Constant mean curvature surfaces in 3-dimensional space forms

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博士論文

Constant Mean Curvature Surfaces in 3-dimensional Space Forms

三次元空間形内の平均曲率一定曲面について

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Introduction

This thesis is research on constant mean curvature (CMC) surfaces in the three-dimensional space forms, $\mathbb{R}^3$, S^3 and H^3. These three space forms are the unique complete simply-connected three dimensional Riemann manifolds of constant sectional curvature 0, 1 and -1, respectively.

Constant mean curvature surfaces are of interest to us because they have a clear physical interpretation as models for soap films. Let us consider a smooth surface f immersed in one of the three-dimensional space forms $\mathbb{R}^3, S^3$ or H^3. Defining H to be the mean curvature of f, we say that f is a constant mean curvature surface

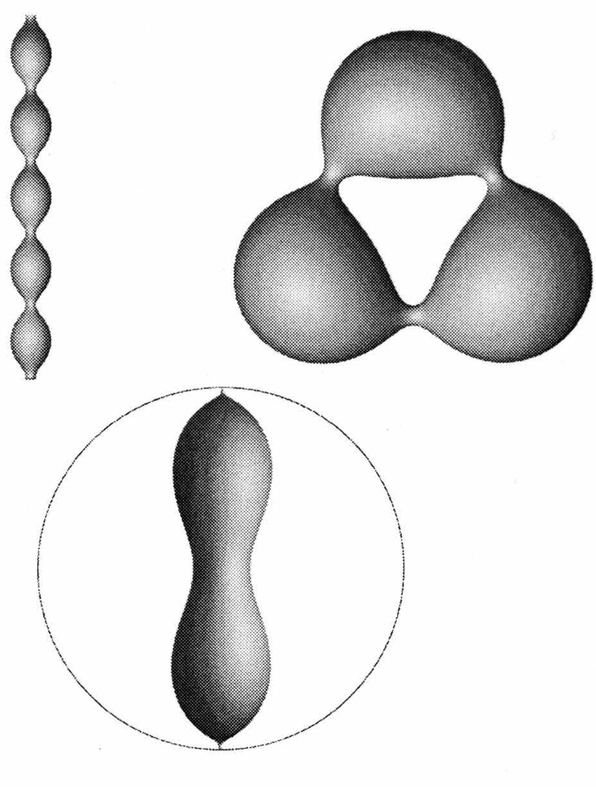

FIGURE 1. Delaunay surfaces in $\mathbb{R}^3$, S^3 and H^3.

if H is constant on f. Soap films have the property of attaining the least area with respect to the fixed volumes they bound, and are examples of CMC surfaces. Mathematically, H being constant implies that compact portions of the surface f are critical values for boundary-preserving, volume-preserving variations.

The sphere is a simple example of a closed CMC surface. For a long time, there were no known closed CMC immersions apart from the sphere, and Hopf asked if such surfaces could exist. It was plausible to believe there were no other surfaces, because:

(i) Hopf showed that the only genus-zero closed CMC surfaces in $\mathbb{R}^3$ are spheres.
(ii) Alexandrov showed that the only embedded closed CMC surfaces in $\mathbb{R}^3$ are spheres.

However, Wente [**67**] showed existence of closed immersed CMC tori in 1984, answering Hopf's question and leading to renewed interest in the field. Triggered by Wente's work, Pinkall, Sterling, and Bobenko found effective ways to construct CMC surfaces that include all CMC tori [**49**], [**5**], [**6**].

The works just mentioned are also indirect triggers of the work by Dorfmeister, Pedit and Wu [**21**]. And in turn, [**21**] is the primary basis upon which the work in this thesis is founded. The method in [**21**] is close in philosophy to Enneper-Weierstraß representation for minimal surfaces, so before saying more about [**21**], let us first mention that representation. Minimal surfaces are the special case of CMC surfaces with $H = 0$, and physically can be interpreted as CMC surfaces that do not trap pockets of air (unlike the sphere).

The classical Enneper-Weierstraß representation. The classical Enneper-Weierstraß representation for minimal surfaces in $\mathbb{R}^3$ can be formulated as follows:

$$\operatorname{Re} \int_{z_0}^{z} \left(\tfrac{1}{2} f(1-g^2), \tfrac{i}{2} f(1+g^2), fg \right) dz \,,$$

where f (resp. g) is a holomorphic (resp. meromorphic) function on a simply-connected domain $\mathfrak{D} \subset \mathbb{C}$, and z_0 is some base point on $\mathfrak{D}$. By this Enneper-Weierstraß representation, minimal surfaces have been intensively studied using complex function theory.

The generalized Weierstraß representation. On the other hand, for non-minimal CMC surfaces such a representation formula had not been known, i.e. a formula using holomorphic and meromorphic functions. In the late 1990's, Dorfmeister, Pedit and Wu formulated a "generalized Weierstraß representation" for harmonic maps into k-symmetric spaces. It is well known that the Gauß map of a CMC surface in $\mathbb{R}^3$ is a harmonic map [**52**]. Therefore the generalized Weierstraß representation formula holds for CMC surfaces, and it is an analogue to the Enneper-Weierstraß representation for minimal surfaces.

Let us now describe the method of Dorfmeister, Pedit and Wu in more detail. According to [**21**] one can construct every non-spherical CMC-immersion of mean curvature $H \neq 0$ from a simply connected domain $\mathfrak{D} \subset \mathbb{C}$ into $\mathbb{R}^3$ in four steps as follows:

Step 1: Choose any holomorphic 2×2-matrix differential form $\eta = A(z,\lambda)dz$, of which the diagonal elements are even functions of $\lambda \in \mathbb{C}^*$, and the off-diagonal elements are odd functions of $\lambda \in \mathbb{C}^*$, and the powers of λ are ≥ -1. Assume $\det A_{-1} \neq 0$, where A_{-1} is the coefficient matrix of the λ^{-1} term of η.

Step 2: Solve the ODE $dC = C\eta$.

Step 3: Perform an Iwasawa splitting: $C = FW_+$, where $F = F(z,\bar{z},\lambda)$ is unitary for all $z \in \mathfrak{D}$, $\lambda \in \mathbb{S}^1$, and W_+ has a Fourier expansion relative to λ without negative exponents.

THEOREM 0.1. ([**21**]) *F is, for every fixed $\lambda \in S^1$, a frame of some immersion of constant mean curvature $H \neq 0$.*

Step 4: Form $\Psi_\lambda(z) = -\frac{1}{2H}\left\{\left(\frac{d}{dt}F\right)F^{-1} + \frac{i}{2}F\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)F^{-1}\right\}$ (the Sym-Bobenko-Formula). Then Ψ_λ is a CMC-immersion of mean curvature $H \neq 0$ from $\mathfrak{D}$ to $\mathbb{R}^3 \cong su(2)$. Moreover, every CMC-immersion of mean curvature $H \neq 0$ different from a Riemann sphere $\mathbb{S}^2$ can be obtained this way.

Obviously, the procedure outlined above leads invariably to a CMC-immersion. The only variable input parameters are the choice of η and the initial condition for C. If one has a certain CMC-immersion in mind, then if η is chosen "right", one can choose the initial condition $C(z_0,\lambda) = \mathrm{Id}$, where $z_0 \in \mathfrak{D}$ is some fixed "base point". In general, however, one does not know precisely what η to choose. Therefore, one chooses η "of reasonable shape" and makes additional "adjustments" via the initial condition of C, to finally construct an immersion with specifically desired properties [**16**], [**15**]. We call η defined in **Step 1** "holomorphic potentials".

At any rate, the choice of η is important. It has been known since [**21**] that one can replace the holomorphic differential form η by some meromorphic differential form ξ of the form $\xi = \lambda^{-1}\hat{\xi}$, where $\hat{\xi}$ is meromorphic on $\mathfrak{D}$. Moreover, if the initial condition at some base point $z_0 \in \mathfrak{D}$ is $C(z_0,\lambda) = \mathrm{Id}$, then there is a bijective relation between the "normalized potentials" $\xi = \lambda^{-1}\hat{\xi}$ and the CMC-immersions ($\neq \mathbb{S}^2$ and $H \neq 0$) from $\mathfrak{D}$ into $\mathbb{R}^3$.

For non-compact $\mathcal{M}$ it has been shown in [**16**], [**15**] that the CMC-immersions from $\mathcal{M}$ to $\mathbb{R}^3$ can be obtained by the generalized Weierstraß representation outlined above from a holomorphic potential η on $\mathcal{M}$, which is invariant under the fundamental group $\pi_1(\mathcal{M}) \hookrightarrow \mathrm{Aut}(\mathfrak{D})$, i.e., from a holomorphic differential $(1,0)$-form well-defined on $\mathcal{M}$.

Global behavior of CMC **surfaces.** We mentioned, just above, a convenient property in the case that $\mathcal{M}$ is non-compact, that is, that η can be chosen to be well-defined on $\mathcal{M}$. This well-definedness of η is useful for studying the global behavior of CMC surfaces, so let us assume that $\mathcal{M}$ is non-compact, and then now move to the global study of CMC surfaces. (We remark that if $\mathcal{M}$ is compact, then η is not a holomorphic differential on $\mathcal{M}$, but rather a meromorphic differential on $\mathcal{M}$, but this is not immediately relevant to the present discussion.) In **Step 3**, solving the initial value problem

$$dC = C\eta\ ,\quad C(z_0) = C_0 \tag{0.1}$$

yields a solution C and corresponding monodromy representation depending on λ. Let M_δ be a monodromy matrix corresponding to some deck transformation $\delta \in \pi_1(\mathcal{M})$. If the monodromy matrices M_δ satisfy, for all deck transformations $\delta \in \pi_1(\mathcal{M})$, the following three conditions

$$M_\delta|_{\mathbb{S}^1} \in \mathrm{SU}_2, \tag{0.2}$$

$$M_\delta|_{\lambda_0} = \pm\mathrm{Id}, \tag{0.3}$$

$$d_\lambda M_\delta|_{\lambda_0} = 0, \tag{0.4}$$

then the resulting associated family Ψ_{λ_0} defined in **Step 4** factors through the fundamental group $\pi_1(M)$ at λ_0 and we thus have a CMC immersion $f : \mathcal{M} \to \mathbb{R}^3$. In Equation (0.4) and throughout this work we denote by $d_{\_}$ the derivative with respect to the subscript, which we omit in the case of the exterior derivative on the Riemann surface, as in (0.1). Condition (0.3) removes the rotational periods while (0.4) removes the translational periods, and both can be ensured by properties on η. The condition (0.2) is harder to satisfy and makes use of varying the initial condition C_0. These three conditions have been used in a number of papers, starting with the work of Dorfmeister and Haak [**16**] and later by the author and others while investigating CMC immersions of the n-punctured Riemann sphere, the so called n-Noids [**36**], [**39**] and [**55**]. (Another separate approach to studying embedded CMC 3-Noids can be found in the work of Große-Brauckmann, Kusner and Sullivan [**26**].)

Therefore we have the first aim of the present thesis as follows:

- **The global existence of constant mean curvature surfaces in 3-dimensional space forms via generalized Weierstraß representation.**

Bianchi and Darboux transformations. One of the central topics in 19'th century differential geometry was the transformation theory of surfaces. The best known example is perhaps the Bäcklund transformation for constant negative curvature (CNC) surfaces in Euclidean 3-space $\mathbb{R}^3$. Originally, a line congruence through a surface was said to be a *Bäcklund transformation* if there is another focal surface such that the congruence is tangent to both focal surfaces and the normal directions of the two focal surfaces keep a constant angle to each other. The existence of a Bäcklund transformation characterizes the negative constancy of Gaussian curvature. Thus, there does not exist a line congruence with Bäcklund property for surfaces of constant positive curvature (CPC).

Instead of line congruences with the Bäcklund property, L. Bianchi [**2**] considered complexified line congruences for CPC surfaces. By two appropriate successive complex line congruences, Bianchi obtained a real CPC surface from a seed CPC surface. This transformation of a CPC surface is referred to as a *Bianchi-Bäcklund transformation* (see Definition 2.1). The Bianchi-Bäcklund transform of a real line–which is regarded as a degenerate surface–is the surface of Sievert [**56**].

In [**59**], I. Sterling and H. Wente studied Bianchi-Bäcklund transformations of constant mean curvature (CMC) surfaces. Their starting point was that each CMC surface corresponds to a CPC surface (Bonnet transform, or parallel surface). In particular, they constructed iterated Bianchi-Bäcklund transforms of the circular cylinder. The resulting CMC surfaces are called *multi-bubbletons* when they are periodic.

One of the fundamental properties of CMC surfaces is that such surfaces are *isothermic*. Namely, on a region free of umbilics in a CMC surface, there exists an isothermal-curvature line coordinate system. Such a coordinate system is traditionally called *isothermic*.

In the field of conformal geometry, transformations of isothermic surfaces in terms of "sphere-congruences" were studied intensively in the 19'th century. Transformations of isothermic surfaces defined by sphere-congruences which preserve the principal directions are called *Darboux transformations* (see Definition 3.1).

It is known that the hyperbolic sine-Gordon equation, which is the Gauß equation of a CMC surface, is an integrable PDE. One of the most important features of integrable PDE's is that they have Soliton solutions, which are obtained by a "Bäcklund transformation". However the relation between this "Bäcklund transformation" and the classical Bäcklund transformation via tangent line congruences as mentioned above has not yet been clarified.

Thus we have the second aim of the present thesis as follows:

- **Characterizations of transformations of constant mean curvature surfaces in 3-dimensional space forms.**

Organization of this thesis. Chapter 1, Chapter 3, Chapter 4 and Chapter 5 relate to the first aim, and Chapter 1 and Chapter 2 relate to the second aim. More precisely, the present thesis is organized as follows:

Chapter 1.

Chapter 1 is based on the paper [**40**]. The bubbletons are CMC surfaces made from Bäcklund transformations (in Bianchi's sense) of round cylinders. They are shaped like cylinders with attached bubbles, thus they are called bubbletons [**59**], [**65**]. The parallel constant positive Gaussian curvature surfaces of bubbletons are well known, and as they were first found by Sievert [**56**], they are called Sievert surfaces. Bubbletons in $\mathbb{R}^3$ have been closely examined by Kilian, Sterling and Wente [**34**], [**59**], [**65**].

In Chapter 1, analogous to Delaunay surfaces in $\mathbb{R}^3$ we define Delaunay surfaces in S^3 and H^3 (see Definition 3.1 in Section 3.1). Using loop group techniques applied to harmonic maps (via the generalized Weierstrass representation), we represent these Delaunay surfaces in space forms. We then define bubbletons based on Delaunay surfaces in space forms by a simple type dressing action, like those of Terng and Uhlenbeck [**61**], on loop groups (see Definition 4.1 in Section 4.1). Then we solve the period problems for these Delaunay bubbletons and additionally find explicit immersion formulas for those bubbletons in space forms based on round cylinders. In the case of $\mathbb{R}^3$, this was originally done in [**34**], [**59**], [**65**]. Furthermore we prove that the cylinder bubbletons produced here (by the DPW method) are the same as those produced in [**59**] in the case of $\mathbb{R}^3$, and that the parallel CMC surface of a round cylinder bubbleton is congruent to the starting bubbleton, in any of the three space forms. Recently, Mahler [**44**] interpreted the simple type dressing as the Bianchi Bäcklund transformation in the case of $\mathbb{R}^3$.

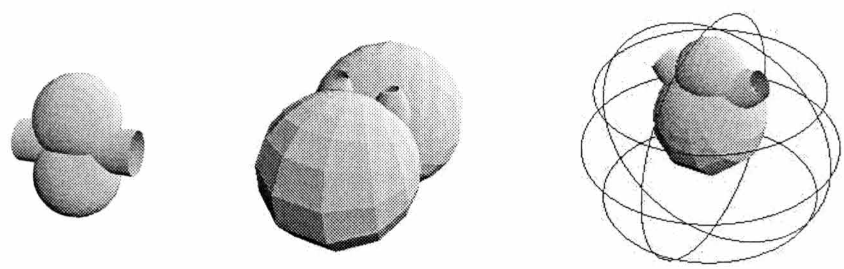

FIGURE 2. CMC bubbletons in $\mathbb{R}^3$, S^3 and H^3. The $\mathbb{R}^3$ bubbleton was first described in [**65**].

So what is new in Chapter 1 is the following:

(i) We show existence of round cylinder bubbletons and Delaunay bubbletons in all three space forms.
(ii) We give explicit parametrizations for round cylinder bubbletons in all three space forms.
(iii) We show the equivalence of Bianchi-Bäcklund transformation and simple type dressing for round cylinders in $\mathbb{R}^3$.

The first two of these three items are new results for the cases of S^3 and H^3, and the third item is a new result in $\mathbb{R}^3$.

Chapter 2.

Chapter 2 is based on the paper [**41**]. U. Hertrich-Jeromin and F. Pedit [**30**] gave a modern approach to Darboux transformation theory via quaternionic calculus. In particular, they showed that Bianchi-Bäcklund transformations of a CMC surface are Darboux transformations such that a positive multiple of the parallel CMC surface (Darboux transformations of positive type, in short). They conjectured that Darboux transformations of a CMC surface such that a negative multiple of the parallel CMC surface (Darboux transformations of negative type) are not Bianchi-Bäcklund transformations.

In Chapter 2, we shall answer the Jeromin-Pedit conjecture negatively, i.e. Darboux transformations of negative type can be realized as Bianchi-Bäcklund transformations. To show this result, we introduce a new parameter into Bianchi-Bäcklund transformations for CMC surfaces. With this new parameter, we call it an "imaginary Bianchi-Bäcklund transformation" (see Definition 2.1), and it is equivalent to a Darboux transformation of negative type. As a consequence of our reformulation, we shall prove the equivalence of the following transformations on CMC surfaces: Bianchi-Bäcklund transformations and Darboux transformations.

Chapter 3.

Chapter 3 is based on the paper [**36**]. To prove the existence of a non-simply-connected CMC surface via the generalized Weierstraß representation, one has to solve period problems and study the monodromy representation of the solution of the ODE as mentioned in (0.2)– (0.4). To show existence of new non-simply connected examples, we provide sufficient conditions which ensure that the conditions (0.2)–(0.4) hold, and apply these methods to when the underlying domain is the n–punctured sphere for $n = 2, 3$. Here the punctures correspond to ends of the surface, where the coefficient matrix of the ODE has poles.

Cylinders, the case of two ends, have been studied via the generalized Weierstraß representation in [**16**], [**34**] and [**38**]. While several classes with various end behavior are now known, the moduli space of all CMC cylinders is not yet well understood. The only classification result was obtained for CMC cylinders of revolution by Delaunay [**24**] in 1841 - the resulting surfaces are the well known Delaunay surfaces. When an end of a CMC surface converges to a Delaunay surface, we call this a Delaunay end. We show how the pole structure at the punctures determines the geometry of the ends of the surface and prove that simple poles can generate Delaunay ends.

Trinoids, the case $n = 3$, constitute the next simplest topology. In analogy to Delaunay surfaces, there are CMC surfaces with three Delaunay ends [**32**]. These have recently been studied in [**22**], [**26**], [**38**] and [**55**]. We shall build on these results and construct CMC trinoids with Delaunay ends via the generalized Weierstraß representation in all three space forms.

Chapter 4.

Chapter 4 is based on the paper [**37**]. The purpose of this chapter is to show the existence of new CMC surfaces by exhibiting Weierstraß data $(\mathcal{M}, \eta, C_0, z_0)$ that fulfill the above requirements (0.2)–(0.4) and to initiate the study of higher genus surfaces via loop group techniques. Briefly summarizing the contents of this chapter, after providing some general sufficient conditions on Weierstraß data to satisfy the condition (0.2), (0.3) and (0.4), we apply these results to prove existence of new examples of CMC surfaces:

(i) of any positive genus and a single end,
(ii) of genus 1 with two ends,
(iii) which are doubly-periodic with infinitely many ends asymptotic to Delaunay ends.

Although the last mentioned surfaces (iii) are immersions of genus zero domains with infinitely many punctures, they have natural quotient surfaces with positive genus.

Chapter 5.

Chapter 5 is based on the paper [**18**]. In the procedure of the generalized Weierstraß representation, it is known that the potentials and CMC surfaces do not have a one-to-one correspondence to each other, i.e. the potentials are not uniquely determined for a given CMC surface. Several types of potentials are known, for example, "normalized potentials", "holomorphic potentials" as mentioned before [**13**], [**16**] and "meromorphic potentials" [**18**].

In Chapter 5 we present a new type of potential, which does exist for every CMC-immersion, from $\mathfrak{D}$ to $\mathbb{R}^3$, where $\mathfrak{D}$ is the universal cover of some Riemann surface $\mathcal{M}$, and which is always invariant under at least one generator of the fundamental group $\pi_1(\mathcal{M})$ of $\mathcal{M}$. As an application we present a "coarse" classification of all CMC-cylinders. (We would like to note that in some special cases the above mentioned new type of potentials has already been used [**34**], [**55**].)

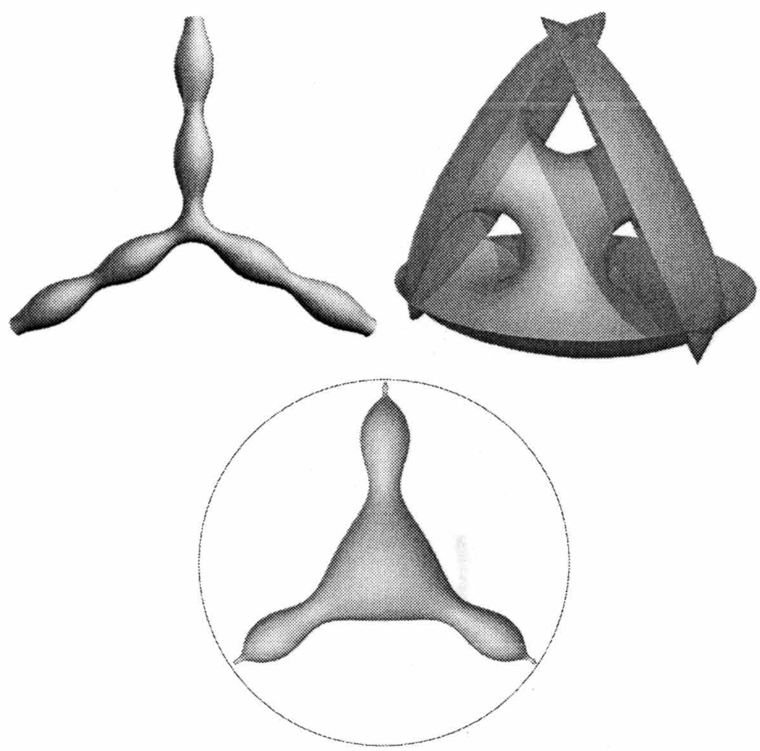

FIGURE 3. Equilateral trinoids in $\mathbb{R}^3$, S^3 and H^3.

Visualization of graphics.

As noted before, $\mathbb{R}^3$ (resp. S^3, H^3) is the unique complete simply-connected three-dimensional Riemannian manifold with constant sectional curvature 0 (resp. 1, -1). $\mathbb{R}^3$ is the standard flat Euclidean three-space. S^3 is the unit three-sphere in $\mathbb{R}^4$ with the metric induced by $\mathbb{R}^4$. To define H^3 we shall use the Lorentz space $\mathbb{R}^{3,1}$:

$$H^3 = \{(t, x, y, z) \in R^{3,1} \mid x^2 + y^2 + z^2 - t^2 = -1\,,\, t > 0\}$$

with the metric induced by $\mathbb{R}^{3,1}$, where $\mathbb{R}^{3,1}$ is the four-dimensional Lorentz space

$$\{(t, x, y, z) \mid t, x, y, z \in \mathbb{R}\}$$

with the Lorentz metric

$$\langle (t_1, x_1, y_1, z_1), (t_2, x_2, y_2, z_2) \rangle = x_1 x_2 + y_1 y_2 + z_1 z_2 - t_1 t_2 \ .$$

To visualize surfaces in S^3 and H^3, we use specific projections. In the case of S^3, we stereographically project S^3 from its north pole to the space $\mathbb{R}^3 \cup \{\infty\}$. In the case of H^3, we use the Poincare model, which is stereographic projection of the Minkowski model in Lorentz space from the point $(0, 0, 0, -1)$ to the 3-ball $\{(0, x, y, z) \in R^{3,1} \mid x^2 + y^2 + z^2 < 1\} \cong \{p = (x, y, z) \in \mathbb{R}^3 \mid |p| < 1\}$.

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CHAPTER 1

Bubbletons in 3-dimensional space forms

1. Lax pairs for CMC surfaces in space forms

1.1. Surfaces in space forms. Let Σ be a simply connected surface with conformal coordinate $z = x + iy$ defined on Σ, and let $f : \Sigma \to \mathcal{M}^3$ be a CMC conformal immersion, where $\mathcal{M}^3$ is either $\mathbb{R}^3$ or S^3 or H^3. We write $f = f(z, \bar{z})$ as a function of both z and $\bar{z}$ to emphasize that f is not holomorphic in z.

Each of the three space forms lies isometrically in a vector space V: $\mathcal{M}^3 = \mathbb{R}^3 \subset V = \mathbb{R}^3$, $\mathcal{M}^3 = S^3 \subset V = \mathbb{R}^4$, or $\mathcal{M}^3 = H^3 \subset V = \mathbb{R}^{3,1}$. Let $\langle \cdot, \cdot \rangle$ be the inner product associated to V, which is the Euclidean inner product in the first two cases, and the Lorentz inner product in the third case. Then the space form metric for each of $\mathbb{R}^3$, S^3 and H^3 is the one induced from the metric of the associated vector space V. Since $\mathcal{M}^3 \subset V$ is an embedding, we may also view f as a C^∞ map into V,

$$f : \Sigma \to \mathcal{M}^3 \subseteq V \quad , \quad \text{where } V \text{ is } \mathbb{R}^3 \text{ or } \mathbb{R}^4 \text{ or } \mathbb{R}^{3,1} \ .$$

The derivatives $f_x = \partial_x f$ and $f_y = \partial_y f$ are vectors in the tangent space $T_{f(z,\bar{z})}V$ of V at $f(z, \bar{z})$. Because V is a vector space, f_x and f_y can be viewed as lying in V itself. We will also use $f_z = (1/2)(f_x - if_y)$ and $f_{\bar{z}} = (1/2)(f_x + if_y)$, defined in the complex extension $V_\mathbb{C} = \{v_1 + iv_2 | v_1, v_2 \in V\}$ of V. The inner product of V extends to a bilinear form $\langle v_1 + iv_2, v_1 + iv_2 \rangle = \langle v_1, v_1 \rangle + 2i\langle v_1, v_2 \rangle + \langle v_2, v_2 \rangle$ (which we also denote by $\langle \cdot, \cdot \rangle$ although it is not actually a true inner product on $V_\mathbb{C}$). Note that f is conformal if and only if

$$\langle f_z, f_z \rangle = \langle f_{\bar{z}}, f_{\bar{z}} \rangle = 0 \ , \ \ \langle f_z, f_{\bar{z}} \rangle = 2e^{2\mu} \ , \tag{1.1}$$

where the right-most equation defines the function $\mu : \Sigma \to \mathbb{R}$.

In each space form, a unit normal vector $N = N(z, \bar{z}) \in T_{f(z,\bar{z})}V \equiv V$ of f is defined by the properties

(i) $\langle N, N \rangle = 1$,
(ii) $N \in T_{f(z,\bar{z})}\mathcal{M}^3$, and
(iii) $\langle N, f_z \rangle = \langle N, f_{\bar{z}} \rangle = 0$.

The mean curvature of f is then given by

$$H = \frac{1}{2e^{2\mu}} \langle f_{z\bar{z}}, N \rangle \ . \tag{1.2}$$

We also define the Hopf differential

$$Q = \langle f_{zz}, N \rangle dz^2 \ . \tag{1.3}$$

1.2. The vector spaces V in terms of quaternions. Define the matrices

$$\sigma_0 = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

We can think of $\boldsymbol{H} = \mathrm{span}_{\mathbb{R}}\{i\sigma_0, i\sigma_1, i\sigma_2, i\sigma_3\}$ as the quaternions because it has the quaternionic algebraic structure.

The $\mathbb{R}^3$ case. When $\mathcal{M}^3 = V = \mathbb{R}^3$, we associate $\mathcal{M}^3$ with the imaginary quaternions $\mathrm{Im}\,\boldsymbol{H} = \mathrm{span}_{\mathbb{R}}\{i\sigma_1, i\sigma_2, i\sigma_3\} \subsetneq \boldsymbol{H}$ by the map

$$(x_1, x_2, x_3) \to x_1 \frac{i}{2}\sigma_1 + x_2 \frac{i}{2}\sigma_2 + x_3 \frac{i}{2}\sigma_3 . \tag{1.4}$$

Then for $X, Y \in \mathrm{Im}\,\boldsymbol{H}$, the inner product inherited from $\mathbb{R}^3$ is

$$\langle X, Y \rangle = -2 \cdot \mathrm{trace}(XY) = 2 \cdot \mathrm{trace}(XY^*) , \tag{1.5}$$

where $Y^* = \bar{Y}^t$. Also, any oriented orthonormal basis $\{X, Y, Z\}$ of vectors of $\mathcal{M}^3 \equiv \mathrm{Im}\,\boldsymbol{H}$ satisfies

$$X = F\left(\frac{i}{2}\sigma_1\right)F^{-1} , \quad Y = F\left(\frac{i}{2}\sigma_2\right)F^{-1} , \quad Z = F\left(\frac{i}{2}\sigma_3\right)F^{-1} \tag{1.6}$$

for some $F \in \mathrm{SU}(2)$, and this F is unique up to sign. In other words, rotations S of $\mathbb{R}^3$ fixing the origin are represented in the quaternionic representation $\mathrm{Im}\,\boldsymbol{H}$ of $\mathbb{R}^3$ by matrices $F \in \mathrm{SU}(2)$. And the image of F under $\mathrm{Im}\,\boldsymbol{H} \to SO(3)$ is the rotation S.

The S^3 case. When $\mathcal{M}^3 = S^3$ and $V = \mathbb{R}^4$, we associate V with $\boldsymbol{H}$ by the map

$$(x_1, x_2, x_3, x_4) \to x_1 i\sigma_0 + x_2 i\sigma_1 + x_3 i\sigma_2 + x_4 i\sigma_3 , \tag{1.7}$$

so points $(x_1, x_2, x_3, x_4) \in V = \mathbb{R}^4$ are matrices of the form

$$X = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} , \tag{1.8}$$

where $a = x_1 + ix_4$ and $b = x_3 + ix_2$. That is, they are matrices X that satisfy

$$X = \sigma_2 \bar{X} \sigma_2 . \tag{1.9}$$

The inner product on $\boldsymbol{H}$ inherited from V is

$$\langle X, Y \rangle = (1/2) \cdot \mathrm{trace}(XY^*) , \tag{1.10}$$

where $Y^* = \bar{Y}^t$. Note that this inner product is the same as in (1.5), up to a factor of 4, and this factor of 4 appears only because we include a factor of 1/2 in (1.4) but not in (1.7).

The H^3 case. When $\mathcal{M}^3 = H^3$ and $V = \mathbb{R}^{3,1}$, we can associate V with the set of self-adjoint 2×2 matrices $\{X \in Mat(2, \mathbb{C}) \mid X^* = X\}$ by the map

$$(x_0, x_1, x_2, x_3) \in R^{3,1} \to X = x_0 i\sigma_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 . \tag{1.11}$$

One can check that $\sigma_2 X^t \sigma_2 = X^{-1} \det X$ and that the inner product inherited from V is

$$\langle X, Y \rangle = (-1/2)\mathrm{trace}(X\sigma_2 Y^t \sigma_2) ,$$

for self-adjoint matrices X, Y. Thus $\langle X, X \rangle = -\det X$.

1.3. The Lax Pair in the space forms. Let f be a conformal immersion, as in Section 1.1. Because f is a surface in $\mathcal{M}^3 = \mathbb{R}^3$ (resp. S^3, H^3), μ and H and Q satisfy the following Gauss and Codazzi equations for $\mathbb{R}^3$ (resp. S^3, or H^3), and the F_1 (resp. F_1, F_2, or F_1), which correspond to the moving frame of a surface f by the map (1.4) (resp. (1.7), (1.11)), satisfies the following Lax pair equations.

$$4\mu_{z\bar{z}} - Q\bar{Q}e^{-2\mu} + 4H_k e^{2\mu} = 0\ , \quad Q_{\bar{z}} = 2H_z e^{2\mu}\ , \tag{1.12}$$

and

$$F_{k,z} = F_k U_k\ , \quad F_{k,\bar{z}} = F_k V_k \tag{1.13}$$

with

$$U_k = \frac{1}{2}\begin{pmatrix} \mu_z & -2H_k e^{\mu}\lambda^{-1} \\ Qe^{-\mu}\lambda^{-1} & -\mu_z \end{pmatrix}\ , \quad V_k = \frac{1}{2}\begin{pmatrix} -\mu_{\bar{z}} & -\bar{Q}e^{-\mu}\lambda \\ 2H_{k+1}e^{\mu}\lambda & \mu_{\bar{z}} \end{pmatrix}\ , \tag{1.14}$$

where H_k is H (resp. $H - (-1)^k i$, $H - (-1)^k$) in the case of $\mathbb{R}^3$ (resp. S^3, H^3), with $k \in \{1, 2\}$. (k will be 2 only in the S^3 case.)

For H constant, we see that the Gauss and Codazzi equations for $\mathcal{M}^3$ remain satisfied when Q is replaced by $\lambda^{-2}Q$ for any $\lambda \in S^1 = \{p \in \mathbb{C}\,|\,|p| = 1\}$. Hence, up to rigid motions, there is a unique surface f_λ with metric determined by μ and with mean curvature H and Hopf differential $\lambda^{-2}Q$. (We use the notation f_λ to state that f depends on λ; it does not denote the derivative $\partial_\lambda f$.) The surfaces f_λ for $\lambda \in S^1$ form a one-parameter family called the *associate family* of f. The parameter λ is called the *spectral parameter* and is essential to the DPW method. From [**48**], we have the following facts. When the ambient space is $\mathbb{R}^3$, the parallel surfaces of f_λ are

$$f_{\lambda,t} = f_\lambda + tN\ \ , \quad t \in \mathbb{R}.$$

When the ambient space is S^3, the parallel surfaces of f_λ are

$$f_{\lambda,t} = \cos(t) f_\lambda + \sin(t) N\ \ , \quad t \in \mathbb{R}.$$

When the ambient space is H^3, the parallel surfaces of f_λ are

$$f_{\lambda,t} = \cosh(t) f_\lambda + \sinh(t) N\ \ , \quad t \in \mathbb{R}.$$

There are special values of t for which the parallel surfaces $f_{\lambda,t}$ also have CMC surfaces. In the case of $\mathbb{R}^3$ (resp. S^3, H^3), this is true when the parallel surface is $f_\lambda^* = f_{\lambda,1/(2H)}$ (resp. $f_\lambda^* = f_{\lambda,\mathrm{arccot}(H)}, f_\lambda^* = f_{\lambda,\mathrm{arccoth}(H)}$).

The $\mathbb{R}^3$ case. In the case of $\mathbb{R}^3$, by applying a homothety if necessary, we may assume $H = 1/2$. We have the following theorem, proven in [**6**] and [**36**] using different notations, with the notations here matching those of [**21**], [**13**]. We also include information on the parallel surfaces f_λ^* here.

THEOREM 1.1. *Let u and Q solve the Gauss-Codazzi equations*

$$4u_{z\bar{z}} - Q\bar{Q}e^{-2u} + e^{2u} = 0\ , \quad Q_{\bar{z}} = 0\ , \tag{1.15}$$

and let $F(z, \bar{z}, \lambda)$ be a solution of the system

$$F_z = FU\ , \quad F_{\bar{z}} = FV \tag{1.16}$$

with

$$(1.17)\qquad U=\frac{1}{2}\begin{pmatrix} u_z & -e^u\lambda^{-1} \\ Qe^{-u}\lambda^{-1} & -u_z \end{pmatrix}\ ,\quad V=\frac{1}{2}\begin{pmatrix} -u_{\bar z} & -\bar{Q}e^{-u}\lambda \\ e^u\lambda & u_{\bar z} \end{pmatrix}$$

such that $F(z,\bar z,\lambda)\in \mathrm{SU}(2)$ *for all* $\lambda\in S^1$ *and* $F(z,\bar z,\lambda)$ *is complex analytic in* λ. *Define*

$$(1.18)\qquad f_\lambda=\left[\frac{-i}{2}F\sigma_3F^{-1}-i\lambda(\partial_\lambda F)\cdot F^{-1}\right]\bigg|_{\lambda=1}\ ,\quad N=\frac{-i}{2}F\sigma_3F^{-1}\ .$$

$$(1.19)\qquad f^*_\lambda=f-2N\ ,\quad N^*=-N\ .$$

Then f_λ *and* f^*_λ *are of the form*

$$(1.20)\qquad r\cdot\frac{i}{2}\sigma_1+s\cdot\frac{i}{2}\sigma_2+t\cdot\frac{i}{2}\sigma_3\ \text{ and }\ r^*\cdot\frac{i}{2}\sigma_1+s^*\cdot\frac{i}{2}\sigma_2+t^*\cdot\frac{i}{2}\sigma_3\ ,$$

where r,s,t,r^*,s^* *and* t^* *are real-valued, and* (r,s,t) *and* (r^*,s^*,t^*) *are both conformal parametrizations of CMC with* $H=1/2$ *surfaces in* $\mathbb{R}^3$, *parametrized by* z. f_λ *and* f^*_λ *are parallel surfaces. Also,* μ *and* $\mathcal{Q}$ *satisfy* $e^{2\mu}=e^{2u}$ *and* $\mathcal{Q}=Q$, *where* μ *and* $\mathcal{Q}$ *are defined as in* (1.1) *and* (1.3). *Furthermore, with* μ^* *and* $\mathcal{Q}^*$ *defined by* $2e^{2\mu^*}=\langle f^*_{\lambda,z},f^*_{\lambda,\bar z}\rangle$ *and* $\mathcal{Q}^*=\langle f^*_{\lambda,zz},N^*\rangle$, *we have* $e^{2\mu^*}=e^{-2u}|Q|^2$ *and* $\mathcal{Q}^*=Q$.

Conversely, for every conformal CMC immersion with $H=1/2$ *into* $\mathbb{R}^3$, *there exists a system* (1.16)-(1.17) *and solution* F *producing the immersion via* (1.18).

The S^3 case. Consider a conformal CMC immersion $f:\Sigma\to\mathcal{M}^3=S^3\subset V=\mathbb{R}^4$, with $\langle\cdot,\cdot\rangle$ as in (1.10). Similar to the $\mathbb{R}^3$ case, we have the following theorem, proven in [**6**] and [**36**] using different notations.

THEOREM 1.2. *Let* u *and* Q *solve* (1.15) *and let* $F(z,\bar z,\lambda)$ *be a solution of the system* (1.16)-(1.17) *such that* $F(z,\bar z,\lambda)\in\mathrm{SU}(2)$ *for all* $\lambda\in S^1$ *and* $F(z,\bar z,\lambda)$ *is complex analytic in* λ. *Define* $F_1=F(z,\bar z,\lambda=e^{i\gamma_1})$ *and* $F_2=F(z,\bar z,\lambda=e^{i\gamma_2})$ *for some fixed* $\gamma_1,\gamma_2\in\mathbb{R}$, *and set*

$$(1.21)\qquad f_\lambda=F_1AF_2^{-1}\ ,\quad N=iF_1A\sigma_3F_2^{-1},\ \text{ where } A=\begin{pmatrix} e^{\frac{i(\gamma_1-\gamma_2)}{2}} & 0 \\ 0 & e^{\frac{i(\gamma_2-\gamma_1)}{2}} \end{pmatrix},$$

$$(1.22)\qquad f^*_\lambda=\cos(\gamma_2-\gamma_1)f+\sin(\gamma_2-\gamma_1)N\ ,\quad N^*=\sin(\gamma_2-\gamma_1)f-\cos(\gamma_2-\gamma_1)N\ .$$

Then f_λ *and* f^*_λ *are conformal CMC immersions with* $H=\cot(\gamma_2-\gamma_1)$ *into* S^3. f_λ *and* f^*_λ *are parallel surfaces. Also,* μ *and* $\mathcal{Q}$ *satisfy* $e^{2\mu}=\sin^2(\gamma_2-\gamma_1)\cdot e^{2u}/4$ *and* $\mathcal{Q}=\sin(\gamma_2-\gamma_1)\cdot Q$, *where* μ *and* $\mathcal{Q}$ *are defined as in* (1.1) *and* (1.3). *Furthermore, with* μ^* *and* $\mathcal{Q}^*$ *defined by* $2e^{2\mu^*}=\langle f^*_{\lambda,z},f^*_{\lambda,\bar z}\rangle$ *and* $\mathcal{Q}^*=\langle f^*_{\lambda,zz},N^*\rangle$, *we have* $e^{2\mu^*}=\sin^2(\gamma_2-\gamma_1)\cdot|Q|^2e^{-2u}/4$ *and* $\mathcal{Q}^*=\sin(\gamma_2-\gamma_1)\cdot Q$.

Conversely, for every conformal CMC immersion with $H=\cot(\gamma_2-\gamma_1)$ *into* S^3, *there exists a system* (1.16)-(1.17) *and solution* F *producing the immersion via* (1.21).

The H^3 case. Let $f:\Sigma\to\mathcal{M}^3=H^3\subset V=\mathbb{R}^{3,1}$ be a conformal CMC immersion with $H>1$ in H^3. Again, similar to the $\mathbb{R}^3$ and S^3 cases, we have the following theorem, proven in [**6**] and [**36**] with different notations.

THEOREM 1.3. *Let u and Q solve* (1.15) *and let $F = F(z, \bar{z}, \lambda = e^{q/2}e^{i\psi})$, for some fixed $q, \psi \in \mathbb{R}$ (with $q \neq 0$), be a solution of the system* (1.16)-(1.17) *such that $F(z, \bar{z}, \lambda) \in \mathrm{SU}(2)$ for all $\lambda \in S^1$ and $F(z, \bar{z}, \lambda)$ is complex analytic in λ. We set*

$$f_\lambda = FAF^* \ , \ N = FA\sigma_3F^* \ , \text{where } A = \begin{pmatrix} e^{q/2} & 0 \\ 0 & e^{-q/2} \end{pmatrix}, F^* = \overline{F}^t . \tag{1.23}$$

$$f_\lambda^* = \cosh(-q)f + \sinh(-q)N \ , \ N^* = \sinh(-q)f - \cosh(-q)N \ . \tag{1.24}$$

Then f_λ and f_λ^* are CMC conformal immersions with $H = \coth(-q) > 1$ into H^3. f_λ and f_λ^* are parallel surfaces. Also, μ and $\mathcal{Q}$ satisfy $e^{2\mu} = \sinh^2(-q) \cdot e^{2u}/4$ and $\mathcal{Q} = \sinh(-q) \cdot Q$, where μ and $\mathcal{Q}$ are defined as in (1.1) *and* (1.3). *Furthermore, with μ^* and $\mathcal{Q}^*$ defined by $2e^{2\mu^*} = \langle f_{\lambda,z}^*, f_{\lambda,\bar{z}}^* \rangle$ and $\mathcal{Q}^* = \langle f_{\lambda,zz}^*, N^* \rangle$, we have $e^{2\mu^*} = \sinh^2(-q) \cdot |Q|^2 e^{-2u}/4$ and $\mathcal{Q}^* = \sinh(-q) \cdot Q$.*

Conversely, for every CMC conformal immersion with $H = \coth(-q)$ into H^3, there exists a system (1.16)-(1.17) *and solution F producing the immersion via* (1.23).

REMARK 1.4. *The parallel surface f_λ^* can have singular points. At points where the Hopf differential $\mathcal{Q}$ of the original CMC surface f_λ is zero, then the metric of f_λ^* is degenerate.*

2. The DPW recipe

We saw in Section 1 that finding CMC surfaces with $H \neq 0$ in $\mathbb{R}^3$ and CMC surfaces in $\mathbb{S}^3$ and CMC surfaces with $H > 1$ in H^3 is equivalent to finding integrable Lax pairs of the form (1.16)-(1.17) and their solutions F. Then the surfaces are found by using the Sym-Bobenko type formulas (1.18), (1.21) and (1.23). So to prove that the DPW recipe finds all of these types of surfaces, it is sufficient to prove that the DPW recipe produces all integrable Lax pairs of the form (1.16)-(1.17) and all their solutions F. Here we describe how these Lax pairs and solutions F are found by the DPW method in [**21**].

2.1. The loop groups and Iwasawa splitting. Let $C_r := \{\lambda \in \mathbb{C} \mid |\lambda| = r\}$ be the circle of radius r with $r \in (0, 1]$, and let $D_r := \{\lambda \in \mathbb{C} \mid |\lambda| < r\}$ be the open disk of radius r. We denote the closure of D_r by $\overline{D_r} := \{\lambda \in \mathbb{C} \mid |\lambda| \leq r\}$.

DEFINITION 2.1. *For any $r \in (0, 1] \subset \mathbb{R}$, we define the following loop algebra and loop groups:*

(i) *The twisted $sl(2, \mathbb{C})$ r-loop algebra is*

$$\Lambda_r sl(2, \mathbb{C}) = \left\{ A : C_r \xrightarrow{C^\infty} sl(2, \mathbb{C}) \mid A(-\lambda) = \sigma_3 A(\lambda) \sigma_3 \right\} .$$

(ii) *The twisted $\mathrm{SL}(2, \mathbb{C})$ r-loop group is*

$$\Lambda_r \mathrm{SL}(2, \mathbb{C}) = \left\{ \phi : C_r \xrightarrow{C^\infty} \mathrm{SL}(2, \mathbb{C}) \mid \phi(-\lambda) = \sigma_3 \phi(\lambda) \sigma_3 \right\} .$$

(iii) *The twisted* SU(2) r*-loop group is*

$$\Lambda_r \mathrm{SU}(2) = \{F \in \Lambda_r \mathrm{SL}(2,\mathbb{C}) \,|F(\lambda)^{-1} = F(\bar{\lambda}^{-1})^*, \text{ and } F(\lambda) \text{ extends holomorphically to } D_{1/r} \setminus \overline{D_r}\ \} .$$

When $r = 1$*, we abbreviate* $\Lambda_1 \mathrm{SU}(2)$ *to* $\Lambda \mathrm{SU}(2)$*, and in this case the condition that* F *extends holomorphically to* $D_{1/r} \setminus \overline{D_r}$ *is vacuous.*

(iv) *The twisted plus* r*-loop group with* $\mathbb{R}^+$ *is*

$$\Lambda_{+,r}SL(2,\mathbb{C}) = \{B \in \Lambda_r \mathrm{SL}(2,\mathbb{C}) \mid B(\lambda) \text{ extends holomorphically to } D_r, \text{ and } B|_{\lambda=0} = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \text{ with } \rho > 0\ \} .$$

When $r = 1$*, we abbreviate* $\Lambda_{+,1}SL(2,\mathbb{C})$ *to* $\Lambda_+ SL(2,\mathbb{C})$*. Here we defined* $\Lambda_{+,r}SL(2,\mathbb{C})$ *such that* $\Lambda_{+,r}SL(2,\mathbb{C}) \cap \Lambda_r \mathrm{SU}(2) = \mathrm{Id}$.

