi_0 = x^{(1)}_{\infty 0}, \quad \eta_{10} = x^{(1)}_{\infty 0} y^{(1)}_{\infty 0}, \quad \eta_{12} = x^{(1)}_{\infty 0} z^{(1)}_{\infty 0},$$
$$\xi_0 = x^{(1)}_{\infty 1} y^{(1)}_{\infty 1}, \quad \eta_{10} = y^{(1)}_{\infty 1}, \quad \eta_{12} = y^{(1)}_{\infty 1} z^{(1)}_{\infty 1},$$
$$\xi_0 = x^{(1)}_{\infty 2} z^{(1)}_{\infty 2}, \quad \eta_{10} = y^{(1)}_{\infty 2} z^{(1)}_{\infty 2}, \quad \eta_{12} = z^{(1)}_{\infty 2},$$

then

$$D^{(1)}_{\infty 1}(s) = Q_{A_\infty(s) \cap W_1}(A_\infty(s) \cap W_1) = \{x^{(1)}_{\infty 0} = 0\} \cup \{y^{(1)}_{\infty 1} = 0\} \cup \{z^{(1)}_{\infty 2} = 0\},$$

the set of accessible singular points is given by

$$A^{(1)}_{\infty 1}(s) = \{(\xi_2, x^{(1)}_{\infty 0}, y^{(1)}_{\infty 0}, z^{(1)}_{\infty 0}) = (\xi_2, 0, 0, -1/\xi_2)\} \subset D^{(1)}_{\infty 1}(s).$$

The second quadratic transformation along $A^{(1)}_{\infty 1}(s)$. Note that $\xi_2 = 0$ is excluded. Let

$$x^{(1)}_{\infty 0} = x^{(2)}_{\infty 0}, \quad y^{(1)}_{\infty 0} = x^{(2)}_{\infty 0} y^{(2)}_{\infty 0}, \quad z^{(1)}_{\infty 0} = -1/\xi_2 + x^{(2)}_{\infty 0} z^{(2)}_{\infty 0},$$
$$x^{(1)}_{\infty 0} = x^{(2)}_{\infty 1} y^{(2)}_{\infty 1}, \quad y^{(1)}_{\infty 0} = y^{(2)}_{\infty 1}, \quad z^{(1)}_{\infty 0} = -1/\xi_2 + y^{(2)}_{\infty 1} z^{(2)}_{\infty 1},$$
$$x^{(1)}_{\infty 0} = x^{(1)}_{\infty 2} z^{(2)}_{\infty 2}, \quad y^{(1)}_{\infty 0} = x^{(2)}_{\infty 2} y^{(2)}_{\infty 2}, \quad z^{(1)}_{\infty 0} = -1/\xi_2 + z^{(2)}_{\infty 2},$$

then

$$D^{(2)}_{\infty 1}(s) = Q_{A^{(1)}_{\infty 1}(s)}(A^{(1)}_{\infty 1}(s)) = \{x^{(2)}_{\infty 0} = 0\} \cup \{y^{(2)}_{\infty 1} = 0\} \cup \{z^{(2)}_{\infty 2} = 0\},$$

the set of accessible singular points is given by

$$A^{(2)}_{\infty 1}(s) = \{(\xi_2, x^{(2)}_{\infty 0}, y^{(2)}_{\infty 0}, z^{(2)}_{\infty 0}) = (\xi_2, 0, 0, -1/\xi_2^3)\} \subset D^{(2)}_{\infty 1}(s).$$

The third quadratic transformation along $A^{(2)}_{\infty 1}(s)$. Let

$$x^{(2)}_{\infty 0} = x^{(3)}_{\infty 0}, \quad y^{(2)}_{\infty 0} = x^{(3)}_{\infty 0} y^{(3)}_{\infty 0}, \quad z^{(2)}_{\infty 0} = -1/\xi_2^3 + x^{(3)}_{\infty 0} z^{(3)}_{\infty 0},$$
$$x^{(2)}_{\infty 0} = x^{(3)}_{\infty 1} y^{(3)}_{\infty 1}, \quad y^{(2)}_{\infty 0} = y^{(3)}_{\infty 1}, \quad z^{(2)}_{\infty 0} = -1/\xi_2^3 + y^{(3)}_{\infty 1} z^{(3)}_{\infty 1},$$
$$x^{(2)}_{\infty 0} = x^{(3)}_{\infty 2} z^{(3)}_{\infty 2}, \quad y^{(2)}_{\infty 0} = y^{(3)}_{\infty 2} z^{(3)}_{\infty 2}, \quad z^{(2)}_{\infty 0} = -1/\xi_2^3 + z^{(3)}_{\infty 2},$$

then

$$D^{(3)}_{\infty 1}(s) = Q_{A^{(2)}_{\infty 1}(s)}(A^{(2)}_{\infty 1}(s)) = \{x^{(3)}_{\infty 0} = 0\} \cup \{y^{(3)}_{\infty 1} = 0\} \cup \{z^{(3)}_{\infty 2} = 0\},$$

the set of accessible singular points is given by

$$A^{(3)}_{\infty 1}(s) = \{(\xi_2, x^{(3)}_{\infty 0}, y^{(3)}_{\infty 0}, z^{(3)}_{\infty 0}) = (\xi_2, 0, 0, -2/\xi_2^5)\} \subset D^{(3)}_{\infty 1}(s).$$

The fourth quadratic transformation along $A^{(3)}_{\infty 1}(s)$. Let

$$x^{(3)}_{\infty 0} = x^{(4)}_{\infty 0}, \quad y^{(3)}_{\infty 0} = x^{(4)}_{\infty 0} y^{(4)}_{\infty 0}, \quad z^{(3)}_{\infty 0} = -2/\xi_2^5 + x^{(4)}_{\infty 0} z^{(4)}_{\infty 0},$$
$$x^{(3)}_{\infty 0} = x^{(4)}_{\infty 1} y^{(4)}_{\infty 1}, \quad y^{(3)}_{\infty 0} = y^{(4)}_{\infty 1}, \quad z^{(3)}_{\infty 0} = -2/\xi_2^5 + y^{(4)}_{\infty 1} z^{(4)}_{\infty 1},$$
$$x^{(3)}_{\infty 0} = x^{(4)}_{\infty 2} z^{(4)}_{\infty 2}, \quad y^{(3)}_{\infty 0} = y^{(4)}_{\infty 2} z^{(4)}_{\infty 2}, \quad z^{(3)}_{\infty 0} = -2/\xi_2^5 + z^{(4)}_{\infty 2},$$

then

$$D^{(4)}_{\infty 1}(s) = Q_{A^{(3)}_{\infty 1}(s)}(A^{(3)}_{\infty 1}(s)) = \{x^{(4)}_{\infty 0} = 0\} \cup \{y^{(4)}_{\infty 1} = 0\} \cup \{z^{(4)}_{\infty 2} = 0\},$$

the set of accessible singular points is given by

$$A^{(4)}_{\infty 1}(s) = \{(\xi_2, x^{(4)}_{\infty 0}, y^{(4)}_{\infty 0}, z^{(4)}_{\infty 0}) = (\xi_2, 0, 0, -(5 + s_1 \xi_2^2)/\xi_2^7)\} \subset D^{(4)}_{\infty 1}(s).$$

The fifth quadratic transformation along $A^{(4)}_{\infty 1}(s)$. Let

$$x^{(4)}_{\infty 0} = x^{(5)}_{\infty 0}, \quad y^{(4)}_{\infty 0} = x^{(5)}_{\infty 0} y^{(5)}_{\infty 0}, \quad z^{(4)}_{\infty 0} = -(5 + s_1\xi_2^2)/\xi_2^7 + x^{(5)}_{\infty 0} z^{(5)}_{\infty 0},$$
$$x^{(4)}_{\infty 0} = x^{(5)}_{\infty 1} y^{(5)}_{\infty 1}, \quad y^{(4)}_{\infty 0} = y^{(5)}_{\infty 1}, \quad z^{(4)}_{\infty 0} = -(5 + s_1\xi_2^2)/\xi_2^7 + y^{(5)}_{\infty 1} z^{(5)}_{\infty 1},$$
$$x^{(4)}_{\infty 0} = x^{(5)}_{\infty 2} z^{(5)}_{\infty 2}, \quad y^{(4)}_{\infty 0} = y^{(5)}_{\infty 2} z^{(5)}_{\infty 2}, \quad z^{(4)}_{\infty 0} = -(5 + s_1\xi_2^2)/\xi_2^7 + z^{(5)}_{\infty 2},$$

then

$$D^{(5)}_{\infty 1}(s) = Q_{A^{(4)}_{\infty 1}(s)}(A^{(4)}_{\infty 1}(s)) = \{x^{(5)}_{\infty 0} = 0\} \cup \{y^{(5)}_{\infty 1} = 0\} \cup \{z^{(5)}_{\infty 2} = 0\},$$

the set of accessible singular points is given by

$$A^{(5)}_{\infty 1}(s) = \{(\xi_2, x^{(5)}_{\infty 0}, y^{(5)}_{\infty 0}, z^{(5)}_{\infty 0}) = (\xi_2, 0, -1/2\xi_2^4, z^{(5)}_{\infty 0})\} \subset D^{(5)}_{\infty 1}(s).$$

The sixth quadratic transformation along $A^{(5)}_{\infty 1}(s)$. Here we insert a change of variables

$$\xi_2 = \xi_2, \quad x^{(5)}_{\infty 0} = x^{(5)}_{\infty 0}, \quad y^{(5)}_{\infty 0} = 1/v^{(5)}_{\infty 0}, \quad z^{(5)}_{\infty 0} = z^{(5)}_{\infty 0},$$

Let

$$x^{(5)}_{\infty 0} = x^{(6)}_{\infty 0}, \quad v^{(5)}_{\infty 0} = -2\xi_2^4 + x^{(6)}_{\infty 0} y^{(6)}_{\infty 0},$$
$$x^{(5)}_{\infty 0} = x^{(6)}_{\infty 1} y^{(6)}_{\infty 1}, \quad v^{(5)}_{\infty 0} = -2\xi_2^4 + y^{(6)}_{\infty 1},$$

then

$$D^{(6)}_{\infty 1}(s) = Q_{A^{(5)}_{\infty 1}(s)}(A^{(5)}_{\infty 1}(s)) = \{x^{(6)}_{\infty 0} = 0\} \cup \{y^{(6)}_{\infty 1} = 0\},$$

the set of accessible singular points is given by

$$A^{(6)}_{\infty 1}(s) = \{(\xi_2, z^{(5)}_{\infty 0}, x^{(6)}_{\infty 0}, y^{(6)}_{\infty 0}) = (\xi_2, z^{(5)}_{\infty 0}, 0, 6\xi_2^2)\} \subset D^{(6)}_{\infty 1}(s).$$

