Renormalization Group Analysis and Hierarchical Models

保阪, 賢資

(Degree)
博士（理学）
(Date of Degree)
2006-03-25
(Date of Publication)
2012-05-25
(Resource Type)
doctoral thesis
(Report Number)
甲3528
(URL)
https://hdl.handle.net/20.500.14094/D1003528

博 士 論 文

Renormalization Group Analysis and Hierarchical Models

平成１８年１月

神戸大学大学院自然科学研究科

保　阪　賢　資

博士論文

Renormalization Group Analysis and Hierarchical Models

（くりこみ群解析と階層模型について）

平成１８年１月

神戸大学大学院自然科学研究科

保阪　賢資

Contents

Chapter 1

Introduction

The statistical physics is the one of the main subject of the modern physics which deals with systems involve a very large number of degrees of freedom, for example phase transition of ferromagnet and so on. The renormalization group analysis is a method for dealing with some of the most difficult problems which involve a large number of degrees of freedom. The historical monograph of Renormalization Group (RG) Analysis appeared in 1974 [13]. In this monograph, They show not only the fundamental idea of RG analysis, but also the concrete calculations and examples. However, these calculations are still not proved in mathematical sense. Our purpose of studying mathematical RG analysis is to justify RG analysis in mathematical sense. However, to establish RG analysis in mathematical sense is a long and winding road for mathematical physicists, in fact mathematical RG analysis has not been accomplishment yet. However, some informative results have appeared. For example, P.M. Bleher and Y. G. Sinai generalized Dyson's hierarchical models, and studied these models by using RG analysis. K. Gawędzki and A. Kupiainen has shown so-called triviality of hierarchical ϕ_4^4 model. Recently, the author succeeded the generalization of Gawędzki and Kupiainen's work. This generarization is the main part of this thesis.

The organization of this thesis is as follows. In Chapter 2, we introduce hierarchical models, and Renormalization Group. Chapter 3 is a review of Gawędzki and Kupiainen's work. And then, we present recent result which the author showed. The main part of Chapter 4 will be published in

Kenshi Hosaka; Triviality of Hierarchical Models with Small Negative ϕ^4 Coupling in Four Dimensions, Journal of Statistical Physics.

Section 4.4 in Chapter 4 is proof of triviality of hierarchical $P(\phi)$ model, will be published in

Kenshi Hosaka; Triviality of Hierarchical $P(\phi)$ model, RIMS kokyuroku.

Finally, we try to show the law of iterated logarithm about the block spin at the origin. Unfortunately, we show only Gaussian case in this thesis, however we expect that this property holds in general cases.

Acknowledgements: The author is grateful to Professor Yasunari Higuchi for his encouragements and his helpful advice. The author would like to thank Professor Tetsuya Hattori, Professor Takashi Hara, Professor Hiroshi Watanabe, and Professor K.R. Ito for emphasizing the importance of renormalization group idea and giving valuable comments. The author is also grateful to Professor Hidenori Sonoda, and Professor Tomotoshi Nishino for their advices. The author would also like to thank Professor Munemi Miyamoto, Professor Jousin Murai, Professor Taizo Chiyonobu, Dr. Masaki Takeshima, Dr. Nobuaki Sugimine, Dr. Masato Takei for giving useful advices at the Tuesday seminar, *Statistical Mechanics in Kobe* .

Chapter 2

Hierarchical Models

In this chapter, we introduce specific spin models called hierarchical spin models. F.J. Dyson introduced new type spin models imitating in many respects the lattice systems with pair wise long range interaction in the end of 1960's [4]. He proved that these models exhibit phase transition under suitable conditions. Now, theses models were named Dyson's hierarchical models after his name. In the first half of 1970's, P.M. Bleher and Y.G. Sinai succeeded to generalize Dyson's hierarchical models, and obtained some interesting results by using RG analysis. We start this chapter with a exact definition of hierarchical models introduced by Gawędzki and Kupiainen [5] [6].

2.1 Gawędzki-Kupiainen's hierarchical models

In this section, we will introduce the Gawędzki and Kupiainen's hierarchical models. In the statistical physics, we must consider the following distribution,

$$\frac{1}{Z}\exp[-V(\phi)]d\nu_G(\phi), \tag{2.1}$$

where $d\nu_G(\phi)$ is the Gaussian measure with mean zero and covariance $G = (-\Delta)^{-1}$. Note that

$$(-\Delta)^{-1}_{xy} \sim \frac{1}{|x-y|^{d-2}} \text{ for } |x-y| \to \infty. \tag{2.2}$$

The hierarchical model replaces the covariance $(-\Delta)^{-1}$ by its (hierarchical) modified version G^h with similar long range behavior. Let L be a positive even integer, $\mathbf{Z}^d$ be a d-dimensional hyper cubic lattice, and G^h be as follows

$$G^h_{xy} = \sum_{k=0}^{\infty} L^{-(d-2)k}\Gamma_{x_k,y_k} \tag{2.3}$$

where $x_k \in \mathbf{Z}^d$ satisfies $[L^{-k}x] = x_k$ where $[x]$ is the integer part of x, and

$$\Gamma_{x_k y_k} = \delta_{x_{k+1}y_{k+1}} A_{x_{k+1}}(x_k) A_{y_{k+1}}(y_k) \tag{2.4}$$

where $\{A_z\}_{z\in\mathbf{Z}^d}$ is defined by

$$\begin{aligned} A_z(x) &= +1 \text{ or } -1, \text{ for } [L^{-1}x] = z, z \in \mathbf{Z}^d && (2.5)\\ \sum_{x:[L^{-1}x]=z} A_z(x) &= 0 && (2.6)\end{aligned}$$

Notice that G^h is a positive but not strictly positive matrix. It is easy to see that

$$\begin{aligned} G^h_{xy} &= L^{-(d-2)k_{xy}}\frac{1}{1-L^{-(d-2)}} \\ &\quad + L^{-(d-2)(k_{xy}-1)} A_{x_{k_{xy}}}(x_{k_{xy}-1}) A_{x_{k_{xy}}}(y_{k_{xy}-1}) \end{aligned} \tag{2.7}$$

where k_{xy} is the smallest positive integer satisfies $[L^{-k}x] = [L^{-k}y]$. In that, the long range behavior of G^h_{xy} mimics that of (2.2), however G^h is not translation invariance which $G = (-\Delta)^{-1}$ satisfies. Let $d\nu_{G_h}$ be the Gaussian distribution with mean 0, covariance G^h. Hereafter, we consider the probability measure formally given by

$$d\nu_V(\phi) = \frac{1}{Z}\exp[-V(\phi)]d\nu_{G_h}(\phi) \tag{2.8}$$

2.2 Block Spin Transformation and Renormalization Group of Hierarchical Models

In this section, we investigate Block Spin Transformation (BST) and Renormalization Group. First of all we define Block Spin Transformation as follows,

$$\phi^{(1)}_z = L^{-\theta} \sum_{x\in B_{L^n}(z)} \phi_x. \tag{2.9}$$

Note that we almost consider $\theta = \frac{(d+2)}{2}$ case in this thesis, Particularly we only use the symbol $\phi^{(1)}_z$ for $\theta = \frac{(d+2)}{2}$ case. We note that this hierarchical (massless) Gaussian measure $d\nu_{G_h}$ is invariant under the block spin transformation (2.9). Due to (2.3), we may also write

$$G^h_{xy} = L^{-(d-2)} G^h_{x_1,y_1} + \Gamma_{xy}. \tag{2.10}$$

Both terms on the right hand side of (2.10) are positive matrices. Hence we can realize the field ϕ_x distributed according to $d\nu_{G_h}$ as a sum of two independent ones:

$$\phi_x = L^{-(d-2)/2}\phi_{x_1} + Z_x, \tag{2.11}$$

where Z according to normal distribution $N_{0,1}$. Since $\{Z_x\}_{x\in \mathbf{Z}^d}$ is a mean 0 Gaussian system with covariance Γ, there exist i.i.d. random variables $\{\tilde{Z}_z\}_{z\in\mathbf{Z}^d}$ with distribution $N_{0,1}$. $Z_x = A_z(x)\tilde{Z}_z$ a.s. for every $x \in B_L(z), z \in \mathbf{Z}^d$. So $\sum_{x\in B_L(x)} Z_x = 0$ a.s.. Note that Z describes the fluctuations of ϕ around the average values. From the property of Gaussian system, (2.10) implies that

$$d\nu_{G_h}(\phi) = d\nu_{G_h}(\phi^{(1)}) \times d\nu_\Gamma(Z). \tag{2.12}$$

Now, if the random field ϕ is distributed according to

$$\begin{aligned}
&\frac{1}{\mathcal{N}}\exp[-\sum_x v(\phi_x)]d\nu_{G_h}(\phi)\\
&= \frac{1}{\mathcal{N}}\exp[-\sum_x v(L^{-(d-2)/2}\phi^{(1)}_{x_1} + Z_x)]d\nu_{G_h}(\phi^{(1)}) \times d\nu_\Gamma(Z),
\end{aligned} \tag{2.13}$$

then the distribution of the block spin field $\phi^{(1)}$ is

$$\begin{aligned}
&\frac{1}{\mathcal{N}}\prod_y\Big(\int \exp[-\sum_{x:x_1=y} v(L^{-(d-2)/2}\phi^{(1)}_{x_1} + Z_x)]d\nu_\Gamma(Z)\Big)d\nu_{G^h}(\phi^{(1)})\\
&= \frac{1}{\mathcal{N}}\prod_y\Big(\int \exp[-\sum_{x:x_1=y} v(L^{-(d-2)/2}\phi^{(1)}_{x_1} + A_{Ly}(x)z)]d\nu(z)\Big)d\nu_{G^h}(\phi^{(1)})
\end{aligned} \tag{2.14}$$

where $d\nu(z) = \frac{1}{(2\pi)^{1/2}}\exp(-\frac{1}{2}z^2)dz$. Let us defined the new potential $\mathcal{R}v = v' = v_1$ by

$$\begin{aligned}
&\exp[-\mathcal{R}v(\phi)]\\
&= \frac{\int \exp[-\frac{1}{2}L^d[v(L^{-(d-2)/2}\phi + z) + v(L^{-(d-2)/2}\phi - z)]]d\nu(z)}{\int \exp[-L^4 v(z)]d\nu(z)}.
\end{aligned} \tag{2.15}$$

We call this transformation $\mathcal{R} : v \to v'$ **Renormalizaton Group Transformation**.

2.3 Renormalization Flow – Dynamical System of RG Transformation

In this section, we consider the property of RG transformation. More precisely, we consider about discrete dynamical system determined by iterations of RG transformation 2.15. First, we define a fixed point of this system.

Definition 2.3.1 *Let $f(\phi)$ be a real valued function of $\phi \in \mathbf{R}$. Then $f(\phi)$ is called a fixed point of RG transformation, if $f(\phi)$ satisfies the following equation*

$$\exp[-f(\phi)] \\ = \frac{\int \exp[-\frac{1}{2}L^d[f(L^{-(d-2)/2}\phi + z) + f(L^{-(d-2)/2}\phi - z)]]d\nu(z)}{\int \exp[-L^4 f(z)]d\nu(z)}. \quad (2.16)$$

From the definition of this hierarchical spin model, we can immediately find the following fixed point.

Proposition 2.3.2 *The function $f(\phi) \equiv 0$ is one of the fixed point.*

In fact, we calculate the RHS of (2.16)

$$\frac{\int \exp[-\frac{1}{2}L^d[f(L^{-(d-2)/2}\phi + z) + f(L^{-(d-2)/2}\phi - z)]]d\nu(z)}{\int \exp[-L^4 f(z)]d\nu(z)} \\ = \frac{\int \exp[0]d\nu(z)}{\int \exp[0]d\nu(z)} = 1 = \exp[0] \equiv \exp[f(\phi)]. \quad (2.17)$$

This fixed point is the one of the most fundamental and important fixed point of the dynamical system defined by RG transformation.

Definition 2.3.3 *We call the fixed point of proposition 2.3.2 the* **Gaussian fixed point.**

It is natural to ask about behavior of this discrete dynamical system in the neighborhood of the Gaussian fixed point, it is the main theme of this thesis. Precisely speaking, we consider the following question.

Question 2.3.4 *When does the RG iterations of a function $v(\phi)$ converge to the Gaussian fixed point?*

In the quantum field theory, the above property is called **trivial**.

Chapter 3

Renormalization Group Analysis for Hierarchical ϕ_4^4 case

3.1 About Gawędzki and Kupiainen's work

In this section, we explain the work of Gawędzki and Kupiainen [7]. They established how to analyze RG flow [7], and succeeded proving triviality of ϕ_4^4 case. Their analysis contains two parts. Roughly speaking, the first part is analyzing precise structure of the iterated potential near small field, second part is analyzing global structure of the iterated potential. In this section, we will use the following notations,

$$(v_n)_k(\phi) = \frac{1}{k!}\frac{d^k v_n(0)}{d\phi^k}\phi^k, \tag{3.1}$$

$$(v_n)_{\geq k}(\phi) = v_n(\phi) - \sum_{l<k}(v_n)_l. \tag{3.2}$$

In this section, single spin potentials v_n are assumed to satisfying the following conditions.

G-Ka $e^{-(v_n)_{\geq 4}(\phi)}$ is analytic in $|\mathbf{Im}\phi| < C_1(n_0+n)^{1/4}$, positive for real ϕ, even and

$$|\exp[-(v_n)_{\geq 4}(\phi)]| \leq \exp[D - \lambda_n^{1/2}|\phi|^2 + A_1\lambda_n(\mathbf{Im}\phi)^4], \tag{3.3}$$

in this region.

G-Kb for $|\phi| < C_1(n_0+n)^{1/4}$, $(v_n)_{\geq 4}(\phi)$ is analytic,

$$(v_n)_{\geq 4}(\phi) = \lambda_n\phi^4 + (v_n)_{\geq 6}(\phi) \tag{3.4}$$

with

$$\begin{aligned} \frac{C_- L^{-4}}{n_0+n} &\le \lambda_n \le \frac{C_+ L^{-4}}{n_0+n}, \; C_- = \frac{1}{48}, \; C_+ = \frac{1}{24}, && (3.5)\\ |(v_n)_{\ge 6}(\phi)| &\le (n_0+n)^{-3/4}. && (3.6) \end{aligned}$$

We write $V_n^{G-K}(L, D, C_1, n_0)$ for the collection of such single spin potentials v_n. If single spin potential satisfies these conditions, a potential obtained by iteration of RG (2.15) also satisfies these conditions. Precise statement is as follows.

Theorem 3.1.1 (Gawędzki and Kupiainen) *There exist constants*

$$D, \bar{C}_1(L, D) \ge 1, \bar{n}_0(L, D, C_1)$$

such that the following holds. Let $C_1 \ge \bar{C}_1(L, D)$, $n_0 \ge \bar{n}_0(L, D, C_1)$, $n \ge 0$ and $\delta > 0$ fixed. Put

$$v_n(\phi) = \mu_n - \frac{6\lambda_n}{1-L^{-2}}\phi^2 + (v_n)_{\ge 4}(\phi) \tag{3.7}$$

where $(v_n)_{\ge 4}(\phi) \in \mathcal{V}_n^{G-K}(L, D, C_1, n_0)$. Then, there exists a closed interval $J_n \subset I_n = [-(n_0+n)^{-1-\delta}, (n_0+n)^{-1-\delta}]$ such that for μ_n running through J_n, $(v_{n+1})_{\ge 4}(\phi) = v_{n+1}(\phi) - \mu_{n+1}\phi^2 + \frac{6\lambda_{n+1}}{1-L^{-2}}\phi^2 \in \mathcal{V}_{n+1}^{G-K}(L, D, C_1, n_0)$. Further, the map $\mu_n \mapsto \mu_{n+1}$ sweeps I_{n+1} continuously.

If we use this theorem inductively, we can prove the following theorem.

Theorem 3.1.2 (Gawedzki-Kupiainen) *If $(v_0)_{\ge 4}(\phi)$ satisfies above conditions* **G-Ka** *and* **G-Kb** *for $n = 0$, then there exists $\mu_{crit}(\lambda_0) \in \cap_{n=1}^{\infty} J_n \subset I_0$ such that the RG trajectory sarting from v_0 with $\mu = \mu_{crit}$ will leads to the Gaussian fixed point, i.e., the hierarchical ϕ_4^4 model is trivial.*

3.2 Proof of Theorem 3.1.1

The Proof of Theorem 3.1.1 is as follows. According to reference [7], this proof involves Small field region, and Large field region analyses corresponding to the cases either $|\phi| < C_1(n_0+n)^{1/4}$, or $|\mathbf{Im}\phi| < C_1(n_0+n)^{1/4}$, respectively. In the small field, we prove that $v_n'(\phi)$ satisfies **G-Kb**, n being replaced by $n+1$. We call it simply **G-Kb'**. On the other hand, we only investigate the global behavior of $v_n'(\phi)$ in the large field region, i.e. we confirm that $v_n'(\phi)$ satisfies with (3.3) of **G-Ka'**.

3.2.1 small field region

In this subsection, we assume that ϕ satisfies $|\phi| < (1/2)LC_1(n_0+n)^{1/4}$, and $\chi(z)$ is the indicator function given by $\chi(z) = \chi(|z| < (n_0+n)^{1/4})$. $e^{-\tilde{v}'(\phi)}$ is defined by

$$e^{-\tilde{v}'_n(\phi)} = \int \exp[-\frac{1}{2}L^d \sum_{\pm} v_n(L^{2-d/2}\phi \pm z)]\chi(z)\frac{d\nu(z)}{(\phi=0)_{small}}, \tag{3.8}$$

where

$$\begin{aligned} d\nu(z) &= e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}, \\ (\phi=0)_{small} &= \int \exp[-L^d v_n(z)]\chi(z)d\nu(z). \end{aligned}$$

We prepare the following notation,

$$\begin{aligned} d\nu_\chi(z) &= \chi(z)e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}, & (3.9)\\ w_n(\phi,z) &= 6L^2\lambda_n\phi^2z^2 + L^4\lambda_n z^4 + \frac{1}{2}L^4\mu_n z^2 - \frac{6L^4\lambda_n}{1-L^2}z^2 \\ &\quad +\frac{1}{2}L^4\sum_{\pm}(v_n)_{\geq 6}(L^{-1}\phi \pm z), & (3.10)\\ \langle - \rangle_t &\equiv \int -e^{-tw_n(\phi,z)}d\nu_\chi(z)/\int e^{-tw_n(\phi,z)}d\nu_\chi(z). & (3.11) \end{aligned}$$

Notice that $e^{v'_n(\phi)}$ is positive if ϕ is in the small field region. So, we take the logarithm.

$$\tilde{v}'_n(\phi) = \frac{1}{2}L^2\mu_n\phi^2 - \frac{6L^2\lambda_n}{1-L^2}\phi^2 + \lambda_n\phi^4 - \log\int e^{-w_n(\phi,z)}d\nu_\chi(z). \tag{3.12}$$

Now, we start expanding $-\log\int e^{-w_n(\phi,z)}d\nu_\chi(z)$, we call this expansion perturbation expansion.

$$\begin{aligned} -\log\int e^{-w_n(\phi,z)}d\nu_\chi(z) &= \langle w_n(\phi,z)\rangle_0 - \frac{1}{2}\langle w_n(\phi,z); w_n(\phi,z)\rangle_0 \\ &\quad +\frac{1}{2}\int_0^1 dt(1-t)^2\langle w_n(\phi,z); w_n(\phi,z); w_n(\phi,z)\rangle_t, \end{aligned} \tag{3.13}$$

where $\langle f;g\rangle_0 = \langle fg\rangle_0 - \langle f\rangle_0\langle g\rangle_0$, and $\langle f;g;h\rangle_t = \langle fgh\rangle_0 - \langle fg\rangle_0\langle h\rangle_0 - -\langle fh\rangle_0\langle g\rangle_0 - \langle gh\rangle_0\langle f\rangle_0 + 2\langle f\rangle_0\langle g\rangle_0\langle h\rangle_0$. Let us estimate each term of the equation of (3.13).

