Spacelike Mean Curvature One Surfaces in de Sitter 3-Space

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Spacelike Mean Curvature One Surfaces in de Sitter 3-Space

3次元ドジッター空間内の平均曲率1を持つ空間的曲面について

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Contents

CHAPTER 1

Introduction

This thesis is concerned with spacelike surfaces of constant mean curvature (CMC) 1 with singularities in de Sitter 3-space.

It is known that there is a representation formula, using holomorphic null immersions into $SL(2,\mathbb{C})$, for spacelike CMC 1 immersions in de Sitter 3-space $\mathbb{S}^3_1$ [**AA**]. Since this formula plays a crucial role in this thesis, we first review the local theory of spacelike immersions in $\mathbb{S}^3_1$ and give a proof of the formula in Chapter 2.

Although this formula is very similar to a representation formula for CMC 1 immersions in hyperbolic 3-space $\mathbb{H}^3$ (the so-called Bryant representation formula [**B, UY1**]), and the global properties of CMC 1 immersions in $\mathbb{H}^3$ have been investigated [**CHR, RUY2, RUY3, UY1, UY2, UY3, Yu**], global properties and singularities of spacelike CMC 1 immersions in $\mathbb{S}^3_1$ are not yet well understood. Perhaps one of the biggest reasons for this is that the only complete spacelike CMC 1 immersion in $\mathbb{S}^3_1$ is the flat totally umbilic immersion [**Ak, Ra**], causing researchers to mistakenly assume there is little to say about the $\mathbb{S}^3_1$ case, or at least to ignore this case because it is lacking in numerous complete embeddings. This situation is somewhat parallel to the relation between minimal immersions in Euclicean 3-space $\mathbb{R}^3$ and spacelike maximal immersions in Lorentz 3-space $\mathbb{R}^3_1$, the $\mathbb{R}^3$ case having been extensively studied while the $\mathbb{R}^3_1$ case still being not so well understood. (It is known that the only complete spacelike maximal immersion in $\mathbb{R}^3_1$ is a plane.)

So to have an interesting global theory about spacelike CMC 1 surfaces in $\mathbb{S}^3_1$, we need to consider a wider class of surfaces than just complete and immersed ones. Recently, Umehara and Yamada defined spacelike maximal surfaces with certain kinds of "admissible" singularities, and named them "maxfaces" [**UY4**]. They then showed that maxfaces are rich objects with respect to global geometry. So in Chapter 3, we introduce admissible singularities of spacelike CMC 1 surfaces in $\mathbb{S}^3_1$ and name these surfaces with admissible singularities "CMC 1 faces", and we extend the representation of Aiyama-Akutagawa to CMC 1 faces.

Then we investigate properties of CMC 1 faces. In particular, we investigate singularities of CMC 1 faces and give criteria for singular points, and investigate the global theory of CMC 1 faces and prove the Osserman-type inequality. Also, we construct numerous examples, using a method for transferring CMC 1 immersions in $\mathbb{H}^3$ to CMC 1 faces in $\mathbb{S}^3_1$. Furthermore, we numerically construct CMC 1 faces of genus 1 with two ends.

Singularities. In Chapter 4, we shall give criteria for cuspidal cross caps in frontals (Theorems 4.1.2 and 4.1.3), and then give applications of that.

CMC 1 faces are frontal maps. In a neighborhood of a singular point, a CMC 1 face is represented by a holomorphic function using the representation formula. In fact, for a holomorphic function $h \in \mathcal{O}(U)$ defined on a simply connected domain $U \subset \mathbb{C}$, there is a CMC 1 face f_h with Weierstrass data $(g = e^h, \omega = dz)$, where $\mathcal{O}(U)$ is the set of holomorphic functions on U and z is a complex coordinate of U. Conversely, for a neighborhood of a singular point of a CMC 1 face f, there exists an $h \in \mathcal{O}(U)$ such that $f = f_h$. For precise descriptions, see Section 4.3.

This is somewhat analogous to the work [**UY4**], where Umehara and Yamada introduced a notion of maxface as a class of spacelike maximal surfaces in $\mathbb{R}^3_1$ with singularities (see Section B.1 and [**UY4**]), and it is shown there that such a map is a frontal map. Like the case of CMC 1 faces, maxfaces near a singular point are represented by holomorphic functions; that is, for $h \in \mathcal{O}(U)$, there is a maxface f_h. Conversely, for a neighborhood of a singular point of a maxface f, there exists a holomorphic function h such that $f = f_h$.

Endowing the set $\mathcal{O}(U)$ of holomorphic functions on U with the compact open C^∞-topology, we shall show that cuspidal edges, swallowtails and cuspidal cross caps are generic singularities of CMC 1 faces in $\mathbb{S}^3_1$ (Theorem 4.3.8) and maxfaces in $\mathbb{R}^3_1$ (Corollary B.1.5).

We remark that conelike singularities of maximal surfaces, although not generic, are still important singularities, which are investigated by O. Kobayashi [**Kob**], Fernández-López-Souam [**FLS1, FLS2**] and others.

In [**SUY**], Saji, Umehara and Yamada investigated the behavior of the Gaussian curvature near cuspidal edges. We shall remark on how the behavior of the Gaussian curvature near a cuspidal cross cap is almost the same as that of a cuspidal edge (see Proposition 4.2.10).

The Osserman-type inequality. In Chapter 5 we investigate the global theory of CMC 1 faces and prove the Osserman-type inequality. For CMC 1 immersions in $\mathbb{H}^3$, the monodromy representation at each end is always diagonalizable with eigenvalues in $\mathbb{S}^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$, but this is not true for CMC 1 faces in general. However, when the monodromy representation at each end of a CMC 1 face is diagonalizable with eigenvalues in $\mathbb{S}^1$, we can directly apply many results for CMC 1 immersions in $\mathbb{H}^3$ to CMC 1 faces. So in Section 5.1, we give a definition of such "elliptic ends" of CMC 1 faces. In Section 5.2, we give a necessary and sufficient condition for elliptic ends to be embedded.

The total curvature of complete minimal immersions in $\mathbb{R}^3$ of finite total curvature satisfies the Osserman inequality [**O1**, Theorem 3.2].

Furthermore, the condition for equality was given in [**JM**, Theorem 4]. This inequality is a stronger version of the Cohn-Vossen inequality. For a minimal immersion in $\mathbb{R}^3$, the total curvature is equal to the degree of its Gauss map, multiplied by -4π. So the Osserman inequality can be viewed as an inequality about the degree of the Gauss map. In the case of a CMC 1 immersion in $\mathbb{H}^3$, the total curvature never satisfies equality of the Cohn-Vossen inequality [**UY1**, Theorem 4.3], and the Osserman inequality does not hold in general. However, using the hyperbolic Gauss map instead of the total curvature, an Osserman type inequality holds for CMC 1 immersions in $\mathbb{H}^3$ [**UY3**]. Also, Umehara and Yamada showed that the Osserman inequality holds for maxfaces [**UY4**]. In Section 5.3, we show that the Osserman inequality holds for complete CMC 1 faces of finite type with elliptic ends (Theorem 5.3.7). The Osserman inequality for complete minimal immersions in $\mathbb{R}^3$ says that twice the degree of the Gauss map is greater than or equal to the number of ends minus the Euler characteristic of the surface, with equality holding if and only if all the ends are embedded. Osserman-type inequalities for CMC 1 immersions in $\mathbb{H}^3$ and maxfaces in $\mathbb{R}^3_1$ can be found in [**UY1, UY3**] and [**UY4**]. We remark that the assumptions of finite type and ellipticity of the ends in Theorem 5.3.7 can actually be removed, because, in fact, any complete CMC 1 face must be of finite type. This will be shown in the forthcoming paper [**FRUYY**].

Constructing examples. In Chapter 6 we construct examples of CMC 1 faces. Lee and Yang were the first to construct a numerous collection of examples of CMC 1 faces [**LY**]. For example, they constructed complete irreducible CMC 1 faces with three elliptic ends, by using hypergeometric functions [**LY**]. We will also give numerous examples here in Section 6.3, by using a method for transferring CMC 1 immersions in $\mathbb{H}^3$ to CMC 1 faces in $\mathbb{S}^3_1$. Applying this method to examples in [**MU, RUY2, RUY3**], we give many examples of complete reducible CMC 1 faces of finite type with elliptic ends.

Positive genus examples. As noted above, in [**UY4**] Umehara and Yamada introduced a notion of maxfaces as a category of spacelike maximal surfaces with certain kinds of singularities. Then they constructed numerous examples by a transferring method from minimal surfaces in $\mathbb{R}^3$. Furthermore, Kim and Yang discovered an interesting example of a maxface, which has genus 1 with two embedded ends, even though there does not exist such an example as a complete minimal immersion in $\mathbb{R}^3$ [**KY1**].

As said above, the author constructed many examples by transferring from reducible CMC 1 surfaces in $\mathbb{H}^3$ [**F2**], and Lee and Yang investigated spacelike CMC 1 surfaces of genus zero with two and three ends [**LY**]. However, every surface constructed in [**F2**] and [**LY**] was topologically a sphere with finitely many points removed. Given all of this, it is natural to consider whether or not there exist examples with positive genus.

For CMC 1 immersions in $\mathbb{H}^3$, Rossman and Sato constructed genus 1 catenoid cousins by a numerical method [**RS**]. Here we will similarly construct genus 1 "catenoids" using a modification of their method; that is, we show the following numerical result: There exist one-parameter families of weakly-complete CMC 1 faces of genus 1 with two elliptic or two hyperbolic ends which satisfy equality in the Osserman-type inequality.

The examples here satisfy equality in the Osserman-type inequality, even though some of them do not have elliptic ends. (We define elliptic and hyperbolic and parabolic ends in Section 5.1.) This is in accordance with the results in the fothcoming paper [**FRUYY**].

For weakly-complete CMC 1 faces, the behavior of ends is investigated in [**FRUYY**].

Outline of this thesis. This thesis consists of 6 chapters and two appendices. Chapter 1 is this introduction. In Chaper 2, we review the local theory of spacelike immersions in $\mathbb{S}^3_1$. Chapter 3 is based on Section 1 of [**F2**]. Chapter 4 is based on Sections 1 and 3 of [**FSUY**]. Chapter 5 is based on Sections 2–4 of [**F2**]. Chapter 6 is based on Section 4 of [**F2**] and Section 2 of [**F3**]. In Appendix A, we review the local theory of (spacelike) immersions in $\mathbb{R}^3$, $\mathbb{H}^3$ and $\mathbb{R}^3_1$. Appendix B is based on Section 2 and Appendix A of [**FSUY**].

CHAPTER 2

Local theory of spacelike surfaces in de Sitter space

2.1. de Sitter space

2.1.1. de Sitter 3-space. Let $\mathbb{R}^4_1$ be the 4-dimensional Lorentz (Minkowski) space with the Lorentz metric

$$\langle (x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3) \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Then de Sitter 3-space is

$$\mathbb{S}^3_1 = \mathbb{S}^3_1(1) = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1 \,|\, -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\},$$

with the metric induced from $\mathbb{R}^4_1$. $\mathbb{S}^3_1$ is a simply-connected 3-dimensional Lorentzian manifold with constant sectional curvature 1.

2.1.2. Geodesics. Let $I \subset \mathbb{R}$ and $\gamma : I \ni t \mapsto \gamma(t) \in \mathbb{S}^3_1$ be a curve in $\mathbb{S}^3_1$ such that $\gamma(0) =: p$ and $\dot\gamma(0) =: v \in T_p\mathbb{S}^3_1$. We have the following:

Proposition 2.1.1. *Assume that γ is a geodesic in $\mathbb{S}^3_1$ with $\gamma(0) = p$ and $\dot\gamma(0) = v \in T_p\mathbb{S}^3_1$.*

(1) *If γ is a spacelike geodesic with $\langle v, v\rangle = 1$, then γ can be given by*

$$\gamma(t) = (\cos t)p + (\sin t)v.$$

(2) *If γ is a timelike geodesic with $\langle v, v\rangle = -1$, then γ can be given by*

$$\gamma(t) = (\cosh t)p + (\sinh t)v.$$

(3) *If γ is a lightlike geodesic, then γ can be given by*

$$\gamma(t) = p + tv.$$

Note that each geodesic is complete.

2.1.3. The Hermitian matrix model. We can consider $\mathbb{R}^4_1$ to be the 2×2 self-adjoint matrices ($X^* = X$), by the identification

(2.1.1)

$$\mathbb{R}^4_1 \ni X = (x_0, x_1, x_2, x_3) \leftrightarrow X = \sum_{k=0}^{3} x_k e_k = \begin{pmatrix} x_0 + x_3 & x_1 + i x_2 \\ x_1 - i x_2 & x_0 - x_3 \end{pmatrix},$$

where

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\mathbb{S}^3_1$ is

$$\mathbb{S}^3_1 = \{X \mid X^* = X \, , \det X = -1\}$$

with the metric

$$\langle X, Y \rangle = -\frac{1}{2}\text{trace}(X e_2 Y^t e_2).$$

In particular, $|X|^2 = \langle X, X \rangle = -\det X$.

LEMMA 2.1.2. *$\mathbb{S}^3_1$ can be written as* $\mathbb{S}^3_1 = \{F e_3 F^* \mid F \in SL(2, \mathbb{C})\}$.

PROOF. We must show

$$\{X \mid X^* = X \, , \det X = -1\} = \{F e_3 F^* \mid F \in SL(2, \mathbb{C})\}.$$

Any element of the right-hand side is obviously an element of the left-hand side. We show the converse. For any X in the left-hand side, There exists a $E \in SU(2)$ so that

$$X = E \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} E^*,$$

where λ_1 and λ_2 are the eigenvalues of X. Since $X = X^*$, both λ_1, λ_2 must be real. Since $\det X = -1$, the eigenvalues of X can be written as λ and $-1/\lambda$ for some $\lambda > 0$. Thus

$$X = E \begin{pmatrix} \lambda & 0 \\ 0 & -1/\lambda \end{pmatrix} E^*,$$

so $X = F e_3 F^*$, where

$$F = E \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 1/\sqrt{\lambda} \end{pmatrix} \in SL(2, \mathbb{C}),$$

because $\lambda > 0$. $\square$

2.1.4. The hollow ball model. To visualize spacelike immersions in $\mathbb{S}^3_1$, we use the *hollow ball model* of $\mathbb{S}^3_1$, as in [**KY2**]. For any point

$$\begin{pmatrix} x_0 + x_3 & x_1 + i x_2 \\ x_1 - i x_2 & x_0 - x_3 \end{pmatrix} \leftrightarrow (x_0, x_1, x_2, x_3) \in \mathbb{S}^3_1,$$

define

$$y_k = \frac{e^{\arctan x_0}}{\sqrt{1 + x_0^2}} x_k, \qquad k = 1, 2, 3.$$

Then $e^{-\pi} < y_1^2 + y_2^2 + y_3^2 < e^{\pi}$. The identification $(x_0, x_1, x_2, x_3) \leftrightarrow (y_1, y_2, y_3)$ is then a bijection from $\mathbb{S}^3_1$ to the hollow ball

$$\mathscr{H} = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid e^{-\pi} < y_1^2 + y_2^2 + y_3^2 < e^{\pi}\}.$$

So $\mathbb{S}^3_1$ is identified with the hollow ball $\mathscr{H}$, and we show the graphics here using this identification to $\mathscr{H}$.

2.1.5. The slab model. The slab model of $\mathbb{S}^3_1$ is defined as

$$\mathscr{S} := \{(t, z) \mid -1 < t < 1,\, z \in (\mathbb{C} \cup \{\infty\})\}$$

with the metric

$$ds^2 = \left(\frac{2}{t^2-1}\right)^2 \left\{-dt^2 + \left(\frac{t^2+1}{z\bar{z}+1}\right)^2 dz d\bar{z}\right\}.$$

The correspondence between $\mathscr{S}$ and $\mathbb{S}^3_1$ is given by

$$\mathscr{S} \ni (t, z) \mapsto \left(\frac{-2t}{t^2-1}, \frac{t^2+1}{t^2-1}\frac{z+\bar{z}}{z\bar{z}+1}, \frac{t^2+1}{t^2-1}\frac{i(\bar{z}-z)}{z\bar{z}+1}, \frac{t^2+1}{t^2-1}\frac{z\bar{z}-1}{z\bar{z}+1}\right) \in \mathbb{S}^3_1$$

and

$$\mathscr{S} \ni \left(\frac{\sqrt{1+x_0^2}-1}{x_0}, \frac{x_1+ix_2}{\sqrt{1+x_0^2}-x_3}\right) \hookleftarrow (x_0, x_1, x_2, x_3) \in \mathbb{S}^3_1.$$

2.2. Spacelike immersions in $\mathbb{S}^3_1$

2.2.1. Fundamental equations. Let $D \subset \mathbb{C}$ be a simply-connected domain (since we only study a local theory in this chapter, we always assume that D is simply-connected) with complex coordinate $z = x + iy$. Let $f : D \to \mathbb{S}^3_1$ be a *spacelike* immersion, that is, f is an immersion and the induced metric $ds^2 = \langle df, df\rangle$ is positive definite. Without loss of generality we may assume f is conformal. Then there exists a smooth function $u : D \to \mathbb{R}$ so that

$$ds^2 = e^{2u} dz d\bar{z} = (dz, d\bar{z}) \mathrm{I} \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix},$$

where

$$\mathrm{I} = \begin{pmatrix} \langle f_z, f_z\rangle & \langle f_z, f_{\bar{z}}\rangle \\ \langle f_{\bar{z}}, f_z\rangle & \langle f_{\bar{z}}, f_{\bar{z}}\rangle \end{pmatrix} = \frac{1}{2} e^{2u} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Namely,

$$\langle f_z, f_z\rangle = \langle f_{\bar{z}}, f_z\rangle = 0 \quad \text{and} \quad \langle f_z, f_{\bar{z}}\rangle = \frac{1}{2} e^{2u},$$

where

$$f_z = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right),$$

and $\langle \cdot, \cdot\rangle$ is extended linearly to the complexification of $T_p\mathbb{S}^3_1$ for $p \in \mathbb{S}^3_1$. For each $p \in D$, let $N(p)$ be a unit normal vector of f at p. Then $N(p) \in T_p\mathbb{S}^3_1$ is orthogonal to the tangent plane $f_*(T_pD)$ of f at p. Note that N is timelike, that is, $\langle N, N\rangle = -1$, since f is spacelike.

We choose N so that it is future pointing, that is, so that the first coordinate of N is positive. Then

(2.2.1)

$$N : D \to \mathbb{H}^3 := \{n = (n_0, n_1, n_2, n_3) \in \mathbb{R}^4_1 \mid \langle n, n\rangle = -1\ ,\ \ n_0 > 0\}.$$

The second fundamental form h of f is defined by

$$h := -\langle df, dN\rangle = (dz, d\bar{z})\mathrm{I\!I}\begin{pmatrix} dz \\ d\bar{z}\end{pmatrix},$$

where

$$\mathrm{I\!I} = \begin{pmatrix} \langle f_{zz}, N\rangle & \langle f_{z\bar{z}}, N\rangle \\ \langle f_{z\bar{z}}, N\rangle & \langle f_{\bar{z}\bar{z}}, N\rangle \end{pmatrix}.$$

Also, the shape operator S of f is defined by $S := \mathrm{I}^{-1}\mathrm{I\!I}$.