The seventh quadratic transformation along $A^{(6)}_{\infty 1}(s)$. Let

$$x^{(6)}_{\infty 0} = x^{(7)}_{\infty 0}, \quad y^{(6)}_{\infty 0} = 6\xi_2^2 + x^{(7)}_{\infty 0} y^{(7)}_{\infty 0},$$
$$x^{(6)}_{\infty 0} = x^{(7)}_{\infty 1} y^{(7)}_{\infty 1}, \quad y^{(6)}_{\infty 0} = 6\xi_2^2 + y^{(7)}_{\infty 1},$$

then

$$D^{(7)}_{\infty 1}(s) = Q_{A^{(6)}_{\infty 1}(s)}(A^{(6)}_{\infty 1}(s)) = \{x^{(7)}_{\infty 0} = 0\} \cup \{y^{(7)}_{\infty 1} = 0\},$$

the set of accessible singular points is given by

$$A^{(7)}_{\infty 1}(s) = \{(\xi_2, z^{(5)}_{\infty 0}, x^{(7)}_{\infty 0}, y^{(7)}_{\infty 0}) = (\xi_2, z^{(4)}_{\infty 0}, 0, -2)\} \subset D^{(7)}_{\infty 1}(s).$$

The eighth quadratic transformation along $A^{(7)}_{\infty 1}(s)$. Let

$$x^{(7)}_{\infty 0} = x^{(8)}_{\infty 0}, \quad y^{(7)}_{\infty 0} = -2 + x^{(8)}_{\infty 0} y^{(8)}_{\infty 0},$$
$$x^{(7)}_{\infty 0} = x^{(8)}_{\infty 1} y^{(8)}_{\infty 1}, \quad y^{(7)}_{\infty 0} = -2 + y^{(8)}_{\infty 1},$$

then

$$D^{(8)}_{\infty 1}(s) = Q_{A^{(7)}_{\infty 1}(s)}(A^{(7)}_{\infty 1}(s)) = \{x^{(8)}_{\infty 0} = 0\} \cup \{y^{(8)}_{\infty 1} = 0\},$$

the set of accessible singular points is given by

$$A^{(8)}_{\infty 1}(s) = \{(\xi_2, z^{(5)}_{\infty 0}, x^{(8)}_{\infty 0}, y^{(8)}_{\infty 0}) = (\xi_2, z^{(5)}_{\infty 0}, 0, 2(s_2\xi_2 + s_1))\} \subset D^{(8)}_{\infty 1}(s).$$

The ninth quadratic transformation along $A^{(8)}_{\infty 1}(s)$. Let

$$x^{(8)}_{\infty 0} = x^{(9)}_{\infty 0}, \quad y^{(8)}_{\infty 0} = 2(s_2\xi_2 + s_1) + x^{(9)}_{\infty 0} y^{(9)}_{\infty 0},$$
$$x^{(8)}_{\infty 0} = x^{(9)}_{\infty 1} y^{(9)}_{\infty 1}, \quad y^{(8)}_{\infty 0} = 2(s_2\xi_2 + s_1) + y^{(9)}_{\infty 1},$$

then

$$D^{(9)}_{\infty 1}(s) = Q_{A^{(8)}_{\infty 1}(s)}(A^{(8)}_{\infty 1}(s)) = \{x^{(9)}_{\infty 0} = 0\} \cup \{y^{(9)}_{\infty 1} = 0\},$$

the set of accessible singular points is given by

$$A^{(9)}_{\infty 1}(s) = \{(\xi_2, z^{(5)}_{\infty 0}, x^{(9)}_{\infty 0}, y^{(9)}_{\infty 0}) = (\xi_2, z^{(5)}_{\infty 0}, 0, -2\alpha)\} \subset D^{(9)}_{\infty 1}(s).$$

The tenth quadratic transformation along $A^{(9)}_{\infty 1}(s)$. Let

$$x^{(9)}_{\infty 0} = x^{(10)}_{\infty 0}, \quad y^{(9)}_{\infty 0} = -2\alpha + x^{(10)}_{\infty 0} y^{(10)}_{\infty 0},$$
$$x^{(9)}_{\infty 0} = x^{(10)}_{\infty 1} y^{(10)}_{\infty 1}, \quad y^{(9)}_{\infty 0} = -2\alpha + y^{(10)}_{\infty 1}$$

then

$$D^{(10)}_{\infty 1}(s) = Q_{A^{(9)}_{\infty 1}(s)}(A^{(9)}_{\infty 1}(s)) = \{x^{(10)}_{\infty 0} = 0\} \cup \{y^{(10)}_{\infty 1} = 0\}.$$

We can verify that the differential system in the cordinates $(\xi_2, z^{(5)}_{\infty 0}, x^{(10)}_{\infty 0}, y^{(10)}_{\infty 0})$ is holomorphic in a neighborhood of $\{x^{(10)}_{\infty 0} = 0\}$ except for $\xi_2 = 0$ and the points $(\xi_2, z^{(5)}_{\infty 0}, x^{(10)}_{\infty 1}, y^{(10)}_{\infty 1}) = (\xi_2, z^{(5)}_{\infty 0}, 0, 0)$ are inaccessible with $\xi_2 \neq 0$.

Thus we have obtained a coordinate system $(\xi_2, z^{(5)}_{\infty 0}, x^{(10)}_{\infty 0}, y^{(10)}_{\infty 0}) \in \mathbb{C}^4$ which separates the solutions passing through $A_\infty(s) \cap W_1 = A_\infty(s) \cap W_{11}$. It is related to the coordinate system $(q^1_1, q^1_2, p^1_1, p^1_2)$ by

$$\begin{aligned}
q^1_1 &= x^{(10)}_{\infty 0}, \quad q^1_2 = \xi_2, \\
p^1_1 &= -\frac{2\xi_2^4}{(x^{(10)}_{\infty 0})^5} + \frac{6\xi_2^2}{(x^{(10)}_{\infty 0})^4} - \frac{2}{(x^{(10)}_{\infty 0})^3} + \frac{2(s_2\xi_2 + s_1)}{(x^{(10)}_{\infty 0})^2} - \frac{2\alpha}{x^{(10)}_{\infty 0}} + y^{(10)}_{\infty 0}, \\
p^1_1 &= \frac{2e_2^3}{(x^{(10)}_{\infty 0})^4} - \frac{4e_2}{(x^{(10)}_{\infty 0})^3} - \frac{2s_2}{x^{(10)}_{\infty 0}} - 2\xi_2^4 z^{(5)}_{\infty 0} + \frac{2\alpha}{\xi_2} - \frac{2s_2}{\xi_2^2} - \frac{8s_1}{\xi_2^3} - \frac{26}{\xi_2^5} \\
&\quad + \Big(6\xi_2^2 z^{(5)}_{\infty 0} - \frac{y^{(10)}_{\infty 0}}{\xi_2} + \frac{2\alpha}{\xi_2^3} - \frac{4s_2}{\xi_2^4} - \frac{2s_1}{\xi_2^5} + \frac{10}{\xi_2^7}\Big) x^{(10)}_{\infty 0} \\
&\quad - \Big(2z^{(5)}_{\infty 0} + \frac{y^{(10)}_{\infty 0}}{\xi_2^3} + \frac{2s_1 s_2}{\xi_2^4} + \frac{2(s_1^2 - 2\alpha)}{\xi_2^5} + \frac{10s_2}{\xi_2^6} + \frac{10s_1}{\xi_2^7}\Big)(x^{(10)}_{\infty 0})^2 \\
&\quad + \Big\{2(s_2\xi_2 + s_1) z^{(5)}_{\infty 0} + \frac{2(\alpha s_1 - y^{(10)}_{\infty 0})}{\xi_2^5} + \frac{10\alpha}{\xi_2^7}\Big\}(x^{(10)}_{\infty 0})^3 \\
&\quad - \Big\{2\alpha z^{(5)}_{\infty 0} + \Big(\frac{s_1}{\xi_2^5} + \frac{5}{\xi_2^7}\Big) y^{(10)}_{\infty 0}\Big\}(x^{(10)}_{\infty 0})^4 + z^{(5)}_{\infty 0} y^{(10)}_{\infty 0} (x^{(10)}_{\infty 0})^5.
\end{aligned}$$

Now we calculate the 2-form $dq^1_1 \wedge dp^1_1 + dq^1_2 \wedge dp^1_2$ in the coordinates $(\xi_2, z^{(5)}_{\infty 0}, x^{(10)}_{\infty 0}, y^{(10)}_{\infty 0})$:

$$\begin{aligned}
&dq^1_1 \wedge dp^1_1 + dq^1_2 \wedge dp^1_2 \\
&= dx^{(10)}_{\infty 0} \wedge dy^{(10)}_{\infty 0} \\
&\quad + \{y^{(10)}_{\infty 0}(x^{(10)}_{\infty 0})^5 - 2\alpha (x^{(10)}_{\infty 0})^4 + 2(s_2\xi_2 + s_1)(x^{(10)}_{\infty 0})^3 - 2(x^{(10)}_{\infty 0})^2 + 6\xi_2^2 x^{(10)}_{\infty 0} - 2\xi_2^4\} d\xi_2 \wedge dz^{(5)}_{\infty 0}
\end{aligned}$$