Estimation of $\langle w_n(\phi,z)\rangle_0$

$$\langle w_n(\phi,z)\rangle_0 = 6L^2\lambda_n\phi^2\langle z^2\rangle_0 + L^4\lambda_n\langle z^4\rangle_0 + \frac{1}{2}L^4\mu_n\langle z^2\rangle_0 - \frac{6L^4\lambda_n}{1-L^2}\langle z^2\rangle_0 + \frac{1}{2}L^4\sum_{\pm}\langle (v_n)_{\geq 6}(L^{-1}\phi\pm z)\rangle_0. \tag{3.14}$$

Notice that

$$\langle z^{2m}\rangle_0 = (2m-1)!!(1+O(e^{-\epsilon(n_0+n)^{1/2}})).$$

So, we estimate $\langle w_n(\phi,z)\rangle_0$ as follows,

$$\langle w_n(\phi,z)\rangle_0 = O_0((n_0+n)^{-1}) + 6L\lambda_n\phi^2 + O_2(e^{-\epsilon(n_0+n)^{1/2}}) + \frac{L^4}{2}\langle\sum_{\pm}(v_n)_{\geq 6}(L^{-1}\phi\pm z)\rangle_t, \tag{3.15}$$

where $O_{2n}(A) = O(\sup_{|\phi|=L/2C_1(n_0+n)^{1/4}}|\phi|^{-2n}A)$, $O(\cdot)$ is usual Landau O. $(v_n)_{\geq 6}(\phi)$ is even analytic function, we get

$$\langle\sum_{\pm}(v_n)_{\geq 6}(L^{-1}\phi\pm z)\rangle_0 = \langle (v_n)_{\geq 6}(z)\rangle_0 + \frac{1}{2}L^{-2}\phi^2\langle\frac{d^2}{dz^2}(v_n)_{\geq 6}(z)\rangle_0 + \frac{1}{4!}L^{-4}\phi^4\langle\frac{d^4}{dz^4}(v_n)_{\geq 6}(z)\rangle_0 + R_6(\phi), \tag{3.16}$$

where

$$R_6(\phi) = \frac{1}{5!}L^{-6}\phi^6\int_0^1 dt(1-t)^5 \langle\frac{d^6}{dz^6}\{(v_n)_{\geq 6}(tL^{-1}\phi+z) - (v_n)_{\geq 6}(-tL^{-1}\phi-z)\}\rangle_0. \tag{3.17}$$

By Cauchy's formula, we get

$$|\langle (v_n)_{\geq 6}(z)\rangle_0| = \frac{1}{5!}\int_0^1 dt(1-t)^5|\langle z^6\frac{d^6}{dz^6}(v_n)_{\geq 6}(z)\rangle_0| \leq \frac{15C_1}{(C_1-1)^7}(n_0+n)^{-9/4}(1+O(e^{-\epsilon(n_0+n)^{-1/2}})), \tag{3.18}$$

$$|\langle\frac{d^2}{dz^2}(v_n)_{\geq 6}(z)\rangle_0| = \frac{1}{3!}\int_0^1 dt(1-t)^3|\langle z^4\frac{d^6}{dz^6}(v_n)_{\geq 6}(z)\rangle_0| \leq \frac{90C_1}{(C_1-1)^7}(n_0+n)^{-9/4}(1+O(e^{-\epsilon(n_0+n)^{-1/2}})), \tag{3.19}$$

$$|\langle\frac{d^4}{dz^4}(v_n)_{\geq 6}(z)\rangle_0| = \int_0^1 dt(1-t)|\langle z^2\frac{d^6}{dz^6}(v_n)_{\geq 6}(z)\rangle_0| \leq \frac{360C_1}{(C_1-1)^7}(n_0+n)^{-9/4}(1+O(e^{-\epsilon(n_0+n)^{-1/2}})). \tag{3.20}$$

From the condition of **G-Kb**, we get

$$
\begin{aligned}
|R_6(\phi)| &\le |\langle (v_n)_{\ge 6}(L^{-1}\phi + z)\rangle_0| + |\frac{1}{2}L^{-2}\phi^2\langle \frac{d^2}{dz^2}(v_n)_{\ge 6}(z)\rangle_0| \\
&\quad + |\frac{1}{4!}L^{-4}\phi^4\langle \frac{d^4}{dz^4}(v_n)_{\ge 6}(z)\rangle_0| \\
&\le (n_0+n)^{-3/4} + O_0((n_0+n)^{-9/4}) + O_2((n_0+n)^{-7/4}) \\
&\quad + O_4((n_0+n)^{-5/4}) \\
&\le (n_0+n)^{-3/4} + O((n_0+n)^{-5/4}). \qquad (3.21)
\end{aligned}
$$

So, we get

$$
\begin{aligned}
\langle w_n(\phi, z)\rangle_0 &= O_0((n_0+n)^{-1}) + 6L^2\lambda_n\phi^2 + O_2((n_0+n)^{-7/4}) \\
&\quad + O_4((n_0+n)^{-5/4}) + R_6(\phi). \qquad (3.22)
\end{aligned}
$$

Especially, Following relation is satisfied

$$
|R_6(\phi)| \le L^4(n_0+n)^{-3/4} + O((n_0+n)^{-5/4}). \qquad (3.23)
$$

Estimation of $\langle w_n(\phi, z); w_n(\phi, z)\rangle_0$ Next we estimate $\langle w_n(\phi, z); w_n(\phi, z)\rangle_0$. By linearity,

$$
\begin{aligned}
&-\frac{1}{2}\langle w_n(\phi, z); w_n(\phi, z)\rangle_0 \\
&= -18L^4\lambda_n^2\phi^4\langle z^2; z^2\rangle_0 - 6L^6\lambda_n^2\phi^2\langle z^2; z^4\rangle_0 - 3L^6\lambda_n\mu_n\phi^2\langle z^2; z^2\rangle_0 \\
&\quad + \frac{36L^6\lambda_n^2}{(1-L^{-2})^2}\phi^2\langle z^2; z^2\rangle_0 - 6L^6\lambda_n\phi^2\langle z^2; (v_n)_{\ge 6}(L^{-1}\phi + z)\rangle_0 \\
&\quad - L^8\lambda_n\langle z^4; (v_n)_{\ge 6}(L^{-1}\phi + z)\rangle_0 - \frac{1}{2}L^8\mu_n\langle z^2; (v_n)_{\ge 6}(L^{-1}\phi + z)\rangle_0 \\
&\quad + \frac{6L^8\lambda_n}{(1-L^{-2})}\langle z^2; (v_n)_{\ge 6}(L^{-1}\phi + z)\rangle_0 \\
&\quad - \frac{L^8}{4}\sum_{\pm}\sum_{\pm}\langle (v_n)_{\ge 6}(L^{-1}\phi \pm z); (v_n)_{\ge 6}(L^{-1}\phi \pm z)\rangle_0 \\
&\quad + O_0((n_0+n)^{-2}), \qquad (3.24)
\end{aligned}
$$

where

$$
\begin{aligned}
&O_0((n_0+n)^{-2}) \\
&= -\frac{1}{2}L^8\lambda_n^2\langle z^4; z^4\rangle_0 - \frac{18L^8\lambda_n^2}{(1-L^{-2})^2}\langle z^2; z^2\rangle_0 - \frac{1}{8}L^8(\mu_n)^2\langle z^2; z^2\rangle_0 \\
&\quad - \frac{1}{2}L^8\lambda_n\mu_n\langle z^2; z^4\rangle_0 + \frac{6L^8\lambda_n^2}{(1-L^{-2})}\langle z^2; z^4\rangle_0 \\
&\quad - \frac{3L^8\lambda_n^2\mu_n}{(1-L^{-2})}\langle z^2; z^2\rangle_0. \qquad (3.25)
\end{aligned}
$$

From following estimations,

$$
\begin{aligned}
&|\langle z^{2n}; z^{2m}\rangle_0| \\
&\quad = ((2(n+m)-1)!! - (2n-1)!!(2m-1)!!)(1+O(e^{-\epsilon(n_0+n)^{1/2}})), \qquad (3.26) \\
&|\langle z^{2n}; (v_n)_{\geq 6}(L^{-1}\phi + z)\rangle_0| = (n_0+n)^{-3/4} + O((n_0+n)^{-5/4}), \qquad (3.27) \\
&|\langle (v_n)_{\geq 6}(L^{-1}\phi \pm z); (v_n)_{\geq 6}(L^{-1}\phi \pm z)\rangle_0| = (n_0+n)^{-3/2} + O_0((n_0+n)^{-5/2}) \\
&\qquad + O_2((n_0+n)^{-4}) + O_4((n_0+n)^{-7/2}) + O_{\geq 6}((n_0+n)^{-6/4}). \qquad (3.28)
\end{aligned}
$$

We use these estimations, we get as follows,

$$
\begin{aligned}
-\frac{1}{2}\langle w_n(\phi,z); w_n(\phi,z)\rangle_0 &= O_0((n_0+n)^{-2}) + O_2((n_0+n)^{-3/2}) \\
&\quad -36L^4\lambda_n^2\phi^4 + O_4((n_0+n)^{-9/4}) + O_{\geq 6}((n_0+n)^{-5/4}). \qquad (3.29)
\end{aligned}
$$

The estimation of $\int_0^1 dt(1-t)^2\langle w_n(\phi,z); w_n(\phi,z); w_n(\phi,z)\rangle_t$ Since we assume that $|\phi| < (1/2)LC_1(n_0+n)^{1/4}$, $|z| < (n_0+n)^{1/4}$, by definition of $w(\phi,z)$, we get

$$|w_n(\phi,z)| \leq O(1).$$

For $t \in (0,1)$, we get

$$
\begin{aligned}
&|\langle w_n(\phi,z); w_n(\phi,z); w_n(\phi,z)\rangle_t| \\
&\quad \leq K(\langle w_n(\phi,z)^3\rangle| + 3\langle w_n(\phi,z)^2\rangle\langle w_n(\phi,z)\rangle_0 + 2\langle w_n(\phi,z)\rangle_0) \qquad (3.30)
\end{aligned}
$$

where K is absolute constant.

For $|\phi| < (1/2)LC_1(n_0+n+1)^{1/4}$, by the above calculation, we have

$$
\begin{aligned}
\tilde{v}'_n(\phi) &= -\frac{1}{2}L^2\mu_n\phi^2 + \frac{6L^2\lambda_n}{1-L^2}\phi^2 + \lambda_n\phi^4 - 36L^4\lambda_n^2\phi^4 \\
&\quad + O_4((n_0+n)^{-4/5}) + \tilde{v}'_{n\geq 6}(\phi), \qquad (3.31)
\end{aligned}
$$

and $\tilde{v}'_{n\geq 6}(\phi)$ satisfies the following estimate,

$$|\tilde{v}'_{n\geq 6}(\phi)| \leq L^4(n_0+n)^{-3/4} + O((n_0+n)^{-5/4}). \qquad (3.32)$$

In the last part of this subsection, we estimate $\tilde{v}'_{n\geq 6}(\phi)$ again. From analyticity of $\tilde{v}'_n(\phi)$ in $|\phi| < (1/2)LC_1(n_0+n+1)^{1/4}$, $\phi^{-6}\cdot\tilde{v}'_{n\geq 6}(\phi)$ is also analytic function. So, we apply the maximum principle, and we get

$$
\begin{aligned}
&|\tilde{v}'_{n\geq 6}(\phi)| \\
&\leq (|\phi|/(1/2)LC_1(n_0+n+1)^{1/4})^6[L^4(n_0+n)^{-3/4} + O((n_0+n)^{-5/4})]. \qquad (3.33)
\end{aligned}
$$

So, we get

$$| \overset{\sim}{v'}_{n\geq 6} (\phi)| \leq \frac{2^6}{L^2}(n_0+n+1)^{-3/4} + O((n_0+n)^{-5/4}) \tag{3.34}$$

in $|\phi| < C_1(n_0+n+1)^{1/4}$. Now, we estimate remainder part of $-v'_n(\phi)$, i.e.,

$$\overset{\approx}{v'_n} (\phi) = \log(1 + \frac{\int \exp[-\frac{1}{2}L^4 \sum_\pm v_n(L^{-1}\phi \pm z) - \frac{1}{2}z^2](1-\chi(z))\frac{dz}{\sqrt{2\pi}}}{\exp[-\overset{\sim}{v'_n} (\phi)]}). \tag{3.35}$$

If n_0 is sufficiently large, this function is sufficiently small analytic function in the small field region. In other words, $v'_n(\phi)$ is almost equal to $\overset{\sim}{v'_n} (\phi)$. To show this, we must estimate $\overset{\approx}{v'_n} (\phi)$ using n_0, for sufficiently large n_0. So, we estimate

$$\frac{\int(1-\chi(z))\exp[-\frac{L^4}{2}\sum_\pm v_n(L^{-1}\pm z) - \frac{1}{2}z^2]\frac{dz}{\sqrt{2\pi}}}{\exp[-\overset{\sim}{v'_n} (\phi)]}, \tag{3.36}$$

the denominator part and the numerator part of separately. We use estimation of $\overset{\sim}{v'_n} (\phi)$.

$$\begin{aligned}
&|\text{denominator}| = |\exp[\overset{\sim}{v'_n} (\phi)]| \\
= \;& |\exp[-\frac{1}{2}L^2\mu_n\phi^2 + \frac{6L^2\lambda_n}{1-L^{-2}}\phi^2 - 6L^2\lambda_n\phi^2 - O_2((n_0+n)^{-3/2})]| \\
& \times|\exp[-\lambda_n\phi^4 + 36L^4\lambda_n^2\phi^4 - O_4((n_0+n)^{-5/4}) - \overset{\sim}{(v_n)'_{\geq 6}} (\phi)]| \\
= \;& \exp[(-\frac{1}{2}L^2\mu_n + \frac{6L^2\lambda_n}{1-L^{-2}} - 6L^2\lambda_n)((\mathbf{Re}\phi)^2 - (\mathbf{Im}\phi)^2)] \\
& \times\exp[+O_2((n_0+n)^{-3/2}) + O_4((n_0+n)^{-5/4})] \\
\times \;& \exp[(-\lambda_n + 36L^4\lambda_n^2)((\mathbf{Re}\phi)^4 - 6(\mathbf{Re}\phi)^2(\mathbf{Im}\phi)^2 + (\mathbf{Im}\phi)^4)].
\end{aligned} \tag{3.37}$$

From the condition of the small field region, $\mathbf{Re}\phi, \mathbf{Im}\phi < |\phi| < \frac{1}{2}LC_1(n_0 + n+1)^{1/4}$. Hence, we estimate as follows.

$$\begin{aligned}
&\text{RHS of (3.37)} \geq \exp[-(\frac{1}{2}L^2|\mu_n| + \frac{6L^2\lambda_n}{1-L^{-2}} + 6L^2\lambda_n)((\mathbf{Re}\phi)^2 + (\mathbf{Im}\phi)^2)] \\
&\quad \times\exp[O_2((n_0+n)^{-3/2}) + O_4((n_0+n)^{-5/4})] \\
&\quad \times\exp[+(\lambda_n + 36L^4\lambda_n^2)((\mathbf{Re}\phi)^4 + 6(\mathbf{Re}\phi)^2(\mathbf{Im}\phi)^2 + (\mathbf{Im}\phi)^4)] \\
&> \exp[-(\frac{1}{2}L^2|\mu_n| + \frac{6L^2\lambda_n}{1-L^{-2}} + 6L^2\lambda_n)\frac{1}{2}L^2(n_0+n+1)^{1/2}] \\
&\quad \times\exp[O_2((n_0+n)^{-3/2}) + (\lambda_n + 36L^4\lambda_n^2)\frac{1}{2}L^4(n_0+n+1)] \\
&\quad \times\exp[+O_4((n_0+n)^{-5/4})].
\end{aligned} \tag{3.38}$$

The last inequality is bucause the copling constants λ_n, μ_n have estimates; $\lambda_n < \frac{L^{-4}}{24(n_0+n)}, |\mu_n| < (n_0+n)^{-3/2}$.

$$\geq L^4 O((n_0+n)^{-1/2}) + O(1) + O_2((n_0+n)^{-3/2}) + O_4((n_0+n)^{-5/4}). \quad (3.39)$$

So, the absolute value of the denominator is bounded from below by a positive absolute constant. Next we estimate the numerator part,

$$\begin{aligned}
&|\text{numerator part}| \\
&< \int |(1-\chi(z)) \exp[-\frac{L^4}{2} \sum_{\pm} v_n(L^{-1} \pm z)]| d\nu(z) \\
&< \exp[(\frac{1}{2} L^2 |\mu_n| + \frac{6L^2 \lambda_n}{1-L^{-2}}) |\phi|^2] \\
&\quad \times \int |(1-\chi(z)) e^{\frac{L^4}{2}|\mu_n| z^2 + \frac{6L^4\lambda_n}{1-L^{-2}} z^2 - \sum_{\pm} (v_n)_{\geq 4}(L^{-1}\phi \pm z)} |^{\frac{L^4}{2}} d\nu(z). \quad (3.40)
\end{aligned}$$

From the condition of **G-Kb**, we get

$$\begin{aligned}
&|\text{numerator part}| \\
&< \exp[(\frac{1}{2} L^2 |\mu_n| + \frac{6L^2 \lambda_n}{1-L^{-2}} - L^2 \lambda_n^{1/2}) |\phi|^2 + L^4 D + 20\lambda_n (\mathbf{Im}\phi)^4] \\
&\quad \times \int |(1-\chi(z)) \exp[(\frac{1}{2} L^4 |\mu_n| + \frac{6L^4 \lambda_n}{1-L^{-2}} - \frac{1}{2}) z^2]| \frac{dz}{\sqrt{2\pi}}. \quad (3.41)
\end{aligned}$$

We estimate the last part of the right hand side of the above inequality as follows,

$$\begin{aligned}
&\int (1-\chi(z)) \exp[(\frac{1}{2} L^4 |\mu_n| + \frac{6L^4 \lambda_n}{1-L^{-2}} - \frac{1}{2}) z^2] \frac{dz}{\sqrt{2\pi}} \\
&= 2 \int_{(n_0+n)^{1/4}}^{\infty} \exp[(\frac{1}{2} L^4 |\mu_n| + \frac{6L^4 \lambda_n}{1-L^{-2}} - \frac{1}{2}) z^2] \frac{dz}{\sqrt{2\pi}} \\
&\leq 2 \exp[(\frac{1}{2} L^4 |\mu_n| + \frac{6L^4 \lambda_n}{1-L^{-2}} - \frac{1}{2}) (n_0+n)^{1/2}] \\
&\quad \times \int_0^{\infty} \exp[(\frac{1}{2} L^4 |\mu_n| + \frac{6L^4 \lambda_n}{1-L^{-2}} - \frac{1}{2}) z^2] \frac{dz}{\sqrt{2\pi}} \\
&= \exp[(\frac{1}{2} L^4 |\mu_n| + \frac{6L^4 \lambda_n}{1-L^{-2}} - \frac{1}{2}) (n_0+n)^{1/2}] \\
&\quad \times \sqrt{\frac{1}{2} - \frac{1}{2} L^4 |\mu_n| - \frac{6L^4 \lambda_n}{1-L^{-2}}}. \quad (3.42)
\end{aligned}$$

So, we get

$$\int (1-\chi(z)) \exp[-\frac{L^4}{2}\sum_{\pm} v_n(L^{-1}\pm z) - \frac{1}{2}z^2]\frac{dz}{\sqrt{2\pi}} / \exp[\tilde{v}'_n(\phi)]$$
$$\leq \text{ const.} \times \exp[(\frac{1}{2}L^4|\mu_n| + \frac{6L^4\lambda_n}{1-L^{-2}} - \frac{1}{2})(n_0+n)^{1/2}]. \quad (3.43)$$

Since n_0 is sufficiently large and ϕ is in the small field region, $\overset{\approx}{v}'_n(\phi)$ is an analytic function and estimated as follows

$$\overset{\approx}{v}'_n(\phi) = O(e^{-\epsilon_1(n_0+n)^{1/2}}). \quad (3.44)$$

By analyticity $\frac{1}{2}\frac{d^2}{d\phi^2}\overset{\approx}{v}'_n(\phi)$ and $\frac{1}{4!}\frac{d^4}{d\phi^4}\overset{\approx}{v}'_n(\phi)$ are estimated as follows,

$$\frac{1}{2}\frac{d}{d\phi}\overset{\approx}{v}'_n(\phi) = O(e^{-\epsilon_1(n_0+n)^{1/2}}) \quad (3.45)$$
$$\frac{1}{4!}\frac{d^4}{d\phi^4}\overset{\approx}{v}'_n(\phi) = O(e^{-\epsilon_1(n_0+n)^{1/2}}). \quad (3.46)$$

Note that $\frac{1}{2}\frac{d^2}{d\phi^2}\overset{\approx}{v}'_n(\phi)$ and $\frac{1}{4!}\frac{d^4}{d\phi^4}\overset{\approx}{v}'_n(\phi)$ are continuous function of μ_n by using the dominated convergence theorem.