Definition 2.2.1. The mean curvature H of f and the Hopf differntial Q of f are defined as

$$H := \frac{1}{2}\mathrm{trace}S = 2e^{-2u}\langle f_{z\bar{z}}, N\rangle \quad \text{and} \quad Q = q dz^2 = \langle f_{zz}, N\rangle dz^2.$$

Note that although q depends on the choice of complex coordinate, Q and H are independent on the choice of complex coordinate.

Let $\mathcal{F} := (N, f_z, f_{\bar{z}}, f)$ be the moving frame of f. Then we have

$$\mathcal{F}_z = \mathcal{F}\mathcal{U} \quad \text{and} \quad \mathcal{F}_{\bar{z}} = \mathcal{F}\mathcal{V},$$

where

$$\mathcal{U} = \begin{pmatrix} 0 & -q & -e^{2u}H/2 & 0 \\ -H & 2u_z & 0 & 1 \\ -2e^{-2u}q & 0 & 0 & 0 \\ 0 & 0 & -e^{2u}/2 & 0 \end{pmatrix}$$

and

$$\mathcal{V} = \begin{pmatrix} 0 & -e^{2u}H/2 & -\bar{q} & 0 \\ -2e^{-2u}\bar{q} & 0 & 0 & 0 \\ -H & 0 & 2u_{\bar{z}} & 1 \\ 0 & -e^{2u}/2 & 0 & 0 \end{pmatrix}.$$

This is equivalent to the following Gauss-Weingarten equations:

$$\begin{cases} f_{zz} = -qN + 2u_z f_z, \\ f_{z\bar{z}} = -\frac{1}{2}e^{2u}HN - \frac{1}{2}e^{2u}f, \\ f_{\bar{z}\bar{z}} = -\bar{q}N + 2u_{\bar{z}}f_{\bar{z}}, \end{cases} \quad \text{and} \quad \begin{cases} N_z = -Hf_z - 2e^{-2u}qf_{\bar{z}}, \\ N_{\bar{z}} = -2e^{-2u}\bar{q}f_z - Hf_{\bar{z}}. \end{cases}$$

Therefore

$$\mathrm{I\!I} = \begin{pmatrix} q & e^{2u}H/2 \\ e^{2u}H/2 & \bar{q} \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} H & 2e^{-2u}\bar{q} \\ 2e^{-2u}q & H \end{pmatrix},$$

and hence

$$h = q dz^2 + \bar{q}d\bar{z}^2 + e^{2u}H dz d\bar{z} = Q + \bar{Q} + H ds^2.$$

The Gauss-Codazzi equation, that is, the integrability condition $(\mathcal{F}_z)_{\bar{z}} = (\mathcal{F}_{\bar{z}})_z$, which is equivalent to

$$\mathcal{U}_{\bar{z}} - \mathcal{V}_z - [\mathcal{U}, \mathcal{V}] = 0,$$

have the following form:

$$2u_{z\bar{z}} + \frac{1}{2}e^{2u}(1 - H^2) + 2e^{-2u}q\bar{q} = 0, \tag{2.2.2}$$

$$H_z + 2e^{-2u}q_{\bar{z}} = 0. \tag{2.2.3}$$

(2.2.2) is called the Gauss equation and (2.2.3) is called the Codazzi equation.

The Gaussian curvature of $ds^2 = e^{2u}dzd\bar{z}$ is defined as

$$K = -e^{-2u}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = -4e^{-2u}u_{z\bar{z}}. \tag{2.2.4}$$

(See (A.1.5).) So the Gauss equation (2.2.2) is written as

$$K = -H^2 + 4e^{-4u}q\bar{q} + 1 = -\det S + 1. \tag{2.2.5}$$

2.2.2. The 2 by 2 Lax pair for f. Now we use the Hermitian matrix model of $\mathbb{S}^3_1$. Namely, we identify a point $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1$ with the matrix

$$\sum_{k=0}^{3} x_k e_k = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}.$$

Then the metric is given by

$$\langle X, Y\rangle = -\frac{1}{2}\text{trace}(Xe_2Y^te_2), \quad \langle X, X\rangle = -\det X$$

for all X, Y in $\mathbb{S}^3_1$. The following proposition is immediate:

PROPOSITION 2.2.2. *If $F \in SL(2,\mathbb{C})$, then $\langle X, Y\rangle = \langle FXF^*, FYF^*\rangle$ for all X, Y in $\mathbb{S}^3_1$.*

We also have the following proposition:

PROPOSITION 2.2.3. *There exists an $F \in SL(2,\mathbb{C})$ (unique up to sign $\pm F$) so that*

$$N = FF^*, \quad \frac{f_x}{|f_x|} = Fe_1F^*, \quad \frac{f_y}{|f_y|} = Fe_2F^*, \quad f = Fe_3F^*,$$

where $z = x + iy$.

Therefore, choosing F as in Proposition 2.2.3, we have

$$f_x = e^uFe_1F^* \quad \text{and} \quad f_y = e^uFe_2F^*,$$

and so

$$f_z = e^uF\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}F^* \quad \text{and} \quad f_{\bar{z}} = e^uF\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}F^*.$$

Define the Lax pair U, V as

$$F^{-1}dF = Udz + Vd\bar{z}.$$

Then we have

$$U = \frac{1}{2}\begin{pmatrix} u_z & -e^u(1+H) \\ -2e^{-u}q & -u_z \end{pmatrix} \quad \text{and} \quad V = \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & -2e^{-u}\bar{q} \\ e^u(1-H) & u_{\bar{z}} \end{pmatrix}.$$

2.2.3. Totally umbilic immersions. A point $p \in D$ is an umbilic point of a spacelike immersion $f : D \to \mathbb{S}^3_1$ if $q(p) = 0$, and an immersion is called totally umbilic if $q \equiv 0$. Let $f : D \to \mathbb{S}^3_1$ be a totally umbilic immersion. Then

$$U = \frac{1}{2}\begin{pmatrix} u_z & -e^u(1+H) \\ 0 & -u_z \end{pmatrix} \quad \text{and} \quad V = \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & 0 \\ e^u(1-H) & u_{\bar{z}} \end{pmatrix}.$$

Defining functions $F_{jk} : D \to \mathbb{C}$ $(j, k = 1, 2)$ so that

$$F = e^{u/2}\begin{pmatrix} \bar{F}_{11} & F_{12} \\ \bar{F}_{21} & F_{22} \end{pmatrix}$$

holds, then $\bar{F}_{11}F_{22} - F_{12}\bar{F}_{21} = e^{-u}$, because $F \in SL(2, \mathbb{C})$.

LEMMA 2.2.4. *All of F_{jk} $(j, k = 1, 2)$ are holomorphic functions.*

PROOF. Since $U = F^{-1}F_z$, we have

$$\frac{1}{2}\begin{pmatrix} u_z & -e^u(1+H) \\ 0 & -u_z \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} u_z + 2e^u(F_{22}(\bar{F}_{11})_z - F_{12}(\bar{F}_{21})_z) & 2e^u(F_{22}(F_{12})_z - F_{12}(F_{22})_z) \\ 2e^u(\bar{F}_{11}(\bar{F}_{21})_z - \bar{F}_{21}(\bar{F}_{11})_z) & u_z + 2e^u(\bar{F}_{11}(F_{22})_z - \bar{F}_{21}(F_{12})_z) \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} F_{22} & -F_{12} \\ -\bar{F}_{21} & \bar{F}_{11} \end{pmatrix}\begin{pmatrix} (\bar{F}_{11})_z \\ (\bar{F}_{21})_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and so $(\bar{F}_{11})_z = (\bar{F}_{21})_z = 0$, that is, F_{11} and F_{21} are holomorphic. Similarly, $V = F^{-1}F_{\bar{z}}$ implies that $(F_{12})_{\bar{z}} = (F_{22})_{\bar{z}} = 0$, that is, F_{12} and F_{22} are holomorphic, proving the proposition. □

2.2.4. CMC 1 immersions. Spacelike immersions of constant mean curvature (CMC) 1 in $\mathbb{S}^3_1$ have the special representation formula, the so-called Aiyama-Akutagawa representation. In this subsection we prove that formula. First, we show the following lemma:

LEMMA 2.2.5. *Let $f : D \to \mathbb{S}^3_1$ be a spacelike immersion. If f has constant mean curvature $H = 1$, then there exists a $B : D \to SU(1, 1)$ so that $(FB)_{\bar{z}} = 0$.*

REMARK. Note that multiplying by B will not change the immersion f, as $f = Fe^*_F = (FB)e_3(FB)^*$, even though the first three properties in Proposition 2.2.3 will no longer hold.

PROOF OF LEMMA 2.2.5. We first assume $H = 1$. Then we need B to satisfy

$$B_{\bar{z}} = -VB. \tag{2.2.6}$$

Consider also the equation

$$B_z = WB. \tag{2.2.7}$$

Then the Lax pair (2.2.6)-(2.2.7) is equivalent to

$$B_x = (W - V)B \quad \text{and} \quad B_y = i(W + V)B. \tag{2.2.8}$$

A simple computation shows that

$$W - V\,,\ i(W + V) \in \mathfrak{su}(1, 1) \quad \text{if and only if} \quad W = e_3V^*e_3,$$

so we set $W = e_3V^*e_3$. Then we have that

$$V_z + (e_3V^*e_3)_{\bar{z}} + [V, e_3V^*e_3] = \frac{1}{2}e^{2u}(1 - H)e_3.$$

So if $H \equiv 1$, the compatibility condition $V_z+(e_3V^*e_3)_{\bar{z}}+[V, e_3V^*e_3] = 0$ for the Lax pair (2.2.8) (or equivalently, for the Lax pair (2.2.6)-(2.2.7)) holds. Thus an analog of [**FKR**, Proposition 3.1.2], with $\mathfrak{sl}(n,\mathbb{C})$ and $SL(n,\mathbb{C})$ replaced by $\mathfrak{su}(1, 1)$ and $SU(1, 1)$, implies that there exists a solution $B \in SU(1, 1)$ of the Lax pair (2.2.6)-(2.2.7). In particular, (2.2.6) has a solution $B \in SU(1, 1)$. □

Now, since f has constant mean curvature $H = 1$, writing that solution B as

$$B = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad a\bar{a} - b\bar{b} = 1,$$

we have

$$(FB)^{-1}(FB)_z = B^{-1}(U + e_3V^*e_3)B = e^u \begin{pmatrix} -\bar{a}\bar{b} & -\bar{a}^2 \\ \bar{b}^2 & \bar{a}\bar{b} \end{pmatrix},$$

which must be holomorphic, because FB is. Define

$$\omega = wdz = e^u\bar{b}^2dz, \quad g = -\frac{\bar{a}}{\bar{b}}, \quad \tilde{F} = FB.$$

(Note that $|g|^2 > 1$, since $|a|^2 - |b|^2 = 1$.) Then $\tilde{F}$ is holomorphic in z and

$$\tilde{F}^{-1}d\tilde{F} = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix}\omega, \tag{2.2.9}$$

as in the Aiyama-Akutagawa representation in [**AA**]. Note that ω is a holomorphic 1-form and g is a meromorphic function, by (2.2.9). Moreover, the set of poles of g is the set of zeros of w, and each pole of g with order k is a zero of w with order $2k$, by definiton of g and w (Note that it is equivalent to

$$d\hat{s}^2 := (1 + |g|^2)^2\omega\bar{\omega} \tag{2.2.10}$$

giving a Riemannian metric on D). Also,

$$(1-|g|^2)^2\omega\bar{\omega} = e^{2u}dzd\bar{z},$$

so we also have the metric determined from the g and ω used in the Aiyama-Akutagawa representation. Furthermore, comparing the upper-right entries of V and $-B_{\bar{z}}B^{-1}$, we have

$$q = e^u(\bar{b}_z\bar{a} - \bar{a}_z\bar{b}) = g_z w,$$

because $V = -B_{\bar{z}}B^{-1}$.

Conversely, starting with any $\tilde{F}$ solving (2.2.9), one can check that a spacelike CMC 1 immersion Fe_3F^* is obtained. We have just shown the folowing:

THEOREM 2.2.6 (The representation of Aiyama-Akutagawa). *Let D be a simply-connected domain in $\mathbb{C}$ with a base point $z_0 \in D$. Let $g : D \to (\mathbb{C} \cup \{\infty\}) \setminus \{z \in \mathbb{C} \,|\, |z| \leq 1\}$ be a meromorphic function and ω a holomorphic 1-form on D so that ω has a zero of order $2k$ if and only if g has a pole of order k and so that ω has no other zeros. Choose the holomorphic immersion $F = (F_{jk}) : D \to SL(2,\mathbb{C})$ so that $\tilde{F}(z_0) = e_0$ and $\tilde{F}$ satisfies (2.2.9). Then $f : D \to \mathbb{S}^3_1$ defined by $f = \tilde{F}e_3\tilde{F}^*$ is a conformal spacelike CMC 1 immersion. The induced metric $ds^2 = f^*(ds^2_{\mathbb{S}^3_1})$ on D and the second fundamental form h, and the Hopf differential Q of f are given as follows:*

$$ds^2 = (1-|g|^2)^2\omega\bar{\omega}, \quad h = Q + \bar{Q} + ds^2, \quad Q = \omega dg. \tag{2.2.11}$$

Conversely, any simply-connected spacelike CMC 1 immersion can be represented this way.

REMARK. We note that choosing $H = 1$ was essential to proving the Aiyama-Akutagawa representation above.

REMARK 2.2.7. We make the following remarks about Theorem 2.2.6:

(1) Equations (2.2.4) and (2.2.5) imply that the Gaussian curvature for a spacelike CMC 1 immersion in $\mathbb{S}^3_1$ is always non-negative. This is like the case in maximal immersions in Lorentz 3-space $\mathbb{R}^3_1$, and unlike the cases in minimal immersions in Euclidean 3-space $\mathbb{R}^3$ and CMC 1 immersions in hyperbolic 3-space $\mathbb{H}^3$.

(2) Following the terminology of Umehara and Yamada, g is called the *secondary* Gauss map. Also, the pair (g,ω) is called the *Weierstrass data.*

(3) The future pointing unit normal vector field N of f is given as

$$N = \frac{1}{|g|^2-1}\tilde{F}\begin{pmatrix} |g|^2+1 & 2g \\ 2\bar{g} & |g|^2+1 \end{pmatrix}\tilde{F}^*.$$

(4) For the holomorphic immersion $\tilde{F}$ satisfying (2.2.11), $\hat{f} := FF^* : D \to \mathbb{H}^3$ is a conformal CMC 1 immersion with first

fundamental form $\hat{f}^*(ds^2_{\mathbb{H}^3}) = d\hat{s}^2 = (1+|g|^2)^2\omega\bar{\omega}$, and with second fundamental form $\hat{h} = -\hat{Q} - \overline{\hat{Q}} + d\hat{s}^2$, where $\hat{Q} = \omega dg$ is the Hopf differential of $\hat{f}$.

(5) By Equation (2.6) in [**UY1**], G and g and Q have the following relation:

$$2Q = S(g) - S(G), \tag{2.2.12}$$

where $S(g) = S_z(g)dz^2$ and

$$S_z(g) = \left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2 \qquad (' = d/dz)$$

is the Schwarzian derivative of g.

2.3. The hyperbolic Gauss map

Let $f : D \to \mathbb{S}^3_1$ be a spacelike immersion and N the future pointing unit normal vector field.

Definition 2.3.1. The hyperbolic Gauss map $G : D \to \mathbb{C} \cup \{\infty\}$ is defined

$$G = \frac{b}{c}, \quad \text{where} \quad f + N = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}. \tag{2.3.1}$$

The hyperbolic Gauss map has the following geometric meaning: The future pointing unit normal vector $N = FF^*$ is both perpendicular to the surface $f = Fe_3F^*$ in $\mathbb{S}^3_1$ and tangent to the space $\mathbb{S}^3_1$ in $\mathbb{R}^4_1$ at each point $f \in \mathbb{S}^3_1$. Let P be the unique 2-dimensional plane in $\mathbb{R}^4_1$ containing the three points $(0,0,0,0)$ and f and $f+N$. Then P contains the geodesic γ in $\mathbb{S}^3_1$ that starts at f and extends in the direction of N (this follows from the fact that all isometries of $\mathbb{S}^3_1$ are of the form AfA^* for all $A \in SL(2,\mathbb{C})$, and there is a rotation of $\mathbb{R}^4_1$ that moves $P \cap \mathbb{S}^3_1$ to the geodesic $\{(\cosh t, \sinh t, 0, 0) \in \mathbb{R}^4_1 \mid t \in \mathbb{R}\}$ that lies in a plane). This geodesic γ has a limiting direction in the upper half-cone $\mathbb{N}^+ = \{(x_0,x_1,x_2,x_3) \in \mathbb{R}^4_1 \mid -x_0^2+x_1^2+x_2^2+x_3^2 = 0,\ x_0 \geq 0\}$ of the light cone of $\mathbb{R}^4_1$. This limiting direction, in the direction of $f+N$, is the hyperbolic Gauss map of f. We identify the set of limiting directions with $\mathbb{S}^2$ by associating the direction $[(x_0,x_1,x_2,x_3)] := \{\alpha(x_0,x_1,x_2,x_3) \mid \alpha \in \mathbb{R}^+\}$ for $(x_0,x_1,x_2,x_3) \in \mathbb{N}^+$ with the point $(x_1/x_0, x_2/x_0, x_3/x_0) \in \mathbb{S}^2$; that is, we identify

$$\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in \mathbb{N}^+ \quad \text{with} \quad \left(\frac{b+\bar{b}}{a+c}, i\frac{\bar{b}-b}{a+c}, \frac{a-c}{a+c}\right) \in \mathbb{S}^2.$$

Moreover, composing with stereographic projection

$$\mathbb{S}^2 \ni (\xi_1,\xi_2,\xi_3) \mapsto \begin{cases} (\xi_1 + i\xi_2)/(1-\xi_3) \in \mathbb{C} & \text{if} \quad \xi_3 \neq 1\ , \\ \infty & \text{if} \quad \xi_3 = 1\ , \end{cases}$$

we can then identify

$$\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in \mathbb{N}^+ \quad \text{with} \quad \frac{b}{c} \in \mathbb{C} \cup \{\infty\}.$$

LEMMA 2.3.2 ([**B, UY1**]). *If $f : D \to \mathbb{S}^3_1$ is a CMC 1 immersion, Then G is holomorphic.*

PROOF. As seen in Lemma 2.2.5, there exists a $B \in SU(1,1)$ so that $(FB)_{\bar{z}} = 0$. We set $\tilde{F} = FB$. Then

$$\begin{aligned} f + N = 2F \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} F^* &= 2\left(\tilde{F}B^{-1}\right)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\left(\tilde{F}B^{-1}\right)^* \\ &= 2\begin{pmatrix} (\tilde{F}_{11}\bar{a} - \tilde{F}_{12}\bar{b})(\overline{\tilde{F}_{11}}a - \overline{\tilde{F}_{12}}b) & (\tilde{F}_{11}\bar{a} - \tilde{F}_{12}\bar{b})(\overline{\tilde{F}_{21}}a - \overline{\tilde{F}_{22}}b) \\ (\tilde{F}_{21}\bar{a} - \tilde{F}_{22}\bar{b})(\overline{\tilde{F}_{11}}a - \overline{\tilde{F}_{12}}b) & (\tilde{F}_{21}\bar{a} - \tilde{F}_{22}\bar{b})(\overline{\tilde{F}_{21}}a - \overline{\tilde{F}_{22}}b) \end{pmatrix}, \end{aligned}$$

where

$$\tilde{F} = \begin{pmatrix} \tilde{F}_{11} & \tilde{F}_{12} \\ \tilde{F}_{21} & \tilde{F}_{22} \end{pmatrix} \quad B = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad a\bar{a} - b\bar{b} = 1.$$