We will give $\Lambda_r SL(2,\mathbb{C})$ an H^s-topology for a fixed $s > 1/2$ and take its completion. Then, $\Lambda_r SL(2,\mathbb{C})$ is a complex Banach Lie group and its elements have Fourier expansions in the loop parameter λ. We quote the following result from [**21**].

LEMMA 2.1. (Iwasawa decomposition) *For any* $r \in (0,1]$*, we have the following globally defined real-analytic diffeomorphism from* $\Lambda_r \mathrm{SL}(2,\mathbb{C})$ *to* $\Lambda_r \mathrm{SU}(2) \times \Lambda_{+,r} \mathrm{SL}(2,\mathbb{C})$ *: For any* $\phi \in \Lambda_r SL(2,\mathbb{C})$*, there exist unique* $F \in \Lambda_r \mathrm{SU}(2)$ *and* $B \in \Lambda_{+,r}SL(2,\mathbb{C})$ *so that*

$$\phi = FB .$$

We call this r-Iwasawa splitting *of* ϕ*. When* $r = 1$*, we call it simply* Iwasawa splitting. *Because the diffeomorphism is real-analytic, if* ϕ *depends real-analytically (resp. smoothly) on some parameter* z*, then* F *and* B *do as well.*

2.2. The DPW method. We now describe the DPW method. Let

$$\xi = A(z,\lambda)dz\ , \quad A(z,\lambda) \in \Lambda sl(2,\mathbb{C})\ \ , \ \ \lambda \in \mathbb{C} \setminus \{0\}\ , \tag{2.1}$$

where $A := A(z,\lambda)$ is holomorphic in both z and λ for $z \in \Sigma$. Furthermore, we assume that A has a pole of order at most 1 at $\lambda = 0$, and the upper-right and lower-left entries of A have poles of order exactly 1 at $\lambda = 0$. We call ξ a *holomorphic potential.*

Let ϕ be the solution to

$$d\phi = \phi\xi\ , \qquad \phi(z_*) = \mathrm{Id}$$

for some base point $z_* \in \Sigma$. Then ϕ is holomorphic in $z \in \Sigma$ and $\lambda \in \mathbb{C}^*$, and

$$\phi \in \Lambda SL(2,\mathbb{C})\ .$$

By Lemma 2.1 above, we can perform an Iwasawa splitting, and write the result as

$$\phi = FB\ . \tag{2.2}$$

From [**22**], we have the following proposition.

PROPOSITION 2.2. *Up to a conformal change of the coordinate z, F is a solution to a Lax pair of the form* (1.16)-(1.17)*, and then the Sym-Bobenko formula* (1.18) *or* (1.21) *or* (1.23) *produces a conformal CMC immersion in the corresponding space form $\mathbb{R}^3$, S^3 or H^3.*

Also from [**22**], conversely the conformal CMC immersion has a holomorphic potential, as the next proposition shows.

PROPOSITION 2.3. *For any solution $F \in \Lambda\,\mathrm{SU}(2)$, defined for all $z \in \Sigma$ and all $\lambda \in \mathbb{C} \setminus \{0\}$ with $F(z_*) = \mathrm{Id}$, to a Lax pair of type* (1.16)-(1.17)*, there exists a holomorphic potential $\xi = Adz$ with A as in* (2.1) *and a solution $\phi \in \Lambda\,\mathrm{SL}(2,\mathbb{C})$ of $d\phi = \phi\xi$ so that ϕ Iwasawa splits into $\phi = FB$ for some $B \in \Lambda_+ SL(2,\mathbb{C})$.*

2.3. Dressing. Let Σ be a simply connected surface and let ϕ be a solution to $d\phi = \phi\xi$ with $\phi(z_*) = \mathrm{Id}$ on Σ, where ξ is defined as in Equation (2.1). If we define

$$\hat{\phi} = h_+ \cdot \phi \ ,$$

for $h_+ = h_+(\lambda) \in \Lambda_{+,r}\,\mathrm{SL}(2,\mathbb{C})$ depending only on λ, then this multiplication on the left by h_+ is called a dressing.

Note that $\hat{\phi}$ satisfies $d\hat{\phi} = \hat{\phi}\xi$, because h_+ is independent of z. Hence the dressing h_+ does not change the potential ξ, and changes only the resulting surface. To see how the surface is changed by h_+, one must Iwasawa split h_+F into $h_+F = \tilde{F}\tilde{B}$, and then $\tilde{F} \in \Lambda SU(2)$ is the frame for the changed surface. This change in the frame from F to $\tilde{F}$ is nontrivial to understand in general, hence the change in the surface is also nontrivial to understand.

2.4. Period problems. Let Σ be a connected Riemann surface with universal cover $\tilde{\Sigma}$ and let Δ denote the group of deck transformations. Let ξ be a holomorphic potential as in Equation (2.1) and ϕ be a solution of $d\phi = \phi\xi$. We assume $\tau^*\xi = \xi$ for any $\tau \in \Delta$. For each $\tau \in \Delta$, we define the monodromy matrix M_τ of ϕ by $M_\tau(\lambda) = (\phi \circ \tau) \cdot \phi^{-1}$.

From [**16**], we have the following theorem.

THEOREM 2.4. *Let M_τ be the monodromy matrix of a solution ϕ, with respect to some deck transformation $\tau \in \Delta$ of $\tilde{\Sigma}$. Then M_τ is unitarizable via dressing for some $r \in (0,1]$ if and only if, for all $\lambda \in S^1$,*

$$\mathrm{trace}(M_\tau) \in (-2,2) \ \ or \ \ M_\tau = \pm id\,. \tag{2.3}$$

We introduce the following theorem to solve the period problems in $\mathbb{R}^3$, S^3 or H^3, respectively, as in [**36**].

THEOREM 2.5. *Let M_τ be as in Theorem 2.4. Assume $M_\tau \in \Lambda_r SU(2)$, so M_τ is also the monodromy matrix of F about τ, where F is as in* (2.2)*. Let f be one of the Sym-Bobenko formulas* (1.18) *or* (1.21) *or* (1.23) *for F, respectively. Then*

- *$\mathbb{R}^3$ case: $f \circ \tau = f$ holds if and only if*

$$M_\tau|_{\lambda=1} = \pm id \ \ and \ \ \partial_\lambda M_\tau|_{\lambda=1} = 0\ , \tag{2.4}$$

- S^3 *case:* $f \circ \tau = f$ *holds if and only if*

$$M_\tau|_{\lambda=e^{i\gamma_1}} = M_\tau|_{\lambda=e^{i\gamma_2}} = \pm \mathrm{Id} \ , \tag{2.5}$$

- H^3 *case:* $f \circ \tau = f$ *holds if and only if*

$$M_\tau|_{\lambda=e^{q/2}e^{i\psi}} = \pm id \ . \tag{2.6}$$

3. Surfaces of Revolution

3.1. Delaunay surfaces via DPW. Delaunay surfaces are periodic surfaces of revolution in $\mathbb{R}^3$ and are described via DPW in detail in [**34**]. The generalization of Delaunay surfaces, which are rotational W-hypersurfaces of σ_1-type in H^{n+1} and S^{n+1}, are studied in [**58**]. We also give a description here. First we give the definition of Delaunay surfaces in space forms.

DEFINITION 3.1. *Let $f : \Sigma = S^2 \setminus p_1, p_2 \to \mathbb{R}^3$ (resp. S^3, H^3) be a CMC immersion. Then f is a Delaunay surface in $\mathbb{R}^3$ (resp. S^3, H^3) if f is a surface of revolution in $\mathbb{R}^3$ (resp. S^3, H^3), i.e. a surface of revolution about a fixed geodesic line in $\mathbb{R}^3$ (resp. S^3, H^3).*

Using stereographic projection and a Moebius transformation, we may assume $\Sigma = \mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Define

$$\xi = D\frac{dz}{z} \ , \qquad \text{where } D = \begin{pmatrix} l & s\lambda^{-1} + t\lambda \\ s\lambda + t\lambda^{-1} & -l \end{pmatrix} , \tag{3.1}$$

with $l, s, t \in \mathbb{R}$.

One solution of $d\phi = \phi\xi$ is

$$\phi = \exp(\ln z \cdot D) \ . \tag{3.2}$$

This ϕ can be split (this is not r-Iwasawa splitting) in the following way:

$$\phi = F_1 B_1 \ , \quad F_1 = \exp(i\theta D) \ , \quad B_1 = \exp(\ln \rho \cdot D) \ ,$$

where $z = \rho e^{i\theta}$, with $\rho = |z|$ and $\theta = \arg(z)$. We note that $F_1 \in \Lambda_r \mathrm{SU}(2)$.

Since $D^2 = X^2 \mathrm{Id}$, where $X = \sqrt{l^2 + (s+t)^2 + st(\lambda - \lambda^{-1})^2}$, we see that

$$F_1 = \begin{pmatrix} \cos(\theta X) + ilX^{-1}\sin(\theta X) & iX^{-1}\sin(\theta X)(s\lambda^{-1} + t\lambda) \\ iX^{-1}\sin(\theta X)(s\lambda + t\lambda^{-1}) & \cos(\theta X) - ilX^{-1}\sin(\theta X) \end{pmatrix} , \tag{3.3}$$

$$B_1 = \begin{pmatrix} \cosh(\ln\rho \cdot X) + lX^{-1}\sinh(\ln\rho \cdot X) & X^{-1}\sinh(\ln\rho \cdot X)(s\lambda^{-1} + t\lambda) \\ X^{-1}\sinh(\ln\rho \cdot X)(s\lambda + t\lambda^{-1}) & \cosh(\ln\rho \cdot X) - lX^{-1}\sinh(\ln\rho \cdot X) \end{pmatrix} .$$

We can now r-Iwasawa split B_1, i.e. $B_1 = F_2 \cdot B$, where $F_2 \in \Lambda_r SU(2)$ and $B \in \Lambda_{+,r} SL(2,\mathbb{C})$. We define $F = F_1 \cdot F_2$. Thus $\phi = FB$ is the r-Iwasawa splitting of ϕ (for any choice of $r \in (0,1]$).

Because F_2 and B depend only on $|z| = \rho$ and F_1 depends only on θ, we have that, under the rotation of the domain

$$z \to R_{\theta_0}(z) = e^{i\theta_0} z \ , \ \theta_0 \in \mathbb{R} \ ,$$

the following transformations occur:

$$F \to M_{\theta_0} F \quad \text{and} \quad B \to B \ , \quad \text{where } M_{\theta_0} = \exp(i\theta_0 D) \ .$$

We note that $M_{\theta_0} \in \Lambda_r \mathrm{SU}(2)$, and that M_{θ_0} is of the same explicit form as F_1 in (3.3), but evaluated at $\theta = \theta_0$. When $\theta = 2\pi$, we have

$$M_\tau = M_{2\pi} \ . \tag{3.4}$$

Clearly M_τ is the monodromy matrix of the generating counterclockwise deck transformation $\tau \in \Delta$ of the universal cover of $\mathbb{C} \setminus \{0\}$.

Now we consider the closing conditions in each of the three space forms:

- When $\mathcal{M}^3 = \mathbb{R}^3$, $M_{2\pi}$ must satisfy (2.4) for $\lambda = 1$, so that the surface will close about the deck transformation τ. This is satisfied if
$$l^2 + (s+t)^2 = 1/4 \ , \tag{3.5}$$
so we impose this condition when $\mathcal{M}^3 = \mathbb{R}^3$.
- When $\mathcal{M}^3 = S^3$, $M_{2\pi}$ must satisfy (2.5), so that the surface will close about τ. With $\lambda_1 = e^{i\gamma}$ and $\lambda_2 = e^{-i\gamma}$, (2.5) is satisfied if
$$l^2 + (s+t)^2 - 4st\sin^2(\gamma) = 1/4 \ , \tag{3.6}$$
so we impose this when $\mathcal{M}^3 = S^3$.
- When $\mathcal{M}^3 = H^3$, $M_{2\pi}$ must satisfy (2.6), so that the surface will close about τ. With $\lambda = q/2 \in \mathbb{R}^+$, (2.6) is satisfied if
$$l^2 + (s+t)^2 + 4st\sinh^2(\frac{q}{2}) = 1/4 \ , \tag{3.7}$$
so we impose this when $\mathcal{M}^3 = H^3$.

With these conditions, Delaunay surfaces are produced in $\mathbb{R}^3$, S^3 and H^3, and this can be seen as follows:

In the case of $\mathbb{R}^3$, under the mapping $z \to R_{\theta_0}(z)$, we have that f as in (1.18) changes as

$$f \to M_{\theta_0} f M_{\theta_0}^{-1} - i(\partial_\lambda M_{\theta_0})|_{\lambda=1} M_{\theta_0}^{-1} \ . \tag{3.8}$$

One can check that Equation (3.8) represents a rotation of angle θ_0 about the line

$$\{x \cdot (-s-t, 0, l) + 2(s-t) \cdot (2l, 0, 2s+2t) \mid x \in \mathbb{R}\} \ ,$$

hence f is a surface of revolution, and thus a Delaunay surface in $\mathbb{R}^3$.

In the case of S^3, under the mapping $z \to R_{\theta_0}(z)$, f as in (1.21) changes by

$$f \to (M_{\theta_0}|_{\lambda=e^{-i\gamma}}) f (M_{\theta_0}^{-1}|_{\lambda=e^{i\gamma}}) \ . \tag{3.9}$$

One can check that Equation (3.9) represents a rotation of angle θ_0 about the geodesic line

$$\{(x_1, x_2, 0, x_4) \in \mathbb{S}^3 \mid \sin(\gamma)(s-t)x_1 + rx_2 - \cos(\gamma)(s+t)x_4 = 0\} \ . \tag{3.10}$$

So we have a surface of revolution in this case also (since the geodesic line (3.10) does not depend on θ_0), and hence a Delaunay surface in S^3.

In the case of H^3, under the mapping $z \to R_{\theta_0}(z)$, f as in (1.23) changes by

$$f \to (M_{\theta_0}|_{\lambda=e^{q/2}}) f (\overline{M_{\theta_0}}^t|_{\lambda=e^{q/2}}) \ . \tag{3.11}$$

One can check that Equation (3.11) represents a rotation of angle θ_0 about the geodesic line

$$\{(x_1, 0, x_3, x_0) \in H^3 \mid \sinh(q)(s-t)x_0 - rx_1 + \cosh(q)(s+t)x_3 = 0\} . \tag{3.12}$$

Therefore f is a surface of revolution (since the geodesic line (3.12) does not depend on θ_0), and hence a Delaunay surface in H^3.

We summarize this in the following theorem.

THEOREM 3.1. *The holomorphic potential ξ defined in Equation* (3.1) *with the condition* (3.5) *(resp.* (3.6), (3.7)*) produces a Delaunay surface in R^3 (resp. S^3, H^3).*

Which Delaunay surface one gets depends on the choice of r, s and t. An unduloid is produced when $st > 0$. A nodoid is produced when $st < 0$, and for the limiting singular case of a sphere, $st = 0$. A cylinder is produced when $s = t$ and $l = 0$. In the next subsection, we will show that we can explicitly compute f in the case of cylinders.

3.2. Cylinders via DPW. We choose $l = 0$ and $s = t$ for D in Equation (3.1). Thus ξ is

$$\xi = (\lambda^{-1} + \lambda) \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} \frac{dz}{z} .$$

By (3.5), (3.6) and (3.7), $s = 1/4$ or $s = 1/(4\cos(\gamma))$ or $s = 1/(4\cosh(q/2))$ in the respective space form. Furthermore, the ϕ in Equation (3.2) is

$$\begin{aligned} \phi &= \exp\left(\log z \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} (\lambda^{-1} + \lambda)\right) \\ &= \begin{pmatrix} \cosh(s(\lambda^{-1}+\lambda)\log z) & \sinh(s(\lambda^{-1}+\lambda)\log z) \\ \sinh(s(\lambda^{-1}+\lambda)\log z) & \cosh(s(\lambda^{-1}+\lambda)\log z) \end{pmatrix} , \end{aligned}$$

which has the explicit r-Iwasawa splitting

$$\phi = FB , \quad \text{where } B = \exp\left(\lambda(\log z + \log \bar{z}) \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix}\right) \text{ and}$$

$$F = \exp\left((\lambda^{-1}\log z - \lambda \log \bar{z}) \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix}\right) ,$$

for any $r \in (0, 1]$.

Inserting this F into equations (1.18), (1.21), and (1.23), one can explicitly compute the parametrizations for the surfaces and see that cylinders are produced. In the case of S^3, the cylinder wraps around onto itself to become of torus, since the geodesic lines in S^3 are closed loops.

In the case of $\mathbb{R}^3$, from Section 4.4 in [**17**] Delaunay surfaces are obtained from the dressing of cylinders. Similar arguments show the following results for S^3 and H^3.

THEOREM 3.2. *Delaunay surfaces in S^3 (resp. H^3) are obtained from the dressing of cylinders in S^3 (resp. H^3).*

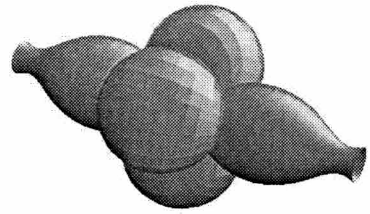

FIGURE 1. A Delaunay bubbleton in $\mathbb{R}^3$. (A cylinder bubbleton in R^3 is shown in Figure 1.) This figure was made by Y. Morikawa.

4. Bubbletons

4.1. Bubbletons via DPW. Let $\tilde{\Sigma}$ be the universal cover of the Riemann surface $\mathbb{C}^*$. Let $\phi(z,\lambda)$ be a solution of $d\phi = \phi\xi$ on $\tilde{\Sigma}$ with some initial condition $\phi(z_*,\lambda)$ at $z = z_*$ and let $\phi = F \cdot B$ be the r-Iwasawa splitting of ϕ, where ξ is as in (3.1). Choose $D/z \in \Lambda_r sl(2,\mathbb{C})$ for some $r \in (0,1]$ satisfying either (3.5) or (3.6) or (3.7), depending on the ambient space form. Let f be as in the Sym-Bobenko formula (1.18) or (1.21) or (1.23), respectively, made from the extended frame F. By Theorem 3.1, f is well defined on $\mathbb{C}^*$.

Consider the dressing $\phi \to \tilde{\phi} = h \cdot \phi$, where h is the matrix

$$h = \begin{pmatrix} \sqrt{\frac{1-\bar{\alpha}^2\lambda^2}{\lambda^2-\alpha^2}} & 0 \\ 0 & \sqrt{\frac{\lambda^2-\alpha^2}{1-\bar{\alpha}^2\lambda^2}} \end{pmatrix}, \quad \alpha \in \mathbb{C}^*. \tag{4.1}$$

Let $\tilde{\phi} = \tilde{F} \cdot \tilde{B}$ be the r-Iwasawa splitting of $\tilde{\phi}$ and let $\tilde{f}$ be the Sym-Bobenko formula (1.18) or (1.21) or (1.23), respectively, made from the extended frame $\tilde{F}$. We note that if $|\alpha| < r$ or $r^{-1} < |\alpha|$, then $h \in \Lambda_r SU(2)$. So the surface $\tilde{f}$ differs from f by only a rigid motion. Therefore we assume $r < |\alpha| < 1$. We note that in general $\tilde{f}$ is not well defined on $\mathbb{C}^*$.

DEFINITION 4.1. *Let $f : \mathbb{C}^* \longrightarrow \mathcal{M}^3$ and $\tilde{f} : \tilde{\Sigma} \longrightarrow \mathcal{M}^3$, where $\mathcal{M}^3$ is $\mathbb{R}^3$ (resp. S^3 or H^3), be CMC immersions derived from the above solutions ϕ and $\tilde{\phi}$. Let M_τ be the monodromy matrix of ϕ defined in Equation* (3.4). *Then $\tilde{f}$ is a* bubbleton *of the Delaunay surface f in $\mathbb{R}^3$ (resp. S^3, H^3) if $hM_\tau h^{-1} \in \Lambda_r SU(2)$.*

LEMMA 4.1. *The bubbleton $\tilde{f}$ satisfies the closing condition* (2.4) *or* (2.5) *or* (2.6), *so is defined on $\mathbb{C}^*$.*

PROOF. In the $\mathbb{R}^3$ case, we show that since $M_\tau|_{\lambda=1} = \pm id$ and $\partial_\lambda M_\tau|_{\lambda=1} = 0$ are satisfied, thus $(hM_\tau h^{-1})|_{\lambda=1} = \pm id$ and $\partial_\lambda(hM_\tau h^{-1})|_{\lambda=1} = 0$ are also satisfied. This follows from the following computations:

$$\left(hM_\tau h^{-1}\right)|_{\lambda=1} = h|_{\lambda=1}\,(\pm id)\,h^{-1}|_{\lambda=1} = \pm id \ ,$$
$$\partial_\lambda\left(hM_\tau h^{-1}\right)|_{\lambda=1} =$$

$$\left((\partial_\lambda h)\,M_\tau h^{-1}\right)|_{\lambda=1} + \left(h\,(\partial_\lambda M_\tau)\,h^{-1}\right)|_{\lambda=1} - \left(hM_\tau\left(h^{-1}\,(\partial_\lambda h)\,h^{-1}\right)\right)|_{\lambda=1} = 0 \ .$$

The H^3 and S^3 cases are similar, in fact they are even simpler, because no derivatives with respect to λ are involved. □

REMARK 4.2. *We saw in Lemma 4.1 that Definition 4.1 implies the bubbleton is topologically a cylinder.*

LEMMA 4.3. *$hM_\tau h^{-1}$ is in $\Lambda_r SU(2)$ if and only if M_τ is a lower triangular matrix at $\lambda = \pm\alpha$ and an upper triangular matrix at $\lambda = \pm\bar{\alpha}^{-1}$.*

PROOF. Let m_{ij} be the entries of M_τ. We have

$$hM_\tau h^{-1} = \begin{pmatrix} m_{11} & \frac{1-\bar{\alpha}^2\lambda^2}{\lambda^2-\alpha^2}m_{12} \\ \frac{\lambda^2-\alpha^2}{1-\bar{\alpha}^2\lambda^2}m_{21} & m_{22} \end{pmatrix}.$$

Thus $hM_\tau h^{-1}$ is in $\Lambda_r SU(2)$ if and only if $\frac{1-\bar{\alpha}^2\lambda^2}{\lambda^2-\alpha^2}m_{12}(\lambda)$ and $\frac{\lambda^2-\alpha^2}{1-\bar{\alpha}^2\lambda^2}m_{21}(\lambda)$ are holomorphic on $r < |\lambda| < r^{-1}$. This happens if and only if M_τ is a lower triangular matrix at $\lambda = \pm\alpha$ and an upper triangular matrix at $\lambda = \pm\bar{\alpha}^{-1}$. □

THEOREM 4.4. *There exist round cylinder bubbleton and Delaunay bubbleton surfaces for all three space forms.*

PROOF. The monodromy matrix M_τ is

$$M_\tau = \begin{pmatrix} \cos(2\pi X) + ilX^{-1}\sin(2\pi X) & iX^{-1}\sin(2\pi X)(s\lambda^{-1} + t\lambda) \\ iX^{-1}\sin(2\pi X)(s\lambda + t\lambda^{-1}) & \cos(2\pi X) - ilX^{-1}\sin(2\pi X) \end{pmatrix},$$

where

$$X = \sqrt{\frac{1}{4} - a + st(\lambda - \lambda^{-1})^2} \ ,$$

and

$$\begin{cases} \mathbb{R}^3 \text{ case: } a = 0 \\ S^3 \text{ case: } a = -4st\sin^2(\gamma) \\ H^3 \text{ case: } a = 4st\sinh^2(q/2) \end{cases} .$$

M_τ is in $\Lambda_r SU(2)$ for all $r \in (0, 1]$ and satisfies the closing conditions. We take

$$\alpha = \frac{\sqrt{\delta+4} - \sqrt{\delta}}{2} \in \mathbb{R} \cup i\mathbb{R} \setminus \{0, \pm 1, \pm i\} \text{ with } \delta = \frac{1}{st}\left(\frac{k^2-1}{4} + a\right) \ , \tag{4.2}$$

$$k^2 \geq \max\{-16st - 4a + 1, -4a + 1, 4\} \text{ and } k \in N \ .$$

We can immediately compute $M_\tau|_{\lambda=\pm\alpha,\pm\bar{\alpha}^{-1}} = -id$. We can choose r so that α satisfies $r < |\alpha| < 1$. Thus Lemma 4.3 and Definition 4.1 imply existence of bubbletons of cylinders and Delaunay surfaces. □

4.2. Computing the change of frame for the simple type dressing. Now we consider the explicit Iwasawa factorization of $h\phi$ with the simple type dressing h defined in Equation (4.1). This will lead to an explicit parametrization of the bubbletons of round cylinders in all three space forms. In this section, we allow ξ to be a general potential. Let Σ be a Riemann surface with coordinate z. Let ϕ be a solution of $d\phi = \phi\xi$ on Σ with some initial condition $\phi(z_*, \lambda)$ at z_* and let $\phi = F \cdot B$ be the r-Iwasawa splitting of ϕ, where $\xi \in \Lambda_r sl(2, \mathbb{C})$ and $r \in (0, 1]$. We assume $r < |\alpha| < 1$, for the same reason as in Section 4.1.

We consider $\mathbb{C}^2$ with the standard inner product $\langle \cdot, \cdot \rangle$, and e_1, e_2 forming the orthonormal basis

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

of $\mathbb{C}^2$. We define two subspaces V_1, V_2 spanned by specific vectors v_1, v_2 in $\mathbb{C}^2$:

$$V_1 = \left\{ a \cdot v_1 \;\middle|\; v_1 = \begin{pmatrix} \bar{A} \\ \lambda^{-1}\bar{\alpha}^{-1}\bar{B} \end{pmatrix}, a \in \mathbb{C} \right\},$$

$$V_2 = \left\{ a \cdot v_2 \;\middle|\; v_2 = \begin{pmatrix} -\lambda\alpha^{-1}B \\ A \end{pmatrix}, a \in \mathbb{C} \right\},$$

where

$$F|_{\lambda=\alpha} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad F \in \Lambda_r SU(2)$$

We now define projections $\pi_1, \pi_2, \tilde{\pi}_1, \tilde{\pi}_2$ and linear combinations $h, \tilde{h}$ of these projections.

$$(4.3) \qquad \begin{cases} \pi_1 & = \text{orthogonal projection to the span of } e_1 \\ \pi_2 & = \text{orthogonal projection to the span of } e_2 \\ h & = f^{-1/2}\pi_1 + f^{1/2}\pi_2 \end{cases},$$

$$(4.4) \qquad \begin{cases} \tilde{\pi}_1 & = \text{projection to } V_1 \text{ parallel to } V_2 \\ \tilde{\pi}_2 & = \text{projection to } V_2 \text{ parallel to } V_1 \\ \tilde{h} & = f^{-1/2}\tilde{\pi}_1 + f^{1/2}\tilde{\pi}_2 \end{cases},$$

where

$$f = \frac{\lambda^2 - \alpha^2}{1 - \bar{\alpha}^2\lambda^2}, \quad \alpha \in \mathbb{C}^* \quad .$$

Note that in general $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are non-orthogonal projections.

We have the following two lemmas, the first of which is obvious.

LEMMA 4.5.

$$\pi_1 \circ \pi_1 = \pi_1 \;, \pi_1 \circ \pi_2 = 0 \;, \pi_2 \circ \pi_1 = 0 \;, \pi_2 \circ \pi_2 = \pi_2 \;.$$

$$\tilde{\pi}_1 \circ \tilde{\pi}_1 = \tilde{\pi}_1 \;, \tilde{\pi}_1 \circ \tilde{\pi}_2 = 0 \;, \tilde{\pi}_2 \circ \tilde{\pi}_1 = 0 \;, \tilde{\pi}_2 \circ \tilde{\pi}_2 = \tilde{\pi}_2 \;.$$

LEMMA 4.6.

$$h^{-1} = f^{1/2}\pi_1 + f^{-1/2}\pi_2 \;, \qquad \tilde{h}^{-1} = f^{1/2}\tilde{\pi}_1 + f^{-1/2}\tilde{\pi}_2 \quad .$$

PROOF.

$$\begin{aligned}&(f^{-1/2}\pi_1 + f^{1/2}\pi_2) \circ (f^{1/2}\pi_1 + f^{-1/2}\pi_2)\\ &= \pi_1 \circ \pi_1 + f^{-1}\pi_1 \circ \pi_2 + f\pi_2 \circ \pi_1 + \pi_2 \circ \pi_2\\ &= \pi_1 + \pi_2 = id ,\end{aligned}$$

by Lemma 4.5. Similarly $(f^{1/2}\pi_1 + f^{-1/2}\pi_2) \circ (f^{-1/2}\pi_1 + f^{1/2}\pi_2) = id$. Replacing π_1 by $\tilde{\pi}_1$ and π_2 by $\tilde{\pi}_2$, we get the analogous result for $\tilde{h}^{-1}$. □

LEMMA 4.7. *In terms of the basis* e_1, e_2, *we can write* $\pi_j, \tilde{\pi}_j$ $(j = 1, 2)$ *in the following matrix forms:*

$$\pi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} ,$$

$$\tilde{\pi}_1 = \frac{1}{|A|^2 + |\alpha|^{-2}|B|^2} \begin{pmatrix} |A|^2 & \lambda\alpha^{-1}\bar{A}B \\ \lambda^{-1}\bar{\alpha}^{-1}A\bar{B} & |\alpha|^{-2}|B|^2 \end{pmatrix} ,$$

$$\tilde{\pi}_2 = \frac{1}{|A|^2 + |\alpha|^{-2}|B|^2} \begin{pmatrix} |\alpha|^{-2}|B|^2 & -\lambda\alpha^{-1}\bar{A}B \\ -\lambda^{-1}\bar{\alpha}^{-1}A\bar{B} & |A|^2 \end{pmatrix} .$$

PROOF. The matrix forms for π_1 and π_2 are evident. Regarding $\tilde{\pi}_1$ and $\tilde{\pi}_2$, we have $\tilde{\pi}_j \cdot w_i = w_i\delta_{ij}$, $\forall w_i \in V_i$ $(i, j = 1, 2)$, where δ_{ij} is the Kronecker δ function, and so for

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{C}^2,$$

we have

$$\tilde{\pi}_1 \cdot x = \frac{Ax_1 + \lambda\alpha^{-1}Bx_2}{|A|^2 + |\alpha|^{-2}|B|^2} \begin{pmatrix} \bar{A} \\ \lambda^{-1}\bar{\alpha}^{-1}\bar{B} \end{pmatrix} ,$$

$$\tilde{\pi}_2 \cdot x = \frac{-\lambda^{-1}\bar{\alpha}^{-1}\bar{B}x_1 + \bar{A}x_2}{|A|^2 + |\alpha|^{-2}|B|^2} \begin{pmatrix} -\lambda\alpha^{-1}B \\ A \end{pmatrix} .$$

Thus $\tilde{\pi}_1$ and $\tilde{\pi}_2$ have matrix forms as in the lemma. □

We now define a matrix $\mathcal{C} \in \Lambda_r \mathrm{SU}(2)$:

$$\mathcal{C} = \frac{1}{\sqrt{|T|^2 + 1}} \begin{pmatrix} e^{i\theta} & \lambda e^{i\theta}T \\ -\lambda^{-1}e^{-i\theta}\bar{T} & e^{-i\theta} \end{pmatrix} , \tag{4.5}$$

where $T = \dfrac{\bar{A}B\,(1 + \bar{\alpha}^2)}{\alpha|A|^2 - \bar{\alpha}|B|^2}$ and $\theta = \arg\left(|A|^2 - \dfrac{\bar{\alpha}}{\alpha}|B|^2\right) + \arg\left(\sqrt{-\alpha^2}\right)$.

THEOREM 4.8. *Let* ϕ *be a solution of* $d\phi = \phi\xi$ *on* Σ *with some initial condition* $\phi(z_*, \lambda) \in \Lambda_r \mathrm{SL}(2, \mathbb{C})$ *at* z_*, *and let* $\phi = FB$ *be the* r-Iwasawa splitting *of* ϕ. *We consider the dressing* $\phi \to h \cdot \phi$. *Then* $h\phi = (hF\tilde{h}^{-1}\mathcal{C}^{-1})(\mathcal{C}\tilde{h}B)$ *is* r-Iwasawa splitting *of* $h\phi$, *i.e.* $hF\tilde{h}^{-1}\mathcal{C}^{-1} \in \Lambda_r \mathrm{SU}(2)$ *and* $\mathcal{C}\tilde{h}B \in \Lambda_{r+} \mathrm{SL}(2, \mathbb{C})$, *where* $h, \tilde{h}, \mathcal{C}$ *are as in* (4.3), (4.4) *and* (4.5).

PROOF. $\mathcal{C}$ is already in $\Lambda_r SU(2)$, so we first show $hF\tilde{h}^{-1} \in \Lambda_r SU(2)$. F satisfies the reality condition $F(\bar{\lambda}^{-1}) = (F^{-1}(\lambda))^*$. We show that h and $\tilde{h}$ also satisfy the same reality condition:

$$\begin{aligned} h(\bar{\lambda}^{-1}) &= f(\bar{\lambda}^{-1})^{-1/2}\pi_1 + f(\bar{\lambda}^{-1})^{1/2}\pi_2 \ , \\ \left(h^{-1}(\lambda)\right)^* &= \left(f(\lambda)^{1/2}\pi_1 + f(\lambda)^{-1/2}\pi_2\right)^* \\ &= f(\bar{\lambda}^{-1})^{-1/2}\pi_1 + f(\bar{\lambda}^{-1})^{1/2}\pi_2 \ . \\ \tilde{h}(\bar{\lambda}^{-1}) &= f(\bar{\lambda}^{-1})^{-1/2}\tilde{\pi}_1(\bar{\lambda}^{-1}) + f(\bar{\lambda}^{-1})^{1/2}\tilde{\pi}_2(\bar{\lambda}^{-1}) \ , \\ \left(\tilde{h}^{-1}(\lambda)\right)^* &= \left(f(\lambda)^{1/2}\tilde{\pi}_1(\lambda) + f(\lambda)^{-1/2}\tilde{\pi}_2(\lambda)\right)^* \\ &= f(\bar{\lambda}^{-1})^{-1/2}\tilde{\pi}_1(\bar{\lambda}^{-1}) + f(\bar{\lambda}^{-1})^{1/2}\tilde{\pi}_2(\bar{\lambda}^{-1}) \ . \end{aligned}$$

Thus we have shown the reality condition for h and $\tilde{h}$. F is holomorphic on $r < |\lambda| < r^{-1}$. h, $\tilde{h}$ are holomorphic on $r < |\lambda| < r^{-1}$ with singularities only at $\lambda = \pm\alpha, \pm\bar{\alpha}^{-1}$. Thus we need only check that $hF\tilde{h}^{-1}$ has no singularities at $\lambda = \pm\alpha, \pm\bar{\alpha}^{-1}$.

$$\begin{aligned} hF\tilde{h}^{-1}\Big|_{\lambda=\pm\alpha,\pm\bar{\alpha}^{-1}} &= \left(\left(f^{-1/2}\pi_1 + f^{1/2}\pi_2\right) F \left(f^{1/2}\tilde{\pi}_1 + f^{-1/2}\tilde{\pi}_2\right)\right)\Big|_{\lambda=\pm\alpha,\pm\bar{\alpha}^{-1}} \\ &= \left(\pi_1 F\tilde{\pi}_1 + f\pi_2 F\tilde{\pi}_1 + f^{-1}\pi_1 F\tilde{\pi}_2 + \pi_2 F\tilde{\pi}_2\right)\big|_{\lambda=\pm\alpha,\pm\bar{\alpha}^{-1}} \ . \end{aligned}$$

The only possible singularities in this sum of four terms can occur in $f\pi_2 F\tilde{\pi}_1$ when $\lambda = \pm\bar{\alpha}^{-1}$ and in $f^{-1}\pi_1 F\tilde{\pi}_2$ when $\lambda = \pm\alpha$. But the reality condition of F and a calculation shows that in fact such singularities do not occur.

Finally we show $\mathcal{C}\tilde{h}B \in \Lambda_{+r}SL(2,\mathbb{C})$. B is in $\Lambda_{+r}SL(2,\mathbb{C})$, so we need only check that $\mathcal{C}\tilde{h}$ is in $\Lambda_{+,r}SL(2,\mathbb{C})$. We can easily see $\mathcal{C}\tilde{h} \in \Lambda_r SL(2,\mathbb{C})$ and is holomorphic on $0 < |\lambda| < r$ and continuous on $0 < |\lambda| \le r$, and a direct computation shows that

$$\mathcal{C}\tilde{h}|_{\lambda=0} = \begin{pmatrix} \rho_1 & 0 \\ 0 & {\rho_1}^{-1} \end{pmatrix} \ ,$$

$$\text{where } \rho_1 = \sqrt{\frac{|\alpha|^{-1}|A|^2 + |\alpha||B|^2}{|\alpha||A|^2 + |\alpha|^{-1}|B|^2}} \in \mathbb{R}^+ \ .$$

Thus the theorem is proven. □

Theorem 4.8 has the following corollary:

COROLLARY 4.9. *We have explicit parametrizations for round cylinder bubbletons in all three space forms using the* r-Iwasawa *splitting in Theorem 4.8, the extended frame in Section 3.2 and the Sym-Bobenko formulas* (1.18), (1.21) *and* (1.23).

REMARK 4.10. *Let* $\xi = \Sigma_{j\ge -1}\lambda^j A_j dz$ *be a holomorphic potential on* Σ *and let* ϕ *be a solution of* $d\phi = \phi\xi$ *with some initial condition* $\phi(z_*,\lambda) \in \Lambda_r \mathrm{SL}(2,\mathbb{C})$ *at* z_*. *Let* $\phi = F \cdot B$ *be r-Iwasawa splitting and let* f *be as in the Sym-Bobenko formula*

(1.18) *or* (1.21) *or* (1.23), *respectively, made from the extended frame* F. *Then the conformal factor of the metric* $4e^{2\mu}dzd\bar{z}$ *of* f *is (see* [**34**]*)*

$$4e^{2\mu} = 16\epsilon^2 e^{2u}|a_{-1}|^2 = 16\epsilon^2\rho^4|a_{-1}|^2 \quad , \tag{4.6}$$

where a_{-1} *is the upper right entry of* A_{-1}, $B|_{\lambda=0} = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}$ *and* $\epsilon = 1$ *(resp.* $\epsilon = \sin((\gamma_2 - \gamma_1)/2)$, $\epsilon = \sinh(-q/2)$*) in the case of* $\mathbb{R}^3$ *(resp.* S^3, H^3*).*

4.3. Equivalence of the simple type dressing and Bianchi's Bäcklund transformation on the round cylinder. In this section we prove the equivalence of the simple type dressing (4.1) and Bianchi's Bäcklund transformation in $\mathbb{R}^3$, when applied to a cylinder. (The latter is a bubbleton surface in the sense of [**59**].) We show that the metric, Hopf differential and mean curvature of Bianchi's Bäcklund transformation of a round cylinder are the same as those resulting from the simple type dressing of (4.1) with real α. For general CMC surfaces in $\mathbb{R}^3$, Burstall [**9**] has proven the equivalence of the simple type dressing (4.1) for α either real or pure imaginary and the Darboux transformation. Hertrich-Jeromin and Pedit [**30**] have proven that any of Bianchi's Bäcklund transformations of a CMC surface is a Darboux transformation of the surface, but not the converse.

In the S^3 and H^3 cases, we have not seen a notion of Bianchi's Bäcklund transformation. So we do not prove the equivalence for the S^3 and H^3 cases.

First we introduce the metric, Hopf differential and mean curvature of Bianchi's Bäcklund transformation using [**59**]. Using the notation in [**59**], we can write the first and second fundamental forms and the principal curvatures of a CMC surface as follows:

$$\begin{cases} ds^2 & = e^{2u}dwd\bar{w} \\ II & = e^u\left(\sinh(u)\,dx^2 + \cosh(u)\,dy^2\right) \\ k_1 & = e^{-u}\sinh(u)\,, \quad k_2 = e^{-u}\cosh(u) \end{cases} \quad ,$$

where $w = x + iy$. The Gauss equation becomes

$$2u_{w\bar{w}} + \sinh(2u) = 0 \quad . \tag{4.7}$$

In particular, in the round cylinder case we have $u = 0$. We do the Bäcklund transformation on the cylinder, and using Bianchi's Permutability formula ([**59**]), we have the new solution u_1 satisfying the Gauss equation (4.7):

$$\tanh\left(\frac{u_1}{2}\right) = \tanh(\beta_1)\frac{\cos(y\cosh(\beta_1))}{\cosh(x\sinh(\beta_1))} \quad ,$$

where $\beta_1 \in \mathbb{R}$. Under Bianchi's Bäcklund transformation, the mean curvature and the Hopf differential do not change. So the mean curvature and the Hopf differential of the bubbleton are $H = 1/2$ and $Q = (-1/4)dw^2$.

We consider the change of coordinate $\log z = w$. Thus we have the following:

$$\tanh\left(\frac{u_1}{2}\right) = \tanh(\beta_1)\frac{\cos(\mathrm{Im}(\log z)\cosh(\beta_1))}{\cosh(\mathrm{Re}(\log z)\sinh(\beta_1))} \quad ,$$
$$H = 1/2 \quad ,$$
$$Q = -\frac{1}{4z^2}dz^2 \quad .$$

THEOREM 4.11. *Bianchi's Bäcklund transformation of the round cylinder and the simple type dressing with real α of the round cylinder are the same surface.*

PROOF. Using Corollary 4.9, and Equations (4.6), (1.2) and (1.3), the simple type dressing by h has the following conformal factor for the metric $4e^{2\mu}dzd\bar{z}$, and the following mean curvature and Hopf differential:

$$\begin{aligned} 4e^{2\mu} &= 16e^{2u_1}|a_{-1}|^2 = 16\rho^4|a_{-1}|^2 \\ &= 16\left(\frac{|\alpha|^{-1}|A|^2+|\alpha||B|^2}{|\alpha||A|^2+|\alpha|^{-1}|B|^2}\right)^2|a_{-1}|^2 \\ &= 16\left(\frac{\alpha^{-1}|\cosh(X)|^2+\alpha|\sinh(X)|^2}{\alpha|\cosh(X)|^2+\alpha^{-1}|\sinh(X)|^2}\right)^2|a_{-1}|^2 \ , \\ H &= 1/2 \ , \\ Q &= -\frac{1}{4z^2}dz^2 \ , \end{aligned}$$

where $X = \frac{\alpha^{-1}\log z - \alpha\log\bar{z}}{4}$ and $a_{-1} = 1/(4z)$. Then $\tanh(u_1/2) = \frac{e^{u_1}-1}{e^{u_1}+1}$ implies that

$$\tanh\left(\frac{u_1}{2}\right) = \frac{(\alpha^{-1}-\alpha)\left(|\cosh(X)|^2 - |\sinh(X)|^2\right)}{(\alpha^{-1}+\alpha)\left(|\cosh(X)|^2 + |\sinh(X)|^2\right)} \ .$$

Using addition properties for the hyperbolic sine and cosine functions, we can rewrite the equation as follows:

$$\tanh\left(\frac{u_1}{2}\right) = \frac{(\alpha^{-1}-\alpha)\cosh\left(X-\bar{X}\right)}{(\alpha^{-1}+\alpha)\cosh\left(X+\bar{X}\right)} \ .$$

We have $X+\bar{X} = \mathrm{Re}(\log z)(\frac{\alpha^{-1}-\alpha}{2})$ and $X-\bar{X} = i\,\mathrm{Im}(\log z)(\frac{\alpha^{-1}+\alpha}{2})$. Thus the equation finally becomes

$$\begin{aligned} \tanh\left(\frac{u_1}{2}\right) &= \frac{\left(\frac{\alpha^{-1}-\alpha}{2}\right)\cosh\left(i\,\mathrm{Im}\,(\log z)\,\frac{\alpha^{-1}+\alpha}{2}\right)}{\left(\frac{\alpha^{-1}+\alpha}{2}\right)\cosh\left(\mathrm{Re}\,(\log z)\,\frac{\alpha^{-1}-\alpha}{2}\right)} \\ &= \frac{\sinh(\beta_1)\cos\left(\mathrm{Im}\,(\log z)\cosh(\beta_1)\right)}{\cosh(\beta_1)\cosh\left(\mathrm{Re}\,(\log z)\sinh(\beta_1)\right)} \ , \end{aligned}$$

where we set $\frac{\alpha^{-1}+\alpha}{2} = \cosh(\beta_1)$ and $\frac{\alpha^{-1}-\alpha}{2} = \sinh(\beta_1)$. Therefore both transformations give the same metric, mean curvature and Hopf differential. So the fundamental theorem of surface theory implies that the two transformations of the round cylinder are the same. □

4.4. Parallel surfaces of the bubbletons. In this section, we prove that the parallel surfaces of the round cylinder bubbletons are the same surface as the original bubbletons.

THEOREM 4.12. *The parallel surface of a round cylinder bubbleton is the same surface as the original cylinder bubbleton, up to a rigid motion, in any of the three space forms $\mathbb{R}^3$, S^3 and H^3.*

PROOF. Using Corollary 4.9 and Equations (4.6), (1.2) and (1.3), we can describe the conformal factor for the metric $4e^{2\mu}dzd\bar{z}$, mean curvature H and Hopf differential Q of the round cylinder bubbletons as follows:

$$4e^{2\mu} = 16\epsilon^2 e^{2u}|a_{-1}|^2 = 16\epsilon^2 \left(\frac{\alpha^{-1}|A|^2 + \alpha|B|^2}{\alpha|A|^2 + \alpha^{-1}|B|^2} \right)^2 |a_{-1}|^2 \ ,$$
$$H = b \ ,$$
$$Q = -\frac{1}{4z^2}\epsilon dz^2 \ ,$$

where $A = \cosh(\frac{\alpha^{-1}\log z - \alpha \log \bar{z}}{4})$, $B = \sinh(\frac{\alpha^{-1}\log z - \alpha\log\bar{z}}{4})$, and $\epsilon = 1$, $a_{-1} = 1/(4z)$ and $b = 1/2$ (resp. $\epsilon = \sin(-2\gamma), a_{-1} = 1/(4z\cos(\gamma))$ and $b = \cot(-\gamma)$, or $\epsilon = \sinh(-q), a_{-1} = 1/(4z\cosh(q/2))$ and $b = \coth(-q/2)$) in the case of $\mathbb{R}^3$ (resp. S^3, or H^3), and where $\alpha \in \mathbb{R}$ as in (4.2).