From (3.44), we get

$$|(v'_n)_{\geq 6}(\phi)| \leq (n_0+n+1)^{-3/4}. \quad (3.47)$$

After all we get the following recursion relation

$$(v'_n)_4(\phi) = \lambda'_n\phi^4 = (\lambda_n - 36L^4\lambda_n^2 + O((n_0+n)^{-5/4}))\phi^4, \quad (3.48)$$

and we know

$$\frac{1}{\lambda'_n} = \frac{1}{\lambda_n} + 36L^4 + O((n_0+n)^{1/4}). \quad (3.49)$$

We solve this recursion relation, and get

$$\frac{C_-L^{-4}}{n_0+n+1} \leq \lambda'_n \leq \frac{C_+L^{-4}}{n_0+n+1}, \; C_- = \frac{1}{48}, \; C_+ = \frac{1}{24}, \quad (3.50)$$

We also solve the recursion relation of $(v'_n)_2(\phi)$,

$$(v'_n)_2(\phi) = (\frac{1}{2}\mu'_n - \frac{6\lambda'_n}{1-L^{-2}})\phi^2$$
$$= (\frac{1}{2}L^2\mu_n + \frac{6L^2\lambda_n}{1-L^{-2}} + 6L^2\lambda_n + O((n_0+n)^{-2}))\phi^2. \quad (3.51)$$

Hence, μ_n' is written by

$$\begin{aligned}\mu_n' &= L^2\mu_n + \frac{12}{1-L^{-2}}(\lambda_n' - \lambda_n) + O((n_0+n)^{-2}) \\ &= L^2\mu_n + O((n_0+n)^{-2}). \end{aligned} \tag{3.52}$$

The RG map $R : \mu_n \rightarrow \mu_n'$is continuous, so the image $R(I_n)$ contains I_{n+1}. Then we chose suitable closed interval J_{n+1} in $R^{-1}(I_{n+1}) \subset I_n$ so that $R(J_{n+1}) = I_{n+1}$. We complete proof of small field region.

3.2.2 large field region

We prove the condition **G-Kb** in $|\phi| > C_1(n_0+n+1)^{1/4}$ and $|\phi| < C_1(n_0+n+1)^{1/4}$ separately. From **G-Kb**, in the region $|\mathbf{Im}\phi| < C_1(n_0+n+1)^{1/4}$, we get

$$\begin{aligned}
&|e^{(-v_n')(\phi)}| \\
&= |(\phi=0)^{-1}\int \exp[-\frac{1}{2}L^4\sum_{\pm} v_n(L^{-1}\phi \pm z)]d\nu(z)| \\
&= |(\phi=0)^{-1}\exp[-\frac{1}{2}L^2\mu_n\phi^2 - \frac{6\lambda_n}{1-L^{-2}}L^2\phi^2] \\
&\quad \times \int \exp[-\frac{1}{2}L^4\mu_n z^2 - \frac{6\lambda_n}{1-L^{-2}}L^4 z^2 - \frac{1}{2}L^4\sum_{\pm}(v_n)_{\geq 4}(L^{-1}\phi \pm z)]d\nu(z)| \\
&\leq |(\phi=0)^{-1}\exp[O((n_0+n)^{-1}) + L^4 D - L^{1/2}\lambda_n^{1/2}|\phi|^2] \\
&\quad \times \exp[+20\lambda_n(\mathbf{Im}\phi)^4 - \frac{1}{2}L^2\mu_n\mathbf{Re}\phi^2 - \frac{6\lambda_n}{1-L^{-2}}L^2\mathbf{Re}\phi^2] \\
&\quad \times \int \exp[-\frac{1}{2}L^4\mu_n z^2 - \frac{6\lambda_n}{1-L^{-2}}L^4 z^2 - \frac{1}{2}L^4 - L^4\lambda_n^{1/2}z^2]d\nu(z)| \\
&\leq \exp[O((n_0+n)^{-1/2}) + L^4 D - \frac{1}{2}L^2(\lambda_n')^{1/2}|\phi|^2 + 20\lambda_n'(\mathbf{Im}\phi)^4] \\
&\leq \exp[D - (\lambda_n')^{1/2}|\phi|^2 + 20\lambda_n'(\mathbf{Im}\phi)^4] \\
&\quad \times \exp[O((n_0+n)^{-1/2}) + (L^4-1)D - (L^2 - \frac{1}{2}L^2(\lambda_n')^{1/2})|\phi|^2].
\end{aligned} \tag{3.53}$$

From $|\phi| > C_1(n_0+n+1)^{1/4}$ and if C_1 is sufficiently large, we estimate

$$\exp[O((n_0+n)^{-1/2}) + (L^4-1)D - (L^2 - \frac{1}{2}L^2(\lambda_n')^{1/2})|\phi|^2] \leq 1. \tag{3.54}$$

So, we get

$$|e^{(v_n')_{\geq 4}(\phi)}| \leq \exp[D - (\lambda_n')^{1/2}|\phi|^2 + 20\lambda_n'(\mathbf{Im}\phi)^4]. \tag{3.55}$$

Finally, we prove the condition **G-Ka** in the $|\phi| < C_1(n_0 + n + 1)^{1/4}$. From the estimation of the small field region, we get

$$|e^{(v'_n)(\phi)}| \leq \exp[O((n_0+n)^{-1})|\phi|^2\lambda'_n(\mathbf{Re}\phi^4) + (n_0+n+1)^{-3/4}]. \quad (3.56)$$

We estimate as follows,

$$\begin{aligned}
&\lambda'_n(\mathbf{Re}\phi^4)\\
&= \lambda'_n(\mathbf{Re}\phi)^4 - 6\lambda'_n(\mathbf{Re}\phi)^2(\mathbf{Im}\phi)^2 + \lambda'_n(\mathbf{Im}\phi)^4\\
&\geq \frac{1}{2}\lambda'_n(\mathbf{Re}\phi)^4 - 17\lambda'_n(\mathbf{Im}\phi)^4\\
&\geq \frac{1}{2}\lambda'_n[(\mathbf{Re}\phi)^4 + (\mathbf{Im}\phi)^4] - 20\lambda'_n(\mathbf{Im}\phi)^4\\
&\geq -\frac{1}{2}D + \frac{1}{2}\lambda'_n[(\mathbf{Re}\phi)^4 + (\mathbf{Im}\phi)^4] - 20\lambda'_n(\mathbf{Im}\phi)^4. \quad (3.57)
\end{aligned}$$

So, we can prove **G-Ka** in the large field region. Finally we chose constants L, $D(L)$, $C_1(L,D)$, and $n_0(L,D,C_1)$ sufficiently large, we prove theorem 3.1.1 completely.

Chapter 4

Main Results

4.1 Introduction and Main Results

Hierarchical spin model is an equilibrium statistical mechanical system introduced by Bleher and Sinai [2] [3] as a model suitable for tracing block spin renormalization group (RG) trajectories. For the model, the RG transformation is reduced to the following nonlinear transformation $\mathcal{R}$ of a function (single spin potential) $v = v(\phi)$:

$$\exp[-\mathcal{R}v(\phi)] = \frac{\int \exp[-\frac{1}{2}L^d[v(L^{-(d-2)/2}\phi + z) + v(L^{-(d-2)/2}\phi - z)]]d\nu(z)}{\int \exp[-L^4 v(z)]d\nu(z)} \tag{4.1}$$

where $d\nu(z) = \frac{1}{(2\pi)^{1/2}}\exp(-\frac{1}{2}z^2)dz$, and $L \geq 10$ is an even integer valued constant. It is easy to see that the trivial function $v(\phi) \equiv 0$ is a fixed point of $\mathcal{R}$, which we call the Gaussian fixed point. If, for a class of single spin potentials, RG trajectories with initial potentials in the class, converge to the Gaussian fixed point, then we say that the class of functions is trivial Gawędzki and Kupiainen studied this recursion in detail, and proved (among other things) the triviality for ϕ^4 models with some small ϕ^4 coupling constant in 4 dimensions [5] [6] [7]. See [7] for a review of their results together with the relation of (2.15) and the hierarchical spin model. The purpose of the present paper is to extend the results of Gawędzki and Kupiainen and prove triviality for a wider class of potentials. To be specific, we consider the following class of single spin potentials:

$$v(\phi) = \mu\phi^2 + (\lambda - \frac{15\rho}{1-L^{-2}})\phi^4 + \rho\phi^6, \qquad \phi \in \mathbf{R}. \tag{4.2}$$

Before stating our results, let us briefly review the relative known results. The triviality of the hierarchical model with (4.2) in 3 dimensions has already been established for small parameters by Müller and Schiemann [10]. A common belief based on power counting type of arguments suggest that the results of [10] would support triviality also for higher dimensions. However, one should note that the role of ϕ^4 term for $d = 3$ is different from that for $d = 4$. It is because ϕ^4 is relevant for $d < 4$ while it is not relevant for $d \geq 4$. The statement in [10] says that for sufficiently small ρ we can find μ and λ such that the RG trajectory converges to the Gaussian fixed point. On the other hand, a triviality statement for $d = 4$ is that for arbitrary (but small) ρ and λ we can find μ such that the same conclusion holds. In the latter case, we have to prove that an arbitrary (small) choices of λ and ρ, in particular, with negative ϕ^4 term, do not distort the standard expectation of the behavior of an RG trajectory. T. Hara, T. Hattori, and H. Watanabe proved the triviality of Dyson's hierarchical Ising model in 4 dimensions [9]. They used characteristic function of single spin distributions and Newman's inequalities on truncated correlations. Their method is useful to analyze in the strong coupling regime. We expect that their method is valid for analyzing general class of initial single spin potentials. However, (4.2) is still out of the range of their method, because these initial single spin potentials do not satisfy the Lee-Yang property when the coefficient of ϕ^4 term is negative [11], so truncated correlations of these potentials do not satisfy Newman's inequalities that are the key estimation which their method needs [9], [12]. Gawędzki and Kupiainen succeeded the construction of the non-trivial Euclidean ϕ^4_4 theory with the complex coupling constant with negative real part [8]. However, the parameter region of the coupling constants which we studied is not included in their studied region.

Let us turn to our proof of triviality for (2.15) with potentials of the form (4.2). We will show that the parameters will enter the region where the Theorem of Gawędzki and Kupiainen [7] can be applied (we call this region "G-K region"), after some iterations (finite time iterations) of the RG by the same techniques of Gawędzki and Kupiainen [7]. The point of our proof is to change the induction hypothesis after some iterations to reflect the dominant terms in the potential. Now, we state the results precisely. We will use the following notation:

$$v_n(\phi) = \mathcal{R}^n v_0(\phi), \tag{4.3}$$

$$(v_n)_k(\phi) = \frac{1}{k!}\frac{d^k(v_n)(0)}{d\phi^k}\phi^k, \tag{4.4}$$

$$(v_n)_{\geq k}(\phi) = v_n(\phi) - \sum_{l<k}(v_n)_l(\phi). \tag{4.5}$$

Let us be given an initial single spin potential

$$v_0(\phi) = (\mu_0 - \frac{1}{2}c_2(v_0))\phi^2 + (\lambda_0 - \frac{1}{2}c_4(v_0))\phi^4 + \rho_0\phi^6 + (v_0)_{\geq 8}(\phi), \quad (4.6)$$

where

$$c_2(v_0) = \frac{12\lambda_0}{1-L^{-2}} - \frac{180\rho_0}{(1-L^{-2})(1-L^{-4})} + \frac{90\rho_0}{1-L^{-4}}, \quad (4.7)$$

$$c_4(v_0) = \frac{30\rho_0}{1-L^{-2}}, \quad (4.8)$$

are the counter terms originating from Wick ordering. Let us define a class of initial single spin potentials $\mathcal{V}_0(L, D, C_1, n_0, \rho_0)$ satisfying the following conditions for constants L, D, C_1, n_0, and ρ_0.

Ta for $|\mathbf{Im}\phi| < C_1((L^{-4}\rho_0^{-1})^{1/6} \wedge n_0^{1/4})$, $\exp[-v_0(\phi)]$ is analytic, positive for real ϕ even, and

$$|e^{-(v_0)(\phi)}| \leq \exp[D - (\lambda_0^{1/2} + \rho_0^{1/3})|\phi|^2 + A_1\lambda_0(\mathbf{Im}\phi)^4 + A_2\rho_0(\mathbf{Im}\phi)^6], \quad (4.9)$$

where $A_1(\geq 20)$ and $A_2(\geq 2004)$ are universal constants.

Tb for $|\phi| < C_1((L^{-4}\rho_0^{-1})^{1/6} \wedge n_0^{1/4})$, $(v_0)_{\geq 4}(\phi)$ is analytic,

$$(v_0)_{\geq 4}(\phi) = (\lambda_0 - \frac{15\rho_0}{1-L^{-2}})\phi^4 + \rho_0\phi^6 + (v_0)_{\geq 8}(\phi), \quad (4.10)$$

with

$$\frac{C_{--}L^{-4}}{n_0} \leq \lambda_0 \leq \frac{C_{++}L^{-4}}{n_0}, \quad C_{--} = \frac{1}{42}, \quad C_{++} = \frac{1}{28}, \quad (4.11)$$

$$|(v_0)_{\geq 8}(\phi)| \leq \rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4}. \quad (4.12)$$

Notice that the coefficient of ϕ^4 is represented as $\lambda_0 - \frac{15\rho_0}{1-L^{-2}}$. If we take a value of ρ_0 suitably large, then the coefficient of ϕ^4 will be negative. So, the class $\mathcal{V}_0(L, D, C_1, n_0, \rho_0)$ includes small negative ϕ^4 case. This case is the main object of our study in this paper. Of course, this class includes potentials which Gawędzki and Kupiainen studied. In fact it is easy to see that $\mathcal{V}_0(L, D, C_1, n_0, 0)$ is a class which are investigated in [7]. We will prove the following for our class.

Theorem 4.1.1 *There exist positive constants:*

$$D, \; \bar{C_1}(L, D) \geq L, \; \bar{n_0}(L, D, C_1) \geq L^{48},$$

such that the following holds.

Let $C_1 \geq \bar{C}_1(L,D)$, $n_0 \geq \bar{n}_0(L,D,C_1)$, and

$$0 \leq \rho_0 \leq L^{-4} n_0^{-1}. \tag{4.13}$$

Define the RG as (2.15). Then there exists $\mu_{crit} \in \mathbf{R}$ such that the iterates v_n of the recursion converge to zero uniformly on compacts in $\mathbf{C}^1$, if we start from $v_0 \in \mathcal{V}_0(L,D,C_1,n_0,\rho_0)$ with $\mu_0 = \mu_{crit}$.

The proof goes along the following line. In the beginning, we are in the region where $(v_n)_{\geq 6}(\phi)$ is dominant. For properly chosen initial data, $(v_n)_{\geq 6}(\phi)$ decreases rapidly, and we then go into the region where ϕ^4 term of $v_n(\phi)$ is comparable to $(v_n)_{\geq 6}(\phi)$. As the recursion proceeds, the ϕ^4 term becomes positive and dominant, after which it eventually decreases, and $v_n(\phi)$ finally enters the G–K region. To trace the trajectory, we will divide up the induction into 2 parts along the trajectory and impose different induction hypothesis for the ρ dominant regime and the λ dominant regime. (Compare the induction hypotheses L1.2a and L1.2b with L1.3a and L1.3b, respectively.)

We will prove this by means of two lemmas. First, for $n \geq 0$, let $\mathcal{V}_n^1(L,D,C_1,n_0,\rho_0)$ be the class of potentials v_n satisfying:

L4.1.2a for $|\mathbf{Im}\phi| < C_1(L^{2n-4}\rho_0^{-1})^{1/6}$, $\exp[-v_n(\phi)]$ is analytic, positive for real ϕ, even, and

$$|e^{-v_n(\phi)}| \leq \exp[D - (\lambda_n^{1/2} + \rho_n^{1/3})|\phi|^2 + A_1\lambda_n(\mathbf{Im}\phi)^4 + A_2\rho_n(\mathbf{Im}\phi)^6], \tag{4.14}$$

L4.1.2b for $|\phi| < C_1(L^{2n-4}\rho_0^{-1})^{1/6}$, $(v_n)_{\geq 4}(\phi)$ is analytic, and

$$(v_n)_{\geq 4}(\phi) = (\lambda_n - \frac{1}{2}c_4(v_n))\phi^4 + \rho_n\phi^6 + (v_n)_{\geq 8}(\phi), \tag{4.15}$$

with

$$|\lambda_n - \lambda_0| \leq n n_0^{-7/4}, \tag{4.16}$$

$$|\rho_n - L^{-2n}\rho_0| \leq n L^{-2n} n_0^{-7/4}, \tag{4.17}$$

$$|(v_n)_{\geq 8}(\phi)| \leq (\rho_0^{2/3} n_0^{1/8} \vee n_0^{-3/4}) L^{-n}. \tag{4.18}$$

Lemma 4.1.2 *There exist constants D, $\bar{C}_1(L,D) \geq L$, $\bar{n}_0(L,D,C_1) \geq L^{48}$ such that the following holds. Let $C_1 \geq \bar{C}_1(L,D)$, $n_0 \geq \bar{n}_0(L,D,C_1)$ and $n \geq 0$ satisfy the inequality*

$$(n_0 + n)^{1/4} \geq (L^{2n-4}\rho_0^{-1})^{1/6}. \tag{4.19}$$

Suppose also that $v_0(\phi) \in \mathcal{V}_0(L,D,C_1,n_0,\rho_0)$ with

$$L^{-4} n_0^{-3/2} \leq \rho_0 \leq L^{-4} n_0^{-1}, \tag{4.20}$$

and $v_n(\phi) \in \mathcal{V}_n^1(L, D, C_1, n_0, \rho_0)$.

Then, there exists a closed interval $J_n \subset I_n = [-(n_0+n)^{-3/2}, (n_0+n)^{-3/2}]$ *such that for* μ_n *running through* J_n, $v_{n+1} \in \mathcal{V}_{n+1}^1(L, D, C_1, n_0, \rho_0)$. *Further, the map* $\mu_n \mapsto \mu_{n+1}$ *sweeps* I_{n+1} *continuously.*

Iterating Lemma 4.1.2, each time we can choose a subinterval $J_{n+1} \subset J_n \subset I_0$ of the initial mass squared values such that μ_n sweeps I_{n+1}. The effect of relatively large coefficient of ϕ^6 provides us with a positive coefficient of ϕ^4 in the next step, and we can get rid of negative ϕ^4 term. However, we can not iterate Lemma 4.1.2 for arbitrary times because of assumption (4.19). In other words, $(v_n)_{\geq 6}(\phi)$ will not be dominant any more compared with ϕ^4 term after some iterations. After applying Lemma 4.1.2 as much as possible, we are still not in the G-K region, and we have to trace the trajectory carefully for some more steps. So, we must prepare new assumptions. Put

$$n_1 = \max\{n \in \mathbf{N} | (L^{2n-4}\rho_0^{-1})^{1/6} \leq (n_0 + n)^{1/4}\} + 1. \tag{4.21}$$

Notice that this number is the first n for which Lemma 4.1.2 can not be applied. Obviously, we have $n_1 \leq \frac{1}{4}\log_L n_0$.

Let us define a class of single spin potentials $\mathcal{V}_{n_1+n}^2(L, D, C_1, n_0, \rho_0)$ satisfying:

L4.1.3a for $|\mathbf{Im}\phi| < C_1(n_0+n_1+n)^{1/4}$, $\exp[-v_{n_1+n}]$ is analytic and positive for real ϕ, even, and

$$\begin{aligned} |e^{-v_{n_1+n}(\phi)}| &\leq \exp[D - (\lambda_{n_1+n}^{1/2} + \rho_{n_1+n}^{1/3})|\phi|^2 + A_1\lambda_{n_1+n}(\mathbf{Im}\phi)^4] \\ &\quad \times \exp[A_2\rho_{n_1+n}(\mathbf{Im}\phi)^6], \end{aligned} \tag{4.22}$$

L4.1.3b for $|\phi| < C_1(n_0 + n_1 + n)^{1/4}$, $(v_{n_1+n})_{\geq 4}(\phi)$ is analytic,

$$\begin{aligned} &(v_{n_1+n})_{\geq 4}(\phi) \\ &\quad = (\lambda_{n_1+n} - \frac{1}{2}c_4(v_{n_1+n}))\phi^4 + \rho_{n_1+n}\phi^6 + (v_{n_1+n})_{\geq 8}(\phi), \end{aligned} \tag{4.23}$$

with

$$|\lambda_{n_1+n} - \lambda_0| \leq (n_1 + n)n_0^{-7/4}, \tag{4.24}$$

$$|\rho_{n_1+n} - L^{-2(n_1+n)}\rho_0| \leq (n_1 + n)L^{-2(n_1+n-1)}n_0^{-7/4}, \tag{4.25}$$

$$|(v_{n_1+n})_{\geq 8}(\phi)| \leq L^{-3n-n_1}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4}). \tag{4.26}$$

Note that $\mathcal{V}_{n_1}^1(L, D, C_1, n_0, \rho_0) \subset \mathcal{V}_{n_1}^2(L, D, C_1, n_0, \rho_0)$, if L, D, C_1, n_0, and ρ_0 are same constants.