So

$$G = \frac{\tilde{F}_{11}\bar{a} - \tilde{F}_{12}\bar{b}}{\tilde{F}_{21}\bar{a} - \tilde{F}_{22}\bar{b}} = \frac{\tilde{F}_{11}g + \tilde{F}_{12}}{\tilde{F}_{21}g + \tilde{F}_{22}},$$

which is holomorphic, since all of $\tilde{F}_{jk}$ and g are holomorphic. □

REMARK 2.3.3. By Equation (2.2.9), we have

$$\begin{aligned} \begin{pmatrix} d\tilde{F}_{11} & d\tilde{F}_{12} \\ d\tilde{F}_{21} & d\tilde{F}_{22} \end{pmatrix} &= \begin{pmatrix} \tilde{F}_{11} & \tilde{F}_{12} \\ \tilde{F}_{21} & \tilde{F}_{22} \end{pmatrix}\begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix}\omega \\ &= \begin{pmatrix} \tilde{F}_{11}g + \tilde{F}_{12} & -g(\tilde{F}_{11}g + \tilde{F}_{12}) \\ \tilde{F}_{21}g + \tilde{F}_{22} & -g(\tilde{F}_{21}g + \tilde{F}_{22}) \end{pmatrix}\omega. \end{aligned}$$

So the hyperbolic Gauss map can be written as

$$G = \frac{\tilde{F}_{11}g + \tilde{F}_{12}}{\tilde{F}_{21}g + \tilde{F}_{22}} = \frac{d\tilde{F}_{11}}{d\tilde{F}_{21}} = \frac{d\tilde{F}_{12}}{d\tilde{F}_{22}}.$$

LEMMA 2.3.4. *If $f : D \to \mathbb{S}^3_1$ is totally umbilic, then G is antiholomorphic.*

PROOF. As seen in Lemma 2.2.4, we can set the frame F of f as

$$F = e^{u/2}\begin{pmatrix} \bar{F}_{11} & F_{12} \\ \bar{F}_{21} & F_{22} \end{pmatrix},$$

where F_{jk} $(j, k = 1, 2)$ are holomorphic functions. Since

$$f + N = 2F\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}F^* = 2e^u\begin{pmatrix} F_{11}\bar{F}_{11} & \bar{F}_{11}F_{21} \\ F_{11}\bar{F}_{21} & F_{21}\bar{F}_{21} \end{pmatrix},$$

we have

$$G = \frac{\bar{F}_{11}}{\bar{F}_{21}} = \overline{\left(\frac{F_{11}}{F_{21}}\right)}.$$

Since both F_{11} and F_{21} are holomorphic, G is antiholomorphic. □

The converse of these two lemmas are also true. In fact, we have the following:

Proposition 2.3.5. *The hyperbolic Gauss map G is holomorphic (resp. antiholomorphic) if and only if f is a CMC 1 immersion (resp. totally umbilic).*

Proof. By (2.3.1), we can write

$$f + N = c\begin{pmatrix} G\bar{G} & G \\ \bar{G} & 1 \end{pmatrix},$$

where $c : D \to \mathbb{R}$ is a real function. If G is holomorphic, then

$$(f+N)_z = \begin{pmatrix} (Gc)_z\bar{G} & (Gc)_z \\ c_z\bar{G} & c_z \end{pmatrix}.$$

So

$$\langle (f+N)_z, (f+N)_z \rangle = -\det[(f+N)_z] = 0.$$

On the other hand,

$$(f+N)_z = f_z + N_z = (1-H)f_z - 2e^{-2u}qf_{\bar{z}},$$

by the Weingarten equation. So

$$\begin{aligned}
&\langle (f+N)_z, (f+N)_z \rangle \\
&\quad = (1-H)^2\langle f_z, f_z\rangle + 4e^{-4u}q^2\langle f_{\bar{z}}, f_{\bar{z}}\rangle + 4e^{-2u}q(H-1)\langle f_z, f_{\bar{z}}\rangle \\
&\quad = 2(H-1)q.
\end{aligned}$$

Thus, if G is holomorphic, then

$$q(H-1) = 0. \tag{2.3.2}$$

Similarly, if G is antiholomorphic, we again have (2.3.2). Hence combining Lemmas 2.3.2 and 2.3.4, we have the conclution. □

CHAPTER 3

CMC 1 faces

In this chapter we extend Theorem 2.2.6 to non-simply-connected CMC 1 surfaces with singularities, along the same lines as in [**UY4**].

3.1. Definition of CMC 1 faces

We first define admissible singularities.

DEFINITION 3.1.1. Let M be an oriented 2-manifold. A smooth (that is, C^∞) map $f : M \to \mathbb{S}^3_1$ is called a *CMC 1 map* if there exists an open dense subset $W \subset M$ such that $f|_W$ is a spacelike CMC 1 immersion. A point $p \in M$ is called a *singular point* of f if the induced metric ds^2 is degenerate at p.

DEFINITION 3.1.2. Let $f : M \to \mathbb{S}^3_1$ be a CMC 1 map and $W \subset M$ an open dense subset such that $f|_W$ is a CMC 1 immersion. A singular point $p \in M \setminus W$ is called an *admissible* singular point if

(1) there exists a C^1-differentiable function $\beta : U \cap W \to \mathbb{R}^+$, where U is a neighborhood of p, such that βds^2 extends to a C^1-differentiable Riemannian metric on U, and
(2) $df(p) \neq 0$, that is, df has rank 1 at p.

We call a CMC 1 map f a *CMC 1 face* if each singular point is admissible.

3.2. Representation formula for CMC 1 faces

To extend Theorem 2.2.6 to CMC 1 faces that are not simply-connected, we prepare two propositions. First, we prove the following proposition:

PROPOSITION 3.2.1. *Let M be an oriented 2-manifold and $f : M \to \mathbb{S}^3_1$ a CMC 1 face, where $W \subset M$ an open dense subset such that $f|_W$ is a CMC 1 immersion. Then there exists a unique complex structure J on M such that*

(1) *$f|_W$ is conformal with respect to J, and*
(2) *there exists an immersion $F : \widetilde{M} \to SL(2,\mathbb{C})$ which is holomorphic with respect to J such that*
$$\det(dF) = 0 \quad \textit{and} \quad f \circ \varrho = Fe_3F^*,$$
where $\varrho : \widetilde{M} \to M$ is the universal cover of M.

This F is called a holomorphic null lift *of f.*

REMARK 3.2.2. The holomorphic null lift F of f is unique up to right multiplication by a constant matrix in $SU(1,1)$. See also (1) of Remark 3.2.7 below.

PROOF OF PROPOSITION 3.2.1. Since the induced metric ds^2 gives a Riemannian metric on W, it induces a complex structure J_0 on W. Let p be an admissible singular point of f and U a local simply-connected neighborhood of p. Then by definition, there exists a C^1-differentiable function $\beta : U\cap W \to \mathbb{R}^+$ such that βds^2 extends to a C^1-differentiable Riemannian metric on U. Then there exists a positively oriented orthonormal frame field $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ with respect to βds^2 which is C^1-differentiable on U. Using this, we can define a C^1-differentiable almost complex structure J on U such that

$$J(\boldsymbol{v}_1) = \boldsymbol{v}_2 \quad \text{and} \quad J(\boldsymbol{v}_2) = -\boldsymbol{v}_1. \tag{3.2.1}$$

Since ds^2 is conformal to βds^2 on $U \cap W$, J is compatible with J_0 on $U\cap W$. There exists a C^1-differentiable decomposition

$$\Gamma\left(T^*M^{\mathbb{C}}\otimes \mathfrak{sl}(2,\mathbb{C})\right) = \Gamma\left(T^*M^{(1,0)}\otimes \mathfrak{sl}(2,\mathbb{C})\right) \oplus \Gamma\left(T^*M^{(0,1)}\otimes \mathfrak{sl}(2,\mathbb{C})\right) \tag{3.2.2}$$

with respect to J, where $\Gamma(E)$ denotes the sections of a vector bundle E on U. Since f is smooth, $df \cdot f^{-1}$ is a smooth $\mathfrak{sl}(2,\mathbb{C})$-valued 1-form. We can take the $(1,0)$-part ζ of $df\cdot f^{-1}$ with respect to this decomposition. Then ζ is a C^1-differentiable $\mathfrak{sl}(2,\mathbb{C})$-valued 1-form which is holomorphic on $U\cap W$ with respect to the equivalent complex structures J_0 and J (which follows from the fact that $f|_W$ is a CMC 1 immersion, so the hyperbolic Gauss map G of f is holomorphic on W, which is equivalent to the holomorphicity of ζ with respect to J and J_0 on $U\cap W$). Hence $d\zeta \equiv 0$ on $U\cap W$. Moreover, since W is an open dense subset and ζ is C^1-differentiable on U, $d\zeta\equiv 0$ on U. Similarly, $\zeta\wedge\zeta\equiv 0$ on U. In particular,

$$d\zeta + \zeta\wedge\zeta = 0. \tag{3.2.3}$$

As U is simply-connected, the existence of a C^1-differentiable map $F_U = (F_{jk})_{j,k=1,2} : U \to SL(2,\mathbb{C})$ such that $dF_U \cdot F_U^{-1} = \zeta$ is equivalent to the condition (3.2.3). Hence such an F exists.

Note that since f takes Hermitian matrix values, we have $df = \zeta f + (\zeta f)^*$. So $df(p) \neq 0$ (that is, p is an admissible singularity) implies $\zeta\neq 0$. Then at least one entry dF_{jk} of dF_U does not vanish at p. Using this F_{jk}, we define the function $z = F_{jk} : U \to \mathbb{C} = \mathbb{R}^2$. Then, z gives a coordinate system on U. Since $z = F_{jk}$ is a holomorphic function on $U\cap W$, it gives a complex analytic coordinate around p compatible with respect to that of $U\cap W$. The other entries of F_U are holomorphic functions with respect to z on $U\cap W$ and are C^1-differentials on U, so each entry of F_U is holomorphic with respect to z on U, by the Cauchy-Riemann equations. Since p is an arbitrary fixed admissible singularity, the complex structure of W extends across each singular point p.

This complex structure can be seen to be well-defined at singular points as follows: Let $p' \in M \setminus W$ be another singular point and U' a neighborhood of p' so that $U \cap U' \neq \emptyset$. Then by the same argument as above, there exists a C^1-differentiable almost complex structure J' on U' and C^1-differentiabe map $F'_{U'} = (F'_{j'k'})_{j',k'=1,2} : U' \to SL(2,\mathbb{C})$ such that $dF'_{U'} \cdot {F'_{U'}}^{-1}$ is the $(1,0)$-part of $df \cdot f^{-1}$ with respect to Equation (3.2.2). Define $z' = F'_{j'k'}$ so that $dF'_{j'k'} \neq 0$. Then by uniqueness of ordinary differential equations, $F_U = F'_{U'}A$ for some constant matrix A. So z and z' are linearly related, and hence they are holomorphically related. Also, because dz and dz' are nonzero, we have $dz/dz' \neq 0$ on $U \cap U'$.

For local coordinates z on M compatible with J, $\partial f \cdot f^{-1} := (f_z dz) \cdot f^{-1}$ (which is equal to ζ) is holomorphic on M and there exists a holomorphic map $F : \widetilde{M} \to SL(2,\mathbb{C})$ such that

$$dF \cdot F^{-1} = \partial f \cdot f^{-1}. \tag{3.2.4}$$

Since $\partial f \cdot f^{-1} \neq 0$, also $dF \neq 0$, and hence F is an immersion. Also, since f is conformal, $0 = \langle \partial f, \partial f \rangle = -\det(\partial f)$. Thus $\det(dF) = 0$.

Finally, we set $\hat{f} = Fe_3F^*$, defined on $\widetilde{M}$. We consider some simply-connected region $V \subset W$. By Theorem 2.2.6, there exists a holomorphic null lift $\hat{F}$ of f,

$$f = \hat{F} e_3 \hat{F}^*, \tag{3.2.5}$$

defined on that same V. Then by Equations (3.2.4) and (3.2.5), we have

$$d\hat{F} \cdot \hat{F}^{-1} = dF \cdot F^{-1},$$

and hence $\hat{F} = FB$ for some constant $B \in SL(2,\mathbb{C})$. We are free to choose the solution F of Equation (3.2.4) so that $B = e_0$, that is, $\hat{F} = F$, so $f = \hat{f}$ on V. By the holomorphicity of F, $\hat{f}$ is real analytic on $\widetilde{M}$. Also, $f \circ \varrho$ is real analytic on $\widetilde{M}$, by Equation (3.2.4) and the holomorphicity of F. Therefore $f \circ \varrho = \hat{f}$ on $\widetilde{M}$, proving the proposition. □

By Proposition 3.2.1, the 2-manifold M on which a CMC 1 face $f : M \to \mathbb{S}^3_1$ is defined always has a complex structure. So throughout this thesis, we will treat M as a Riemann surface with a complex structure induced as in Proposition 3.2.1.

The next proposition is the converse to Proposition 3.2.1:

PROPOSITION 3.2.3. *Let M be a Riemann surface and $F : M \to SL(2,\mathbb{C})$ a holomorphic null immersion. Assume the symmetric* $(0,2)$-*tensor*

$$\det[d(Fe_3F^*)] \tag{3.2.6}$$

is not identically zero. Then $f = Fe_3F^* : M \to \mathbb{S}^3_1$ *is a CMC* 1 *face, and* $p \in M$ *is a singular point of* f *if and only if* $\det[d(Fe_3F^*)]_p = 0$. *Moreover,*

$$-\det[d(FF^*)] \quad \textit{is positive definite} \tag{3.2.7}$$

on M.

Proof. Since (3.2.6) is not identically zero, the set

$$W := \{p \in M \,|\, \det[d(Fe_3F^*)]_p \neq 0\}$$

is open and dense in M. Since $F^{-1}dF$ is a $\mathfrak{sl}(2,\mathbb{C})$-valued 1-form, there exist holomorphic 1-forms a_1, a_2 and a_3 such that

$$F^{-1}dF = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}.$$

Since F is a null immersion, that is, $\operatorname{rank}(dF) = 1$, a_j $(j = 1,2,3)$ satisfy

$$a_1^2 + a_2a_3 = 0 \qquad \text{and} \qquad |a_1|^2 + |a_2|^2 + |a_3|^2 > 0. \tag{3.2.8}$$

Since

$$\begin{aligned} d(Fe_3F^*) &= F\left(F^{-1}dFe_3 + (F^{-1}dFe_3)^*\right)F^* \\ &= F\begin{pmatrix} a_1 + \overline{a_1} & -a_2 + \overline{a_3} \\ a_3 - \overline{a_2} & a_1 + \overline{a_1} \end{pmatrix} F^*, \end{aligned}$$

we have

$$\begin{aligned} -\det[d(Fe_3F^*)] &= -2|a_1|^2 + |a_2|^2 + |a_3|^2 \\ &= -2|a_2a_3| + |a_2|^2 + |a_3|^2 \\ &= |a_2|^2 - 2|a_2||a_3| + |a_3|^2 = (|a_2| - |a_3|)^2 \geq 0. \end{aligned}$$

So $-\det[d(Fe_3F^*)]$ is positive definite on W. Set $f = Fe_3F^*$. Then $f|_W : W \to \mathbb{S}^3_1$ determines a conformal immersion with induced metric

$$ds^2 = f^*ds^2_{\mathbb{S}^3_1} = \langle df, df \rangle = -\det[d(Fe_3F^*)].$$

Furthermore, f is CMC 1 by Theorem 2.2.6. Also, by (3.2.8),

$$-\det[d(FF^*)] = 2|a_1|^2 + |a_2|^2 + |a_3|^2$$

is positive definite on M. Thus if we set

$$\beta = \frac{\det[d(FF^*)]}{\det[d(Fe_3F^*)]}$$

on W, β is a positive function on W such that

$$\beta ds^2 = -\det[d(FF^*)]$$

extends to a Riemannian metric on M. Also,

$$\partial f \cdot f^{-1} = dF \cdot F^{-1} = F(F^{-1}dF)F^{-1} \neq 0,$$

and so $df \neq 0$. This completes the proof. $\square$

REMARK 3.2.4. Even if F is a holomorphic null immersion, $f = Fe_3F^*$ might not be a CMC 1 face. For example, for the holomorphic null immersion

$$F : \mathbb{C} \ni z \mapsto \begin{pmatrix} z+1 & -z \\ z & -z+1 \end{pmatrix} \in SL(2,\mathbb{C}),$$

$f = Fe_3F^*$ degenerates everywhere on $\mathbb{C}$. Note that $\det[d(Fe_3F^*)]$ is identically zero here.

Using Propositions 3.2.1 and 3.2.3, we can now extend the representation of Aiyama-Akutagawa for simply-connected CMC 1 immersions to the case of CMC 1 faces with possibly non-simply-connected domains.

THEOREM 3.2.5. *Let M be a Riemann surface with a base point $z_0 \in M$. Let g be a meromorphic function and ω a holomorphic 1-form on the universal cover $\widetilde{M}$ such that $d\hat{s}^2 := (1+|g|^2)^2\omega\bar{\omega}$ is a Riemannian metric on $\widetilde{M}$ and $|g|$ is not identically 1. Choose the holomorphic immersion $F = (F_{jk}) : \widetilde{M} \to SL(2,\mathbb{C})$ so that $F(z_0) = e_0$ and F satisfies*

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega. \tag{3.2.9}$$

Then $f : \widetilde{M} \to \mathbb{S}^3_1$ defined by

$$f = Fe_3F^* \tag{3.2.10}$$

is a CMC 1 face that is conformal away from its singularities. The induced metric ds^2 on M and the second fundamental form h of f are given as in Equation (2.2.11). Also, the hyperbolic Gauss map G of f are given by

$$G = \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}} \tag{3.2.11}$$

The singularities of the CMC 1 face occur at points where $|g| = 1$.

Conversely, let M be a Riemann surface and $f : M \to \mathbb{S}^3_1$ a CMC 1 face. Then there exists a meromorphic function g (so that $|g|$ is not identically 1) and holomorphic 1-form ω on $\widetilde{M}$ such that $d\hat{s}^2$ is a Riemannian metric on $\widetilde{M}$, and such that Equation (3.2.10) holds, where $F : \widetilde{M} \to SL(2,\mathbb{C})$ is an immersion which satisfies Equation (3.2.9).

PROOF. First we prove the first paragraph of the theorem. Since

$$d(Fe_3F^*) = F\left(F^{-1}dF \cdot e_3 + (F^{-1}dF \cdot e_3)^*\right)F^*,$$

we have

$$-\det[d(Fe_3F^*)] = (1-|g|^2)^2\omega\bar{\omega}.$$

Also, since $d\hat{s}^2$ gives a Riemannian metric on $\widetilde{M}$, ω has a zero of order k if and only if g has a pole of order $k/2 \in \mathbb{N}$. Therefore $\det[d(Fe_3F^*)] = 0$ if and only if $|g| = 1$. Hence by Proposition 3.2.3, $f = Fe_3F^* : \widetilde{M} \to \mathbb{S}^3_1$ is a CMC 1 face, and $p \in \widetilde{M}$ is a singular point of f if and only if $|g(p)| = 1$, proving the first half of the theorem.