$$
\begin{aligned}
&+\Big[5z_{\infty0}^{(5)}y_{\infty0}^{(10)}(x_{\infty0}^{(10)})^4-4\Big(2\alpha z_{\infty0}^{(5)}+\frac{s_1y_{\infty0}^{(10)}}{\xi_2^5}+\frac{5y_{\infty0}^{(10)}}{\xi_2^7}\Big)(x_{\infty0}^{(10)})^3\\
&\qquad+6\Big\{(\xi_2s_2+s_1)z_{\infty0}^{(5)}+\frac{\alpha s_1-y_{\infty0}^{(10)}}{\xi_2^5}+\frac{5\alpha}{\xi_2^7}\Big\}(x_{\infty0}^{(10)})^2\\
&\qquad-2\Big(2z_{\infty0}^{(5)}+\frac{y_{\infty0}^{(10)}}{\xi_2^3}+\frac{2s_1s_2}{\xi_2^4}+\frac{2(s_1^2-2\alpha)}{\xi_2^5}+\frac{10s_2}{\xi_2^6}+\frac{10s_1}{\xi_2^7}\Big)x_{\infty0}^{(10)}\\
&\qquad+6\xi_2^2z_{\infty0}^{(5)}-\frac{y_{\infty0}^{(10)}}{\xi_2}+\frac{2\alpha}{\xi_2^3}-\frac{4s_2}{\xi_2^4}-\frac{2s_1}{\xi_2^5}+\frac{10}{\xi_2^7}\Big]d\xi_2\wedge dx_{\infty0}^{(10)}\\
&+\Big\{z_{\infty0}^{(5)}(x_{\infty0}^{(10)})^5-\Big(\frac{s_1}{\xi_2^5}+\frac{5}{\xi_2^7}\Big)(x_{\infty0}^{(10)})^4-\frac{(x_{\infty0}^{(10)})^3}{\xi_2^5}-\frac{(x_{\infty0}^{(10)})^2}{\xi_2^3}-\frac{x_{\infty0}^{(10)}}{\xi_2}\Big\}d\xi_2\wedge dy_{\infty0}^{(10)}\\
=\;&dx_{\infty0}^{(10)}\wedge dy_{\infty0}^{(10)}\\
&+d\xi_2\wedge d\Big[\{y_{\infty0}^{(10)}(x_{\infty0}^{(10)})^5-2\alpha(x_{\infty0}^{(10)})^4+2(s_2\xi_2+s_1)(x_{\infty0}^{(10)})^3-2(x_{\infty0}^{(10)})^2+6\xi_2^2x_{\infty0}^{(10)}-2\xi_2^4\}z_{\infty0}^{(5)}\\
&\qquad-\Big\{\Big(\frac{s_1}{\xi_2^5}+\frac{5}{\xi_2^7}\Big)(x_{\infty0}^{(10)})^4+\frac{(x_{\infty0}^{(10)})^3}{\xi_2^5}+\frac{(x_{\infty0}^{(10)})^2}{\xi_2^3}+\frac{x_{\infty0}^{(10)}}{\xi_2}\Big\}y_{\infty0}^{(10)}\\
&\qquad+2\Big(\frac{\alpha s_1}{\xi_2^5}+\frac{5\alpha}{\xi_2^7}\Big)(x_{\infty0}^{(10)})^3-\Big(\frac{2s_1s_2}{\xi_2^4}+\frac{2(s_1^2-2\alpha)}{\xi_2^5}+\frac{10s_2}{\xi_2^6}+\frac{10s_1}{\xi_2^7}\Big)(x_{\infty0}^{(10)})^2\\
&\qquad+\Big(\frac{2\alpha}{\xi_2^3}-\frac{4s_2}{\xi_2^4}-\frac{2s_1}{\xi_2^5}+\frac{10}{\xi_2^7}\Big)x_{\infty0}^{(10)}\Big]\\
=\;&dx_{\infty0}^{(10)}\wedge dy_{\infty0}^{(10)}\\
&+d\xi_2\wedge d\Big[\{y_{\infty0}^{(10)}(x_{\infty0}^{(10)})^5-2\alpha(x_{\infty0}^{(10)})^4+2(s_2\xi_2+s_1)(x_{\infty0}^{(10)})^3-2(x_{\infty0}^{(10)})^2+6\xi_2^2x_{\infty0}^{(10)}-2\xi_2^4\}z_{\infty0}^{(5)}\\
&\qquad-\Big\{\Big(\frac{s_1}{\xi_2^5}+\frac{5}{\xi_2^7}\Big)(x_{\infty0}^{(10)})^4+\frac{(x_{\infty0}^{(10)})^3}{\xi_2^5}+\frac{(x_{\infty0}^{(10)})^2}{\xi_2^3}+\frac{x_{\infty0}^{(10)}}{\xi_2}\Big\}y_{\infty0}^{(10)}\\
&\qquad+2\Big(\frac{\alpha s_1}{\xi_2^5}+\frac{5\alpha}{\xi_2^7}\Big)(x_{\infty0}^{(10)})^3-\Big(\frac{2s_1s_2}{\xi_2^4}+\frac{2(s_1^2-2\alpha)}{\xi_2^5}+\frac{10s_2}{\xi_2^6}+\frac{10s_1}{\xi_2^7}\Big)(x_{\infty0}^{(10)})^2\\
&\qquad+\Big(\frac{2\alpha}{\xi_2^3}-\frac{4s_2}{\xi_2^4}-\frac{2s_1}{\xi_2^5}+\frac{10}{\xi_2^7}\Big)x_{\infty0}^{(10)}+\frac{2\alpha}{\xi_2}-\frac{2s_2}{\xi_2^2}-\frac{8s_1}{\xi_2^3}-\frac{26}{\xi_2^5}\Big].
\end{aligned}
$$

Therefore, setting

$$
\begin{aligned}
w_{\infty0}^{(5)}=&\{y_{\infty0}^{(10)}(x_{\infty0}^{(10)})^5-2\alpha(x_{\infty0}^{(10)})^4+2(s_2\xi_2+s_1)(x_{\infty0}^{(10)})^3-2(x_{\infty0}^{(10)})^2+6\xi_2^2x_{\infty0}^{(10)}-2\xi_2^4\}z_{\infty0}^{(5)}\\
&-\Big\{\Big(\frac{s_1}{\xi_2^5}+\frac{5}{\xi_2^7}\Big)(x_{\infty0}^{(10)})^4+\frac{(x_{\infty0}^{(10)})^3}{\xi_2^5}+\frac{(x_{\infty0}^{(10)})^2}{\xi_2^3}+\frac{x_{\infty0}^{(10)}}{\xi_2}\Big\}y_{\infty0}^{(10)}\\
&+2\Big(\frac{\alpha s_1}{\xi_2^5}+\frac{5\alpha}{\xi_2^7}\Big)(x_{\infty0}^{(10)})^3-\Big(\frac{2s_1s_2}{\xi_2^4}+\frac{2(s_1^2-2\alpha)}{\xi_2^5}+\frac{10s_2}{\xi_2^6}+\frac{10s_1}{\xi_2^7}\Big)(x_{\infty0}^{(10)})^2\\
&+\Big(\frac{2\alpha}{\xi_2^3}-\frac{4s_2}{\xi_2^4}-\frac{2s_1}{\xi_2^5}+\frac{10}{\xi_2^7}\Big)x_{\infty0}^{(10)}+\frac{2\alpha}{\xi_2}-\frac{2s_2}{\xi_2^2}-\frac{8s_1}{\xi_2^3}-\frac{26}{\xi_2^5},
\end{aligned}
$$

we have symplectic coordinates $(\xi_2, w_{\infty0}^{(5)}, x_{\infty0}^{(10)}, y_{\infty0}^{(10)})$. Writing

$$
q_1^{\infty1}=x_{\infty0}^{(10)},\quad q_2^{\infty1}=\xi_2,\quad p_1^{\infty1}=y_{\infty0}^{(10)},\quad p_2^{\infty1}=w_{\infty0}^{(5)},
$$

we have

$$
\begin{aligned}
&q_1^1 = q_1^{\infty 1}, \quad q_2^1 = q_2^{\infty 1},\\
&p_1^1 = -\frac{2(q_2^{\infty 1})^4}{(q_1^{\infty 1})^5} + \frac{6(q_2^{\infty 1})^2}{(q_1^{\infty 1})^4} - \frac{2}{(q_1^{\infty 1})^3} + \frac{2(s_1 + s_2 q_2^{\infty 1})}{(q_1^{\infty 1})^2} - \frac{2\alpha}{q_1^{\infty 1}} + p_1^{\infty 1},\\
&p_2^1 = \frac{2(q_2^{\infty 1})^3}{(q_1^{\infty 1})^4} - \frac{4q_2^{\infty 1}}{(q_1^{\infty 1})^3} - \frac{2s_2}{q_1^{\infty 1}} + p_2^{\infty 1}.
\end{aligned}
\tag{10.1}
$$

Thus we have a simplectic coordinate system $(q^{\infty 1}, p^{\infty 1}) \in \mathbb{C}^4$ in which Hamiltonians have no singularity on $\xi_2 = 0$.

10.2.2 Coordinate system for $A_\infty(s) \cap W_2$

The first quadratic transformation along $A_\infty(s) \cap W_2$. Let

$$
\begin{aligned}
&\xi_0 = X_{\infty 0}^{(1)}, \quad \eta_{20} = X_{\infty 0}^{(1)} Y_{\infty 0}^{(1)}, \quad \eta_{21} = X_{\infty 0}^{(1)} Z_{\infty 0}^{(1)},\\
&\xi_0 = X_{\infty 1}^{(1)} Y_{\infty 1}^{(1)}, \quad \eta_{20} = Y_{\infty 1}^{(1)}, \quad \eta_{21} = Y_{\infty 1}^{(1)} Z_{\infty 1}^{(1)},\\
&\xi_0 = X_{\infty 2}^{(1)} Z_{\infty 2}^{(1)}, \quad \eta_{20} = Y_{\infty 2}^{(1)} Z_{\infty 2}^{(1)}, \quad \eta_{21} = Z_{\infty 2}^{(1)},
\end{aligned}
$$

then

$$
D_{\infty 2}^{(1)}(s) = Q_{A_\infty(s)\cap W_2}(A_\infty(s) \cap W_2) = \{X_{\infty 0}^{(1)} = 0\} \cup \{Y_{\infty 1}^{(1)} = 0\} \cup \{Z_{\infty 2}^{(1)} = 0\},
$$

the set of accessible singular points is given by

$$
A_{\infty 2}^{(1)}(s) = \{(\xi_1, X_{\infty 0}^{(1)}, Y_{\infty 0}^{(1)}, Z_{\infty 0}^{(1)}) = (\xi_1, 0, 0, 0)\} \subset D_{\infty 2}^{(1)}(s).
$$

The second quadratic transformation along $A_{\infty 2}^{(1)}(s)$. Let

$$
\begin{aligned}
&X_{\infty 0}^{(1)} = X_{\infty 0}^{(2)}, \quad Y_{\infty 0}^{(1)} = X_{\infty 0}^{(2)} Y_{\infty 0}^{(2)}, \quad Z_{\infty 0}^{(1)} = X_{\infty 0}^{(2)} Z_{\infty 0}^{(2)},\\
&X_{\infty 0}^{(1)} = X_{\infty 1}^{(2)} Y_{\infty 1}^{(2)}, \quad Y_{\infty 0}^{(1)} = Y_{\infty 1}^{(2)}, \quad Z_{\infty 0}^{(1)} = Y_{\infty 1}^{(2)} Z_{\infty 1}^{(2)},\\
&X_{\infty 0}^{(1)} = X_{\infty 2}^{(1)} Z_{\infty 2}^{(2)}, \quad Y_{\infty 0}^{(1)} = X_{\infty 2}^{(2)} Y_{\infty 2}^{(2)}, \quad Z_{\infty 0}^{(1)} = Z_{\infty 2}^{(2)},
\end{aligned}
$$

then

$$
D_{\infty 2}^{(2)}(s) = Q_{A_{\infty 2}^{(1)}(s)}(A_{\infty 2}^{(1)}(s)) = \{X_{\infty 0}^{(2)} = 0\} \cup \{Y_{\infty 1}^{(2)} = 0\} \cup \{Z_{\infty 2}^{(2)} = 0\},
$$

the set of accessible singular points is given by

$$
A_{\infty 2}^{(2)}(s) = \{(\xi_1, X_{\infty 0}^{(2)}, Y_{\infty 0}^{(2)}, Z_{\infty 0}^{(2)}) = (\xi_1, 0, 0, 1)\} \subset D_{\infty 2}^{(2)}(s).
$$

The third quadratic transformation along $A_{\infty 2}^{(2)}(s)$. Let

$$
\begin{aligned}
&X_{\infty 0}^{(2)} = X_{\infty 0}^{(3)}, \quad Y_{\infty 0}^{(2)} = X_{\infty 0}^{(3)} Y_{\infty 0}^{(3)}, \quad Z_{\infty 0}^{(2)} = 1 + X_{\infty 0}^{(3)} Z_{\infty 0}^{(3)},\\
&X_{\infty 0}^{(2)} = X_{\infty 1}^{(3)} Y_{\infty 1}^{(3)}, \quad Y_{\infty 0}^{(2)} = Y_{\infty 1}^{(3)}, \quad Z_{\infty 0}^{(2)} = 1 + Y_{\infty 1}^{(3)} Z_{\infty 1}^{(3)},\\
&X_{\infty 0}^{(2)} = X_{\infty 2}^{(3)} Z_{\infty 2}^{(3)}, \quad Y_{\infty 0}^{(2)} = Y_{\infty 2}^{(3)} Z_{\infty 2}^{(3)}, \quad Z_{\infty 0}^{(2)} = 1 + Z_{\infty 2}^{(3)},
\end{aligned}
$$

then

$$
D_{\infty 2}^{(3)}(s) = Q_{A_{\infty 2}^{(2)}(s)}(A_{\infty 2}^{(2)}(s)) = \{X_{\infty 0}^{(3)} = 0\} \cup \{Y_{\infty 1}^{(3)} = 0\} \cup \{Z_{\infty 2}^{(3)} = 0\},
$$

the set of accessible singular points is given by

$$
A_{\infty 2}^{(3)}(s) = \{(\xi_1, X_{\infty 0}^{(3)}, Y_{\infty 0}^{(3)}, Z_{\infty 0}^{(3)}) = (\xi_1, 0, 0, 2\xi_1)\} \subset D_{\infty 2}^{(3)}(s).
$$