Lemma 4.1.3 *There exist constants D, $\bar{C}_1(L,D) \geq L$, $\bar{n_0}(L,D,C_1) \geq L^{48}$ such that the following holds.*

Let $C_1 \geq \bar{C}_1(L,D)$, $n_0 \geq \bar{n_0}(L,D,C_1)$, $\log_L n_0 \geq n \geq 0$. $v_0(\phi) \in \mathcal{V}_0(L,D,C_1,n_0,\rho_0)$, and $v_{n_1+n}(\phi) \in \mathcal{V}^2_{n_1+n}(L,D,C_1,n_0,\rho_0)$.

Then, there exists a closed interval

$$J_{n_1+n} \subset I_{n_1+n} = [-(n_0+n_1+n)^{-3/2}, (n_0+n_1+n)^{-3/2}]$$

such that for μ_{n_1+n} running through J_{n_1+n}, $v_{n_1+n+1} \in \mathcal{V}^2_{n_1+n+1}$. Further, the map $\mu_{n_1+n} \mapsto \mu_{n_1+n+1}$ sweeps I_{n_1+n+1} continuously.

The proof of Lemma 4.1.3 is close to the proof of Lemma 4.1.2. A different point from Lemma 4.1.2 is the difference in the condition of the region where $v_{n_1+n}(\phi)$ satisfies analyticity. In fact we require that $\exp[-v_{n_1+n}(\phi)]$ is analytic for $|\mathbf{Im}\phi| < C_1(n_0+n_1+n)^{1/4}$ in Lemma 4.1.3. The reason why that there is such a difference because ϕ^4 term becomes dominant compared with $(v_{n_1+n})_{\geq 6}(\phi)$. If we notice that the conditions of Lemma 4.1.3 are different from the conditions of Lemma 4.1.2, we can prove Lemma 4.1.3 in a similar way as Lemma 4.1.2. So, we omit the proof of Lemma 4.1.3 from this paper.

With Lemma 4.1.3 we can continue iterations, and we can make sure that after a finite number of iterations, this potential is in the G-K region. More precisely, Gawędzki and Kupiainen introduced a class of potentials $\mathcal{V}^{G-K}_n(L,D,C_1,n_0)$, which is defined to satisfy:

G-Ka $e^{-(v_n)_{\geq 4}(\phi)}$ is analytic in $|\mathbf{Im}\phi| < C_1(n_0+n)^{1/4}$, positive for real ϕ, even and

$$|\exp[-(v_n)_{\geq 4}(\phi)]| \leq \exp[D - \lambda_n^{1/2}|\phi|^2 + A_1\lambda_n(\mathbf{Im}\phi)^4], \tag{4.27}$$

G-Kb for $|\phi| < C_1(n_0+n)^{1/4}$, $(v_n)_{\geq 4}(\phi)$ is analytic,

$$(v_n)_{\geq 4}(\phi) = \lambda_n\phi^4 + (v_n)_{\geq 6}(\phi) \tag{4.28}$$

with

$$\frac{C_-L^{-4}}{n_0+n} \leq \lambda_n \leq \frac{C_+L^{-4}}{n_0+n}, \; C_- = \frac{1}{48}, \; C_+ = \frac{1}{24}, \tag{4.29}$$

$$|(v_n)_{\geq 6}(\phi)| \leq (n_0+n)^{-3/4} \tag{4.30}$$

In this class $\mathcal{V}^{G-K}_n(L,D,C_1,n_0)$, Gawędzki and Kupiainen proved the following,

Theorem 4.1.4 (Gawędzki and Kupiainen) *There exist constants D, $\bar{C}_1(L,D), \bar{n}_0(L,D,C_1)$ such that the following holds. Let $C_1 \geq \bar{C}_1(L,D)$, $n_0 \geq \bar{n}_0(L,D,C_1)$ and $n \geq 0$.*

Put

$$v_n(\phi) = \mu_n - \frac{6\lambda_n}{1-L^{-2}}\phi^2 + (v_n)_{\geq 4}(\phi) \qquad (4.31)$$

where $(v_n)_{\geq 4}(\phi) \in \mathcal{V}_n^{G-K}(L,D,C_1,n_0)$. Then, there exists a closed interval $J_n \subset I_n$ such that for μ_n running through J_n, $(v_{n+1})_{\geq 4}(\phi) = v_{n+1}(\phi) - \mu_{n+1}\phi^2 + \frac{6\lambda_{n+1}}{1-L^{-2}}\phi^2 \in \mathcal{V}_{n+1}^{G-K}(L,D,C_1,n_0)$. Further, the map $\mu_n \mapsto \mu_{n+1}$ sweeps I_{n+1} continuously.

Finally, let us explain differences between the work of Gawędzki and Kupiainen [7] and this paper. First of all, we study contribution of the term of ϕ^6 to the term of ϕ^4 rigorously to control the term of ϕ^4 even with negative coefficient. Secondly, our class of single spin potentials permits us to take $v_{\geq 6}(\phi)$ in wider class than that Gawędzki and Kupiainen studied.

4.2 Proof of Lemma 4.1.2

We will prove that $v_n'(\phi) = v_{n+1}(\phi)$ is in $\mathcal{V}_{n+1}^1(L,D,C_1,n_0,\rho_0)$, if μ_n is in I_n. We prove this lemma according to Gawędzki and Kupiainen [7]. The proof involves the small field region analysis, and the large field region analysis corresponding to the cases either $|\phi| < C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$, or $|\mathbf{Im}\phi| < C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$ respectively.

In the small field, we prove that $v_n'(\phi)$ satisfies **L4.1.2b$'$**, the condition **L4.1.2b** with n being replaced by $n+1$, by using the Taylor expansion, and some estimation of the Gaussian integrals as in [7].

As for the large field region, we only investigate global behavior of $v_n'(\phi)$, i.e., we confirm that $v_n'(\phi)$ satisfies (4.18) of **L4.1.2a$'$**, the condition **L4.1.2a** with n being replaced by $n+1$.

We use K for calculable absolute constants, whose values will vary in each occurrence.

4.2.1 Small field region analysis

Let $v_n \in \mathcal{V}_n^1$. Write $\chi_1(z) = \chi(|z| < (L^{2n-4}\rho_0^{-1})^{1/6})$ and throughout this subsection, we assume that ϕ is in the region $|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$.

Note that we have to put C_1 to satisfy the inequality $|L^{-1}\phi \pm z| < C_1(L^{2n-4}\rho_0^{-1})^{1/6}$ for $|z| < (L^{2n-4}\rho_0^{-1})^{1/6}$ and $|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$.

Next, decompose $v_{n+1}(\phi)$ as follows,

$$
\begin{aligned}
v_{n+1}(\phi) &= v'_n(\phi) = \tilde{v}'_n(\phi) + \tilde{\tilde{v}}'_n(\phi), & (4.32)\\
e^{-\tilde{v}'_n(\phi)} &= \int \exp[-\frac{L^4}{2}\sum_{\pm} v_n(L^{-1}\phi \pm z)]d\nu_1(z)/(\phi = 0)_{small}, & (4.33)
\end{aligned}
$$

where

$$
\begin{aligned}
(\phi = 0)_{small} &= \int \exp[-L^4 v_n(z)]d\nu_1(z), & (4.34)\\
d\nu_1(z) &\equiv \chi_1(z)e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}. & (4.35)
\end{aligned}
$$

First of all, we estimate $\tilde{v}'_n(\phi)$ in 4.2.1, and then complete the analysis in the small field region by looking into the Taylor coefficients in 4.2.1.

Estimation of $\tilde{v}'_n(\phi)$

Let us take a logarithm of (4.33).

$$
\begin{aligned}
\tilde{v}'_n(\phi) &= L^2(\mu_n - \frac{1}{2}c_2(v_n))\phi^2 + (\lambda_n - \frac{1}{2}c_4(v_n))\phi^4 + L^{-2}\rho_n\phi^6\\
&\quad - \log\int e^{-w_\phi(z)}d\nu_1(z) + \log(\phi = 0)_{small}. & (4.36)\\
w_\phi(z) &= w_0(z) + w_2(z)\phi^2 + w_4(z)\phi^4 + w_6(z)\phi^6 + w_{\geq 8}(\phi, z), & (4.37)
\end{aligned}
$$

and

$$
\begin{aligned}
w_0(z) &= L^4 v_n(z)\\
&= L^4\{(\mu_n - \frac{1}{2}c_2(v_n))z^2 + (\lambda_n - \frac{1}{2}c_4(v_n))z^4 + \rho_n z^6 + (v_n)_{\geq 8}(z)\},\\
w_2(z) &= L^2(6(\lambda_n - \frac{1}{2}c_4(v_n))z^2 + 15\rho_n z^4 + \frac{d^2}{2dz^2}(v_n)_{\geq 8}(z)), & (4.38)\\
w_4(z) &= 15\rho_n z^2 + \frac{1}{4!}\frac{d^4}{dz^4}(v_n)_{\geq 8}(z), & (4.39)\\
w_6(z) &= L^{-2}\frac{d^6}{6!dz^6}(v_n)_{\geq 8}(z), & (4.40)\\
w_{\geq 8}(\phi, z) &= \frac{L^{-4}\phi^8}{7!}\{\int_0^1 dt(1-t)^7\frac{d^8}{dz^8}(v_n)_{\geq 8}(L^{-1}t\phi + z)\\
&\quad + \int_0^1 dt(1-t)^7\frac{d^8}{dz^8}(v_n)_{\geq 8}(L^{-1}t\phi - z)\}. & (4.41)
\end{aligned}
$$

From the conditions **L4.1.2a** - **L4.1.2b**, $v_n(\phi)$ is even and analytic.

We can estimate $\frac{d^8}{dz^8}(v_n)_{\geq 8}(\phi)$ on the support of $d\nu_1(z)$ as follows by using the Cauchy formula and (4.18),

$$\begin{aligned}
|(v_n)_{\geq 8}(z)| &\leq \frac{1}{7!}\int_0^1 dt(1-t)^7 |z^8 \frac{d^8}{dz^8}(v_n)_{\geq 8}(tz)| \\
&\leq \frac{C_1}{8!(C_1-1)^9}(\rho_0^{2/3} n_0^{1/8} \vee n_0^{-3/4})(L^4\rho_0)^{4/3} L^{-11n/3} z^8.
\end{aligned} \tag{4.42}$$

$\frac{d^2}{dz^2}(v_n)_{\geq 8}(z)$ to $\frac{d^6}{dz^6}(v_n)_{\geq 8}(z)$ can be estimated as (4.42).

From the perturbation expansion:

$$\begin{aligned}
&-\log\int e^{-w_\phi(z)} d\nu_1(z) \\
&\quad = -\log\int d\nu_1(z) + \langle w_\phi(z)\rangle_0 - \int_0^1 dt(1-t)\langle w_\phi(z); w_\phi(z)\rangle_t,
\end{aligned} \tag{4.43}$$

where

$$\langle \cdots \rangle_t \equiv \int \cdots e^{-tw_\phi(z)} d\nu_1(z) \Big/ \int e^{-tw_\phi(z)} d\nu_1(z). \tag{4.44}$$

Now, we shall estimate each part of (4.43). Using the estimation of the Gaussian integrations, we get

$$\begin{aligned}
\langle w_\phi(z)\rangle_0 &= L^4\langle v_n(z)\rangle_0 + 6L^2(\lambda_n - \frac{1}{2}c_4(v_n))\phi^2 + 45L^2\rho_n\phi^2 \\
&\quad + \overset{\sim 0,0}{R_2}(L, n_0, \rho_0, n)\phi^2 + 15\rho_n\phi^4 + \overset{\sim 0,0}{R_4}(L, n_0, \rho_0, n)\phi^4 \\
&\quad + \overset{\sim 0,0}{R_6}(L, n_0, \rho_0, n)\phi^6 + \langle w_{\geq 8}(\phi, z)\rangle_0,
\end{aligned} \tag{4.45}$$

where, the terms $\overset{\sim 0,0}{R_{2i}}(L, n_0, \rho_0, n), i = 1, \cdots, 3$ satisfy

$$|\overset{\sim 0,0}{R_{2i}}(L, n_0, \rho_0, n)| \leq (\rho_0^{2/3} n_0^{1/8} \vee n_0^{-3/4})(L^4\rho_0)^{4/3} L^{-11n/3}. \tag{4.46}$$

Therefore from (4.45), we can estimate $\langle w_{\geq 8}(\phi, z)\rangle_0$ as follows,

$$|\langle w_{\geq 8}(\phi, z)\rangle_0| \leq L^{4-n}(1 + (L^4\rho_0)^{1/3} L^{-2n/3})(\rho_0^{2/3} n_0^{1/8} \vee n_0^{-3/4}). \tag{4.47}$$

Next we estimate

$$\begin{aligned}
&\int_0^1 dt(1-t)\langle w_\phi(z); w_\phi(z)\rangle_t = \int_0^1 dt(1-t)\sum_{i,j}\langle \tilde{w}_{2i}; \tilde{w}_{2j}\rangle_t \\
&\quad = \int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t + \int_0^1 dt(1-t)\sum_{i,j\neq 0}\langle \tilde{w}_{2i}; \tilde{w}_{2j}\rangle_t,
\end{aligned} \tag{4.48}$$

where

$$\tilde{w}_{2i} = \begin{cases} w_{2i}(z)\phi^{2i} & i = 0,1,2,3 \\ w_{\geq 8}(\phi, z) & i = 4. \end{cases}$$

The cumulants are

$$\begin{aligned} \langle \tilde{w}_{2i}; \tilde{w}_{2j} \rangle_t &= \langle e^{-tw_\phi(z)} \rangle_0^{-1} \langle \tilde{w}_{2i}\tilde{w}_{2j} e^{-tw_\phi(z)} \rangle_0 \\ &\quad - \langle e^{-tw_\phi(z)} \rangle_0^{-2} \langle \tilde{w}_{2i} e^{-tw_\phi(z)} \rangle_0 \langle \tilde{w}_{2j} e^{-tw_\phi(z)} \rangle_0. \end{aligned} \tag{4.49}$$

Note that the support of $d\nu_1(z)$ is $|z| < (L^{2n-4}\rho_0^{-1})^{1/6}$ and the coefficients are as small as the inverse of the diameter of the small field region.

We get the uniform estimate $|w_\phi(z)| \leq K \cdot C_1^4$ for $|z| < (L^{2n-4}\rho_0^{-1})^{1/6}$ and $|\phi| < \frac{10LC_1}{11}(L^{2n-4}\rho_0^{-1})^{1/6}$. We have

$$\begin{aligned} &|\sum_{(i,j)\neq(0,0)} \langle \tilde{w}_{2i}; \tilde{w}_{2j} \rangle_t | \\ &\quad \leq e^{K\cdot C_1^4} \sum_{(i,j)\neq(0,0)} (\langle |\tilde{w}_{2i}||\tilde{w}_{2j}| \rangle_0 + \langle |\tilde{w}_{2i}| \rangle_0 \langle |\tilde{w}_{2j}| \rangle_0). \end{aligned} \tag{4.50}$$

So, we can estimate $| \int_0^1 dt(1-t) \sum_{(i,j)\neq(0,0)} \langle \tilde{w}_{2i}; \tilde{w}_{2j} \rangle_t |$ similarly as in (4.42), we obtain

$$\begin{aligned} &|\text{2nd term of RHS of (4.48)}| \\ &\quad \leq K e^{K\cdot C_1^4} L^{-2} n_0^{-2} (|\phi|^2 + L^{-2}|\phi|^4 + L^{-2n-4}|\phi|^6) \\ &\qquad + |\text{higher order terms}|. \end{aligned} \tag{4.51}$$

The higher order terms are estimated as follows,

$$|\text{higher order terms}| \leq K e^{K\cdot C_1^4} L^{12-n} C_1^{12} n_0^{-1/8} (\rho_0^{2/3} n_0^{1/8} \vee n_0^{-3/4}). \tag{4.52}$$

Next, we estimate $\int_0^1 dt(1-t) \langle w_0(z); w_0(z) \rangle_t$. From the Taylor expansion of ϕ, and we can use the Cauchy formula because $\langle w_0(z); w_0(z) \rangle_t$ is analytic function in $|\phi| < \frac{10}{11} LC_1 (L^{2n-4}\rho_0^{-1})^{1/6}$, we get

$$\begin{aligned} &| \int_0^1 dt(1-t) \langle w_0(z); w_0(z) \rangle_t - \int_0^1 dt(1-t) \langle w_0(z); w_0(z) \rangle_t |_{\phi=0}| \\ &\qquad \leq K \exp(K\cdot C_1^4) \cdot L^{-2} n_0^{-2} |\phi|^2. \end{aligned} \tag{4.53}$$

So we have,

$$\begin{aligned} &| \int_0^1 dt(1-t) \langle w_\phi(z); w_\phi(z) \rangle_t - \int_0^1 dt(1-t) \langle w_0(z); w_0(z) \rangle_t |_{\phi=0}| \\ &\quad \leq K \exp(K\cdot C_1^4) L^{-2} n_0^{-2} (|\phi|^2 + L^{-2}|\phi|^4 + L^{-2n-4}|\phi|^6) \\ &\qquad + |\text{higher order terms}|, \end{aligned} \tag{4.54}$$

$$|\text{higher order terms}| \leq K e^{K\cdot C_1^4} L^{12-n} C_1^{12} n_0^{-1/8} (\rho_0^{2/3} n_0^{1/8} \vee n_0^{-3/4}). \tag{4.55}$$

These coefficients are large, but not terrible, because we can take n_0 sufficiently large. In the following, we put $n_0^{1/8} \geq K \cdot C_1^{12} L^{12} e^{K \cdot C_1^4}$.

From (4.36) and (4.43), we infer that

$$\begin{aligned}\tilde{v}_n'(\phi) &= L^2(\mu_n - \frac{1}{2}c_2(v_n))\phi^2 + 6L^2(\lambda_n - \frac{1}{2}c_4(v_n))\phi^2 + 45L^2\rho_n\phi^2 \\ &\quad + \tilde{R}_2(L, n_0, \rho_0, n)\phi^2 + (\lambda_n - \frac{1}{2}c_4(v_n))\phi^4 + 15\rho_n\phi^4 \\ &\quad + \tilde{R}_4(L, n_0, \rho_0, n)\phi^4 + L^{-2}\rho_n\phi^6 + \tilde{R}_6(L, n_0, \rho_0, n)\phi^6 + (\tilde{v}_n)'_{\geq 8}(\phi),\end{aligned} \tag{4.56}$$

where, the terms $\tilde{R}_{2i}(L, n_0, \rho_0, n), i = 1, 2, 3$ satisfy

$$| \tilde{R}_{2i}(L, n_0, \rho_0, n)| \leq L^{-10-2i} n_0^{-2+1/8} + | \tilde{R}_{2i}^{0,0}(L, n_0, \rho_0, n)|, i = 1, 2, \tag{4.57}$$

$$| \tilde{R}_6(L, n_0, \rho_0, n)| \leq L^{-2n-18} n_0^{-2+1/8} + | \tilde{R}_6^{0,0}(L, n_0, \rho_0, n)|, \tag{4.58}$$

and from (4.47) and (4.55), $(\tilde{v}_n)'_{\geq 8}(\phi)$ satisfy

$$| (\tilde{v}_n)'_{\geq 8}(\phi)| \leq L^{4-n}(1 + L^{-2n/3}(L^4\rho_0)^{1/3} + L^{-4})(\rho_0^{2/3} n_0^{1/8} \vee n_0^{-3/4}), \tag{4.59}$$

for $|\phi| < \frac{10}{11} L C_1 (\rho_0^{-1} L^{2n})^{1/6}$. Notice that

$$(\phi = 0)_{small} = \log \int d\nu_1(z) - \langle w_0(z) \rangle_0 + \int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t|_{\phi=0}.$$

So we can check that the constant term $(\phi = 0)_{small}$ vanishes. The estimate (4.59) is a little weaker than what we want (see (4.18)). So, we need a stronger estimate. Since $\tilde{v}_n'(\phi)$ is analytic in $|\phi| < \frac{10}{11} L C_1 (L^{2n-4}\rho_0^{-1})^{1/6}$, $\phi^{-8}(\tilde{v}_n)'_{\geq 8}(\phi)$ is also analytic in $|\phi| < \frac{10}{11} L C_1 (L^{2n-4}\rho_0^{-1})^{1/6}$. We obtain from the maximum principle

$$\begin{aligned}| (\tilde{v}_n)'_{\geq 8}(\phi)| &\leq (\frac{|\phi|}{(10L/11)C_1(L^{2n-4}\rho_0^{-1})^{1/6}})^8 (\rho_0^{2/3} n_0^{1/8} \vee n_0^{-3/4}) \\ &\quad \times (L^{4-n}(1 + L^{-2n/3}(L^4\rho_0)^{1/3} + L^{-4})),\end{aligned} \tag{4.60}$$

so that for $|\phi| < C_1(L^{2(n+1)-4}\rho_0)^{1/6}$,

$$\begin{aligned}| (\tilde{v}_n)'_{\geq 8}(\phi)| &\leq (\frac{11}{10})^8 L^{-16/3}(L^{4-n}(1 + L^{-2n/3}(L^4\rho_0)^{1/3} + L^{-4}) \\ &\quad \times (\rho_0^{2/3} n_0^{1/8} \vee n_0^{-3/4})).\end{aligned} \tag{4.61}$$

Estimation of $\overset{\approx}{v'_n}(\phi)$ for $|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$

Represent (4.32) as

$$\begin{aligned}\overset{\approx}{v'_n}(\phi) &= \log\Big(1 + \frac{\int \exp[-\frac{1}{2}L^4\sum_{\pm} v_n(L^{-1}\phi \pm z)](1-\chi_1(z))d\nu(z)}{e^{-\tilde{v}'_n(\phi)}(\phi = 0)_{small}}\Big) \\ &\quad + \log(\phi = 0)_{small} - \log(\phi = 0). \end{aligned} \tag{4.62}$$

We want to prove that $\overset{\approx}{v'_n}(\phi)$ is analytic in$|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$ and sufficiently smaller than $\tilde{v}'_n(\phi)$. To prove these properties, we have only to prove that

$$\frac{\int \exp[-\frac{1}{2}L^4\sum_{\pm} v_n(L^{-1}\phi \pm z)](1-\chi_1(z))d\nu(z)}{e^{-\tilde{v}'_n(\phi)}(\phi = 0)_{small}} \tag{4.63}$$

is analytic and sufficiently small in $|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$.