We now prove the second paragraph of the theorem. By Proposition 3.2.1, there exists a holomorphic null lift $F : \widetilde{M} \to SL(2, \mathbb{C})$ of the CMC 1 face f. Then by the same argument as in the proof of Proposition 3.2.3, we may set

$$F^{-1}dF = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix},$$

where a_j $(j = 1, 2, 3)$ are holomorphic 1-forms such that (3.2.8) holds. By changing F into FB for some constant $B \in SU(1, 1)$, if necessary, we may assume that a_3 is not identically zero. We set

$$\omega := a_3, \qquad g := \frac{a_1}{a_3}.$$

Then ω is a holomorphic 1-form and g is a meromorphic function. Since $a_2/a_3 = a_2a_3/a_3^2 = -(a_1/a_3)^2 = -g^2$, we see that Equation (3.2.9) holds. Since $|a_2| - |a_3|$ is not identically zero (by the same argument as in the proof of Proposition 3.2.3), $|g|$ is not identically one. Also, since $g^2\omega = a_2$ is holomorphic, (3.2.7) implies $-\det[d(FF^*)] = (1 + |g|^2)^2\omega\bar{\omega} = d\hat{s}^2$ is positive definite, so $d\hat{s}^2$ gives a Riemannian metric on $\widetilde{M}$, proving the converse part of the theorem. □

Remark 3.2.6. Since the hyperbolic Gauss map G has the geometric meaning explained in Section 2.3, G is single-valued on M itself, although F might not be. We also note that, by Equation (2.2.11), the Hopf differential Q is single-valued on M as well.

Remark 3.2.7. (1) Let F be a holomorphic null lift of a CMC 1 face f with Weierstrass data (g, ω). For any constant matrix

$$B = \begin{pmatrix} \bar{p} & -q \\ -\bar{q} & p \end{pmatrix} \in SU(1, 1), \qquad p\bar{p} - q\bar{q} = 1, \tag{3.2.12}$$

FB is also a holomorphic null lift of f. The Weierstrass data $(\hat{g}, \hat{\omega})$ corresponding to $(FB)^{-1}d(FB)$ is given by

$$\hat{g} = B^{-1} \star g := \frac{pg + q}{\bar{q}g + \bar{p}} \quad \text{and} \quad \hat{\omega} = (\bar{q}g + \bar{p})^2\omega. \tag{3.2.13}$$

Two Weierstrass data (g, ω) and $(\hat{g}, \hat{\omega})$ are called *equivalent* if they satisfy Equation (3.2.13) for some B as in Equation (3.2.12). See Equation (1.6) in [**UY1**]. We shall call the equivalence class of the Weierstrass data (g, ω) the Weierstrass data associated to f. When we wish to emphasis that (g,ω) is determined by F, not FB for some B, we call (g,ω) the Weierstrass data associated to F.

On the other hand, the Hopf differential Q and the hyperbolic Gauss map G are independent of the choice of F, because

$$\hat{\omega}d\hat{g} = \omega dg \quad \text{and} \quad \frac{\bar{p}dF_{11} + \bar{q}dF_{12}}{\bar{p}dF_{21} + \bar{q}dF_{22}} = \frac{dF_{11}}{dF_{21}}, \quad \text{where} \quad F = (F_{jk}).$$

This can also be seen from Section 2.3, which implies that G is determined just by f. Then $S(g) = S(\hat{g})$ and (5) of Remark 2.2.7 imply Q is independent of the choice of F as well.

(2) Let F be a holomorphic null lift of a CMC 1 face f with the Hopf differential Q and the hyperbolic Gauss map G. Then F satisfies

$$dF \cdot F^{-1} = \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \frac{Q}{dG}. \tag{3.2.14}$$

See [**UY3, RUY1**].

(3) For a CMC 1 face f, if we find both the hyperbolic Gauss map G and the secondary Gauss map g, we can explicitly find the holomorphic null lift F, by using the so-called Small formula:

$$F = \begin{pmatrix} G\dfrac{da}{dG} - a & G\dfrac{db}{dG} - b \\ \dfrac{da}{dG} & \dfrac{db}{dG} \end{pmatrix}, \quad a = \sqrt{\frac{dG}{dg}}, \quad b = -ga.$$

See [**Sm, KUY1**].

CHAPTER 4

Singularities of CMC 1 faces

In this chapter we shall give simple criteria for a given singular point on a surface to be a cuspidal cross cap. As an application, we show that the singularities of CMC 1 faces generically consist of cuspidal edges, swallowtails and cuspidal cross caps. This chapter is based on Sections 1 and 3 of [**FSUY**].

4.1. Criteria for singular points

Let U be a domain in $\mathbb{R}^2$ and $f\colon U \to (M^3, g)$ a C^∞-map from U into a Riemannian 3-manifold (M^3, g). The map f is called a *frontal map* if there exists a unit vector field N on M^3 along f such that N is perpendicular to $f_*(TU)$. Identifying the unit tangent bundle T_1M^3 with the unit cotangent bundle $T_1^*M^3$, the map N is identified with the map

$$L = g(N, *) : U \longrightarrow T_1^*M^3.$$

The unit cotangent bundle $T_1^*M^3$ has a canonical contact form μ and L is an *isotropic map*, that is, the pull back of μ by L vanishes. Namely, a frontal map is the projection of an isotropic map. We call L the *Legendrian lift* (or *isotropic lift*) of f. If L is an immersion, the projection f is called a *front*. Whitney [**W**] proved that the generic singularities of C^∞-maps of 2-manifolds into 3-manifolds can only be cross caps. (For example, $f_{\mathrm{CR}}(u, v) = (u^2, v, uv)$ gives a cross cap.) On the other hand, a cross cap is not a frontal map, and it is also well-known that cuspidal edges and swallowtails are generic singularities of fronts (see, for example, [**AGV**], Section 21.6, page 336). The typical examples of a cuspidal edge f_{C} and a swallowtail f_{S} are given by

$$f_{\mathrm{C}}(u, v) := (u^2, u^3, v), \qquad f_{\mathrm{S}}(u, v) := (3u^4 + u^2v, 4u^3 + 2uv, v).$$

See figure 4.1.1

A cuspidal cross cap is a singular point which is $\mathcal{A}$-equivalent to the C^∞-map (see Figure 4.1.1 again)

$$f_{\mathrm{CCR}}(u, v) := (u, v^2, uv^3), \tag{4.1.1}$$

which is not a front but a frontal map with unit normal vector field

$$N_{\mathrm{CCR}} := \frac{1}{\sqrt{4 + 9u^2v^2 + 4v^6}}(-2v^3, -3uv, 2).$$

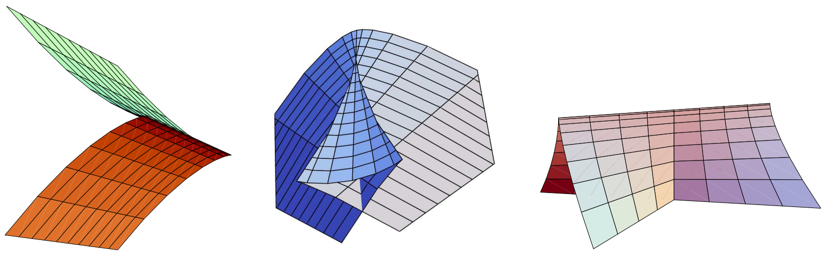


FIGURE 4.1.1. Singularities on a frontal map.

Here, two C^∞-maps $f\colon (U,p) \to M^3$ and $g\colon (V,q) \to M^3$ are $\mathcal{A}$-*equivalent* (or *right-left equivalent*) at the points $p \in U$ and $q \in V$ if there exists a local diffeomorphism φ of $\mathbb{R}^2$ with $\varphi(p) = q$ and a local diffeomorphism Φ of M^3 with $\Phi(f(p)) = g(q)$ such that $g = \Phi \circ f \circ \varphi^{-1}$.

In this section, we shall give simple criteria for a given singular point on the surface to be a cuspidal cross cap. Let (M^3, g) be a Riemannian 3-manifold and Ω the Riemannian volume element on M^3. Let $f : U \to M^3$ be a frontal map defined on a domain U on $\mathbb{R}^2$. Then we can take the unit normal vector field $N\colon U \to T_1M^3$ of f as mentioned above. The smooth function $\lambda\colon U \to \mathbb{R}$ defined by

$$\lambda(u,v) := \Omega(f_u, f_v, N) \tag{4.1.2}$$

is called *the signed area density function*, where (u,v) is a local coordinate system of U. The singular points of f are the zeros of λ. A singular point $p \in U$ is called *non-degenerate* if the exterior derivative $d\lambda$ does not vanish at p. When p is a non-degenerate singular point, the singular set $\{\lambda = 0\}$ consists of a regular curve near p, called the *singular curve*, and we can express it as a parametrized curve $\gamma(t) : (-\varepsilon, \varepsilon) \to U$ such that $\gamma(0) = p$ and

$$\lambda\bigl(\gamma(t)\bigr) = 0 \qquad (t \in (-\varepsilon, \varepsilon)).$$

We call the tangential direction $\gamma'(t)$ the *singular direction*. Since $d\lambda \neq 0$, f_u and f_v do not vanish simultaneously. So the kernel of df is 1-dimensional at each singular point p. A nonzero tangential vector $\eta \in T_pU$ belonging to the kernel is called the *null direction*. There exists a smooth vector field $\eta(t)$ along the singular curve $\gamma(t)$ such that $\eta(t)$ is the null direction at $\gamma(t)$ for each t. We call it the *vector field of the null direction*. In [**KRSUY**], the following criteria for cuspidal edges and swallowtails are given:

FACT 4.1.1. *Let $f\colon U \to M^3$ be a front and $p \in U$ a non-degenerate singular point. Take a singular curve $\gamma(t)$ with $\gamma(0) = 0$ and a vector field of null directions $\eta(t)$. Then*

(1) *The germ of* f *at* $p = \gamma(0)$ *is* $\mathcal{A}$*-equivalent to a cuspidal edge if and only if the null direction* $\eta(0)$ *is transversal to the singular direction* $\gamma'(0)$.

(2) *The germ of* f *at* $p = \gamma(0)$ *is* $\mathcal{A}$*-equivalent to a swallowtail if and only if the null direction* $\eta(0)$ *is proportional to the singular direction* $\gamma'(0)$ *and*

$$\left.\frac{d}{dt}\right|_{t=0} \det\bigl(\eta(t), \gamma'(t)\bigr) \neq 0,$$

where $\eta(t)$ *and* $\gamma'(t) \in T_{\gamma(t)}U$ *are considered as column vectors, and* det *denotes the determinant of a* 2×2*-matrix.*

In the next section, we shall prove the following:

THEOREM 4.1.2. *Let* $f\colon U \to (M^3, g)$ *be a frontal map with unit normal vector field* N*, and* $\gamma(t)$ *a singular curve on* U *passing through a non-degenerate singular point* $p = \gamma(0)$*. We set*

$$\psi(t) := \Omega(\tilde{\gamma}', D^f_\eta N, N),$$

where $\tilde{\gamma} = f \circ \gamma$, $D^f_\eta N$ *is the canonical covariant derivative along a map* f *induced from the Levi-Civita connection on* (M^3, g)*, and* $' = d/dt$*. Then the germ of* f *at* $p = \gamma(0)$ *is* $\mathcal{A}$*-equivalent to a cuspidal cross cap if and only if*

(i) $\eta(0)$ *is transversal to* $\gamma'(0)$*,*

(ii) $\psi(0) = 0$ *and* $\psi'(0) \neq 0$*.*

Note that these criteria for cuspidal edges, swallowtails and cuspidal cross caps are applied in Ishikawa-Machida [**IM**] and Izumiya-Saji-Takeuchi [**IST**].

We also give a metric-free varsion of the above theorem as follows:

THEOREM 4.1.3. *Let* $f\colon U \to M^3$ *be a frontal map and* $L\colon U \to (T^*M^3)^\circ$ *an admissible lift of* f*. Let* D *be an arbitrary linear connection on* M^3*. Suppose that* $\gamma(t)$ $(|t| < \varepsilon)$ *is a singular curve on* U *passing through a non-degenerate singular point* $p = \gamma(0)$*, and that* $X\colon (-\varepsilon, \varepsilon) \to TM^3$ *is an arbitrarily fixed vector field along* γ *such that*

(1) $L(X)$ *vanishes on* U*, and*

(2) X *is transversal to the subspace* $f_*(T_pU)$ *at* p*.*

We set

$$\tilde{\psi}(t) := L\bigl(D^f_{\eta(t)} X_{\gamma(t)}\bigr),$$

where $\gamma(t)$ *is the singular curve at* p*,* $\eta(t)$ *is a null vector field along* γ *and* $' = d/dt$*. Then the germ of* f *at* $p = \gamma(0)$ *is* $\mathcal{A}$*-equivalent to a cuspidal cross cap if and only if*

(i) $\eta(0)$ *is transversal to* $\gamma'(0)$*, and*

(ii) $\tilde{\psi}(0) = 0$ *and* $\tilde{\psi}'(0) \neq 0$

hold.

Remark 4.1.4. This criterion for cuspidal cross caps is independent of the metric of the ambient space. This property will play a crucial role in Section 4.3, where we investigate singular points on CMC 1 faces in $\mathbb{S}^3_1$.

4.2. Proof of Theorem 4.1.3

In this section we shall prove Theorem 4.1.3 and we shall show that Theorem 4.1.2 follows from Theorem 4.1.3.

We denote by $(T^*M^3)^\circ$ the complement of the zero section in T^*M^3.

Lemma 4.2.1. *Let $f : U \to (M^3, g)$ be a frontal map. Then there exists a C^∞-section $L\colon U \to (T^*M^3)^\circ$ along f such that $(\pi\circ L)_*(T_pU) \subset \operatorname{Ker} L_p$ for all $p \in U$, where $\pi\colon (T^*M^3)^\circ \to M^3$ is the canonical projection, and $\operatorname{Ker} L_p \subset T_{f(p)}M^3$ is the kernel of $L_p\colon T_pM^3 \to \mathbb{R}$. We shall call such a map L the admissible lift of f. Conversely, let $L : U \to (T^*M^3)^\circ$ be a smooth section satisfying $(\pi \circ L)_*(T_pU) \subset \operatorname{Ker} L_p$. Then $f := \pi \circ L$ is a frontal map and L is a lift of f.*

By this lemma, we know that the concept of frontal map does not depend on the Riemannian metric of M^3. Frontal maps can be interpreted as a projection of a mapping L into M^3 satisfying $(\pi \circ L)_*(T_pU) \subset \operatorname{Ker} L_p$ for all $p \in U$. (The projection of such an L into the unit cotangent bundle $T^*_1M^3$ gives the Legendrian lift of f. An admissible lift of f is not uniquely determined, since multiplication of L by non-constant functions also gives admissible lifts.)

Proof of Lemma 4.2.1. Let N be the unit normal vector field of f. Then the map

$$L : U \ni p \longmapsto g_p(N, *) \in T^*M^3$$

gives an admissible lift of f. Conversely, let $L\colon U \to (T^*M^3)^\circ$ be a non-vanishing smooth section with $(\pi \circ L)_*(T_pU) \subset \operatorname{Ker} L_p$. Then a non-vanishing section of the orthogonal complement $(\operatorname{Ker} L)^\perp$ gives a normal vector field of f. □

Let $TM^3|_{f(U)}$ be the restriction of the tangent bundle TM^3 to $f(U)$. The subbundle of $TM^3|_{f(U)}$ perpendicular to the unit normal vector N is called the *limiting tangent bundle*.

As pointed out in [**SUY**], the non-degeneracy of the singular points is also independent of the Riemannian metric g of M^3. In fact, Proposition 1.3 in [**SUY**] can be proved under the weaker assumption that f is only a frontal map. In particular, we can show the following:

Proposition 4.2.2. *Let $f\colon U \to M^3$ be a frontal map and $p \in U$ a singular point of f. Let Ω be a nowhere vanishing 3-form on M^3 and E a vector field on N^3 along f which is transversal to the limiting tangent bundle. Then p is a non-degenerate singular point of f if and only if $\lambda = \Omega(f_u, f_v, E)$ satisfies $d\lambda \neq 0$.*

We shall recall the covariant derivative along a map. Let D be an arbitrarily fixed linear connection of TM^3 and $f : U \to M^3$ a C^∞-map. We take a local coordinate system $(V; x_1, x_2, x_3)$ on M^3 and write the connection as

$$D_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{3} \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

We assume that $f(U) \subset V$. Let $X : U \to TM^3$ be an arbitrary vector field of M^3 along f given by

$$X = \xi_1(u,v)\left(\frac{\partial}{\partial x_1}\right)_{f(u,v)} + \xi_2(u,v)\left(\frac{\partial}{\partial x_2}\right)_{f(u,v)} + \xi_3(u,v)\left(\frac{\partial}{\partial x_3}\right)_{f(u,v)}.$$

Then its covariant derivative along f is defined by

$$D^f_{\frac{\partial}{\partial u_l}} X := \sum_{k=1}^{3} \left(\frac{\partial \xi_k}{\partial u_l} + \sum_{i,j=1}^{3} (\Gamma_{ij}^k \circ f)\, \xi_j \frac{\partial f_i}{\partial u_l} \right) \frac{\partial}{\partial x_k}, \qquad (l = 1, 2),$$

where $(u_1, u_2) = (u, v)$ is the coordinate system of U and $f = (f_1, f_2, f_3)$. Let

$$\eta = \eta_1 \frac{\partial}{\partial u_1} + \eta_1 \frac{\partial}{\partial u_2} \in TU$$

be a null vector of f, that is, $f_* \eta = 0$. In this case, we have

$$\eta_1 \frac{\partial f_k}{\partial u} + \eta_2 \frac{\partial f_k}{\partial v} = 0$$

for $k = 1, 2, 3$, and thus

$$D^f_\eta X := \sum_{k=1}^{3} \left(\sum_{l=1}^{2} \eta_l \frac{\partial \xi_k}{\partial u_l} \right) \frac{\partial}{\partial x_k}$$

holds, which implies the following:

LEMMA 4.2.3. *The derivative $D^f_\eta X$ does not depend on the choice of the linear connection D if η is a null vector of f.*

The purpose of this section is to prove Theorem 4.1.3:

Firstly, we shall show that Theorem 4.1.2 immediately follows from this assertion.

PROOF OF THEOREM 4.1.2. We set $X_0 := \tilde{\gamma}' \times_g N$, where $\tilde{\gamma} = f \circ \gamma$, $' = d/dt$ and $\times_g$ is the vector product of TM^3 with respect to the Riemannian metric g. Since X_0 is perpendicular to N, we have $L(X_0) = 0$. Moreover, X_0 is obviously transversal to $\tilde{\gamma}'$, and then it satisfies the conditions (1) and (2) in Theorem 4.1.3. On the other hand, $L := g(N, *)$ gives an admissible lift of f and we have

$$\begin{aligned} \psi(t) &= \Omega(\tilde{\gamma}', D^f_\eta N, N) = g(N \times_g \tilde{\gamma}', D^f_\eta N) \\ &= -g(X_0, D^f_\eta N) = g(D^f_\eta X_0, N) = L(D^f_\eta X_0) = \tilde{\psi}(t). \end{aligned} \tag{4.2.1}$$

This proves the assertion. □

To prove Theorem 4.1.3, we prepare two lemmas:

LEMMA 4.2.4. *Under the same assumptions as in Theorem 4.1.3, $\tilde{\psi}(0) \neq 0$ holds if and only if f is a front on a sufficiently small neighborhood of p.*

PROOF. We take a Riemannian metric g on M^3. Let N be the unit normal vector field of f. Since the covariant derivative D^f_η does not depend on the connection D, we may assume D is the Levi-Civita connection. Then we have that

$$\tilde{\psi}(t) = g(N, D^f_\eta X) = \eta g(N, X) - g(D^f_\eta N, X) = -g(D^f_\eta N, X). \tag{4.2.2}$$

Here, the differential of the map $L\colon U \to (T^*M^3)^\circ$ at $p \in U$ in the direction of γ' is given by

$$(dL)_p(\gamma') = (df)_p(\gamma') + g_p(D^f_{\gamma'}N, *) \in T_{f(p)}M^3 \oplus T^*_{f(p)}M^3 = T_{L(p)}\bigl(T^*M^3\bigr).$$

On the other hand,

$$(dL)_p(\eta) = (df)_p(\eta) + g_p(D^f_\eta N, *) = g_p(D^f_\eta N, *)$$

Hence L is an immersion at p, that is, $(dL)_p(\gamma')$ and $(dL)_p(\eta)$ are linearly independent, if and only if $D^f_\eta N \neq 0$.