The fourth quadratic transformation along $A^{(3)}_{\infty 2}(s)$. Let

$$X^{(3)}_{\infty 0} = X^{(4)}_{\infty 0}, \quad Y^{(3)}_{\infty 0} = X^{(4)}_{\infty 0} Y^{(4)}_{\infty 0}, \quad Z^{(3)}_{\infty 0} = 2\xi_1 + X^{(4)}_{\infty 0} Z^{(4)}_{\infty 0},$$
$$X^{(3)}_{\infty 0} = X^{(4)}_{\infty 1} Y^{(4)}_{\infty 1}, \quad Y^{(3)}_{\infty 0} = Y^{(4)}_{\infty 1}, \quad Z^{(3)}_{\infty 0} = 2\xi_1 + Y^{(4)}_{\infty 1} Z^{(4)}_{\infty 1},$$
$$X^{(3)}_{\infty 0} = X^{(4)}_{\infty 2} Z^{(4)}_{\infty 2}, \quad Y^{(3)}_{\infty 0} = Y^{(4)}_{\infty 2} Z^{(4)}_{\infty 2}, \quad Z^{(3)}_{\infty 0} = 2\xi_1 + Z^{(4)}_{\infty 2},$$

then

$$D^{(4)}_{\infty 2}(s) = Q_{A^{(3)}_{\infty 2}(s)}(A^{(3)}_{\infty 2}(s)) = \{X^{(4)}_{\infty 0} = 0\} \cup \{Y^{(4)}_{\infty 1} = 0\} \cup \{Z^{(4)}_{\infty 2} = 0\},$$

the set of accessible singular points is given by

$$A^{(4)}_{\infty 2}(s) = \{(\xi_1, X^{(4)}_{\infty 0}, Y^{(4)}_{\infty 0}, Z^{(4)}_{\infty 0}) = (\xi_1, 0, 0, 5\xi_1^2 + s_1)\} \subset D^{(4)}_{\infty 2}(s).$$

The fifth quadratic transformation along $A^{(4)}_{\infty 2}(s)$. Let

$$X^{(4)}_{\infty 0} = X^{(5)}_{\infty 0}, \quad Y^{(4)}_{\infty 0} = X^{(5)}_{\infty 0} Y^{(5)}_{\infty 0}, \quad Z^{(4)}_{\infty 0} = 5\xi_1^2 + s_1 + X^{(5)}_{\infty 0} Z^{(5)}_{\infty 0},$$
$$X^{(4)}_{\infty 0} = X^{(5)}_{\infty 1} Y^{(5)}_{\infty 1}, \quad Y^{(4)}_{\infty 0} = Y^{(5)}_{\infty 1}, \quad Z^{(4)}_{\infty 0} = 5\xi_1^2 + s_1 + Y^{(5)}_{\infty 1} Z^{(5)}_{\infty 1},$$
$$X^{(4)}_{\infty 0} = X^{(5)}_{\infty 2} Z^{(5)}_{\infty 2}, \quad Y^{(4)}_{\infty 0} = Y^{(5)}_{\infty 2} Z^{(5)}_{\infty 2}, \quad Z^{(4)}_{\infty 0} = 5\xi_1^2 + s_1 + Z^{(5)}_{\infty 2},$$

then

$$D^{(5)}_{\infty 2}(s) = Q_{A^{(4)}_{\infty 2}(s)}(A^{(4)}_{\infty 2}(s)) = \{X^{(5)}_{\infty 0} = 0\} \cup \{Y^{(5)}_{\infty 1} = 0\} \cup \{Z^{(5)}_{\infty 2} = 0\},$$

the set of accessible singular points is given by

$$A^{(5)}_{\infty 2}(s) = \{(\xi_1, X^{(5)}_{\infty 0}, Y^{(5)}_{\infty 0}, Z^{(5)}_{\infty 0}) = (\xi_1, 0, -1/2, Z^{(5)}_{\infty 0})\} \subset D^{(5)}_{\infty 2}(s).$$

The sixth quadratic transformation along $A^{(5)}_{\infty 2}(s)$. We insert here the transformations

$$\xi_1 = \xi_1, \quad X^{(5)}_{\infty 0} = X^{(5)}_{\infty 0}, \quad Y^{(5)}_{\infty 0} = 1/V^{(5)}_{\infty 0}, \quad Z^{(5)}_{\infty 0} = Z^{(5)}_{\infty 0},$$

Let

$$X^{(5)}_{\infty 0} = X^{(6)}_{\infty 0}, \quad V^{(5)}_{\infty 0} = -2 + X^{(6)}_{\infty 0} Y^{(6)}_{\infty 0},$$
$$X^{(5)}_{\infty 0} = X^{(6)}_{\infty 1} Y^{(6)}_{\infty 1}, \quad V^{(5)}_{\infty 0} = -2 + Y^{(6)}_{\infty 1},$$

then

$$D^{(6)}_{\infty 2}(s) = Q_{A^{(5)}_{\infty 2}(s)}(A^{(5)}_{\infty 2}(s)) = \{X^{(6)}_{\infty 0} = 0\} \cup \{Y^{(6)}_{\infty 1} = 0\},$$

the set of accessible singular points is given by

$$A^{(6)}_{\infty 2}(s) = \{(\xi_1, Z^{(5)}_{\infty 0}, X^{(6)}_{\infty 0}, Y^{(6)}_{\infty 0}) = (\xi_1, Z^{(5)}_{\infty 0}, 0, 6\xi_1)\} \subset D^{(6)}_{\infty 2}(s).$$

The seventh quadratic transformation along $A^{(6)}_{\infty 2}(s)$. Let

$$X^{(6)}_{\infty 0} = X^{(7)}_{\infty 0}, \quad Y^{(6)}_{\infty 0} = 6\xi_1 + X^{(7)}_{\infty 0} Y^{(7)}_{\infty 0},$$
$$X^{(6)}_{\infty 0} = X^{(7)}_{\infty 1} Y^{(7)}_{\infty 1}, \quad Y^{(6)}_{\infty 0} = 6\xi_1 + Y^{(7)}_{\infty 1},$$

then

$$D^{(7)}_{\infty 2}(s) = Q_{A^{(6)}_{\infty 2}(s)}(A^{(6)}_{\infty 2}(s)) = \{X^{(7)}_{\infty 0} = 0\} \cup \{Y^{(7)}_{\infty 1} = 0\},$$

the set of accessible singular points is given by

$$A^{(7)}_{\infty 2}(s) = \{(\xi_1, Z^{(5)}_{\infty 0}, X^{(7)}_{\infty 0}, Y^{(7)}_{\infty 0}) = (\xi_1, Z^{(4)}_{\infty 0}, 0, -2\xi_1^2)\} \subset D^{(7)}_{\infty 2}(s).$$

The eighth quadratic transformation along $A^{(7)}_{\infty 2}(s)$. Let

$$X^{(7)}_{\infty 0} = X^{(8)}_{\infty 0}, \quad Y^{(7)}_{\infty 0} = -2\xi_1^2 + X^{(8)}_{\infty 0} Y^{(8)}_{\infty 0},$$
$$X^{(7)}_{\infty 0} = X^{(8)}_{\infty 1} Y^{(8)}_{\infty 1}, \quad Y^{(7)}_{\infty 0} = -2\xi_1^2 + Y^{(8)}_{\infty 1},$$

then

$$D^{(8)}_{\infty 2}(s) = Q_{A^{(7)}_{\infty 2}(s)}(A^{(7)}_{\infty 2}(s)) = \{X^{(8)}_{\infty 0} = 0\} \cup \{Y^{(8)}_{\infty 1} = 0\},$$

the set of accessible singular points is given by

$$A^{(8)}_{\infty 2}(s) = \{(\xi_1, Z^{(5)}_{\infty 0}, X^{(8)}_{\infty 0}, Y^{(8)}_{\infty 0}) = (\xi_1, z^{(5)}_{\infty 0}, 0, 2(s_1\xi_1 + s_2))\} \subset D^{(8)}_{\infty 2}(s).$$

The ninth quadratic transformation along $A^{(8)}_{\infty 2}(s)$. Let

$$X^{(8)}_{\infty 0} = X^{(9)}_{\infty 0}, \quad Y^{(8)}_{\infty 0} = 2(s_1\xi_1 + s_2) + X^{(9)}_{\infty 0} Y^{(9)}_{\infty 0},$$
$$X^{(8)}_{\infty 0} = X^{(9)}_{\infty 1} Y^{(9)}_{\infty 1}, \quad Y^{(8)}_{\infty 0} = 2(s_1\xi_1 + s_2) + Y^{(9)}_{\infty 1},$$

then

$$D^{(9)}_{\infty 2}(s) = Q_{A^{(8)}_{\infty 2}(s)}(A^{(8)}_{\infty 2}(s)) = \{X^{(9)}_{\infty 0} = 0\} \cup \{Y^{(9)}_{\infty 1} = 0\},$$

the set of accessible singular points is given by

$$A^{(9)}_{\infty 2}(s) = \{(\xi_1, Z^{(5)}_{\infty 0}, X^{(9)}_{\infty 0}, Y^{(9)}_{\infty 0}) = (\xi_1, Z^{(5)}_{\infty 0}, 0, -2\alpha)\} \subset D^{(9)}_{\infty 2}(s).$$

The tenth quadratic transformation along $A^{(9)}_{\infty 2}(s)$. Let

$$X^{(9)}_{\infty 0} = X^{(10)}_{\infty 0}, \quad Y^{(9)}_{\infty 0} = -2\alpha + X^{(10)}_{\infty 0} Y^{(10)}_{\infty 0},$$
$$X^{(9)}_{\infty 0} = X^{(10)}_{\infty 1} Y^{(10)}_{\infty 1}, \quad Y^{(9)}_{\infty 0} = -2\alpha + Y^{(10)}_{\infty 1},$$

then

$$D^{(10)}_{\infty 2}(s) = Q_{A^{(9)}_{\infty 2}(s)}(A^{(9)}_{\infty 2}(s)) = \{X^{(10)}_{\infty 0} = 0\} \cup \{Y^{(10)}_{\infty 1} = 0\}.$$

We see that, in the $(\xi_1, Z^{(5)}_{\infty 0}, X^{(10)}_{\infty 0}, Y^{(10)}_{\infty 0}, s)$-space $\mathbb{C}^4 \times B$, the differential system is holomorphic in a neighborhood of $\{X^{(6)}_{10} = 0\}$, moreover, the points $(\xi_1, Z^{(5)}_{\infty 0}, X^{(6)}_{\infty 1}, Y^{(6)}_{\infty 1}) = (\xi_0, Z^{(5)}_{\infty 0}, 0, 0)$ are inaccessible.