Using Theorem 1.1, Theorem 1.2 and Theorem 1.3, we can also describe the conformal factor for the metric $4e^{2\mu^*}dzd\bar{z}$, mean curvature H^* and Hopf differential Q^* of the bubbleton parallel surface as follows:

$$4e^{2\mu^*} = 16\epsilon^2 e^{-2u}|a_{-1}| = 16\epsilon^2 \left(\frac{\alpha|A|^2 + \alpha^{-1}|B|^2}{\alpha^{-1}|A|^2 + \alpha|B|^2} \right)^2 |a_{-1}|^2 \ ,$$
$$H^* = b \ ,$$
$$Q^* = -\frac{1}{4z^2}\epsilon dz^2 \ .$$

We consider the conformal change of the coordinate $z \to z\exp(\frac{2\pi i}{\alpha+\alpha^{-1}})$ on the parallel surface. Under this change, the mean curvature and Hopf differential do not change. For the metric, $|A|^2$ and $|B|^2$ change to $|B|^2$ and $|A|^2$, respectively, thus the conformal factor $4e^{2\mu^*}$ of the metric changes to

$$16\epsilon^2 \left(\frac{\alpha^{-1}|A|^2 + \alpha|B|^2}{\alpha|A|^2 + \alpha^{-1}|B|^2} \right)^2 |a_{-1}|^2 = 4e^{2\mu} \ .$$

Thus both surfaces have the same metric, mean curvature and Hopf differential up to this change of coordinate. Hence the fundamental theorem of surface theory implies the two surfaces are the same. $\square$

REMARK 4.13. *The parallel surface of a Delaunay bubbleton in general is not the same surface as the original Delaunay bubbleton. For example, if the bubbles of Delaunay bubbleton attach at a neck of the Delaunay surface, then the bubbles of the parallel Delaunay bubbleton attach at a bulge of the Delaunay surface.*

CHAPTER 2

Characterizations of Bianchi-Bäcklund transformations of constant mean curvature surfaces

1. Bianchi-Bäcklund transformations for CPC surfaces

Let Σ be a simply connected domain in $\mathbb{R}^2$, and let $f_\kappa : \Sigma \to \mathbb{R}^3$ be a constant positive gaussian curvature CPC immersion with $K = +1$ parametrized by lines of curvature such that

$$\left\{ \begin{array}{lcl} I & = & \cosh(u)^2 dx^2 + \sinh(u)^2 dy^2 \\ II & = & -\sinh(u)\cosh(u)(dx^2 + dy^2) \end{array} \right. , \tag{1.1}$$

where $u : \Sigma \to \mathbb{R}$ is a solution of the sinh-Gordon (Gauss) equation

$$\Delta u = -\sinh(u)\cosh(u) \ .$$

We consider a complex tangential line congruence of a CPC surface f_κ in the sense of Bianchi [2] (page 492):

$$f_\kappa^\lambda = f_\kappa + \lambda(\cos\varphi e_1 + \sin\varphi e_2) \ , \tag{1.2}$$

where $[e_1, e_2, e_3]$ is an orthonormal frame of f_κ, λ is a nonzero complex constant and $\varphi : \Sigma \to \mathbb{C}$ is a complex valued function. We denote by $(e_1^\lambda, e_2^\lambda, e_3^\lambda)^t$ the orthonormal frame of f_κ^λ. We now impose the following two conditions for the real two dimensional surface f_κ^λ in $\mathbb{C}^3$:

(i) The vector $f_\kappa^\lambda - f_\kappa$ is tangent to both surfaces, i.e. $\langle f_\kappa^\lambda - f_\kappa, e_3\rangle = \langle f_\kappa^\lambda - f_\kappa, e_3^\lambda\rangle = 0$, where $\langle\cdot,\cdot\rangle$ is the $\mathbb{C}$-bilinear extension of the standard inner product of $\mathbb{R}^3$.

(ii) The normal vectors e_3 and e_3^λ have constant angle at corresponding points, i.e. $\langle e_3, e_3^\lambda\rangle = c$, where c is a some constant complex number.

To determine λ and φ, we compute the Frenet equations of the CPC surface f_κ as follows:

$$\begin{array}{c} F_x = AF \\ F_y = BF \end{array} , \tag{1.3}$$

where

$$A = \begin{pmatrix} 0 & -u_y & -\sinh u \\ u_y & 0 & 0 \\ \sinh u & 0 & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 0 & u_x & 0 \\ -u_x & 0 & -\cosh u \\ 0 & \cosh u & 0 \end{pmatrix} \tag{1.4}$$

and $F = (e_1, e_2, e_3)^t = (f_{\kappa,x}/\cosh u, f_{\kappa,y}/\sinh u, e_1 \times e_2)^t$. We can compute the new orthonormal frame $(e_1^\lambda, e_2^\lambda, e_3^\lambda)^t$ of f_κ^λ as follows:

$$\begin{pmatrix} e_1^\lambda \\ e_2^\lambda \\ e_3^\lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\sigma & \sin\sigma \\ 0 & -\sin\sigma & \cos\sigma \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} . \tag{1.5}$$

We note that the right-hand 3 by 3 matrix represents a rotation around e_3, and the left-hand 3 by 3 matrix represents a rotation around $\cos\varphi e_1 + \sin\varphi e_2$, which is the direction vector of the tangential line congruence defined in Equation (1.2). Thus σ is an angle between the normal vector for f_κ and the normal vector for f_κ^λ. Therefore σ is constant by condition (ii). Constancy of σ and condition (i), i.e. $\langle df_\kappa^\lambda, e_3^\lambda\rangle = 0$, imply that

$$\sin\sigma\sin\varphi\mathcal{A} - \sin\sigma\cos\varphi\mathcal{B} + \cos\sigma\mathcal{C} = 0 \ , \tag{1.6}$$

where $\mathcal{A} = (\cosh u + \lambda u_y \sin\varphi)dx - \lambda u_x \sin\varphi dy - \lambda\sin\varphi d\varphi$, $\mathcal{B} = -\lambda u_y\cos\varphi dx + (\sinh u + \lambda u_x\cos\varphi)dy + \lambda\cos\varphi d\varphi$ and $\mathcal{C} = -\lambda\sinh u\cos\varphi dx - \lambda\cosh u\sin\varphi dy$. A calculation shows that

$$(\cot\sigma)^2 + \frac{1}{\lambda^2} = -1 \ . \tag{1.7}$$

We set the parameters $\cot\sigma = -i\cosh\beta$, $1/\lambda = \sinh\beta$ and $\varphi = i\theta_\beta$. Then we can rewrite Equation (1.2) as follows:

$$f_\kappa^\lambda = f_\kappa^\beta = f_\kappa + 1/\sinh(\beta)(\cosh(\theta_\beta)e_1 + i\sinh(\theta_\beta)e_2) \ , \tag{1.8}$$

where β is a non-zero complex constant and $\theta_\beta : \Sigma \to \mathbb{C}$ is a complex valued function. From now on we will use the notation $f_\kappa^\beta = f_\kappa^\lambda$ and $[e_1^\beta, e_2^\beta, e_3^\beta] = [e_1^\lambda, e_2^\lambda, e_3^\lambda]$.

Then, from Equation (1.6), we have the following differential equations for θ_β with parameter β:

$$\begin{cases} 2(\theta_\beta - u)_z &= e^\beta \sinh(\theta_\beta + u) \\ 2(\theta_\beta + u)_{\bar z} &= -e^{-\beta}\sinh(\theta_\beta - u) \end{cases} , \tag{1.9}$$

where $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$.

REMARK 1.1. *The solutions θ_β of the differential equation* (1.9), *which are produced by Bianchi-Bäcklund transformations, satisfy the sinh-Gordon equation* $\Delta\theta_\beta = -\sinh(\theta_\beta)\cosh(\theta_\beta)$.

We again consider a Bianchi-Bäcklund transformation of the surface f_κ^β defined in Equation (1.8) and define a new surface as follows:

$$f_\kappa^{\beta,\beta^*} = f_\kappa^\beta + 1/\sinh(\beta^*)(\cosh(\hat\theta_{\beta,\beta^*})e_1^\beta + i\sinh(\hat\theta_{\beta,\beta^*})e_2^\beta) \ , \tag{1.10}$$

where $\hat\theta_{\beta,\beta^*}$ is a solution of the following differential equation:

$$\begin{cases} 2(\hat\theta_{\beta,\beta^*} - \theta_\beta)_z &= e^{\beta^*}\sinh(\hat\theta_{\beta,\beta^*} + \theta_\beta) \\ 2(\hat\theta_{\beta,\beta^*} + \theta_\beta)_{\bar z} &= -e^{-\beta^*}\sinh(\hat\theta_{\beta,\beta^*} - \theta_\beta) \end{cases} . \tag{1.11}$$

Again we note that $\hat\theta_{\beta,\beta^*}$ is a solution of the complex sinh-Gordon equation $\Delta\hat\theta_{\beta,\beta^*} = -\sinh(\hat\theta_{\beta,\beta^*})\cosh(\hat\theta_{\beta,\beta^*})$.

Then we have the so-called Bianchi-permutability theorem:

THEOREM 1.2. (Bianchi [**2**], page 494) *Let f_κ be a* CPC *surface with $K = +1$ and a frame $[e_1, e_2, e_3]$ such that the first and second fundamental forms are as in Equation* (1.1). *Let θ_β (resp. θ_{β^*}) be a solution of Equation* (1.9) *with some initial condition $(x_0, y_0) \in \mathbb{R}^2$ and a complex parameter β (resp. β^*), and let $\hat{\theta}_{\beta,\beta^*}$ (resp. $\hat{\theta}_{\beta^*,\beta}$) be a solution of Equation* (1.11) *with complex parameter β^* (resp. β). Further, let $\hat{f}_\kappa^{\beta,\beta^*}$ (resp. $\hat{f}_\kappa^{\beta^*,\beta}$) be the surface defined by Equation* (1.10). *Then we have the following:*

$$f_\kappa^{\beta,\beta^*} = f_\kappa^{\beta^*,\beta} \quad . \tag{1.12}$$

Schematically,

$$\begin{tikzcd}
 & \theta \arrow[dr, "\beta^*"] & \\
u \arrow[ur, "\beta"] \arrow[dr, "\beta^*"'] & & \hat{u} \\
 & \theta^* \arrow[ur, "\beta"'] &
\end{tikzcd}$$

where $\hat{u} = \hat{\theta}_{\beta,\beta^*} = \hat{\theta}_{\beta^*,\beta}$. Furthermore, we can obtain a solution $\hat{u} = \hat{\theta}_{\beta,\beta^*} = \hat{\theta}_{\beta^*,\beta}$ of the sinh-Gordon equation, in terms of u, θ_β, θ_{β^*}, β and β^*, via the superposition formula:

$$\tanh\left(\frac{\hat{u} - u}{2}\right) = \tanh\left(\frac{\beta - \beta^*}{2}\right) \tanh\left(\frac{\theta_\beta - \theta_{\beta^*}}{2}\right) \quad , \tag{1.13}$$

where $\hat{u} = \hat{\theta}_{\beta,\beta^*} = \hat{\theta}_{\beta^*,\beta}$.

From now on, we will use the notation $\hat{f}_\kappa = f_\kappa^{\beta,\beta^*} = f_\kappa^{\beta^*,\beta}$. In general, $\hat{f}_\kappa$ is a surface in $\mathbb{C}^3$. To find a real surface, we consider the following ansatz:

$$\beta^* = -\bar{\beta} \quad . \tag{1.14}$$

Then from Equation (1.9), we can show $\theta_{\beta^*} = \pi i - \overline{\theta_\beta}$. Finally we have the following theorem:

THEOREM 1.3. (Bianchi [**2**], page 496) *Let f_κ be a* CPC *surface with $K = +1$ and frame $[e_1, e_2, e_3]$ such that the first and second fundamental forms are as in Equation* (1.1). *Let θ_β be a solution of Equation* (1.9) *with some initial condition $(x_0, y_0) \in \mathbb{R}^2$ and a nonzero complex parameter β. Let β^* be a parameter defined in Equation* (1.14), *and let $\theta_{\beta^*} = \pi i - \overline{\theta_\beta}$ be a solution of Equation* (1.9) *with the same initial condition $(x_0, y_0) \in \mathbb{R}^2$. Futher, let $\hat{f}_\kappa$ be the twice successive Bianchi-Bäcklund transformation defined by the above procedure. Then $\hat{f}_\kappa$ is a real* CPC *surface with $K = +1$, and $\hat{f}_\kappa = f_\kappa + \Lambda_\kappa \alpha_\kappa$, where*

$$\Lambda_\kappa = -\frac{\sinh(2\,\mathrm{Re}\,\beta)}{|\sinh(\beta)|^2[\cosh(2\,\mathrm{Re}\,\beta) + \cosh(2\,\mathrm{Re}\,\theta_\beta)]} \quad , \tag{1.15}$$

$$
\begin{aligned}
(1.16)\quad \alpha_\kappa = {} & [-\cosh(\theta_\beta)\cosh(\bar\beta) - \cosh(\overline{\theta_\beta})\cosh(\beta)]e_1 \\
& + i[-\cosh(\bar\beta)\sinh(\theta_\beta) + \cosh(\beta)\sinh(\overline{\theta_\beta})]e_2 \\
& - \sinh(2\operatorname{Re}\theta_\beta)e_3 \ .
\end{aligned}
$$

2. Bianchi-Bäcklund transformations for CMC surfaces

We now consider the Bianchi-Bäcklund transformation of a CMC surface. Let $f = f_\kappa + e_3$ be the parallel surface of a CPC surface f_κ, where e_3 is the unit normal of f_κ. It is well known that f is a surface with constant mean curvature $H = 1/2$, and with unit normal $n = -e_3$, and that (x, y) are isothermic coordinates of f, i.e. the first and second fundamental forms are as follows:

$$
(2.1)\qquad \left\{ \begin{array}{lcl} I & = & e^{2u}(dx^2 + dy^2) \\ II & = & e^u(\sinh(u)dx^2 + \cosh(u)dy^2) \end{array} \right. .
$$

Analogous to the Bianchi-Bäcklund transformation of a CPC surface, we have the following:

THEOREM 2.1. *Let $f = f_\kappa + e_3$ be a parallel* CMC *surface with $H = 1/2$ of a* CPC *surface f_κ, and let θ_β, β, θ_{β^*} and β^* be the functions and parameters defined in Theorem 1.3. Then $\hat f = f + g$ is a real* CMC *surface with $H = 1/2$, where*

$$
\begin{aligned}
(2.2)\quad g = {} & |\operatorname{csch}\beta|^2 \operatorname{sech}(\operatorname{Re}(\theta+\beta))\,\{\sinh(2\operatorname{Re}\beta)\cos(\operatorname{Im}(\theta+\beta))e_1 \\
& - \sinh(2\operatorname{Re}\beta)\sin(\operatorname{Im}(\theta+\beta))e_2 \\
& + [\cos(2\operatorname{Im}\beta)\cosh(\operatorname{Re}(\theta+\beta)) - \cosh(\operatorname{Re}(\theta-\beta))]e_3\} \ ,
\end{aligned}
$$

where $\operatorname{csch} x = 1/\sinh x$ *and* $\operatorname{sech} x = 1/\cosh x$.

PROOF. We can compute the normal $\hat e_3$ of $\hat f_\kappa$ by using Equation (1.5). Then we can rewrite the new CPC surface $\hat f_\kappa = f_\kappa + \Lambda_\kappa \alpha_\kappa$ as follows:

$$
(2.3)\qquad \hat f_\kappa + \hat e_3 = (f_\kappa + e_3) + (-e_3 + \Lambda_\kappa\alpha_\kappa + \hat e_3) \ .
$$

We set $\hat f = \hat f_\kappa + \hat e_3$, $f = f_\kappa + e_3$. Then $\Lambda_\kappa\alpha_\kappa - e_3 + \hat e_3$ is g defined as in Equation (2.2). And clearly f and $\hat f$ define CMC surfaces. □

Naturally, from the above theorem, we can define a Bianchi-Bäcklund transformation of a CMC surface:

DEFINITION 2.1. *Let f and $\hat f$ be as in Theorem 2.1 with paremeter either $\beta \in \mathbb{R}$ and $\beta = \beta_1 + i\pi/2$ with $\beta_1 \in \mathbb{R}$. Then $\hat f$ is called a Bianchi-Bäcklund transform of the* CMC *surface f. In particular, we call $\hat f$ defined by the parameter $\beta \in \mathbb{R}$ (resp. $\beta = \beta_1 + i\pi/2$ with $\beta_1 \in \mathbb{R}$) a real Bianchi-Bäcklund transform of a* CMC *surface (resp. imaginary Bianchi-Bäcklund transform of a* CMC *surface).*

In Figure 1, a new surface, which is an imaginary Bianchi-Bäcklund transformation of a nodoidal Delaunay surface, is again topologically a cylinder, i.e. the surface is well-defined on $\mathbb{S}^2 \setminus \{p_1, p_2\}$. In the case of a real Bianchi-Bäcklund transformation of a nodoidal Delaunay surface, for any parameter $\beta \in \mathbb{R}$ and $(x_0, y_0) \in \mathbb{R}^2$, the surface cannot be well-defined on $\mathbb{S}^2 \setminus \{p_1, p_2\}$.

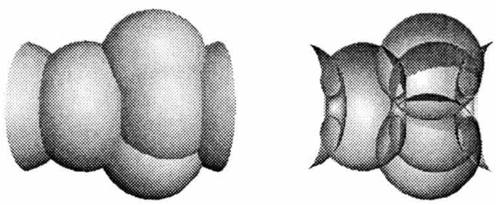

FIGURE 1. Imaginary Bianchi-Bäcklund transformation of a nodoidal Delaunay surface. Figures are constructed using the CMC-Lab program [**53**].

3. The equivalence of Bianchi-Bäcklund transformations, Darboux transformations and the simple type dressings of CMC surfaces

In this section we prove the equivalence of three transformations on CMC surfaces: Bianchi-Bäcklund transformations (Definition 2.1), Darboux transformations and the simple type dressings. These transformations arise from different contexts: the Bianchi-Bäcklund transformation is defined by complex tangential line congruence, the Darboux transformation is defined by sphere congruence, and the simple type dressing is defined by loop group actions.

From [**30**], we recall the definition of a Darboux transformation of a CMC surface.

DEFINITION 3.1. (Hertrich-Jeromin, Pedit [**30**] page 316) *If a congruence of 2-spheres is enveloped by two isothermic surfaces, the correspondence between its two envelopes being conformal and curvature line preserving, the surfaces are said to form a Darboux pair. Each of the two surfaces is called a Darboux transform of the other.*

Here we identify $\mathbb{R}^3$ with the imaginary part $\operatorname{Im}\mathbb{H}$ of the quaternion algebra $\mathbb{H}$ (see [**30**]). Then, also from [**30**], we have necessary and sufficient conditions for above definitions to be satisfied.

THEOREM 3.1. (Hertrich-Jeromin, Pedit [**30**]) *Let f be a CMC surface with mean curvature H, and let $f^c = f + \frac{1}{H}n$ be the parallel CMC surface of f, where n is the unit normal vector of f. Let $r \in \mathbb{R} \cup i\mathbb{R} \setminus \{0\}$ be a parameter and let ξ be a solution of the following Riccati equation with some initial condition $(x_1, y_1) \in \mathbb{R}^2$:*

$$d\xi = r^2 \xi (d\bar{f^c}) \xi - df \ . \tag{3.1}$$

Then $\hat{f} = f + \xi$ provides a Darboux transform of f. Converseoly, every Darboux transform $\hat{f}$ of f is obtained this way. In particular, we call $\hat{f}$ defined by $r \in \mathbb{R} \setminus \{0\}$ (resp. $r \in i\mathbb{R} \setminus \{0\}$) a Darboux transform of positive (resp. negative) type.

Bianchi-Bäcklund	Darboux	simple type
real BB	positive DB	$\alpha \in i\mathbb{R}$ with (l_1, l_2)
imaginary BB	negative DB	$\alpha \in \mathbb{R}$ with (l_1, l_2)

TABLE 1. Classification table. Imaginary BB is the new transformation.

REMARK 3.2. *We consider the distantce between $H\hat{f}$ and Hf^c. Then a computation shows that*

$$|H(\hat{f} - f^c)|^2 = |Hg - n|^2 = 1 + \frac{H^2}{r^2} \geq 0 \ , \tag{3.2}$$

where n and r^2 are defined in Theorem 3.1. Therefore we have the condition for r:

$$r^2 \geq 0 \ \text{ or } \ r^2 \leq -H^2 \ . \tag{3.3}$$

Then we have the following theorem:

THEOREM 3.3. (Hertrich-Jeromin, Pedit [**30**]) *Any Bianchi-Bäcklund transformation of a* CMC *surface is a Darboux transformation.*

The converse of Theorem 3.3 was not proven in [**30**]. Furthermore, Hertrich-Jeromin and Pedit conjectured that Darboux transformations of negative type are not Bianchi-Bäcklund transformations.

The simple type dressings on CMC surfaces arise from integrable systems (see [**9**]). Roughly speaking, the simple type dressing is a loop group action to obtain the new extended frame of a new CMC surface from a known extended frame of a CMC surface. From [**9**], the equivalence of Darboux transformations and the simple type dressings of a CMC surface is known.

THEOREM 3.4. (Burstall [**9**]) *Any transformation of a* CMC *surface is a Darboux transformation if and only if it is a simple type dressing.*

Now we prove the main theorem in this paper:

THEOREM 3.5. *Let f be a* CMC *surface with $H = 1/2$ in $\mathbb{R}^3$ parametrized by isothermic coordinates, i.e. the first and second fundamental forms are as in Equation* (2.1). *Then the following are all the same:*

(i) *the collection of all Bianchi-Bäcklund transformations,*
(ii) *the collection of all Darboux transformations,*
(iii) *the collection of all simple type dressings.*

PROOF. By the previous theorem, we need only show the equivalence of (1) and (2). From now on we identify $\mathbb{R}^3$ with $\operatorname{Im}\mathbb{H}$, thus the immersion f into $\mathbb{R}^3$ is considered as an immersion into $\operatorname{Im}\mathbb{H}$. Let g be the matrix valued function defined by Equation (2.2). and let $f^c = f + 2n$ be the prallel surface of f. We set $\xi = g$. Using a quaternionic calculation, e.g. $a \cdot b = -\langle a, b\rangle + a \times b$ with $a, b \in \operatorname{Im}\mathbb{H}$, and the differential equation (1.9) and the Frenet equations (1.3) of the CMC surface f, we can show the following equation: when using a parameter $\beta \in \mathbb{R}$, i.e. a real

Bianchi-Bäcklund transformation, we have

$$d\xi = \frac{\sinh(\beta)^2}{4}\xi(d\bar{f}^c)\xi - df \ . \tag{3.4}$$

When using a parameter $\beta = \beta_1 + i\pi/2$ with $\beta_1 \in \mathbb{R}$, i.e. a imaginary Bianchi-Bäcklund transformations, we have

$$d\xi = -\frac{\cosh(\beta_1)^2}{4}\xi(d\bar{f}^c)\xi - df \ . \tag{3.5}$$

Clearly Equation (3.4) (resp. Equation (3.5)) is a Darboux transformation of positive (resp. negative) type. Therefore all Darboux transformations can be obtained in this way. $\square$

REMARK 3.6. *In Theorem 3.5, we considered* CMC *surfaces with* $H = 1/2$. *Thus Equation* (3.3) *implies that* $r^2 \geq 0$ *or* $r^2 \leq -1/4$. *This condition can be satisfied by the coefficient* $\sinh(\beta)^2/4$ *in Equation* (3.4) *or the coefficient* $-\cosh(\beta)^2/4$ *in Equation* (3.5).

REMARK 3.7. *The Darboux transform in Definition 3.1 comes in a real three parameter family depending on* $r \in \mathbb{R} \setminus \{0\}$ *or* $r \in i\mathbb{R} \setminus \{0\}$ *and* $(x_1, y_1) \in \mathbb{R}^2$. *These parameters correspond to the parameters* $\beta \in \mathbb{R} \setminus \{0\}$ *or* $\beta = \beta_1 + i\pi/2$ *with* $\beta_1 \in \mathbb{R} \setminus \{0\}$ *and* $(x_0, y_0) \in \mathbb{R}^2$ *of the Bianchi-Bäcklund transform in Theorem 2.1 (and also to the* $\alpha \in i\mathbb{R} \setminus \{0\}$ *or* $\alpha \in \mathbb{R} \setminus \{0\}$ *and* $L = (l_1, l_2) \in \mathbb{R}^2$ *in the simple type dressing (see* [**9**]*)).*

CHAPTER 3

Constant mean curvature surfaces with Delaunay ends in 3-dimensional space forms

1. Conformal immersions into three dimensional space forms

1.1. Preliminaries and notation. Let M be a Riemann surface and G a matrix Lie group with Lie algebra $(\mathfrak{g}, [\,,\,])$. For α, $\beta \in \Omega^1(M, \mathfrak{g})$ smooth 1-forms on M with values in $\mathfrak{g}$, we define the $\mathfrak{g}$-valued 2-form $[\alpha \wedge \beta](X, Y) = [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)]$, X, $Y \in TM$. Let $L_g : h \mapsto gh$ be left multiplication in G, and $\theta : TG \to \mathfrak{g}$, $v_g \mapsto (dL_{g^{-1}})_g v_g$ the (left) Maurer–Cartan form. It satisfies the Maurer–Cartan equation

$$2\,d\theta + [\theta \wedge \theta] = 0. \tag{1.1}$$

For a map $F : M \to G$, the pullback $\alpha = F^*\theta$ also satisfies (1.1). The Maurer–Cartan Lemma asserts that if N is a connected and simply connected smooth manifold, then every solution $\alpha \in \Omega^1(N, \mathfrak{g})$ of (1.1) integrates to a smooth map $F : N \to G$ with $\alpha = F^*\theta$.

We complexify the tangent bundle TM and decompose $TM^{\mathbb{C}} = T'M \oplus T''M$ into $(1,0)$ and $(0,1)$ tangent spaces and write $d = \partial + \overline{\partial}$. Dually, we decompose

$$\Omega^1(M, \mathfrak{g}^{\mathbb{C}}) = \Omega'(M, \mathfrak{g}^{\mathbb{C}}) \oplus \Omega''(M, \mathfrak{g}^{\mathbb{C}}),$$

and accordingly split $\Omega^1(M, \mathfrak{g}^{\mathbb{C}}) \ni \omega = \omega' + \omega''$ into $(1,0)$ part ω' and $(0,1)$ part ω''. We set the $*$-operator on $\Omega^1(M, \mathfrak{g}^{\mathbb{C}})$ to

$$*\omega = -i\omega' + i\omega''.$$

1.2. Euclidean three space. We fix the following basis of $\mathfrak{sl}_2(\mathbb{C})$ as

$$\epsilon_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \; \epsilon_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \epsilon = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{1.2}$$

and will denote by $\langle \cdot\,, \cdot \rangle$ the bilinear extension of the Ad-invariant inner product of $\mathfrak{su}_2$ to $\mathfrak{su}_2^{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$ such that $\langle \epsilon,\, \epsilon \rangle = 1$. We further have

$$\begin{aligned} &\langle \epsilon_-,\, \epsilon_- \rangle = \langle \epsilon_+,\, \epsilon_+ \rangle = 0,\; \epsilon_-^* = -\epsilon_+, \\ &[\epsilon,\, \epsilon_-] = 2i\epsilon_-,\; [\epsilon_+,\, \epsilon] = 2i\epsilon_+ \text{ and } [\epsilon_-,\, \epsilon_+] = i\epsilon. \end{aligned} \tag{1.3}$$

We identify Euclidean three space $\mathbb{R}^3$ with the matrix Lie algebra $\mathfrak{su}_2$. The double cover of the isometry group under this identification is $\mathrm{SU}_2 \ltimes \mathfrak{su}_2$. Let $\mathbb{T}$ denote the stabiliser of $\epsilon \in \mathfrak{su}_2$ under the adjoint action of SU_2 on $\mathfrak{su}_2$. We shall view the two-sphere as $S^2 = \mathrm{SU}_2/\mathbb{T}$.

LEMMA 1.1. *The mean curvature H of a conformal immersion $f : M \to \mathfrak{su}_2$ is given by $2\,d * df = H\,[df \wedge df]$.*

PROOF. Let $U \subset M$ be an open simply connected set with coordinate $z : U \to \mathbb{C}$. Writing $df' = f_z dz$ and $df'' = f_{\bar z} d\bar z$, conformality is equivalent to $\langle f_z, f_z\rangle = \langle f_{\bar z}, f_{\bar z}\rangle = 0$ and the existence of a function $v \in C^\infty(U, \mathbb{R}^+)$ such that $2\langle f_z, f_{\bar z}\rangle = v^2$. Let $N : U \to \mathrm{SU}_2/\mathbb{T}$ be the Gauss map with lift $F : U \to \mathrm{SU}_2$ such that $N = F\,\epsilon\,F^{-1}$ and $df = vF(\epsilon_- dz + \epsilon_+ d\bar z)F^{-1}$. The mean curvature is $H = 2v^{-2}\langle f_{z\bar z}, N\rangle$ and the Hopf differential is $Q\,dz^2$ with $Q = \langle f_{zz}, N\rangle$. Hence $[df \wedge df] = 2iv^2 N dz \wedge d\bar z$. Then $\alpha = F^{-1}dF$ is given by

$$\alpha = \tfrac{1}{2v}\left((-v^2 H dz - 2\overline{Q}d\bar z)i\epsilon_- + (2Qdz + v^2 H d\bar z)i\epsilon_+ - (v_z dz - v_{\bar z} d\bar z)i\epsilon\right). \tag{1.4}$$

This allows us to compute $d * df = iv^2 H N dz \wedge d\bar z$ and proves the claim. □

The Maurer-Cartan equation $2d\alpha + [\alpha \wedge \alpha] = 0$ decomposes into $\epsilon_\pm$ components

$$v^2 H_{\bar z} = 2\overline{Q}_z,\; v^2 H_z = 2Q_{\bar z}, \tag{1.5}$$

called the Codazzi equations, and into an ϵ component

$$v^{-1}v_{z\bar z} - v^{-2}v_z v_{\bar z} + \tfrac14 v^2 H^2 - v^{-2}|Q|^2 = 0, \tag{1.6}$$

the Gauss equation. For $u = 2\log v$, (1.6) reads $u_{z\bar z} + \frac12 e^u H^2 - 2e^{-u}|Q|^2 = 0$.

1.3. The three sphere. We identify the three-sphere $S^3 \subset \mathbb{R}^4$ with $S^3 \cong \mathrm{SU}_2 \times \mathrm{SU}_2/\mathrm{D}$, where D is the diagonal. The double cover of the isometry group SO(4) is $\mathrm{SU}_2 \times \mathrm{SU}_2$ via the action $X \mapsto FXG^{-1}$. Let $\langle\cdot\,,\cdot\rangle$ denote the bilinear extension of the Euclidean inner product of $\mathbb{R}^4$ to $\mathbb{C}^4$ under this identification.

LEMMA 1.2. *Let $f : M \to S^3$ be a conformal immersion and $\omega = f^{-1}df$. The mean curvature H of f is given by $2\,d * \omega = H\,[\omega \wedge \omega]$.*

PROOF. Let $U \subset M$ be an open simply connected set with coordinate $z : U \to \mathbb{C}$. Writing $df' = f_z dz$ and $df'' = f_{\bar z} d\bar z$, conformality is equivalent to $\langle f_z, f_z\rangle = \langle f_{\bar z}, f_{\bar z}\rangle = 0$ and the existence of a function $v \in C^\infty(U, \mathbb{R}^+)$ such that $2\langle f_z, f_{\bar z}\rangle = v^2$. By left invariance, $\langle \omega', \omega'\rangle = \langle df', df'\rangle$, so conformality is $\langle\omega', \omega'\rangle = 0$. Take a smooth lift, that is, a pair of smooth maps $F, G : U \to \mathrm{SU}_2$ such that $f = FG^{-1}$, $df = vF\,(\epsilon_- dz + \epsilon_+ d\bar z)\,G^{-1}$ and $N = F\,\epsilon\,G^{-1}$. Setting $\alpha = F^{-1}dF$, $\beta = G^{-1}dG$, a computation gives

$$\begin{aligned} \alpha = (&-\tfrac12 v(H+i)dz - v^{-1}\overline{Q}d\bar z)i\epsilon_- & \beta = (&-\tfrac12 v(H-i)dz - v^{-1}\overline{Q}d\bar z)i\epsilon_- \\ &+(v^{-1}Qdz + \tfrac12 v(H-i)d\bar z)i\epsilon_+ & &+(v^{-1}Qdz + \tfrac12 v(H+i)d\bar z)i\epsilon_+ \\ &-(\tfrac12 v^{-1}(v_z dz - v_{\bar z}d\bar z))i\epsilon, & &-(\tfrac12 v^{-1}(v_z dz - v_{\bar z}d\bar z))i\epsilon. \end{aligned} \tag{1.7}$$

Using $\omega = G(\alpha - \beta)G^{-1}$ we obtain $d * \omega = iv^2 H G\,\epsilon\,G^{-1} dz \wedge d\bar z$. On the other hand, $[\omega \wedge \omega] = 2iv^2 G\,\epsilon\,G^{-1} dz \wedge d\bar z$, proving the claim. □

The Codazzi equations are the same as in (1.5), while the Gauss equation becomes

$$v^{-1}v_{z\bar z} - v^{-2}v_z v_{\bar z} + \tfrac14 v^2(H^2+1) - v^{-2}|Q|^2 = 0. \tag{1.8}$$

1.4. Hyperbolic three space. We identify hyperbolic three-space H^3 with the symmetric space $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2$ embedded in the real 4-space of Hermitian symmetric matrices as $[g] \hookrightarrow g\,g^*$, where g^* denotes the complex conjugate transpose of g. The double cover of the isometry group $\mathrm{SO}(3,1)$ of H^3 is $\mathrm{SL}_2(\mathbb{C})$ via the action $X \mapsto FXF^*$.

LEMMA 1.3. *For a conformal immersion $f : M \to H^3$ and $\omega = f^{-1}df$, the mean curvature H is given by $2\,d * \omega = i\,H\,[\omega \wedge \omega]$.*

PROOF. Let $U \subset M$ be an open simply connected set with coordinate $z : U \to \mathbb{C}$. Writing $df' = f_z dz$ and $df'' = f_{\bar z} d\bar z$, conformality is equivalent to $\langle f_z,\, f_z\rangle = \langle f_{\bar z},\, f_{\bar z}\rangle = 0$ and the existence of a function $v \in C^\infty(U, \mathbb{R}^+)$ such that $2\langle f_z,\, f_{\bar z}\rangle = v^2$. Take a smooth lift $F : U \to \mathrm{SL}_2(\mathbb{C})$ such that $f = F\,F^*$, $df = vF(\epsilon_- dz - \epsilon_+ d\bar z)F^*$, and $N = -F\,i\epsilon\,F^*$ for the normal. Setting $\alpha = F^{-1}dF$, a computation gives

$$
\begin{aligned}
\alpha = {} & (\tfrac{1}{2}v(H+1)dz + v^{-1}\overline{Q}d\bar z)\epsilon_- + (v^{-1}Qdz + \tfrac{1}{2}v(H-1)d\bar z)\epsilon_+ \\
& -(\tfrac{1}{2}v^{-1}(v_z dz - v_{\bar z} d\bar z))i\epsilon.
\end{aligned} \tag{1.9}
$$

Then $\omega = F^{*-1}(\alpha + \alpha^*)F^*$ together with $\alpha + \alpha^* = v(\epsilon_- dz - \epsilon_+ d\bar z)$ gives $[\omega \wedge \omega] = -2iv^2 F^{*-1}\,\epsilon\,F^* dz \wedge d\bar z$. On the other hand, $d * \omega = v^2 H F^{*-1}\,\epsilon\,F^* dz \wedge d\bar z$. $\square$

The Codazzi equations are the same as in (1.5), while the Gauss equation becomes

$$
v^{-1}v_{z\bar z} - v^{-2}v_z v_{\bar z} + \tfrac{1}{4}v^2(H^2 - 1) - v^{-2}|Q|^2 = 0. \tag{1.10}
$$

A direct consequence of the Codazzi equation is that when the mean curvature is constant, the Hopf differential is holomorphic. If (v, H, Q) is a solution of a Gauss equation, then so is $(v, H, \lambda^{-1}Q)$ for all $|\lambda| = 1$. Since the invariants (v, H, Q) determine an immersion up to isometries, a CMC surface comes in an S^1–family, the *associated family.*

2. Loop groups

We introduce various loop groups, state the Iwasawa decomposition and recall the dressing action [**10**] in our context. For expository accounts consult [**1**], [**27**] and [**51**].

For each real $0 < r \le 1$, the circle, open disk (interior) and open annulus are denoted respectively by

$C_r = \{\lambda \in \mathbb{C} : |\lambda| = r\}$, $\quad I_r = \{\lambda \in \mathbb{C} : |\lambda| < r\}$ and $A_r = \{\lambda \in \mathbb{C} : r < |\lambda| < 1/r\}$.

The *r-loop group* of $\mathrm{SL}_2(\mathbb{C})$ are the smooth maps of C_r into $\mathrm{SL}_2(\mathbb{C})$:

$$
\Lambda_r\mathrm{SL}_2(\mathbb{C}) = \mathcal{C}^\infty(C_r, \mathrm{SL}_2(\mathbb{C})).
$$

The Lie algebras of these groups are $\Lambda_r\mathfrak{sl}_2(\mathbb{C}) = \mathcal{C}^\infty(C_r, \mathfrak{sl}_2(\mathbb{C}))$. We will use the following two subgroups of $\Lambda_r\mathrm{SL}_2(\mathbb{C})$:

(i) For ϵ_+ defined in (1.2) let

$$
\mathcal{B} = \left\{B \in \mathrm{SL}_2(\mathbb{C}) : \mathrm{tr}(B) > 0 \text{ and } \mathrm{Ad}\,B(\epsilon_+) = \rho\,\epsilon_+,\ \rho \in \mathbb{R}^*_+\right\} \quad ,
$$

and define the *positive r-loops*

$$\Lambda_r^+\mathrm{SL}_2(\mathbb{C}) = \{B \in \Lambda_r\mathrm{SL}_2(\mathbb{C}) : B \text{ extends analytically to } B : I_r \to \mathrm{SL}_2(\mathbb{C}) \text{and } B(0) \in \mathcal{B}\}.$$

(ii) For $F : A_r \to \mathrm{SL}_2(\mathbb{C})$ define $F^* : A_r \to \mathrm{SL}_2(\mathbb{C})$ by $F^* : \lambda \mapsto \overline{F(1/\bar{\lambda})}^t$, and denote the *r-unitary loops* by

$$\Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C}) = \{F \in \Lambda_r\mathrm{SL}_2(\mathbb{C}) : F \text{ extends analytically to } F : A_r \to \mathrm{SL}_2(\mathbb{C}) \text{ and } F^* = F^{-1}\}.$$

Note that $F \in \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ implies $F|_{S^1} \in \mathrm{SU}_2$. For $r = 1$ we omit the subscript. Replacing $\mathrm{SL}_2(\mathbb{C})$ by $\mathrm{GL}_2(\mathbb{C})$, we define the analogous loop Lie subgroups of $\Lambda_r\mathrm{GL}_2(\mathbb{C})$. In this case, the subgroup $\mathcal{B} \subset \mathrm{GL}_2(\mathbb{C})$ consists of matrices with $\det B > 0$, $\mathrm{tr} B > 0$ and $\mathrm{Ad}\, B(\epsilon_+) = \rho\, \epsilon_+$ for some positive real number ρ, and $\Lambda_r^{\mathbb{R}}\mathrm{GL}_2(\mathbb{C})$ consists of $F \in \Lambda_r\mathrm{GL}_2(\mathbb{C})$ that extend analytically to $F : A_r \to \mathrm{GL}_2(\mathbb{C})$ and satisfy $F^* = \det(F)\, F^{-1}$. Corresponding to all the above subgroups, we analogously define Lie subalgebras of $\Lambda_r\mathfrak{gl}_2(\mathbb{C})$.

2.1. Iwasawa decomposition. Multiplication $\Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C}) \times \Lambda_r^+\mathrm{SL}_2(\mathbb{C}) \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ is a real-analytic diffeomorphism onto [**46**] (with respect to the natural smooth manifold structure, as in [**47**] and Chapter 3 of [**51**]). The unique splitting of an element $\Phi \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$

$$\Phi = FB,$$

with $F \in \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ and $B \in \Lambda_r^+\mathrm{SL}_2(\mathbb{C})$, will be called Iwasawa (or r-Iwasawa) decomposition. Since $\mathcal{B} \cap \mathrm{SU}_2 = \{\mathrm{Id}\}$, also $\Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C}) \cap \Lambda_r^+\mathrm{SL}_2(\mathbb{C}) = \{\mathrm{Id}\}$. The normalization $B(0) \in \mathcal{B}$ is a choice to ensure uniqueness of the Iwasawa factorization. We call F the r-unitary part of Φ.

2.2. Dressing action. The dressing action of $\Lambda_r^+\mathrm{SL}_2(\mathbb{C})$ on $\Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ is the composition of left multiplication and Iwasawa decomposition. For $h \in \Lambda_r^+\mathrm{SL}_2(\mathbb{C})$ and $F \in \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ let

$$hF = (h\#_r F)\, B \tag{2.1}$$

be the Iwasawa decomposition in $\Lambda_r\mathrm{SL}_2(\mathbb{C})$ with $h\#_r F \in \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$. We say $h\#_r F$ was obtained by r-dressing F by h.

3. Holomorphic potentials

Replacing $Q \mapsto \lambda^{-1}Q$ in the Maurer–Cartan forms (1.4), (1.7) and (1.9) gives $\Lambda\mathfrak{sl}_2(\mathbb{C})$-valued 1-forms of the form

$$\alpha_\lambda = (\alpha_1' + \lambda\alpha_1'')\, \epsilon_- + (\lambda^{-1}\alpha_2' + \alpha_2'')\, \epsilon_+ + (\alpha_3' + \alpha_3'')\, \epsilon. \tag{3.1}$$

Then $2\, d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda] = 0$ decouples into ϵ_-, ϵ_+ and ϵ components

$$\lambda d\alpha_1'' + 2i\lambda\alpha_3' \wedge \alpha_1'' = 2i\alpha_1' \wedge \alpha_3'' - d\alpha_1', \tag{3.2}$$

$$\lambda^{-1} d\alpha_2' + 2i\lambda^{-1}\alpha_2' \wedge \alpha_3'' = 2i\alpha_3' \wedge \alpha_2'' - d\alpha_2'', \tag{3.3}$$

$$d\alpha_3' + i\alpha_1' \wedge \alpha_2'' = i\alpha_2' \wedge \alpha_1'' - d\alpha_3''. \tag{3.4}$$

As the left sides of (3.2) and (3.3) are λ dependent while their right sides are not, both sides of (3.2) and (3.3) must be identically zero. Maps $F_\lambda : M \to \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ for which $F_\lambda^{-1}dF_\lambda$ is of the form (3.1) will be called r-unitary frames. Define

$$\mathcal{F}_r(M) := \left\{F_\lambda : M \to \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C}) : F_\lambda \text{ is an } r\text{-unitary frame}\right\}.$$

If $F_\lambda \in \mathcal{F}_r(M)$ then $\alpha_\lambda = F_\lambda^{-1}dF_\lambda$ is $\mathfrak{su}_2$-valued for $\lambda \in S^1$ and hence $\bar{\alpha}_1'' = \alpha_2'$, $\bar{\alpha}_1' = \alpha_2''$ and $\bar{\alpha}_3' = \alpha_3''$. In the following Lemma we recall a method from [**21**] that generates r-unitary frames. Define

$$\Lambda_r^{-1}\mathfrak{sl}_2(\mathbb{C}) = \left\{\xi \in \Lambda_r\mathfrak{sl}_2(\mathbb{C}) : \xi = \sum \xi_j\lambda^j,\, j \geq -1,\, \xi_{-1} \in \mathbb{C}\otimes\epsilon_+\right\}$$

and denote the holomorphic 1–forms on M with values in $\Lambda_r^{-1}\mathfrak{sl}_2(\mathbb{C})$ by

$$\Lambda_r\Omega(M) = \{\xi \in \Omega'(M, \Lambda_r^{-1}\mathfrak{sl}_2(\mathbb{C})) : d\xi = 0\}.$$

LEMMA 3.1. *Let M be a simply connected Riemann surface and $\xi \in \Lambda_r\Omega(M)$ and Φ be the solution of $d\Phi = \Phi\xi$ with initial condition $\Phi_0 \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$ at $z_0 \in M$. Then the map F obtained by Iwasawa decomposing $\Phi = FB$ pointwise on M is an r-unitary frame.*

PROOF. Expand $B = \sum B_j\lambda^j$, $j \geq 0$ and define $\alpha = F^{-1}dF$. Then $\alpha = B\xi B^{-1} - dBB^{-1}$. Now $\alpha' = B\xi B^{-1} - dB'B^{-1}$ and $\alpha'' = -dB''B^{-1}$, so the λ^{-1} coefficient of α' can only come from $\mathrm{Ad}B_0(\xi_{-1})$. If we set $\xi_{-1} = a\,\epsilon_+$ for $a \in \Omega^1(M,\mathbb{C})$, then $\mathrm{Ad}B_0(\xi_{-1}) = \rho a\epsilon_+$ for some function $\rho : M \to \mathbb{R}_+^*$. So equation (3.1) will hold if α'' has no ϵ_- component. But $\alpha'' = -dB''B^{-1}$ and thus the λ^0 coefficient comes from $-dB_0''B_0^{-1}$, which has no ϵ_- component. This proves the claim. □

4. The Sym–Bobenko Formulas

Given an r-unitary frame, an immersion can be obtained by formulas first found by Sym [**60**] for pseudo–spherical surfaces in $\mathbb{R}^3$ and extended by Bobenko [**6**] to CMC immersions in the three space forms. Our formulas differ from these, since we work in untwisted loop groups. Let $\partial_\lambda = \partial/\partial\lambda$.

THEOREM 4.1. *Let M be a simply connected Riemann surface and $F_\lambda \in \mathcal{F}_r(M)$ an r-unitary frame for some $r \in (0,1]$.*
(i) *Let $H \in \mathbb{R}^*$. Then for each $\lambda \in C_1$, the map $f : M \times C_1 \to \mathbb{R}^3$ defined by*

$$f_\lambda = -2i\lambda H^{-1}(\partial_\lambda F_\lambda)F_\lambda^{-1} \tag{4.1}$$

is a (possibly branched) conformal immersion $M \to \mathbb{R}^3$ with constant mean curvature H.
(ii) *Let $\mu \in C_1$, $\mu \neq 1$. Then for each $\lambda \in C_1$, the map $f : M \times C_1 \to S^3$ defined by*

$$f_\lambda = F_{\mu\lambda}F_\lambda^{-1} \tag{4.2}$$

is a (possibly branched) conformal CMC *immersion into S^3 with $H = i(1+\mu)/(1-\mu)$.*
(iii) *For $s \in [r, 1)$ and any $\lambda \in C_s$, the map $f : M \times C_s \to H^3$ defined by*

$$f_\lambda = F_\lambda\overline{F_\lambda}^t \tag{4.3}$$

is a (possibly branched) conformal CMC *immersion of M into H^3 with $H = (1+s^2)/(1-s^2)$.*

PROOF. (i) Expand $\alpha_\lambda = F_\lambda^{-1} dF_\lambda$ as in (3.1). Differentiating (4.1) gives

$$df_\lambda = 2iH^{-1} F_\lambda (\lambda^{-1}\alpha_2'\,\epsilon_+ - \lambda\alpha_1''\,\epsilon_-) F_\lambda^{-1}.$$

Hence $\langle df_\lambda', df_\lambda' \rangle = 0$ by (1.3), proving conformality.