Since $D^f_\eta N$ is parpendicular to both N and $df(\gamma')$, we have the conlusion. □

LEMMA 4.2.5. *Let $f\colon U \to M^3$ be a frontal map and p a non-degenerate singular point of f satisfying* (i) *of Theorem 4.1.3. Then the condition $\tilde{\psi}(0) = \tilde{\psi}'(0) = 0$ is independent of the choice of vector field X along f satisfying* (1) *and* (2)*.*

PROOF. By (i), we may assume that the null vector field $\eta(t)$ ($|t| < \varepsilon$) is transversal to $\gamma'(t)$. Then we may take a coordinate system (u, v) with the origin at p such that the u-axis corresponds to the singular curve and $\eta(u) = (\partial/\partial v)|_{(u,0)}$. We fix an arbitrary vector field X_0 satisfying (1) and (2). By (2), X_0 is transversal to the vector field $V := f_*(\partial/\partial u) (\neq 0)$ along f. Take an arbitrary vector field X along f satisfying (1) and (2). Then it can be expressed as a linear combination

$$X = a(u, v)X_0 + b(u, v)V \qquad \bigl(a(0, 0) \neq 0\bigr).$$

Then we have

$$D^f_\eta X = da(\eta)X_0 + db(\eta)V + aD^f_\eta X_0 + bD^f_\eta V.$$

Now $L(V) = 0$ holds, since L is an admissible lift of f. Moreover, (1) implies that $L(X) = 0$, and we have

$$L(D^f_\eta X) = aL(D^f_\eta X_0) + bL(D^f_\eta V).$$

Since D^f_η does not depend on the choice of a connection D, we may assume that D is a torsion-free connection. Then we have

$$D^f_\eta V = D^f_{\frac{\partial}{\partial v}} f_*\left(\frac{\partial}{\partial u}\right) = D^f_{\frac{\partial}{\partial u}} f_*\left(\frac{\partial}{\partial v}\right) = 0,$$

since $f_*(\partial/\partial v) = f_*\eta = 0$. Thus we have

$$L(D^f_\eta X) = aL(D^f_\eta X_0) = a\tilde{\psi}(u).$$

Since $a(0,0) \neq 0$, the conditions (1) and (2) for X are the same as those of X_0. □

The following two lemmas are well-known (see [**GG**]). They plays a crucial role in Whitney [**W**] to give a criterion for a given C^∞-map to be a cross cap. Let $h(u,v)$ be a C^∞-function defined around the origin.

Fact 4.2.6 (Division Lemma). *If $h(u,0)$ vanishes for sufficiently small u, then there exists a C^∞-function $\tilde{h}(u,v)$ defined around the origin such that $h(u,v) = v\tilde{h}(u,v)$ holds.*

Fact 4.2.7 (Whitney Lemma). *If $h(u,v) = h(-u,v)$ holds for sufficiently small (u,v), then there exists a C^∞-function $\tilde{h}(u,v)$ defined around the origin such that $h(u,v) = \tilde{h}(u^2,v)$ holds.*

Proof of Theorem 4.1.3. As the assertion is local in nature, we may assume that $M^3 = \mathbb{R}^3$ and let g_0 be the canonical metric. We denote the inner product associated with g_0 by $\langle\ ,\ \rangle$. The canonical volume form Ω is nothing but the determinant: $\Omega(X,Y,Z) = \det(X,Y,Z)$. Then the signed area density function λ defined in (4.1.2) is written as

$$\lambda(u,v) = \det(f_u, f_v, N).$$

Let $f\colon U \to \mathbb{R}^3$ be a frontal map and N the unit normal vector field of f. Take a coordinate system (u,v) centered at the singular point p such that the u-axis is a singular curve and the vector field $\partial/\partial v$ gives the null direction along the u-axis. If we set $X = V \times_g N$, then it satisfies (1) and (2) of Theorem 4.1.3, and by (4.2.1) we have

$$\tilde{\psi}(u) = \det(\gamma', D^f_\eta N, N) = \det(f_u, N_v, N).$$

Thus we now suppose that $\tilde{\psi}(0) = 0$ and $\tilde{\psi}'(0) \neq 0$. It is sufficient to show that f is $\mathcal{A}$-equivalent to the standard cuspidal cross cap as in (4.1.1).

Without loss of generality, we may set $f(0,0) = (0,0,0)$. Since f satisfies (1), $f(u,0)$ is a regular space curve. Since $f_u(u,0) \neq 0$, we may assume $\partial_u f_1(u,0) \neq 0$ for sufficiently small u, where we set $f = (f_1, f_2, f_3)$. Then the map

$$\Phi\colon (y_1,y_2,y_3) \longmapsto \big(f_1(y_1,0), f_2(y_1,0) + y_2, f_3(y_1,0) + y_3\big)$$

is a local diffeomorphism of $\mathbb{R}^3$ at the origin. Replacing f by $\Phi^{-1}\circ f(u,v)$, we may assume $f(u,v)=\bigl(u,f_2(u,v),f_3(u,v)\bigr)$, where f_2 and f_3 are smooth functions around the origin such that $f_2(u,0)=f_3(u,0)=0$ for sufficiently small u.

Then by the division lemma (Fact 4.2.6), there exist C^∞-functions $\tilde f_2(u,v)$, $\tilde f_3(u,v)$ such that $f_j(u,v)=v\tilde f_j(u,v)$ $(j=2,3)$. Moreover, since $f_v=\partial_v f=0$ along the u-axis, we have $\tilde f_2(u,0)=\tilde f_3(u,0)=0$. Applying the division lemma again, there exist C^∞-functions $a(u,v)$, and $b(u,v)$ such that

$$f(u,v)=\bigl(u,v^2a(u,v),v^2b(u,v)\bigr).$$

Since $f_v(u,0)=0$, $\lambda_u(u,0)=0$ and $d\lambda\neq 0$, we have

$$\begin{aligned}0\neq\lambda_v(u,0)&=\det(f_{uv},f_v,N)+\det(f_u,f_{vv},N)+\det(f_u,f_v,N_v)\\&=\det(f_u,f_{vv},N).\end{aligned}$$

In particular, we have

$$0\neq f_{vv}(0,0)=2\bigl(0,a(0,0),b(0,0)\bigr).$$

Hence, changing the y-coordinate to the z-coordinate if necessary, we may assume that $a(0,0)\neq 0$. Then the map

$$(u,v)\mapsto(\tilde u,\tilde v)=(u,v\sqrt{a(u,v)})$$

defined near the origin gives a new local coordinate around $(0,0)$ by the inverse function theorem. Thus we may assume that $a(u,v)=1$, namely

$$f(u,v)=\bigl(u,v^2,v^2b(u,v)\bigr).$$

Now we set

$$\alpha(u,v):=\frac{b(u,v)+b(u,-v)}{2},\qquad \beta(u,v):=\frac{b(u,v)-b(u,-v)}{2}.$$

Then $b=\alpha+\beta$ holds, and α (resp. β) is an even (resp. odd) function. By applying the Whitney lemma, there exist smooth functions $\tilde\alpha(u,v)$ and $\tilde\beta(u,v)$ such that

$$\alpha(u,v)=\tilde\alpha(u,v^2),\qquad \beta(u,v)=v\tilde\beta(u,v^2).$$

Then we have

$$f(u,v)=\bigl(u,v^2,v^2\tilde\alpha(u,v^2)+v^3\tilde\beta(u,v^2)\bigr).$$

Here,

$$\Phi_1:(x,y,z)\longmapsto\bigl(x,y,z-y\tilde\alpha(x,y)\bigr)$$

gives a local diffeomorphism at the origin. Replacing f by $\Phi_1\circ f$, we may set

$$f(u,v)=\bigl(u,v^2,v^3\tilde\beta(u,v^2)\bigr).$$

Then by a straightforward calculation, the unit normal vector field N of f is obtained as

$$N := \frac{1}{\Delta}\left(-v^3\tilde{\beta}_u, -\frac{3}{2}v\tilde{\beta} - v^3\tilde{\beta}_v, 1\right),$$
$$\Delta = \left[1 + v^2\left(\left(\frac{3}{2}\tilde{\beta} + v^2\tilde{\beta}_v\right)^2 + v^4(\tilde{\beta}_u)^2\right)\right]^{1/2}.$$

Since $N_v(u,0) = \bigl(0, -3\tilde{\beta}(u,0)/2, 0\bigr)$, we have

$$\tilde{\psi}(u) = \det(f_u, N_v, N) = \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3\tilde{\beta}(u,0)/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -\frac{3}{2}\tilde{\beta}(u,0).$$

Thus (ii) of Theorem 4.1.3 holds if and only if

$$\tilde{\beta}(0,0) = 0, \qquad \tilde{\beta}_u(0,0) \neq 0. \tag{4.2.3}$$

Then by the implicit function theorem, there exists a C^∞-function $\delta(u,v)$ such that $\delta(0,0) = 0$, and

$$\tilde{\beta}\bigl(\delta(u,v), v\bigr) = u \tag{4.2.4}$$

holds. Using this, we have a local diffeomorphism on $\mathbb{R}^2$ as $\varphi : (u,v) \mapsto \bigl(\delta(u,v^2), v\bigr)$, and

$$f \circ \varphi(u,v) = \bigl(\delta(u,v^2), v^2, uv^3\bigr).$$

Since $\delta_u \neq 0$ by (4.2.4), $\Phi_2 : (x,y,z) \mapsto (\delta(x,y), y, z)$ gives a local diffeomorphism on $\mathbb{R}^3$, and

$$\Phi_2^{-1} \circ f \circ \varphi = (u, v^2, uv^3)$$

gives the standard cuspidal cross cap f_{CCR} mentioned in (4.1.1). □

It is well-known that a slice of a cuspidal edge by a plane is generically a 3/2-cusp. We shall show a similar result for a cuspidal cross cap. Consider a typical cuspidal cross cap $f_{\mathrm{CCR}} = (u, v^2, uv^3)$. Then the set of self-intersections corresponds to the v-axis. Since $(f_{\mathrm{CCR}})_v = (0, 2v, 3uv^2)$, its limiting direction $(0,1,0)$ is called the *direction of self-intersections*. Since any cuspidal cross cap is $\mathcal{A}$-equivalent to the standard one, such a direction is uniquely determined for a given cuspical cross cap.

PROPOSITION 4.2.8. *Let $f : U \to \mathbb{R}^3$ be a C^∞-map which has a cuspidal cross cap at $(u_0, v_0) \in U$, and S an embedded surface in $\mathbb{R}^3$ passing through $f(u_0,v_0)$. Then the intersection of the image of f and S gives a 5/2-cusp if the tangent plane of S at $f(u_0,v_0)$ does not contain the direction of self-intersections and the singular direction.*

This assertion is in [**Po**], and is a fundamental property of cuspidal cross caps. Since the proof is not given there, we shall give it:

PROOF. Since any cuspidal cross cap is $\mathcal{A}$-equivalent to the standard one, we may assume that $(u_0, v_0) = (0,0)$ and $f(u,v) = f_{\mathrm{CCR}}(u,v) = (u, v^2, uv^3)$. In this case, the singular curve is $f(u,0) = (u,0,0)$, and, in particular, the singular direction is $(1,0,0)$. If a regular surface S does not contain this direction at the origin $(0,0,0)$, we can express S as a graph on the yz-plane:

$$x = G(y,z) \qquad \bigl(G(0,0) = 0\bigr).$$

The intersection of S and the image of f is given by an implicit function

$$u - G(v^2, uv^3) = 0; \qquad x = G(v^2, uv^3), \quad y = v^2, \quad z = uv^3.$$

Since

$$\left.\frac{\partial}{\partial u}\right|_{(u,v)=0} \bigl(u - G(v^2, uv^3)\bigr) = 1 - v^3 G_z(v^2, uv^3)\big|_{(u,v)=0} = 1,$$

the implicit function theorem implies that u can be considered as a C^∞-function of v and can be expressed as $u = u(v)$, and the projection into the yz-plane of the intersection with S is given as a plane curve

$$\sigma(v) = \bigl(v^2, u(v)v^3\bigr).$$

It is sufficient to show that $\sigma(v)$ forms a 5/2-cusp at $v = 0$. By a straightforward calculation, we have

$$\begin{aligned}
\sigma'(v) &= (2v, u'v^3 + 3uv^2), \qquad \sigma''(v) = (2, u''v^3 + 6u'v^2 + 6uv),\\
\sigma'''(v) &= (0, u'''v^3 + 9u''v^2 + 18u'v + 6u),\\
\sigma^{(4)}(v) &= (0, u^{(4)}v^3 + 12u'''v^2 + 36u''v + 24u'),\\
\sigma^{(5)}(v) &= (0, u^{(5)}v^3 + 15u^{(4)}v^2 + 60u'''v + 60u''),
\end{aligned}$$

where $' = d/dv$. In particular, we have

$$\begin{aligned}
\sigma'(0) = \sigma'''(0) &= (0,0), &\qquad \sigma''(0) &= (2,0),\\
\sigma^{(4)}(0) &= (0, 24u'(0)), &\qquad \sigma^{(5)}(0) &= (0, 60u''(0)).
\end{aligned}$$

On the other hand, differentiating $u = G(v^2, uv^3)$, we have

$$u'(0) = 0, \qquad u''(0) = 2G_y(0,0),$$

and

$$\sigma^{(4)}(0) = (0,0), \quad \sigma^{(5)}(0) = \bigl(0, 120G_y(0,0)\bigr).$$

Then we have

$$\begin{aligned}
&3\sigma''(0)\det(\sigma''(0), \sigma^{(5)}(0)) - 10\sigma'''(0)\det(\sigma''(0), \sigma^{(4)}(0))\\
&= (1440G_y(0,0), 0).
\end{aligned}$$

Thus by Proposition B.2.2 in Appendix B.2, $\sigma(v)$ has a 5/2-cusp at $v = 0$ if and only if $G_y(0,0) \neq 0$. Since $f(u,v)$ has a self-intersection along the v-axis, the direction of self-intersection is $(0,1,0)$. The tangent plane of the graph $x = G(y,z)$ does not contain this direction if and only if $G_y(0,0) \neq 0$, which proves the assertion. □

In [**SUY**], Saji, Umehara and Yamada introduced the notion of *singular curvature* of cuspidal edges, and studied the behavior of the Gaussian curvature near a cuspidal edge:

FACT 4.2.9. *Let $f\colon U \to \mathbb{R}^3$ be a front, $p \in U$ a cuspidal edge, and $\gamma(t)$ ($|t| < \varepsilon$) a singular curve consisting of non-degenerate singular points with $\gamma(0) = p$. Then the Gaussian curvature K is bounded on a sufficiently small neighborhood of $J := \gamma\big((-\varepsilon,\varepsilon)\big)$ if and only if the second fundamental form vanishes on J. Moreover, if the Gaussian curvature K is non-negative on $U \setminus J$ for a neighborhood of U of p, then the singular curvature is non-positive.*

The singular curvature at a cuspidal cross cap is also defined in a similar way to the cuspidal edge case. Since the unit normal vector field N is well-defined at a cuspidal cross caps, the second fundamental form is well-defined. Since singular points sufficiently close to a cuspidal cross cap are cuspidal edges, the following assertion immediately follows from the above fact.

PROPOSITION 4.2.10. *Let $f\colon U \to \mathbb{R}^3$ be a frontal map, $p \in U$ a cuspidal cross cap, and $\gamma(t)$ ($|t| < \varepsilon$) a singular curve consisting of non-degenerate singular points with $\gamma(0) = p$. Then the Gaussian curvature K is bounded on a sufficiently small neighborhood of $J := \gamma\big((-\varepsilon,\varepsilon)\big)$ if and only if the second fundamental form vanishes on J. Moreover, if the Gaussian curvature K is non-negative on $U \setminus J$ for a neighborhood of U of p, then the singular curvature is non-positive.*

Now, we give an example of a surface with umbilic points accumulating at a cuspidal cross cap point. For a space curve $\gamma(t)$ with arc-length parameter, we take $\{\xi_1(t), \xi_2(t), \xi_3(t)\}$, $\kappa(t) > 0$ and $\tau(t)$ as the Frenet frame, the curvature and the torsion functions of γ. We consider a tangent developable surface $f(t,u) = \gamma(t) + u\xi_1(t)$ of γ. The set of singular points of f is $\{(t,0)\}$.

We remark that this surface is frontal, since $N(t,u) = \xi_3(t)$ gives the unit normal vector. By a direct calculation, the first fundamental form ds^2 and the second fundamental form h are written as

$$ds^2 = \Big(1 + u^2\big(\kappa(t)\big)^2\Big)\, dt^2 + 2\,dt\,du + du^2, \qquad h = u\kappa(t)\tau(t)\,dt^2,$$

and the Gaussian curvature K and the mean curvature are

$$K = 0, \qquad H = \frac{\tau(t)}{2u\kappa(t)}.$$

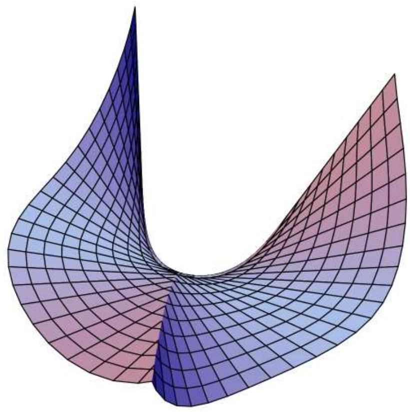


FIGURE 4.2.1. Cuspidal cross cap with accumulating umbilic points.

So a regular point (t, u) is an umbilic point if and only if $\tau(t) = 0$. On the other hand, it is easy to show that f is a front at $(t, 0)$ if and only if $\tau(t) \neq 0$. Moreover, Cleave [**C**] showed that a tangent developable surface f at $(t, 0)$ is $\mathcal{A}$-equivalent to a cuspidal cross cap if and only if $\tau(t) = 0$ and $\tau'(t) \neq 0$, which also follows from our criterion directly. Hence we consider a tangent developable surface with space curve $\gamma(t)$ with $\tau(t) = 0$ and $\tau'(t) \neq 0$, and then we have the desired example.

EXAMPLE 4.2.11. Let $\gamma(t) = (t, t^2, t^4)$ and consider a tangent developable surface f of γ. Since $\tau(0) = 0$ and $\tau'(0) = 12 \neq 0$, all points on the ruling passing through $\gamma(0)$ are umbilic points and f at $(0, 0)$ is a cuspidal cross cap (see Figure 4.2.1).