Thus we have obtained a coordinate system $(\xi_1, Z^{(5)}_{\infty 0}, X^{(10)}_{\infty 0}, Y^{(10)}_{\infty 0}) \in \mathbb{C}^4$ which is related to the coordinate system $(q^2_1, q^2_2, p^2_1, p^2_2)$ as

$$
\begin{aligned}
q^2_1 &= \xi_1, \quad q^2_2 = X^{(10)}_{\infty 0},\\
p^2_1 &= -\frac{2}{(X^{(10)}_{\infty 0})^3} + \frac{2\xi_1}{(X^{(10)}_{\infty 0})^2} - \frac{2s_1}{X^{(10)}_{\infty 0}} - 2Z^{(5)}_{\infty 0} + 26\xi_1^3 + 8s_1\xi_1 + 2s_2\\
&\quad + \{-10\xi_1^4 + 2s_1\xi_1^2 + 2(2s_2 + 3Z^{(5)}_{\infty 0})\xi_1 - 2\alpha\} X^{(10)}_{\infty 0}\\
&\quad - \{10s_1\xi_1^3 + 2(5s_2 - Z^{(5)}_{\infty 0})\xi_1^2 + 2(s_1^2 - 2\alpha)\xi_1 + Y^{(10)}_{\infty 0} + 2s_1s_2\}(X^{(10)}_{\infty 0})^2\\
&\quad + \{-10\alpha\xi_1^2 + 2(Z^{(5)}_{\infty 0}s_1 + Y^{(10)}_{\infty 0})\xi_1 + 2(Z^{(5)}_{\infty 0} - \alpha s_1)\}(X^{(10)}_{\infty 0})^3\\
&\quad - \{2\alpha Z^{(5)}_{\infty 0} + (5\xi_1^2 + s_1)Y^{(10)}_{\infty 0}\}(X^{(10)}_{\infty 0})^4 + Z^{(5)}_{\infty 0} Y^{(10)}_{\infty 0} (X^{(10)}_{\infty 0})^5,\\
p^2_2 &= -\frac{2}{(X^{(4)}_{10})^5} + \frac{6\xi_1}{(X^{(4)}_{10})^4} - \frac{2\xi_1^2}{(X^{(4)}_{10})^3} + \frac{2(s_1\xi_1 + s_2)}{(X^{(4)}_{10})^2} - \frac{2\alpha}{X^{(10)}_{\infty 0}} + y^{(10)}_{\infty 0},
\end{aligned}
$$

Here we calculate the 2-form $dq_1^2 \wedge dp_1^2 + dq_2^2 \wedge dp_2^2$ in the coordinates $(\xi_1, Z_{\infty 0}^{(5)}, X_{\infty 0}^{(10)}, Y_{\infty 0}^{(10)})$:

$$
\begin{aligned}
& dq_1^2 \wedge dp_1^2 + dq_2^2 \wedge dp_2^2 \\
&\quad = \quad dX_{\infty 0}^{(10)} \wedge dY_{\infty 0}^{(10)} \\
&\qquad + \{Y_{\infty 0}^{(10)}(X_{\infty 0}^{(10)})^5 - 2\alpha(X_{\infty 0}^{(10)})^4 + 2(s_1\xi_1 + s_2)(X_{\infty 0}^{(10)})^3 \\
&\qquad\qquad -2\xi_1^2(X_{\infty 0}^{(10)})^2 + 6\xi_1 X_{\infty 0}^{(10)} - 2\} d\xi_2 \wedge dZ_{\infty 0}^{(5)} \\
&\qquad + [5Z_{\infty 0}^{(5)} Y_{\infty 0}^{(10)}(X_{\infty 0}^{(10)})^4 + 4\{-2\alpha Z_{\infty 0}^{(5)} + (5\xi_1^2 + s_1)Y_{\infty 0}^{(10)}\}(X_{\infty 0}^{(10)})^3 \\
&\qquad\qquad +6\{(s_1\xi_1 + s_2)Z_{\infty 0}^{(5)} + \xi_1 Y_{\infty 0}^{(10)} - \alpha(5\xi_1^2 + s_1)\}(X_{\infty 0}^{(10)})^2 \\
&\qquad\qquad +2\{-2\xi_1^2 Z_{\infty 0}^{(5)} + Y_{\infty 0}^{(10)} + 10s_1\xi_1^3 + 10s_2\xi_1^2 + 2(s_1^2 - 2\alpha)\xi_1 + 2s_1 s_2\} X_{\infty 0}^{(10)} \\
&\qquad\qquad +6\xi_1 Z_{\infty 0}^{(5)} - 10\xi_1^4 + 2s_1\xi_1^2 + 4s_2\xi_1 - 2\alpha] d\xi_1 \wedge dX_{\infty 0}^{(10)} \\
&\qquad + \{z_{\infty 0}^{(5)}(X_{\infty 0}^{(10)})^5 + (5\xi_1^2 + s_1)(X_{\infty 0}^{(10)})^4 + 2\xi_1(X_{\infty 0}^{(10)})^3 + (X_{\infty 0}^{(10)})^2\} d\xi_2 \wedge dY_{\infty 0}^{(10)} \\
&\quad = \quad dX_{\infty 0}^{(10)} \wedge dY_{\infty 0}^{(10)} \\
&\qquad + d\xi_2 \wedge d[\{Y_{\infty 0}^{(10)}(X_{\infty 0}^{(10)})^5 - 2\alpha(X_{\infty 0}^{(10)})^4 + 2(s_1\xi_1 + s_2)(X_{\infty 0}^{(10)})^3 \\
&\qquad\qquad\qquad -2\xi_1^2(X_{\infty 0}^{(10)})^2 + 6\xi_1 X_{\infty 0}^{(10)} - 2\} Z_{\infty 0}^{(5)} \\
&\qquad\qquad\qquad +\{(5\xi_1^2 + s_1)(X_{\infty 0}^{(10)})^4 + 2\xi_1(X_{\infty 0}^{(10)})^3 + (X_{\infty 0}^{(10)})^2\} Y_{\infty 0}^{(10)} \\
&\qquad\qquad\qquad -2\alpha(5\xi_1^2 + s_1)(X_{\infty 0}^{(10)})^3 \\
&\qquad\qquad\qquad +(10s_1\xi_1^3 + 10s_2\xi_1^2 + 2(s_1^2 - 2\alpha)\xi_1 + 2s_1 s_2)(X_{\infty 0}^{(10)})^2 \\
&\qquad\qquad\qquad +(-10\xi_1^4 + 2s_1\xi_1^2 + 4s_2\xi_1 - 2\alpha) X_{\infty 0}^{(10)}] \\
&\quad = \quad dX_{\infty 0}^{(10)} \wedge dY_{\infty 0}^{(10)} \\
&\qquad + d\xi_1 \wedge d[\{Y_{\infty 0}^{(10)}(X_{\infty 0}^{(10)})^5 - 2\alpha(X_{\infty 0}^{(10)})^4 + 2(s_1\xi_1 + s_2)(X_{\infty 0}^{(10)})^3 \\
&\qquad\qquad\qquad -2\xi_1^2(X_{\infty 0}^{(10)})^2 + 6\xi_1 X_{\infty 0}^{(10)} - 2\} Z_{\infty 0}^{(5)} \\
&\qquad\qquad\qquad +\{(5\xi_1^2 + s_1)(X_{\infty 0}^{(10)})^4 + 2\xi_1(X_{\infty 0}^{(10)})^3 + (X_{\infty 0}^{(10)})^2\} Y_{\infty 0}^{(10)} \\
&\qquad\qquad\qquad -2\alpha(5\xi_1^2 + s_1)(X_{\infty 0}^{(10)})^3 \\
&\qquad\qquad\qquad +(10s_1\xi_1^3 + 10s_2\xi_1^2 + 2(s_1^2 - 2\alpha)\xi_1 + 2s_1 s_2)(X_{\infty 0}^{(10)})^2 \\
&\qquad\qquad\qquad +(-10\xi_1^4 + 2s_1\xi_1^2 + 4s_2\xi_1 - 2\alpha) X_{\infty 0}^{(10)} + 26\xi_1^3 + 8s_1\xi_1 + 2s_2].
\end{aligned}
$$

Therefore, setting

$$
\begin{aligned}
W_{\infty 0}^{(5)} = & \{Y_{\infty 0}^{(10)}(X_{\infty 0}^{(10)})^5 - 2\alpha(X_{\infty 0}^{(10)})^4 + 2(s_1\xi_1 + s_2)(X_{\infty 0}^{(10)})^3 \\
& -2\xi_1^2(X_{\infty 0}^{(10)})^2 + 6\xi_1 X_{\infty 0}^{(10)} - 2\} Z_{\infty 0}^{(5)} \\
& +\{(5\xi_1^2 + s_1)(X_{\infty 0}^{(10)})^4 + 2\xi_1(X_{\infty 0}^{(10)})^3 + (X_{\infty 0}^{(10)})^2\} Y_{\infty 0}^{(10)} \\
& -2\alpha(5\xi_1^2 + s_1)(X_{\infty 0}^{(10)})^3 \\
& +(10s_1\xi_1^3 + 10s_2\xi_1^2 + 2(s_1^2 - 2\alpha)\xi_1 + 2s_1 s_2)(X_{\infty 0}^{(10)})^2 \\
& +(-10\xi_1^4 + 2s_1\xi_1^2 + 4s_2\xi_1 - 2\alpha) X_{\infty 0}^{(10)} + 26\xi_1^3 + 8s_1\xi_1 + 2s_2,
\end{aligned}
$$

we have symplectic coordinates $(\xi_0, W_{\infty 0}^{(5)}, X_{\infty 0}^{(10)}, Y_{\infty 0}^{(10)})$. Writing

$$
q_1^{\infty 2} = \xi_1, \quad q_2^{\infty 2} = X_{\infty 0}^{(10)}, \quad p_1^{\infty 2} = W_{\infty 0}^{(5)}, \quad p_2^{\infty 2} = Y_{\infty 0}^{(10)},
$$

we have

$$
(10.2)\qquad
\begin{aligned}
&q_1^2 = q_1^{\infty 2}, \quad q_2^2 = q_2^{\infty 2},\\
&p_1^2 = -\frac{2}{(q_2^{\infty 2})^3} + \frac{2q_1^{\infty 2}}{(q_2^{\infty 2})^2} - \frac{2s_1}{q_2^{\infty 2}} + p_1,\\
&p_2^2 = -\frac{2}{(q_2^{\infty 2})^5} + \frac{6q_1^{\infty 2}}{(q_2^{\infty 2})^4} - \frac{2(q_1^{\infty 2})^2}{(q_2^{\infty 2})^3} + \frac{2(s_1 q_1^{\infty 2} + s_2)}{(q_2^{\infty 2})^2} - \frac{2\alpha}{q_2^{\infty 2}} + p_2^{\infty 2},
\end{aligned}
$$

The system $(q^{\infty 2}, p^{\infty 2}) \in \mathbb{C}^4$ separates solution curves passing through $A_\infty(s) \cap W_2$.

Thus we have obtained two symplectic coordinate systems $(q_1^*, q_2^*, p_1^*, p_2^*)$ each of which separates solution curves passing through the accessible singular points (see (10.1)-(10.2)). We notice that the Hamiltonians of each coordinate system is also a polynomial whose coefficients are rational functions of s holomorphic in B.