First of all, we estimate the denominator of (4.63). We can show that the denominator is bounded from below by a constant which depends on C_1, but not on n_0. From **L4.1.2b**, and (4.57)-(4.58) together with uniform estimate of $w_0(z)$ under the condition that $(n_0+n)^{1/4} \geq (L^{2n-4}\rho_0^{-1})^{1/6}$, we estimate denominator as follows,

$$|\text{denominator of } (4.63)| \;\geq\; \exp[-K \cdot C_1^6]. \tag{4.64}$$

Next, we estimate the numerator part of (4.63),

$$\begin{aligned}&|\text{numerator of } (4.63)| \\ &\quad \leq \int (1-\chi_1(z)) \prod_{\pm} |\exp[-v_n(L^{-1}\phi \pm z)]|^{L^4/2} d\nu(z).\end{aligned} \tag{4.65}$$

Using (4.14) of **L4.1.2a** for $|L^{-1}\phi \pm z| < C_1(L^{2n-4}\rho_0)^{1/6}$, we have

$$\begin{aligned}&|\text{numerator of } (4.63)| \\ &\quad \leq \exp[K + L^4D + A_1C_1^4 + 2A_2C_1^6 - \frac{1}{4}(L^{2n-4}\rho_0^{-1})^{1/3}].\end{aligned} \tag{4.66}$$

So,

$$|(4.63)| < \exp[K \cdot C_1^6 + L^4D + A_1C_1^4 + 2A_2C_1^6 - \frac{1}{4}(L^{2n-4}\rho_0^{-1})^{1/3}]. \tag{4.67}$$

For given L, D and C_1, we can take n_0 large enough to obtain

$$\text{RHS of } (4.67) \leq \exp[-\frac{1}{8}(L^{2n-4}\rho_0^{-1})^{1/3}]. \tag{4.68}$$

This estimate is also valid for $\log(\phi = 0) - \log(\phi = 0)_{small}$. According to (4.68), we can show that $\overset{\approx}{v'_n}(\phi)$ is analytic and

$$|\overset{\approx}{v'_n}(\phi)| \leq 2e^{-1/8(L^{2n-4}\rho_0^{-1})^{1/3}}. \tag{4.69}$$

Estimation of coefficients

Now, we assume that $|\phi| < C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$ i.e. ϕ is in the small field region of $v'_n(\phi)$. Notice that the small field region is in the region $|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$, so we can use the result in 4.2.1. Thus, $\overset{\approx}{v'_n}(\phi)$ is analytic in the small field region, and we can obtain power series expansion of $\overset{\approx}{v'_n}(\phi)$. With the use of Cauchy's estimate, we see that coefficients of ϕ^2, ϕ^4 and ϕ^6 satisfy,

$$|\frac{1}{k!}\frac{d^k}{d\phi^k}\overset{\approx}{v'_n}(0)| \leq e^{-1/8(L^{2n-4}\rho_0^{-1})^{1/3}}, k = 2, 4, \text{and } 6. \tag{4.70}$$

Using the bounded convergence theorem, we see that $\frac{1}{2}\frac{d^2}{d\phi^2}\overset{\approx}{v'_n}(0)$, $\frac{1}{4!}\frac{d^4}{d\phi^4}\overset{\approx}{v'_n}(0)$, and $\frac{1}{6!}\frac{d^6}{d\phi^6}\overset{\approx}{v'_n}(0)$ are continuous functions of μ_n on I_n. From (4.61) and (4.69), if n_0 is sufficiently large, then we have

$$|(v_n)'_{\geq 8}(\phi)| \leq L^{-(n+1)}(\rho_0^{2/3}n_0^{1/8} \vee n_0^{-3/4}), \tag{4.71}$$

for $|\phi| < C_1(L^{2(n+1)-4}\rho_0)^{1/6}$.

From (4.56), (4.58), and (4.70), we know that

$$|\rho'_n - L^{-2}\rho_n| = |R_6(L, n_0, \rho_0, n) + \frac{1}{6!}\frac{d^6}{d\phi^6}\overset{\approx}{v'_n}(0)| \leq L^{-2n}n_0^{-15/8}. \tag{4.72}$$

Thus, if n_0 is sufficiently large, we have

$$|\rho'_n - L^{-2(n+1)}\rho_0| < (n+1)L^{-2n}n_0^{-7/4} \tag{4.73}$$

which proves (4.18) of **L4.1.2b′**.

From (4.56), (4.57), we know

$$\begin{aligned} |\lambda'_n - \lambda_n| &= |R_4(L, n_0, \rho_0, n) + \frac{d^4}{4!d\phi^4}\overset{\approx}{v'_n}(0) + \frac{15(\rho'_n - L^{-2}\rho_n)}{1 - L^{-2}}| \\ &\leq n_0^{-15/8}. \end{aligned} \tag{4.74}$$

Thus, we have $$|\lambda'_n - \lambda_0| < (n+1)n_0^{-7/4}, \tag{4.75}$$

which completes the proof of **L4.1.2b′**.

Similarly, we get estimation of coefficient μ_n' as follows,

$$|\mu_n' - L^2\mu_n| \leq |(\frac{6(\lambda_n - \lambda_n')}{1-L^{-2}} - \frac{90(L^{-2}\rho_n - \rho_n')}{(1-L^{-2})(1-L^{-4})} + \frac{45(L^{-2}\rho_n - \rho_n')}{1-L^{-4}})|$$
$$+|R_2(L, n_0, \rho_0, n) + \frac{1}{2}\frac{d^2}{d\phi^2}\overset{\approx}{v_n'}(0)| \leq K \times n_0^{-15/8}. \quad (4.76)$$

We know that map $R : \mu \mapsto \mu'$ is continuous, and image $R(I_n)$ include I_{n+1}. So that we can take for J_{n+1} a connected component of this inverse image $R^{-1}(I_{n+1}) \subset I_n$.

This ends the analysis of the small field properties.

4.2.2 Large field region analysis

Next, we prove that $e^{-(v_n)'(\phi)}$ satisfy the condition **L4.1.2a′**.First, we prove it in the case where $|\mathbf{Re}\phi| > C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$. Next, we prove it in $|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$ i.e. this region includes the small field region of $v'(\phi)$.

The case where $|\mathbf{Re}\phi| > C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$

Note that the definition of the RG (2.15) has the following expression

$$e^{-v_n'(\phi)} = \int \prod_{\pm} \exp[-v_n(L^{-1}\phi \pm z)]^{L^4/2} d\nu(z)/(\phi = 0). \quad (4.77)$$

$|\mathbf{Im}(L^{-1}\phi \pm z)| < C_1(L^{2n-4}\rho_0^{-1})^{1/6}$, if $|\mathbf{Im}\phi| < C_1(L^{2n-2}\rho_0^{-1})^{1/6}$. From the condition **L4.1.2a**,

$$|e^{-(v_n)'(\phi)}| \leq \exp[L^4 D - L^2(\lambda_n^{1/2} + \rho_n^{1/3})|\phi|^2 + A_1\lambda_n(\mathbf{Im}\phi)^4]$$
$$\times \exp[A_2 L^{-2}\rho_n(\mathbf{Im}\phi)^6] \int_{-\infty}^{\infty} e^{-L^4(\lambda_n^{1/2} - \rho_n^{1/3})z^2} d\nu(z)/(\phi = 0). \quad (4.78)$$

Note that, λ_n and ρ_n are positive and sufficiently small, hence, this integral part and $(\phi = 0)$ estimated as absolute constants, so we get

$$\text{RHS of (4.78)} \leq \exp[L^4 D - L^2(\lambda_n^{1/2} + \rho_n^{1/3})|\phi|^2 + A_1\lambda_n(\mathbf{Im}\phi)^4]$$
$$\times \exp[A_2 L^{-2}\rho_n(\mathbf{Im}\phi)^6 + K]. \quad (4.79)$$

If D and L are given, we take C_1 sufficiently large and then we take n_0 sufficiently large. Thus, we obtain

$$|\exp(-v'(\phi))|$$
$$< \exp[D - ({\lambda_n'}^{1/2} + {\rho_n'}^{1/3})|\phi|^2 + A_1\lambda_n'(\mathbf{Im}\phi)^4 + A_2\rho_n'(\mathbf{Im}\phi)^6], \quad (4.80)$$

for $|\mathbf{Im}\phi| < C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$, $|\mathbf{Re}\phi| > C_1(L^{2(n+1)-4}\rho_0^{-1})^{1/6}$.

The case where $|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0)^{1/6}$

Let $(n_0+n)^{1/4} \geq (L^{2n-4}\rho_0^{-1})^{1/6}$, $\mu_n \in I_n$, and $|\phi| < \frac{10}{11}LC_1(L^{2n-4}\rho_0^{-1})^{1/6}$. From (4.59), (4.73), (4.75), and (4.76), we have

$$\begin{aligned} |e^{-(v_n)'(\phi)}| &\leq \exp[K \cdot L^{-2}C_1^2 n_0^{-1/2}] \\ &\quad \times \exp[+K \cdot C_1^4 L^{4/3}(L^{-2(n+1)}\rho_0)^{1/3}] \\ &\quad \times \exp[-\lambda'_n(\mathbf{Re}\phi^4) - \rho'_n(\mathbf{Re}\phi^6) + L^4 n_0^{-1/2}]. \end{aligned} \tag{4.81}$$

And, we estimate $\rho'_n(\mathbf{Re}\phi^6)$ as follows,

$$\begin{aligned} \rho'_n(\mathbf{Re}\phi^6) &\geq \rho'_n(\frac{1}{4}(\mathbf{Re}\phi)^6 - 2001(\mathbf{Im}\phi)^6) \\ &\geq -\frac{1}{2}D_6 + 2(\rho'_n)^{1/3}|\phi|^2 - A_2\rho'_n(\mathbf{Im}\phi)^6. \end{aligned} \tag{4.82}$$

Notice that D_6 does not depend on C_1, n_0 or n. Similarly, we can estimate, $\lambda'_n\mathbf{Re}\phi^4 \geq -\frac{1}{2}D_4 + 2(\lambda'_n)^{1/2}|\phi|^2 - A_1\lambda'_n(\mathbf{Im}\phi)^4$. Notice that D_4 does not depend on C_1, n_0 or n, either. Put $D = D_4 + D_6$. From (4.81) to (4.82),

$$\begin{aligned} |e^{-(v_n)'(\phi)}| &\leq \exp[D - ((\lambda'_n)^{1/2} + (\rho'_n)^{1/3})|\phi|^2 + A_1\lambda'_n(\mathbf{Im}\phi)^4] \\ &\quad \times \exp[+A_2\rho'_n(\mathbf{Im}\phi)^6 - \frac{1}{2}D + K \cdot L^{-2}C_1^2 n_0^{-1/2}] \\ &\quad \times \exp[K \cdot L^{4/3}C_1^4(L^{-2(n+1)}\rho_0)^{1/3} + L^4 n_0^{-1/2}], \end{aligned} \tag{4.83}$$

which is smaller than

$$\exp[D - ((\lambda'_n)^{1/2} + (\rho'_n)^{1/3})|\phi|^2 + A_1\lambda'_n(\mathbf{Im}\phi)^4 + A_2\rho'_n(\mathbf{Im}\phi)^6], \tag{4.84}$$

if n_0 is sufficiently large. Proof of Lemma 4.1.2 is completed.

4.3 Proof of Theorem 4.1.1

Finally, we prove Theorem 4.1.1, using Lemma 4.1.2, Lemma 4.1.3 and Theorem 4.1.4. First of all, we notice that it is possible to take constants L, D, $C_1(L,D)$, $n_0(L,D,C_1)$ to satisfy Lemma 4.1.2, Lemma 4.1.3, and Theorem 4.1.4. We can check that potential $v(\phi)$ can be iterated n_1 times if initial parameters satisfy the conditions **Ta** and **Tb** because of Lemma 4.1.2. Notice that $v_{n_1}(\phi)$, the potential after n_1 iterations, satisfies the conditions **L4.1.3a** and **L4.1.3b** with $n = 0$, and so Lemma 4.1.3 can be applied to this potential. We have to iterate $\mathcal{R}$ using Lemma 4.1.3, sufficiently many times so that the iterated potentials satisfy the G-K conditions. Put

$$n_2 = \min\{n \in \mathbf{N} : |\rho_{n_1+n}||\phi|^6 + |(v_{n_1+n})_{\geq 8}(\phi)| < (n_0+n_1+n)^{-3/4} \text{ for } |\phi| < C_1(n_0+n_1+n)^{1/4}\} + 1. \quad (4.85)$$

Then, $$\rho_{n_1+n_2-1} < (n_0+n_1+n_2-1)^{-9/4}. \quad (4.86)$$

By calculation, n_2 can be estimated as $n_2 < \log_L n_0$. Since, $\rho_{n_1+n_2} \geq 0$, and by (4.24)

$$\begin{aligned}\lambda_{n_1+n_2} - \frac{15\rho_{n_1+n_2}}{1-L^{-2}} &< \lambda_0 + (n_1+n_2)n_0^{-7/4} \\ &< \frac{C_{++}}{L^4}n_0^{-1} + 2(\log_L n_0)n_0^{-7/4} < \frac{C_+}{L^4}(n_0+n_1+n_2)^{-1}. \quad (4.87)\end{aligned}$$

Similarly, by (4.86) we have

$$\lambda_{n_1+n_2} - \frac{15\rho_{n_1+n_2}}{1-L^{-2}} > \frac{C_-}{L^4}(n_0+n_1+n_2)^{-1}. \quad (4.88)$$

So, we checked the condition **G-Kb** completely.

Next, let us check the condition **G-Ka**. Notice that analyticity, positivity for real ϕ, and even function of $v_{n_1+n_2}(\phi)$ are checked easily. Now, We check the bound of $v_{n_1+n_2}(\phi)$

$$\begin{aligned}|\exp[-v_{n_1+n_2}(\phi)]| &\leq \exp[D - (\lambda_{n_1+n_2}^{1/2} + \rho_{n_1+n_2}^{1/3})|\phi|^2] \\ &\times \exp[+A_1\lambda_{n_1+n_2}(\mathbf{Im}\phi)^4 + A_2\rho_{n_1+n_2}(\mathbf{Im}\phi)^6]. \quad (4.89)\end{aligned}$$

From the definitions of n_1 and n_2, $-\rho_{n_1+n_2}^{1/3}|\phi|^2 + A_2\rho_{n_1+n_2}(\mathbf{Im}\phi)^6$ is nonpositive for $(\mathbf{Im}\phi) < C_1(n_0+n_1+n_2)^{1/4}$, so we have

$$|\exp[-v_{n_1+n_2}(\phi)]| \leq \exp[D - \lambda_{n_1+n_2}^{1/2}|\phi|^2 + A_1\lambda_{n_1+n_2}(\mathbf{Im}\phi)^4]. \quad (4.90)$$

We have checked all of the G-K conditions. Since $\rho_{n_1+n_2-1}$ is sufficiently small by (4.86), we know

$$\begin{aligned}&|\mu_{n_1+n_2} - L^2(\mu_{n_1+n_2-1} + \frac{90\rho_{n_1+n_2-1}}{(1-L^{-2})(1-L^{-4})} - \frac{45\rho_{n_1+n_2-1}}{1-L^{-4}})| \\ &\leq 16n_0^{-15/8} + |\frac{15L^2\rho_{n_1+n_2-1}}{1-L^{-2}}| \leq K \cdot n_0^{-15/8}. \quad (4.91)\end{aligned}$$

As in the proof lemma 4.1.2 and Lemma 4.1.3, we can take for $J_{n_1+n_2}$ a suitable connected component. So, we can adapt Theorem 4.1.4. Theorem 4.1.1 is now proved.

4.4 Remark and another result –triviality of hierarchical $P(\phi)$ model –

Theorem 4.1.1 can be easily extended $P(\phi)$ models. However, we require that a polynomial $P(\phi)$ is even and satisfies an additional condition (4.98) below. We consider the following class of single spin potentials:

$$v_0(\phi) = \mu\phi^2 + \lambda P(\phi), \tag{4.92}$$

$$P(\phi) = \sum_{k=2}^{N} a_{2k} : \phi^{2k} :, \tag{4.93}$$

where $: \phi^{2k} :$ is given by

$$\int_{-\infty}^{\infty} L^d \sum_{\pm} : (L^{-(d-2)/2}\phi \pm z)^{2k} : d\nu(z) = L^{2k-(k-1)d} : \phi^{2k} : . \tag{4.94}$$

(For example : $\phi^6 := \phi^6 - \frac{15}{1-L^{-2}}\phi^4 - \frac{45}{1-L^{-4}}\phi^2 + \frac{90}{(1-L^{-2})(1-L^{-4})}\phi^2 +$ "const".) Let us define a class of initial single spin potentials $\mathcal{V}_0(N, L, D, C_1, n_0)$ satisfying the following conditions for constants L, D, C_1, and n_0,

(Pa) for $|\mathbf{Im}\phi| < C_1 n_0^{1/2N}$, $\exp[-v_0(\phi)]$ is analytic, positive for real ϕ, even, and satisfies

$$|e^{-(v_0)_{\geq 4}(\phi)}| \leq \exp[D - \sum_{k=2}^{N} a_{2k,0}^{1/k}|\phi|^2 + \sum_{k=2}^{N} A_{2k}a_{2k,0}(\mathbf{Im}\phi)^{2k}], \tag{4.95}$$

where $\{A_{2k}\}$ are universal constants, and $a_{2k,0} = \lambda \cdot a_{2k}$

(Pb) for $|\phi| < C_1 n_0^{1/2N}$, $(v_0)_{\geq 4}(\phi)$ is analytic,

$$(v_0)_{\geq 4}(\phi) = \lambda_0 \sum_{k=2}^{N} : \phi^{2k} : + (v_0)_{\geq 8}(\phi) \tag{4.96}$$

with

$$\frac{C_{--}L^{-4}}{n_0} \leq a_{4,0} \leq \frac{C_{++}L^{-4}}{n_0}, \quad C_{--}(N) > \frac{1}{48}, C_{++}(N) < \frac{1}{24}, \tag{4.97}$$

$$C_0 L^{-4} n_0^{-1} < a_{2k,0} < C_0' L^{-4} n_0^{-1}, C_0 > 0 \tag{4.98}$$

$$|(v_0)_{\geq 2N+2}(\phi)| \leq n_0^{-3/2N}. \tag{4.99}$$

We will prove the following for our class.

Theorem 4.4.1 *In $d \geq 4$, there exist positive constants:*

$$D(N),\ \bar{C_1}(N,L,D) \geq L,\ \bar{n_0}(N,L,D,C_1) \geq L^{48},$$

such that the following holds. Let $C_1 \geq \bar{C_1}(N,L,D)$, $n_0 \geq \bar{n_0}(N,L,D,C_1)$. Define the RG as (2.15). Then there exists $\mu_{crit} \in \mathbf{R}$ such that the iterates v_n of the recursion converge to zero uniformly on compacts in $\mathbf{C}^1$, if we start from $v_0 \in \mathcal{V}_0(N,L,D,C_1,n_0)$ with $\mu_0 = \mu_{crit}$.