4.3. Singularities of CMC 1 faces

In this section, we shall give criteria for a given singular point on a CMC 1 face to be a cuspidal edge or a swallowtail or a cuspidal cross cap (Theorem 4.3.3). Then we show that the singularities of CMC 1 faces generically consist of these three singularities (Theorem 4.3.8). Though the assertion is the same as the case of maxfaces in $\mathbb{R}^3_1$ (see Corollary B.1.5), the method is not parallel: For maxfaces, one can easily write down the Euclidean normal vector explicitly, as well as the Lorentzian normal, in terms of the Weierstrass data. However, the case of CMC 1 faces in $\mathbb{S}^3_1$ is different, as it is difficult to express the Euclidean normal vector, and we apply Theorem 4.1.3 instead of Theorem 4.1.2, since Theorem 4.1.3 is independent of the metric of the ambient space.

Let $f : U \to \mathbb{S}^3_1$ be a CMC 1 face and F a holomorphic null lift of f with Weierstrass data (g, ω). We set

$$\beta := \begin{pmatrix} 1 & g \\ \bar{g} & 1 \end{pmatrix}.$$

Then

$$N := F\beta^2 F^* \tag{4.3.1}$$

gives the Lorentzian normal vector field of f on the regular set of f. This N is not a unit vector, but extends smoothly across the singular sets. Let $T\mathbb{S}^3_1|_{f(U)}$ be the restriction of the tangent bundle of $\mathbb{S}^3_1$ to $f(U)$. Then

$$L := \langle *, N \rangle$$

gives a section of U into T^*U, and gives an admissible lift of f. In particular, f is a frontal map and the subbundle

$$E := \{X \in T\mathbb{S}^3_1|_{f(U)} \mid \langle X, N \rangle = 0\}$$

coincides with the limiting tangent bundle. Moreover, we have the following:

LEMMA 4.3.1. *Any section X of the limiting tangent bundle E is parametrized as*

$$X = F \begin{pmatrix} \bar{\zeta} g + \zeta \bar{g} & \zeta(|g|^2 + 1) \\ \bar{\zeta}(|g|^2 + 1) & \bar{\zeta} g + \zeta \bar{g} \end{pmatrix} F^* \tag{4.3.2}$$

for some $\zeta : U \to \mathbb{C}$.

PROOF. Let p be an arbitrary point in U. Since $X_p = X_p^*$, $X_p \in T_p\mathbb{R}^4_1$. Because $\langle f_p, X_p \rangle = 0$, $X_p \in T_p\mathbb{S}^3_1$. Since $\langle N_p, X_p \rangle = 0$, and $\langle\ ,\ \rangle$ is a non-degenerate inner product, we get the conclusion. $\square$

The above lemma will play a crucial role in giving a criterion for cuspidal cross caps in terms of the Weierstrass data.

PROPOSITION 4.3.2. *Let $f : U \to \mathbb{S}^3_1$ be a CMC 1 face and F a holomorphic null lift of f with Weierstrass data (g, ω). Then a singular point $p \in U$ is non-degenerate if and only if $dg(p) \neq 0$.*

PROOF. Define $\xi \in T_{f(p)}\mathbb{R}^4_1$ as

$$\xi := FF^*. \tag{4.3.3}$$

Then $\xi \in T_{f(p)}\mathbb{S}^3_1$, because $\langle f, \xi \rangle = 0$. Define a 3-form Ω on $\mathbb{S}^3_1$ as

$$\Omega(X_1, X_2, X_3) := \det(f, X_1, X_2, X_3) \tag{4.3.4}$$

for arbitrary vector fields X_1, X_2, X_3 of $\mathbb{S}^3_1$. Then Ω gives a volume element on $\mathbb{S}^3_1$. Since

$$\begin{aligned}\Omega(f_u, f_v, \xi) &= \det(f, f_u, f_v, \xi) \\ &= \det\begin{pmatrix} 0 & 2\operatorname{Re}(g\hat{\omega}) & -2\operatorname{Im}(g\hat{\omega}) & 1 \\ 0 & \operatorname{Re}\{(1+g^2)\hat{\omega}\} & -\operatorname{Im}\{(1+g^2)\hat{\omega}\} & 0 \\ 0 & -\operatorname{Im}\{(1-g^2)\hat{\omega}\} & -\operatorname{Re}\{(1-g^2)\hat{\omega}\} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &= (1-|g|^2)(1+|g|^2)|\hat{\omega}|^2,\end{aligned}$$

we see that

$$\begin{aligned}d\big(\Omega(f_u, f_v, N)\big) &= -\frac{1}{2}\Big(d(g\bar{g})(1+|g|^2)|\hat{\omega}|^2 - (1-|g|^2)d\big((1+|g|^2)|\hat{\omega}|^2\big)\Big) \\ &= -d(g\bar{g})|\hat{\omega}|^2\end{aligned}$$

at p, because $|g(p)| = 1$, proving the proposition by Proposition 4.2.2. □

We shall now prove the following:

THEOREM 4.3.3. *Let U be a domain of the complex plane $(\mathbb{C}, z)$ and $f : U \to \mathbb{S}^3_1$ a CMC 1 face constructed from the Weierstrass data $(g, \omega = \hat{\omega}\, dz)$, where $\hat{\omega}$ is a holomorphic function on U. Then:*

(1) *A point $p \in U$ is a singular point if and only if $|g(p)| = 1$.*
(2) *f is $\mathcal{A}$-equivalent to a cuspidal edge at a singular point p if and only if*
$$\operatorname{Re}\left(\frac{g'}{g^2\hat{\omega}}\right) \neq 0 \qquad \textit{and} \qquad \operatorname{Im}\left(\frac{g'}{g^2\hat{\omega}}\right) \neq 0$$
hold at p, where $' = d/dz$.
(3) *f is $\mathcal{A}$-equivalent to a swallowtail at a singular point p if and only if*
$$\frac{g'}{g^2\hat{\omega}} \in \mathbb{R} \setminus \{0\} \qquad \textit{and} \qquad \operatorname{Re}\left\{\frac{g}{g'}\left(\frac{g'}{g^2\hat{\omega}}\right)'\right\} \neq 0$$
hold at p.
(4) *f is $\mathcal{A}$-equivalent to a cuspidal cross cap at a singular point p if and only if*
$$\frac{g'}{g^2\hat{\omega}} \in i\mathbb{R} \setminus \{0\} \qquad \textit{and} \qquad \operatorname{Im}\left\{\frac{g}{g'}\left(\frac{g'}{g^2\hat{\omega}}\right)'\right\} \neq 0$$
hold at p.

In particular, the criteria for cuspidal edges, swallowtails and cuspidal cross caps in terms of (g, ω) are exactly the same as in the case of maxfaces (Fact B.1.2 and Theorem B.1.3).

To prove Theorem 4.3.3, we prepare the following lemma:

LEMMA 4.3.4. *Let $f : U \to \mathbb{S}^3_1$ be a CMC 1 face and F a holomorphic null lift of f with Weierstrass data (g, ω). Let X be a section of the limiting tangent bundle E defined as in Equation* (4.3.2). *Take a singular point $p \in U$. Then*

$$\tilde{\psi} := \left\langle N, D^f_\eta X \right\rangle = 2\operatorname{Re}\left(\frac{g'}{g^2\hat{\omega}}\right)\operatorname{Im}(\bar{\zeta}g)$$

holds, where D is the canonical connection of $\mathbb{S}^3_1$, N is the vector field given in (4.3.1), *and η denotes the null direction of f at p.*

PROOF. We set

$$T = \begin{pmatrix} \bar{\zeta}g + \zeta\bar{g} & \zeta(|g|^2+1) \\ \bar{\zeta}(|g|^2+1) & \bar{\zeta}g + \zeta\bar{g} \end{pmatrix}.$$

Then $X = FTF^*$. On the other hand, the null direction η is given by

$$\eta = \frac{i}{g\hat{\omega}}\frac{\partial}{\partial z} - \frac{i}{\bar{g}\bar{\hat{\omega}}}\frac{\partial}{\partial \bar{z}} \tag{4.3.5}$$

at a singular point $p \in U$. Thus

$$\begin{aligned} &D^{\mathbb{R}^4_1}_\eta X \\ &= \frac{i}{g\hat{\omega}}F(F^{-1}F_zT)F^* - \frac{i}{\bar{g}\bar{\hat{\omega}}}F(F^{-1}F_zT)^*F^* + \frac{i}{g\hat{\omega}}FT_zF^* - \frac{i}{\bar{g}\bar{\hat{\omega}}}FT_{\bar{z}}F^*, \end{aligned}$$

where $D^{\mathbb{R}^4_1}$ is the canonical connection of $\mathbb{R}^4_1$. Since $\bar{g} = g^{-1}$ at any singular point p, and by (3.2.9), we see that

$$\begin{aligned} \frac{i}{g\hat{\omega}}F(F^{-1}F_zT)F^* &= -\frac{i}{\bar{g}\bar{\hat{\omega}}}F(F^{-1}F_zT)^*F^* \\ &= iF\begin{pmatrix} \zeta\bar{g} - \bar{\zeta}g & \zeta - \bar{\zeta}g^2 \\ \zeta\bar{g}^2 - \bar{\zeta} & \zeta\bar{g} - \bar{\zeta}g \end{pmatrix}F^* \\ &= i(\zeta\bar{g} - \bar{\zeta}g)\begin{pmatrix} 1 & g \\ \bar{g} & 1 \end{pmatrix}. \end{aligned}$$

Thus

$$\left\langle D^{\mathbb{R}^4_1}_\eta X, N \right\rangle = \left\langle \frac{i}{g\hat{\omega}}FT_zF^* - \frac{i}{\bar{g}\bar{\hat{\omega}}}FT_{\bar{z}}F^*, N \right\rangle.$$

Since

$$\begin{aligned} \left\langle \frac{i}{g\hat{\omega}}FT_zF^*, N \right\rangle &= -\frac{1}{2}\operatorname{trace}\left[\frac{ig'}{g\hat{\omega}}\begin{pmatrix} \bar{\zeta} & \zeta\bar{g} \\ \bar{\zeta}\bar{g} & \zeta \end{pmatrix}\begin{pmatrix} 1 & -g \\ -\bar{g} & 1 \end{pmatrix}\right] \\ &= \frac{g'}{2g^2\hat{\omega}}i(\zeta\bar{g} - \bar{\zeta}g) \end{aligned}$$

and

$$\left\langle \frac{i}{\bar{g}\bar{\hat{\omega}}} F T_{\bar{z}} F^*, N \right\rangle = -\frac{1}{2}\operatorname{trace}\left[\frac{ig'}{\bar{g}\bar{\hat{\omega}}}\begin{pmatrix} \zeta & \zeta g \\ \bar{\zeta} g & \zeta \end{pmatrix}\begin{pmatrix} 1 & -g \\ -\bar{g} & 1 \end{pmatrix}\right]$$
$$= -\overline{\left(\frac{g'}{2g^2\hat{\omega}}\right)} i(\zeta\bar{g} - \bar{\zeta} g),$$

we have

$$\tilde{\psi} = \left\langle D_\eta^{\mathbb{R}^4_1} X, N \right\rangle = 2\operatorname{Re}\left(\frac{g'}{g^2\hat{\omega}}\right)\operatorname{Im}(\bar{\zeta} g),$$

proving the lemma. □

Now assume that X defined as in (4.3.2) satisfies (2) in Theorem 4.1.3. Then by the definition of X, $\operatorname{Im}(\bar{\zeta} g)$ cannot be zero at a singular point. Thus Lemmas 4.2.4 and 4.3.4 imply the following:

Corollary 4.3.5. *Let $f : M \to \mathbb{S}^3_1$ be a CMC 1 face and F a holomorphic null lift of f with Weierstrass data (g, ω). Assume that $p \in M$ is a singular point. Then f is a front on a neighborhood of p if and only if*

$$\operatorname{Re}\left(\frac{dg}{g^2\omega}\right) \neq 0 \tag{4.3.6}$$

holds at p.

Proof of Theorem 4.3.3. Since the criteria for cuspidal edges and swallowtails are described intrinsically, and the first fundamental form of f is the same as in the case of maxfaces, so the assertions (1), (2) and (3) are parallel to the case of maxfaces in $\mathbb{R}^3_1$. See [**UY4**]. So it is sufficient to show the last assertion: Let γ be the singular curve with $\gamma(0) = p$. Since the induced metric ds^2 is in the same form as for the maxface case, we can parametrize γ as

$$\dot{\gamma}(t) = i\overline{\left(\frac{g'}{g}\right)}\bigl(\gamma(t)\bigr),$$

where $\dot{} = d/dt$. Here, we identify T_pU with $\mathbb{R}^2$ and $\mathbb{C}$ with

$$\zeta = a + ib \in \mathbb{C} \;\leftrightarrow\; (a, b) \in \mathbb{R}^2 \;\leftrightarrow\; a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v} = \zeta\frac{\partial}{\partial z} + \bar{\zeta}\frac{\partial}{\partial \bar{z}}, \tag{4.3.7}$$

where $z = u + iv$.

On the other hand, the null direction is given as in (4.3.5). Assume X satisfies (2) in Theorem 4.1.3. Then the necessary and sufficient condition for a cuspidal cross cap is $\tilde{\psi} = 0$ and $d\tilde{\psi}/dt \neq 0$, by

Thereom 4.1.3. Thus, Lemma 4.3.4 implies the last assertion, since

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0}\left\langle D^{\mathbb{R}^4_1}_{\eta}X, N\right\rangle &= 2\operatorname{Im}(\bar\zeta g)\operatorname{Re}\left[\left(\frac{g'}{g^2\hat\omega}\right)'\frac{d\gamma}{dt}\right] \\ &= -2\operatorname{Im}(\bar\zeta g)\operatorname{Im}\left[\left(\frac{g'}{g^2\hat\omega}\right)'\overline{\left(\frac{g'}{g}\right)}\right] \\ &= -2|g'|^2\operatorname{Im}(\bar\zeta g)\operatorname{Im}\left[\left(\frac{g'}{g^2\hat\omega}\right)'\frac{g}{g'}\right]. \qquad \square\end{aligned}$$

Since $d\hat s^2 = (1+|g|^2)^2\omega\bar\omega$ gives a Riemannian metric on U, ω does not vanish at a singular point p. Hence there exists a complex coordinate system z such that $\omega = dz$. On the other hand, $g \neq 0$ at the singular point p. Hence there exists a holomorphic function h in z such that $g = e^h$. We denote by f_h the CMC 1 face defined by the Weierstrass data $(g,\omega) = (e^h, dz)$. Let $\mathcal{O}(U)$ be the set of holomorphic functions defined on U, which is endowed with the *compact open* C^∞-*topology*. Then we have the induced topology on the set of CMC 1 faces $\{f_h\}_{h\in\mathcal{O}(U)}$. We shall prove Theorem 4.3.8. To prove the theorem, we rewrite the criteria in Theorem 4.3.3 in terms of h.

Lemma 4.3.6. *Let $h \in \mathcal{O}(U)$ and set*

$$\alpha_h := e^{-h}h', \qquad \beta_h := e^{-2h}\big(h'' - (h')^2\big),$$

where $' = d/dz$. Then

1. *a point $p \in U$ is a singular point of f_h if and only if* $\operatorname{Re} h = 0$,
2. *a singular point p is non-degenerate if and only if $\alpha_h \neq 0$,*
3. *a singular point p is a cuspidal edge if and only if* $\operatorname{Re}\alpha_h \neq 0$ *and* $\operatorname{Im}\alpha_h \neq 0$,
4. *a singular point p is a swallowtail if and only if* $\operatorname{Re}\alpha_h \neq 0$, $\operatorname{Im}\alpha_h = 0$, *and* $\operatorname{Re}\beta_h \neq 0$,
5. *a singular point p is a cuspidal cross cap if and only if* $\operatorname{Re}\alpha_h = 0$, $\operatorname{Im}\alpha_h \neq 0$, *and* $\operatorname{Re}\beta_h \neq 0$.

Proof. Since $g = e^h$, (1) is obvious. Moreover, a singular point p is non-degenerate if and only if $g' = e^{2h}\alpha_h$ does not vanish. Hence we have (2). Since $g'/(g^2\hat\omega) = \alpha_h$, the criterion for a front (Corollary 4.3.5) is $\operatorname{Re}\alpha_h \neq 0$. Then by Theorem 4.3.3, we have (3). Here,

$$\frac{g}{g'}\left(\frac{g'}{g^2\hat\omega}\right)' = \frac{e^{-h}}{h'}\left(h'' - (h')^2\right).$$

Then, if $\operatorname{Im}\alpha_h = 0$ and $\alpha_h \neq 0$,

$$\operatorname{Re}\left[\frac{g}{g'}\left(\frac{g'}{g^2\hat\omega}\right)'\right] = \operatorname{Re}\left[e^{-2h}\left(h'' - (h')^2\right)\right]\frac{1}{\alpha_h} = \frac{1}{\alpha_h}\operatorname{Re}\beta_h.$$

Then by Theorem 4.3.3, we have (4). On the other hand, if $\operatorname{Re}\alpha_h = 0$ and $\alpha_h \neq 0$,

$$\operatorname{Im}\left[\frac{g}{g'}\left(\frac{g'}{g^2\hat{\omega}}\right)'\right] = \operatorname{Im}\left[\frac{e^{-2h}}{\alpha_h}\left(h'' - (h')^2\right)\right] = -\frac{i}{\alpha_h}\operatorname{Re}\beta_h.$$

Thus we have (5). □

Let $J^2_H(U)$ be the space of 2-jets of holomorphic functions on U, which is identified with an 8-dimensional manifold

$$J^2_H(U) = U \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} = U \times \mathcal{F} \times \mathcal{F}_1 \times \mathcal{F}_2,$$

where $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ correspond to h, h' and h'' for $h \in \mathcal{O}(U)$. Then, the canonical map $j^2_h \colon U \to J^2_H(U)$ is given by $j_h(p) = \bigl(p, h(p), h'(p), h''(p)\bigr)$. The point $P \in J^2_H(U)$ is expressed as

$$P = (p, \hat{h}, \hat{h}_1, \hat{h}_2) = (p, \hat{u}, \hat{v}, \hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2), \tag{4.3.8}$$

where $\hat{h} = \hat{u} + i\hat{v}$, $\hat{h}_j = \hat{u}_j + i\hat{v}_j$ $(j = 1, 2)$. We set

$$A := \{P \in J^2_H(U) \mid \operatorname{Re}\hat{h} = 0, \hat{\alpha} = 0\},$$
$$B := \{P \in J^2_H(U) \mid \operatorname{Re}\hat{h} = 0, \operatorname{Im}\hat{\alpha} = 0, \operatorname{Re}\hat{\beta} = 0\},$$
$$C := \{P \in J^2_H(U) \mid \operatorname{Re}\hat{h} = 0, \operatorname{Re}\hat{\alpha} = 0, \operatorname{Re}\hat{\beta} = 0\},$$

where

$$\hat{\alpha} = e^{-\hat{h}}\hat{h}_1, \qquad \hat{\beta} = e^{-2\hat{h}}(\hat{h}_2 - \hat{h}_1^2).$$

LEMMA 4.3.7. *Let $S = A \cup B \cup C$ and*

$$\mathcal{G} := \{h \in \mathcal{O}(U) \mid j^2_h(U) \cap S = \emptyset\}.$$

Then all singular points of f_h are cuspidal edges, swallowtails or cuspidal cross caps if $h \in \mathcal{G}$.