Branch points occur when $\mathrm{Ad}F_\lambda(\lambda^{-1}\alpha_2'\,\epsilon_+ - \lambda\alpha_1''\,\epsilon_-)$ vanishes. Clearly f_λ takes values in $\mathfrak{su}_2$ for $|\lambda| = 1$. Further, $[df_\lambda \wedge df_\lambda] = (8i/H^2)\,\alpha_1'' \wedge \alpha_2' \mathrm{Ad}F_\lambda(\epsilon)$. Using (3.2) and (3.3) we obtain $d * df_\lambda = (4i/H)\,\alpha_1'' \wedge \alpha_2' \mathrm{Ad}F_\lambda(\epsilon)$. By Lemma (1.1), this proves (i).
(ii) Write $\alpha_\lambda = F_\lambda^{-1} dF_\lambda$ as in (3.1). Then for $\omega_\lambda = f_\lambda^{-1} df_\lambda$ we obtain

$$\omega_\lambda = \mathrm{Ad}F_\lambda(\alpha_{\mu\lambda} - \alpha_\lambda) = \mathrm{Ad}F_\lambda\left(\lambda^{-1}(\mu^{-1}-1)\alpha_2'\epsilon_+ + \lambda(\mu-1)\alpha_1''\epsilon_-\right).$$

Thus $\langle \omega_\lambda', \omega_\lambda' \rangle = 0$ by (1.3), proving conformality. Further, using (3.2) and (3.3) gives $d * \omega_\lambda = (\mu - \mu^{-1})\alpha_2' \wedge \alpha_1'' \mathrm{Ad}F_\lambda(\epsilon)$ while $[\omega_\lambda \wedge \omega_\lambda] = -2i(1-\mu^{-1})(1-\mu)\alpha_2' \wedge \alpha_1'' \mathrm{Ad}F_\lambda(\epsilon)$. Using Lemma 1.2 yields the formula for H.
(iii) Let $\omega_\lambda = f_\lambda^{-1} df_\lambda$. Since F_λ satisfies $F^* = F^{-1}$, we have $\overline{F_\lambda}^t = F_{1/\bar\lambda}^{-1}$ and

$$df_\lambda = F_\lambda(\alpha_\lambda - \alpha_{1/\bar\lambda})\overline{F_\lambda}^t = (\lambda - \bar\lambda^{-1}) F_\lambda(\alpha_1''\epsilon_- - \alpha_2'\epsilon_+)\overline{F_\lambda}^t,$$

proving conformality $\langle df_\lambda', df_\lambda' \rangle = 0$ by (1.3). Further $[\omega_\lambda \wedge \omega_\lambda] = 2i(\lambda - \bar\lambda^{-1})(\bar\lambda - \lambda^{-1})\alpha_2' \wedge \alpha_1'' \mathrm{Ad}F_{1/\bar\lambda}(\epsilon)$ while $d*\omega_\lambda = (\lambda\bar\lambda - \lambda^{-1}\bar\lambda^{-1})\alpha_2' \wedge \alpha_1'' \mathrm{Ad}F_{1/\bar\lambda}(\epsilon)$. Using Lemma 1.3 yields the formula for H and concludes the proof of the theorem. □

5. The generalized Weierstraß representation

Summarizing the above, by combining Lemma 3.1 and Theorem 4.1, CMC surfaces can be constructed in the following three steps: Let $\xi \in \Lambda_r\Omega(M)$, $z_0 \in M$ and $\Phi_0 \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$.
1. Solve the initial value problem

$$d\Phi = \Phi\xi,\ \Phi(z_0) = \Phi_0 \tag{5.1}$$

to obtain a unique *holomorphic frame* $\Phi : M \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$.
2. Iwasawa decompose $\Phi = FB$ pointwise on M to obtain a unique $F \in \mathcal{F}_r(M)$.
3. Insert F into one of the Sym–Bobenko formulas (4.1), (4.2) or (4.3).

We call a triple (ξ, Φ_0, z_0) *Weierstraß data*. A *potential* $\xi \in \Lambda_r\Omega(M)$ has an expansion, with closed forms a, $b \in \Omega'(M, \mathbb{C})$, of the form

$$\xi = (b + O(\lambda))\,\epsilon_- + (\lambda^{-1}a + O(1))\,\epsilon_+ + O(1)\,\epsilon. \tag{5.2}$$

The metric of the resulting CMC immersions is a nowhere vanishing multiple of $|a|^2$. The Hopf differential of the resulting CMC immersion is a constant multiple of the quadratic differential ab. To avoid branch points in examples we choose a closed form $a \in \Omega'(M, \mathbb{C}^*)$ and prescribe the umbilics as the roots of a closed form $b \in \Omega'(M, \mathbb{C})$.

If the initial condition $\Phi_0 \in \Lambda_r\mathrm{GL}_2(\mathbb{C})$, then $\Phi : M \to \Lambda_r\mathrm{GL}_2(\mathbb{C})$. The corresponding Iwasawa decomposition [**51**] of $\Lambda_r\mathrm{GL}_2(\mathbb{C})$ yields a unique map $F : M \to$

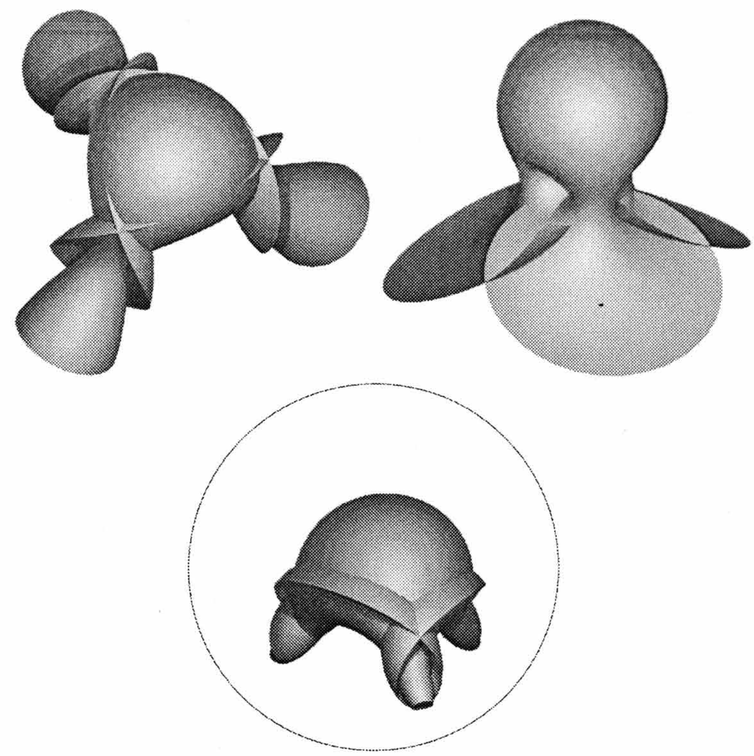

FIGURE 1. Smyth surfaces in $\mathbb{R}^3$, S^3 and H^3.

$\Lambda_r^{\mathbb{R}}\mathrm{GL}_2(\mathbb{C})$ and the Sym-Bobenko formulas (4.1), (4.2) and (4.3) must be modified and become, respectively,

$$f_\lambda = -2i\lambda H^{-1}\left((\partial_\lambda F_\lambda)F_\lambda^{-1} - \mathrm{tr}((\partial_\lambda F_\lambda)F_\lambda^{-1})\mathrm{Id}\right), \tag{5.3}$$

$$f_\lambda = \sqrt{\det(F_\lambda F_{\mu\lambda}^{-1})}\, F_{\mu\lambda}F_\lambda^{-1} \text{ and} \tag{5.4}$$

$$f_\lambda = |\det F_\lambda|^{-1} F_\lambda \overline{F_\lambda}^t. \tag{5.5}$$

The map $(\xi, \Phi_0, z_0) \mapsto \mathcal{F}_r(M)$ is surjective [**21**]. Injectivity fails, since the gauge group

$$\mathcal{G}_r(M) = \{g : M \to \Lambda_r^+\mathrm{SL}_2(\mathbb{C}) \text{ holomorphic }\} \tag{5.6}$$

acts by right multiplication on the fibers of this map: Indeed, on the holomorphic potential level, the gauge action $\Lambda_r\Omega(M) \times \mathcal{G}_r(M) \to \Lambda_r\Omega(M)$ is

$$\xi.g = g^{-1}\xi\, g + g^{-1}dg. \tag{5.7}$$

By Proposition 2.9 of [**10**], the dressing action (2.1) descends to $\mathcal{F}_r(M)$ and in the context of the generalized Weierstraß representation is the variation of the initial condition in (5.1) by left multiplication of $\Lambda_r\mathrm{SL}_2(\mathbb{C})$ or $\Lambda_r\mathrm{GL}_2(\mathbb{C})$.

Examples. We give the Weierstraß data for well known simply connected surfaces.

• **Spheres.** The triple $(\lambda^{-1}\epsilon_+ dz, \mathrm{Id}, 0)$ yields $F = (1+z\bar{z})^{-1/2}(\mathrm{Id}+z\lambda^{-1}\epsilon_+ + \bar{z}\lambda\epsilon_-)$, which inserted into (4.1), (4.2) or (4.3) yields round spheres in $\mathbb{R}^3$, S^3 or H^3.

• **Smyth surfaces.** Define $\xi = (\lambda^{-1}\epsilon_+ - cz^k\epsilon_-)dz$ on $M = \mathbb{C}$, for $c \in \mathbb{C}^*$ and $k \in \mathbb{N} \cup \{0\}$. Then $(\xi, \mathrm{Id}, 0)$ yields surfaces with intrinsic rotational symmetry [**7**], called Smyth surfaces [**57**]. In the $\mathbb{R}^3$ case, they are proper and complete [**62**], and their asymptotics are investigated in [**6**]. In Figure 1 we display Smyth surfaces in $\mathbb{R}^3$, S^3 and H^3 with $c = \frac{1}{5}$ and $k = 1$. When the target space is $\mathbb{R}^3$ (respectively S^3, H^3), we chose $\lambda_0 = 1$ (respectively $\mu = \lambda_0^{-2} = -1$, $\lambda_0 = 8/25$). In all three target spaces, $\Phi_0 = \mathrm{Id}$ and $z_0 = 0$.

For viewing surfaces in S^3, we stereographically project S^3 from its north pole to $\mathbb{R}^3 \cup \{\infty\}$. For surfaces in H^3, we use the Poincare model, which is stereographic projection of the Minkowski model in Lorentz space from the point $(0,0,0,-1)$ to the 3-ball of points $(0,x,y,z) \in \mathbb{R}^{3,1}$ such that $x^2+y^2+z^2 < 1$, equivalent to the 3-ball of points $(x,y,z) \in \mathbb{R}^3$ such that $x^2+y^2+z^2 < 1$. The graphics are produced by the fourth author's `cmclab` [**53**].

6. Invariant potentials & Monodromy

Let M be a connected Riemann surface with universal cover $\widetilde{M} \to M$ and let Δ denote the group of deck transformations. Let $\xi \in \Lambda_r\Omega(M)$ be a potential on M. Then $\gamma^*\xi = \xi$ for all $\gamma \in \Delta$. Let $\Phi : \widetilde{M} \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ be a solution of the differential equation $d\Phi = \Phi\xi$. Writing $\gamma^*\Phi = \Phi \circ \gamma$ for $\gamma \in \Delta$, we define $\chi(\gamma) \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$ by $\chi(\gamma) = (\gamma^*\Phi)\,\Phi^{-1}$. The matrix $\chi(\gamma)$ is called the *monodromy* matrix of Φ with respect to γ. If $\Psi : \widetilde{M} \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ is another solution of $d\Phi = \Phi\xi$ and $\widehat{\chi}(\gamma) = (\gamma^*\Psi)\,\Psi^{-1}$, then there exists a constant $C \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$ such that $\Psi = C\Phi$. Hence $\widehat{\chi}(\gamma) = C\chi(\gamma)C^{-1}$ and different solutions give rise to mutually conjugate monodromy matrices.

A choice of base point $\tilde{z}_0 \in \widetilde{M}$ and initial condition $\Phi_0 \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$ gives the *monodromy representation* $\chi : \Delta \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ of a holomorphic potential $\xi \in \Lambda_r\Omega(M)$. Henceforth, when we speak of the monodromy representation, or simply monodromy, we tacitly assume that it is induced by an underlying triple $(\xi, \Phi_0, \tilde{z}_0)$. Note that the invariance $\gamma^*\xi = \xi$ for all $\gamma \in \Delta$ is equivalent to $d\chi = 0$, ensuring that the monodromy is z–independent and thus well defined. It is shown in [**16**] that CMC immersions of open Riemann surfaces M can always be generated by such invariant holomorphic potentials.

If $\Phi = FB$ is the pointwise Iwasawa decomposition of $\Phi : \widetilde{M} \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$, then we shall need to study the monodromy of F to control the periodicity of the resulting CMC immersion given by (4.1), (4.2) or (4.3). A priori, we are not assured that the quantity $\mathcal{H}(\gamma) = (\gamma^*F)F^{-1}$ is z–independent for all $\gamma \in \Delta$. When χ is $\Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$–valued for all $\gamma \in \Delta$, then by uniqueness of the Iwasawa decomposition, one sees that $\chi = \mathcal{H}$. Assuming $\mathcal{H}$ is z–independent, there are well known closing conditions for the CMC immersions, first formulated in [**15**] for the target $\mathbb{R}^3$.

THEOREM 6.1. *Let M be a Riemann surface with universal cover $\widetilde{M}$, and let $F \in \mathcal{F}_r(\widetilde{M})$ be an r-unitary frame on $\widetilde{M}$ with monodromy $\mathcal{H} : \Delta \to \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$.*
(i) *Let f_λ be as in (4.1) and $\lambda_0 \in C_1$. Then $\gamma^* f_{\lambda_0} = f_{\lambda_0}$ for all $\gamma \in \Delta$ if and only if $\mathcal{H}(\gamma)|_{\lambda_0} = \pm\mathrm{Id}$ and $\partial_\lambda \mathcal{H}(\gamma)|_{\lambda_0} = 0$ for all $\gamma \in \Delta$.*
(ii) *Let f_λ be as in (4.2) and $\mu,\ \lambda_0 \in C_1$. Then $\gamma^* f_{\lambda_0} = f_{\lambda_0}$ for all $\gamma \in \Delta$ if and only if $\mathcal{H}(\gamma)|_{\lambda_0} = \mathcal{H}(\gamma)|_{\mu\lambda_0} = \pm\mathrm{Id}$ for all $\gamma \in \Delta$.*
(iii) *Let f_λ be as in (4.3) for $s \in [r, 1)$ and $\lambda_0 \in C_s$. Then $\gamma^* f_{\lambda_0} = f_{\lambda_0}$ for all $\gamma \in \Delta$ if and only if $\mathcal{H}(\gamma)|_{\lambda_0} = \pm\mathrm{Id}$ for all $\gamma \in \Delta$.*

PROOF. The above *closing conditions* (i) - (iii) are immediate consequences of the Sym-Bobenko formulas (4.1), (4.2) and (4.3). □

The closing conditions in Theorem 6.1 are invariant under conjugation. Furthermore, Theorem 6.1 also holds when (4.1), (4.2) or (4.3) are replaced respectively by (5.3), (5.4) or (5.5), and F and $\mathcal{H}$ take values in $\Lambda_r^{\mathbb{R}}\mathrm{GL}_2(\mathbb{C})$.

7. Cylinders and Delaunay surfaces

We apply the preceeding ideas and derive Weierstraß data for CMC cylinders in the three dimensional space forms. The domain will be the Riemann surface $M = \mathbb{C}^*$. Hence the group of deck transformations is generated by

$$\tau : \log z \mapsto \log z + 2\pi i\ . \tag{7.1}$$

7.1. Round cylinders. Define $\xi = (\lambda^{-1} a\epsilon_+ - a\epsilon_-) z^{-1} dz$, $a \in \mathbb{R}$. Then $(\xi, \mathrm{Id}, 1)$ yields the unitary frame

$$F = \exp\left(\log z(\lambda^{-1} a\epsilon_+ - a\epsilon_-) - \log \bar{z}(a\epsilon_+ - \lambda a\epsilon_-)\right)$$

with monodromy $\mathcal{H}(\tau) = \exp\left(2\pi i a((1 + \lambda^{-1})\epsilon_+ - (1 + \lambda)\epsilon_-)\right)$.

When the target is $\mathbb{R}^3$ we choose $\lambda_0 = 1$ and $a = 1/4$; for S^3 we choose $\lambda_0 = e^{-i\theta}$, $\mu = e^{2i\theta}$ for $\theta \in \mathbb{R}$ and $a = 1/(2\sqrt{2 + 2\cos\theta})$; for H^3 we choose $\lambda_0 = e^q$ for $q \in \mathbb{R}^*$ and $a = 1/(2\sqrt{2 + 2\cosh q})$. In each case, the appropriate condition of Theorem 6.1 holds and thus the resulting maps f_{λ_0} given by the corresponding Sym-Bobenko formula are conformal CMC immersions defined on $\mathbb{C}^*$. Inserting F into equations (4.1), (4.2) or (4.3) yields explicit parametrizations of round cylinders. In S^3, each cylinder becomes a covering of a torus, since its geodesic axis in S^3 is closed. Round cylinders are members of the family of Delaunay surfaces to which we turn next.

7.2. Delaunay surfaces. Delaunay surfaces are CMC surfaces of revolution. In the Euclidean case, these surfaces are described via the generalized Weierstraß representation in detail in [**35**]. Analogous arguments to those in [**35**] show that the Weierstraß data below generate also Delaunay surfaces in S^3 and H^3. Alternatively, since we compute Delaunay frames in section 11, one can explicitly verify this. For more details on Delaunay surfaces in the space forms we refer to [**42**], [**43**] and [**58**], to name just a few references. We give the Weierstraß data here: Define for $a,\, b \in \mathbb{C},\ c \in \mathbb{R}$

$$A = ic\epsilon + (a\lambda^{-1} + \bar{b})\epsilon_{+} - (\bar{a}\lambda + b)\epsilon_{-}. \tag{7.2}$$

Then $(z^{-1}Adz, \mathrm{Id}, 1)$ gives $\Phi = \exp(A\log z)$ with unitary monodromy $\exp(2\pi i A)$. The closing conditions (i) - (iii) of Theorem 6.1 are respectively:

- For $\mathbb{R}^3$, we choose $\lambda_0 = 1$ and $a,\, b \in \mathbb{C},\ c \in \mathbb{R}$ so that

$$c^2 + |a + \bar{b}|^2 = 1/4 \text{ and } ab \in \mathbb{R}. \tag{7.3}$$

- For S^3, we choose $\lambda_0 = e^{-i\theta}$, $\mu = e^{2i\theta}$ for $\theta \in (0, \frac{\pi}{2}] \subset \mathbb{R}$ and $a,\, b \in \mathbb{C},\ c \in \mathbb{R}$ so that

$$c^2 + |a + \bar{b}|^2 - 4ab\sin^2(\theta/2) = 1/4 \text{ and } ab \in \mathbb{R}. \tag{7.4}$$

- For H^3, we choose $\lambda_0 = e^q$ for $q \in \mathbb{R}^*$ and $a,\, b \in \mathbb{C},\ c \in \mathbb{R}$ so that

$$c^2 + |a + \bar{b}|^2 + 4ab\sinh^2(q/2) = 1/4 \text{ and } ab \in \mathbb{R}. \tag{7.5}$$

EXAMPLE 7.1. *Figure 1 depicts Delaunay surfaces in $\mathbb{R}^3$, S^3 and H^3. These are obtained from Weierstraß data $(z^{-1}Adz$, Id, $1)$ with $c = 0$ and the following data: $\mathbb{R}^3$: $a = .4$, $b = .1$ and $\lambda_0 = 1$; S^3: $a = .077$, $b = .442579$ and $\mu = \lambda_0^{-2} = e^{i\pi/2}$; H^3: $a = 1/(2\sqrt{34})$, $b = \sqrt{2/17}$ and $\lambda_0 = 1/4$.*

7.3. Weights. Let γ be an oriented loop about an annular end of a CMC surface in $\mathbb{R}^3$ or S^3 or H^3, and let $\mathcal{Q}$ be an immersed disk with boundary γ. Let η be the unit conormal of the surface along γ and let ν be the unit normal of $\mathcal{Q}$, the signs of both of them determined by the orientation of γ, and let H denote the mean curvature. Then the *flux* of the end with respect to a Killing vector field Y (in $\mathbb{R}^3$ or S^3 or H^3) is

$$w(Y) = \frac{2}{\pi}\left(\int_\gamma \langle \eta, Y\rangle - 2H\int_{\mathcal{Q}} \langle \nu, Y\rangle\right). \tag{7.6}$$

When the end is asymptotic to a Delaunay surface with axis ℓ and Y is the Killing vector field associated to unit translation along the direction of ℓ, we abbreviate $w(Y)$ to w and say that w is the *weight* of the end. This weight w changes sign when the orientation of γ is reversed, but otherwise is independent of the choices of γ and $\mathcal{Q}$ and hence a homology invariant [**43**], [**42**]. (Since the mean curvature in [**43**] and [**42**] is defined as the sum of the principal curvatures, rather than the average, we must replace H by $2H$ in the formulas for the weights there.)

As the unitary Delaunay frame has the simple form $F|_{S^1} = \exp(iA\arg z)$ for $z \in S^1$, the image of S^1 under the resulting immersion is a geodesic circle which can be explicitly computed. Also, one can explicitly compute the normal vector to the surface along the image of S^1, allowing us to compute the weights of Delaunay surfaces:

LEMMA 7.1. *The weights of Delaunay surfaces in $\mathbb{R}^3$, S^3 and H^3 generated by Weierstraß data $(z^{-1}Adz, \mathrm{Id}, 1)$ are given respectively by the following quantities:*

$$
\begin{aligned}
w &= \frac{16ab}{|H|}, \\
w &= \frac{16ab}{\sqrt{H^2+1}}, \\
w &= \frac{16ab}{\sqrt{H^2-1}}.
\end{aligned}
\tag{7.7}
$$

PROOF. Because $\Phi = \exp(\log z\, A) \in \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ when $|z| = 1$, by uniqueness of the Iwasawa factorization, also $F = \exp(\log z\, A)$ on S^1. Thus by the Sym-Bobenko formulas for the CMC immersion f and by the formulas for the normal N in the proofs of Lemmas 1.1, 1.2 and 1.3, we also know f and N explicitly for $z \in S^1$. Hence, defining γ to be the counterclockwise loop about the circle $f(\{|z| = 1\})$, and choosing Q to be the totally geodesic disk with boundary γ, we can explicitly compute the weight (7.6). In the case of $\mathbb{R}^3$, the computation is as follows: To simplify the computations, we may assume without loss of generality that $a, b \in \mathbb{R}$ and $c = 0$, and that both b and H are positive, see [**35**]. Then

$$
\begin{aligned}
F(z \in S^1, \lambda = 1) &= \mathrm{Re}(\sqrt{z})\,\mathrm{Id} + i\,\mathrm{Im}(\sqrt{z})(\epsilon_+ - \epsilon_-), \\
\partial_\lambda F(z \in S^1)\big|_{\lambda=1} &= -2ia\,\mathrm{Im}(\sqrt{z})(\epsilon_+ + \epsilon_-),
\end{aligned}
$$

and the resulting immersion (4.1) and normal are given by

$$
\begin{aligned}
f(z \in S^1, \lambda = 1) &= -4a\,H^{-1}\,\mathrm{Im}(\sqrt{z})\,\left(\mathrm{Re}(\sqrt{z})(\epsilon_+ + \epsilon_-) + \mathrm{Im}(\sqrt{z})\,\epsilon\right), \\
N(z \in S^1, \lambda = 1) &= 2(\mathrm{Re}^2(\sqrt{z}) - 1)\,\epsilon - 2\,\mathrm{Re}(\sqrt{z})\,\mathrm{Im}(\sqrt{z})(\epsilon_+ + \epsilon_-).
\end{aligned}
$$

It follows that the circular disk Q with boundary γ has radius $2|a/H|$ and normal $\nu = i(\epsilon_+ - \epsilon_-)$. Furthermore, $Y = \nu$, and $\eta = \nu$ or $\eta = -\nu$ when $a > 0$ respectively $a < 0$. Then

$$
w(Y) = \frac{2}{\pi}\left(\int_\gamma \langle \eta, Y \rangle - 2H \int_Q \langle \nu, Y \rangle\right) = \frac{8a}{H} - \frac{16a^2}{H} = \frac{16ab}{H}.
$$

□

COROLLARY 7.2. *The weights of Delaunay surfaces in $\mathbb{R}^3$, S^3 or H^3 are subject to the following bounds, respectively:*

$$
\begin{aligned}
w &\le \frac{1}{|H|}, \\
-2(\sqrt{H^2+1} + |H|) \le w &\le 2(\sqrt{H^2+1} - |H|), \\
w &\le 2(|H| - \sqrt{H^2-1}).
\end{aligned}
\tag{7.8}
$$

PROOF. For the $\mathbb{R}^3$ case, by (7.3) it follows that $|a|^2 + |b|^2 + 2ab \le \frac{1}{4}$ and so $4ab \le \frac{1}{4}$. Thus $w \le \frac{1}{|H|}$ by the first equation of (7.7). The arguments are similar for the other two space forms, using (7.4) and (7.5) and the formulas for the mean curvature in Theorem 4.1. □

An unduloid (respectively nodoid, twice punctured round sphere, round cylinder) is produced when $ab > 0$ (respectively $ab < 0$, $ab = 0$, $c = 0$ and $|a| = |b|$).

8. Asymptotics for Delaunay Ends

We give a sketch of how we prove that a surface has a Delaunay end and defer the technical proof to section 12. We must first define some norms: Let $X(\lambda) : E \to \mathfrak{gl}_2(\mathbb{C})$ for $E \subset \mathbb{C}$ and define the operator norm $||X||_E$ by

$$||X||_E := \sup_{\lambda \in E} |X| \,, \quad \text{where } |X| := \max_{w \in \mathbb{C}^2; |w|=1} \sqrt{\langle X\, w,\, \overline{X\, w} \rangle}$$

with $\langle \cdot, \cdot \rangle$ the bilinear extension of the standard $\mathbb{R}^2$ inner product. This is a multiplicative norm and is equivalent to the Euclidean norm on matrices in $\mathfrak{gl}_2(\mathbb{C})$.

To show that some associated family g_λ has a Delaunay end for $\lambda = \lambda_0$, say at a puncture $z = 0$, we compare it to an associated family of Delaunay surfaces f_λ^{D} and show that

$$(8.1) \qquad \begin{array}{ll} \lim_{z\to 0} |g_{\lambda_0} - f^{\mathrm{D}}_{\lambda_0}| = 0 & \text{for the targets } \mathbb{R}^3,\ S^3, \\ \lim_{z\to 0} |g_{\lambda_0} - f^{\mathrm{D}}_{\lambda_0}|\, |f^{\mathrm{D}}_{\lambda_0}|^{-1} = 0 & \text{for the target } H^3. \end{array}$$

Assume the immersions f^{D}_λ, g_λ are generated by unitary frames F_λ and G_λ respectively. Using the Sym-Bobenko formulas (4.1) (4.3) (4.2), the convergence in (8.1) is equivalent to

$$(8.2) \qquad \left.\begin{array}{lc} \mathbb{R}^3 \text{ case}: & \left|F_\lambda\, \partial_\lambda(F_\lambda^{-1} G_\lambda)\, G_\lambda^{-1}\right| \\ S^3 \text{ case}: & \left|G_{\mu\lambda}\left(G_\lambda^{-1} F_\lambda - G_{\mu\lambda}^{-1} F_{\mu\lambda}\right) F_\lambda^{-1}\right| \\ H^3 \text{ case}: & \left|G_\lambda \left(\overline{G_\lambda}^t \overline{F_\lambda}^{t^{-1}} - G_\lambda^{-1} F_\lambda\right) \overline{F_\lambda}^t\right| \left|F_\lambda \overline{F_\lambda}^t\right|^{-1} \end{array}\right\} \xrightarrow[z\,\to\,0]{\lambda\,=\,\lambda_0} 0.$$

Sufficient conditions for (8.2) to hold are: $||\partial_\lambda(F_\lambda^{-1}G_\lambda)||_{C_1} \to 0$ as $z \to 0$ for $\mathbb{R}^3$; $||F_\lambda^{-1}G_\lambda - \mathrm{Id}||_{C_1} \to 0$ as $z \to 0$ for S^3; $||F_\lambda^{-1}G_\lambda - \mathrm{Id}||_{C_{|\lambda_0|}} \to 0$ as $z \to 0$ for H^3.

REMARK 8.1. *We prove only a weaker "relative" convergence in the H^3 case. Since $|\lambda_0| \neq 1$ in this case, $|F_\lambda|$ and $|G_\lambda|$ do not remain uniformly bounded on $C_{|\lambda_0|}$, as they will for the other two targets, restricting us to the weaker result in H^3. Here is only one of several places where $|\lambda_0| \neq 1$ makes the H^3 case signifigantly different from the other two cases.*

While the extended unitary frame of a Delaunay surface can be computed, see section 11, these conditions above remain hard to check since G_λ is generally not explicitly known. If the unitary frames are somehow related, one can say more. A sufficient condition for this to be the case is if the potentials are related as follows: Let A be as in (7.2) and assume that g_λ and f_λ^{D} are generated by potentials η respectively $Az^{-1}dz$, and that $\eta = Az^{-1}dz + \vartheta$ with ϑ holomorphic in a neighbourhood U of $z = 0$. If we write $z^A = \exp(A \log z)$, then we modify in Lemma 8.2 a well-known result from the theory of regular singular points which asserts that there exists a solution of $d\Psi = \Psi\, \eta$ of the form

$$(8.3) \qquad \Psi = z^A\, P$$

with a map $P = P(z, \lambda)$ that is holomorphic in U and $P(0, \lambda) = \mathrm{Id}$.

LEMMA 8.2. *Let U be an open neighbourhood of $z_e \in \mathbb{C}$, $U^* = U \setminus \{z_e\}$ and $\rho \in (0,1)$. Suppose that $\xi \in \Lambda_r\Omega(U^*)$ for all $r \in (\rho, 1/\rho)$ has a simple pole at z_e and the following expansion in z at z_e:*

$$\xi = A\,(z - z_e)^{-1}dz + \sum_{j\geq 0} \xi_j(\lambda)(z - z_e)^j dz\,,$$

where $A = A(\lambda)$ and λ_0 are as in Subsection 7.2 and either (7.3) *or* (7.4) *or* (7.5) *holds. Then there exists an $r_0 \in (\rho, 1)$ and an open set V with $z_e \in V \subset U$ and a map $P : V \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ for all $r \in (r_0, 1) \cup (1, 1/r_0)$ such that $P = I + \mathrm{O}((z - z_e)^1)$ at z_e and*

$$(A\,(z - z_e)^{-1}dz).P = \xi\,. \tag{8.4}$$

Furthermore, P is holomorphic in A_{r_0} except at a finite number of points in C_1. In the case that $\xi_0(\lambda) \equiv 0$ and $R < \min(|\lambda_0|^{\pm 1})$ and that λ_0 and A are chosen so that (7.5) *holds, then r_0 can be chosen strictly less than* $\min(|\lambda_0|^{\pm 1})$.

PROOF. As this result is well known (see [**17**], for example), we merely outline the proof. Setting $P = \sum_{k\geq 0} P_k (z - z_e)^k$ yields the recursive equations

$$kP_k - [P_k, \xi_{-1}] = \sum_{j=0}^{k-1} P_j \xi_{k-1-j}, \quad k \geq 1, \quad P_0 = \mathrm{Id}. \tag{8.5}$$

The solvability for all P_k follows from the fact that the difference of the two eigenvalues of A over C_1 is an integer at only isolated points. The series for P converges by standard ODE arguments, see [**29**], [**22**].

In fact, P is defined and holomorphic in λ and $\det P = 1$, for any nonzero $\lambda \neq \lambda_{k,\pm}$, where

$$\lambda_{k,\pm} = \frac{1}{2}\left(\tfrac{k^2-1}{4ab} - 4\Upsilon + 2 \pm \sqrt{\left(\tfrac{k^2-1}{4ab} - 4\Upsilon + 2\right)^2 - 4}\right),\ k \in \mathbb{N}.$$

with $\Upsilon = 0$ or $\Upsilon = \sin^2(\theta/2)$ or $\Upsilon = -\sinh^2(q/2)$ for $\mathbb{R}^3$ or S^3 or H^3, respectively. Defining $\mathcal{P} = \{\lambda_{k,\pm} : |\lambda_{k,\pm}| < 1\}$, then r_0 can be chosen to be the maximum norm of elements in $\mathcal{P}$.

When $\xi_0 = 0$, then, with $k = 1$ in Equation (8.5), we find that $P_1 = 0$, and hence P is nonsingular at $\lambda_{1,\pm}$. Since λ_0 is in the set $\{\lambda_{1,+}, \lambda_{1,-}\}$, the final statement of the lemma follows. □

Returning to the asymptotics, let us put Lemma 8.2 to work. Let Ψ be the special solution in (8.3). If $\Psi = G\,B$ and $z^A = F_{\mathrm{D}}B_{\mathrm{D}}$ are the pointwise r-Iwasawa decompositions in $U^* = U \setminus \{0\}$ for r close to 1, then (8.3) yields an r-Iwasawa splitting

$$B_{\mathrm{D}} P B_{\mathrm{D}}^{-1} = F_{\mathrm{D}}^{-1} G\,B\,B_{\mathrm{D}}^{-1}. \tag{8.6}$$

Since Iwasawa splitting is an analytic diffeomorphism that preserves C^1 convergence (see [**51**]), it suffices to show

$$\lim_{z\to 0} \|B_{\mathrm{D}} P B_{\mathrm{D}}^{-1} - \mathrm{Id}\|_{C_r} = 0\,, \quad \lim_{z\to 0} \|\partial_\lambda (B_{\mathrm{D}} P B_{\mathrm{D}}^{-1})\|_{C_r} = 0 \tag{8.7}$$

to conclude that

$$\lim_{z\to 0} \|F_{\mathrm{D}}^{-1} G - \mathrm{Id}\|_{A_r} = 0\,, \quad \lim_{z\to 0} \|\partial_\lambda (F_{\mathrm{D}}^{-1} G)\|_{A_r} = 0\,. \tag{8.8}$$

Then if $\lambda_0 \in A_r$, the convergence in (8.1) is shown. (The H^3 case is more difficult, because then $\lambda_0 \notin A_r$, but we remedy this in Theorem 8.3 and its proof in section 12.)

The convergence in (8.7) can be analysed since we know enough about the map $P(z, \lambda)$ and we explicitly know B_{D} from Theorem 11.1. This is the essential idea behind the following asymptotics result, which we state and prove for general solutions Ψ of $d\Psi = \Psi\xi$ in $\Lambda_r \mathrm{GL}_2(\mathbb{C})$.

THEOREM 8.3. *Let U be an open neighborhood of $z_e \in \mathbb{C}$, and A as in* (7.2) *with $ab \neq 0$. Let*

$$\xi = A(z - z_e)^{-1} dz + \sum_{j\ge 0} \xi_j(\lambda)(z - z_e)^j dz \in \Lambda_r \Omega(U).$$

Let P be as in Lemma 8.2 satisfying (8.4). *Suppose $r < 1$ is chosen so that P is nonsingular on $A_r \setminus C_1$. Let G be the r-unitary frame obtained from r-Iwasawa splitting a solution $\Psi \in \Lambda_r \mathrm{GL}_2(\mathbb{C})$ of $d\Psi = \Psi\,\xi$ that is defined on A_r with* $\det(\Psi) \neq 0$ *on $A_r \setminus C_1$, and let F be the r-unitary frame obtained from splitting $\Psi\, P^{-1}$.*

Suppose that the monodromy matrices of F and G around z_e both satisfy the appropriate closing condition of Theorem 6.1. Denoting by f and g the corresponding CMC *immersions* (5.3) (5.4) (5.5) *for the appropriate spaceform, we conclude in each of the following cases* (i) – (iii) *below that f is an end of a Delaunay surface and*

$$\lim_{z\to z_e} \|g - f\|_{\{\lambda_0\}} = 0 \textit{ for } \mathbb{R}^3 \textit{ and } S^3, \quad \lim_{z\to z_e} \tfrac{\|g-f\|_{\{\lambda_0\}}}{\|f\|_{\{\lambda_0\}}} = 0 \textit{ for } H^3, \tag{8.9}$$

for any r sufficiently close to 1. Furthermore, there exists a small radius ε ball $B_\varepsilon(\lambda_0)$ about λ_0 such that

$$\lim_{z\to z_e} \|F^{-1} G - \mathrm{Id}\|_{B_\varepsilon(\lambda_0)} = \lim_{z\to z_e} \|\partial_\lambda (F^{-1} G)\|_{B_\varepsilon(\lambda_0)} = 0\,. \tag{8.10}$$

(i) *For the $\mathbb{R}^3$ case assume that* (7.3) *and choose $\lambda_0 = 1$. Suppose that the $\mathbb{R}^3$ weight associated to A satisfies*

$$w > -3/|H|\,. \tag{8.11}$$

(ii) *For the S^3 case, assume that A satisfies* (7.4) *and $\lambda_0^{-2} = \mu = e^{2i\theta}$ for some $\theta \in (0, \frac{\pi}{2}] \subset \mathbb{R}$. · Suppose that the S^3 weight associated to A and the S^3 mean curvature $H = -\cot\theta$ satisfy*

$$w > 6(H - \sqrt{H^2 + 1})\,. \tag{8.12}$$

(iii) *For the H^3 case, assume that A satisfies* (7.5) *and $\lambda_0 = e^q$ for some $q < 0$. In addition we require that*

$$[A, \xi_0] = 0 \tag{8.13}$$

holds for all $\lambda \in C_1$. Suppose there exists an $s \in (0, |\lambda_0|)$ so that $\Psi \in \Lambda_r\mathrm{GL}_2(\mathbb{C})$ for all $r \in (s, 1)$. Suppose that the H^3 weight associated to A and the H^3 mean curvature $H = -\coth q$ satisfy

$$w > 6(\sqrt{H^2 - 1} - H)\,. \tag{8.14}$$

REMARK 8.4. *In the case of H^3, we have $|\lambda_0| < r$, in which case it is not clear that F, G, f and g are even definable at λ_0. However, the conditions in case (iii) of the theorem imply that they indeed are defined, as we will see in the proof of this theorem.*

The inequalities (8.11), (8.12) and (8.14) seem to be essential, since numerical experiments suggest they are precisely the interval of weights for which the corresponding Delaunay surfaces are not bifurcating in the sense of Mazzeo and Pacard [**45**].

We prove Theorem 8.3 in section 12, and first discuss examples to which this result applies. We present cylinders with one Delaunay end and an arbitrary number of umbilics and then turn our attention to trinoids with Delaunay ends in section 9.

8.1. Perturbed Delaunay surfaces. The period problem for the cylinders we describe here was solved for the $\mathbb{R}^3$ case in [**38**], but the asymptotics question is resolved here. Let $M = \mathbb{C}^*$ and $\xi = Az^{-1}dz$ with A as in (7.2). Assume that A and λ_0 satisfy (7.3) or (7.4) or (7.5) for the respective target. Let $\eta \in \Lambda_r\Omega(\mathbb{C})$. By Theorem 8.2, there is a solution of $d\Phi = \Phi(\xi + \eta)$ of the form $\Phi = z^A P$ with entire $P : \mathbb{C} \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ for some $r \in (0, 1)$ arbitrarily close to 1. By (8.5), we may assume $P_0 = \mathrm{Id}$. For τ as in (7.1), the monodromy of Φ is $\chi(\tau) = \exp(2\pi i A) \in \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$, because $\tau^*P = P$. So the monodromy of the unitary part F of Φ under r-Iwasawa decomposition satisfies $\mathcal{H}(\tau) = \chi(\tau)$, and, by (7.3) (7.4) (7.5), the closing conditions (i), (ii) and (iii) of Theorem 6.1 are satisfied. This yields the existence of many types of CMC cylinders in $\mathbb{R}^3$, H^3, and S^3. When the target is H^3, we must further ensure that F is nonsingular at $\lambda_0 = e^q = \lambda_{1,+}$ or $\lambda_{1,-}$. Choosing $C = \mathrm{Id}$ and $z_e = 0$ in Theorem 8.3, and retaining all other notations of Theorem 8.3 we arrive at the following:

COROLLARY 8.5. *When A satisfies* (8.11) *or* (8.12) *and the target is $\mathbb{R}^3$ or S^3 respectively, the* CMC *surface obtained from $\Phi = \exp(A \log z)P$ has an asymptotically Delaunay end at $z = 0$. If A satisfies* (8.14) *and additionally $[A, \xi_0] = 0$, then this is also true for the target H^3.*

One example of such a CMC surface, looking like a Smyth surface with a Delaunay end added to its head, can be produced with the potential $\xi = Az^{-1}dz + \alpha z^k \epsilon_- dz$, for any $\alpha \in \mathbb{C}^*$ and $k \in \mathbb{N}$. See Figure 2 (see also [**34**] and [**38**] when the target space is $\mathbb{R}^3$). We note that ξ is asymptotically the same as the potential

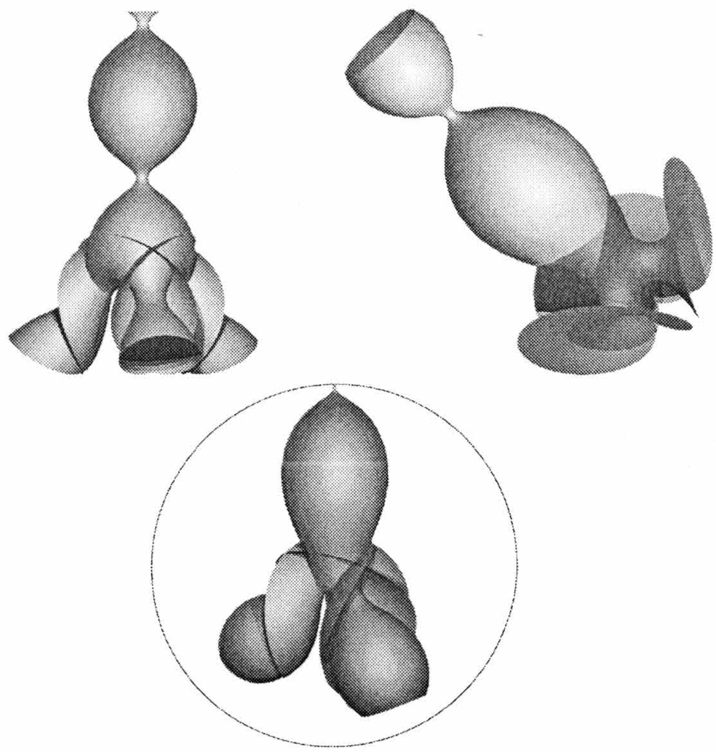

FIGURE 2. Perturbed Delaunay surfaces in $\mathbb{R}^3$, S^3 and H^3.

for the Smyth surfaces at $z = \infty$. Furthermore, $\xi_0 = 0$ and so Corollary 8.5 is applicable for all three target space forms, implying that all these surfaces have an asymptotically Delaunay end at $z = 0$.

EXAMPLE 8.1. *Figure 2 displays perturbed Delaunay surfaces in* $\mathbb{R}^3$, S^3 *and* H^3. *These surfaces have three umbilics and are generated by triples* $(Az^{-1}dz + \alpha z^k \epsilon_- dz, P(z=1), 1)$ *with* $\alpha = -1/5$, $c = 0$, $k = 2$ *and in* $\mathbb{R}^3$: $a = 9/20$, $b = 1/20$ *and* $\lambda_0 = 1$; *in* S^3: $a = 0.077$, $b = 0.442579$ *and* $\mu = \lambda_0^{-2} = e^{i\pi/2}$; *in* H^3: $a = 1/(2\sqrt{34})$, $b = \sqrt{2/17}$ *and* $\lambda_0 = 1/4$. *The initial condition* $P(1)$ *is obtained by solving the corresponding equation* (8.5).

9. Trinoids in $\mathbb{R}^3$, S^3 and H^3

To construct trinoids, we find a family of potentials defined on the thrice punctured Riemann sphere and dressing elements that unitarize their monodromy representations. First we choose a potential whose monodromy representation is pointwise unitarizable on C_1, and then apply Theorem 9.4 to construct a dressing that closes the resulting CMC surface. Then we show that the conditions of Theorem 8.3 are satisfied, ensuring that the ends of the surface are asymptotically Delaunay.

This family of trinoids is a three-parameter family of surfaces for each choice of target space and each choice of constant mean curvature, parametrized by the three end-weights.

Normalization: We make the following normalizations, for each of the target spaces:

$\mathbb{R}^3: \lambda_0 = 1$	$S^3: \lambda_0 \in C_1 \cap \{-\frac{\pi}{2} \leq \mathrm{arc}(\lambda) < 0\}$, $\mu = \lambda_0^{-2}$	$H^3: \lambda_0 \in (0,1)$ in $\mathbb{R}^+$

For S^3 and H^3, the mean curvature H is determined as in Theorem 4.1. For $\mathbb{R}^3$, H can be any positive real number. We next extend a result of [**55**] to S^3 and H^3.

THEOREM 9.1. *Let $M = \mathbb{C} \setminus \{0, 1\}$. Let H and λ_0 be as in the normalization above. For $k \in \{0, 1, \infty\}$, let $v_k \in \mathbb{R} \setminus \{0\}$ satisfy and define $v_k(\lambda_0 + \lambda_0^{-1} \pm 2) \leq 4$*

$$\nu_k(\lambda) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{v_k}{4}\left((\lambda_0 + \lambda_0^{-1}) - (\lambda + \lambda^{-1})\right)}\,. \tag{9.1}$$

With $m_k = \nu_k(+1)$ and $n_k = \nu_k(-1)$ and $\{i, j, k\} = \{0, 1, \infty\}$, suppose that

$$\begin{aligned} &|n_0| + |n_1| + |n_\infty| \leq 1 \\ &|n_i| \leq |n_j| + |n_k| \\ &\left.\begin{aligned} |m_0| + |m_1| + |m_\infty| \leq 1 \\ |m_i| \leq |m_j| + |m_k| \end{aligned}\right\} \text{ for the ambient spaces } S^3,\ H^3 \\ &|v_i| \leq |v_j| + |v_k| \text{ for the ambient space } \mathbb{R}^3. \end{aligned} \tag{9.2}$$

Let

$$\xi = \begin{pmatrix} 0 & \lambda^{-1}dz \\ \lambda(\lambda + \lambda^{-1} - \lambda_0 - \lambda_0^{-1})Q/dz & 0 \end{pmatrix},$$

with

$$Q = \frac{v_\infty z^2 + (v_1 - v_0 - v_\infty)z + v_0}{16z^2(z-1)^2}dz^2.$$

Then ξ generates a conformal CMC *immersion $f : M \to \mathbb{R}^3$ (respectively S^3, H^3) with three asymptotically Delaunay ends having weights $w_0 = cv_0$, $w_1 = cv_1$, $w_\infty = cv_\infty$, where c is respectively $1/H$, $1/\sqrt{H^2+1}$, $1/\sqrt{H^2-1}$.*

We defer the proof of Theorem 9.1 to the end of this section, and first discuss examples and prerequisite technicalities.