To prove of the triviality for (2.15) with potentials of the form (Pa)-(Pb), we will show that the parameters will enter the region where the Theorem of Gawędzki and Kupiainen [7] can be applied (i.e. G-K region), after some iterations (finite time iterations) of the RG. The point of our proof is to change the induction hypothesis after some iterations to reflect the dominant terms in the potential. The proof goes along the following line. In the beginning, we are in the region where $(v_n)_{\geq 2N}(\phi)$ is dominant. For properly chosen initial data, $(v_n)_{\geq 2N}(\phi)$ decreases rapidly, and we then go into the region where ϕ^{2N-2} term of $v_n(\phi)$ is comparable to $(v_n)_{\geq 2N}(\phi)$. As the recursion proceeds, the ϕ^{2N-2} term becomes positive and dominant, and then ϕ^{2N-4} becomes positive and dominant etc. After all, $v_n(\phi)$ enters the G–K region. To trace the trajectory, we will divide up the induction into $N+1$ parts along the trajectory and impose different induction hypothesis for the $a_{2k,n}$ dominant regime for $k = N, N-1, \cdots, 2, 1$. (Compare the induction hypotheses L4.4.2a and L4.4.2b with L1.3a and L1.3b, respectively.) We will prove this by means of two lemmas. First, for $N > m > 2, n \geq 0$, let $\mathcal{V}_n^m(N,L,D,C_1,n_0)$ be the class of potentials v_n satisfying:

L4.4.2a for $|\mathbf{Im}\phi| < C_1(L^{(2m-4)n}n_0)^{1/2m}$, $\exp[-v_n(\phi)]$ is analytic, positive for real ϕ, even, and

$$|e^{-(v_n)_{\geq 4}(\phi)}| \leq \exp[D - \sum_{k=2}^{N} a_{2k,n}^{1/k}|\phi|^2 + \sum_{k=2}^{N} A_{2k}a_{2k,n}(\mathbf{Im}\phi)^{2k}], \tag{4.100}$$

L4.4.2b for $|\phi| < C_1(L^{(2m-4)n}n_0)^{1/2m}$, $(v_n)_{\geq 4}(\phi)$ is analytic, and

$$(v_n)_{\geq 4}(\phi) \quad = \quad \sum_{k=2}^{N} a_{2k,n}\phi^{2k} + (v_n)_{\geq 2N+2}(\phi), \tag{4.101}$$

with

$$|a_{4,n} - L^{(d-2k)n}a_{2k,0}| \leq nL^{(d-2k)n}n_0^{-1-2/N}, \text{ for } k = 1, \cdots, N \tag{4.102}$$

$$|(v_n)_{\geq 2N+2}| \leq (n_0^{-3/2N})L^{-n/N}. \tag{4.103}$$

Lemma 4.4.2 *Let $3 \leq m \leq N$ There exist constants*

$$D(N), \bar{C_1}(N,L,D) \geq L, \bar{n_0}(N,L,D,C_1) \geq L^{48} \tag{4.104}$$

such that the following holds. Let $1/2N > \delta > 0$, $C_1 \leq \bar{C_1}(N,L,D)$, $n_0 \geq \bar{n_0}(N,L,D,C_1)$ and $n \geq 0$ satisfy the inequality

$$(L^{(d-2m)n}n_0^{-1})^{1/2m} \geq \begin{cases} (L^{(d-2m+2)n}n_0^{-1})^{1/(2m-2)} & \text{if } m > 3, \\ (n_0+n)^{-1/4} & \text{if } m = 3. \end{cases} \tag{4.105}$$

Suppose also that $v_0 \in \mathcal{V}_0(N,L,D,C_1,n_0)$, and $v_n \in \mathcal{V}_n^m(N,L,D,C_1,n_0)$. Then, there exists a closed interval $J_n \subset I_n = [-(n_0+n)^{-1-\delta}, (n_0+n)^{-1-\delta}]$ such that for μ_n running through J_n, $v_{n+1} \in \mathcal{V}_{n+1}^m(N,L,D,C_1,n_0)$. Further, the map $\mu_n \mapsto \mu_{n+1}$ sweeps I_{n+1} continuously.

Since $\mathcal{V}_0(N,L,D,C_1,n_0) = \mathcal{V}_0^N(N,L,D,C_1,n_0)$, we can iterate Lemma 4.4.2 for $m = N$, and for $n \geq 0$ as long as (4.105) is satisfied. For $3 \leq m \leq N-1$, put

$$n_m = \min\{n \in \mathbf{N} | (L^{(d-2m)n}n_0^{-1})^{1/2m} \leq (L^{(d-2m+2)n}n_0^{-1})^{1/(2m-2)}\}. \tag{4.106}$$

Obviously, $\frac{1}{d}\log_L n_0 \leq n_m < \log_L n_0$. By Lemma 4.4.2 for $m = N$,

$$v_{n_{N-1}} \in \mathcal{V}_{n_{N-1}}^N(N,L,D,C_1,n_0) = \mathcal{V}_{n_{N-1}}^{N-1}(N,L,D,C_1,n_0). \tag{4.107}$$

Therefore we can restart applying Lemma 4.4.2 for $m = N-1$. Since

$$\mathcal{V}_{n_{m-1}}^m(N,L,D,C_1,n_0) = \mathcal{V}_{n_{m-1}}^{m-1}(N,L,D,C_1,n_0) \tag{4.108}$$

for each m, this can be continued until $n = n_3$. Let

$$n_2 = \min\{n : (n_0+n)^{1/4} \leq (L^{2n}n_0)^{1/6}\}, \tag{4.109}$$

and let us define a class of single spin potentials $\mathcal{V}_{n_2+n}^2(N,L,D,C_1,n_0)$ satisfying:

L4.4.3a for $|\mathbf{Im}\phi| < C_1(n_0+n_2+n)^{1/4}$, $\exp[-v_{n_2+n}]$ is analytic and positive for real ϕ, even, and

$$|e^{-(v_{n_2+n})_{\geq 4}(\phi)}| \leq \exp[D - \sum_{k=2}^{N} a_{k,n_2}^{1/2k}|\phi|^2 + \sum_{k=2}^{N} A_{2k}a_{2k,n_2+n}(\mathbf{Im}\phi)^{2k}], \tag{4.110}$$

L4.4.3b for $|\phi| < C_1(n_0+n_2+n)^{1/4}$, $(v_{n_2+n})_{\geq 4}(\phi)$ is analytic,

$$(v_{n_2+n})_{\geq 4}(\phi) = \sum_{k=2}^{N} a_{2k,n} : \phi^{2k} : + (v_{n_2+n})_{\geq 2N+2}(\phi), \tag{4.111}$$

with

$$|a_{4,n_2+n} - a_{4,0}| \leq (n_2+n) n_0^{-1-2/N}, \tag{4.112}$$

$$|a_{2k,n_2+n} - L^{(d-2k)(n_2+n)} n_0| \leq (n_2+n) L^{(d-2k)(n_2+n-1)} n_0^{-1-2/N}, \tag{4.113}$$

$$|(v_{n_1+n})_{\geq 2N+2}(\phi)| \leq L^{-3n-n_2/N} n_0^{-3/2N}. \tag{4.114}$$

Lemma 4.4.3 *There exist constants*

$$N, D(N), \bar{C_1}(N,L,D) \geq L, \bar{n_0}(N,L,D,C_1) \geq L^{48}$$

such that the following holds. Let $N^{-1} > \delta > 0$, $C_1 \geq \bar{C_1}(N,L,D)$, $n_0 \geq \bar{n_0}(N,L,D,C_1)$, $\log_L n_0 \geq n \geq 0$. $v_0(\phi) \in \mathcal{V}_0(N,L,D,C_1,n_0)$, and $v_{n_2+n} \in \mathcal{V}^2_{n_2+n}(N,L,D,C_1,n_0)$. Then, there exists a closed interval $J_{n_2+n} \subset I_{n_2+n} = [-(n_0+n_2+n)^{-1-\delta}, (n_0+n_2+n)^{-1-\delta}]$ such that for μ_{n_2+n} running through J_{n_2+n}, $v_{n_2+n+1} \in \mathcal{V}^2_{n_2+n+1}$. Further, the map $\mu_{n_2+n} \mapsto \mu_{n_2+n+1}$ sweeps I_{n_2+n+1} continuously.

The proof of Lemma 4.4.3 is close to the proof of Lemma 4.4.2. A different point from Lemma 4.4.2 is the difference in the condition of the region where $v_{n_2+n}(\phi)$ satisfies analyticity. In fact we require that $\exp[-v_{n_2+n}(\phi)]$ is analytic for $|\mathbf{Im}\phi| < C_1(n_0+n_2+n)^{1/4}$ in Lemma 4.4.3. Because ϕ^4 term becomes dominant compared with $(v_{n_1+n})_{\geq 6}(\phi)$ this time.

With Lemma 4.4.3 we can continue iterations, and we can make sure that after a finite number of iterations, this potential is in the G-K region. More precise information about G–K region is in section 3.1 and section 4.1.

4.5 Proof of Lemma 4.4.2

Now we start to prove Lemma 4.4.2. The proof is quite similar to that of lemma 4.1.2. Let $2 < m < N$, we will only prove that $v'_n(\phi) = v_{n+1}(\phi)$ is in $\mathcal{V}^m_{n+1}(N,L,D,C_1,n_0)$, if μ_n is in I_n. As before, we separate the cases into two ; small field case or large field case corresponding to the cases either $|\phi| < C_1(L^{(2m-4)(n+1)} n_0)^{1/2m}$, or $|\mathbf{Im}\phi| < C_1(L^{(2m-4)(n+1)} n_0)^{1/2m}$ respectively. In the small field case, we prove that $v'_n(\phi)$ satisfies **L4.4.2b$'$**, the condition **L4.4.2b** with n being replaced by $n+1$, by using the Taylor expansion, and some estimation of the Gaussian integrals as in [7]. As for the large field region, we only investigate global behavior of $v'_n(\phi)$, i.e., we confirm that $v'_n(\phi)$ satisfies (4.103) of **L4.4.2a′**, the condition **L4.4.2a** with n being replaced by $n+1$. We use K for calculable absolute constants, whose values will vary in each occurrence.

4.5.1 Small field region analysis

Let $v_n \in \mathcal{V}_n^m$. We must also prepare some notations. Write $\chi_1(z) = \chi(|z| < (L^{(2m-4)n}n_0)^{1/2m})$ and throughout this subsection, we assume that ϕ is in the region $|\phi| < \frac{10}{11}LC_1(L^{(2m-4)n}n_0)^{1/2m}$. Note that we have to put C_1 to satisfy the inequality $|L^{-1}\phi \pm z| < C_1(L^{(2m-4)n}n_0)^{1/2m}$ for $|z| < (L^{(2m-4)n}n_0)^{1/2m}$ and $|\phi| < \frac{10}{11}LC_1(L^{(2m-4)n}n_0)^{1/2m}$. Next, decompose $v_{n+1}(\phi)$ as follows,

$$v_{n+1}(\phi) = v'_n(\phi) = \tilde{v}'_n(\phi) + \approx{v}'_n(\phi), \tag{4.115}$$

$$e^{-\tilde{v}'_n(\phi)} = \int \exp[-\frac{L^4}{2}\sum_{\pm} v_n(L^{-1}\phi \pm z)]d\nu_1(z)/(\phi = 0)_{small}, \tag{4.116}$$

where

$$(\phi = 0)_{small} = \int \exp[-L^4 v_n(z)]d\nu_1(z), \tag{4.117}$$

$$d\nu_1(z) \equiv \chi_1(z)e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}. \tag{4.118}$$

Estimation of $\tilde{v}'_n(\phi)$

Let us take a logarithm of (4.116).

$$\begin{aligned}\tilde{v}'_n(\phi) &= \sum_{k=1}^N L^{4-2k}(a_{2k,n} - c_{2k,n})\phi^{2k} \\ &\quad - \log\int e^{-w_\phi(z)}d\nu_1(z) + \log(\phi = 0)_{small},\end{aligned} \tag{4.119}$$

where $c_{2k,n}, w_\phi(z)$ are given by

$$\sum_{k=1}^N a_{2k,n} : \phi^{2k} := \sum_{k=1}^N (a_{2k,n} - c_{2k,n})\phi^{2k}, \tag{4.120}$$

$$w_\phi(z) = w_0(z) + w_2(z)\phi^2 + w_4(z)\phi^4 + w_6(z)\phi^6 + w_{\geq 8}(\phi, z), \tag{4.121}$$

$$w_0(z) = L^4 v_n(z)$$
$$w_{2p}(z) = \sum_{k=1}^{N} L^{4-2p}\{\binom{2k}{2p}(a_{2k,n} - c_{2k,n})z^{2p} + \frac{d^{2(N-p)}}{dz^{2(N-p)}}(v_n)_{\geq 2N+2}(z)\}\phi^{2N-2p}, \quad (4.122)$$

for $p = 0, \cdots, N-1$ and

$$w_{\geq 2N+2}(\phi, z) = \frac{L^{-4}\phi^{2N+2}}{(2N+1)!}\{\int_0^1 dt(1-t)^{2N+1}\frac{d^{2N+2}}{dz^{2N+2}}(v_n)_{\geq 2N+2}(L^{-1}t\phi + z) + \int_0^1 dt(1-t)^{2N+1}\frac{d^{2N+2}}{dz^{2N+2}}(v_n)_{\geq 2N+2}(L^{-1}t\phi - z)\}. \quad (4.123)$$

From the conditions **L4.4.2a** - **L4.4.2b**, $v_n(\phi)$ is even and analytic. We can estimate $\frac{d^{2N+2}}{dz^{2N+2}}(v_n)_{\geq 2N+2}(\phi)$ on the support of $d\nu_1(z)$ as follows by using the Cauchy formula and (4.103),

$$\begin{aligned} &|(v_n)_{\geq 2N+2}(z)| \\ &\leq \frac{1}{(2N+2)!}\int_0^1 dt(1-t)^{2N+1}|z^{2N+2}\frac{d^{2N+2}}{dz^{2N+2}}(v_n)_{\geq 2N+2}(tz)| \\ &\leq \frac{C_1}{(2N+2)!(C_1-1)^{2N+3}} n_0^{-\frac{3}{2N}} n_0^{-\frac{(N+1)}{m}} L^{-(\frac{1}{N}+\frac{(N+1)(m-2)}{m})n}|z|^{2N+2}. \end{aligned} \quad (4.124)$$

$\frac{d^2}{dz^2}(v_n)_{\geq 2N+2}(z)$ to $\frac{d^{2N}}{dz^{2N}}(v_n)_{\geq 2N+2}(z)$ can be estimated as (4.124). From the perturbation expansion:

$$-\log\int e^{-w_\phi(z)}d\nu_1(z) = -\log\int d\nu_1(z) + \langle w_\phi(z)\rangle_0 - \int_0^1 dt(1-t)\langle w_\phi(z); w_\phi(z)\rangle_t, \quad (4.125)$$

where

$$\langle\cdots\rangle_t \equiv \int \cdots e^{-tw_\phi(z)}d\nu_1(z)/\int e^{-tw_\phi(z)}d\nu_1(z). \quad (4.126)$$

Now, we shall estimate each part of (4.125). Using the estimation of the Gaussian integrations, we get

$$\begin{aligned} \langle w_\phi(z)\rangle_0 &= L^4\langle v_n(z)\rangle_0 \\ &+ \sum_{p=0}^{N-1}\sum_{k=1}^{N} L^{4-2k}\binom{2k}{2p}(a_{2k,n} - c_{2k,n})\phi^{2N-2p}(2p-1)!! \\ &+ \sum_{k=2}^{N} \overset{\sim 0,0}{R_{2k}}(L, n_0, n)\phi^{2k} + \langle w_{\geq 2N+2}(\phi, z)\rangle_0, \end{aligned} \quad (4.127)$$

where, the terms $\overset{\sim 0,0}{R_{2i}}(L, n_0, n), i = 1, \cdots, N$ satisfy

$$| \overset{\sim 0,0}{R_{2i}}(L, n_0, n)| \leq (n_0^{-3/2N}) n_0^{-(N+1)/m} L^{-(1/N+(N+1)(m-2)/m)n}. \quad (4.128)$$

From (4.124) and the similar estimates for $\frac{d^2}{dz^2}(v_n)_{\geq 2N+2}, \cdots,$ $\frac{d^{2N}}{dz^{2N}}(v_n)_{\geq 2N+2}$, we obtain,

$$|\langle w_{\geq 2N+2}(\phi, z)\rangle_0| \leq L^{4-n/N}(1 + (n_0)^{-1/m} L^{(4-2m)n/m})(n_0^{-3/2N}). \quad (4.129)$$

Next we estimate

$$\begin{aligned}\int_0^1 dt(1-t)\langle w_\phi(z); w_\phi(z)\rangle_t &= \int_0^1 dt(1-t)\sum_{i,j}\langle \tilde{w}_{2i}; \tilde{w}_{2j}\rangle_t \\ &= \int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t + \int_0^1 dt(1-t)\sum_{i,j\neq 0}\langle \tilde{w}_{2i}; \tilde{w}_{2j}\rangle_t, \quad (4.130)\end{aligned}$$

where

$$\tilde{w}_{2i} = \begin{cases} w_{2i}(z)\phi^{2i} & i = 0, \cdots, 2N \\ w_{\geq 2N+2}(\phi, z) & i = N+1. \end{cases}$$

The cumulants are

$$\begin{aligned}\langle \tilde{w}_{2i}; \tilde{w}_{2j}\rangle_t &= \langle e^{-tw_\phi(z)}\rangle_0^{-1}\langle \tilde{w}_{2i}\tilde{w}_{2j}e^{-tw_\phi(z)}\rangle_0 \\ &\quad -\langle e^{-tw_\phi(z)}\rangle_0^{-2}\langle \tilde{w}_{2i}e^{-tw_\phi(z)}\rangle_0\langle \tilde{w}_{2j}e^{-tw_\phi(z)}\rangle_0. \quad (4.131)\end{aligned}$$

Note that the support of $d\nu_1(z)$ is $|z| < (L^{(2m-4)n}n_0)^{1/2m}$. From (4.105), we get the uniform estimate $|w_\phi(z)| \leq K \cdot L^{2N}C_1^{2N}$ for $|z| < (L^{(2m-4)n}n_0)^{1/2m}$ and $|\phi| < \frac{10LC_1}{11}(L^{(2m-4)n}n_0)^{1/2m}$. Hence,

$$\begin{aligned}&|\sum_{(i,j)\neq(0,0)}\langle \tilde{w}_{2i}; \tilde{w}_{2j}\rangle_t| \\ &\quad \leq e^{K\cdot L^{2N}C_1^{2N}}\sum_{(i,j)\neq(0,0)}(\langle|\tilde{w}_{2i}||\tilde{w}_{2j}|\rangle_0 + \langle|\tilde{w}_{2i}|\rangle_0\langle|\tilde{w}_{2j}|\rangle_0). \quad (4.132)\end{aligned}$$

From (4.122)-(4.123), we can estimate $|\int_0^1 dt(1-t)\sum_{(i,j)\neq(0,0)}\langle \tilde{w}_{2i}; \tilde{w}_{2j}\rangle_t|$ similarly as in (4.124), and we obtain

$$\begin{aligned}&|\text{2nd term of RHS of (4.130)}| \\ &\quad \leq Ke^{K\cdot C_1^{2N}}L^{-2}n_0^{-2}(|\phi|^2 + \sum_{k=2}^{N}L^{-(4-2k)n-2}|\phi|^{2k}) \\ &\qquad + |\text{higher order terms}|. \quad (4.133)\end{aligned}$$

The higher order terms are estimated as follows,

$$|\text{higher order terms}| \le Ke^{K\cdot L^N C_1^{2N}} L^{4(N-1)-n/N} C_1^{4(N-1)} (n_0^{-4/2N}). \quad (4.134)$$

Next, we estimate $\int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t$. Since $\langle w_0(z); w_0(z)\rangle_t$ is analytic function in $|\phi| < \frac{10}{11} L C_1 (L^{(2m-4)} n_0)^{1/2m}$, by Cauchy formula we get

$$\begin{aligned}
&|\int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t - \int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t|_{\phi=0}| \\
&\qquad \le K \exp(K\cdot L^{2N} C_1^{2N}) \cdot L^{-2} n_0^{-2} |\phi|^2. \quad (4.135)
\end{aligned}$$