PROOF. We set

$$\mathcal{S}_A := \{h \in \mathcal{O}(U) \mid j^2_h(U) \cap A \neq \emptyset\},$$
$$\mathcal{S}_B := \{h \in \mathcal{O}(U) \mid j^2_h(U) \cap B \neq \emptyset\},$$
$$\mathcal{S}_C := \{h \in \mathcal{O}(U) \mid j^2_h(U) \cap C \neq \emptyset\}.$$

Then we have $\mathcal{G} = (\mathcal{S}_A)^c \cap (\mathcal{S}_B)^c \cap (\mathcal{S}_C)^c$. Let $h \in \mathcal{G}$, and let $p \in U$ be a singular point of f_h. Since $h \notin \mathcal{S}_A$, p is a non-degenerate singular point. If f_h is not a front at p, then $\operatorname{Re}\alpha_h = 0$. Since $h \notin \mathcal{S}_C$, this implies that $\operatorname{Re}\beta_h \neq 0$, and hence p is a cuspidal cross cap. If f_h is a front at p and not a cuspidal edge, p is a swallowtail since $h \notin \mathcal{S}_B$. □

THEOREM 4.3.8. *Let $U \subset \mathbb{C}$ be a simply connected domain and K an arbitrary compact set, and let $S(K)$ be the subset of $\mathcal{O}(U)$ consisting of $h \in \mathcal{O}(U)$ such that the singular points of the CMC 1 face f_h are cuspidal edges, swallowtails or cuspidal cross caps. Then $S(K)$ is an open and dense subset of $\mathcal{O}(U)$.*

Theorem 4.3.8 can be proved in a similar way to Theorem 3.4 of [**KRSUY**] using the following lemma.

LEMMA 4.3.9. *$S = A \cup B \cup C$ is the union of a finite number of submanifolds in $J^2_H(U)$ of codimension 3.*

PROOF. Using parameters in (4.3.8), we can write

$$A = \{\hat{u} = \hat{u}_1 = \hat{v}_1 = 0\},$$

which is a codimension 3 submanifold in $J^2_H(U)$. Moreover, one can write

$$\begin{aligned}
&B = \{\zeta_1 = 0, \zeta_2 = 0, \zeta_3 = 0\}, \quad \text{where} \\
&\quad \zeta_1 = \hat{u}, \\
&\quad \zeta_2 = e^{-\hat{u}}(\hat{v}_1 \cos\hat{v} - \hat{u}_1 \sin\hat{v}), \\
&\quad \zeta_3 = e^{-2\hat{u}}\big(\cos 2\hat{v}(\hat{u}_2 - \hat{u}_1^2 + \hat{v}_1^2) + \sin 2\hat{v}(\hat{v}_2 - 2\hat{u}_1\hat{v}_1)\big).
\end{aligned}$$

Here, we can compute that

$$\frac{\partial(\zeta_1, \zeta_2, \zeta_3)}{\partial(\hat{u}, \hat{u}_1, \hat{v}_1)} = 2e^{-3\hat{u}}(\hat{u}_1 \cos\hat{v} + \hat{v}_1 \sin\hat{v}). \tag{4.3.9}$$

Since $(\hat{u}_1, \hat{v}_1) \neq (0, 0)$ and $\hat{v}_1 \cos\hat{v} - \hat{u}_1 \sin\hat{v} = 0$ hold on $B \setminus A$, (4.3.9) does not vanish on $B \setminus A$. Hence by the implicit function theorem, $B \setminus A$ is a submanifold of codimension 3.

Similarly, C is written as

$$\begin{aligned}
&C = \{\xi_1 = 0, \xi_2 = 0, \xi_3 = 0\}, \quad \text{where} \\
&\quad \xi_1 = \hat{u}, \qquad \xi_2 = e^{-\hat{u}}(\hat{u}_1 \cos\hat{v} + \hat{v}_1 \sin\hat{v}), \\
&\quad \xi_3 = e^{-2\hat{u}}\big(\cos 2\hat{v}(\hat{u}_2 - \hat{u}_1^2 + \hat{v}_1^2) + \sin 2\hat{v}(\hat{v}_2 - 2\hat{u}_1\hat{v}_1)\big).
\end{aligned}$$

Then we have

$$\frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(\hat{u}, \hat{u}_1, \hat{v}_1)} = 2e^{-3\hat{u}}(\hat{v}_1 \cos\hat{v} - \hat{u}_1 \sin\hat{v}).$$

Thus, $C \setminus A$ is a submanifold of codimension 3.

Hence $S = A \cup (B \setminus A) \cup (C \setminus A)$ is a union of submanifolds of codimension 3. $\square$

CHAPTER 5

Global theory of CMC 1 faces with elliptic ends

5.1. CMC 1 faces with elliptic ends

It is known that the only complete spacelike CMC 1 immersion is a flat totally umbilic immersion [**Ak, Ra**] (see Example 6.1.1). In the case of non-immersed CMC 1 faces in $\mathbb{S}^3_1$, we now define the notions of completeness and finiteness of total curvature away from singularities, like in [**KUY2, UY4**].

DEFINITION 5.1.1. Let M be a Riemann surface and $f : M \to \mathbb{S}^3_1$ a CMC 1 face. Set $ds^2 = f^*(ds^2_{\mathbb{S}^3_1})$. f is *complete* (resp. of *finite type*) if there exists a compact set C and a symmetric $(0, 2)$-tensor T on M such that T vanishes on $M \setminus C$ and $ds^2 + T$ is a complete (resp. finite total curvature) Riemannian metric.

REMARK 5.1.2. For CMC 1 immersions in $\mathbb{S}^3_1$, the Gauss curvature K is non-negative. So the total curvature is the same as the total absolute curvature. However, for CMC 1 faces with singular points the total curvature is never finite, not even on neighborhoods of singular points, as can be seen from the form $K = 4dgd\bar{g}/(1 - |g|^2)^4\omega\bar{\omega}$ for the Gaussian curvature, see also [**ER**]. Hence the phrase "finite type" is more appropriate in Definition 5.1.1.

REMARK 5.1.3. The universal covering of a complete (resp. finite type) CMC 1 face might not be complete (resp. finite type), because the singular set might not be compact on the universal cover.

Let $f : M \to \mathbb{S}^3_1$ be a complete CMC 1 face of finite type. Then $(M, ds^2 + T)$ is a complete Riemannian manifold of finite total curvature. So by [**H**, Theorem 13], M has finite topology, where we define a manifold to be of finite topology if it is diffeomorphic to a compact manifold with finitely many points removed. The *ends* of f correspond to the removed points of that Riemann surface.

Let $\varrho : \widetilde{M} \to M$ be the universal cover of M, and $F : \widetilde{M} \to SL(2, \mathbb{C})$ a holomorphic null lift of a CMC 1 face $f : M \to \mathbb{S}^3_1$. We fix a point $z_0 \in M$. Let $\gamma : [0, 1] \to M$ be a loop so that $\gamma(0) = \gamma(1) = z_0$. Then there exists a unique deck transformation τ of $\widetilde{M}$ associated to the homotopy class of γ. We define the monodromy representation Φ_γ of F as

$$F \circ \tau = F\Phi_\gamma. \tag{5.1.1}$$

Note that $\Phi_\gamma \in SU(1,1)$ for any loop γ, since f is well-defined on M. So Φ_γ is conjugate to either

(5.1.2)
$$\mathcal{E} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \text{ or } \mathcal{H} = \pm \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix} \text{ or } \mathcal{P} = \pm \begin{pmatrix} 1+i & 1 \\ 1 & 1-i \end{pmatrix}$$

for $\theta \in [0, 2\pi)$, $s \in \mathbb{R} \setminus \{0\}$.

Definition 5.1.4. Let $f : M \to \mathbb{S}^3_1$ be a complete CMC 1 face of finite type with holomorphic null lift F. An end of f is called an *elliptic end* or *hyperbolic end* or *parabolic end* if its monodromy representation is conjugate to $\mathcal{E}$ or $\mathcal{H}$ or $\mathcal{P}$ in $SU(1,1)$, respectively.

Remark 5.1.5. A matrix
$$X = \begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix} \in SU(1,1)$$
acts on the hyperbolic plane in the Poincaré model $\mathbb{H}^2 = (\{w \in \mathbb{C} \,|\, |w| < 1\}, ds^2_{\mathbb{H}^2} = 4dwd\bar{w}/(1-|w|^2)^2)$ as an isometry:
$$\mathbb{H}^2 \ni w \mapsto X \star w = \frac{pw+q}{\bar{q}w+\bar{p}} \in \mathbb{H}^2.$$
X is called *elliptic* if this action has only one fixed point which is in $\mathbb{H}^2$. X is called *hyperbolic* if there exist two fixed points, both in the ideal boundary $\partial\mathbb{H}^2$. X is called *parabolic* if there exists only one fixed point which is in $\partial\mathbb{H}^2$. This is what motivates the terminology in Definition 5.1.4.

Since any matrix in $SU(2)$ is conjugate to $\mathcal{E}$ in $SU(2)$, CMC 1 immersions in $\mathbb{H}^3$ and CMC 1 faces with elliptic ends in $\mathbb{S}^3_1$ share many analogous properties. So in this chapter we consider CMC 1 faces with only elliptic ends. We leave the study of hyperbolic ends and parabolic ends for another occasion.

Proposition 5.1.6. *Let V be a neighborhood of an end of f and $f|_V$ a spacelike CMC 1 immersion of finite total curvature which is complete at the end. Suppose the end is elliptic. Then there exists a holomorphic null lift $F : \widetilde{V} \to SL(2,\mathbb{C})$ of f with Weierstrass data (g, ω) associated to F such that*
$$d\hat{s}^2|_V = (1+|g|^2)^2\omega\bar{\omega}$$
is single-valued on V. Moreover, $d\hat{s}^2|_V$ has finite total curvature and is complete at the end.

Proof. Let $\gamma : [0,1] \to V$ be a loop around the end and τ the deck transformation associated to γ. Take a holomorphic null lift $F_0 : \widetilde{V} \to SL(2,\mathbb{C})$ of f. Then by definition of an elliptic end, there exists a $\theta \in [0, 2\pi)$ such that
$$F_0 \circ \tau = F_0 P E_\theta P^{-1},$$

where $P \in SU(1,1)$ and

$$E_\theta = \mathrm{diag}(e^{i\theta}, e^{-i\theta}) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Defining the holomorphic null lift $F = F_0 P$ of f and defining (g, ω) to be the Weierstrass data associated to F, we have

$$F \circ \tau = F E_\theta, \quad g \circ \tau = e^{-2i\theta} g \quad \text{and} \quad \omega \circ \tau = e^{2i\theta}\omega.$$

Thus, $|g \circ \tau| = |g|$ and $|\omega \circ \tau| = |\omega|$. This implies $d\hat{s}^2|_V$ is single-valued on V. Let T be a $(0,2)$-tensor as in Definition 5.1.1. Then by Equation (2.2.11), we have $ds^2 + T \leq d\hat{s}^2|_V$ on $V \setminus C$. Thus, if $ds^2 + T$ is complete, $d\hat{s}^2|_V$ is also complete. We denote the Gaussian curvature of the metric $d\hat{s}^2|_V$ by $K_{d\hat{s}^2|_V}$ (note that $K_{d\hat{s}^2|_V}$ is non-positive). Then we have

$$(-K_{d\hat{s}^2|_V}) d\hat{s}^2|_V = \frac{4 dg d\bar{g}}{(1+|g|^2)^2} \leq \frac{4 dg d\bar{g}}{(1-|g|^2)^2} = K ds^2$$

on $V \setminus C$. Thus, if $ds^2 + T$ is of finite total curvature, the total curvature of $d\hat{s}^2|_V$ is finite, proving the proposition. □

Proposition 5.1.7. *Let $f : M \to \mathbb{S}^3_1$ be a complete CMC 1 face of finite type with elliptic ends. Then there exists a compact Riemann surface $\overline{M}$ and finite number of points $p_1, \dots, p_n \in \overline{M}$ so that M is biholomorphic to $\overline{M} \setminus \{p_1, \dots, p_n\}$. Moreover, the Hopf differential Q of f extends meromorphically to $\overline{M}$.*

Proof. Since f is of finite type, M is finitely connected, by [**H**, Theorem 13]. Consequently, there exists a compact region $M_0 \subset M$, bounded by a finite number of regular Jordan curves $\gamma_1, \dots, \gamma_n$, such that each component M_j of $M \setminus M_0$ can be conformally mapped onto the annulus $D_j = \{z \in \mathbb{C} \,|\, r_j < |z| < 1\}$, where γ_j corresponds to the set $\{|z| = 1\}$. Then by Proposition 5.1.6, there exists $d\hat{s}^2|_{M_j}$ which is single-valued on M_j and is of finite total curvature and is complete at the end, and so that $K_{d\hat{s}^2|_{M_j}}$ is non-positive. Therefore by [**La**, Proposition III-16] or [**O2**, Theorem 9.1], $r_j = 0$, and hence each M_j is biholomorphic to the punctured disk $\{z \in \mathbb{C} \,|\, 0 < |z| < 1\}$. We can, using the biholomorphism from M_j to D_j, replace M_j in M with D_j without affecting the conformal structure of M. Thus, without loss of generality, $M = \overline{M} \setminus \{p_1, \dots, p_n\}$ for some compact Riemann surface $\overline{M}$ and a finite number of points $p_1, \dots, p_n$ in $\overline{M}$, and each M_j becomes a punctured disk about p_j. Hence, by (4) of Remark 2.2.7, we can apply [**B**, Proposition 5] to $\hat{f}_j := \hat{f}|_{M_j}$ to see that $Q = \hat{Q}$ extends meromorphically to $M_j \cup \{p_j\}$, proving the proposition. □

Let $\Delta = \{z \in \mathbb{C} \,|\, |z| < 1\}$ and $\Delta^* = \Delta \setminus \{0\}$. Let $f : \Delta^* \to \mathbb{S}^3_1$ be a conformal spacelike CMC 1 immersion of finite total curvature which has a complete elliptic end at the origin. Then we can take the Weierstrass data associated to f in the following form:

$$\begin{cases} g = z^\mu \tilde{g}(z), & \mu > 0, \quad \tilde{g}(0) \neq 0, \\ \omega = w(z)dz = z^\nu \tilde{w}(z)dz, & \nu \leq -1, \quad \tilde{w}(0) \neq 0, \end{cases} \tag{5.1.3}$$

where $\tilde{g}$ and $\tilde{w}$ are holomorphic functions on Δ and $\mu + \nu \in \mathbb{Z}$. (See [**UY1**] and also [**B**, Proposition 4] for case that $\mu, \nu \in \mathbb{R}$. Then applying a transformation as in Equation (3.2.13) if necessary, we may assume $\mu > 0$. Completeness of the end then gives $\nu \leq -1$.)

Definition 5.1.8. The end $z = 0$ of $f : \Delta^* \to \mathbb{S}^3_1$ is called *regular* if the hyperbolic Gauss map G extends meromorphically across the end. Otherwise, the end is called *irregular*.

Since Q extends meromorphically to each end, we have the following proposition, by (4) and (5) of Remark 2.2.7:

Proposition 5.1.9. [**B**, Proposition 6] *An end $f : \Delta^* \to \mathbb{S}^3_1$ is regular if and only if the order at the end of the Hopf differential of f is at least -2.*

Remark 5.1.10. In [**LY**], Lee and Yang define *normal ends* and *abnormal ends*. Both normal and abnormal ends are biholomorphic to a punctured disk Δ^*, and the Hopf differential has a pole of order 2 at the origin. Normal ends are elliptic ends, and abnormal ends are hyperbolic ends. Moreover, the Lee-Yang catenoids with normal ends are complete in our sense. However, the Lee-Yang catenoids with abnormal ends include incomplete examples, because the singular set of these examples accumulates at the ends. (In fact, CMC 1 face with hyperbolic ends cannot be complete, which is shown in [**FRUYY**]).

5.2. Embeddedness of elliptic ends

In this section we give a criterion for embeddedness of elliptic ends, which is based on results in [**UY1**]. This criterion will be applied in the next section.

Let $f : \Delta^* \to \mathbb{S}^3_1$ be a conformal spacelike CMC 1 immersion of finite total curvature with a complete regular elliptic end at the origin. Let $\gamma : [0, 1] \to \Delta^*$ be a loop around the origin and τ the deck transformation of $\widetilde{\Delta^*}$ associated to the homotopy class of γ. Then by the same argument as in the proof of Proposition 5.1.6, there exists the holomorphic null lift F of f such that $F \circ \tau = F E_\theta$ for some $\theta \in [0, 2\pi)$, where $E_\theta = \mathrm{diag}(e^{i\theta}, e^{-i\theta})$. Since $E_\theta \in SU(2)$, $\hat{f} := FF^*$ in $\mathbb{H}^3$ is single-valued on Δ^*.

Let (g, ω) be the Weierstrass data associated to F, defined as in (5.1.3). Then by (4) in Remark 2.2.7, $\hat{f} : \Delta^* \to \mathbb{H}^3$ is a conformal CMC 1 immersion with the induced metric $d\hat{s}^2 = (1 + |g|^2)^2 \omega\bar{\omega}$. Thus by the final sentence of Proposition 5.1.6, $\hat{f}$ has finite total curvature and is complete at the origin.

Since $\hat{f}$ has the same Hopf differential Q as f, f having a regular end immediately implies that $\hat{f}$ has a regular end.

Furthermore we show the following theorem:

THEOREM 5.2.1. *Let $f : \Delta^* \to \mathbb{S}^3_1$ be a conformal spacelike CMC 1 immersion of finite total curvature with a complete regular elliptic end at the origin. Then there exists a holomorphic null lift $F : \widetilde{\Delta^*} \to SL(2,\mathbb{C})$ of f (that is, $f = Fe_3F^*$) such that $\hat{f} = FF^*$ is a conformal CMC 1 finite-total-curvature immersion from Δ^* into $\mathbb{H}^3$ with a complete regular end at the origin. Moreover, f has an embedded end if and only if $\hat{f}$ has an embedded end.*

REMARK 5.2.2. The converse of the first part of Theorem 5.2.1 is also true, that is, the following holds: Let $\hat{f} : \Delta^* \to \mathbb{H}^3$ be a conformal CMC 1 immersion of finite total curvature with a complete regular end at the origin. Take a holomorphic null lift $F : \widetilde{\Delta^*} \to SL(2,\mathbb{C})$ of $\hat{f}$ (that is, $\hat{f} = FF^*$) such that the associated Weierstrass data (g, ω) is written as in (5.1.3). Then $f = Fe_3F^*$ is a conformal spacelike CMC 1 finite-total-curvature immersion from Δ^* into $\mathbb{S}^3_1$ with a complete regular elliptic end at the origin. Moreover, $\hat{f}$ has an embedded end if and only if f has an embedded end. See Proposition 6.3.1 below.

We already know that such an F exists. So we must prove the equivalency of embeddedness between the ends of f and $\hat{f}$. To prove this we prepare three lemmas.