REMARK 9.2. *Note that Q in Theorem 9.1 is proportional to the Hopf differential (by a z-independent constant), and that the condition $v_k(\lambda_0 + \lambda_0^{-1} - 2) \leq 4$ is redundant when the target space is $\mathbb{R}^3$ or H^3.*

The following lemma is required for proving Theorem 9.1. The lemma shows that ξ can be gauged to a form with a simple pole that is asymptotically equal to a potential for a Delaunay surface, at any given end. Furthermore, the constant term of the potential becomes zero, which ensures that (8.13) holds for the H^3 case.

LEMMA 9.3. *Let ξ be a trinoid potential as in Theorem 9.1. Then for each end $p \in \{0, 1, \infty\}$ there exists a gauge on a punctured neighborhood $V \setminus \{p\}$ of p, which is nonsingular on I_1, so that in some conformal coordinate ζ with $\zeta(p) = 0$, the expansion of $\xi.g$ is*

$$\xi.g = \begin{pmatrix} 0 & a\lambda^{-1} + b \\ b + a\lambda & 0 \end{pmatrix} \frac{d\zeta}{\zeta} + O(\zeta)d\zeta;\ a,\, b \in \mathbb{R}, \tag{9.3}$$

where $O(\zeta^1)$ denotes a holomorphic term that is zero at $\zeta = 0$.

PROOF. We consider the end $z = 0$; the calculation at the other ends is analogous. Let $\mu = \frac{1}{2} - \nu_0(\lambda)$. There exist $a, b \in \mathbb{R}$ with $|a| \geq |b|$ so that $\varphi\varphi^* = \mu^2$ for $\varphi = a\lambda^{-1} + b$. With

$$k = \frac{v_0 + v_1 - v_\infty}{2v_0},$$

let

$$g(z, \lambda) = \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}\lambda & \lambda\varphi \end{pmatrix} \left(\mathrm{Id} + \frac{k}{2} \begin{pmatrix} -1 & 0 \\ \varphi^{-1} & 1 \end{pmatrix} z \right). \tag{9.4}$$

On a small punctured neighborhood $V \setminus \{0\}$ of $z = 0$, $g : (V \setminus \{0\}) \times I_1 \to \mathrm{SL}_2(\mathbb{C})/\{\pm \mathrm{Id}\}$ is a nonsingular analytic map such that $g(z, 0)$ is upper triangular. (This is sufficient to guarantee that gauging by g has no effect in the Sym-Bobenko formulas.) A calculation shows that the expansion of $\xi.g$ at $z = 0$ is

$$\xi.g = A\, z^{-1}dz + kAdz + O(z)dz, \quad A = \begin{pmatrix} 0 & \varphi \\ \varphi^* & 0 \end{pmatrix}. \tag{9.5}$$

In the coordinate $\zeta = z - kz^2$, the expansion of $\xi.g$ at $\zeta = 0$ is $\xi.g = A\,\zeta^{-1}d\zeta + O(\zeta)d\zeta$. □

9.1. Pointwise unitarisation. Let us denote the ends by $(z_0, z_1, z_\infty) = (0, 1, \infty)$. Let $\Phi(z, \lambda) : \widetilde{M} \times \mathbb{C}^* \to \mathrm{SL}_2(\mathbb{C})$ be a holomorphic solution of $d\Phi = \Phi\xi$ for some trinoid potential ξ. Let $\mathcal{H}$ be the monodromy representation of Φ. Let τ_j be the deck transformation associated to a once-wrapped counterclockwise loop about the end z_j, and let $\mathcal{H}_j = \mathcal{H}(\tau_j) : \mathbb{C}^* \to \mathrm{SL}_2(\mathbb{C})$ be the monodromy of τ_j. Note that $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_\infty = \mathrm{Id}$.

Fix some $\lambda_1 \in C_1$. Assume that each $\mathcal{H}_j(\lambda_1)$ is individually conjugate to a matrix in SU_2, that is, either $-1 < t_j(\lambda_1) < 1$ or $\mathcal{H}_j(\lambda_1) = \pm\mathrm{Id}$, where $2t_j = \mathrm{tr}(\mathcal{H}_j)$. The three matrices $\mathcal{H}_j(\lambda_1)$ are then simultaneously unitarisable, that is, there exists a $C \in \mathrm{GL}_2(\mathbb{C})$ so that $C\,\mathcal{H}_j(\lambda_1)\,C^{-1} \in \mathrm{SU}_2$ for $j = 0, 1, \infty$, if and only if, see for example [**25**], [**64**],

$$\left(1 - t_0^2 - t_1^2 - t_\infty^2 + 2t_0t_1t_\infty\right)\big|_{\lambda_1} \geq 0\,.$$

Let $T_0 \subset \{(\nu_0, \nu_1, \nu_\infty) \in \mathbb{R}^3\}$ be the set with tetrahedral boundary defined by

$$\nu_0 + \nu_1 + \nu_\infty \leq 1 \quad \text{and} \quad \nu_i \leq \nu_j + \nu_k \tag{9.6}$$

for all distinct $\{i, j, k\} = \{0, 1, \infty\}$ and $\nu_j \in \mathbb{R}$. Let T be the orbit of T_0 by the action of the group generated by the transformations $\nu_k \to \nu_k + 1$ and $\nu_k \to -\nu_k$. The inequalities (9.6) are the case $n = 3$ of Biswas' inequalities for n punctures [**3**], and are the spherical triangle inequalities in a 2-sphere of radius $1/(2\pi)$. For a triangle with side lengths ν_j in such a sphere, $\nu_0+\nu_1+\nu_\infty \le 1$ means that the triangle is never bigger than the maximal triangle, and $\nu_i \le \nu_j + \nu_k$ is the triangle inequality.

Define ν_j by $t_j = \cos(2\pi\nu_j)$ for $\nu_j \in \mathbb{R}$. The matrices $\mathcal{H}_j(\lambda_1)$ are simultaneously unitarisable if and only if, see for instance [**3**], [**55**] or [**64**],

$$(\nu_0, \nu_1, \nu_\infty)|_{\lambda_1} \in T\,. \tag{9.7}$$

Furthermore, $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_\infty$ do not all commute at λ_1 if $(\nu_0,\nu_1,\nu_\infty)|_{\lambda_1}$ is not in the boundary ∂T of T [**64**].

9.2. Global unitarisation. We cite a result from [**55**], which implies that once the monodromies of a trinoid are simultaneously pointwise unitarised, they can be simultaneously unitarised globally by a loop element. A similar result can be found in [**22**].

THEOREM 9.4. [**55**] *(Gluing theorem) Let $\mathcal{H}_1$,...,$\mathcal{H}_n : \mathbb{C}^* \to \mathrm{SL}_2(\mathbb{C})$ and S be a finite set of points in C_1 such that*

(i) *there exist a j and k so that $[\mathcal{H}_j, \mathcal{H}_k] \neq 0$ on $C_1 \setminus S$, and*
(ii) *$\mathcal{H}_1$,...,$\mathcal{H}_n$ are simultaneously unitarisable pointwise on $C_1 \setminus S$.*

Then there exists an $r \in (0,1)$ and a $C : I^*_{1/r} \to M_{2\times 2}(\mathbb{C})$ so that

(i) *$\{\det(C) = 0\} \cap A_r$ is a finite subset of C_1,*
(ii) *$C \in \Lambda_s^+\mathrm{GL}_2(\mathbb{C})$ for all $s \in (0,1)$,*
(iii) *$C\mathcal{H}_jC^{-1} \in \Lambda_r^{\mathbb{R}}\mathrm{GL}_2(\mathbb{C})$ for all $j = 1, ..., n$.*

EXAMPLE 9.1. *Examples of equilateral trinoids in $\mathbb{R}^3$, S^3 and H^3 are displayed in Figure 3. The values (end-weight, H, λ_0) are respectively $(\frac{8}{9}, 1, 1)$, $(\frac{16}{9}, 0, -i)$ and $(\frac{2\sqrt{7}}{9}, \frac{4}{\sqrt{7}}, \frac{1}{3}(4-\sqrt{7}))$. For these graphics, we computed appropriate dressing matrices according to Theorem 9.4 to close the surfaces.*

LEMMA 9.5. *Let ξ be as in Theorem 9.1, and $\Phi(z,\lambda) : \widetilde{M} \times \mathbb{C}^* \to \mathrm{SL}_2(\mathbb{C})$ a holomorphic solution of $d\Phi = \Phi\xi$. Then the monodromy representation $\mathcal{H}$ of Φ satisfies the assumptions (i) and (ii) of the gluing theorem 9.4. Furthermore, the inequalities* (9.2) *imply that the lower bounds on the weights* (8.11) *and* (8.12) *and* (8.14) *hold.*

PROOF. We first compute the eigenvalues of $\mathcal{H}_k$. For the gauge g and new coordinate ζ in Lemma 9.3 at the end z_k, one solution of $d\hat{\Phi} = \hat{\Phi}(\xi.g)$ is $\hat{\Phi} = \Phi g$ with monodromies $-\mathcal{H}_k$ for each τ_k. Since $\xi.g$ has a simple pole at z_k, Lemma 8.2 implies that $-\mathcal{H}_k$ is conjugate to $\exp(2\pi i A_k)$, where $A_k = \mathrm{Res}_{z_k}\,\xi.g$. Hence the eigenvalues of $\mathcal{H}_k$ are $\exp(\pm 2\pi i \nu_k)$.

We now show that the curve

$$(\nu_0(\lambda),\, \nu_1(\lambda),\, \nu_\infty(\lambda)) \in T \cap [-1/2, 1/2]^3 \subset \mathbb{R}^3$$

for all $\lambda \in C_1$ if and only if the inequalities (9.2) are satisfied. One direction is clear. To prove the other direction, assume equations (9.2) are satisfied. Define

$$\begin{aligned} \rho_k &= \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - v_k x}, \quad k \in \{0, 1, \infty\} \\ f &= |\rho_0| + |\rho_1| + |\rho_\infty| \\ f_i &= -|\rho_i| + |\rho_j| + |\rho_k|, \quad \{i, j, k\} = \{0, 1, \infty\} , \end{aligned}$$

for $x \in \mathcal{I} := [\frac{\lambda_0+\lambda_0^{-1}-2}{4}, \frac{\lambda_0+\lambda_0^{-1}+2}{4}]$. All three terms of f are increasing when $x \geq 0$ and decreasing when $x \leq 0$, so the first two inequalities in (9.2) imply that $f \leq 1$ on $\mathcal{I}$. Hence $|\nu_1| + |\nu_2| + |\nu_3| \leq 1$ on C_1.

In the case $0 < v_0 \leq v_1$ or $v_1 \leq v_0 < 0$, f_0 is increasing, so $|n_0| \leq |n_1| + |n_\infty|$ implies f_1 is non-negative on $\mathcal{I}$. Hence $|\nu_0| \leq |\nu_1| + |\nu_\infty|$ on C_1.

It is easy to prove the following fact: The function ρ_1/ρ_0 extends to a smooth function at $x = 0$, and, if $v_1 > v_0$, then $|\rho_1/\rho_0|$ is strictly increasing.

In the case $v_0 \geq v_1$, the above fact implies that $f_0/|\rho_0|$ is non-increasing. Further, $|n_0| \leq |n_1| + |n_\infty|$ implies that $f_0/|\rho_0|$ is non-negative at the upper endpoint of $\mathcal{I}$, so $f_0/|\rho_0|$, and hence f_0, is non-negative on $\mathcal{I}$. Hence $|\nu_0| \leq |\nu_1| + |\nu_\infty|$ on C_1.

In the case $v_0 \leq v_1$, the above fact implies that that $f_0/|\rho_0|$ is non-decreasing. But $(f_0/|\rho_0|)(0) = -1 + |v_2/v_1| + |v_3/v_1| \geq 0$, so $f_0/|\rho_0|$ is non-negative on $\mathcal{I}$. Hence $|\nu_1| \leq |\nu_2| + |\nu_3|$ on C_1. Symmetric arguments for the other cases imply that $(\nu_0, \nu_1, \nu_\infty) \in T \cap [-1/2, 1/2]^3$ for all $\lambda \in C_1$.

To prove the first part of the lemma, suppose the inequalities (9.2) hold. Because $(\nu_0, \nu_1, \nu_\infty) \in T \cap [-1/2, 1/2]^3$, Equation (9.6) holds, and so $\mathcal{H}_0$, $\mathcal{H}_1$, $\mathcal{H}_\infty$ are simultaneously unitarizable on $C_1 \backslash S$, where S is the finite set $\{\lambda \in C_1$ such that $(\nu_0, \nu_1, \nu_\infty) \in \partial T\}$. Assumption (i) of Theorem 9.4 holds on $C_1 \setminus S$ as well.

To prove the final sentence of the lemma, we note that $n_k \geq -1/2$ and $m_k \geq -1/2$ imply that $v_k \geq -12/(\lambda_0 + \lambda_0^{-1} + 2)$ in all three space forms, and that $v_k \geq -12/(\lambda_0 + \lambda_0^{-1} - 2)$ in S^3. Thus the weights w_k satisfy

$$\begin{aligned} & w_k \geq -3/H \text{ in } \mathbb{R}^3 , \\ & 6(\sqrt{H^2+1} + H) \geq w_k \geq -6(\sqrt{H^2+1} - H) \text{ in } S^3 , \qquad (9.8) \\ & w_k \geq -6(H - \sqrt{H^2-1}) \text{ in } H^3 , \end{aligned}$$

where c is defined as in Theorem 9.1. We must show that the inequalities in (9.8) are strict. If v_0 attains its lower bound in (9.8), then $|n_1| + |n_\infty| = 1/2$ and $v_0 < v_1$ and $v_0 < v_\infty$. Therefore $f_0/|\rho_0|$ is an increasing function which is zero at the upper endpoint of $\mathcal{I}$, and so $|m_0| > |m_1| + |m_\infty|$, contradicting one of the inequalities in (9.2). All other cases of equality in (9.8) can be dealt with similarly. □

REMARK 9.6. *When the target space is S^3, there exists a $\lambda \in C_1$ so that $\nu_k(\lambda) = 0$ for all $k = 0, 1, \infty$ and thus $|v_i| < |v_j| + |v_k|$ automatically holds in S^3, if all the other inequalities in (9.2) hold. However, $|v_i| \leq |v_j| + |v_k|$ will not hold in general when the target is H^3, since then there exist trinoids that satisfy (9.2) but not $|v_i| \leq |v_j| + |v_k|$: $\lambda_0 \in (0,1)$ with $\lambda_0 + \lambda_0^{-1} = 10$ and $v_0 = -3/5$ and $v_1 = v_\infty = 1/4$ in Theorem 9.1 is such an example.*

Proof of Theorem 9.1: Let $\Phi(z,\lambda) : \widetilde{M} \times \mathbb{C}^* \to \mathrm{SL}_2(\mathbb{C})$ be a solution of $d\Phi = \Phi\xi$. By Lemma 9.5 and the gluing theorem 9.4, there exists an r-dressing matrix C for r close to 1 which unitarizes the monodromy representation. We have that $\nu_k(\lambda_0) = 0$ for $k = 0,\, 1,\, \infty$, which is equivalent to the conditions in Theorem 6.1 for $\mathbb{R}^3$, S^3 and H^3 respectively, for the dressed monodromy representation. (In the case of $\mathbb{R}^3$, one must also use that $\partial_\lambda \nu_k|_{\lambda_0} = 0$.)

In the case of H^3, with F and B the r-Iwasawa factors of Φ, F extends holomorphically to λ_0 and $\det F_{\lambda_0} \neq 0$. This is because $F = C\Phi B^{-1}$ and C, Φ, B are all nonsingular at λ_0. Thus we are able to apply the Sym-Bobenko formula (5.5) in the case of H^3. In the other space forms, $\lambda_0 \in C_1$, so F is clearly nonsingular at λ_0.

We conclude that the resulting CMC immersion satisfies $\tau^* f_{\lambda_0} = f_{\lambda_0}$ for all $\tau \in \Delta$, for all three space forms.

Finally, to show that each end z_j is asymptotic to an end of a Delaunay surface, we note that all the conditions of Theorem 8.3 are satisfied for $\xi.g$ in the coordinate ζ, where g and ζ are as in Lemma 9.3: λ_0 equals $\lambda_{1,+}$ or $\lambda_{1,-}$, the commutativity condition (8.13) holds by the choice of g, and the lower bounds on the weights (8.11), (8.12), (8.14) hold by Lemma 9.5. □

10. Dressed n-noids

We come to a brief discussion of examples of dressed cylinders and trinoids by the simple factors of Terng and Uhlenbeck [**61**]. A simple factor is determined by a choice of a line in $\mathbb{C}^2$ and a complex number, and by suitably choosing this data, the dressed surface will retain its topology. We show how to obtain dressed cylinders and trinoids in all three space forms, building on results obtained in [**39**].

10.1. Simple factors. Let $\pi_{\mathrm{L}} : \mathbb{C}^2 \to \mathrm{L}$ be the hermitian projection onto a line $\mathrm{L} \in \mathbb{CP}^1$. For $\alpha \in \mathbb{C}$, simple factors [**61**] are loops of the form

$$\psi_{\mathrm{L},\alpha} = \pi_{\mathrm{L}} + \frac{\alpha - \lambda}{1 - \overline{\alpha}\lambda}\, \pi_{\mathrm{L}}^{\perp}. \tag{10.1}$$

To obtain elements of $\Lambda_r^+\mathrm{SL}_2(\mathbb{C})$, we write

$$(\det \psi_{\mathrm{L},\alpha}(0))^{-1/2}\, \psi_{\mathrm{L},\alpha}(0) = Q\, R \tag{10.2}$$

with $Q \in \mathrm{SU}_2$, $R \in \mathcal{B}$. A simple factor of $\Lambda_r^+\mathrm{SL}_2(\mathbb{C})$ with $r < |\alpha|$ is a loop of the form

$$h_{\mathrm{L},\alpha} = (\det \psi_{\mathrm{L},\alpha})^{-1/2}\, Q^{-1} \psi_{\mathrm{L},\alpha} \tag{10.3}$$

with $\psi_{\mathrm{L},\alpha}$ and Q defined as in (10.1) respectively (10.2). By construction

$$h_{\mathrm{L},\alpha} \in \Lambda_s^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C}) \tag{10.4}$$

for $s > |\alpha|$. By Proposition 4.2 in [**61**] dressing by simple factors is explicit: In fact, for $F(z,\lambda) \in \mathcal{F}_r(\widetilde{M})$ and $r \in (0,1)$ and $h_{\mathrm{L},\alpha}$ a simple factor with $\alpha \in \mathbb{C}$ and $r < |\alpha| < 1$, we have

$$h_{\mathrm{L},\alpha} \,\#_r\, F = h_{\mathrm{L},\alpha}\, F\, h_{\mathrm{L}',\alpha}^{-1} \text{ with } \mathrm{L}' = \overline{F(z,\alpha)}^{\,t}\mathrm{L}. \tag{10.5}$$

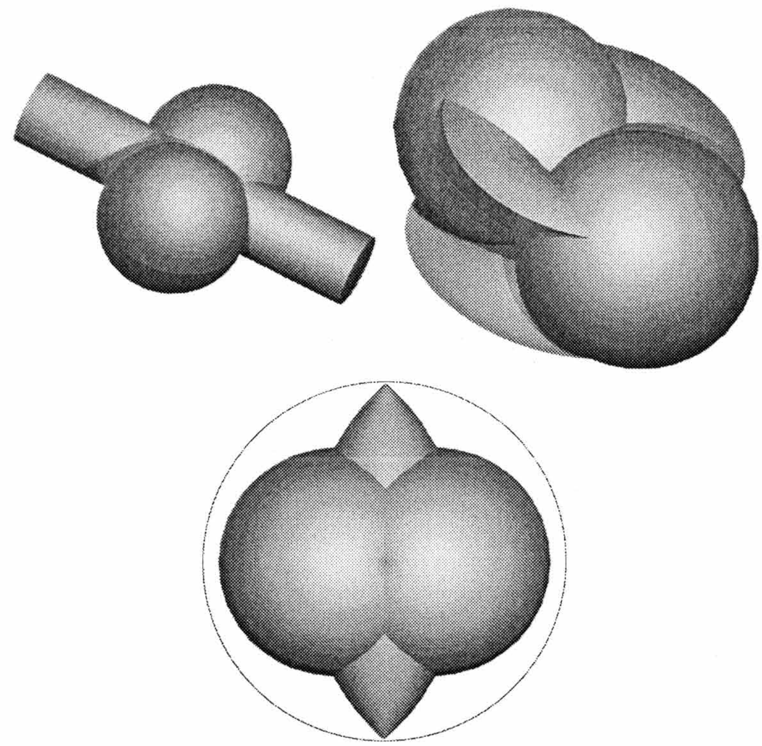

FIGURE 3. Bubbletons in $\mathbb{R}^3$, S^3 and H^3.

The proof of this may be found in [**39**], where furthermore, the following result is obtained:

THEOREM 10.1. [**39**] *Let* $F \in \mathcal{F}_r(\widetilde{M})$ *with monodromy* $\mathcal{H}(\lambda) : \Delta \to \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$. *Assume there exists* $\alpha \in \mathbb{C}$, $r < |\alpha| < 1$ *such that* $\mathcal{H}(\alpha)$ *is reducible and let* $\mathrm{L} \in \mathbb{CP}^1$ *such that*

$$\overline{\mathcal{H}(\alpha)}^t \mathrm{L} = \mathrm{L} \tag{10.6}$$

for all $\tau \in \Delta$. *Then* $h_{\mathrm{L},\alpha} \#_r F$ *has monodromy*

$$h_{\mathrm{L},\alpha} \mathcal{H}\, h_{\mathrm{L},\alpha}^{-1}. \tag{10.7}$$

10.2. Bubbletons. Let $\Phi = z^A$ with A as in (7.2) be the holomorphic frame of a Delaunay surface and λ_0 (and μ in the S^3 case) such that (7.3) or (7.4) or (7.5) hold. As the monodromy of Φ is unitary, it coincides with the monodromy of the unitary Delaunay frame F with respect to the deck transformation (7.1) and is given by $\mathcal{H} = \exp(2\pi i A)$. Now let $\alpha \in \mathbb{C}$ be a solution of

$$2\alpha - 2 - \delta + \sqrt{(2+\delta)^2 - 4} = 0,$$

where $\delta = (ab)^{-1}(k^2/4 - c^2 - |a + \bar{b}|^2)$, for $k \in \mathbb{Z}$ such that $\alpha \notin C_1$. Using the closing conditions (7.3), (7.4) and (7.5) for Delaunay surfaces, we see that

$\delta = (k^2-1)(4ab)^{-1} + \Upsilon$, where $\Upsilon = 0$ or $\Upsilon = -4\sin^2(\theta/2)$ or $\Upsilon = 4\sinh^2(q/2)$ for the target space $\mathbb{R}^3$ or S^3 or H^3, respectively. Then $\exp(2\pi i\, A(\alpha)) = \pm\mathrm{Id}$, so for the line $\mathrm{L} = [1:0] \in \mathbb{CP}^1$ and the simple factor $h_{\mathrm{L},\alpha}$, the dressed frame $\hat{F} = h_{\mathrm{L},\alpha}\#_r F$ with $r < |\alpha|$ has monodromy $h_{\mathrm{L},\alpha}\exp(2\pi i A)\, h_{\mathrm{L},\alpha}^{-1}$ and satisfies one of the conditions (i), (ii) or (iii) of Theorem 6.1 since $\exp(2\pi i A)$ does. Hence the resulting surfaces are again CMC cylinders. A computation shows that $|\alpha| < \lambda_0 < |\alpha^{-1}|$ when the target space is H^3, so $\lambda_0 \in A_r$ in the H^3 case, and so $\det F_{\lambda_0} \neq 0$, allowing us to apply the generalized Sym-Bobenko formula (5.5).

These ideas where used in [**34**] to construct bubbletons in $\mathbb{R}^3$ and have recently been utilised in [**39**] to dress CMC n-noids. In [**59**], bubbletons in $\mathbb{R}^3$ with ends asymptotic to round cylinders were explicitly constructed. In [**65**] and [**50**] less explicit constructions of bubbletons in $\mathbb{R}^3$ with Delaunay ends where obtained. One can easily show that for the surfaces in [**59**] the ends are asymptotic to round cylinders, but it was unproven that dressing a Delaunay frame with an appropriate $h_{\mathrm{L},\alpha}$ gives a surface with asymptotically Delaunay ends. We prove this in Theorem 10.3. (Note that one cannot apply part (i) of Theorem 8.3, since $r < |\alpha|$ may not be sufficiently close to 1.)

REMARK 10.2. *Let* $\mathrm{L} = [1:0]$ *and* $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, $\alpha_i \neq \alpha_j$ *for* $i \neq j$ *as above, and define* $h = \prod_{j=1}^n h_{\mathrm{L},\alpha_j}$. *By similar arguments as above, dressing with* h *produces multibubbletons.*

When the initial surface is a round cylinder, one can find explicit parametrisations for the resulting bubbletons (and multibubbletons) in all three space forms [**40**]. When the target space is $\mathbb{R}^3$, these explicit parametrisations coincide with those in [**59**], which were produced by Bianchi's Bäcklund transformation of a round cylinder. This can also be seen by combining results of [**30**] and [**9**].

EXAMPLE 10.1. *The examples in Figure 3 are generated by triples* $(Az^{-1}dz, h_{\mathrm{L},\alpha}, 1)$ *where* A *is as in* (7.2) *with* $c = 0$, $\mathrm{L} = [1:0]$ *and the following data:* R^3*:* $a = b = 1/4$, $\lambda_0 = 1$, $\alpha = 7 + 2\sqrt{12}$*;* S^3*:* $a = b = 1/3$, $\mu = \lambda_0^{-2} = \exp(2i\tan^{-1}(3\sqrt{7}))$, $\alpha = (7+3\sqrt{5})/2$*;* H^3*:* $a = b = 1/6$, $\lambda_0 = (7-3\sqrt{5})/2$, $\alpha = 17 - 12\sqrt{2}$.

In [**39**] it was shown that trinoids and symmetric n-noids in $\mathbb{R}^3$ can be dressed by simple factors. The potentials we use in Theorem 9.1 for constructing trinoids in $\mathbb{R}^3$, H^3 and S^3 are in a slightly more general form than in [**39**], and are identical when the target space is $\mathbb{R}^3$. In the more general form, the only difference in the potentials is the change in the constant $\lambda_0 + \lambda_0^{-1}$ ($= 2$, < 2 and > 2 for $\mathbb{R}^3$, S^3 and H^3, respectively.). This change has little effect on the computations in [**39**], and only deforms the infinite discrete set of values of λ where the monodromy is reducible. Then Lemma 3.2 of [**39**] can be applied verbatim, and we conclude existence of trinoids dressed by simple factors in S^3 and H^3 as well.

EXAMPLE 10.2. *In Figure 4 we display three equilateral trinoids dressed by suitable simple factors. For the* $\mathbb{R}^3$ *model, the potential in Theorem 9.1 has entries* $v_0 = v_1 = v_\infty = 8/9$ *and* $\lambda_0 = 1$. *The* S^3 *model is displayed with the front bubble cut away to show the trinoidal structure. The potential data is* $v_0 = v_1 = v_\infty = 16/9$ *and* $\lambda_0 = i$. *For the* H^3 *surface, in the potential we take* $v_0 = v_1 = v_\infty = 1/2$ *and*

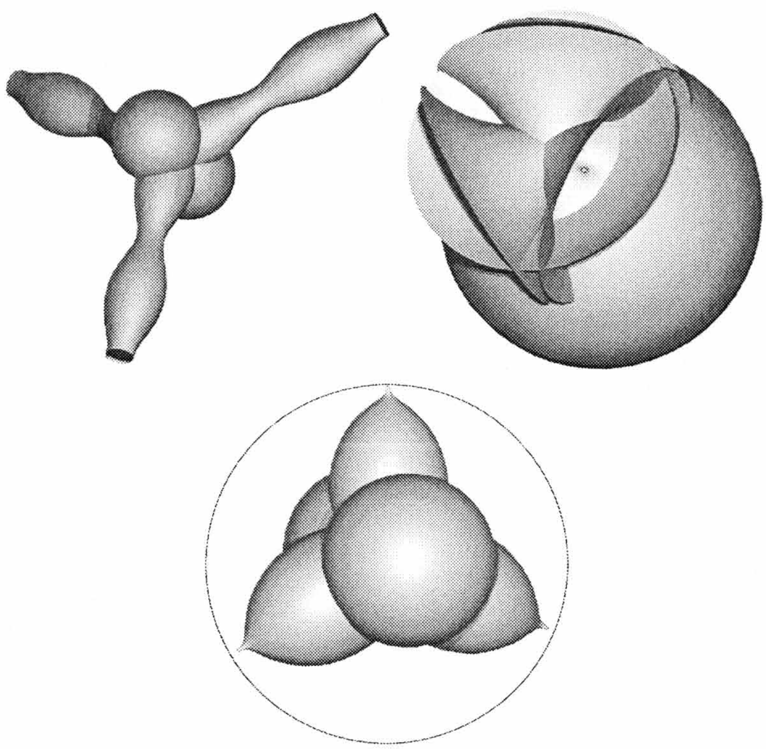

FIGURE 4. Dressed equilateral trinoids in $\mathbb{R}^3$, S^3 and H^3.

$\lambda_0 = 1/4$. *For each picture, the dressing was computed to close the surface, and then the resulting surface was further dressed by a simple factor, having the effect of adding two bubbles to each surface at its center.*

It was not shown in [**39**] that dressing by simple factors preserves the asymptotic behaviour of a Delaunay end of a surface. We prove that any asymptotically Delaunay end, which is dressed by simple factors to another closed end, gives again an asymptotically Delaunay end. This result applies to all examples in [**39**] and to the bubbletons in this section, as well as to dressed trinoids in all three target space forms $\mathbb{R}^3$, S^3 and H^3.

10.3. Asymptotics of dressed CMC **surfaces.** For the line $L = [a_0 : b_0] \in \mathbb{CP}^1$, the projections π_L and $\pi_L^\perp$ have the following matrix forms:

$$\pi_L = \frac{1}{|a_0|^2 + |b_0|^2}\begin{pmatrix} |a_0|^2 & a_0\bar{b}_0 \\ \bar{a}_0 b_0 & |b_0|^2 \end{pmatrix}, \quad \pi_L^\perp = \frac{1}{|a_0|^2 + |b_0|^2}\begin{pmatrix} |b_0|^2 & -a_0\bar{b}_0 \\ -\bar{a}_0 b_0 & |a_0|^2 \end{pmatrix}. \tag{10.8}$$

We set

$$\mathfrak{A} = \{\alpha \in I_1^* : |\alpha|^2 < \tfrac{b^2}{a^2} \text{ and } 1/4 - \Upsilon + \alpha^{-1}ab(1-\alpha)^2 > 0\}\ , \tag{10.9}$$

where a, $b \in \mathbb{R}$ with $|a| \geq |b| > 0$ satisfy equations (7.3) or (7.4) or (7.5) with $c = 0$, and $\Upsilon = 0$ or $\Upsilon = -4ab\sin^2(\theta)$ or $\Upsilon = 4ab\sinh^2(q/2)$ for the target space $\mathbb{R}^3$ or S^3 or H^3. (Note that by switching a and b if necessary, we may assume without loss of generality that $|b| \leq |a|$.) In the following theorem, we denote by

$$\mathcal{U}_\varepsilon = \{z \in \mathbb{C} : 0 < |z| < \varepsilon\} \tag{10.10}$$

an open punctured disk in a Riemann surface M, where z is a centered coordinate about a point $p \in M$. Let $\gamma : [0, 1] \to \mathcal{U}_\varepsilon$ be a simple closed curve and denote by δ the corresponding deck transformation of the universal cover of $\mathcal{U}_\varepsilon$. Denoting a small radius ε ball about λ_0 by $B_\varepsilon(\lambda_0)$, we have the following:

THEOREM 10.3. *Let $G \in \mathcal{F}_r(\mathcal{U}_\varepsilon)$ such that $\lim_{z\to 0} \|F^{-1}G - \mathrm{Id}\|_{B_\varepsilon(\lambda_0)} = 0$ for some extended unitary Delaunay frame F (producing a Delaunay surface at λ_0) in any of the three targets. Assume that G has reducible monodromy with respect to γ at some $\alpha \in \mathfrak{A} \setminus \{\lambda_0\}$ with invariant subspace $\mathrm{L} \in \mathbb{CP}^1$ and let $h_{\mathrm{L},\alpha}$ be the corresponding simple factor. Let f_{λ_0} be the* CMC *surface obtained from $h_{\mathrm{L},\alpha}\#_r G$ with $r < |\alpha|$. Then f_{λ_0} converges to an end of a Delaunay surface as $z \to 0$ in the sense of* (8.9).

We defer the proof of Theorem 10.3 to section 12, and give a corollary here:

COROLLARY 10.4. *All of the dressed Delaunay surfaces and dressed trinoids in all three space forms described in this section, as well as the dressed symmetric n-noids in $\mathbb{R}^3$ in* **[39]**, *have asymptotically Delaunay ends in the sense of* (8.9).

PROOF. We need only show that the conditions regarding α and the convergence of $F^{-1}G$ in Theorem 10.3 are satisfied. Although the condition of reducible monodromy about γ at some $\alpha \in \mathfrak{A}$ in Theorem 10.3 appears restrictive, it is in fact satisfied by all of these surfaces. The condition $\lim_{z\to 0} \|F^{-1}G - \mathrm{Id}\|_{B_\varepsilon(\lambda_0)} = 0$ is trivial for the dressed Delaunay surfaces, because in that case $F = G$. This condition $\lim_{z\to 0} \|F^{-1}G - \mathrm{Id}\|_{B_\varepsilon(\lambda_0)} = 0$ holds for dressed trinoids and symmetric n-noids, by equation (8.10) in Theorem 8.3. □

11. The Delaunay Extended Frame

We compute the extended unitary frame F of a Delaunay surface in terms of elliptic functions. We restrict to the case that A is off-diagonal in Theorems 11.1 and 11.4, but return to the more general form for A in Corollary 11.5.

THEOREM 11.1. *Let $\Phi : \mathbb{C} \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ be defined by $\Phi = \exp((x+iy)A)$, with A given as in* (7.2) *where a, $b \in \mathbb{R}^*$ and $c = 0$. The r-Iwasawa factorization $\Phi = FB$ for any $r \in (0, 1]$ is given by*

$$F = \Phi\exp(-\mathbf{f}A)B_1^{-1}, \quad B = B_1\exp(\mathbf{f}A), \tag{11.1}$$

and the functions $v = v(x)$, $\mathbf{f} = \mathbf{f}(x)$ *and the matrices* B_0, B_1 *satisfy*

$$\begin{aligned} &v'^2 = -\left(v^2 - 4a^2\right)\left(v^2 - 4b^2\right),\ v(0) = 2b, \\ &\mathbf{f} = \int_0^x \frac{2\,dt}{1 + (4ab\lambda)^{-1} v^2(t)}, \\ &B_0 = \begin{pmatrix} 2v(b + a\lambda) & -v' \\ 0 & 4ab\lambda + v^2 \end{pmatrix}, \quad B_1 = (\det B_0)^{-1/2} B_0\,. \end{aligned} \tag{11.2}$$

PROOF. Choose $H \in \mathbb{R}^*$ and set $Q = -2abH^{-1}\lambda^{-1}$. Let v be the nonconstant solution of (11.2) when $|a| \neq |b|$ or the constant solution $v = 2b$ when $|a| = |b|$, and set $v_1^2 = H^{-2}v^2$. Let $\Theta = \Theta_1 dx + \Theta_2 dy$, where

$$\Theta_1 = \begin{pmatrix} 0 & -v_1^{-1}Q - \frac{1}{2}v_1 H \\ v_1^{-1}Q^* + \frac{1}{2}v_1 H & 0 \end{pmatrix}$$

$$\Theta_2 = i\begin{pmatrix} -\frac{1}{2}v_1^{-1}v_1' & -v_1^{-1}Q + \frac{1}{2}v_1 H \\ -v_1^{-1}Q^* + \frac{1}{2}v_1 H & \frac{1}{2}v_1^{-1}v_1' \end{pmatrix}.$$

Let F and B be as in (11.1). We will show that that $F \in \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ and $B \in \Lambda_r^+\mathrm{SL}_2(\mathbb{C})$. A calculation shows that B satisfies the gauge equation

$$dB + \Theta B = BA(dx + idy), \quad B(0, \lambda) = \mathrm{Id}, \tag{11.3}$$

or equivalently, $\Theta_2 B - iBA = 0$ and $B' + (\Theta_1 + i\Theta_2)B = 0$ with $B(0, \lambda) = \mathrm{Id}$. Since $\Theta_1 + i\Theta_2$ is smooth on $\mathbb{C}$ with holomorphic parameter λ on $\mathbb{C}$, the same is true of B. Since $\Theta_1 + i\Theta_2$ is tracefree and $B(0, \lambda) = \mathrm{Id}$, then $\det B = 1$. Also,

$$B(x, 0) = \sqrt{\frac{2b}{v}}\begin{pmatrix} 1 & 8a^2 b \int_0^x \frac{dt}{v^2(t)} - \frac{v'}{2bv} \\ 0 & \frac{v}{2b} \end{pmatrix},$$

which is upper-triangular with diagonal elements in $\mathbb{R}^+$. Hence $B : \mathbb{C} \to \Lambda_r^+\mathrm{SL}_2(\mathbb{C})$ is smooth. From $\Phi = FB$ it follows that $F : \mathbb{C} \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ is smooth. Equivalently to equation (11.3), F satisfies $F^{-1}dF = \Theta$ with $F(0, \lambda) = \mathrm{Id}$. The symmetry $\Theta^* = -\Theta$ together with $F(0, \lambda) = \mathrm{Id}$ imply that F satisfies $F^* = F^{-1}$. Hence $F : \mathbb{C} \to \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ is smooth. □

REMARK 11.2. *The functions* v *and* $\mathbf{f}$ *have explicit forms in terms of elliptic functions (we follow the notational conventions of the Mathematica programming language here). For example, when* $0 < |b| < |a|$, *we have*

$$v(x) = 2b \cdot \mathrm{sn}\left(2ia\left(x - \frac{iK}{2a}\right), \frac{b^2}{a^2}\right) = \frac{2b}{\mathrm{dn}(2ax,\, 1 - b^2/a^2)}, \tag{11.4}$$

where sn, dn *are Jacobi elliptic functions and* $K = K(b^2/a^2)$ *is the quarter-period of* $\mathrm{sn}(x, b^2/a^2)$.

COROLLARY 11.3. *The unitary frame of any Delaunay surface up to isometry can be written as* $F_{\mathrm{D}} : \mathbb{C} \to \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ *with*

$$F_{\mathrm{D}} = \exp\left((x + iy - \mathbf{f}(x))\begin{pmatrix} 0 & a\lambda^{-1} + b \\ b + a\lambda & 0 \end{pmatrix}\right) B_1^{-1} \tag{11.5}$$

for suitable a, $b \in \mathbb{R}$ *and* $\mathbf{f}$ *and* B_1 *as in* (11.2).

PROOF. A Delaunay surface is determined up to rigid motion by its weight w. There exist a, $b \in \mathbb{R}$ satisfying (7.7) and (7.3), (7.4) or (7.5) with $c = 0$. The claim now follows from Theorem 11.1. $\square$

The associated family of surfaces obtained from F_{D} has constant mean curvature H, $\sqrt{H^2-1}$, $\sqrt{H^2+1}$ respectively in $\mathbb{R}^3$, S^3 and H^3. One can verify that v_1^2 is the conformal factor and $Q\,dz^2$ the Hopf differential using Equations (1.4), (1.7) and (1.9).

11.1. Growth Estimates. The following results, which compute the growth rate of the gauge B that gauges the Delaunay Maurer-Cartan form to the Delaunay potential, will be used to prove Theorem 8.3 to show that the CMC immersion resulting from a perturbation of a Delaunay potential is asymptotic to the base Delaunay immersion.

We will use the following estimate on $\exp(\mathbf{f}A)$ for $A \in \mathfrak{sl}_2(\mathbb{C})$ as in Theorem 11.1, with $\mu = \mu(\lambda)$ an eigenvalue of A:

$$|\exp(\mathbf{f}A)| \le \max(|\mathrm{Id} \pm \mu^{-1}A|)\exp(|\operatorname{Re}\mathbf{f}\mu|), \tag{11.6}$$

which is computable using $\exp(\mathbf{f}A) = \frac{1}{2}\exp(\mathbf{f}\mu)(\mathrm{Id} + \mu^{-1}A) + \frac{1}{2}\exp(-\mathbf{f}\mu)(\mathrm{Id} - \mu^{-1}A)$.

THEOREM 11.4. *With B and A as in Theorem 11.1, let $\mu(\lambda)$ be an eigenvalue of A and let*

$$c = \max_{\lambda \in C_1} |\mu(\lambda)|. \tag{11.7}$$

Then there exists an $\mathcal{X} \in \mathbb{R}^+$ and $c_0 : \{0 < |\lambda| \le 1\} \to \mathbb{R}^+$ such that for all $|x| > \mathcal{X}$ and all $\lambda \in \{0 < |\lambda| \le 1\}$, we have

$$|B(x, \lambda)| \le c_0(\lambda)\exp(c\,|x|). \tag{11.8}$$

PROOF. We compute the growth rate of B using its periodicity properties. Let $\rho \in \mathbb{R}^+$ be the period of the function v of Theorem 11.1. For any $x \in \mathbb{R}$ there exist $x_0 \in [0, \rho)$ and $n \in \mathbb{Z}$ such that $x = x_0 + n\rho$. Then $v(x_0 + n\rho) = v(x_0)$, $\mathbf{f}(x_0 + n\rho) = \mathbf{f}(x_0) + n\mathbf{f}(\rho)$ and $B(x_0 + n\rho) = B(x_0)\exp(n\mathbf{f}\rho A)$ and consequently

$$B(x) = B(x_0)\exp((x - x_0)\mathbf{f}(\rho)/\rho A). \tag{11.9}$$

Let d be the point in the set $\{-a/b, -b/a\} \cap \overline{I_1}$, and assume first that $\lambda \in \overline{I_1} \setminus \{0, d\}$. Equation (11.9) and formula (11.6) yield the estimate

$$|B(x)| \le |B(x_0)|\exp(|\operatorname{Re}\mu\mathbf{f}(\rho)/\rho|\,|x - x_0|) \le c_0\exp(|\operatorname{Re}\mu\mathbf{f}(\rho)/\rho|\,|x|), \tag{11.10}$$

where $c_0 = c_1c_2c_3$, $c_1 = \sup_{x_0 \in [0,\rho)}|B(x_0)|$, $c_2 = \max|\mathrm{Id} \pm \mu^{-1}A|$ and $c_3 = e^{|\operatorname{Re}\mu\mathbf{f}(\rho)|}$.

Next we compute the upper bound of $|\operatorname{Re}\mu\mathbf{f}(\rho)/\rho|$ on $\overline{I_1}$. Since $M = \exp(\mathbf{f}(\rho)A)$, then $\mathbf{f}(\rho)A$ is defined on $\overline{I_1}$ modulo the additive quantity $\pi i\mu^{-1}A$. Let $J = [0, d]$ be the closed interval. Then μ is single-valued on $\overline{I_1} \setminus J$, so $\mu\mathbf{f}(\rho)$ is defined on $\overline{I_1} \setminus J$ modulo the additive quantity πi. Hence $h = \operatorname{Re}\mu\mathbf{f}(\rho)/\rho$ is well-defined on $\overline{I_1} \setminus J$. Moreover, h is a harmonic function, since it is locally the real part of a holomorphic function. A calculation shows that $h = \mu$ for $|\lambda| = 1$, and $h = 0$ on

J. The harmonicity of h then implies that $|h| \leq c$ for all $\lambda \in I_1$. This bound, together with the estimate (11.10), yields the estimate (11.8) for all $x \in \mathbb{R}$ and all $\lambda \in \overline{I_1} \setminus \{0, d\}$.

In the exceptional case $\lambda_1 \in \{0, d\}$, $\mathbf{f}A$ is nilpotent at λ_1, so $B(x, \lambda_1)$ has linear growth in $|x|$. Define $c_1(\lambda_1) = 1$. Since $c > 0$, and because exponential growth eventually supersedes linear growth, there exists an $\mathcal{X} \in \mathbb{R}^+$ such that (11.8) holds at λ_1 for all $|x| > \mathcal{X}$. $\square$

This next result extends Theorem 11.4 to the case when $c \neq 0$ in (7.2).

COROLLARY 11.5. *Let $\Phi : \mathbb{C} \to \Lambda_r \mathrm{SL}_2(\mathbb{C})$ be defined by $\Phi = \exp((x+iy)A)$, with*

$$A = \begin{pmatrix} \tilde{c} & a\lambda^{-1} + \bar{b} \\ b + \bar{a}\lambda & -\tilde{c} \end{pmatrix} \quad \text{where } a,\, b \in \mathbb{C}^* \text{ and } ab,\, \tilde{c} \in \mathbb{R}.$$

Let $\Phi = FB$ be the Iwasawa factorization of Φ, and let $\mu(\lambda)$ be an eigenvalue of A and c as in (11.7). Then there exists an $\mathcal{X} \in \mathbb{R}^+$ and $c_0 : \{0 < |\lambda| \leq 1\} \to \mathbb{R}^+$ such that for all $|x| > \mathcal{X}$ and all $\lambda \in \{0 < |\lambda| \leq 1\}$ the inequality (11.8) holds.

PROOF. To prove the inequality, we first show that there exists a $\theta \in \mathbb{R}$ and a g such that the following two properties hold:

(i) $gD \in \Lambda_r^+ \mathrm{SL}_2(\mathbb{C})$ with $D = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$, and g and D are independent of $x + iy$.

(ii) $g^{-1}Ag = \tilde{A}$, where $\tilde{A} := \begin{pmatrix} 0 & \tilde{a}\lambda^{-1} + \tilde{b} \\ \tilde{b} + \tilde{a}\lambda & 0 \end{pmatrix}$ for some $\tilde{a},\, \tilde{b} \in \mathbb{R}^*$.

Choose $\tilde{a},\, \tilde{b} \in \mathbb{R}^*$ such that $\tilde{c}^2 + |a|^2 + |b|^2 = \tilde{a}^2 + \tilde{b}^2$ and $ab = \tilde{a}\tilde{b}$ and so that $|\tilde{a}/\tilde{b}| \geq 1$ if $|a/b| \geq 1$, respectively $|\tilde{a}/\tilde{b}| \leq 1$ if $|a/b| \leq 1$. If $|a/b| \geq 1$, define g by

$$g := \frac{1}{\sqrt{(\tilde{a} + \tilde{b}\lambda)(a + \bar{b}\lambda)}} \begin{pmatrix} a + \bar{b}\lambda & 0 \\ -\tilde{c}\lambda & \tilde{a} + \tilde{b}\lambda \end{pmatrix}. \tag{11.11}$$

If $|a/b| \leq 1$, define g by

$$g := \frac{1}{\sqrt{(\tilde{a}\lambda + \tilde{b})(\bar{a}\lambda + b)}} \begin{pmatrix} \tilde{a}\lambda + \tilde{b} & \tilde{c} \\ 0 & \bar{a}\lambda + b \end{pmatrix}. \tag{11.12}$$

With the proper choice of θ, we have the two above properties.

Let $\tilde{\Phi} = \exp((x+iy)\tilde{A})$, and let $\tilde{\Phi} = \tilde{F}\tilde{B}$ be its Iwasawa decomposition. By Theorem 11.4, $\tilde{B}$ satisfies the growth rate (11.8). We will use $\tilde{B}$ to show that B also satisfies (11.8).