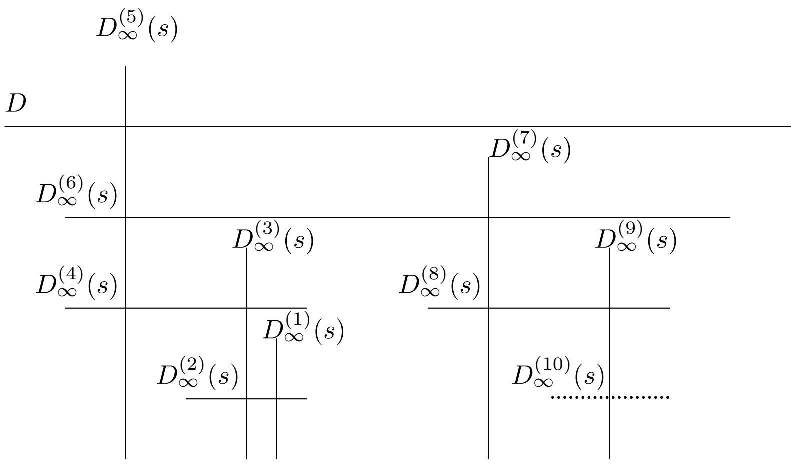

Figure 7. J=5

11 Description of spaces of initial conditions for all systems

In this section, we summarize the results obtained in the preceding sections, namely we give the description of the fiber spaces E_J for the systems $\mathcal{H}_J$. For every J, $E_J(s)$ is covered by finite number of $V^* \times B_J \ni (q_1^*, q_2^*, p_1^*, p_2^*, s_1, s_2)$ each of which is isomorphic to $\mathbb{C}^4 \times B_J$. Remark that $V^0 \times B_J$ is the original space in which the original Hamiltonians $H_{Ji}(q, p, s),\ i = 1, 2$ are defined and so that the coordinate system of $V^0 \times B_J$ is denoted by $(q_1, q_2, p_1, p_2, s_1, s_2)$. In the following theorems, we use the notation

$$V(x_i = 0) = \{(x_1, ..., x_n) \in \mathbb{C}^n \mid x_i = 0\}.$$

for an affine space $V = \mathbb{C}^n \ni (x_1, ..., x_n)$.

Theorem 1. *The space E_{11111} for the system $\mathcal{H}_{11111}$ is obtained by glueing thirteen copies of $\mathbb{C}^4 \times B_{11111}$*

$$V^* \times B_{11111} \ni (q_1^*, q_2^*, p_1^*, p_2^*, s_1, s_2), \quad * = 0, 1, 2, 01, 02, 11, 12, 21, 22, 31, 32, \infty 1, \infty 2.$$

via the following symplectic transformations

$$q_1 = \frac{1}{q_1^1}, \quad q_2 = \frac{q_2^1}{q_1^1}, \quad p_1 = -q_1^1(\nu + q_1^1 p_1^1 + q_2^1 p_2^1), \quad p_2 = q_1^1 p_2^1,$$

$$q_1 = \frac{q_1^2}{q_2^2}, \quad q_2 = \frac{1}{q_2^2}, \quad p_1 = q_2^2 p_1^2, \quad p_2 = -q_2^2(\nu + q_1^2 p_1^2 + q_2^2 p_2^2),$$

$$q_1 = q_1^{01}, \quad q_2 = p_2^{01}(\alpha_0 - q_2^{01} p_2^{01}) - \frac{s_2 q_1^{01}}{s_1} + s_2, \quad p_1 = \frac{s_2}{s_1 p_2^{01}} + p_1^{01}, \quad p_2 = \frac{1}{p_2^{01}},$$

$$q_1^1 = q_1^{02}, \quad q_2^1 = p_2^{02}(\alpha_0 - q_2^{02} p_2^{02}) + s_2 q_1^{02} - \frac{s_2}{s_1}, \quad p_1^1 = -\frac{s_2}{p_2^{02}} + p_1^{02}, \quad p_2^1 = \frac{1}{p_2^{02}},$$

$$q_1 = q_1^{11}, \quad q_2 = p_2^{11}(\alpha_1 - q_2^{11} p_2^{11}) - q_1^{11} + 1, \quad p_1 = \frac{1}{p_2^{11}} + p_1^{11}, \quad p_2 = \frac{1}{p_2^{11}},$$

$$q_1^1 = q_1^{12}, \quad q_2^1 = p_2^{12}(\alpha_1 - q_2^{12} p_2^{12}) + q_1^{12} - 1, \quad p_1^1 = -\frac{1}{p_2^{12}} + p_1^{12}, \quad p_2^1 = \frac{1}{p_2^{12}},$$

$$q_1 = p_1^{21}(\alpha_2 - q_1^{21} p_1^{21}), \quad q_2 = q_2^{21}, \quad p_1 = \frac{1}{p_1^{21}}, \quad p_2 = p_2^{21},$$

$$q_1^2 = p_1^{22}(\alpha_2 - q_1^{22} p_1^{22}), \quad q_2^2 = q_2^{22}, \quad p_1^2 = \frac{1}{p_1^{22}}, \quad p_2^2 = p_2^{22},$$

$$q_1 = q_1^{31}, \quad q_2 = p_2^{31}(\alpha_3 - q_2^{31} p_2^{31}), \quad p_1 = p_1^{31}, \quad p_2 = \frac{1}{p_2^{31}},$$

$$q_1^1 = q_1^{32}, \quad q_2^1 = p_2^{32}(\alpha_3 - q_2^{32} p_2^{32}), \quad p_1^1 = p_1^{32}, \quad p_2^1 = \frac{1}{p_2^{32}},$$

$$q_1^1 = p_1^{\infty 1}(\alpha_\infty - q_1^{\infty 1} p_1^{\infty 1}), \quad q_2^1 = q_2^{\infty 1}, \quad p_1^1 = \frac{1}{p_1^{\infty 1}}, \quad p_2^1 = p_2^{\infty 1},$$

$$q_1^2 = q_1^{\infty 2}, \quad q_2^2 = p_2^{\infty 2}(\alpha_\infty - q_2^{\infty 2} p_2^{\infty 2}), \quad p_1^2 = p_1^{\infty 2}, \quad p_2^2 = \frac{1}{p_2^{\infty 2}},$$

where

$$B = B_{11111} = \mathbb{C}^2 \setminus \cup\{s_1(s_1-1)s_2(s_2-1) = 0\},$$
$$\nu = -\frac{1}{2}(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 - 1 + \alpha_\infty).$$

Each fiber $E_{11111}(s)$ is a disjoint union of $V^0 = \mathbb{C}^4$ and

$$V^1(q_1^1=0)\cup V^2(q_2^2=0),\quad V^{01}(p_2^{01}=0)\cup V^{02}(p_2^{02}=0),\quad V^{11}(p_2^{11}=0)\cup V^{12}(p_2^{12}=0),$$
$$V^{21}(p_1^{21}=0)\cup V^{22}(p_1^{22}=0),\quad V^{31}(p_2^{31}=0)\cup V^{32}(p_2^{32}=0),\quad V^{\infty 1}(p_1^{\infty 1}=0)\cup V^{\infty 2}(p_2^{\infty 2}=0),$$

where each of the last six sets is a $\mathbb{C}^2$-bundle over $\mathbb{P}^1$ which can be decomposed as a disjoint union of $\mathbb{C}^3$ and $\mathbb{C}^2$:

$$E_{11111}(s) = \mathbb{C}^4 \sqcup 6(\mathbb{C}^3 \sqcup \mathbb{C}^2).$$

The Hamiltonians $H_i(*) = H_i(*; q^*, p^*, s)$ $i = 1, 2$ in every chart $V^* \times B_{11111}$ are polynomials of $q^* = (q_1^*, q_2^*)$ and $p^* = (p_1^*, p_2^*)$ whose coefficiants are rational functions of $s = (s_1, s_2)$ holomorphic in B_{11111}.

Theorem 2. *The space E_{1112} for the system $\mathcal{H}_{1112}$ is obtained by glueing eleven copies of $\mathbb{C}^4 \times B_{1112}$*

$$V^* \times B_{1112} \ni (q_1^*, q_2^*, p_1^*, p_2^*, s_1, s_2), \quad * = 0, 1, 2, 01, 02, 11, 12, 21, 22, \infty 1, \infty 2.$$

via the following symplectic transformations

$$q_1 = \frac{1}{q_1^1},\quad q_2 = \frac{q_2^1}{q_1^1},\quad p_1 = -q_1^1(\nu + q_1^1 p_1^1 + q_2^1 p_2^1),\quad p_2 = q_1^1 p_2^1,$$

$$q_1 = \frac{q_1^2}{q_2^2},\quad q_2 = \frac{1}{q_2^2},\quad p_1 = q_2^2 p_1^2,\quad p_2 = -q_2^2(\nu + q_1^2 p_1^2 + q_2^2 p_2^2),$$

$$q_1 = q_1^{01},\quad q_2 = p_2^{01}(\alpha_0 - q_2^{01} p_2^{01}) - \frac{s_2 q_1^{01}}{s_1} + s_2,\quad p_1 = \frac{s_2}{s_1 p_2^{01}} + p_1^{01},\quad p_2 = \frac{1}{p_2^{01}},$$

$$q_1^1 = q_1^{02},\quad q_2^1 = p_2^{02}(\alpha_0 - q_2^{02} p_2^{02}) + s_2 q_1^{02} - \frac{s_2}{s_1},\quad p_1^1 = -\frac{s_2}{p_2^{02}} + p_1^{02},\quad p_2^1 = \frac{1}{p_2^{02}},$$

$$q_1 = q_1^{11},\quad q_2 = q_2^{11},\quad p_1 = -\frac{\eta(q_2^{11}-1)}{(q_1^{11})^2} + \frac{\alpha_1}{q_1^{11}} + p_1^{11},\quad p_2 = \frac{\eta}{q_1^{11}} + p_2^{11},$$

$$q_1^2 = q_1^{12},\quad q_2^2 = q_2^{12},\quad p_1^2 = \frac{\eta(q_2^{12}-1)}{(q_1^{12})^2} + \frac{\alpha_1}{q_1^{12}} + p_1^{12},\quad p_2^2 = -\frac{\eta}{q_1^{12}} + p_2^{12},$$

$$q_1 = q_1^{21},\quad q_2 = p_2^{21}(\alpha_2 - q_2^{21} p_2^{21}),\quad p_1 = p_1^{21},\quad p_2 = \frac{1}{p_2^{21}},$$

$$q_1^1 = q_1^{22},\quad q_2^1 = p_2^{22}(\alpha_2 - q_2^{22} p_2^{22}),\quad p_1^1 = p_1^{22},\quad p_2^1 = \frac{1}{p_2^{22}},$$

$$q_1^1 = p_1^{\infty 1}(\alpha_\infty - q_1^{\infty 1} p_1^{\infty 1}),\quad q_2^1 = q_2^{\infty 1},\quad p_1^1 = \frac{1}{p_1^{\infty 1}},\quad p_2^1 = p_2^{\infty 1},$$

$$q_1^2 = q_1^{\infty 2}, \quad q_2^2 = p_2^{\infty 2}(\alpha_\infty - q_2^{\infty 2} p_2^{\infty 2}), \quad p_1^2 = p_1^{\infty 2}, \quad p_2^2 = \frac{1}{p_2^{\infty 2}},$$

where

$$B = B_{1112} = \mathbb{C}^2 \setminus \cup\{s_1 s_2(s_2 - 1) = 0\},$$
$$\nu = -\frac{1}{2}(\alpha_0 + \alpha_1 + \alpha_2 - 1 + \alpha_\infty).$$

Each fiber $E_{1112}(s)$ is a disjoint union of $V^0 = \mathbb{C}^4$ and

$$V^1(q_1^1 = 0) \cup V^2(q_2^2 = 0), \quad V^{01}(p_2^{01} = 0) \cup V^{02}(p_2^{02} = 0), \quad V^{11}(q_1^{11} = 0) \cup V^{12}(q_1^{12} = 0),$$
$$V^{21}(p_2^{21} = 0) \cup V^{22}(p_2^{22} = 0), \quad V^{\infty 1}(p_1^{\infty 1} = 0) \cup V^{\infty 2}(p_2^{\infty 2} = 0),$$

where each of the last five sets is a $\mathbb{C}^2$-bundle over $\mathbb{P}^1$ which can be decomposed as a disjoint union of $\mathbb{C}^3$ and $\mathbb{C}^2$, then

$$E_{1112}(s) = \mathbb{C}^4 \sqcup 5(\mathbb{C}^3 \sqcup \mathbb{C}^2).$$

The Hamiltonians $H_i(*) = H_i(*; q^*, p^*, s)$ $i = 1, 2$ in every chart $V^* \times B_{1112}$ are polynomials of $q^* = (q_1^*, q_2^*)$ and $p^* = (p_1^*, p_2^*)$ whose coefficiants are rational functions of $s = (s_1, s_2)$ holomorphic in B_{1112}.