So we have,

$$\begin{aligned}
&|\int_0^1 dt(1-t)\langle w_\phi(z); w_\phi(z)\rangle_t - \int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t|_{\phi=0}| \\
&\quad \le K \exp(K\cdot L^{2N} C_1^{2N}) L^{-2} n_0^{-2} (|\phi|^2 + \cdots + L^{-(4-2N)n} |\phi|^{2N}) \\
&\qquad + |\text{higher order terms}|, \quad (4.136)
\end{aligned}$$

$$|\text{higher order terms}| \le Ke^{K\cdot L^{2N} C_1^{2N}} L^{4(N-1)-n/N} C_1^{4(N-1)} (n_0^{-4/2N}). \quad (4.137)$$

These coefficients are large, but not terrible, because we can take n_0 sufficiently large. In the following, we put $n_0^{1/2N} \ge K\cdot C_1^{4(N-1)} L^{4(N-1)} e^{K\cdot L^{2N} C_1^{2N}}$. From (4.119) and (4.125), we infer that

$$\begin{aligned}
\tilde{v}'_n(\phi) &= \sum_{k=1}^N L^{4-2k}(a_{2k,n} - c_{2k,n})\phi^{2k} \\
&+ \sum_{p=1}^N \sum_{k=1}^N L_{2k}^{4-2k} C_{2p}(a_{2k,n} - c_{2k,n})(2N-2p-1)!!\phi^{2p} \\
&+ \sum_{k=1}^{2N} \tilde{R}_{2k}(N, L, n_0, n)\phi^{2k} + (\tilde{v}_n)'_{\ge 2N+2}(\phi), \quad (4.138)
\end{aligned}$$

where, the terms $\tilde{R}_{2i}(N, L, n_0, n), i = 1, \cdots, N$ satisfy

$$\begin{aligned}
|\tilde{R}_{2i}(N, L, n_0, n)| &\le L^{-10-(4-2i)n} n_0^{-2+1/2N} + |\tilde{R}_{2i}^{0,0}(N, L, n_0, n)|, \\
& i = 1, \cdots, N, \quad (4.139)
\end{aligned}$$

and from (4.129) and (4.137), $(\tilde{v}_n)'_{\ge 2N+2}(\phi)$ satisfy

$$|(\tilde{v}_n)'_{\ge 8}(\phi)| \le L^{4-n/N}(1 + L^{-(4-2m)n/m}(n_0)^{-1/m} + L^{-4})(n_0^{-3/2N}), \quad (4.140)$$

for $|\phi| < \frac{10}{11} L C_1 (L^{(4-2m)n} n_0)^{1/2m}$. Notice that

$$(\phi = 0)_{small} = \log \int d\nu_1(z) - \langle w_0(z)\rangle_0 + \int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t|_{\phi=0}.$$

So we can check that the constant term $(\phi = 0)_{small}$ vanishes. The estimate (4.140) is a little weaker than what we want (see (4.103)). So, we need a stronger estimate. Since $\tilde{v}'_n(\phi)$ is analytic in $|\phi| < \frac{10}{11}LC_1(L^{-(4-2m)n}n_0)^{1/2m}$, $\phi^{-2N-2}\,(\tilde{v}_n)'_{\geq 8}\,(\phi)$ is also analytic in $|\phi| < \frac{10}{11}LC_1(L^{-(4-2m)n}n_0)^{1/2m}$. We obtain from the maximum principle

$$\begin{aligned} |\,(\tilde{v}_n)'_{\geq 2N+2}\,(\phi)| \leq & \left(\frac{|\phi|}{(10L/11)C_1(L^{-(4-2m)n}n_0)^{1/2m}}\right)^{2N+2}(n_0^{-3/2N}) \\ & \times(L^{4-n/N}(1+L^{-(4-2m)n/m}(n_0)^{-1/m}+L^{-4})), \end{aligned} \tag{4.141}$$

so that for $|\phi| < C_1(L^{-(4-2m)(n+1)}n_0)^{1/2m}$,

$$\begin{aligned} |\,(\tilde{v}_n)'_{\geq 2N+2}\,(\phi)| \leq & \left(\frac{11}{10}\right)^{2N+2}L^{-(2N+2)(1-(4-2m)/2m)} \\ & \times(L^{4-n/N}(1+L^{-(4-2m)n/m}(n_0)^{-1/m}+L^{-4})(n_0^{-3/2N})). \end{aligned} \tag{4.142}$$

Estimation of $\approx{v}'_n\,(\phi)$ for $|\phi| < \frac{10}{11}LC_1(L^{-(4-2m)n}n_0)^{1/2m}$

Represent (4.115) as

$$\begin{aligned} \overset{\approx}{v}'_n\,(\phi) \;=\; & \log\left(1+\frac{\int\exp[-\frac{1}{2}L^4\sum_{\pm}v_n(L^{-1}\phi\pm z)](1-\chi_1(z))d\nu(z)}{e^{-\tilde{v}'_n(\phi)}(\phi=0)_{small}}\right) \\ & +\log(\phi=0)_{small}-\log(\phi=0). \end{aligned} \tag{4.143}$$

We want to prove that $\overset{\approx}{v}'_n\,(\phi)$ is analytic in$|\phi| < \frac{10}{11}LC_1(L^{(2m-4)n}n_0)^{1/2m}$ and sufficiently smaller than $\tilde{v}'_n\,(\phi)$. To prove these properties, we have only to prove that

$$\frac{\int\exp[-\frac{1}{2}L^4\sum_{\pm}v_n(L^{-1}\phi\pm z)](1-\chi_1(z))d\nu(z)}{e^{-\tilde{v}'_n(\phi)}(\phi=0)_{small}} \tag{4.144}$$

is analytic and sufficiently small in $|\phi| < \frac{10}{11}LC_1(L^{(2m-4)n}n_0)^{1/2m}$. First of all, we estimate the denominator of (4.144). We can show that the denominator is bounded from below by a constant which depends on C_1, but not on n_0. From **L4.4.2b**, and (4.139) together with uniform estimate of $w_0(z)$ under the condition of (4.105), we estimate denominator as follows,

$$|\text{denominator of (4.144)}| \;\geq\; \exp[-K\cdot L^{2N}C_1^{2N}]. \tag{4.145}$$

Next, we estimate the numerator part of (4.144),

$$|\text{numerator of } (4.144)| \leq \int (1-\chi_1(z)) \prod_{\pm} |\exp[-v_n(L^{-1}\phi \pm z)]|^{L^4/2} d\nu(z). \quad (4.146)$$

Using (4.100) of **L4.4.2a** for $|L^{-1}\phi \pm z| < C_1(L^{(2m-4)n}n_0)^{1/2m}$, we have

$$|\text{numerator of } (4.144)| \leq \exp[K + L^4 D + \sum_{k=2}^{N} A_{2k}C_0' C_1^{2k} - \frac{1}{4}(L^{(2m-4)n}n_0)^{1/m}]. \quad (4.147)$$

So,

$$|(4.144)| < \exp[K \cdot L^{2N}C_1^{2N} + L^4 D + \sum_{k=2}^{2N} A_{2k}C_0' C_1^{2k} - \frac{1}{4}(L^{(2m-4)n}n_0)^{\frac{1}{m}}]. \quad (4.148)$$

For given L, D and C_1, we can take n_0 large enough to obtain

$$\text{RHS of } (4.148) \leq \exp[-\frac{1}{8}(L^{(2m-4)n}n_0)^{1/m}]. \quad (4.149)$$

This estimate is also valid for $\log(\phi = 0) - \log(\phi = 0)_{small}$. According to (4.149), we can show that $\overset{\approx}{v_n'}(\phi)$ is analytic and

$$|\overset{\approx}{v_n'}(\phi)| \leq 2e^{-1/8(L^{(2m-4)n}n_0)^{1/m}}. \quad (4.150)$$

Estimation of coefficients

Now, we assume that $|\phi| < C_1(L^{(2m-4)(n+1)}n_0)^{1/2m}$ i.e. ϕ is in the small field region of $v_n'(\phi)$. Notice that the small field region is in the region $|\phi| < \frac{10}{11}LC_1(L^{(2m-4)n}n_0)^{1/2m}$, so we can use the argument above. Thus, $\overset{\approx}{v_n'}(\phi)$ is analytic in the small field region of v_n', and we can obtain power series expansion of $\overset{\approx}{v_n'}(\phi)$. With the use of Cauchy's estimate, we see that coefficients of ϕ^2 to ϕ^{2N} satisfy,

$$|\frac{1}{k!}\frac{d^k}{d\phi^k}\overset{\approx}{v_n'}(0)| \leq e^{-1/8(L^{(2m-4)n}n_0)^{1/m}}, k = 2, 4, \cdots, 2N. \quad (4.151)$$

Using the bounded convergence theorem, we see that $\frac{1}{2}\frac{d^2}{d\phi^2}\overset{\approx}{v_n'}(0), \frac{1}{4!}\frac{d^4}{d\phi^4}\overset{\approx}{v_n'}(0), \cdots \frac{1}{2N!}\frac{d^{2N}}{d\phi^{2N}}\overset{\approx}{v_n'}(0)$ are continuous functions of μ_n on I_n. From (4.142) and

(4.150), if n_0 is sufficiently large, then we have

$$|(v_n)'_{\geq 2N+2}(\phi)| \leq L^{-(n+1)/N}(n_0^{-3/2N}), \tag{4.152}$$

for $|\phi| < C_1(L^{(2m-4)(n+1)}n_0)^{1/2m}$. From (4.94), (4.138), (4.139), and (4.151), we know that

$$\begin{aligned} |a_{2k,n+1} - L^{4-2k}a_{2k,n}| &= |R_{2k}(N,L,n_0,n) + \frac{1}{2k!}\frac{d^{2k}}{d\phi^{2k}} \overset{\approx}{v'_n}(0)| \\ &\leq L^{(4-2k)n}n_0^{-1-2/N}, k = 3, \cdots 2N. \end{aligned} \tag{4.153}$$

Thus, if n_0 is sufficiently large, we have

$$|a_{2k,n+1} - L^{(4-2k)(n+1)}a_{2k,0}| < (n+1)L^{(4-2k)n}n_0^{-1-2/N} \tag{4.154}$$

which proves (4.103) of **L4.4.2b$'$**. From (4.138), (4.139), we know

$$|a_{4,n+1} - a_{4,n}| \leq n_0^{-1-2/N}. \tag{4.155}$$

Thus, we have

$$|a_{4,n+1} - a_{4,0}| < (n+1)n_0^{-1-2/N}, \tag{4.156}$$

which completes the proof of **L4.4.2b$'$**. Similarly, we get estimation of coefficient μ'_n as follows,

$$|\mu'_n - L^2\mu_n| \leq K \times n_0^{-1-2/N}. \tag{4.157}$$

We know that map $R : \mu \mapsto \mu'$ is continuous, and image $R(I_n)$ include I_{n+1}. So that we can take for J_{n+1} a connected component of this inverse image $R^{-1}(I_{n+1}) \subset I_n$.

This ends the analysis of the small field properties.

4.5.2 Large field region analysis

Next, we prove that $e^{-(v_n)'(\phi)}$ satisfy the condition **L4.4.2a$'$**. First, we prove it in the case where $|\mathbf{Re}\phi| > C_1(L^{(2m-4)(n+1)}n_0)^{1/2m}$. Next, we prove it in $|\phi| < \frac{10}{11}LC_1(L^{(2m-4)n}n_0)^{1/2m}$ i.e. this region includes the small field region of $v'_n(\phi)$.

The case where $|\mathbf{Re}\phi| > C_1(L^{(2m-4)(n+1)}n_0)^{1/2m}$

Note that the definition of the RG (2.15) has the following expression

$$e^{-v_n'(\phi)} = \int \prod_{\pm} \exp[-v_n(L^{-1}\phi \pm z)]^{L^4/2} d\nu(z)/(\phi = 0). \quad (4.158)$$

$|\mathbf{Im}(L^{-1}\phi \pm z)| < C_1(L^{(2m-4)n}n_0)^{1/2m}$, if $|\mathbf{Im}\phi| < C_1(L^{(2m-4)(n+1)}n_0)^{1/2m}$. From the condition **L4.4.2a**,

$$\begin{aligned} |e^{-(v_n)'_{\geq 4}(\phi)}| &\leq \exp[L^4 D - L^2 \sum_{k=2}^{N} a_{2k,n}^{1/2k}|\phi|^2 + \sum_{k=2}^{N} L^{4-2k} A_{2k} a_{2k,n} (\mathbf{Im}\phi)^{2k}] \\ &\times \int_{-\infty}^{\infty} e^{-L^4\mu_n z^2 - L^4 \sum_{k=2}^{2N} a_{2k,n}^{1/2k} z^2} d\nu(z)/(\phi = 0). \end{aligned} \quad (4.159)$$

Note that, $\{a_{2k,n}\}$ are positive and sufficiently small, hence, this integral part and $(\phi = 0)$ estimated as absolute constants, so we get

$$\begin{aligned} &\text{RHS of (4.159)} \\ &\leq \exp[L^4 D - L^2 \sum_{k=2}^{N} a_{2k,n}^{1/2k}|\phi|^2 + \sum_{k=2}^{N} L^{4-2k} A_{2k} a_{2k,n} (\mathbf{Im}\phi)^{2k} + K]. \end{aligned} \quad (4.160)$$

If D and L are given, we take C_1 sufficiently large and then we take n_0 sufficiently large. Thus, we obtain

$$\begin{aligned} &|\exp(-(v_n')_{\geq 4}(\phi))| \\ &< \exp[D - \sum_{k=2}^{2N} a_{2k,n+1}^{1/2k}|\phi|^2 + \sum_{k=2}^{2N} A_{2k} a_{2k,n+1} (\mathbf{Im}\phi)^{2k}], \end{aligned} \quad (4.161)$$

for $|\mathbf{Im}\phi| < C_1(L^{(2m-4)(n+1)}n_0)^{1/2m}$, $|\mathbf{Re}\phi| > C_1(L^{(2m-4)(n+1)}n_0)^{1/2m}$.

The case where $|\phi| < \frac{10}{11} L C_1(L^{(2m-4)n}n_0)^{1/2m}$

Now we prove remainder part of large field region. Let $\mu_n \in I_n$, and $|\phi| < \frac{10}{11} L C_1(L^{(2m-4)n}n_0)^{1/2m}$. From (4.140), (4.154), (4.156), (4.157), and $K(n_0 + n)^{1/4} > (L^{(2m-4)n}n_0)^{1/2m}$ for $m \geq 3$, we have

$$\begin{aligned} |e^{-((v_n)')_{\geq 4}(\phi)}| &\leq \exp[K \sum_{k=2}^{2N} L^{-2} C_1^{2k} n_0^{-1/k}] \\ &\times \exp[-\sum_{k=2}^{N} a_{2k,n+1}(\mathbf{Re}\phi^{2k}) + L^4 n_0^{-1/2}]. \end{aligned} \quad (4.162)$$

And, we estimate $a_{2k,n+1}(\mathbf{Re}\phi^{2k})$ as follows,

$$a_{2k,n+1}(\mathbf{Re}\phi^{2k}) \geq a_{2k,n+1}(\frac{1}{4}(\mathbf{Re}\phi)^{2k} - K(\mathbf{Im}\phi)^{2k})$$
$$\geq -\frac{1}{2}D_{2k} + 2(a_{2k,n+1})^{1/k}|\phi|^2 - A_{2k}a_{2k,n+1}(\mathbf{Im}\phi)^{2k}. \quad (4.163)$$

Notice that D_{2k} does not depend on C_1, n_0 or n. Put $D = \sum_{k=2}^{N} D_{2k}$. From (4.162) to (4.163),

$$|e^{-((v_n)')_{\geq 4}(\phi)}| \leq \exp[D - \sum_{k=2}^{N}(a_{2k,n+1})^{1/k}|\phi|^2 + \sum_{k=2}^{N} A_{2k}a_{2k,n+1}(\mathbf{Im}\phi)^{2k}]$$
$$\times \exp[-\frac{1}{2}D + K \cdot L^{-2}C_1^2 n_0^{-1/2}]$$
$$\times \exp[K \cdot L^{4/3}C_1^{2N}(L^{(4-2m)(n+1)}n_0)^{1/m} + L^4 n_0^{-1/2}], \quad (4.164)$$

which is smaller than

$$\exp[D - \sum_{k=2}^{N} a_{2k,n+1}^{1/k}|\phi|^2 + \sum_{k=2}^{N} A_{2k}a_{2k,n+1}(\mathbf{Im}\phi)^{2k}], \quad (4.165)$$

if n_0 is sufficiently large. Proof of Lemma 4.4.2 is completed.

4.6 Proof of Theorem 4.4.1

Finally, we prove Theorem 4.4.1, using Lemma 4.4.2, Lemma 4.4.3 and Theorem 4.1.4. First of all, we notice that it is possible to take constants L, $D(N)$, $C_1(N, L, D)$, $n_0(N, L, D, C_1)$ to satisfy Lemma 4.4.2, Lemma 4.4.3, and Theorem 4.1.4. We can check that potential $v(\phi)$ can be iterated n_2 times if initial parameters satisfy the conditions **(Pa)** and **(Pb)** because of Lemma 4.4.2. Notice that $v_{n_2}(\phi)$, the potential after n_2 iterations, satisfies the conditions **L4.4.3a** and **L4.4.3b** with $n = 0$, and so Lemma 4.4.3 can be applied to this potential. We have to iterate $\mathcal{R}$ using Lemma 4.4.3, sufficiently many times so that the iterated potentials satisfy the G-K conditions. Put

$$n_1 = \min\{n \in \mathbf{N} : |(v_{n_2+n})_{\geq 6}(\phi)| < (n_0 + n_2 + n)^{-3/4}$$
$$\text{for } |\phi| < C_1(n_0 + n_2 + n)^{1/4}\}. \quad (4.166)$$

Then,

$$a_{6,n_1+n_2-1} < (n_0 + n_1 + n_2 - 1)^{-9/4}. \quad (4.167)$$

By calculation, n_1 can be estimated as $n_1 < K \log_L n_0$. Since, $a_{2k,n_1+n_2} \geq 0$, and by (4.112)

$$a_{4,n_1+n_2} - c_{4,n_1+n_2} < a_{4,0} + (n_1+n_2)n_0^{-1-2/N}$$
$$< \frac{C_{++}}{L^4} n_0^{-1} + 2(\log_L n_0) n_0^{-1-2/N} < \frac{C_+}{L^4}(n_0+n_1+n_2)^{-1}. \quad (4.168)$$

Similarly, by (4.167) we have

$$a_{4,n_1+n_2} - c_{4,n_1+n_2} > \frac{C_-}{L^4}(n_0+n_1+n_2)^{-1}. \quad (4.169)$$

So, we checked the condition **G-Kb** completely. Next, let us check the condition **G-Ka**. Notice that analyticity, positivity for real ϕ, and even function of $v_{n_1+n_2}(\phi)$ are checked easily. Now, We check the bound of $v_{n_1+n_2}(\phi)$

$$|\exp[-v_{n_1+n_2}(\phi)]| \leq \exp[D - \sum_{k=2}^{2N} a_{2k,n_1+n_2}^{1/k} |\phi|^2]$$
$$\times \exp[+\sum_{k=2}^{2N} A_{2k} a_{2k,n_1+n_2} (\mathbf{Im}\phi)^{2k}]. \quad (4.170)$$

Notice that $-\sum_{k=3}^{2N} a_{2k,n_1+n_2}^{1/2k}|\phi|^2 + \sum_{k=3}^{2N} A_{2k} a_{2k,n_1+n_2}(\mathbf{Im}\phi)^{2k}$ is nonpositive for $(\mathbf{Im}\phi) < C_1(n_0+n_1+n_2)^{1/4}$ from the definitions of n_1 and n_2. So we have the following inequality

$$|\exp[-v_{n_1+n_2}(\phi)]| \leq \exp[D - a_{4,n_1+n_2}^{1/2}|\phi|^2 + A_4 a_{4,n_1+n_2}(\mathbf{Im}\phi)^4]. \quad (4.171)$$

We have checked all of the G-K conditions. Since $a_{2k,n_1+n_2-1}, k \geq 3$ is sufficiently small by (4.167), we know

$$|\mu_{n_1+n_2} - L^2(\mu_{n_1+n_2-1} - c_{2,n_1+n_2-1} + \frac{6\lambda_{n_1+n_2-1}}{1-L^{-2}})| \leq K \cdot n_0^{-1-2/N}. (4.172)$$

As in the proof lemma 4.4.2 and Lemma 4.4.3, we can take for $J_{n_1+n_2}$ a suitable connected component. So, we can adapt Theorem 4.1.4. Now, Theorem 4.4.1 is finished.