The gauge g can be explicitly decomposed into $g = UR$, where $U \in \Lambda_r^{\mathbb{R}} \mathrm{SL}_2(\mathbb{C})$ and $R : C_r \to \mathfrak{gl}_2(\mathbb{C})$ and $[R, \tilde{A}] = 0$. In fact, when g takes the form in (11.11), then

$$U = \mathfrak{T} \begin{pmatrix} (\tilde{a} + \tilde{b}\lambda^{-1})\mathfrak{R} + (a + \bar{b}\lambda)\mathfrak{R}^* & \tilde{c}\mathfrak{R}\lambda^{-1} \\ -\tilde{c}\mathfrak{R}^*\lambda & (\tilde{a} + \tilde{b}\lambda)\mathfrak{R}^* + (\bar{a} + b\lambda^{-1})\mathfrak{R} \end{pmatrix}$$

and

$$R = \frac{(\mathfrak{R}\mathfrak{R}^* + \mathfrak{S})\mathrm{Id} - \tilde{c}\tilde{A}}{\sqrt{2\mathfrak{S}(\mathfrak{R}\mathfrak{R}^* + \mathfrak{S} - \tilde{c}^2)}} = \exp\left(\mathfrak{S}^{-1/2}\log(\mathfrak{F})\tilde{A}\right),$$

where

$$\mathfrak{R} = \sqrt{(\tilde{a} + \tilde{b}\lambda)(a + \bar{b}\lambda)}, \quad \mathfrak{S} = (\tilde{a}\lambda^{-1} + \tilde{b})(\tilde{a}\lambda + \tilde{b}),$$

$$\mathfrak{T} = (2\mathfrak{R}\mathfrak{R}^*(\mathfrak{R}\mathfrak{R}^* + \mathfrak{S}))^{-1/2} \text{ and } \mathfrak{F} = \frac{\sqrt{\mathfrak{S}} + \sqrt{(a + \bar{b}\lambda)(\bar{a} + b\lambda^{-1})} - \tilde{c}}{\sqrt{2(\mathfrak{S} + \mathfrak{R}\mathfrak{R}^* - \tilde{c}^2)}}.$$

Furthermore, $\mathfrak{S}^{-1/2}\log\mathfrak{F}$ can be algebraically decomposed into

$$\mathfrak{S}^{-1/2}\log\mathfrak{F} = (\mathfrak{S}^{-1/2}\log\mathfrak{F})_u + (\mathfrak{S}^{-1/2}\log\mathfrak{F})_0 + (\mathfrak{S}^{-1/2}\log\mathfrak{F})_+,$$

where $(\mathfrak{S}^{-1/2}\log\mathfrak{F})_u$ satisfies $(\mathfrak{S}^{-1/2}\log\mathfrak{F})_u = -(\mathfrak{S}^{-1/2}\log\mathfrak{F})_u^*$ and $(\mathfrak{S}^{-1/2}\log\mathfrak{F})_+|_{\lambda=0}$ is zero and $(\mathfrak{S}^{-1/2}\log\mathfrak{F})_0$ is independent of λ. Although R is single-valued in λ, $\mathfrak{S}^{-1/2}\log\mathfrak{F}$ can be multiple valued, with values in $\mathbb{C}\cup\{\infty\}/\Omega$, where

$$\Omega = \{2\pi ik\mathfrak{S}^{-1/2} \mid k \in \mathbb{Z}\}.$$

In other words, different values of $\mathfrak{S}^{-1/2}\log\mathfrak{F}$ can differ by terms of the form $2\pi ik\mathfrak{S}^{-1/2}$ for $k \in \mathbb{Z}$. Since $2\pi ik\mathfrak{S}^{-1/2} = -(2\pi ik\mathfrak{S}^{-1/2})^*$, we conclude that $(\mathfrak{S}^{-1/2}\log\mathfrak{F})_0$ and $(\mathfrak{S}^{-1/2}\log\mathfrak{F})_+$ are single-valued, and that $\exp((\mathfrak{S}^{-1/2}\log\mathfrak{F})_u\tilde{A})$ is single-valued although $(\mathfrak{S}^{-1/2}\log\mathfrak{F})_u$ might not be. Thus we have the single-valued decomposition

$$R = R_u\, R_0\, R_+,$$

where $R_u = \exp((\mathfrak{S}^{-1/2}\log\mathfrak{F})_u\tilde{A}) \in \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$, $R_0 = \exp((\mathfrak{S}^{-1/2}\log\mathfrak{F})_0\tilde{A})$ is constant, and $R_+ = \exp((\mathfrak{S}^{-1/2}\log\mathfrak{F})_+\tilde{A}) \in \Lambda_r^+\mathrm{SL}_2(\mathbb{C})$, and R_u, R_0 and R_+ all commute with $\tilde{A}$. Since

$$R_0\tilde{\Phi} = \exp((x + \hat{x} + iy)\tilde{A}) = \tilde{F}|_{x\to x+\hat{x}} \cdot \tilde{B}|_{x\to x+\hat{x}}$$

with $\hat{x} = (\mathfrak{S}^{-1/2}\log\mathfrak{F})_0$, we have

$$\Phi = (UR_u\tilde{F}|_{x\to x+\hat{x}}D)(D^{-1}\tilde{B}|_{x\to x+\hat{x}}D)(D^{-1}R_+D)(gD)^{-1},$$

where $UR_u\tilde{F}|_{x\to x+\hat{x}}D \in \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ and $D^{-1}\tilde{B}|_{x\to x+\hat{x}}D, D^{-1}R_+D, gD \in \Lambda_r^+\mathrm{SL}_2(\mathbb{C})$. Thus

$$B = D^{-1}\tilde{B}|_{x\to x+\hat{x}}R_+g^{-1},$$

and so B satisfies the same growth rate (11.8) as $\tilde{B}$, since D and R_+ and g are all independent of $x+iy$. When g takes the form in (11.12), the argument is similar. □

12. Delaunay Asymptotics

We conclude by proving Theorems 8.3 and 10.3 and begin with two technical results, which are proven in [**55**] for the target space $\mathbb{R}^3$. The proofs for the targets S^3 and H^3 are analogous and thus omitted.

LEMMA 12.1. *Let A be as in (7.2) satisfying $ab \neq 0$. Let λ_0 (and also μ in the case of S^3) be as in Subsection 7.2, and suppose that (7.3) or (7.4) or (7.5) is satisfied (with $|\lambda_0| \leq 1$) for the respective target space $\mathbb{R}^3$ or S^3 or H^3.*

Let $s \in (0, |\lambda_0|)$ and let $C : A_s \to \mathfrak{gl}_2(\mathbb{C})$ be an analytic map, where $\det C = 0$ only at a finite number of points $p_1, ..., p_k \in C_1$, and suppose that $C \exp(2\pi i A) C^{-1} \in \Lambda_s^{\mathbb{R}} \mathrm{SL}_2(\mathbb{C})$.

Then there exists a $\rho \in (0,1)$ such that for every $r \in (\rho, 1)$, the CMC immersion from $\mathbb{C} \setminus \{z_e\}$ to $\mathbb{R}^3$ or S^3 or H^3 produced by (5.3) or (5.4) or (5.5) respectively from the unitary part of the r-Iwasawa splitting of $C(z-z_e)^A$ is an end of a Delaunay surface with the same weight as the Delaunay surface produced from $(z-z_e)^A$.

LEMMA 12.2. *Let A, λ_0, s and C (and μ in the case of S^3) be as in Lemma 12.1. Then there exists a $\rho \in (0,1)$ such that, for every $r \in (\rho, 1)$, there exist a $U \in \Lambda_r^{\mathbb{R}} \mathrm{GL}_2(\mathbb{C})$ and an $R : A_r \to \mathfrak{gl}_2(\mathbb{C})$ so that $C = UR$ and $[R, A] = 0$. Furthermore, $\hat{A} := UAU^{-1}$ is also of the Delaunay potential form in (7.2), and has the same weight as A, and satisfies (7.3) or (7.4) or (7.5) at λ_0 if A does.*

Proof of Theorem 8.3: The proof of (i) can be found in [**55**], where it is shown that one can assume $\xi_0 = 0$, ensuring that P is nonsingular at $\lambda_{1,\pm}$. In that proof in [**55**], condition (8.11) implies that P has no singularities on C_1. Thus P is nonsingular on A_s with s as in Lemma 12.1. Since $\Phi := \Psi \cdot P^{-1} = C(z-z_e)^A$ for some $C = C(\lambda)$, and since C is finite wherever P is nonsingular, C is finite on A_s. Then Lemma 12.1 implies that f is a portion (containing one end) of a Delaunay surface in $\mathbb{R}^3$ with the same weight w. Therefore, g has an asymptotically Delaunay end at z_e of weight w, in the sense of (8.9).

(ii) The proof is almost analogous to (i), since λ_0 and μ still lie in C_1. Just like in the $\mathbb{R}^3$ case, using condition (8.12) and Lemma 12.1, f is an end of a Delaunay surface in S^3 with weight w, and g has an asymptotically Delaunay end at z_e of weight w. There are only two parts of the proof in [**55**] of the $\mathbb{R}^3$ Delaunay asymptotics theorem that require minor modifications in order to prove (ii):

A. The weight condition in equation (8.12) must be computed by taking into account that A now satisfies (7.4) rather than (7.3).

B. It follows from $\lim_{z \to z_e} ||F^{-1}G - \mathrm{Id}||_{C_1} = 0$ that (8.9) holds. When the target space is S^3 rather than $\mathbb{R}^3$, a slightly different computation is required: Given $\lambda, \mu \in C_1$, both $G_\lambda^{-1} F_\lambda$ and $G_{\mu\lambda}^{-1} F_{\mu\lambda}$ converge to Id as $z \to 0$. Hence both $\alpha = \det(F_\lambda G_\lambda^{-1})$ and $\beta = \det(G_{\mu\lambda} F_{\mu\lambda}^{-1})$ converge to 1 as $z \to 0$. Then $g \to f$ as $z \to 0$ follows from (5.4) by rewriting

$$g_\lambda - f_\lambda = G_{\mu\lambda} \left(G_\lambda^{-1} F_\lambda - \sqrt{\alpha\beta}\, G_{\mu\lambda}^{-1} F_{\mu\lambda} \right) F_\lambda^{-1} \sqrt{\det(G_\lambda G_{\mu\lambda}^{-1})}.$$

(iii) The case of Delaunay asymptotics in H^3 requires a significantly different proof, using the explicit Delaunay frame computed in Corollary 11.3, because now $\lambda_0 \notin C_1$. Let ξ, Ψ, G, F, P and $ab \neq 0$ all be as in Theorem 8.3, with r-Iwasawa splittings $\Phi := \Psi P^{-1} = F B_D$, $\Psi = G B$.

Step 1: *Changing ξ_0 to zero.* Recall from (8.13) that we now assume $[A, \xi_0] = 0$ for all $\lambda \in C_1$. This condition is included for the following technical reason:

In the proofs of Delaunay asymptotics for $\mathbb{R}^3$ in [**22**], [**34**] and [**55**] first a gauge transformation and a conformal change of parameter z fixing z_e is applied to achieve $\xi_0 = 0$, and hence $\xi = A(z-z_e)^{-1}dz + \mathrm{O}((z-z_e)^1)dz$. To accomplish this gauging, the condition that $\det A \neq -1/4$ in $I_r^* := I_r \setminus \{0\}$ for $r<1$ and r close to 1 is used. However, $\det A = -1/4$ does occur in I_r^* when $\lambda_0 \notin C_1$. To avoid this problem, we include the condition (8.13), and then a mere conformal change of parameter z fixing z_e can force $\xi_0 = 0$. Hence $\xi_0 = P_1 = 0$. So P is defined at $\lambda_0 = \lambda_{1,-}$, and there exists a positive $s < |\lambda_0|$ such that P is nonsingular on A_s. We cannot weaken (8.13) to the assumption that $[A, \xi_0] = 0$ at only λ_0, since then one can still find examples where P_1 becomes singular at λ_0. (In fact, the trinoid potentials in section 9 all satisfy condition (8.13), regardless of the target.)

Step 2: *F, G, f, g are well defined.* Now G and F are defined and nonsingular for all $\lambda \in \cup_{r\in(s,1)} C_r$, because Ψ, P, B and B_D are. In particular, this is so at λ_0, and so f and g are defined. (See Remark 8.4.)

Step 3: *Sufficient conditions for convergence.* As in the asymptotics theorem in [**55**], there exists a $\rho > s$ such that equation (8.8) (with F_D replaced by F) holds for all $r \in (\rho, 1)$. It will suffice to strengthen this to

$$\lim_{z\to z_e} ||F^{-1}G - \mathrm{Id}||_{\mathcal{C}} = 0\,, \quad \lim_{z\to z_e} ||\partial_\lambda(F^{-1}G)||_{\mathcal{C}} = 0 \tag{12.1}$$

where $\mathcal{C} = \{\lambda \,|\, s < |\lambda| < 1\}$. Then (8.9) will hold at $\lambda_0 = e^q$, by an argument analogous to part **B** in the proof of (ii).

Step 4: *f is a Delaunay surface.* Since $d\Phi = \Phi A(z-z_e)^{-1}dz$, we have $\Phi = C(z-z_e)^A$ for some $C = C(\lambda)$, where C is singular at the same points as P. Condition (8.14) and $\xi_0 = 0$ imply that P is nonsingular on A_s, so C is as well. Then Lemma 12.1 implies that f is an end of a Delaunay surface in H^3 with weight w.

Step 5: *Showing* (12.1). Let $C = C_u C_+$ be the r-Iwasawa decomposition of C. By Lemma 12.2, there exists a $\rho \in (s,1)$ such that for all $r \in (\rho,1)$, there exist a $U \in \Lambda_r^{\mathbb{R}}\mathrm{GL}_2(\mathbb{C})$ and an $R : A_r \to \mathfrak{gl}_2(\mathbb{C})$ commuting with A such that $C_+ = UR$ and so that $\hat{A} := UAU^{-1}$ has the form in (7.2). A and $\hat{A}$ have the same weight w, and $\hat{A}$ satisfies (7.5) at λ_0. It follows that

$$\Phi = C_u (z-z_e)^{\hat{A}} C_+ .$$

Consider the r-Iwasawa decomposition $(z-z_e)^{\hat{A}} = F_0 B_0$ for $r \in (\rho, 1)$. It follows that $C_u F_0 = F$ and $B_0 C_+ = B_D$. To show (12.1), we first show that

$$\lim_{z\to z_e} ||B_D P B_D^{-1} - \mathrm{Id}||_{\mathcal{C}} = 0\,, \quad \lim_{z\to z_e} ||\partial_\lambda(B_D P B_D^{-1})||_{\mathcal{C}} = 0. \tag{12.2}$$

By Corollary 11.5, there exists an $\mathcal{X} \in \mathbb{R}^+$ and a function $c_0(\lambda)$ such that $\log|z - z_e| < -\mathcal{X}$ implies

$$|B_0(z,\lambda)| \leq c_0(\lambda)\,|z-z_e|^{-c}$$

for all $\lambda \in \overline{I_1}$, where $c = \max_{\lambda\in C_1} |\mu(\lambda)|$ with $\mu(\lambda)$ an eigenvalue of A and $\hat{A}$.

Since $C_+ \in \Lambda_r^+\mathrm{GL}_2(\mathbb{C})$ is independent of z, we have that $|C_+|$ is universally bounded on I_r, and so B_0 and B_D have the same growth rate as $z \to z_e$. Thus

$$||B_D P B_D^{-1} - \mathrm{Id}||_{\mathcal{C}} \leq ||B_D||_{\mathcal{C}}\, ||P - \mathrm{Id}||_{\mathcal{C}}\, ||B_D^{-1}||_{\mathcal{C}} \approx |z-z_e|^{2-2c}$$

asymptotically. The weight conditions on w in equation (8.14) imply that $c < 1$, proving the first half of (12.2).

As $B_D P B_D^{-1}$ is holomorphic in λ, shifting s to be slightly larger but still in $(0, |\lambda_0|)$ and applying the Cauchy integral formula, we also have that

$$\lim_{z \to z_e} ||\partial_\lambda (B_D P B_D^{-1})||_{\mathcal{C} \cap I_r} = 0.$$

Thus (12.2) holds. This C^1 convergence of $B_D P B_D^{-1}$ to Id with respect to λ suffices to conclude that the s-unitary part of $B_D P B_D^{-1}$ converges to Id on $C_s \cup C_{1/s}$ (see [**51**]). Then the maximum modulus principle for holomorphic functions implies that the s-unitary part of $B_D P B_D^{-1}$ C^1-converges to Id on A_s, and in particular on $\mathcal{C}$.

Since $B_D P B_D^{-1}$ is nonsingular on $A_s \cap I_1$, ℓ-Iwasawa decomposition is the same for all $\ell \in [s, 1)$. Thus the s-unitary part of $B_D P B_D^{-1}$ is equal to the r-unitary part $F^{-1}G$ of $B_D P B_D^{-1}$, and thus (12.1) holds. Hence g has an asymptotically Delaunay end at z_e of weight w in the sense of (8.9). This concludes the proof of Theorem 8.3. □

Proof of Theorem 10.3: We use the explicit form of the unitary frame of a Delaunay surface, computed in Theorem 11.1. After a rigid motion we may assume that $F = UF_{\mathrm{D}}$, with F_{D} as in (11.5) and $U \in \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$. Let f^{D} be the immersion obtained from the frame $h_{\mathrm{L},\alpha} U F_{\mathrm{D}} h_\infty^{-1}$, where h_∞ will be defined below after some prerequisite functions are defined. We will show that f^{D} is a Delaunay surface at the end of this proof (Step 3). We recall from equation (10.5) that f_{λ_0} has extended frame $h_{\mathrm{L},\alpha} \#_r G = h_{\mathrm{L},\alpha} G h_{\mathrm{L}',\alpha}^{-1}$. Define x, y by $\log z = -(x + iy)$ and let λ_0 be as in Section 7.2.

Step 1: *Reducing the convergence $f_{\lambda_0} \to f_{\lambda_0}^{\mathrm{D}}$ as $z \to 0$ to $h_{\mathrm{L}'',\alpha} \to h_\infty$.* As in the proof of Theorem 8.3, it suffices to show that $||\mathrm{Id} - (h_{\mathrm{L}',\alpha} G^{-1} h_{\mathrm{L},\alpha}^{-1})(h_{\mathrm{L},\alpha} U F_{\mathrm{D}} h_\infty^{-1})||_{B_\varepsilon(\lambda_0)} \to 0$ for some sufficiently small $\varepsilon > 0$. (This suffices because the Cauchy integral formula will additionally give $||\partial_\lambda((h_{\mathrm{L}',\alpha} G^{-1} h_{\mathrm{L},\alpha}^{-1})(h_{\mathrm{L},\alpha} U F_{\mathrm{D}} h_\infty^{-1}))||_{B_{\varepsilon/2}(\lambda_0)} \to 0$.) Setting

$$\mathrm{L}'' = \overline{U(\alpha) F_{\mathrm{D}}(z, \alpha)}^t \mathrm{L},$$

we consider, for $\lambda \in B_\varepsilon(\lambda_0)$,

$$\begin{aligned} |\mathrm{Id} - (h_{\mathrm{L}',\alpha} G^{-1} h_{\mathrm{L},\alpha}^{-1})(h_{\mathrm{L},\alpha} U F_{\mathrm{D}} h_\infty^{-1})| \leq{}& |h_{\mathrm{L}',\alpha}|\, |\mathrm{Id} - G^{-1} U F_{\mathrm{D}}|\, |h_{\mathrm{L}',\alpha}^{-1}| \\ &+ |h_{\mathrm{L}',\alpha} G^{-1} U F_{\mathrm{D}} h_{\mathrm{L}',\alpha}^{-1}|\, |\mathrm{Id} - h_{\mathrm{L}',\alpha} h_{\mathrm{L}'',\alpha}^{-1}| \\ &+ |h_{\mathrm{L}',\alpha} G^{-1} U F_{\mathrm{D}} h_{\mathrm{L}'',\alpha}^{-1}|\, |\mathrm{Id} - h_{\mathrm{L}'',\alpha} h_\infty^{-1}|. \end{aligned}$$

Note that $|h_{\mathrm{L}',\alpha}|$ is bounded, because $\alpha \notin B_\varepsilon(\lambda_0)$ for small ε. By assumption,

$$\lim_{x \to \infty} ||\mathrm{Id} - G^{-1} U F_{\mathrm{D}}||_{B_\varepsilon(\lambda_0)} = 0.$$

Writing $\mathrm{L}'' = \overline{G^{-1}\, U\, F_{\mathrm{D}}}^t \mathrm{L}'$, we conclude that $\lim_{x \to \infty} ||\mathrm{Id} - h_{\mathrm{L}',\alpha} h_{\mathrm{L}'',\alpha}^{-1}||_{B_\varepsilon(\lambda_0)} = 0$. It thus only remains to show that $h_{\mathrm{L}'',\alpha}$ converges to the map h_∞ defined below.

Step 2: *Showing $h_{\mathrm{L}'',\alpha} \to h_\infty$ as $z \to 0$.* Using equation (11.5), we write

$$F_{\mathrm{D}} = \frac{1}{\mathcal{L}} \begin{pmatrix} 2v(b+a\lambda)C & -v'C + (4ab\lambda + v^2)(a\lambda^{-1}+b)X^{-1}S \\ 2v(b+a\lambda)(a\lambda+b)X^{-1}S & -v'(a\lambda+b)X^{-1}S + (4ab\lambda+v^2)C \end{pmatrix}, \tag{12.3}$$

where $\mathcal{L} = \sqrt{2v(b+a\lambda)(4ab\lambda+v^2)}$, $C = \cosh((x+iy-\mathbf{f})X)$, $S = \sinh((x+iy-\mathbf{f})X)$, $X = \sqrt{1/4 - \Upsilon + \lambda^{-1}ab(1-\lambda)^2}$, and a, b are defined in equation (7.3) or (7.4) or (7.5), and $\Upsilon = 0$ or $\Upsilon = -4ab\sin^2(\theta)$ or $\Upsilon = 4ab\sinh^2(q/2)$ for the target spaces $\mathbb{R}^3$, S^3 or H^3 respectively. We set

$$F_{\mathrm{D}}|_{\lambda=\alpha} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B}^* & \mathcal{A}^* \end{pmatrix}. \tag{12.4}$$

The upper-left entry of F_{D} defined from equation (12.3) with $\lambda = \alpha$ has no zeroes near $x = \infty$, so $\mathcal{A}$ has no zeroes near $x = \infty$. Thus from equation (10.8), the entries of $\psi_{\mathrm{L}'',\alpha}$ (equation (10.1)) can be written as rational functions of $\mathcal{B}/\mathcal{A}$, $\mathcal{B}^*/\mathcal{A}$, $\mathcal{A}^*/\mathcal{A}$ and their complex conjugates, with coefficients independent of z. Thus the entries of $h_{\mathrm{L}'',\alpha}$ are rational in terms of $\mathcal{B}/\mathcal{A}$, $\mathcal{B}^*/\mathcal{A}$, $\mathcal{A}^*/\mathcal{A}$ and their complex conjugates, with coefficients independent of z. We show that $\mathcal{B}/\mathcal{A}$ respectively $\mathcal{B}^*/\mathcal{A}$, $\mathcal{A}^*/\mathcal{A}$ converge to the following periodic functions P_1 respectively P_2, P_3 as $x \to \infty$:

$$P_1 = \frac{-v' + (4ab\alpha + v^2)(a\alpha^{-1}+b)X_\alpha^{-1}}{2v(b+a\alpha)}, \tag{12.5}$$

$$P_2 = -\frac{2v(b+a\alpha)(a\alpha+b)X_\alpha^{-1}}{2v(b+a\alpha)}, \tag{12.6}$$

$$P_3 = -\frac{v'(a\alpha+b)X_\alpha^{-1} + 4ab\alpha + v^2}{2v(b+a\alpha)}, \tag{12.7}$$

where X_α denotes the value of X at $\lambda = \alpha$. We show only that $\mathcal{B}/\mathcal{A}$ converges to P_1. (To show convergence of $\mathcal{B}^*/\mathcal{A}$ and $\mathcal{A}^*/\mathcal{A}$ to P_2 respectively P_3 is similar.) We have

$$\begin{aligned} \frac{\mathcal{B}}{\mathcal{A}} &= \frac{-v' + (4ab\alpha+v^2)(a\alpha^{-1}+b)X_\alpha^{-1}S_\alpha/C_\alpha}{2v(b+a\alpha)} \\ &= \frac{-v' + (4ab\alpha+v^2)(a\alpha^{-1}+b)X_\alpha^{-1}\frac{1-\exp(-2(x+iy-\mathbf{f}_\alpha)X_\alpha)}{1+\exp(-2(x+iy-\mathbf{f}_\alpha)X_\alpha)}}{2v(b+a\alpha)}, \end{aligned}$$

where C_α, S_α and $\mathbf{f}_\alpha$ denote the values of C, S respectively $\mathbf{f}$ at $\lambda = \alpha$. It thus suffices to show

$$\lim_{x\to\infty} \mathrm{Re}((x+iy-\mathbf{f}_\alpha)X_\alpha) = \infty, \tag{12.8}$$

to conclude that $\mathcal{B}/\mathcal{A}$ converges to P_1. When $|a| = |b|$, then the function as in (11.4) is constant $v = 2b$, and (12.8) is clear. When $|a| > |b|$, because $\alpha \in \mathfrak{A}$, $\mathrm{Re}((x+iy-\mathbf{f}_\alpha)X_\alpha) = (x-\mathbf{f}_\alpha)X_\alpha$. Now v is not constant, and since $|\alpha| < |b/a|$ and $\mathrm{dn}(2ax, 1-b^2/a^2) \in (0,1]$, it follows that $\partial_x(x-\mathbf{f}_\alpha)X_\alpha$ is positive and uniformly bounded away from zero for all $x \in \mathbb{R}$. Thus (12.8) holds.

We set h_∞ to be $h_{\mathrm{L}'',\alpha}$ with $\mathcal{B}/\mathcal{A}$ (respectively $\mathcal{B}^*/\mathcal{A}, \mathcal{A}^*/\mathcal{A}$) replaced by P_1 (respectively P_2, P_3) in equation (12.5) (respectively (12.6), (12.7)). Thus $\lim_{x\to\infty} ||\mathrm{Id}-h_{\mathrm{L}'',\alpha}h_\infty^{-1}||_{B_\epsilon(\lambda_0)} = 0$. Therefore $f_{\lambda_0} \to f_{\lambda_0}^{\mathrm{D}}$ as $x \to \infty$ in the sense of (8.9).

Step 3: *Showing that f^{D} is a Delaunay surface.* (We show only the $\mathbb{R}^3$ case, as the arguments for S^3 and H^3 are similar.) The immersion f^{D} has frame $h_{\mathrm{L},\alpha}\, U\, F_{\mathrm{D}} h_\infty^{-1}$ and $h_{\mathrm{L},\alpha}\, U$ affects a rigid motion of the surface generated by the frame $F_{\mathrm{D}} h_\infty^{-1}$, and h_∞ on the right is independent of y, hence vertical translation of the domain is equivalent to rotating the surface about a fixed line. Thus f^{D} is a Delaunay surface. More rigorously, we consider the translation of the domain

$$\mathcal{T}_\theta : x + iy \mapsto x + i(y+\theta), \text{ for } \theta \in \mathbb{R} .$$

As $g := (h_{\mathrm{L},\alpha}U)^{-1}\left(f^{\mathrm{D}} + 2i\lambda H^{-1}(\partial_\lambda(h_{\mathrm{L},\alpha}U))(h_{\mathrm{L},\alpha}U)^{-1}\right)(h_{\mathrm{L},\alpha}U)$ is a rigid motion of f^{D}, we conclude that $f^{\mathrm{D}}\big|_{\lambda_0=1}$ is a Delaunay surface if $g|_{\lambda_0=1}$ is. Defining A as in (7.2) with $a, b \in \mathbb{R}$ and $c = 0$, we have

$$\begin{aligned}
\mathcal{T}_\theta^*\, g|_{\lambda_0} &= \mathcal{T}_\theta^*\, \left(-2i\lambda H^{-1}(\partial_\lambda(F_{\mathrm{D}}h_\infty^{-1}))(F_{\mathrm{D}}h_\infty^{-1})^{-1}\right)\big|_{\lambda_0} \\
&= \left[\exp(i\theta A)\, g\, \exp(-i\theta A) - 2i\lambda H^{-1}(\partial_\lambda \exp(i\theta A))\exp(-i\theta A)\right]_{\lambda_0},
\end{aligned} \tag{12.9}$$

since h_∞ is independent of the variable y. Equation (12.9) represents a rotation of $g|_{\lambda_0}$ by the angle θ about a fixed axis (independent of θ). Hence $g|_{\lambda_0}$ is a surface of revolution and so f^{D} is a Delaunay surface. This concludes the proof of Theorem 10.3. $\square$

CHAPTER 4

Constant mean curvature surfaces of any positive genus

1. Preliminary results

We denote an annular neighbourhood of the unit circle $\mathbb{S}^1$ for some real $r \in (0,\,1]$ by $A_r = \{\lambda \in \mathbb{C} : r \leq |\lambda| \leq 1/r\}$. It is common abuse to call a map $M : A_r \to \mathrm{SL}_2(\mathbb{C})$ *unitary* if $M|_{\mathbb{S}^1} \in \mathrm{SU}_2$.

DEFINITION 1.1. *We shall call a map $M : A_r \to \mathrm{SL}_2(\mathbb{C})$* unitarisable *on A_s for some $s \in [r,\,1]$ if there exists a map $h : A_s \to \mathrm{GL}_2(\mathbb{C})$ for some $s \in [r,\,1]$ such that $h\,M\,h^{-1} : A_s \to \mathrm{SL}_2(\mathbb{C})$ is unitary.*

We use the following notation for diagonal and off–diagonal 2×2 matrices:

$$\mathrm{diag}[u,\,v] = \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right),\ \mathrm{off}[u,\,v] = \left(\begin{smallmatrix} 0 & u \\ v & 0 \end{smallmatrix}\right).$$

In preparation for Theorems 2.1 and 3.1 we first provide some technical results. The next lemma gives conditions on a matrix which ensure that after unitarisation, it satisfies the closing conditions at $\lambda_0 = e^{i\,x_0}$.

LEMMA 1.1. *Let $J \subset \mathbb{R}$ be an open interval, and let $M : J \to \mathrm{SL}_2(\mathbb{C})$, $U : J \to \mathrm{SU}_2$ be smooth maps with $\mathrm{tr}M = \mathrm{tr}U$. If $M(x_0) = \pm\mathrm{Id}$ and $d_x M(x_0)$ is nilpotent for $x_0 \in J$, then $U(x_0) = \pm\mathrm{Id}$ and $d_x U(x_0) = 0$.*

PROOF. Since $\mathrm{tr}U(x_0) = \pm 2$ and $U(x_0) \in \mathrm{SU}_2$, we have $U(x_0) = \pm\mathrm{Id}$. Let $\tau = \frac{1}{2}\mathrm{tr}M = \frac{1}{2}\mathrm{tr}U$. We differentiate the Cayley-Hamilton equations $M^2 - 2\tau M = U^2 - 2\tau U = -\mathrm{Id}$ twice and evaluate at x_0 to get $\pm d_x M(x_0)^2 = d_x^2\tau(x_0)\mathrm{Id} = \pm d_x U(x_0)^2$. So $d_x M(x_0)$ nilpotent implies $d_x U(x_0)$ is also nilpotent. Since $U \in \mathrm{SU}_2$, we have $d_x U \in \mathfrak{su}_2$, and so nilpotency implies $d_x U(x_0) = 0$. $\square$

Lemma 1.2 computes the derivatives of a solution to a linear ODE with respect to a parameter. From this the series expansion of the trace of the monodromy with respect to the parameter can be computed, and hence the trace can be estimated in a small interval, as we will see in Lemma 1.4. See [**16**] for a related theorem.

LEMMA 1.2. *Let Σ be a simply connected Riemann surface with coordinate z. Let $A(z,\,x) : \Sigma \times \mathbb{R} \to \mathfrak{gl}_2(\mathbb{C})$ be analytic in z and smooth in x and $B(x) : \mathbb{R} \to \mathfrak{gl}_2(\mathbb{C})$ be smooth. Let $X(z,\,x) : \Sigma \times \mathbb{R} \to \mathfrak{gl}_2(\mathbb{C})$ be the solution of the initial value problem*

$$(d_z X)(z,\,x) = X(z,\,x)A(z,\,x)\,, \quad X(z_0,\,x) = B(x). \tag{1.1}$$

Let $X_k(z) : \Sigma \to \mathfrak{gl}_2(\mathbb{C})$, for integers $k \geq 0$, be the solutions to the sequence of initial value problems

$$(d_z X_k)(z) = \sum_{i,j\geq 0, i+j=k} \frac{k!}{i!(k-i)!} X_i(z) A_j(z)\,, \quad X_k(z_0) = B_k,$$

where $A_j(z) := \left(d_x^j A\right)(z, x_0)$ and $B_k := \left(d_x^k B\right)(x_0)$. Then

$$\left(d_x^k X\right)(z, x_0) = X_k(z). \tag{1.2}$$

PROOF. Differentiate (1.1) repeatedly with respect to x. □

In the following, let Σ be a connected Riemann surface with universal cover $\widetilde{\Sigma}$ and Δ its group of deck transformations. We denote the holomorphic 1-forms on Σ by $\Omega'(\Sigma, \mathbb{C})$.

In the next Lemma we show that a certain class of potentials always ensures the closing conditions (0.3) and (0.4). Such potentials will be used in later examples (Theorems 2.1 and 3.1) to show the existence of new CMC surfaces.

LEMMA 1.3. *Let $f, g \in \Omega'(\Sigma, \mathbb{C})$ and $t = \lambda^{-1}(\lambda - 1)^2$ and*

$$A = \begin{pmatrix} 0 & f\,t \\ g & 0 \end{pmatrix}. \tag{1.3}$$

Let $w_0 \in \widetilde{\Sigma}$ and X be the solution to the initial value problem

$$dX = X\,A\,, \quad X(w_0, t) = \mathrm{Id}\,. \tag{1.4}$$

Let $\gamma \in \Delta$ and $M(t) := X(\gamma(w_0), t)$. Suppose that

$$\int_{w_0}^{\gamma(w_0)} g = 0. \tag{1.5}$$

Then $\tilde{M}(1) = \mathrm{Id}$ and $d_\lambda \tilde{M}(1) = 0$, where $\tilde{M}(\lambda) = M(t)$.

PROOF. Note that $X(w, 0) = \mathrm{Id} + \mathrm{off}[0, \int_{w_0}^{w} g]$. Hence $X(\gamma(w_0), 0) = \mathrm{Id}$, and so $\tilde{M}(1) = \mathrm{Id}$. Then with $z_0 = w_0$, $e^{ix} = \lambda$, $e^{ix_0} = \lambda_0 = 1$, $B(x) = \mathrm{Id}$, A as in (1.3) and z fixed to $\gamma(w_0)$ in (1.2), it follows that $A_1(z)$ is identically zero, and Lemma 1.2 implies that $(d_\lambda \tilde{M})(1) = 0$. □

The next lemma will be used in the proofs of Theorems 2.1 and 3.1 to show that certain monodromy groups can be unitarised.

LEMMA 1.4. *Take the same notations and conditions as in Lemma 1.3, with t replaced by ct for some constant $c \in \mathbb{R} \setminus \{0\}$. Suppose that $\tau(ct) = \frac{1}{2}\mathrm{tr}M(ct)$ is real for all $t \in [-4, 0]$ and that*

$$I_0 I_2 + I_1^2 < 0\,, \tag{1.6}$$

where

$$I_0 = \int_{w_0}^{\gamma(w_0)} f\,, \quad I_1 = \int_{w_0}^{\gamma(w_0)} \left(f \int g\right) \quad \text{and} \quad I_2 = \int_{w_0}^{\gamma(w_0)} \left(g \int \left(f \int g\right)\right).$$

Then for $\tilde{\tau}(\lambda, c) = \tau(ct) = \tau(c\lambda^{-1}(\lambda - 1)^2)$ *there exists a* $c_0 > 0$ *such that for all* $|c| \in (0, c_0)$ *we have*

$$|\tilde{\tau}(\lambda, c)| < 1 \text{ for all } \lambda \in \mathbb{S}^1 \setminus \{1\}\,.$$

PROOF. Let X be the solution of equation (1.4) with f replaced by cf, and write $X_{ij} = X_{ij}(w, t, c)$ for the entries of X. Defining $\hat{X}_{ij} = \frac{d}{d(ct)} X_{ij}(\gamma(w_0), 0, c)$, Lemma 1.2 implies

$$\hat{X}_{11} = \int_{w_0}^{\gamma(w_0)} \Big(g \int f \Big)\,,\ \hat{X}_{12} = \int_{w_0}^{\gamma(w_0)} f$$
$$\hat{X}_{21} = \int_{w_0}^{\gamma(w_0)} \Big(g \int f \Big(\int g \Big) \Big) \text{ and } \hat{X}_{22} = \int_{w_0}^{\gamma(w_0)} \Big(f \int g \Big)\,.$$

Note that the $\hat{X}_{ij}$ are independent of c. Considering the first two derivatives of $\det X = 1$ with respect to ct and evaluating at $t = 0$ and $w = \gamma(w_0)$ yields

$$(d_{ct}\tau)(0, c) = 0\,, \quad (d_{ct}^2\tau)(0, c) = \hat{X}_{12}\hat{X}_{21} - \hat{X}_{11}\hat{X}_{22} = \hat{X}_{12}\hat{X}_{21} + \hat{X}_{22}^2\,.$$

Note that the first of these two equations implies $\hat{X}_{11} = -\hat{X}_{22}$, which is used in the second of these two equations. Equation (1.6) implies that $\hat{X}_{12}\hat{X}_{21} + \hat{X}_{22}^2 < 0$, and so the second derivative with respect to ct of τ is negative, and τ attains a maximum of 1 at $t = 0$. Thus there exists a $k_0 > 0$ such that $|ct| \in (0, k_0]$ implies $|\tau(ct)| \in [0, 1)$. Let $c_0 = k_0/4$. Then for all c such that $|c| \in (0, c_0)$ and for all $t \in [-4, 0]$, we have $|\tau(ct)| \in [0, 1)$ and the lemma follows. □

Applying Lemma 1.2, similarly to the proofs of Lemmas 1.3 and 1.4, we obtain

COROLLARY 1.5. *With notations and conditions as in Lemmas 1.3 and 1.4, we have* $d_\lambda^2 \tilde{M}(1) = 2(\mathrm{diag}[\,-I_1,\, I_1\,] + \mathrm{off}[\,I_0,\, I_2\,])$.

2. Singly-punctured CMC surfaces of arbitrary genus

We construct a family of CMC immersions of a singly-punctured genus g Riemann surface into $\mathbb{R}^3$ with umbilics, for any positive g. The closing problem is solved by imposing symmetries so that the monodromy group can be shown to be unitarisable.

THEOREM 2.1. *Let* $n \geq 2$ *be an even integer. Let* Σ *be the singly-punctured hyperelliptic genus* $n/2$ *Riemann surface defined by* $\Sigma = \{(z, w) \in \mathbb{C}^2 \mid w^2 = z(1 - z^n)\}$. *Let*

$$\xi = \begin{pmatrix} 0 & c\lambda^{-1}(\lambda - 1)^2 w^{-1} dz \\ d(z^{n-1}w) & 0 \end{pmatrix}, \quad c \in \mathbb{R}^*\,.$$

Then for c *sufficiently close to zero,* ξ *induces a conformal* CMC *immersion* $\Sigma \to \mathbb{R}^3$ *with order* $2n$ *dihedral symmetry.*

PROOF. Choose a basepoint $(z_0, w_0) \in \widetilde{\Sigma}$ in the fibre of $(0, 0) \in \Sigma$ and let Φ be the solution to the initial value problem (5.1) with $\Phi_0 = \Phi(z_0, w_0) = \mathrm{I}\alpha$. We must show that there exists a unitariser for the monodromy of Φ and verify the closing conditions.

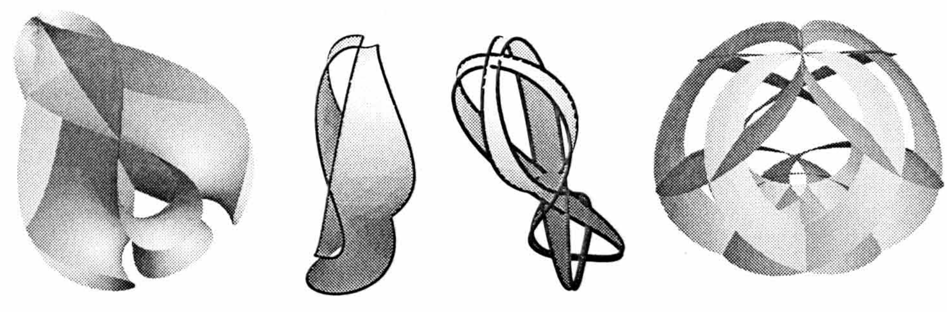

FIGURE 1. The three figures on the left are parts of a CMC singly-punctured torus, as in Theorem 2.1 with $n = 2$. The left-most image shows the torus with a neighborhood of the end removed. The surface has 90° rotation symmetry and reflection symmetry. The second image shows one-fourth of the surface, which extends to the full surface (again with a neighborhood of the end removed) by these symmetries. The third image is a skeletal portion of the surface. The Hopf differential has a pole of order 6 at the end. The right-most figure is a CMC doubly-punctured torus, as in Theorem 3.1, and the image here shows a skeleton of this torus (with a doubly-punctured disk containing the ends removed).

With $\alpha = \exp(\pi i/n)$, for $k \in \{0, \dots, n-1\}$ let $\gamma_k : [0,1] \to \Sigma$ be the curve from $(0,0)$ to $(\alpha^{2k}, 0)$ and back to $(0,0)$ along the straight line in the z-plane from 0 to α^{2k}, defined by

$$\gamma_k(s) = \begin{cases} \left(2s\alpha^{2k}, (-\alpha)^k \left|\sqrt{2s(1-2^n s^n)}\right|\right), & 0 \le s \le 1/2 \\ \left(2(1-s)\alpha^{2k}, -(-\alpha)^k \left|\sqrt{2(1-s)(1-2^n(1-s)^n)}\right|\right) & 1/2 \le s \le 1. \end{cases}$$

Let $\hat{\gamma}_k$ be the lifted curves originating at (z_0, w_0) and $M_k(\lambda) := \Phi(\hat{\gamma}_k(1), \lambda)$. Then $M_0, \dots, M_{n-1}$ generate the monodromy group of Φ, since Σ has only one puncture at $(z,w) = (\infty, \infty)$, and the monodromy about the puncture is $N = \prod_{k=0}^{n-1} M_k$. Lemma 1.3 then implies

$$M_k(1) = \mathrm{Id}, \quad d_\lambda M_k(1) = 0\,, \quad k \in \{0, \dots, n-1\}\,. \tag{2.1}$$

Hence we also have $N(1) = \mathrm{Id}$, $d_\lambda N(1) = 0$. It remains to show that there exists a unitariser for the monodromy group. For this, we compute the monodromy group's symmetries. We define the following maps on Σ:

$$\sigma(z,w) = (\alpha^2 z, \alpha w)\,,\ \rho(z,w) = (z, -w) \text{ and } \theta(z,w) = (\overline{z}, \overline{w})\,. \tag{2.2}$$

Then for $g_\sigma = \mathrm{diag}[\sqrt{\alpha}^{-1}, \sqrt{\alpha}]$ and $g_\rho = \mathrm{diag}[-i, i]$ we have

$$\sigma^*\xi = g_\sigma^{-1}\,\xi\, g_\sigma,\ \rho^*\xi = g_\rho^{-1}\,\xi\, g_\rho \text{ and } \overline{\theta^*\xi(1/\overline{\lambda})} = \xi(\lambda),$$

where expressions like $\sigma^*\xi$ denote $\sigma^*\xi((z,w),\lambda) = \xi(\sigma(z,w),\lambda)$. Since $(0,0)$ is a fixed point of σ, ρ and θ, we define the lifts $\hat{\sigma}$, $\hat{\rho}$, $\hat{\theta}$ that map $(0,0)$ to (z_0, w_0).

Then using $\Phi(z_0, w_0) = \mathrm{Id}$, we obtain

$$\hat{\sigma}^*\Phi = g_\sigma^{-1}\Phi\, g_\sigma\ , \quad \hat{\rho}^*\Phi = g_\rho^{-1}\Phi\, g_\rho\ , \quad \overline{\hat{\theta}^*\Phi(1/\bar{\lambda})} = \Phi(\lambda)\ .$$

Hence the monodromy group has the following symmetries:

$$\begin{aligned} &M_k^{(-1)^k} = g_\sigma^{-k} M_0\, g_\sigma^k\ , \quad k \in \{0, \ldots, n-1\}\ , \\ &M_0^{-1} = g_\rho^{-1} M_0\, g_\rho\ , \\ &\overline{M_0} = M_0 \quad \text{for all } \lambda \in \mathbb{S}^1\ . \end{aligned} \tag{2.3}$$

Note that the third of these symmetries also follows from the facts that $\lambda^{-1}(\lambda-1)^2 \in \mathbb{R}$ for all $\lambda \in \mathbb{S}^1$ and $\gamma_0(s) \in \mathbb{R}^2$ for all $s \in [0,1]$. Denoting the entries of M_0 by M_{ij}, the third symmetry in (2.3) implies that the M_{ij} are all real, and the second symmetry in (2.3) implies that $M_{11} = M_{22}$, for all $\lambda \in \mathbb{S}^1$. In particular, $\mathrm{tr}(M_0)$ is real for all $\lambda \in \mathbb{S}^1$. The integrals I_j in equation (1.6) are then

$$I_0 = 2c \int_{\hat{\gamma}} |w|^{-1} dz\ , \quad I_1 = 0\ , \quad \text{and } I_2 = \frac{2c}{n} \int_{\hat{\gamma}} z^n d(z^{n-1}|w|)\ ,$$

for the curve $\hat{\gamma}(s) = (s, |\sqrt{s(1-s^n)}|) \in \Sigma$, $s \in [0,1]$. Using the formula, valid for $\mathrm{Re}\, n > 0$, $\mathrm{Re}\, r > 0$ and $\mathrm{Re}\, s > 0$,

$$\int_{\hat{\gamma}} z^{r-1}(1-z^n)^{s-1}\, dz = \frac{\Gamma(\frac{r}{n})\,\Gamma(s)}{n\,\Gamma(\frac{r}{n}+s)}\ ,$$

where $\Gamma(\ell) = \int_0^\infty y^{\ell-1}e^{-y}dy$ is the Euler gamma function, we get

$$I_0 I_2 + I_1^2 = -\frac{8\pi c^2(2n-1)\cot(\frac{\pi}{2n})}{(n-1)(3n-1)(5n-1)} < 0.$$

Hence by Lemma 1.4, with $\tau(\lambda, c) = \frac{1}{2}\mathrm{tr}(M_0(\lambda))$, there exists a $c_0 > 0$ such that for all c satisfying $|c| \in (0, c_0)$,

$$|\tau(\lambda, c)| < 1 \text{ for all } \lambda \in \mathbb{S}^1 \setminus \{1\}\ , \tag{2.4}$$

and $\tau(1, c) = 1$. Then $\tau = M_{11} = M_{22} \in \mathbb{R}$ has modulus at most 1 for all $\lambda \in \mathbb{S}^1$. Thus $-M_{12}M_{21} = 1 - M_{11}^2 \geq 0$ on $\mathbb{S}^1$, so

$$v := -\frac{M_{21}}{M_{12}} \geq 0 \text{ on } \mathbb{S}^1\ .$$

Furthermore, v is finite and strictly positive on $\mathbb{S}^1 \setminus \{1\}$, by (2.4).