Theorem 3. *The space E_{113} for the system $\mathcal{H}_{113}$ is obtained by glueing nine copies of $\mathbb{C}^4 \times B_{113}$*

$$V^* \times B_{113} \ni (q_1^*, q_2^*, p_1^*, p_2^*, s_1, s_2), \quad * = 0, 1, 2, 01, 02, 11, 12, \infty 1, \infty 2.$$

via the following symplectic transformations

$$q_1 = \frac{1}{q_1^1}, \quad q_2 = \frac{q_2^1}{q_1^1}, \quad p_1 = -q_1^1(\nu + q_1^1 p_1^1 + q_2^1 p_2^1), \quad p_2 = q_1^1 p_2^1,$$

$$q_1 = \frac{q_1^2}{q_2^2}, \quad q_2 = \frac{1}{q_2^2}, \quad p_1 = q_2^2 p_1^2, \quad p_2 = -q_2^2(\nu + q_1^2 p_1^2 + q_2^2 p_2^2),$$

$$q_1 = q_1^{01}, \quad q_2 = p_2^{01}(\alpha_0 - q_2^{01} p_2^{01}) - \Big(\frac{2s_1 + s_2^2}{2s_2}\Big) q_1^{01} - \frac{1}{s_2}, \quad p_1 = \frac{2s_1 + s_2^2}{2s_2 p_2^{01}} + p_1^{01}, \quad p_2 = \frac{1}{p_2^{01}}.$$

$$q_1^1 = q_1^{02}, \quad q_2^1 = p_2^{02}(\alpha_0 - p_2^{02} q_2^{02}) - \frac{q_1^{02}}{s_2} - \frac{2s_1 + s_2^2}{2s_2}, \quad p_1^1 = \frac{1}{s_2 p_2^{02}} + p_1^{02}, \quad p_2^1 = \frac{1}{p_2^{02}}.$$

$$q_1 = q_1^{11}, \quad q_2 = q_2^{11}, \quad p_1 = -\frac{\eta (q_2^{11})^2}{(q_1^{11})^3} + \frac{\eta}{(q_1^{11})^2} + \frac{\alpha_1}{q_1^{11}} + p_1^{11}, \quad p_2 = \frac{\eta q_2^{11}}{(q_1^{11})^2} + p_2^{11},$$

$$q_1^2 = q_1^{12}, \quad q_2^2 = q_2^{12}, \quad p_1^2 = -\frac{\eta}{(q_1^{12})^3} + \frac{\eta q_2^{12}}{(q_1^{12})^2} + \frac{\alpha_1}{q_1^{12}} + p_1^{12}, \quad p_2^2 = -\frac{\eta}{q_1^{12}} + p_2^{12},$$

$$q_1^1 = p_1^{\infty 1}(\alpha_\infty - q_1^{\infty 1} p_1^{\infty 1}), \quad q_2^1 = q_2^{\infty 1}, \quad p_1^1 = \frac{1}{p_1^{\infty 1}}, \quad p_2^1 = p_2^{\infty 1},$$

$$q_1^2 = q_1^{\infty 2}, \quad q_2^2 = p_2^{\infty 2}(\alpha_\infty - q_2^{\infty 2} p_2^{\infty 2}), \quad p_1^2 = p_1^{\infty 2}, \quad p_2^2 = \frac{1}{p_2^{\infty 2}},$$

where

$$B = B_{113} = \mathbb{C}^2 \setminus \{s_2 = 0\},$$
$$\nu = -\frac{1}{2}(\alpha_0 + \alpha_1 - 1 + \alpha_\infty).$$

Each fiber $E_{113}(s)$ is a disjoint union of $V^0 = \mathbb{C}^4$ and

$$V^1(q_1^1 = 0) \cup V^2(q_2^2 = 0), \quad V^{01}(p_2^{01} = 0) \cup V^{02}(p_2^{02} = 0),$$
$$V^{11}(q_1^{11} = 0) \cup V^{12}(q_1^{12} = 0), \quad V^{\infty 1}(p_1^{\infty 1} = 0) \cup V^{\infty 2}(p_2^{\infty 2} = 0),$$

where each of the last five sets is a $\mathbb{C}^2$-bundle over $\mathbb{P}^1$ which can be decomposed as a disjoint union of $\mathbb{C}^3$ and $\mathbb{C}^2$, then

$$E_{113}(s) = \mathbb{C}^4 \sqcup 4(\mathbb{C}^3 \sqcup \mathbb{C}^2).$$

The Hamiltonians $H_i(*) = H_i(*; q^*, p^*, s)$ $i = 1, 2$ in every chart $V^* \times B_{113}$ are polynomials of $q^* = (q_1^*, q_2^*)$ and $p^* = (p_1^*, p_2^*)$ whose coefficiants are rational functions of $s = (s_1, s_2)$ holomorphic in B_{113}.

Theorem 4. *The space E_{122} for the system $\mathcal{H}_{122}$ is obtained by glueing nine copies of $\mathbb{C}^4 \times B_{122}$*

$$V^* \times B_{122} \ni (q_1^*, q_2^*, p_1^*, p_2^*, s_1, s_2), \quad * = 0, 1, 2, 01, 02, 11, 12, \infty 1, \infty 2.$$

via the following symplectic transformations

$$q_1 = \frac{1}{q_1^1}, \quad q_2 = \frac{q_2^1}{q_1^1}, \quad p_1 = -q_1^1(\nu + q_1^1 p_1^1 + q_2^1 p_2^1), \quad p_2 = q_1^1 p_2^1,$$

$$q_1 = \frac{q_1^2}{q_2^2}, \quad q_2 = \frac{1}{q_2^2}, \quad p_1 = q_2^2 p_1^2, \quad p_2 = -q_2^2(\nu + q_1^2 p_1^2 + q_2^2 p_2^2),$$

$$q_1 = q_1^{01}, \quad q_2 = q_2^{01}, \quad p_1 = \frac{\eta_0 s_2}{s_1 q_2^{01}} + p_1^{01}, \quad p_2 = -\frac{\eta_0 s_2 (q_1^{01} - s_1)}{s_1 (q_2^{01})^2} + \frac{\alpha_0}{q_2^{01}} + p_2^{01},$$

$$q_1^1 = q_1^{02}, \quad q_2^1 = q_2^{02}, \quad p_1^1 = -\frac{\eta_0 s_2}{q_2^{02}} + p_1^{02}, \quad p_2^1 = \frac{\eta_0 s_2 (s_1 q_1^{02} - 1)}{s_1 (q_2^{02})^2} + \frac{\alpha_0}{q_2^{02}} + p_2^{02},$$

$$q_1 = q_1^{11}, \quad q_2 = q_2^{11}, \quad p_1 = -\frac{\eta_1 (q_2^{11} - 1)}{(q_1^{11})^2} + \frac{\alpha_1}{q_1^{11}} + p_1^{11}, \quad p_2 = \frac{\eta_1}{q_1^{11}} + p_2^{11},$$

$$q_1^2 = q_1^{12}, \quad q_2^2 = q_2^{12}, \quad p_1^2 = \frac{\eta_1 (q_2^{12} - 1)}{(q_1^{12})^2} + \frac{\alpha_1}{q_1^{12}} + p_1^{12}, \quad p_2^2 = -\frac{\eta_1}{q_1^{12}} + p_2^{12},$$

$$q_1^1 = p_1^{\infty 1}(\alpha_\infty - q_1^{\infty 1} p_1^{\infty 1}), \quad q_2^1 = q_2^{\infty 1}, \quad p_1^1 = \frac{1}{p_1^{\infty 1}}, \quad p_2^1 = p_2^{\infty 1},$$

$$q_1^2 = q_1^{\infty 2}, \quad q_2^2 = p_2^{\infty 2}(\alpha_\infty - q_2^{\infty 2} p_2^{\infty 2}), \quad p_1^2 = p_1^{\infty 2}, \quad p_2^2 = \frac{1}{p_2^{\infty 2}},$$

where

$$B = B_{122} = \mathbb{C}^2 \setminus \{s_1 = 0\},$$
$$\nu = -\frac{1}{2}(\alpha_0 + \alpha_1 - 1 + \alpha_\infty).$$

Each fiber $E_{122}(s)$ is a disjoint union of $V^0 = \mathbb{C}^4$ and

$$V^1(q_1^1 = 0) \cup V^2(q_2^2 = 0), \quad V^{01}(q_2^{01} = 0) \cup V^{02}(q_2^{02} = 0),$$
$$V^{11}(q_1^{11} = 0) \cup V^{12}(q_1^{12} = 0), \quad V^{\infty 1}(p_1^{\infty 1} = 0) \cup V^{\infty 2}(p_2^{\infty 2} = 0),$$

where each of the last five sets is a $\mathbb{C}^2$-bundle over $\mathbb{P}^1$ which can be decomposed as a disjoint union of $\mathbb{C}^3$ and $\mathbb{C}^2$, then

$$E_{122}(s) = \mathbb{C}^4 \sqcup 4(\mathbb{C}^3 \sqcup \mathbb{C}^2).$$

The Hamiltonians $H_i(*) = H_i(*; q^*, p^*, s)$ $i = 1, 2$ in every chart $V^* \times B_{122}$ are polynomials of $q^* = (q_1^*, q_2^*)$ and $p^* = (p_1^*, p_2^*)$ whose coefficiants are rational functions of $s = (s_1, s_2)$ holomorphic in B_{122}.