Chapter 5

Central Limit Theorem and Law of Iterated Logarithm for Hierarchical Models

In this Chapter, we consider some problems related to hierarchical models. First, we consider the central limit theorem (CLT) for the block spin see [2] for a related work for the central limit theorem for hierarchical models. Finally, we show the Law of Iterated Logarithm for Block Spin at the origin.

5.1 Central Limit Theorem for Block Spin at the origin

In this section, we consider the central limit theorem for the block spin at the origin. This is a natural application of RG analysis, because we already investigated the limit probability measure for iterations of RG. First, we consider Gaussian cases.

5.1.1 massless hierarchical Gaussian case

Proposition 5.1.1 *Let $\phi_0^{(n)}$ be the n'th iterated block spin at the origin, i.e.,*

$$\phi_0^{(n)} = L^{-\frac{(d+2)n}{2}} \sum_{x \in B_{L^n}(0)} \phi_x. \tag{5.1}$$

Then, $\phi_0^{(n)}$ has the normal distribution with mean 0 and variance $\sigma^2 = \frac{1}{1-L^{-(d-2)}}$ for all n. Especially, limit block spin at the origin also has the same distribution.

Proof of Proposition 5.1.1

The massless hierarchical spin ϕ_x is defined by the summation of independent Gaussian random variables $\{Z_x^k\}$, so $\{\phi_x\}$ are Gaussian system. Obviously, $E[\phi_x] = 0$ for all x, and it is easy to see that $E[(\phi_x)^2] = \frac{1}{1-L^{-(d-2)}}$. From the definition of block spin, $\phi_x^{(n)}$ has the same distribution of ϕ_x. So, statement is shown.

5.1.2 massive hierarchical Gaussian case

Here, we consider the central limit theorem for massive hierarchical Gaussian case, i.e., single spin is $v(\phi) = \mu\phi^2, \mu > 0$. In this case, $\{\phi_x\}$ is still a Gaussian system. However the order of scaling is different from massless case.

Lemma 5.1.2 *Let $v(\phi) = \mu\phi^2$, and $P_v[\cdot]$ be formaly written by*

$$\begin{aligned} P_v[A] &= \frac{1}{Z_v}\int_A e^{-\sum v(\phi)} d\nu_{G_h}(\phi), A \in \mathcal{B}(\mathbf{R}^\infty) &(5.2)\\ Z_v &= \int_{\mathbf{R}^\infty} e^{-\sum v(\phi)} d\nu_{G_h}(\phi) &(5.3)\end{aligned}$$

Then, this system is also a Gaussian System, whose means $\{m_\mu(\phi_x)\}$, variances $\{V_\mu(\phi_x)\}$, and covariances $\{Cov_\mu(\phi_x,\phi_y)\}$ are given by

$$\begin{aligned} m_\mu(\phi_x) &= 0, &(5.4)\\ V_\mu(\phi_x) &= \sum_{k=0}^{k_0-1} \frac{L^{-(d-2)k}}{2L^{d+2k}\mu+1} + \sum_{k=k_0}^{\infty} \frac{L^{-(d-2)k}}{4dL^{d+2k}\mu+1}, &(5.5)\\ Cov_\mu(\phi_x,\phi_y) &= \sum_{k=\underline{k}(x,y)}^{k_0-1} \frac{L^{-(d-2)k}}{2L^{d+2k}\mu+1} + \sum_{k=k_0}^{\infty} \frac{L^{-(d-2)k}}{4dL^{d+2k}\mu+1}, &(5.6)\end{aligned}$$

where $\underline{k}(x,y)$ is the smallest number which satisfies $[L^{-k}x] = [L^{-k}y]$, for each $x,y \in \mathbf{Z}^d$ and $k_0 = \underline{k}_0(x)$ is the smallest number which satisfies $[L^{-k}x] = 0$.

proof of Lemma 5.1.2

Let ϕ_x^N be the L^N size cut off spin variables, i.e.,

$$\phi_x^N = \sum_{k=0}^{N-1} A_{[L^{-k}x]} Z_{[L^{-k-1}x],k}, x \in \Lambda_N \qquad (5.7)$$

where $\Lambda_N = \{x \in \mathbf{Z}^d; [L^{-N}x] = 0\}$. We first compute the characteristic function of ϕ_x^N for the massive hierarchical Gaussian system $\{\phi_x^N\}_{x\in\Lambda_N}$.

$$\begin{aligned}
&E_\mu^N[\exp(it\sum_{l=1}^{m} t_l\phi_{x_l})] \\
&= \frac{1}{Z_{\Lambda_N}^N}\int_{\mathbf{R}^{\Lambda_N}} \exp(i\sum_{l=1}^{m} t_l \sum_{k=0}^{N-1} L^{-(d-2)k/2} A_{[L^{-k}x]} z_{[L^{-k-1}x],k}) \\
&\qquad\qquad \exp[-\sum_{x\in\Lambda_{N-1}\setminus\{0\}}\sum_{k=0}^{N-1}(1/2 + L^{(k+1)d}L^{-(d-2)k}\mu)z_{x,k}^2] \\
&\qquad \times \exp[-\sum_{k=0}^{N-1}(1/2 + 2dL^{(k+1)d}L^{-(d-2)k}\mu)z_{0,k}^2] \prod_{0\le k\le N-1, x\in\Lambda_{N-1}} dz_{x,k}. \\
&= \exp[-\sum_{p=1}^{m}\sum_{l_1<l_2<\cdots<l_q}\sum_{k=\underline{\mathrm{k}}(x_{l_1},\cdots,x_{l_q})}^{\bar{k}(x_{l_1},\cdots,x_{l_q})} \frac{(t_{l_1}+\cdots+t_{l_p})^2}{2}\frac{L^{-(d-2)k}}{1+2L^{2k+d}\mu}] \\
&\quad \times \exp[-\sum_{p=1}^{m}\sum_{l_1<l_2<\cdots<l_q}\sum_{k=\underline{\mathrm{k}}_0(x_{l_1},\cdots,x_{l_q})}^{\bar{k}_0(x_{l_1},\cdots,x_{l_q})} \frac{(t_{l_1}+\cdots+t_{l_p})^2}{2}\frac{L^{-(d-2)k}}{1+4dL^{2k+d}\mu}]
\end{aligned} \tag{5.8}$$

where $\underline{\mathrm{k}}(x_{l_1},\cdots,x_{l_p})$ is the smallest number which satisfies $[L^{-k}x_{l_1}] = \cdots = [L^{-k}x_{l_p}] \neq 0$ for only $x_{l_1},\ldots,x_{l_p}$, and $\bar{k}(x_{l_1},\cdots,x_{l_p})$ is the largest number which satisfies $[L^{-k}x_{l_1}] = \cdots = [L^{-k}x_{l_p}] \neq 0$ for only $x_{l_1},\ldots,x_{l_p}$, $\underline{\mathrm{k}}_0(x_{l_1},\cdots,x_{l_p})$ is the smallest number, satisfies $[L^{-k}x_{l_1}] = \cdots = [L^{-k}x_{l_p}] = 0$ for only $x_{l_1},\ldots,x_{l_p}$, and $\bar{k}_0(x_{l_1},\cdots,x_{l_p})$ is the largest number which satisfies $[L^{-k}x_{l_1}] = \cdots = [L^{-k}x_{l_p}] = 0$ for only $x_{l_1},\ldots,x_{l_p}$. Notice that cross term in $\sum_{x\in\Lambda_N} v(\phi_x^N)$ vanish because of the factor $\{A_x\}$. Let take limit $N\to\infty$, we get the assertion of the lemma.

CLT of massive hierarchical Gaussian Case

We show the central limit theorem of block spin at the origin. We must calculate the characteristic function of $\phi_0^{(n)}$. Recall that $\phi_0^{(n)}$ is the n'th iterated block spin, i.e., $\phi_0^{(n)} = L^{-(d+2)n/2}\sum_{x:[L^{-n}x]=0}\phi_x$, Hence we have

$$E_\mu[\exp[it\phi_0^{(n)}]] = \exp[-\frac{t^2}{2}\sum_{k=0}^{\infty}\frac{L^{-(d-2)k}}{1+4dL^{2(k+n)+d}\mu}]. \tag{5.9}$$

In this formula, $E_\mu[\exp[it\phi_0^{(n)}]]$ goes to 1 as $n\to\infty$. Obviously, the limit distribution is degenerated. It means that this scaling is incorrect. The correct scaling is $L^{dn/2}$. In fact,

$$\begin{aligned} E_\mu[\exp[itL^{-\frac{d}{2}n}\sum_{x:[L^{-n}x]=0}\phi_x]] &= E_\mu[\exp[itL^n\phi_x^{(n)}]] \\ &= \exp[-\frac{t^2}{2}\sum_{k=0}^{\infty}\frac{L^{-(d-2)k}}{L^{-2n}+4dL^{2k+d}\mu}]. \quad (5.10) \end{aligned}$$

So, the limit of $E_\mu[\exp[itL^{-\frac{d}{2}n}\sum_{x:[L^{-n}x]=0}\phi_x]]$ exists, and we know

$$\begin{aligned} \lim_{n\to\infty} E_\mu[\exp[itL^{-dn/2}\sum_{x:[L^{-n}x]=0}\phi_x]] &= \exp[-\frac{t^2}{2}\sum_{k=0}^{\infty}\frac{1}{4dL^{-(k+1)d}\mu}] \\ &= \exp[-\frac{t^2}{2}\frac{1}{L^d-1}\cdot\frac{1}{4d\mu}]. \quad (5.11) \end{aligned}$$

Theorem 5.1.3 *Let $\phi_0^{(n)}$ be the n'th iterated block spin at the origin, i.e.,*

$$\tilde{\phi_0}^{(n)} = L^{-\frac{dn}{2}}\sum_{x\in B_{L^n}(0)}\phi_x. \quad (5.12)$$

Then, the limit distribution of $\tilde{\phi_0}^{(n)}$ is as follows

$$\lim_{n\to\infty} P_\mu[a\le\tilde{\phi_0}^{(n)}<b] = \frac{1}{\sqrt{2\pi\sigma^2}}\int_a^b \exp[-\frac{x^2}{2\sigma^2}]dx \quad (5.13)$$

where

$$\sigma^2 = \frac{1}{L^d-1}\cdot\frac{1}{4d\mu}. \quad (5.14)$$

5.2 The Hartman-Wintner Law of the Iterated Logarithm for Hierarchical Spin Systems

In this section, we investigate into the Law of the Iterated Logarithm (LIL) of this system. First of all, we prove the LIL of hierarchical massless Gaussian Case, next we consider another situation.

5.2.1 massless hierarchical Gaussian case

The opportunity of this section, we prove the following assertion,

Theorem 5.2.1 *Let $\phi_0^{(n)}$ be the block spin of the n th scale at the origin, and $\sigma^2 = \frac{1}{1-L^{-(d-2)}}$. Then if $K_\sigma = [-\sigma, \sigma]$,*

$$P[\lim_{n\to\infty} \mathrm{dist}(\phi_0^{(n)}/(2\log\log L^{dn})^{1/2}, K_\sigma) = 0] = 1, \tag{5.15}$$

$$P[C(\phi_0^{(n)}/(2\log\log L^{dn})^{1/2}) = K_\sigma] = 1, \tag{5.16}$$

where $C(a_n)$ is the all of the cluster points of $\{a_n\}$. In particular, with probability 1

$$\limsup_{n\to\infty} \phi_0^{(n)}/(2\log\log L^{dn})^{1/2} = \sigma, \liminf_{n\to\infty} \phi_0^{(n)}/(2\log\log L^{dn})^{1/2} = -\sigma. \tag{5.17}$$

Proof of 5.2.1

We follow the argument by as de Acosta [1]. We put $a_n = \sqrt{2\log\log L^{dn}}$, and, we will show

$$P[\limsup_{n\to\infty} \frac{\phi_0^{(n)}}{a_n} \leq \sigma] = 1. \tag{5.18}$$

For arbitrary $\delta > 0$, from proposition 5.1.1 we calculate easily as follows,

$$\begin{aligned} P[\phi_0^{(n)} > a_n\sigma(1+\delta)] &= \int_{a_n\sigma(1+\delta)}^{\infty} \exp[-x^2/2\sigma^2]\frac{dz}{\sqrt{2\pi\sigma^2}} \\ &< \text{const } \exp[-a_n^2\sigma^2(1+\delta)^2/2\sigma^2] \\ &< \text{const}(nd\log L)^{-(1+\delta)^2}. \end{aligned} \tag{5.19}$$

So, $\sum_{n=0}^{\infty} P[\phi_0^{(n)} > a_n(\sigma+\delta)] < \infty$. By Borel-Cantelli's first lemma, we get

$$P[\limsup_{n\to\infty} \frac{\phi_0^{(n)}}{a_n} \leq \sigma + \delta] = 1. \tag{5.20}$$

Since δ is arbitrary, this proves (5.18). We prove next

$$P[\lim_{n\to\infty} \mathrm{dist}(\phi_0^{(n)}/a_n, K_\sigma) = 0] = 1. \tag{5.21}$$

In fact, (5.18) applied to $-\phi_0^{(n)}$ yields that $P[\liminf_{n\to\infty} \frac{\phi_0^{(n)}}{a_n} \geq -\sigma] = 1$, and since

$$\begin{aligned} &\{\limsup_{n\to\infty} \mathrm{dist}(\phi_0^{(n)}/a_n, K_\sigma) > 0\} \\ &\qquad \subset \{\limsup_{n\to\infty} \phi_0^{(n)}/a_n > \sigma\} \cup \{\liminf_{n\to\infty} \phi_0^{(n)}/a_n < -\sigma\}, \end{aligned}$$

(5.21) follows. It remains to show

$$P[C(\phi_0^{(n)}/a_n) = K_\sigma] = 1 \tag{5.22}$$

In order to prove (5.22), it is enough to show that for every $b \in (-\sigma, \sigma)$

$$P[b \in C(\phi_0^{(n)}/a_n)] = 1. \tag{5.23}$$

In fact, taking a countable dense set $D \subset (-\sigma, \sigma)$ it follows from (5.23) that $P[D \subset C(\phi_0^{(n)}/a_n)] = 1$. Since $C(\phi_0^{(n)}/a_n)$ is closed and as a consequence of (5.21), $P[C(\phi_0^{(n)}/a_n) \subset K_\sigma] = 1$, (5.22) follows. Let $n_k = (L+k)\log_L(L+k)$ and write

$$\begin{aligned} |a_{n_k}^{-1}\phi_0^{(n_k)} - b| &\leq |a_{n_k}^{-1}L^{-\frac{(d-2)}{2}n_{k+1}}\phi_0^{(n_{k+1})}| \\ &\quad + |a_{n_k}^{-1}(\phi_0^{(n_k)} - L^{-\frac{(d-2)}{2}n_{k+1}}\phi_0^{(n_{k+1})}) - b|. \end{aligned} \tag{5.24}$$

Since $\frac{a_{n_{k+1}}}{a_{n_k}}L^{-\frac{(d-2)}{2}n_{k+1}} \longrightarrow 0$, it follows from (5.21) that

$$\begin{aligned} \limsup_{k\to\infty} |a_{n_k}^{-1}L^{-\frac{(d-2)}{2}n_{k+1}}\phi_0^{(n_{k+1})}| &= \limsup_{k\to\infty} \frac{a_{n_{k+1}}}{a_{n_k}} L^{-\frac{(d-2)}{2}n_{k+1}} |\phi_0^{(n_{k+1}))}/a_{n_{k+1}}| \\ &= 0, \text{a.s.} \end{aligned} \tag{5.25}$$

Given $\epsilon > 0$, let $A_k = \{|a_{n_k}^{-1}(\phi_0^{(n_k)} - L^{-\frac{(d-2)}{2}n_{k+1}}\phi_0^{(n_{k+1})}) - b| < \epsilon\}$. Let $|b| < \sigma$, put $\alpha = |b/\sigma|$ and choose $\delta > 0$ so that $\alpha + \delta < 1$. From a property of Gaussian system, there exists $k_\delta = k(\delta)$ such that for $k \geq k_\delta$

$$P(A_k) \geq (n_k d \log L)^{-(\alpha+\delta)} \tag{5.26}$$

and therefore $\sum_{k=1}^\infty P(A_k) = \infty$. Since $\{\phi_0^{(n_k)} - L^{-\frac{d-2}{2}n_{k+1}}\phi_0^{(n_{k+1})}; k \in \mathbf{N}\}$ is independent, Borel-Cantelli's second lemma yields:

$$P[\liminf_{k\to\infty} |a_{n_k}^{-1}(\phi_0^{(n_k)} - L^{-\frac{d-2}{2}n_{k+1}}\phi_0^{(n_{k+1})}) - b| \leq \epsilon] = 1 \tag{5.27}$$

Since ϵ is arbitrary it follows (5.24) - (5.27) that

$$P[\liminf_{k\to\infty} |a_{n_k}^{-1}\phi_0^{(n_k)} - b| = 0] = 1, \tag{5.28}$$

which implies (5.23).

5.2.2 massive hierarchical Gaussian case

In this section, we consider the LIL of massive hierarchical Gaussian case. From the argument of 5.1.2, massive hierarchical Gaussian model has similar system as massless hierarchical Gaussian model. So, we can show the LIL of massive hierarchical Gaussian model as same as massless case.

Theorem 5.2.2 *Let $\tilde{\phi}_0^{(n)} = L^n \phi_0^{(n)}$ be the block spin of the n th scale at the origin, and $\sigma^2 = \frac{1}{L^d-1} \cdot \frac{1}{4d\mu}$. Then if $K_\sigma = [-\sigma, \sigma]$,*

$$P[\lim_{n\to\infty} \operatorname{dist}(\tilde{\phi}_0^{(n)}/(2\log\log L^{dn})^{1/2}, K_\sigma) = 0] = 1, \tag{5.29}$$

$$P[C(\tilde{\phi}_0^{(n)}/(2\log\log L^{dn})^{1/2}) = K_\sigma] = 1, \tag{5.30}$$

where $C(a_n)$ is the all of the cluster points of $\{a_n\}$. In paticular, with probability 1

$$\limsup_{n\to\infty} \tilde{\phi}_0^{(n)}/(2\log\log L^{dn})^{1/2} = \sigma, \liminf_{n\to\infty} \tilde{\phi}_0^{(n)}/(2\log\log L^{dn})^{1/2} = -\sigma,$$
$$\limsup_{n\to\infty} |\tilde{\phi}_0^{(n)}/(2\log\log L^{dn})^{1/2}| = \sigma.$$

Bibliography

[1] Alejandro. de Acosta. : A New Proof of the Hartman-Wintner Law of the Iterated Logarithm. Ann of Prob, Vol. 11. No. 2, 270-276 (1983)

[2] P. M. Bleher. and Y. G. Sinai. : Investigation of the critical point in models of the type of Dyson's hierarchical model. Commun. Math. Phys. 33, 23-42 (1973)

[3] P. M. Bleher. and Y. G. Sinai. : Critical indices for Dyson's asymptotically hierarchical models. Commun. Math. Phys. 45, 247-278 (1975)

[4] F.J. Dyson.: Existence of a phase-transition in a one-dimensional Ising ferromagnet.Commun. Math. Phys. 12,91-107(1969)

[5] K. Gawędzki. and A. Kupiainen. : Renormalization Group Study of a Critical Lattice Moedel. I. Convergence to the Line of Fixed Points. Commun. Math. Phys. 82, 407-433 (1981)

[6] K. Gawędzki. and A. Kupiainen. : Renormalization Group Study of a Critical Lattice Moedel. II. The Correlation Functions. Commun. Math. Phys. 83, 469-492 (1982)

[7] K. Gawędzki. and A. Kupiainen. : Triviality of ϕ_4^4 and all that in a hierarchical model approximation. J. Stat. Phys. 29, 683-699 (1982)

[8] K. Gawędzki. and A. Kupiainen. : Non-Trivial Cnotinuum Limit of a ϕ_4^4 Model with Negative Coupling Constant. Nuclear Physics B257 [FS14] (1985) 474-504

[9] T. Hara. T. Hattori. and H. Watanabe. : Triviality of Hierarchical Ising model in Four Dimensions. Commun. Math. Phys. 220, 13-40 (2001)

[10] V. F. Müller. and J. Schiemann. : Infrared Asymptotic Freedom of a Hierarchical ϕ_3^6 Lattice Theory. J. Stat. Phys. 43, 123-142 (1986)

[11] C. Newman. : Zeros of the Partition Function for Generalized Ising Systems. Commun. Pure Appl. Math. 27, 143-159 (1974)

[12] C. Newman. : Inequalities for Ising Models and Field Theories which Obey the Lee-Yang Theorem. Commun. Math. Phys. 41, 1-9 (1975)

[13] K. G. Wilson. and J. Kogut. : The Renormalization Group and The ϵ-Expansion. Physics Reports 12, No.2, 75-200 (1974)