Let us now consider the behavior of v at $\lambda = 1$. By (2.1), we know that

$$M_{12}|_{\lambda=1} = M_{21}|_{\lambda=1} = d_\lambda M_{12}|_{\lambda=1} = d_\lambda M_{21}|_{\lambda=1} = 0\ .$$

Applying Corollary 1.5, we have $d_\lambda^2 M_{12}|_{\lambda=1} = 2I_0$ and $d_\lambda^2 M_{21}|_{\lambda=1} = 2I_2$. Since $I_1 = 0$ and $I_0 I_2 + I_1^2 < 0$, we conclude that I_0 and I_2 are both nonzero, so M_{12} and M_{21} both have zeroes of order exactly two at $\lambda = 1$. Hence v is nonzero and finite at $\lambda = 1$. Thus v is a strictly positive finite function on all of $\mathbb{S}^1$, and therefore $\sqrt[4]{v}$

can be globally and smoothly defined on $\mathbb{S}^1$. Then by the first symmetry of (2.3), the diagonal unitariser is given by

$$h = \begin{pmatrix} \sqrt[4]{v} & 0 \\ 0 & (\sqrt[4]{v})^{-1} \end{pmatrix},$$

which simultaneously unitarizes $M_0, \ldots, M_{n-1}$ on $\mathbb{S}^1$, i.e. $hM_jh^{-1} \in \mathrm{SU}_2$ for all $\lambda \in \mathbb{S}^1$. Therefore the monodromy group of $h\Phi$ is unitarized on all of $\mathbb{S}^1$.

By Equation (2.1) and Lemma 1.1, the monodromy group hM_jh^{-1} still satisfies the closing conditions (0.3) and (0.4) at $\lambda = 1$. Hence the resulting CMC immersion is well-defined on Σ.

Since the coefficient $cw^{-1}dz$ of the λ^{-1} term of the upper-right entry of the potential ξ has no zeros or poles on the singly-punctured Riemann surface Σ, the CMC immersion is unbranched, see [**16**], Theorem 3.1.

We now consider the symmetries of the CMC immersion resulting from $h\Phi$. Since (0, 0) is fixed by the map σ and h is independent of (z, w) and $[h, g_\sigma] = 0$ and also $\hat{\sigma}^*\Phi = g_\sigma^{-1}\Phi g_\sigma$, we have that $(\hat{\sigma}^k)^*(h\Phi) = g_\sigma^{-k}(h\Phi)g_\sigma^k$, where $\hat{\sigma}^k$ is the composition of $\hat{\sigma}$ with itself k times.

Let $h\Phi = FB$ be the Iwasawa decomposition with respect to $\mathbb{S}^1$ (Theorem 8.1.1 [**51**]), pointwise on $\widetilde{\Sigma}$. Since $g_\sigma^{-k}Fg_\sigma^k$ is unitary and $g_\sigma^{-k}Bg_\sigma^k$ positive, the unitary part of $(\hat{\sigma}^k)^*(h_1\Phi)$ is $g_\sigma^{-k}Fg_\sigma^k$. The symmetry $(\hat{\sigma}^k)^*F = g_\sigma^{-k}Fg_\sigma^k$ descends to the immersion via the Sym-Bobenko formula [**5**], see also [**36**] section 4, and results in a rotation of angle $k\pi/n$ about an axis independent of k. Hence the surface has an order $2n$ rotational symmetry.

To show dihedral symmetry, we now need only show that the surface has at least one reflective symmetry across a plane parallel to the common axis of the rotational symmetries. We will show that the map $\theta(z, w) = (\bar{z}, \bar{w})$ is such a reflective symmetry, by showing that the immersion generated by F via the Sym-Bobenko formula [**5**], and denoted by f, satisfies

$$\theta^* f = -\bar{f}\,.$$

Because $\xi|_\lambda = \xi|_{\lambda^{-1}}$, we have $\Phi(z, w, \lambda) = \Phi(z, w, \lambda^{-1})$ and consequently

$$\overline{\Phi(\bar{z}, \bar{w}, \bar{\lambda})} = \overline{\Phi(\bar{z}, \bar{w}, \bar{\lambda}^{-1})} = \overline{\hat{\theta}^*\Phi(\bar{\lambda}^{-1})} = \Phi(z, w, \lambda).$$

This further implies that $\overline{\Phi(\gamma_0(s), \bar{\lambda})} = \Phi(\gamma_0(s), \lambda)$ and so $\overline{M(\bar{\lambda})} = M(\lambda)$, and in turn $\overline{h(\bar{\lambda})} = h(\lambda)$, since $\theta^*\gamma_0(s) = \gamma_0(s)$. Thus

$$\overline{h(\bar{\lambda})\,\Phi(\bar{z}, \bar{w}, \bar{\lambda})} = h(\lambda)\Phi(z, w, \lambda)$$

and consequently $\overline{F(\bar{z}, \bar{w}, \bar{\lambda})}\,\overline{B(\bar{z}, \bar{w}, \bar{\lambda})} = F(z, w, \lambda)\,B(z, w, \lambda)$. Uniqueness of the Iwasawa decomposition yields $\overline{F(\bar{z}, \bar{w}, \bar{\lambda})} = F(z, w, \lambda)$ and implies $\theta^* f = -\bar{f}$. □

REMARK 2.2. *Note that the end of any surface in Theorem 2.1 is not asymptotically Delaunay, because the order of the Hopf differential there is strictly less than* -2. *This is also implied by* [**43**], *since Delaunay ends have non-zero weight, but the balancing formula implies that the single end of any surface in Theorem 2.1 must have zero weight.*

3. CMC immersions of a doubly-punctured torus

In this section, we construct immersions of a doubly-punctured genus 1 Riemann surface into $\mathbb{R}^3$ with umbilics.

THEOREM 3.1. *Let $\mathcal{T} = \{[z] \in \mathbb{C}/\Gamma \,|\, z \in \mathbb{C}\}$ be the square torus, where Γ is the 2-dimensional lattice generated by $2\omega_1 \in \mathbb{R}^+$ and $2\omega_2 = 2i\omega_1$. Let $\omega_3 = \omega_1 + \omega_2$. On the twice-punctured torus $\Sigma = \mathcal{T} \setminus \{[\omega_3/2], [-\omega_3/2]\}$, let ξ be the potential*

$$\xi = \begin{pmatrix} 0 & c\lambda^{-1}(\lambda-1)^2 \\ \wp''''(z+\omega_3/2) + \wp''''(z-\omega_3/2) & 0 \end{pmatrix} dz\,, \quad c \in \mathbb{R}^*\,,$$

where $\wp$ is the Weierstrass $\wp$-function with respect to $\mathcal{T}$ satisfying $(\wp')^2 = 4\wp(\wp^2-1)$ and $'$ denotes the derivative with respect to z. Then for c sufficiently close to zero, ξ induces a conformal CMC *immersion $\Sigma \to \mathbb{R}^3$ with order 4 dihedral symmetry.*

REMARK 3.2. *Note that $\wp'''' = 120\wp^3 - 72\wp$. Then, since $\wp(-z) = \wp(z)$ and $\wp(iz) = -\wp(z)$, it follows that also $\wp''''(-z) = \wp''''(z)$ and $\wp''''(iz) = -\wp''''(z)$. These properties will be used in the following proof. One other particular property that we will need is, defining*

$$\begin{aligned}\mathcal{I}(z) &= \wp'''(z+\omega_3/2) + \wp'''(z-\omega_3/2) - \wp'''(\omega_3/2) - \wp'''(-\omega_3/2) \\ &= \wp'''(z+\omega_3/2) + \wp'''(z-\omega_3/2),\end{aligned}$$

that the integral $\int_0^{2\omega_1} (\mathcal{I}(z))^2 dz > 0$ along the real axis from 0 to $2\omega_1$ is positive. This integral is real because of the relations $(\mathcal{I}(\omega_1 \pm \bar{z}))^2 = \overline{(\mathcal{I}(z))}^2$, and then one can check that it is positive for any choice of $\omega_1 > 0$.

PROOF. Choose a basepoint $w_0 \in \widetilde{\Sigma}$ in the fibre of $z_0 = 0 \in \mathbb{C}$, and let Φ be the solution to the initial value problem $d\Phi = \Phi\xi$, $\Phi(w_0) = \mathrm{Id}$. Let $\gamma_k = \gamma_k(s) \in \mathbb{C}$ be the straight-line curve from z_0 to $2\omega_k$ ($k \in \{1, 2\}$) defined by $\gamma_k(s) = 2s\omega_k$ for $s \in [0,1]$. Let $\delta_1 = \delta_1(s) \in \mathbb{C}$ for $s \in [0,1]$ be a curve from z_0 around $\omega_3/2$ in the counterclockwise direction and back to z_0 lying in a small neighborhood of the straight line from z_0 to $\omega_3/2$, and let $\delta_2 = \delta_2(s) = -\delta_1(s)$ be the curve from z_0 around $-\omega_3/2$ in the counterclockwise direction and back to z_0 that is the reflection of δ_1 through the point z_0.

Let $M_k = M_k(\lambda)$ be the respective global monodromies of Φ over the torus along γ_k, and let $A_k = A_k(\lambda)$ be the monodromies of Φ about the two punctures of Σ along δ_k ($k \in \{1, 2\}$). Then M_1, M_2, A_1, A_2 generate the monodromy group of Φ.

This proof follows the same strategy as the proof of Theorem 2.1. First, we note that all the generating elements M_1, M_2, A_1, A_2 of the monodromy group of Φ satisfy the closing conditions (0.3) and (0.4). This follows from Lemma 1.3, since the lower-left entry in ξ is the derivative with respect to z of a function that is well-defined on Σ and hence Equation (1.5) will be satisfied. Our main effort again goes into showing that there exists an initial condition that unitarises the monodromy group of Φ. To accomplish this, we first compute the symmetries of the monodromy group and define the following transformations of Σ:

$$\sigma(z) = z + \omega_3\,, \quad \rho(z) = iz + \omega_1\,, \quad \theta(z) = \bar{z} + \omega_1\,.$$

Then with $g = \mathrm{diag}[\,1/\sqrt{i},\ \sqrt{i}\,]$, the potential ξ has the symmetries

$$\sigma^*\xi = \xi\ , \quad \rho^*\xi = g^{-1}\,\xi\, g\ , \text{ and } \overline{\theta^*\xi(1/\bar\lambda)} = \xi\ .$$

Hence $\hat\sigma^*\Phi = V_\sigma\Phi$, $\hat\rho^*\Phi = V_\rho\Phi g$ and $\overline{\hat\theta^*\Phi(1/\bar\lambda)} = V_\theta\Phi$ for some z-independent V_σ, V_ρ and V_θ. Since $z_0 = 0$ is a fixed point of the two maps

$$\sigma^{-1}\rho^2 : z \mapsto -z\ , \quad \sigma^{-1}\rho\,\theta : z \mapsto i\bar z$$

(we interpret these compositions as being applied in order from rightmost first to leftmost last), and since $\Phi(z_0) = \mathrm{Id}$, we have $V_\rho^2 V_\sigma^{-1} = g^{-2}$ and $V_\theta V_\rho V_\sigma^{-1} = g^{-1}$. It follows that

$$\begin{aligned} &M_1^{-1} = g^{-2}M_1g^2\ , \quad A_2 = g^{-2}A_1g^2\ , \\ &\overline{M_2(1/\bar\lambda)} = g^{-1}M_1(\lambda)g\ , \quad \overline{A_1(1/\bar\lambda)}^{-1} = g^{-1}A_1(\lambda)g\ . \end{aligned} \tag{3.1}$$

The first and third equations in (3.1) imply that also $M_2^{-1} = g^{-2}M_2g^2$.

Because the potential is real-valued along the curve γ_1 when $\lambda \in \mathbb{S}^1$, we conclude that M_1 is a real-valued matrix for all $\lambda \in \mathbb{S}^1$. This fact, combined with the first equation in (3.1), implies that M_1 has the form

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & a_1 \end{pmatrix},$$

where $a_1 = a_1(\lambda) = \overline{a_1(1/\bar\lambda)}$, $b_1 = b_1(\lambda) = \overline{b_1(1/\bar\lambda)}$ and $c_1 = c_1(\lambda) = \overline{c_1(1/\bar\lambda)}$. Furthermore, the fourth equation in (3.1) implies

$$A_1 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \quad \text{where}$$

$$\overline{a_2(1/\bar\lambda)} = d_2(\lambda)\ , \ \overline{b_2(1/\bar\lambda)} = -ib_2(\lambda)\ , \ \overline{c_2(1/\bar\lambda)} = ic_2(\lambda)\ . \tag{3.2}$$

From this it is clear that $\tau_1 = \frac{1}{2}\mathrm{tr}M_1$ and $\tau_2 = \frac{1}{2}\mathrm{tr}A_1$ are real for all $\lambda \in \mathbb{S}^1$.

We will now show that M_1 and A_1 are simultaneously unitarizable for small $|c|$. Toward this goal, we first apply Lemma 1.4 to show that for small $|c|$ we have $|\tau_1(\lambda)| < 1$ for all $\lambda \in \mathbb{S}^1 \setminus \{1\}$: We take Σ and ξ as in Theorem 3.1 and take $f = dz$ and $g = (\wp''''(z+\omega_3/2) + \wp''''(z-\omega_3/2))dz$ and the curve $\gamma = \gamma_1$. Then, in Lemma 1.4, we have $I_0 = 2\omega_1 > 0$ and $I_1 = 0$. To compute I_2, integration by parts yields

$$I_2 = -\int_{\gamma_1}\left(f\left(\int g\right)^2\right) = -\int_{\gamma_1}(\mathcal{I}(z))^2dz\ ,$$

where $\mathcal{I}(z)$ is as defined in Remark 3.2. Then by Remark 3.2, we have $I_2 < 0$. Thus the conditions of Lemma 1.4 hold and we conclude that for all $c \in \mathbb{R}$ sufficiently close to 0, we have

$$|\tau_1(\lambda)| < 1 \quad \text{for all } \lambda \in \mathbb{S}^1 \setminus \{1\}\ . \tag{3.3}$$

From (3.3) and the fact that $b_1c_1 = a_1^2 - 1 \in \mathbb{R}$ on $\mathbb{S}^1$, we have

$$(b_1c_1)|_{\lambda=1} = 0 \ \text{ and } \ (b_1c_1)|_{\mathbb{S}^1\setminus\{1\}} < 0\ . \tag{3.4}$$

Thus we can define a function

$$v = -\frac{c_1}{b_1}$$

that is finite and nonzero on $\mathbb{S}^1 \setminus \{1\}$. Furthermore, by (3.4) and the fact that $b_1 \in \mathbb{R}$ on $\mathbb{S}^1$, we conclude that

$$0 < v < \infty \tag{3.5}$$

for all $\lambda \in \mathbb{S}^1 \setminus \{1\}$. Similar to the arguments in proving Theorem 2.1, Corollary 1.5 implies that b_1 and c_1 both have zeroes of order exactly two, hence v is nonzero and finite at $\lambda = 1$ as well. It follows that $\sqrt[4]{v} > 0$, representing the positive fourth root of v, is globally and smoothly defined on $\mathbb{S}^1$. We define

$$h = \begin{pmatrix} \sqrt[4]{v} & 0 \\ 0 & (\sqrt[4]{v})^{-1} \end{pmatrix} . \tag{3.6}$$

Because $b_1 c_1 \leq 0$ and $\sqrt{v} \geq 0$, the conjugate $hM_1h^{-1} \in \mathrm{SU}_2$ for $|\lambda| = 1$.

The image of the path $\gamma_2\delta_1$ under the map ρ is homotopic to the path $\gamma_1^{-1}\delta_1$. Hence A_1M_2 and $A_1M_1^{-1}$ are conjugate and so have the same trace. Hence

$$\mathrm{tr}(A_1(M_2 - M_1^{-1})) = b_2c_1(1+i) + c_2b_1(1-i) = 0 .$$

It follows that

$$c_2b_1 + ic_1b_2 = 0 . \tag{3.7}$$

In (3.7), either both b_2 and c_2 are identically zero, or neither of them are identically zero. If b_2 and c_2 are identically zero, then A_1 is diagonal and $A_1 \in \mathrm{SU}_2$ for all $\lambda \in \mathbb{S}^1$. Hence we have succeeded in simultaneously unitarizing both M_1 and A_1 on $\mathbb{S}^1$ by conjugating by h. We may then proceed to the final paragraph of this proof, which gives the concluding argument for proving Theorem 3.1. Therefore, without loss of generality, let us assume that neither b_2 nor c_2 is identically zero.

Under the assumption that b_2 and c_2 are not identically zero, by (3.7) we also have

$$v = -\frac{\overline{c_1(1/\bar{\lambda})}}{b_1(\lambda)} = -\frac{\overline{c_2(1/\bar{\lambda})}}{b_2(\lambda)} .$$

Furthermore, (3.2) and (3.5) then imply that $b_2 = r_1(1+i)$ and $c_2 = r_2(1-i)$ with $r_1, r_2 \in \mathbb{R}$ and $r_1^{-1}r_2 \leq 0$, on $\mathbb{S}^1$. These facts together show that also the conjugate $hA_1h^{-1} \in \mathrm{SU}_2$ for $|\lambda| = 1$.

Thus we have simultaneously unitarised M_1 and A_1 on $\mathbb{S}^1$. Since $g \in \mathrm{SU}_2$ and commutes with h, conjugation by h also unitarizes M_2 and A_2, so the full monodromy group is unitarized on $\mathbb{S}^1$. Now, like in the proof of Theorem 2.1, using Iwasawa splitting on $\mathbb{S}^1$ and noting that the monodromy of $h\,\Phi$ still satisfies (0.3) and (0.4), we conclude that the resulting CMC surface given by the Sym-Bobenko formula is defined on Σ. Finally, analogous to the arguments at the end of the proof of Theorem 2.1, the order 4 dihedral symmetry of the resulting CMC immersions can be shown, and since the coefficient cdz of the λ^{-1} term of the upper-right entry of the potential ξ has no zeros or poles on the twice-punctured Riemann surface Σ, the resulting CMC immersion is unbranched [**16**]. This completes the proof. $\square$

4. Doubly-periodic CMC surfaces in $\mathbb{R}^3$ with ends that are asymptotically Delaunay

In this section, we provide a third class of Weierstraß data for which the monodromy can be unitarised. The potential is of interest to us because, although the monodromy can be unitarised, the monodromy does not satisfy (0.3) at $\lambda_0 = 1$. The relaxing of this closing condition is what allows the resulting CMC immersions to extend to doubly-periodic surfaces (when $n = 3, 4, 6$ in the theorem).

THEOREM 4.1. *Let $n \geq 3$ be an integer, and define the Riemann surface $\Sigma = (\mathbb{C} \setminus \mathcal{P}) \cup \{\infty\}$ with $\mathcal{P} = \{z \in \mathbb{C} \,|\, z^n = 1\}$. Let*

$$\xi = \begin{pmatrix} 0 & \lambda^{-1} dz \\ v(\lambda) \frac{n^2 z^{n-2}}{(z^n - 1)^2} dz & 0 \end{pmatrix} , \tag{4.1}$$

where

$$v(\lambda) = \frac{(n-2)^2 w}{16 n^2} (1 - \lambda)^2 + \frac{1-n}{n^2} \lambda \, , \quad w \in \left[\frac{-8n}{(n-2)^2} , 0 \right) . \tag{4.2}$$

Let $w_0 \in \tilde{\Sigma}$ be in the fibre of $z_0 = 0 \in \Sigma$ and let Φ be the solution of $d\Phi = \Phi\xi$, $\Phi(w_0) = \mathrm{Id}$. Then there exists an initial condition that unitarises the monodromy of Φ.

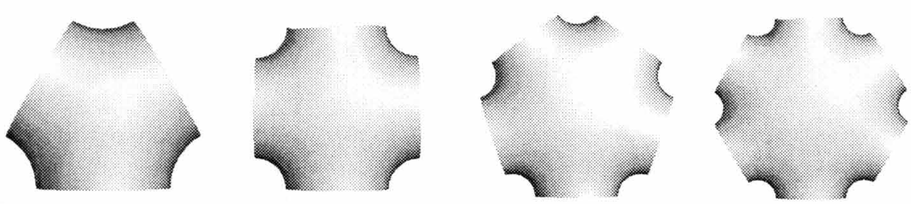

FIGURE 2. CMC surfaces with 3-, 4-, 5- and 6-fold symmetry (the CMC immersions f in (4.5) produced from Theorem 4.1) in the upper row. In the case of 3-, 4- and 6-fold symmetry, doubly periodic CMC surfaces can be constructed by reflection in planes perpendicular to the plane of this page. The annulur ends of each surface are nodoidal with equal weights and parallel same-directed axes.

PROOF. Let $\alpha = \exp(\pi i / n)$, and define the closed polygonal loop $\gamma_0 : [0,1] \to \Sigma$ as follows:

$$\gamma_0(t) = \begin{cases} 4t\alpha^{-1} & \text{for} \quad 0 \leq t \leq 1/4, \\ (2 - 4t)\alpha^{-1} + 8t - 2 & \text{for} \quad 1/4 \leq t \leq 1/2, \\ 6 - 8t + (4t - 2)\alpha & \text{for} \quad 1/2 \leq t \leq 3/4, \\ (4 - 4t)\alpha & \text{for} \quad 3/4 \leq t \leq 1. \end{cases}$$

Then define the loops

$$\gamma_j(t) = \alpha^{2j} \gamma_0(t) \, , \quad j = 1, 2, ..., n-1 \, .$$

Let M_j be the monodromy of Φ along γ_j. Then M_0, M_1, ..., M_{n-1} generate the monodromy group of Φ. Under the transformation $\rho : z \to \alpha^2 z$ of Σ, we have $\rho^* \xi = g^{-1} \xi g$, where $g = \mathrm{diag}[\alpha^{-1}, \alpha]$. Because $\rho(z_0) = z_0$ and $\Phi(z_0) = \mathrm{Id}$, we have $M_j = g^{-j} M_0 g^j$.

Changing variables to $\tilde{z} = 1/z$ and gauging ($\xi \mapsto \xi.g = g^{-1} \xi g + g^{-1} dg$) by $\tilde{g} = \mathrm{diag}[\tilde{z}^{-1}, \tilde{z}]$, we have

$$\xi.\tilde{g} = \begin{pmatrix} -\tilde{z}^{-1} & -\lambda^{-1} \\ \frac{-v(\lambda) n^2 \tilde{z}^{n-2}}{(\tilde{z}^n - 1)^2} & \tilde{z}^{-1} \end{pmatrix} d\tilde{z} \,.$$

Then one solution of $d\tilde{\Phi} = \tilde{\Phi} \cdot (\xi.\tilde{g})$ is $\tilde{\Phi} = \exp\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \log \tilde{z} \right) \tilde{P}(\tilde{z}, \lambda)$, where $\tilde{P}(\tilde{z}, \lambda)$ is well-defined and holomorphic with respect to $\tilde{z}$ and is nonsingular at $\tilde{z} = 0$. Furthermore, $\tilde{P}(\tilde{z}, \lambda)$ is defined for all $\lambda \in \mathbb{S}^1$. (This follows from a well-known result in the theory of ordinary differential equations, see **[36]** section 8.) It follows that the monodromy of Φ along the loop $\gamma_{n-1} ... \gamma_1 \gamma_0$ (here again composition of these loops is from rightmost first to leftmost last) encircling $z = \infty$ is $(M_0)(g^{-1} M_0 g)...(g^{1-n} M_0 g^{n-1}) = \mathrm{Id}$. Thus $(M_0 g^{-1})^n = -\mathrm{Id}$. Hence the eigenvalues of $M_0 g^{-1}$ are constant and are n-th roots of -1. Since

$$\Phi|_{\lambda=1} = \begin{pmatrix} d_z B & B \\ d_z D & D \end{pmatrix} \,,$$

with

$$B = \alpha \sqrt[n]{z^n - 1} \int_0^z \left(\sqrt[n]{\zeta^n - 1} \right)^{-2} d\zeta \,, \quad D = \alpha^{-1} \sqrt[n]{z^n - 1} \,,$$

we have that M_0 is upper-triangular at $\lambda = 1$ and the upper-left (resp. lower-right) entry of its diagonal is α^{-2} (resp. α^2). So the eigenvalues of $M_0 g^{-1}$ are the same as the eigenvalues of g^{-1}:

$$\text{(eigenvalues of } g^{-1}) = \text{(eigenvalues of } M_0 g^{-1}) = \alpha^{\pm 1} \,. \tag{4.3}$$

Now we determine the eigenvalues of M_0 for general λ:
For $\hat{g} = \mathrm{diag}[\sqrt{z-1}, \frac{1}{\sqrt{z-1}}]$ we have

$$\xi.\hat{g} = A \frac{dz}{z-1} + O((z-1)^0) \,, \quad A = \begin{pmatrix} \frac{1}{2} & \lambda^{-1} \\ v(\lambda) & \frac{-1}{2} \end{pmatrix} \,,$$

in a neighborhood of $z = 1$. Applying Lemma 9.1 in **[36]**, we have that one solution of $d\hat{\Phi} = \hat{\Phi} \cdot (\xi.\hat{g})$ is $\hat{\Phi} = \exp(A \log(z-1)) \cdot \hat{P}(z, \lambda)$, where $\hat{P}(z, \lambda)$ is holomorphic and well-defined at $z = 1$. Furthermore, $\hat{P}(z, \lambda)$ is defined for any $\lambda \in \mathbb{S}^1$ at which the difference of the eigenvalues of A is not an integer. Hence $\hat{P}(z, \lambda)$ is defined on $\mathbb{S}^1$ minus a finite set of points.

Hence one solution of $d\check{\Phi} = \check{\Phi} \xi$ is $\check{\Phi} = \hat{\Phi} \hat{g}^{-1}$. Therefore any solution of $d\check{\Phi} = \check{\Phi} \xi$ has monodromy along γ_0 that is conjugate to $-\exp(2\pi i A)$. In particular, the eigenvalues of M_0 are $-\exp(\pm i\pi \sqrt{1 + 4\lambda^{-1} v(\lambda)})$, and so

$$\text{(eigenvalues of } M_0) = -\exp\left(\pm \tfrac{\pi i (n-2)}{n} \sqrt{1 + \tfrac{w}{4} \tfrac{(\lambda-1)^2}{\lambda}} \right) \,. \tag{4.4}$$

We now show that M_0 and g can be simultaneously unitarised at every point in $\mathbb{S}^1$ where $\hat{P}(z,\lambda)$ is defined: We define the half-traces $t_1 = (1/2)\mathrm{tr}(g^{-1})$, $t_2 = (1/2)\mathrm{tr}(M_0 g^{-1})$ and $t_3 = (1/2)\mathrm{tr}(M_0)$. Then, since $M_0\, g^{-1}\,(M_0 g^{-1})^{-1} = \mathrm{Id}$, the condition for simultaneous unitarizability [**25**], see also [**4**], of M_0 and g^{-1} and $(M_0 g^{-1})^{-1}$ is

$$1 - t_1^2 - t_2^2 - t_3^2 + 2t_1t_2t_3 \geq 0\ .$$

By Equations (4.3) and (4.4), this condition holds for all $\lambda \in \mathbb{S}^1$ (where $\hat{P}(z,\lambda)$ is defined) if and only if

$$-\cos\left(\tfrac{\pi(n-2)}{n}\sqrt{1+\tfrac{w}{4}\tfrac{(\lambda-1)^2}{\lambda}}\right) \in \left[\cos\left(\tfrac{2\pi}{n}\right),\, 1\right]\ ,$$

and this in turn holds if and only if $w \in [\frac{-8n}{(n-2)^2}, 0]$, as in Equation (4.2).

It follows that the full monodromy group can be unitarized at all but a finite number of points in $\mathbb{S}^1$.

Note that if M_0 and g^{-1} commute for all $\lambda \in \mathbb{S}^1$, then M_0 must be diagonal, and hence $M_0(\lambda) \in \mathrm{SU}_2$ for all $\lambda \in \mathbb{S}^1$. In this case, Lemma 4.1 is then clearly true, so without loss of generality we may assume that $[M_0, g^{-1}] \neq 0$. Thus we can apply the gluing theorem [**55**] (see also [**36**]) to conclude there exists an initial condition h such that the monodromy group of $h\Phi$ is unitary. $\square$

EXAMPLE 4.1. *Now let n, $\mathcal{P}$, Σ, ξ and w be as in Theorem 4.1. Let $\mathcal{D} = \{z \in \mathbb{C}\,|\,|z| \leq 1\}$ be the closed unit disk in $\mathbb{C}$. Let $h = h(\lambda)$ be the unitariser of the monodromy of the solution Φ of $d\,\Phi = \Phi\,\xi$ given by Theorem 4.1. Let*

$$f : \mathcal{D} \setminus \mathcal{P} \to \mathbb{R}^3 \tag{4.5}$$

be the CMC *immersion generated by the data $(\Sigma,\, \xi,\, h,\, 0)$. By the gluing theorem* [**55**], *the immersion f via the Sym-Bobenko formula* [**5**], *in* (4.5) *is defined when using r-Iwasawa splitting* [**46**] *for $r < 1$ and r sufficiently close to* 1. *Then, up to a rigid motion and homothety of $\mathbb{R}^3$, we find numerically that f has the following properties (see Figure 2):*

- *the image of f has order n dihedral symmetry,*
- *the boundary of the image of f consists of n complete planar geodesics that are congruent to each other, each lieing in a different plane*
$$\left\{(x_1,x_2,x_3) \in \mathbb{R}^3 \;\middle|\; \cos\left(\tfrac{2\pi j}{n}\right)x_1 + \sin\left(\tfrac{2\pi j}{n}\right)x_2 = 1\right\}$$
for $j = 0, 1, ..., n-1$,
- *f has n ends at the punctures in $\mathcal{P}$, and the image of each end is asymptotic to a $(\pi(n-2)/n)$-angle arc of a Delaunay nodoid,*
- *the axes of the asymptotically Delaunay ends are all vertical (i.e. parallel to the line $\{(0,0,x_3) \in \mathbb{R}^3\}$) and the third coordinate x_3 of f satisfies $\lim_{z\in\mathcal{D}, z\to p} x_3 = +\infty$ for all $p \in \mathcal{P}$.*
- *If $n \in \{3,\, 4,\, 6\}$, the complete surface built by reflection across boundary planar geodesics is doubly periodic; in particular, it is invariant with respect to two independent translations of $\mathbb{R}^3$ parallel to the plane $\{(x_1,x_2,0) \in \mathbb{R}^3\}$.*

REMARK 4.2. *In the cases $n = 3, 4, 6$, the image $f(\mathcal{D} \setminus \mathcal{P})$ can be repeatedly reflected to produce a doubly-periodic surface with closed ends. By the asymptotics theorem* [**55**], *the annular ends are asymptotically Delaunay with negative weight w.*

5. Open problems

(i) Can one prove that the surfaces in Theorems 2.1 and 3.1 are complete and properly immersed? Could one further prove the asymptotic behavior of their ends? In particular, are the ends of the examples in Theorem 2.1 asymptotic to ends of $2n$-legged Smyth surfaces?

(ii) By techniques like those used here, can one prove existence of a CMC surface with finite topology and asymptotically Delaunay ends and positive genus?

CHAPTER 5

Coarse classification of constant mean curvature cylinders

1. Basic definitions and results

1.1. Loop groups: In this chapter we introduce a loop group, a loop algebra and two splitting theorems. Let $C_r := \{\lambda \in \mathbb{C} \mid |\lambda| = r\}$ be the circle of radius r with $r \in (0,1]$, and let $D_r := \{\lambda \in \mathbb{C} \mid |\lambda| < r\}$ be the open disk of radius r. We denote the closure of D_r by $\overline{D_r} := \{\lambda \in \mathbb{C} \mid |\lambda| \leq r\}$. Also, let $A_r = \{\lambda \in \mathbb{C} \mid r < |\lambda| < 1/r\}$. This is an open annulus containing S^1. Let $\overline{A_r}$ denote the closure of A_r.

Furthermore, let $E_r = \{\lambda \in \mathbb{C} \mid r < |\lambda|\}$ be the exterior of the circle C_r. For any $r \in (0,1] \subset \mathbb{R}$, we consider the twisted loop algebra and loop group:

$$\Lambda_r sl(2,\mathbb{C})_\sigma = \{\alpha : C_r \to sl(2,\mathbb{C}) \mid \alpha \text{ is continous and } \alpha(-\lambda) = \sigma_3 \alpha(\lambda) \sigma_3\} ,$$

$$\Lambda_r SL(2,\mathbb{C})_\sigma = \{g : C_r \to SL(2,\mathbb{C}) \mid g \text{ is continous and } g(-\lambda) = \sigma_3 g(\lambda) \sigma_3\} ,$$

where $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

We need to define special subgroups of $\Lambda SL(2,\mathbb{C})_\sigma$. First we consider the twisted $SU(2)$ r-loop group:

$$\Lambda_r SU(2)_\sigma = \{F(\lambda) \in \Lambda_r SL(2,\mathbb{C})_\sigma \mid F(\lambda) \in SU(2), \text{ for all } \lambda \in S^1, \\ F(\lambda) \quad \text{extends holomorphically to} A_r\} .$$

Note that the definition of $\Lambda_r SU(2)_\sigma$ implies that F is continuous on $\overline{A_r}$ and holomorphic on A_r. Next, we define the twisted "plus r-loop group" and "minus r-loop group":

$$\Lambda^+_{r,B} SL(2,\mathbb{C})_\sigma = \{W_+ \in \Lambda_r SL(2,\mathbb{C})_\sigma \mid W_+(\lambda) \text{ extends holomorphically} \\ \text{to } D_r \text{ and } B(0) \in \boldsymbol{B}\} ,$$

$$\Lambda^-_{r,B} SL(2,\mathbb{C})_\sigma = \{W_+ \in \Lambda_r SL(2,\mathbb{C})_\sigma \mid W_+(\lambda) \text{ extends holomorphically} \\ \text{to } E_r \text{ and } B(\infty) \in \boldsymbol{B}\} ,$$

where $\boldsymbol{B}$ is a group of diagonal matrices in $SL(2,\mathbb{C})$. If $\boldsymbol{B} = \{\mathrm{Id}\}$ we write the subscript $*$ instead of $\boldsymbol{B}$, if $\boldsymbol{B} = \{\text{all diagonal matrices}\}$ we abbreviate $\Lambda^+_{r,B} SL(2,\mathbb{C})_\sigma$ and $\Lambda^-_{r,B} SL(2,\mathbb{C})_\sigma$ by $\Lambda^+_r SL(2,\mathbb{C})_\sigma$ and $\Lambda^-_r SL(2,\mathbb{C})_\sigma$ respectively. From now on we will use the subscript $\boldsymbol{B}$ as above only if $\boldsymbol{B} \cap SU(2) = \{\mathrm{Id}\}$ holds, in particular if $\boldsymbol{B}$ is the group of all diagonal matrices with positive real entries. When $r = 1$, we always omit the 1. In order to make the above groups and algebras complex Banach Lie groups and Lie algebras, we restrict the occurring matrix coefficients to the "Wiener algebra" (see [**31**], page 5)

$$\mathcal{A} = \left\{ f(\lambda) = \sum_{n\in\mathbb{Z}} f_n \lambda^n \ : \ C_r \to \mathbb{C} \ ; \ \sum_{n\in\mathbb{Z}} |f_n| < \infty \right\} . \tag{1.1}$$

We will assume from here on that all matrix coefficients are contained in the Wiener algebra $\mathcal{A}$. It is well known that the Wiener algebra is a Banach algebra relative to the norm $\|f\| = \sum |f_n|$, and that $\mathcal{A}$ consists of continous functions. Moreover, with coefficients in $\mathcal{A}$, the loop groups and loop algebras defined above are Banach Lie groups and Banach Lie algebras.

From [**21**], we quote the following two splitting Theorems:

THEOREM 1.1. (Birkhoff decomposition) *For any $r \in (0, 1]$, we have the disjoint union*

$$\Lambda_r SL(2,\mathbb{C})_\sigma = \bigcup \Lambda_r^- SL(2,\mathbb{C})_\sigma \cdot w_n \cdot \Lambda_r^+ SL(2,\mathbb{C})_\sigma \ ,$$

where $w_n = \begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{pmatrix}$ if $n = 2k$ and $\begin{pmatrix} 0 & \lambda^n \\ -\lambda^{-n} & 0 \end{pmatrix}$ if $n = 2k+1$. The loops, for which $n = 0$, form an open dense subset of $\Lambda_r SL(2,\mathbb{C})_\sigma$, and the multiplication map

$$\Lambda_{r,*}^- SL(2,\mathbb{C})_\sigma \times \Lambda_r^+ SL(2,\mathbb{C})_\sigma \longrightarrow \Lambda_r SL(2,\mathbb{C})_\sigma$$

is an analytic diffeomorphism onto its image.

THEOREM 1.2. (Iwasawa decomposition) *For any $r \in (0, 1]$ and each $\boldsymbol{B}$ of diagonal matrices of $SL(2,\mathbb{C})$, which satisfies $U(1) \cdot \boldsymbol{B} = \{$all diagonal matrices$\}$ and $SU(1) \cap \boldsymbol{B} = \{\mathrm{Id}\}$, the multiplication map*

$$\Lambda_r SU(2)_\sigma \times \Lambda_{r,B}^+ SL(2,\mathbb{C})_\sigma \to \Lambda_r SL(2,\mathbb{C})_\sigma$$

is a real analytic diffeomorphism onto.

1.2. Holomorphic and normalized potentials: Let $\Psi_\lambda : \mathfrak{D} \to \mathbb{R}^3$, $\lambda \in S^1$, be the associated family of (conformal) constant mean curvature $H = 1/2$ immersions, where $\mathfrak{D}$ = disk in $\mathbb{C}$ or $\mathfrak{D} = \mathbb{C}$. Moreover, let $F(z,\lambda) : \mathfrak{D} \to \Lambda SU(2)_\sigma$, $\lambda \in S^1$, be the extended framing of Ψ_λ (see [**13**]). It is well known (see also the introduction) that Ψ_λ can be obtained from F by the Sym-Bobenko-Formula

$$\Psi_\lambda = -\left(\frac{\mathrm{d}}{\mathrm{d}t} F \cdot F^{-1} + \frac{i}{2} F \sigma_3 F^{-1} \right) , \tag{1.2}$$

where we have set $\lambda = e^{it}$.

Next we quote Lemma 4.5 in [**21**].

THEOREM 1.3. (Existence of holomorphic potentials) *Let $\Psi_\lambda : \mathfrak{D} \to \mathbb{R}^3$ be a* CMC*-immersion with $H = 1/2$, and let $F \in \Lambda_r SU(2)_\sigma$ be the extended framing of Ψ_λ. Then there exists a holomorphic 1-form η on $\mathfrak{D}$ of the form:*

$$\eta(z,\lambda) = \sum_{j\geq -1} \eta_j(z) \lambda^j dz \ , \tag{1.3}$$

where $\sum \eta_j(z)\lambda^j \in \Lambda sl(2,\mathbb{C})_\sigma$, *such that a holomorphic solution* $C \in \Lambda_r SL(2,\mathbb{C})_\sigma$ *of* $dC = C\eta$ *has an Iwasawa splitting* $C = F \cdot W_+$, *that is, with* F *as given above and with* $W_+ \in \Lambda^+_{r,B} SL(2,\mathbb{C})_\sigma$.

We call any holomorphic solution $C \in \Lambda_r SL(2,\mathbb{C})_\sigma$ to $dC = C\eta$, η as above, a *holomorphic extended framing.* Also from Theorem 4.10 in [**21**], we have the following:

THEOREM 1.4. (Existence of normalized potentials) *We retain the assumptions of Theorem 1.3. Then there exists a meromorphic 1-form* ξ *on* $\mathfrak{D}$ *of the form:*

$$\xi(z,\lambda) = \lambda^{-1}\begin{pmatrix} 0 & f \\ Q/f & 0 \end{pmatrix} dz \ , \tag{1.4}$$

where f *is a nonvanishing meromorphic function and* Q *is a holomorphic function such that there exists a meromorphic solution* $g_- \in \Lambda_r SL(2,\mathbb{C})_\sigma$ *to* $dg_- = g_-\xi$ *with Iwasawa splitting* $g_- = Fg_+$, *that is, with* F *as given above and with* $g_+ \in \Lambda^+_{r,B} SL(2,\mathbb{C})_\sigma$.

1.3. Dressing and Symmetries: Let $\mathcal{F}$ be the set of extended framings of CMC-immersions. For $F(z,\bar z,\lambda) \in \mathcal{F}$ and $h_+ \in \Lambda_r^+ SL(2,\mathbb{C})_\sigma$ we define

$$h_+(\lambda)F(z,\bar z,\lambda) = (h_+\#F)(z,\bar z,\lambda)g_+(z,\bar z,\lambda) \ , \tag{1.5}$$

where $(h_+\#F)(z,\bar z,\lambda)$ is the unitary part of the unique Iwasawa decomposition in $\Lambda_r SL(2,\mathbb{C})_\sigma$ and g_+ its positive part. Let $\mathcal{F}_{\mathrm{Id}}$ be the set of normalized extended framings with base point $z_0 \in \mathfrak{D}$ i.e., $F \in \mathcal{F}_{\mathrm{Id}}$ if and only if $F(z_0,\bar z_0,\lambda) = \mathrm{Id}$. Then $h_+(\lambda) = h_+(\lambda)F(z_0,\bar z_0,\lambda) = (h_+\#F)(z_0,\bar z_0,\lambda)g_+(z_0,\bar z_0,\lambda)$ implies $h_+\#F(z_0,\bar z_0,\lambda) = \mathrm{Id}$. Thus $h_+\#F$ is again in $\mathcal{F}_{\mathrm{Id}}$. We will say that $h_+\#F$ is obtained from F by dressing with h_+.

We will also define the dressing on the level of holomorphic extended framings. Let C be the solution of $dC = C\eta$ in Theorem 1.3 with some initial condition $C(z_0,\lambda) = \mathrm{Id}$, then we define

$$\hat C = h_+(\lambda)\cdot C \cdot h_+^{-1}(\lambda) \ ,$$

where $h_+(\lambda) \in \Lambda_r^+ SL(2,\mathbb{C})_\sigma$. We will say that $\hat C$ is obtained from C by dressing with h_+. To see how the surface is changed by dressing with h_+, one needs to perform an Iwasawa decomposition of $h_+F = \tilde F\tilde W_+$, where $C = F\cdot W_+$ is the Iwasawa splitting of C. Then $\tilde F \in \Lambda_r SU(2)_\sigma$ is the frame for a new CMC-immersion. This change in the frame from F to $\tilde F$ is nontrivial. Moreover, also the associated change on the surface level is nontrivial.

1.4. Invariant potentials and the monodromy problem: In this subsection, we will consider the construction of CMC-immersions with $H = 1/2$ from surfaces with nontrivial fundamental group. The essential point in the construction of CMC-immersion here is the "Monodromy problem". Before we state the definition of a monodromy matrix, we recall [**17**].

PROPOSITION 1.5. *Let* $\mathcal{M} = \Gamma \setminus \mathfrak{D}$ *be a connected Riemann surface with universal cover* $\mathfrak{D}$ *and Fuchsian group* Γ *and* $\Psi : \mathcal{M} \to \mathbb{R}^3$ *a* CMC*-immersion. Let*

Ψ_λ denote the associated family of Ψ and F and C an extended framing and a holomorphic extended framing respectively. Then for every $\gamma \in \Gamma$ there exists some $\chi_\gamma \in \Lambda SU(2)_\sigma$ such that $\gamma^* C = \chi_\gamma(\lambda) C G_+$ and $\gamma^* F = \chi_\gamma(\lambda) F k(z, \bar{z})$ for some $G_+ \in \Lambda^+ SL(2, \mathbb{C})$ and $k \in U(1)$.

More generally we have the following definition.

DEFINITION 1.6. *Let $\mathcal{M} = \Gamma \backslash \mathfrak{D}$ be a connected Riemann surface with universal cover $\mathfrak{D}$ and Fuchsian group Γ. Let η or ξ be a holomorphic potential or a normalized potential on $\mathfrak{D}$ as in Theorem 1.3 or Theorem 1.4, and let C be a solution to $dC = C\eta$ or $dC = C\xi$. And let F be the unitary part of the Iwasawa splitting of $C = FW_+$, and let $\gamma \in \Gamma$ be some deck transformation. A matrix $M_\gamma \in \Lambda_r SL(2, \mathbb{C})_\sigma$ (resp. $\Lambda_r SU(2)_\sigma$) is called a monodromy matrix for γ and C (resp. F) if $\gamma^* C = M_\gamma C G_+$ (resp. $\gamma^* F = M_\gamma F k$), for some $G_+ \in \Lambda SL_r^+(2, \mathbb{C})_\sigma$ (resp. $k \in U(1)$).*

REMARK 1.7. *Here the k defined above is the change of coordinate of a* CMC *surface. Note that:*

(i) *If $\mathcal{M}$ is $\mathbb{C}$, then from* [**14**], *without loss of generality we can assume $k = \mathrm{Id}$.*
(ii) *Moreover, if $\mathcal{M}$ is non-compact, then for the same argument above, we also can assume $k = \mathrm{Id}$.*
(iii) *Actually k is irrelevant for the resulting* CMC *surface, since k goes away in the Sym-Bobenko-Formula.*

It will be particularly convenient to have $G_+ = \mathrm{Id}$. We have the following necessary and sufficient condition for such a monodromy.

PROPOSITION 1.8. *Let $\mathcal{M} = \Gamma \backslash \mathfrak{D}$ be a connected Riemann surface with universal cover $\mathfrak{D}$ and Fuchsian group Γ. Let η be a holomorphic or normalized potential on $\mathfrak{D}$ and C be a solution of $dC = C\eta$. Then η satisfies $\gamma^* \eta = \eta$ for some $\gamma \in \Gamma$ if and only if there exists a monodromy M_γ of C such that $\gamma^* C = M_\gamma C$.*

PROOF. "$\Rightarrow$" Let C be the solution to $dC = C\eta$, and let $\gamma^* C$ be the solution to $d(\gamma^* C) = (\gamma^* C)(\gamma^* \eta) = (\gamma^* C)\eta$ with $(\gamma^* C)(z_0, \lambda) = M_\gamma \in \Lambda_r SL(2, \mathbb{C})_\sigma$. Then the uniqueness of the solutions to ODE's implies $\gamma^* C = M_\gamma C$ (see [**11**]). The converse statement is clear. □

We now introduce the notion of a "meromorphic potential" on a Riemann surface $\mathcal{M}$, which is locally equivalent to the notion of a "holomorphic potential" as described in Theorem 1.3. These two notions are, however, globally different, i.e., in general holomorphic potentials are only well defined on a universal cover of $\mathcal{M}$, while meromorphic potentials as defined below are always well defined on $\mathcal{M}$.