Theorem 5. *The space E_{14} for the system $\mathcal{H}_{14}$ is obtained by glueing seven copies of $\mathbb{C}^4 \times B_{14}$*

$$V^* \times B_{14} \ni (q_1^*, q_2^*, p_1^*, p_2^*, s_1, s_2), \quad * = 0, 1, 2, 01, 02, \infty 1, \infty 2.$$

via the following symplectic transformations

$$q_1 = \frac{1}{q_1^1}, \quad q_2 = \frac{q_2^1}{q_1^1}, \quad p_1 = -q_1^1(\nu + q_1^1 p_1^1 + q_2^1 p_2^1), \quad p_2 = q_1^1 p_2^1,$$

$$q_1 = \frac{q_1^2}{q_2^2}, \quad q_2 = \frac{1}{q_2^2}, \quad p_1 = q_2^2 p_1^2, \quad p_2 = -q_2^2(\nu + q_1^2 p_1^2 + q_2^2 p_2^2),$$

$$q_1 = q_1^{01}, \quad q_2 = p_2^{01}(\alpha_0 - q_2^{01} p_2^{01}) - s_2 q_1^{01} - \frac{2s_1 + s_2^2}{2}, \quad p_1 = \frac{s_2}{p_2^{01}} + p_1^{01}, \quad p_2 = \frac{1}{p_2^{01}},$$

$$q_1^1 = q_1^{02}, \quad q_2^1 = p_2^{02}(\alpha_0 - q_2^{02} p_2^{02}) - \left(\frac{2s_1 + s_2^2}{2}\right) q_1^{02} - s_2, \quad p_1^1 = \frac{2s_1 + s_2^2}{2p_2^{02}} + p_1^{02}, \quad p_2^1 = \frac{1}{p_2^{02}},$$

$$q_1^1 = q_1^{\infty 1}, \quad q_2^1 = q_2^{\infty 1},$$
$$p_1^1 = -\frac{1}{(q_1^{\infty 1})^4} + \frac{2q_2^{\infty 1}}{(q_1^{\infty 1})^3} + \frac{1 - \alpha_0 + 2\alpha_\infty}{q_1^{\infty 1}} + p_1^{\infty 1}, \quad p_2^1 = -\frac{1}{(q_1^{\infty 1})^2} + p_2^{\infty 1},$$

$$q_1^2 = q_1^{\infty 2}, \quad q_2^2 = q_2^{\infty 2}, \quad p_1^2 = \frac{(q_1^{\infty 2})^2}{(q_2^{\infty 2})^3} - \frac{1}{(q_2^{\infty 2})^2} + p_1^{\infty 2},$$
$$p_2^2 = -\frac{(q_1^{\infty 2})^3}{(q_2^{\infty 2})^4} + \frac{2q_1^{\infty 2}}{(q_2^{\infty 2})^3} + \frac{1 - \alpha_0 + 2\alpha_\infty}{q_2^{\infty 2}} + p_2^{\infty 2},$$

where

$$B = B_{14} = \mathbb{C}^2,$$
$$\nu = -\alpha_\infty.$$

Each fiber $E_{14}(s)$ is a disjoint union of $V^0 = \mathbb{C}^4$ and

$$V^1(q_1^1 = 0) \cup V^2(q_2^2 = 0), \quad V^{01}(p_2^{01} = 0) \cup V^{02}(p_2^{02} = 0), \quad V^{\infty 1}(q_1^{\infty 1} = 0) \cup V^{\infty 2}(q_2^{\infty 2} = 0),$$

where each of the last three sets is a $\mathbb{C}^2$-bundle over $\mathbb{P}^1$ which can be decomposed as a disjoint union of $\mathbb{C}^3$ and $\mathbb{C}^2$, then

$$E_{14}(s) = \mathbb{C}^4 \sqcup 3(\mathbb{C}^3 \sqcup \mathbb{C}^2).$$

The Hamiltonians $H_i(*) = H_i(*; q^*, p^*, s)$ $i = 1, 2$ in every chart $V^* \times B_{14}$ are polynomials of $q^* = (q_1^*, q_2^*)$ and $p^* = (p_1^*, p_2^*)$ whose coefficiants are rational functions of $s = (s_1, s_2)$ holomorphic in B_{14}.

Theorem 6. *The space E_{23} for the system $\mathcal{H}_{23}$ is obtained by glueing seven copies of $\mathbb{C}^4 \times B_{23}$*

$$V^* \times B_{23} \ni (q_1^*, q_2^*, p_1^*, p_2^*, s_1, s_2), \quad * = 0, 1, 2, 01, 02, \infty 1, \infty 2.$$

via the following symplectic transformations

$$q_1 = \frac{1}{q_1^1}, \quad q_2 = \frac{q_2^1}{q_1^1}, \quad p_1 = -q_1^1(\nu + q_1^1 p_1^1 + q_2^1 p_2^1), \quad p_2 = q_1^1 p_2^1,$$

$$q_1 = \frac{q_1^2}{q_2^2}, \quad q_2 = \frac{1}{q_2^2}, \quad p_1 = q_2^2 p_1^2, \quad p_2 = -q_2^2(\nu + q_1^2 p_1^2 + q_2^2 p_2^2),$$

$$q_1 = q_1^{01}, \quad q_2 = q_2^{01}, \quad p_1 = -\frac{\eta s_2}{q_2^{01}} + p_1^{01}, \quad p_2 = \frac{\eta s_2(q_1^{01} - s_1)}{(q_2^{01})^2} + \frac{\alpha_0}{q_2^{01}} + p_2^{01},$$

$$q_1^1 = q_1^{02}, \quad q_2^1 = q_2^{02}, \quad p_1^1 = \frac{\eta s_1 s_2}{q_2^{02}} + p_1^{02}, \quad p_2^1 = -\frac{\eta s_2(s_1 q_1^{02} - 1)}{(q_2^{02})^2} + \frac{\alpha_0}{q_2^{02}} + p_2^{02},$$

$$\begin{aligned} &q_1^1 = q_1^{\infty 1}, \quad q_2^1 = q_2^{\infty 1}, \\ &p_1^1 = -\frac{1}{2(q_1^{\infty 1})^3} + \frac{q_2^{\infty 1}}{2(q_1^{\infty 1})^2} + \frac{1 - \alpha_0 + 2\alpha_\infty}{q_1^{\infty 1}} + p_1^{\infty 1}, \quad p_2^1 = -\frac{1}{2q_1^{\infty 1}} + p_2^{\infty 1}, \end{aligned}$$

$$\begin{aligned} &q_1^2 = q_1^{\infty 2}, \quad q_2^2 = q_2^{\infty 2}, \quad p_1^2 = \frac{q_1^{\infty 2}}{2(q_2^{\infty 2})^2} + p_1^{\infty 2}, \\ &p_2^2 = -\frac{(q_1^{\infty 2})^2}{2(q_2^{\infty 2})^3} + \frac{1}{2(q_2^{\infty 2})^2} + \frac{1 - \alpha_0 + 2\alpha_\infty}{q_2^{\infty 2}} + p_2^{\infty 2}, \end{aligned}$$

where

$$B = B_{23} = \mathbb{C}^2,$$
$$\nu = -\alpha_\infty.$$

Each fiber $E_{23}(s)$ *is a disjoint union of* $V^0=\mathbb{C}^4$ *and*

$$V^1(q_1^1=0)\cup V^2(q_2^2=0),\quad V^{01}(q_2^{01}=0)\cup V^{02}(q_2^{02}=0),\quad V^{\infty 1}(q_1^{\infty 1}=0)\cup V^{\infty 2}(q_2^{\infty 2}=0),$$

where each of the last three sets is a $\mathbb{C}^2$*-bundle over* $\mathbb{P}^1$ *which can be decomposed as a disjoint union of* $\mathbb{C}^3$ *and* $\mathbb{C}^2$*, then*

$$E_{23}(s)=\mathbb{C}^4\sqcup 3(\mathbb{C}^3\sqcup\mathbb{C}^2).$$

The Hamiltonians $H_i(*)=H_i(*;q^*,p^*,s)$ $i=1,2$ *in every chart* $V^*\times B_{23}$ *are polynomials of* $q^*=(q_1^*,q_2^*)$ *and* $p^*=(p_1^*,p_2^*)$ *whose coefficiants are rational functions of* $s=(s_1,s_2)$ *holomorphic in* B_{23}.

Theorem 7. *The space* E_5 *for the system* $\mathcal{H}_5$ *is obtained by glueing five copies of* $\mathbb{C}^4\times B_5$

$$V^*\times B_5\ni(q_1^*,q_2^*,p_1^*,p_2^*,s_1,s_2),\quad *=0,1,2,\infty 1,\infty 2.$$

via the following symplectic transformations

$$q_1=\frac{1}{q_1^1},\quad q_2=\frac{q_2^1}{q_1^1},\quad p_1=-q_1^1(\nu+q_1^1p_1^1+q_2^1p_2^1),\quad p_2=q_1^1p_2^1,$$

$$q_1=\frac{q_1^2}{q_2^2},\quad q_2=\frac{1}{q_2^2},\quad p_1=q_2^2p_1^2,\quad p_2=-q_2^2(\nu+q_1^2p_1^2+q_2^2p_2^2),$$

$$\begin{aligned}
&q_1^1=q_1^{\infty 1},\quad q_2^1=q_2^{\infty 1},\\
&p_1^1=-\frac{2(q_2^{\infty 1})^4}{(q_1^{\infty 1})^5}+\frac{6(q_2^{\infty 1})^2}{(q_1^{\infty 1})^4}-\frac{2}{(q_1^{\infty 1})^3}+\frac{2(s_1+s_2q_2^{\infty 1})}{(q_1^{\infty 1})^2}-\frac{2\alpha}{q_1^{\infty 1}}+p_1^{\infty 1},\\
&p_2^1=\frac{2(q_2^{\infty 1})^3}{(q_1^{\infty 1})^4}-\frac{4q_2^{\infty 1}}{(q_1^{\infty 1})^3}-\frac{2s_2}{q_1^{\infty 1}}+p_2^{\infty 1},
\end{aligned}$$

$$\begin{aligned}
&q_1^2=q_1^{\infty 2},\quad q_2^2=q_2^{\infty 2},\quad p_1^2=-\frac{2}{(q_2^{\infty 2})^3}+\frac{2q_1^{\infty 2}}{(q_2^{\infty 2})^2}-\frac{2s_1}{q_2^{\infty 2}}+p_1^{\infty 2},\\
&p_2^2=-\frac{2}{(q_2^{\infty 2})^5}+\frac{6q_1^{\infty 2}}{(q_2^{\infty 2})^4}-\frac{2(q_1^{\infty 2})^2}{(q_2^{\infty 2})^3}+\frac{2(s_1q_1^{\infty 2}+s_2)}{(q_2^{\infty 2})^2}-\frac{2\alpha}{q_2^{\infty 2}}+p_2^{\infty 2},
\end{aligned}$$

where

$$B=B_5=\mathbb{C}^2,\qquad \nu=\alpha+\frac{1}{2}.$$

Each fiber $E_5(s)$ *is a disjoint union of* $V^0=\mathbb{C}^4$ *and*

$$V^1(q_1^1=0)\cup V^2(q_2^2=0),\quad V^{\infty 1}(q_1^{\infty 1}=0)\cup V^{\infty 2}(q_2^{\infty 2}=0),$$

where each of the last two sets is a $\mathbb{C}^2$*-bundle over* $\mathbb{P}^1$ *which can be decomposed as a disjoint union of* $\mathbb{C}^3$ *and* $\mathbb{C}^2$*, then*

$$E_5(s)=\mathbb{C}^4\sqcup 2(\mathbb{C}^3\sqcup\mathbb{C}^2).$$

The Hamiltonians $H_i(*)=H_i(*;q^*,p^*,s)$ $i=1,2$ *in every chart* $V^*\times B_5$ *are polynomials of* $q^*=(q_1^*,q_2^*)$ *and* $p^*=(p_1^*,p_2^*)$ *whose coefficiants are rational functions of* $s=(s_1,s_2)$ *holomorphic in* B_5.

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