THEOREM 1.9. (Existence of meromorphic potentials) *Let $\mathcal{M} = \Gamma \setminus \mathfrak{D}$ be a connected Riemann surface with Fuchsian group Γ. And let $\Psi_\lambda : \mathcal{M} \to \mathbb{R}^3$ be a* CMC *immersion with $H = 1/2$, and let $F \in \Lambda_r SU(2)_\sigma$ be the extended frame of Ψ_λ. Then there exists a meromorphic 1-form η on $\mathcal{M}$ of the form:*

$$\eta(z, \lambda) = \sum_{j \geq -1} \eta_j(z) \lambda^j dz \ , \tag{1.6}$$

satisfying

$$\gamma^*\eta = \eta \ \textit{for all}\ \gamma \in \Gamma \tag{1.7}$$

such that there exists a meromorphic solution $C \in \Lambda_r SL(2,\mathbb{C})_\sigma$ *to* $dC = C\eta$ *with Iwasawa splitting* $C = F \cdot W_+$, *that is, with* F *as given above and with* $W_+ \in \Lambda^+_{r,B} SL(2,\mathbb{C})_\sigma$.

PROOF. We follow the proof of Theorem 3.2 in [**16**]. We would like to find a $W_+ \in \Lambda^+_r SL(2,\mathbb{C})_\sigma$ such that $C = FW_+$ is meromorphic and satisfies $\gamma^*C = \chi_\gamma(\lambda)C$ for $\gamma \in \Gamma$. By Lemma 4.5 in [**21**], there exists $\tilde{W}_+ : \mathfrak{D} \to \Lambda^+ SL(2,\mathbb{C})_\sigma$ such that

$$\tilde{C} = F\tilde{W}_+ : \mathfrak{D} \to \Lambda SL(2,\mathbb{C})_\sigma$$

is holomorphic. Hence

$$\gamma^*\tilde{C} = (\gamma^*F)(\gamma^*\tilde{W}_+) = \chi_\gamma(\lambda)Fk(\gamma^*\tilde{W}_+) = \chi_\gamma(\lambda)\tilde{C}\tilde{W}_+^{-1}k(\gamma^*\tilde{W}_+) \ , \tag{1.8}$$

where $P_+ = \tilde{W}_+^{-1}k(\gamma^*\tilde{W}_+)$ is holomorphic. Since k satisfies the cocycle condition (see Theorem 2.3 in [**14**]), $P_+ = P_+(\gamma, z)$ satisfies the cocycle condition

$$P_+(\gamma_2\gamma_1, z) = P_+(\gamma_2, z)P_+(\gamma_1, \gamma_2 z) \ . \tag{1.9}$$

From Theorem 12 in [**28**], P_+ can be represented in the form

$$P_+ = \pm\alpha(z,\lambda)\left(\gamma^*\alpha^{-1}(z,\lambda)\right) \ , \tag{1.10}$$

where $\alpha : \mathfrak{D} \to SL(2,\mathbb{C})/\pm\mathrm{Id}$ is meromorphic. Moreover if $\mathcal{M}$ is non-compact, then α can be chosen holomorphic (see Corollary 4 in [**28**]). Let $\alpha = \alpha_+\alpha_-$ be the Birkhoff decomposition of α. Equation (1.10) now implies that $\gamma^*\alpha_- = \alpha_-$ and $\gamma^*\alpha_+ = P_+^{-1}\alpha_+$. Thus $P_+ = \pm\alpha_+(\gamma^*\alpha_+^{-1})$, and we set

$$C(z,\lambda) = \tilde{C} \cdot \alpha_+ \ .$$

Then C satisfies $\gamma^*C = C$. □

REMARK 1.10. *As pointed out in the proof above, and as known from* [**16**], *if* $\mathcal{M}$ *is a non-compact Riemann surface* $\mathcal{M}$, *then there exists a holomorphic 1-form, not only a meromorphic 1-form, for every* CMC*-immersion from* $\mathcal{M}$ *to* $\mathbb{R}^3$, *i.e., there exists a holomorphic potential on* $\mathcal{M}$ *for every* CMC*-immersion from* $\mathcal{M}$ *to* $\mathbb{R}^3$.

In general, the monodromy matrix M_γ in Definition 1.6 is not uniquely determined. We have the two possibilities: either the associated family of an immersion Ψ_λ has umbilic points or the associated family of an immersion Ψ_λ does not have umbilic points. If Ψ_λ has umbilic points, then the monodromy matrix M_γ considered in Definition 1.6 is uniquely determined up to a sign, because the isotropy group of Ψ_λ consists of $\pm\mathrm{Id}$ only. If Ψ_λ does not have umbilic points, then the monodromy matrix M_γ considered in Definition 1.6 is not uniquely determined, because the isotropy group is (in general) not trivial. If we take an element B_+ of the isotropy group, then $M_\gamma B_+$ is also a monodromy matrix in the sense of Definition 1.6.

However, if η is an invariant potential on $\mathcal{M}$, meromorphic or holomorphic, then for $\gamma \in \Gamma$ the matrix M_γ satisfying $\gamma^*C = M_\gamma C$ is uniquely determined and

$\gamma \to M_\gamma$ is a group homomorphism. We are still interested in the more general definition given in Definition 1.6, since in general M_γ may not be unitary on S^1, while $M_\gamma B_+$ is unitary on S^1 for some B_+ in the isotropy group (see e.g. [**15**]). For more detail on this issue we refer to [**17**].

As pointed out just above, the monodromy matrix M_γ obtained in Proposition 1.8 is, in general, not unitary on S^1. However, in order to construct CMC-immersions on $\mathcal{M} = \Gamma \setminus \mathfrak{D}$ we need unitary monodromy (together with the closing conditions). If $M_\gamma B_+$ is unitary, we can continue with the construction of an immersion defined on $\mathcal{M}$. If there is no B_+ in the isotropy group such that $M_\gamma B_+$ is unitary, then one can try to find at least some dressing transformation, which changes the given monodromy matrix $M_\gamma B_+$ into a unitary monodromy matrix.

Let $0 < r < 1$ and $M : A_r \to Sl(2, \mathbb{C})$ be holomorphic. Then M is called s*−unitarizable* for $0 < r < s < 1$ if there exists some $h \in \Lambda_s SL(2,\mathbb{C})_\sigma$ such that $hMh^{-1} \in \Lambda_s SU(2)_\sigma$. If M is just defined on the unit circle, then M is called unitarizable if and only if there exists some $h \in \Lambda SL(2,\mathbb{C})_\sigma$ such that $hMh^{-1} \in \Lambda SU(2)_\sigma$.

THEOREM 1.11. ([**16**]) *Let M be an element of $\Lambda_r SL(2,\mathbb{C})_\sigma$. Then M is unitarizable via dressing for some $s \in (r, 1]$ if and only if, for all $\lambda \in S^1$,*

$$\mathrm{Tr}(M) \in (-2, 2) \quad or \quad M = \pm\mathrm{Id}\,. \tag{1.11}$$

As a corollary of the above theorem, for a monodromy matrix of C defined in Proposition 1.8, we have the necessary and sufficient condition for being unitarizable:

COROLLARY 1.12. *We retain the assumptions of Proposition 1.8. Then there exists $h_+(\lambda) \in \Lambda_r^+ SL(2,\mathbb{C})_\sigma$ such that $\hat{C} = h_+(\lambda)C$ is a solution of $d\hat{C} = \hat{C}\eta$ and a monodromy matrix of $\hat{C}$ is $\hat{M}_\gamma = h_+ M_\gamma h_+^{-1} \in \Lambda_r SU(2)_\sigma$ if and only if M_γ satisfies the condition* (1.11).

Note, if a monodromy matrix for a holomorphic extended framing is unitary, then after Iwasawa splitting, the extended framing admits the same monodromy matrix. By the Sym-Bobenko-Formula this yields an immersion Ψ_λ satisfying $\gamma^*\Psi_\lambda = R(\lambda)\Psi_\lambda(z)$, where $R(\lambda)$ is a rigid motion.

We let $M_\gamma \in \Lambda SU(2)_\sigma$ be a monodromy matrix of F. Then we obtain the following necessary and sufficient condition for the closing of a surface $\Psi_{\lambda=\lambda_0}$ with respect to γ.

THEOREM 1.13. *We retain the assumptions of Proposition 1.8. Furthermore, if $M_\gamma \in \Lambda_r SU(2)$, then M_γ is also a monodromy matrix for F with respect to γ, where F is the unitary part of the Iwasawa splitting of $C = FW_+$. Let Ψ_λ be defined from F via the Sym-Bobenko-Formula* (1.2), *then $\gamma^*\Psi_{\lambda=\lambda_0} = \Psi_{\lambda=\lambda_0}$ for some $\lambda_0 \in S^1$ holds if and only if*

$$M_\gamma(\lambda_0) = \pm\mathrm{Id} \quad and \quad \partial_\lambda M_\gamma(\lambda_0) = 0\,. \tag{1.12}$$

2. CMC-immersions and two complex variables

2.1. Double loop groups and the Iwasawa decomposition: In the context of Wu's-Formula [**68**], it turned out that the extended framings $F(z, \bar{z}, \lambda)$ of some CMC-immersion are restrictions of meromorphic maps $F(z, w, \lambda)$, defined on $\mathfrak{D} \times \overline{\mathfrak{D}}$, where $\mathfrak{D}$ =disk in $\mathbb{C}$ or $\mathbb{C}$ and $\overline{\mathfrak{D}}$ is the complex conjugate domain of $\mathfrak{D}$. Also for the main result of this paper (Theorem 4.8) it will be useful to consider meromorphic extensions to two complex variables.

To explain this in detail one needs to consider double loop groups. Following [**23**], but interchanging "+" and "−" and "R"and "r", we set

$$\mathcal{H} = \Lambda_r SL(2,\mathbb{C})_\sigma \times \Lambda_R SL(2,\mathbb{C})_\sigma \quad ,$$

where $0 < r < R$. Moreover

$$\mathcal{H}_+ = \Lambda_r^+ SL(2,\mathbb{C})_\sigma \times \Lambda_R^- SL(2,\mathbb{C})_\sigma \quad ,$$

$$\mathcal{H}_{+,*} = \Lambda_{r,*}^+ SL(2,\mathbb{C})_\sigma \times \Lambda_R^- SL(2,\mathbb{C})_\sigma \quad ,$$

$\mathcal{H}_- = \{(g_1,\ g_2) \in \mathcal{H};\ g_1$ and g_2 extend holomorphically to A_r and $g_1|_{A_r} = g_2|_{A_r}\ \}$.

We quote Theorem 2.6 in [**23**].

THEOREM 2.1. *We have the disjoint union*

$$\mathcal{H} = \bigcup_{n=0}^{\infty} \mathcal{H}_- w_n \mathcal{H}_+ \quad ,$$

where $w_n = \left(\mathrm{Id},\ \begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{pmatrix}\right)$ *if* $n = 2k$ *and* $\left(\mathrm{Id},\ \begin{pmatrix} 0 & \lambda^n \\ -\lambda^{-n} & 0 \end{pmatrix}\right)$ *if* $n = 2k+1$. *The loops, for which* $n = 0$, *form an open dense subset of* $\mathcal{H}$, *and the multiplication map*

$$\mathcal{H}_- \times \mathcal{H}_{+,*} \to \mathcal{H}$$

is an analytic diffeomorphism onto its image.

We would like to point out that the proof of the theorem above is almost verbatim the proof given in the basic splitting paper [**1**]. Below we show how this implies the well-known r-Iwasawa decomposition Theorem 1.2 (see also [**20**]):

Proof of Theorem 1.2. Let $g \in \Lambda_r SL(2,\mathbb{C})_\sigma$. Consider the map

$$\Lambda_r SL(2,\mathbb{C})_\sigma \to \Lambda_r SL(2,\mathbb{C})_\sigma \times \Lambda_R SL(2,\mathbb{C})_\sigma \quad , \quad g \mapsto \left(g(\lambda),\ \left(\overline{g(1/\bar{\mu})}^t\right)^{-1}\right) ,$$

where $R = 1/r$, $|\lambda| = r$ and $|\mu| = R$. From Theorem 2.1 above we infer that one can write

$$\left(g(\lambda),\ \left(\overline{g(1/\bar{\mu})}^t\right)^{-1}\right) = \left(U^{(r)},\ U^{(R)}\right)(\mathrm{Id},\ W)\left(h_+^{(r)}(\lambda),\ h_-^{(R)}(\mu)\right) . \tag{2.1}$$

Spelling this out we obtain

$$g(\lambda) = U^{(r)}(\lambda) h_+^{(r)}(\lambda) \tag{2.2}$$

$$\left(\overline{g(1/\bar{\mu})}^t\right)^{-1} = U^{(R)}(\mu) W h_-^{(R)}(\mu) \ , \tag{2.3}$$

where $U^{(r)}$ and $U^{(R)}$ denote the boundary values of U on C_r and C_R respectively. The second equation is equivalent with

$$g(1/\bar{\mu}) = \left(\overline{U^{(R)}(\mu)}^t\right)^{-1} \left(\overline{W}^t\right)^{-1} \left(\overline{h_-^{(R)}(\mu)}^t\right)^{-1} . \tag{2.4}$$

Replacing μ by $1/\bar{\lambda}$ we obtain

$$g(\lambda) = \left(\overline{U^{(R)}(1/\bar{\lambda})}^t\right)^{-1} \left(\overline{W}^t\right)^{-1} \left(\overline{h_-^{(R)}(1/\bar{\lambda})}^t\right)^{-1} . \tag{2.5}$$

A comparsion with Equation (2.2) shows

$$\left(\overline{U^{(R)}(1/\bar{\lambda})}^t\right) U^{(r)}(\lambda) = \left(\overline{W}^t\right)^{-1} \left(\overline{h_-^{(R)}(1/\bar{\lambda})}^t\right)^{-1} \left(h_+^{(r)}(\lambda)\right)^{-1} . \tag{2.6}$$

Equation (2.6) is the Birkhoff decomposition of a self-adjoint loop $q(\lambda) = \left(\overline{U^{(R)}(1/\bar{\lambda})}^t\right) U^{(r)}(\lambda)$. Therefore $W = \mathrm{Id}$, and the right hand side loop of Equation (2.6) is defined on $\mathbb{C} \cup \{\infty\}$. Hence $q(\lambda)$ is constant. Since $q(\lambda)$ is also a positive definite matrix, we obtain $q(\lambda) = k$, where k is a λ independent diagonal matrix with entries $k_0, k_0^{-1} > 0$. We set $\tilde{U} = U\tilde{k}$, where $\tilde{k}$ is the λ independent diagonal matrix with entries $\sqrt{k_0}$, $\sqrt{k_0}^{-1}$. We have $\left(g(\lambda),\ \left(\overline{g(1/\bar{\mu})}^t\right)^{-1}\right) = \left(\tilde{U}^{(r)},\ \tilde{U}^{(R)}\right)(\mathrm{Id},\ \mathrm{Id})\left(\tilde{k}^{-1}h_+^{(r)}(\lambda),\ \tilde{k}^{-1}h_-^{(R)}(\mu)\right)$. Moreover, from Equation (2.6) we obtain $\tilde{U}^{(r)}(\lambda) = \left(\overline{\tilde{U}^{(R)}(1/\bar{\lambda})}^t\right)^{-1}$. If $r = 1$, the claim follows. Assume now $r < 1$. Since $\tilde{U}^{(r)}$ and $\tilde{U}^{(R)}$ are the boundary values of a holomorphic function $\tilde{U}$ in $r < |\lambda| < 1/r$, we can restrict the equation above to the unit circle and obtain $\tilde{U}(\lambda) = \left(\overline{\tilde{U}(\lambda)}^t\right)^{-1}$, whence $\tilde{U}(\lambda)$ is unitary on S^1.

2.2. CMC-potentials in the double loop group picture: In this subsection, since the extended framing $F(z,\bar{z},\lambda)$ of a CMC surface is defined on the unit circle $|\lambda| = 1$, we use $r = R = 1$. Let $C = C(z,\lambda) = FW_+$ be a holomorphic extended framing of some CMC-immersion. Using the embedding discussed in the last section we consider

$$C \to \left(C, \left(\overline{C}^t\right)^{-1}\right) . \tag{2.7}$$

If $\eta = \eta(z,\lambda)$ denotes the Maurer-Cartan-Form of C, $\eta = C^{-1}dC$, then the Maurer-Cartan-Form of the image under the map (2.7) is given by

$$\left(\eta(z,\lambda),\ -\overline{\eta(z,\lambda)}^t\right) . \tag{2.8}$$

We would like to point out

$$\begin{cases} \eta(z,\lambda) &= \sum_{k=-1}^{\infty} \eta_k(z)\lambda^k \\ -\overline{\eta(z,\lambda)}^t &= \sum_{k=-1}^{\infty} -\overline{\eta_k(z)}^t \bar{\lambda}^k \end{cases} . \tag{2.9}$$

2.3. Extended framings in two complex variables: In this subsection, we consider a procedure that is converse to the construction discussed in the previous section. To motivate the approach below we rephrase the second equation of (2.9) as

$$\tau(w,\mu) = -\overline{\eta(\bar{w}, 1/\bar{\mu})}^t \ , \tag{2.10}$$

where $w = \bar{z}$ and $\mu = 1/\bar{\lambda}$. Thus

$$\tau(w,\mu) = \sum_{m=-\infty}^{1} \tau_m(w)\mu^m \ , \tag{2.11}$$

where $\tau_m(w) = -\overline{\eta_{-m}(\bar{w})}^t$. Using the equations above and setting $R = \left(\overline{C}^t\right)^{-1}$ we obtain

$$\left\{ \begin{array}{lcl} dC & = & C\eta, \ \ C(z_0,\lambda) = \mathrm{Id} \\ dR & = & R\tau, \ \ R(\bar{z}_0,\lambda) = \mathrm{Id} \end{array} \right. . \tag{2.12}$$

where and z_0 and $\bar{z}_0$ are the base points chosen in $\mathfrak{D}$ and $\overline{\mathfrak{D}}$ respectively.

We now consider a more general setting, i.e., w, μ and $\tau(w,\mu)$ are independent of z, λ and $\eta(z,\lambda)$ and $z_0 \in \mathfrak{D}$, $w_0 \in \overline{\mathfrak{D}}$ are arbitary, but fixed. Let's start from the potential:

$$\check{\eta} = (\eta(z,\lambda), \ \ \tau(w,\mu)) = \left(\sum_{k=-1}^{\infty} \eta_k(z)\lambda^k, \ \ \sum_{m=-\infty}^{1} \tau_m(w)\mu^m \right) \ , \tag{2.13}$$

where $z \in \mathfrak{D}$, $w \in \overline{\mathfrak{D}}$, $\lambda \in \mathbb{C}$, $|\lambda| = r$, $\mu \in \mathbb{C}$ and $|\mu| = R$ and η_k and τ_m are holomorphic differential 1-forms. Let C and R denote the solutions to the differential equations

$$\left\{ \begin{array}{lcl} dC & = & C\eta \ , \ \ C(z_0,\lambda) = \mathrm{Id} \\ dR & = & R\tau \ , \ \ R(w_0,\mu) = \mathrm{Id} \end{array} \right. , \tag{2.14}$$

where $z_0 \in \mathfrak{D}$ and $w_0 \in \overline{\mathfrak{D}}$. We consider the generalized Iwasawa decomposition of Theorem 2.1.

$$(C, \ R) = (U, \ U)(\mathrm{Id}, \ W)(V_+, \ V_-) \ . \tag{2.15}$$

In view of the initial conditions and the fact that the big cell in the double loop group is open, we can assume $W = \mathrm{Id}$, if (z,w) is sufficiently close to (z_0, w_0). Thus

$$C = U^{(r)}V_+ \ , \ \ R = U^{(R)}V_- \ , \tag{2.16}$$

and $U = U(z,w,\lambda)$ is meromorphic in two complex variables z and w (see [**33**]).

2.4. Meromorphic extensions of extended framings: In this section, we show that extended framings have unique meromorphic extensions. This follows essentially from the previous two sections.

THEOREM 2.2. *Let $F(z,\bar{z},\lambda)$, $z \in \mathfrak{D}$ be a extended framing of any CMC-immersion. Then there exists a λ independent diagonal matrix $l(z,\bar{z}) \in SL(2,\mathbb{C})$, and $F(z,\bar{z},\lambda)l(z,\bar{z})$ has a meromorphic extension $U(z,w,\lambda)$ to $\mathfrak{D} \times \overline{\mathfrak{D}}$.*

PROOF. Let $F = F(z, \bar{z}, \lambda)$ be the extended framing of some CMC-immersion. Let $C(z, \lambda) = F(z, \bar{z}, \lambda)W_+(z, \bar{z}, \lambda)$ be a holomorphic extended framing. We note that $|\lambda| = 1$. Set $R(w, \lambda) = \left(\overline{C(\bar{w}, \lambda)}^t\right)^{-1}$, where $w \in \overline{\mathfrak{D}}$ is independent of z. Then the pair $(C(z, \lambda),\ R(w, \lambda))$ corresponds to a potential $(\eta,\ \tau)$ as considered in the previous section,

$$\eta(z, \lambda) = C^{-1}dC \quad , \quad \tau(w, \lambda) = R^{-1}dR \ .$$

We thus obtain $C(z, \lambda)$ and $R(w, \lambda)$ as the solutions to $dC = C\eta$, $z \in \mathfrak{D}$, and $dR = R\tau$, $w \in \overline{\mathfrak{D}}$, where we also use $C(z_0, \lambda) = \mathrm{Id}$ and $R(w_0 = \bar{z}_0, \lambda) = \mathrm{Id}$. As in the previous section we obtain (using the unique generalized Iwasawa decomposition of Theorem 2.1, i.e., $\mathcal{H}_- \times \mathcal{H}_{+,*} \to \mathcal{H}$)

$$\begin{cases} C(z, \lambda) &= U(z, w, \lambda)V_+(z, w, \lambda) \\ R(w, \lambda) &= U(z, w, \lambda)V_-(z, w, \lambda) \end{cases} . \tag{2.17}$$

Therefore

$$\begin{aligned} U(z, w, \lambda) &= C(z, \lambda)V_+(z, w, \lambda)^{-1} \\ &= R(w, \lambda)V_-(z, w, \lambda)^{-1} \ . \end{aligned} \tag{2.18}$$

Substituting $C(z, \lambda) = F(z, \bar{z}, \lambda)W_+(z, \bar{z}, \lambda)$ and $R(w, \lambda) = \left(\overline{C(\bar{w}, \lambda)}^t\right)^{-1} = F(\bar{w}, w, \lambda)\left(\overline{W_+(\bar{w}, w, \lambda)}^t\right)^{-1}$, we obtain

$$\begin{aligned} U(z, w, \lambda) &= F(z, \bar{z}, \lambda)W_+(z, \bar{z}, \lambda)V_+(z, w, \lambda)^{-1} \\ &= F(\bar{w}, w, \lambda)\left(\overline{W_+(\bar{w}, w, \lambda)}^t\right)^{-1} V_-(z, w, \lambda)^{-1} \ . \end{aligned} \tag{2.19}$$

For $w = \bar{z}$, Equations (2.19) imply that

$$W_+(z, \bar{z}, \lambda)V_+(z, \bar{z}, \lambda)^{-1} = \left(\overline{W_+(z, \bar{z}, \lambda)}^t\right)^{-1} V_-(z, \bar{z}, \lambda)^{-1} \ . \tag{2.20}$$

The left hand side is in $\Lambda^+SL(2, \mathbb{C})_\sigma$ and the right hand side is in $\Lambda^-SL(2, \mathbb{C})_\sigma$, thus $W_+V_+^{-1} = \left(\overline{W_+}^t\right)^{-1} V_-^{-1} = l(z, \bar{z})$, where l is a λ independent diagonal matrix with entries $l_0, l_0^{-1} > 0$. We note that $l(z, \bar{z})^2 = V_{-,0}^{-1}(z, \bar{z})$ by Equation (2.20), where $V_{-,0}$ is the first coefficient matrix of the expansion of $V_-(z, \bar{z}, \lambda)$ with respect to λ. From Equation (2.19), we have

$$U(z, \bar{z}, \lambda) = F(z, \bar{z}, \lambda)l(z, \bar{z}) \ . \tag{2.21}$$

Therefore $F(z, \bar{z}, \lambda)l(z, \bar{z})$ has a meromorphic extension $U(z, w, \lambda)$ to $\mathfrak{D} \times \overline{\mathfrak{D}}$. □

REMARK 2.3. *In general, there will be many meromorphic extensions of $F \cdot l$ to $\mathfrak{D} \times \overline{\mathfrak{D}}$. However, if U and T both are such meromorphic extensions to $\mathfrak{D} \times \overline{\mathfrak{D}}$ and also satisfy Equation (2.17), then $U = T$, since the generalized Iwasawa decomposition of Theorem 2.1 is unique.*

The meromorphic extension $U(z, w, \lambda)$ constructed in the proof will be called "the unique meromorphic extension " of $F \cdot l$.

REMARK 2.4. *In view of the Sym-Bobenko-Formula for* CMC*-immersions it is tempting to consider the 1-parameter family of complex surfaces (with singularities)*

$$(2.22)\quad \Psi_\lambda = \left(\frac{\mathrm{d}}{\mathrm{dt}} U(z,w,\lambda) \cdot U(z,w,\lambda)^{-1} + U(z,w,\lambda) \cdot \frac{i}{2}\sigma_3 U(z,w,\lambda)^{-1} \right)\Big|_{\lambda=e^{it}} ,$$

where $U(z,w,\lambda)$ is the meromorphic extension of the extended framing $F(z,\bar z,\lambda)l(z,\bar z)$. Even for arbitary potentials $(\eta(z)dz, \tau(w)dw)$, we expect these immersions to be of great interest to the theory of integrable surface equations. We plan to pursue this topic in a separate publication [**19**].

3. Natural potentials for CMC-immersions

3.1. Existence of skew-hermitian potentials: In this section we consider a new type of potentials, so-called "skew-hermitian potentials", derived from a CMC-immersion from $\mathcal{M} = \Gamma \setminus \mathfrak{D}$ to $\mathbb{R}^3$. These potentials are meromorphic 1-forms on $\mathfrak{D}$. From Equations (2.17), we obtain the equation:

$$(3.1)\qquad C(z,\lambda) = U(z,w,\lambda) \cdot V_+(z,w,\lambda) ,$$

where $(z,w) \in \mathfrak{D} \times \overline{\mathfrak{D}}$ are independent variables and $U(z,w,\lambda)$ is the unique meromorphic extension of the extended framing $F(z,\bar z,\lambda)l(z,\bar z)$. We note that in Equation (3.1) we can replace w by any function of z and $\bar z$. First we set $w = z$, then the entries of $U(z,z,\lambda) = C(z,\lambda)V_+(z,z,\lambda)^{-1}$ are meromorphic functions with respect to z. Thus, $U(z,z,\lambda)$ can be considered as a meromorphic extended framing, which is obtained from C by gauging with some ${V_+}^{-1}$. Moreover, for $z \in \mathbb{R} \cap \mathfrak{D}$, we have $F(z,\bar z,\lambda)l(z,\bar z) = U(z,z,\lambda)$. Therefore

$$\begin{aligned} \alpha = F^{-1}dF &= lU^{-1}dUl^{-1} - (dl)\cdot l^{-1} \\ (3.2)\qquad &= \begin{pmatrix} u_{11} & l_0^2 u_{12} \\ l_0^{-2}u_{21} & u_{22} \end{pmatrix} - \begin{pmatrix} l_0'/l_0 & 0 \\ 0 & -l_0'/l_0 \end{pmatrix} , \quad \text{for } z \in \mathbb{R}\cap\mathfrak{D}. \end{aligned}$$

where $u_{ij}, \{i,j\} = \{1,2\}$ are the entries of $U^{-1}dU$ and $l = \mathrm{diag}(l_0, l_0^{-1})$. We note that l_0'/l_0 is the half of the logarithmic derivative of l_0^2, and l_0^2 can be extended to all $(z,w) \in \mathfrak{D} \times \overline{\mathfrak{D}}$ (see the proof of Theorem 2.2). Therefore the right hand side of Equation (3.2) can be extended to all $z \in \mathfrak{D}$. Clearly $U(z,z,\lambda)l(z,\bar z)^{-1}$ is a unitary matrix for $z \in \mathbb{R}\cap\mathfrak{D}$. We denote by ζ the unique extension of α, which is obtained by choosing the natural meromorphic extension of 3.2 via the unique meromorphic extension of Ul^{-1}. Therefore we have proved the following.

THEOREM 3.1. (Existence of skew-hermitian potentials) *Let $\mathcal{M} = \Gamma\setminus\mathfrak{D}$ be a connected Riemann surface with universal cover $\mathfrak{D}$ and Fuchsian group Γ. Assume that $\Psi : \mathcal{M} \to \mathbb{R}^3$ is an immersion of constant mean curvature $H = 1/2$ with associated family $\Psi_\lambda : \mathfrak{D} \to \mathbb{R}^3$. Then Ψ_λ can be derived from a meromorphic 1-form ζ on $\mathfrak{D}$ such that*

$$(3.3)\qquad \zeta(z,\lambda) = \lambda^{-1}\zeta_{-1}(z) + \zeta_0(z) + \lambda\zeta_1(z) ;$$

where each $\zeta_j, j = -1,0,1$, is a meromorphic 1-form on $\mathfrak{D}$ and ζ is skew-hermitian for $z \in \mathbb{R}\cap\mathfrak{D}$.

Potentials of the form just stated will be called *skew-hermitian potentials.*

REMARK 3.2. *We note that Theorem 3.1 implies that the Maurer-Cartan form $\alpha = F^{-1}dF$ of some extended framing F has a unique meromorphic extension, while the extended framing F may not have a unique meromorphic extension.*

3.2. Periods of skew-hermitian potentials: We have shown in Theorem 2.2 that $F(z,\bar{z},\lambda)$ has, up to a λ-independent diagonal factor, a meromorphic extension $U(z,w,\lambda)$ to $\mathfrak{D}\times\overline{\mathfrak{D}}$. Moreover, the square of this diagonal factor has a meromorphic extension to $\mathfrak{D}\times\overline{\mathfrak{D}}$ as well. Hence if α is the Maurer-Cartan form of the extended coordinate framing of some CMC-immersion, then $\alpha(z,\bar{z},\lambda)$ has a meromorphic extension $\hat{\zeta}(z,w,\lambda)$. We set $w=z$, then the corresponding skew-hermitian potential $\zeta = \hat{\zeta}(z,w=z,\lambda)$ is

$$\zeta(z,z,\lambda) = \left\{\lambda^{-1}\begin{pmatrix} 0 & h(z) \\ \frac{Q(z)}{h(z)} & 0 \end{pmatrix} + \zeta_0 + \lambda\begin{pmatrix} 0 & -\frac{\overline{Q(\bar{z})}}{h(z)} \\ -h(z) & 0 \end{pmatrix}\right\} dz \quad , \tag{3.4}$$

where $h(z) = \exp(1/2u(z,w))|_{w=z} = l_0^2(z,z)$ with l_0 defined in the proof of Theorem 2.2, which is the unique extension of the square root of the conformal factor $e^{u(z,\bar{z})}$ of a CMC-immersion. We note that $h(z)$ is real for $z \in \mathfrak{D}\cap\mathbb{R}$. And Q is the coefficient of the Hopf differential of the CMC immersion and

$$\zeta_0 = \begin{pmatrix} \frac{1}{4}(u_z(z,w) - u_w(z,w))|_{w=z} & 0 \\ 0 & -\frac{1}{4}(u_z(z,w)-u_w(z,w))|_{w=z} \end{pmatrix} .$$

COROLLARY 3.3. *If u has real coefficients, i.e., if $u(z,\bar{z}) = u(\bar{z},z)$, then $\zeta_0(z,z) = 0$.*

Clearly if the fundamental group $\Gamma = \pi_1(\mathcal{M})$ contains the real translation $z \to z+p$, then the conformal factor $e^{u(z,\bar{z})}$ and the coefficient Q of the Hopf differential of the CMC-immersion are periodic with period p.

COROLLARY 3.4. *If the fundamental group $\Gamma = \pi_1(\mathcal{M})$ contains the real translation $z \to z+p$, $p\in\mathbb{R}$, then the corresponding skew-hermitian potential ζ defined in Equation* (3.4) *is periodic with period p.*

3.3. Uniqueness of skew-hermitian potentials: We know that normalized potentials are uniquely determined while holomorphic potentials are not (see Introduction and [**21**]). It is thus natural to ask whether the new type of potential defined above is uniquely determined by a given CMC-immersion. From Section 3.2, we will consider the skew-hermitian potentials defined in Equation (3.4). The following theorem implies that the skew-hermitian potential is almost uniquely determined by a CMC-immersion.

THEOREM 3.5. *Let $\zeta = \lambda^{-1}\zeta_{-1} + \zeta_0 + \lambda\zeta_1$, $\lambda\in S^1$ be a skew-hermitian potential defined in Equation* (3.4). *Assume that the same assumptions hold for $\hat{\zeta}$. Then ζ and $\hat{\zeta}$ induce the same* CMC*-immersion if and only if $\hat{\zeta} = L_0^{-1}\zeta L_0 + L_0^{-1}dL_0$, where $L_0(z) = \pm\mathrm{Id}$ or $L_0(z) = \mathrm{diag}(\pm i, \mp i)$.*

PROOF. "$\Rightarrow$" Since ζ and $\hat{\zeta}$ are meromorphic potentials for the same immersion, we know that the solutions to $dC = C\zeta$ and $d\hat{C} = \hat{C}\hat{\zeta}$ satisfy the relation

$$\hat{C} = CL_+ \ ,$$

where $L_+ \in \Lambda^+ SL(2,\mathbb{C})_\sigma$ is meromorphic in $z \in \mathfrak{D}$ and holomorphic for $\lambda \in \mathbb{C}\setminus\{0\}$. For $z \in \mathbb{R}\cap\mathfrak{D}$ we know that $L_+(z,\lambda)$ is unitary. Hence $\left(\overline{L_+(z,\lambda)}^t\right)^{-1} = L_+(z,\lambda)$ for $z \in \mathbb{R}\cap\mathfrak{D}$. Therefore, $L_+(z,\lambda) = L_0(z)$ is independent of λ for $z \in \mathbb{R}\cap\mathfrak{D}$, and $\hat{C} = CL_0$ implies $\hat{\zeta} = L_0^{-1}\zeta L_0 + L_0^{-1}dL_0$ for $z \in \mathbb{R}\cap\mathfrak{D}$, where $L_0 = \mathrm{diag}(l_0, l_0^{-1})$ and $|l_0| = 1$. Moreover $\hat{h}(z)$, which is an entry of $\hat{\zeta}$ defined by Equation (3.4), is real for $z \in \mathfrak{D}\cap\mathbb{R}$, and thus $L_+ = \pm\mathrm{Id}$ or $L_+ = \mathrm{diag}(\pm i, \mp i)$, and by analytic continuation, $L_+ = \pm\mathrm{Id}$ or $\mathrm{diag}(\pm i, \mp i)$ for all $z \in \mathfrak{D}$.
"$\Leftarrow$" Let $L_0 = \pm\mathrm{Id}$ or $\mathrm{diag}(\pm i, \mp i)$. We set $\hat{C} = CL_0$, $\hat{\zeta} = \hat{C}^{-1}d\hat{C}$ and $\zeta = C^{-1}dC$. We consider the Iwasawa decomposition of $\hat{C} = \left(F\tilde{U}\right)\left(\tilde{U}^{-1}W_+L_0\right)$, where $C = FW_+$ is the Iwasawa decomposition of C and $\tilde{U} \in U(1)$. $\tilde{U}$ goes away in the Sym-Bobenko-Formula, thus $F\tilde{U}$ and F define the same CMC surface. □

4. CMC-cylinders

4.1. Necessary and sufficient condition for CMC surfaces with a period: In this section, we consider CMC-cylinders. These are homeomorphic to $\mathbb{S}^2 \setminus \{p_1, p_2\}$, where p_1 and p_2 are different, but otherwise arbitary, points in $\mathbb{S}^2$. Using a Möbius transformation, we can move these two points to the north pole and the south pole of $\mathbb{S}^2$, and using stereographic projection from the north pole we can assume that every CMC-cylinder is an immersion from $\mathcal{M} = \mathbb{C}\setminus\{0\}$ to $\mathbb{R}^3$. Then using a change of coordinates we can assume that every CMC-cylinder is an immersion from $\mathcal{M} = \mathbb{C}/p\mathbb{Z}$, where p is a real number. Since the fundamental group of a CMC-cylinder is generated by a single period, $p \in \mathbb{R}$, we can, by Corollary 3.4, assume that the corresponding skew-hermitian potential is well defined on $\mathcal{M} = \mathbb{C}/p\mathbb{Z}$.

PROPOSITION 4.1. *Let $\zeta = \lambda^{-1}\zeta_{-1} + \zeta_0 + \lambda\zeta_1$, $\lambda \in S^1$, be a meromorphic potential on $\mathfrak{D} = \mathbb{C}$ such that ζ is skew-hermitian for $z \in \mathbb{R}$. Let $C(z,\lambda)$ be the unique solution to $dC = C\zeta$, $C(0,\lambda) = \mathrm{Id}$. Assume ζ is periodic with period $p \in \mathbb{R}$. Then the CMC surface associated with ζ for $\lambda = 1$ has the translation by $p \in \mathbb{R}$ in its fundamental group if and only if the unitary monodromy matrix $C(p,\lambda)$ satisfies the closing conditions (1.12) for $\lambda = 1$.*

PROOF. Let C be the solution to $dC = C\zeta$, $C(0,\lambda) = \mathrm{Id}$, where $0 \in \mathfrak{D} = \mathbb{C}$. Since ζ is skew-hermitian for $z \in \mathbb{R}$ we obtain that $C(z,\lambda) \in \Lambda_r SU(2)_\sigma$ for $z \in \mathbb{R}$. Since ζ is periodic with period p we have $C(p+z,\lambda) = \chi(\lambda)C(z,\lambda)$ by Proposition 1.8. The initial condition $C(0,\lambda) = \mathrm{Id}$ implies $\chi(\lambda) = C(p,\lambda)$. Since p is a real number, $\chi(\lambda)$ is a unitary matrix, and thus $\chi(\lambda) = C(p,\lambda)$ is also a monodromy matrix for $F(z,\bar{z},\lambda)$, which is the unitary part of the Iwasawa decomposition of $C(z,\lambda) = F(z,\bar{z},\lambda)W_+(z,\bar{z},\lambda)$. The claim now follows from Theorem 1.13. □

COROLLARY 4.2. *We retain the assumptions of Proposition 4.1. Then, for a cylinder, $C(z,1)$ and $F(z,\bar{z},1)$ are periodic with period p.*

Now we will give the definition of a *frame periodic surface.*

DEFINITION 4.3. *We call a surface frame periodic for $\lambda = \lambda_0 \in S^1$ if the frame $F(z, \bar{z}, \lambda = \lambda_0) \in SU(2)$ of a CMC-immersion $\Psi_{\lambda=\lambda_0}$ is periodic with period p.*

COROLLARY 4.4. *We retain the assumptions of Proposition 4.1. Then a surface is frame periodic at $\lambda = \lambda_0 \in S^1$ if and only if $C(z, \lambda_0)$ is periodic.*

4.2. Frame periodic CMC-**immersions:** To illustrate the discussion above, we consider the construction of a frame periodic surface from a skew-hermitian periodic potential. It turns out that it is easy to modify the "generalized Weierstrass representation" ([**21**]) so that the first closing condition for a monodromy matrix is trivially satisfied, i.e., $M|_{\lambda=1} = \pm\mathrm{Id}$. To motivate our approach, assume first that ζ is the skew-hermitian potential as in Equation (3.4) associated with a frame periodic surface. Let C be the solution to $dC = C\zeta$, $C(0, \lambda) = \mathrm{Id}$. Then $C(p, \lambda)$ is a unitary monodromy matrix and $C_0(z) = C(z, \lambda = 1)$ is periodic.

We use this last fact as a starting point for a "sufficiently nice potential". By the remark just made, every frame periodic CMC surface can be obtained in this way. Let's start, conversely, from a meromorphic and periodic matrix $C_0(z)$ of the form

$$C_0(z) = \begin{pmatrix} a_0(z) & b_0(z) \\ -\overline{b_0(\bar{z})} & \overline{a_0(\bar{z})} \end{pmatrix} , \tag{4.1}$$

where $a_0(z), b_0(z)$ are periodic functions of period $p \in \mathbb{R}$ satisfying $\det C_0(z) = a_0(z)\overline{a_0(\bar{z})} + b_0(z)\overline{b_0(\bar{z})} = 1$. In particular C_0 is unitary for $z \in \mathbb{R}$. Set

$$\begin{aligned} \zeta_0(z) &= C_0^{-1} C_0' \\ &= \begin{pmatrix} \nu(z) & \kappa(z) \\ -\overline{\kappa(\bar{z})} & -\nu(z) \end{pmatrix} , \end{aligned} \tag{4.2}$$

where ν and κ are periodic 1-forms of period $p \in \mathbb{R}$ and $\overline{\nu(\bar{z})} = -\nu(z)$. We take meromorphic 1-forms $h(z)$ and $\overline{g(\bar{z})}$ on $\mathbb{C}$, which are periodic of period p, and satisfy $\kappa(z) = h(z) - \overline{g(\bar{z})}$ for $z \in \mathbb{C}$. Set

$$\zeta(z, \lambda) = \left\{ \lambda^{-1} \begin{pmatrix} 0 & h(z) \\ g(z) & 0 \end{pmatrix} + \begin{pmatrix} \nu(z) & 0 \\ 0 & -\nu(z) \end{pmatrix} + \lambda \begin{pmatrix} 0 & -\overline{g(\bar{z})} \\ -\overline{h(\bar{z})} & 0 \end{pmatrix} \right\} dz . \tag{4.3}$$

Then $\zeta(z, \lambda = 1) = \zeta_0(z)$ and ζ is meromorphic for $z \in \mathfrak{D}$ and skew-hermitian for $z \in \mathbb{R}$. Therefore ζ satisfies the assumptions of Proposition 4.1. Let $C = C(z, \lambda)$ denote the solution to $dC = C\zeta$, $C(0, \lambda) = \mathrm{Id}$. Our construction implies $C(z, 1) = C_0(z)$, whence the first closing condition is satisfied, i.e., $\chi(1) = \pm\mathrm{Id}$. Therefore we have shown

THEOREM 4.5. *Let $C_0(z)$ be given by Equation* (4.1) *and assume $C_0(z)$ is periodic with period p. Set $\zeta_0 = C_0^{-1} dC_0$ and define ζ by Equation* (4.3). *Then ζ satisfies the assumptions of Proposition 4.1 and defines a frame periodic* CMC-*immersion. Moreover, every frame periodic* CMC-*immersion can be obtained this way.*

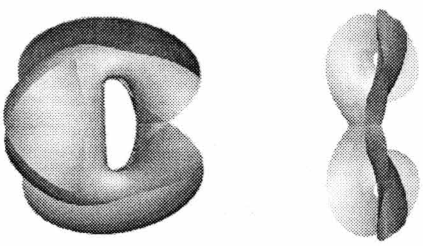

FIGURE 1. New example of a CMC-cylinder. Figures are constructed using [**53**].

REMARK 4.6. *The construction of* ζ *from* ζ_0 *carried out above shows that there is a lot of freedom. Actually, different choices of* h *and* g *yield by and large different surfaces.*

4.3. Second closing conditions: Finally, we consider CMC-cylinders. By the discussion of the last section it only remains to consider the second closing condition, i.e., $\partial_\lambda M|_{\lambda=1} = 0$, where M is the monodromy matrix defined in the sense of Definition 1.6. There is a nice trick reducing the second closing condition to the vanishing of some integral [**34**], [**38**].

PROPOSITION 4.7. *If* $C_0(z)$ *is as in Equation* (4.1) *and if* ζ *is defined as in Equation* (4.3). *Then* ζ *satisfies the assumptions of Proposition 4.1 and the first closing condition, i.e.,* $M|_{\lambda=1} = \pm\mathrm{Id}$, *is satisfied. The second closing condition, i.e.,* $\partial_\lambda M|_{\lambda=1} = 0$, *is satisfied if and only if*

$$\int_0^p \{C_0 \partial_\lambda \zeta|_{\lambda=1} C_0^{-1}\} dw = 0 \ . \tag{4.4}$$

We summarize the discussion of the last few sections:

THEOREM 4.8. *The construction outlined in Section 4.2 and 4.3 produces all frame periodic* CMC*-immersions. Moreover, the construction above also produces all* CMC*-cylinders.*

4.4. New examples of CMC**-cylinders:** Below we illustrate how the theory presented above can be applied to concrete constructions of CMC-cylinders. Let C_0 be the following matrix:

$$C_0 = \begin{pmatrix} \cos(z) & \sin(z) \\ -\sin(z) & \cos(z) \end{pmatrix} \ .$$

Clearly C_0 is holomorphic in $\mathbb{C}$, periodic with period 2π and unitary for $z \in \mathbb{R}$. Then for the Maurer-Cartan form ζ_0, $\zeta_0 = C_0^{-1} dC_0$ of C_0, we obtain:

$$\zeta_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ .$$

We decompose $1 = h(z) - \overline{g(\bar{z})} = 1/2e^{ik\cos(z)} + 1/2(2 - e^{ik\cos(z)})$ as in Section 4.2, and we introduce λ according to Equation (4.3). Then we obtain the periodic skew-hermitian potential

$$\zeta(z,\lambda) = \begin{pmatrix} 0 & h(z)\lambda^{-1} - \overline{g(\bar{z})}\lambda \\ g(z)\lambda^{-1} - \overline{h(\bar{z})}\lambda & 0 \end{pmatrix} ,$$

where $h(z) = 1/2e^{ik\cos(z)}$, $g(z) = -1/2(2 - e^{-ik\cos(z)})$ and k is some real number. To determine k, we evaluate Equation (4.4).

$$\int_0^{2\pi} \{C_0 \partial_\lambda \zeta|_{\lambda=1} C_0^{-1}\} dw = \begin{pmatrix} 0 & 2\pi J_2(k) \\ 2\pi J_2(k) & 0 \end{pmatrix} , \tag{4.5}$$

where $J_2(z)$ is the Bessel function of the first kind (see [**8**]). If we choose k such that $J_2(k) = 0$. Then we obtain an example of a CMC-cylinder. In Figure 1, we have the CMC-cylinder corresponding to two different values of k. In the left picture in Figure 1, we use $k = 5.13$, the square domain $0 < x < 2\pi$ and $-0.4 < y < 0.4$. In the right picture in Figure 1, we use $k = 8.41$, the square domain $0 < x < 2\pi$ and $-0.1 < y < 0.1$, where $z = x + iy$.

REMARK 4.9. (i) *In* [**34**], [**38**], *many examples are given starting from skew-hermitian potentials. However a comparsion of the Hopf differentials of these surfaces with the example above shows that the surface given above is different from the examples given in* [**34**] *and* [**38**] *and therefore yields a new example of a* CMC*-cylinder.*

(ii) *The main result of this paper shows how one can construct* CMC*-cylinders from skew-hermitian matrices, and even better, from matrices of type* C_0*. Theorem 3.5 shows that different potentials yield (by and large) different immersions. A fine classification of* CMC*-cylinders would be highly desirable.*

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