LEMMA 5.2.3 ([**UY1**, Lemma 5.3]). *There exists a $\Lambda \in SL(2,\mathbb{C})$ such that*

$$\Lambda F = \begin{pmatrix} z^{\lambda_1} a(z) & z^{\lambda_2} b(z) \\ z^{\lambda_1 - m_1} c(z) & z^{\lambda_2 - m_2} d(z) \end{pmatrix}, \tag{5.2.1}$$

where a, b, c, d are holomorphic functions on Δ that do not vanish at the origin, and $\lambda_j \in \mathbb{R}$ and $m_j \in \mathbb{N}$ $(j = 1, 2)$ are defined as follows:

(1) *If* $\mathrm{Ord}_0 Q = \mu + \nu - 1 = -2$, *then*

$$m_1 = m_2, \quad \lambda_1 = \frac{-\mu + m_j}{2} \quad \textit{and} \quad \lambda_2 = \frac{\mu + m_j}{2}. \tag{5.2.2}$$

(2) *If* $\mathrm{Ord}_0 Q = \mu + \nu - 1 \geq -1$, *then*

$$m_1 = -(\nu + 1), \quad m_2 = 2\mu + \nu + 1, \quad \lambda_1 = 0 \quad \textit{and} \quad \lambda_2 = m_2. \tag{5.2.3}$$

Note that in either case we have $\nu < -1$ and

$$\lambda_1 < \lambda_2, \quad \lambda_1 - m_1 < \lambda_2 - m_2 \quad \text{and} \quad \lambda_1 - m_1 < 0. \tag{5.2.4}$$

Note also that in the second case we have $m_1 < m_2$.

PROOF OF LEMMA 5.2.3. F satisfies Equation (3.2.9), which is precisely Equation (1.5) in [**UY1**]. So we can apply [**UY1**, Lemma 5.3], since that lemma is based on Equation (1.5) in [**UY1**]. This gives the result. $\square$

It follows that

$$\begin{aligned}\Lambda f \Lambda^* &= (\Lambda F)e_3(\Lambda F)^* \\ &= \begin{pmatrix} |z|^{2\lambda_1}|a|^2 - |z|^{2\lambda_2}|b|^2 & |z|^{2\lambda_1}\bar{z}^{-m_1}a\bar{c} - |z|^{2\lambda_2}\bar{z}^{-m_2}b\bar{d} \\ |z|^{2\lambda_1}z^{-m_1}\bar{a}c - |z|^{2\lambda_2}z^{-m_2}\bar{b}d & |z|^{2(\lambda_1-m_1)}|c|^2 - |z|^{2(\lambda_2-m_2)}|d|^2 \end{pmatrix}.\end{aligned}$$

Note that $\Lambda f \Lambda^*$ is congruent to $f = Fe_3F^*$, because $(\Lambda F)^{-1}d(\Lambda F) = F^{-1}dF$ determines both the first and second fundamental forms as in Equation (2.2.11).

To study the behavior of the elliptic end f, we present the elliptic end in a 3-ball model as follows: We set

$$\begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} = \Lambda f \Lambda^*.$$

Since

$$\begin{aligned} x_0 &= \frac{1}{2}\mathrm{trace}(\Lambda f \Lambda^*) \\ &= \frac{1}{2}\left(|z|^{2\lambda_1}|a|^2 - |z|^{2\lambda_2}|b|^2 + |z|^{2(\lambda_1-m_1)}|c|^2 - |z|^{2(\lambda_2-m_2)}|d|^2\right), \end{aligned}$$

(5.2.4) implies that $\lim_{z\to 0} x_0(z) = \infty$. So we may assume that $x_0(z) > 1$ for any $z \in \Delta^*$. So we can define a bijective map

$$p : \{(x_0, x_1, x_2, x_3) \in \mathbb{S}^3_1 \,|\, x_0 > 1\} \to \mathbb{D}^3 := D^3_1 \setminus \overline{D^3_{1/\sqrt{2}}}$$

as

$$p(x_0, x_1, x_2, x_3) := \frac{1}{1+x_0}(x_1, x_2, x_3),$$

where D^3_r denotes the open ball of radius r in $\mathbb{R}^3$ and $\overline{D^3_r} = D^3_r \cup \partial D^3_r$ (See Figure 5.2.1).

We set $(X_1, X_2, X_3) = p \circ (\Lambda f \Lambda^*)$. Then we have

$$X_1 + iX_2 = \frac{2a\bar{c}|z|^{2(\lambda_1-m_1)}z^{m_1} - 2b\bar{d}|z|^{2(\lambda_2-m_2)}z^{m_2}}{2 + |a|^2|z|^{2\lambda_1} + |c|^2|z|^{2(\lambda_1-m_1)} - |b|^2|z|^{2\lambda_2} - |d|^2|z|^{2(\lambda_2-m_2)}}, \tag{5.2.5}$$

$$X_3 = \frac{|a|^2|z|^{2\lambda_1} - |c|^2|z|^{2(\lambda_1-m_1)} - |b|^2|z|^{2\lambda_2} + |d|^2|z|^{2(\lambda_2-m_2)}}{2 + |a|^2|z|^{2\lambda_1} + |c|^2|z|^{2(\lambda_1-m_1)} - |b|^2|z|^{2\lambda_2} - |d|^2|z|^{2(\lambda_2-m_2)}}. \tag{5.2.6}$$

We now define a function $U : \Delta^* \to \mathbb{C}$ that will be useful for proving Theorem 5.2.1:

$$U(z) = z^{-m_1}(X_1 + iX_2). \tag{5.2.7}$$

Then by making just a few sign changes to the argument in [**UY1**, Lemma 5.4], we have the following lemma:

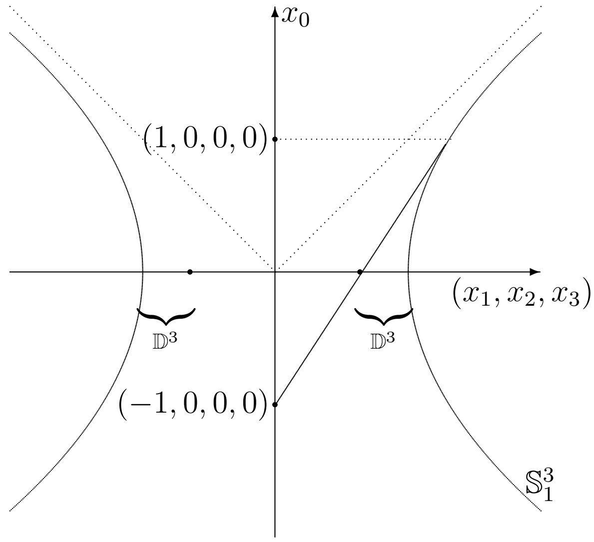


FIGURE 5.2.1. The projection p.

LEMMA 5.2.4. *The function U is a C^∞ function on Δ^* that extends continuously to Δ with $U(0) \neq 0$. Moreover,*

$$\lim_{z\to 0} z\frac{\partial U}{\partial z} = 0 \quad \textit{and} \quad \lim_{z\to 0} z\frac{\partial U}{\partial \bar{z}} = 0. \tag{5.2.8}$$

PROOF. If $\mathrm{Ord}_0 Q = -2$, then Equation (5.2.7) is reduced to

$$U(z) = \frac{2a\bar{c} - 2b\bar{d}|z|^{2\mu}}{2|z|^{\mu+m_1} + |a|^2|z|^{2m_1} + |c|^2 - |b|^2|z|^{2(\mu+m_1)} - |d|^2|z|^{2\mu}}. \tag{5.2.9}$$

Then we have $U(0) = 2a(0)/c(0) \neq 0$, because $\mu > 0$ and $m_1 \geq 1$. Also, by a straightforward calculation, we see that Equation (5.2.8) holds.

If $\mathrm{Ord}_0 Q \geq -1$, then Equation (5.2.7) is reduced to

$$U(z) = \frac{2a\bar{c} - 2b\bar{d}z^{m_2}\bar{z}^{m_1}}{(2 + |a|^2 - |d|^2)|z|^{2m_1} - |b|^2|z|^{2(m_1+m_2)} + |c|^2}. \tag{5.2.10}$$

Then we have $U(0) = 2a(0)/c(0) \neq 0$, because $m_1, m_2 \geq 1$. Also, since U is C^1-differentiable at the origin, we see that Equation (5.2.8) holds. □

Also, we have the following lemma, analogous to Remark 5.5 in [**UY1**]:

LEMMA 5.2.5. *$p \circ (\Lambda f \Lambda^*) = (X_1, X_2, X_3)$ converges to the single point $(0,0,-1) \in \partial D^3_1$. Moreover, $p \circ (\Lambda f \Lambda^*)$ is tangent to ∂D^3_1 at the end $z = 0$.*

PROOF. By (5.2.2)–(5.2.6), we see that

$$\lim_{z\to 0}(X_1, X_2, X_3) = (0,0,-1).$$

Define a function $V : \Delta^* \to \mathbb{C}$ by $V := z^{-m_1}(X_3+1)$. Then from either case of the proof of Lemma 5.2.4, we see that

$$\lim_{z\to 0} V = 0 \quad \text{and} \quad \lim_{z\to 0} z\frac{\partial V}{\partial z} = 0.$$

Therefore

$$0 = \lim_{z\to 0} z\frac{\partial V}{\partial z} = \lim_{z\to 0}\left(z^{-m_1+1}\frac{\partial X_3}{\partial z}\right) - m_1 V(0),$$

$$0 = \lim_{z\to 0} z\frac{\partial U}{\partial z} = \lim_{z\to 0}\left(z^{-m_1+1}\frac{\partial (X_1+iX_2)}{\partial z}\right) - m_1 U(0)$$

imply that

$$\lim_{X_1+iX_2\to 0}\frac{\partial X_3}{\partial(X_1+iX_2)} = \frac{V(0)}{U(0)} = 0,$$

proving the lemma. □

PROOF OF THEOREM 5.2.1. [**UY1**, Theorem 5.2] shows that $\hat{f}$ has an embedded end if and only if $m_1 = 1$, so the theorem will be proven by showing that f also has an embedded end if and only if $m_1 = 1$. We now show this:

Since $U(0) \neq 0$, by taking a suitable branch we can define the function $u : \Delta \to \mathbb{C}$ by

$$u(z) = z\,(U(z))^{1/m_1}.$$

Then u is a C^1 function such that

$$\frac{\partial u}{\partial z}(0) \neq 0 \quad \text{and} \quad \frac{\partial u}{\partial \bar{z}} = 0. \tag{5.2.11}$$

Assume that $m_1 = 1$. By Equation (5.2.7), $X_1 + iX_2 = u$ holds. Then by Equation (5.2.11), $X_1 + iX_2$ is one-to-one on some neighborhood of the origin $z = 0$. Hence f has an embedded end.

Conversely, assume that f has an embedded end. Let $p_3 : \mathbb{D}^3 \to \Delta$ be the projection defined by $p_3(X_1, X_2, X_3) = X_1 + iX_2$. By Equation (5.2.7) we have

$$p_3 \circ p \circ (\Lambda f \Lambda^*) = u^{m_1}. \tag{5.2.12}$$

By Equations (5.2.11) and (5.2.12) the map $p_3\circ p\circ(\Lambda f\Lambda^*)$ is an m_1-fold cover of $\Delta^*_\varepsilon = \{z \in \mathbb{C} \,|\, 0 < |z| < \varepsilon\}$ for a sufficiently small $\varepsilon > 0$. Thus, by Lemma 5.2.5, m_1 must be 1, by the same topological arguments as at the end of the proof of Theorem 5.2 in [**UY1**].

Therefore, we have that f has an embedded end if and only if $m_1 = 1$. □

5.3. The Osserman-type inequality

Let $f : M = \overline{M} \setminus \{p_1, \ldots, p_n\} \to \mathbb{S}^3_1$ be a complete CMC 1 face of finite type with hyperbolic Gauss map G and Hopf differential Q.

DEFINITION 5.3.1. We set

$$d\hat{s}^{\sharp 2} := (1+|G|^2)^2 \frac{Q}{dG}\overline{\left(\frac{Q}{dG}\right)} \tag{5.3.1}$$

and call it the *lift-metric* of the CMC 1 face f. We also set

$$d\hat{\sigma}^{\sharp 2} := (-K_{d\hat{s}^{\sharp 2}})d\hat{s}^{\sharp 2} = \frac{4dGd\overline{G}}{(1+|G|^2)^2}.$$

REMARK 5.3.2. Since G and Q are defined on M, both $d\hat{s}^{\sharp 2}$ and $d\hat{\sigma}^{\sharp 2}$ are also defined on M.

We define the order of pseudometrics as in [**UY2**, Definition 2.1], that is:

DEFINITION 5.3.3. A pseudometric $d\varsigma^2$ on $\overline{M}$ is of order m_j at p_j if $d\varsigma^2$ has a local expression

$$d\varsigma^2 = e^{2u_j} dz d\bar{z}$$

around p_j such that $u_j - m_j \log|z - z(p_j)|$ is continuous at p_j. We denote m_j by $\mathrm{Ord}_{p_j}(d\varsigma^2)$. In particular, if $d\varsigma^2$ gives a Riemannian metric around p_j, then $\mathrm{Ord}_{p_j}(d\varsigma^2) = 0$.

We now apply [**UY3**, Lemma 3] for regular ends in $\mathbb{H}^3$ to regular elliptic ends in $\mathbb{S}^3_1$, that is, we show the following proposition:

PROPOSITION 5.3.4. *Let $f : \Delta^* \to \mathbb{S}^3_1$ be a conformal spacelike CMC 1 immersion of finite total curvature with a complete regular elliptic end at the origin $z = 0$. Then the following inequality holds:*

$$\mathrm{Ord}_0(d\hat{\sigma}^{\sharp 2}) - \mathrm{Ord}_0(Q) \geq 2. \tag{5.3.2}$$

Moreover, equality holds if and only if the end is embedded.

PROOF. By Theorem 5.2.1, there exists a holomorphic null lift $F : \widetilde{\Delta^*} \to SL(2,\mathbb{C})$ of f such that $\hat{f} = FF^* : \Delta^* \to \mathbb{H}^3$ is a conformal CMC 1 immersion of finite total curvature with a complete regular end at the origin. Then by [**UY3**, Lemma 3], we have (5.3.2). Moreover, equality holds if and only if the end of $\hat{f}$ is embedded, by [**UY3**, Lemma 3]. This is equivalent to the end of f being embedded, by Theorem 5.2.1, proving the proposition. $\square$

The following lemma is a variant on known results in [**Yu, KTUY**]. In fact, [**Yu**] showed that $d\hat{s}^2$ is complete if and only if $d\hat{s}^{\sharp 2}$ is complete, see also [**KTUY**, Lemma 4.1].

LEMMA 5.3.5. *Let $f : M \to \mathbb{S}^3_1$ be a CMC 1 face. Assume that each end of f is regular and elliptic. If f is complete and of finite type, then the lift-metric $d\hat{s}^{\sharp 2}$ is complete and of finite total curvature on M. In particular,*

$$\operatorname{Ord}_{p_j}(d\hat{s}^{\sharp 2}) \le -2 \tag{5.3.3}$$

holds at each end p_j $(j = 1, \dots, n)$.

PROOF. Since f is complete and of finite type, each end is complete and has finite total curvature. Then by (5.3.2) and the relation

$$\operatorname{Ord}_{p_j}(d\hat{s}^{\sharp 2}) + \operatorname{Ord}_{p_j}(d\hat{\sigma}^{\sharp 2}) = \operatorname{Ord}_{p_j}(Q) \tag{5.3.4}$$

(which follows from the Gauss equation $d\hat{\sigma}^{\sharp 2} d\hat{s}^{\sharp 2} = 4Q\overline{Q}$), we have (5.3.3) at each end p_j. Hence $d\hat{s}^{\sharp 2}$ is a complete metric. Also, again by (5.3.2), we have

$$\operatorname{Ord}_{p_j}(d\hat{\sigma}^{\sharp 2}) \ge 2 + \operatorname{Ord}_{p_j}(Q) \ge 0,$$

because p_j is regular (that is, $\operatorname{Ord}_{p_j}(Q) \ge -2$, by Proposition 5.1.9). This implies that the total curvature of $d\hat{s}^{\sharp 2}$ is finite. $\square$

REMARK 5.3.6. Consider a CMC 1 face with regular elliptic ends. If it is complete and of finite type, then, by Lemma 5.3.5, the lift-metric is complete and of finite total curvature. But the converse is not true. See [**FRUYY**].

THEOREM 5.3.7 (Osserman-type inequality). *Let $f : M \to \mathbb{S}^3_1$ be a complete CMC 1 face of finite type with n elliptic ends and no other ends. Let G be its hyperbolic Gauss map. Then the following inequality holds:*

$$2\deg(G) \ge -\chi(M) + n, \tag{5.3.5}$$

where $\deg(G)$ *is the mapping degree of G (if G has essential singularities, then we define* $\deg(G) = \infty$*). Furthermore, equality holds if and only if each end is regular and embedded.*

REMARK 5.3.8. As we remarked in the introduction, the completeness of a CMC 1 face f implies that f must be of finite type. See the forthcoming paper [**FRUYY**].

PROOF OF THEOREM 5.3.7. Recall that we can set $M = \overline{M} \setminus \{p_1, \dots, p_n\}$, where $\overline{M}$ is a compact Riemann surface and $p_1, \dots, p_n$ is a set of points in $\overline{M}$, by Proposition 5.1.7. If f has irregular ends, then G has essential singularities at those ends. So $\deg(G) = \infty$ and then (5.3.5) automatically holds. Hence we may assume f has only regular ends. Using the Riemann-Hurwicz formula and the Gauss equation

$d\hat{s}^{\sharp 2} d\hat{\sigma}^{\sharp 2} = 4Q\overline{Q}$, we have

$$\begin{aligned}
2\deg(G) &= \chi(\overline{M}) + \sum_{p\in\overline{M}} \mathrm{Ord}_p d\hat{\sigma}^{\sharp 2} \\
&= \chi(\overline{M}) + \sum_{p\in\overline{M}} \left(\mathrm{Ord}_p Q - \mathrm{Ord}_p d\hat{s}^{\sharp 2}\right) \\
&= \chi(\overline{M}) + \sum_{p\in\overline{M}} \mathrm{Ord}_p Q - \sum_{p\in M} \mathrm{Ord}_p d\hat{s}^{\sharp 2} - \sum_{j=1}^{n} \mathrm{Ord}_{p_j} d\hat{s}^{\sharp 2} \\
&= -\chi(\overline{M}) - \sum_{j=1}^{n} \mathrm{Ord}_{p_j} d\hat{s}^{\sharp 2} \\
&\geq -\chi(\overline{M}) + 2n \quad (\text{because } d\hat{s}^{\sharp 2} \text{ is complete, by (5.3.3)}) \\
&= -\chi(M) + n.
\end{aligned}$$

Equality in (5.3.5) holds if and only if equality in (5.3.3) holds at each end, which is equivalent to equality in (5.3.2) holding at each end, by Equation (5.3.4). Thus by Proposition 5.3.4, we have the conclusion. $\square$

CHAPTER 6

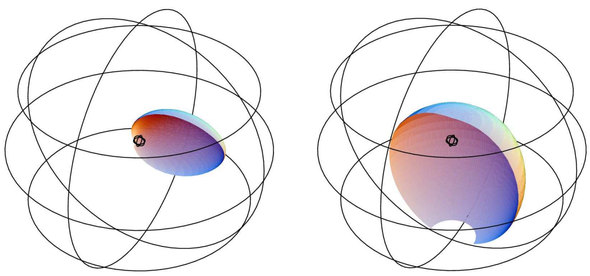
Examples

To visualize CMC 1 faces, we use the hollow ball model $\mathscr{H}$ of $\mathbb{S}^3_1$, as introduced in Subsection 2.1.4.

6.1. Basic examples

We shall first introduce three simply-connected examples, using the same Weierstrass data as for CMC 1 immersions in $\mathbb{H}^3$.

Example 6.1.1. The CMC 1 face associated to horosphere in $\mathbb{H}^3$ is given by the Weierstrass data $(g, \omega) = (c_1, c_2 dz)$ with $c_1 \in \mathbb{C} \setminus \{z \in \mathbb{C} \,|\, |z| = 1\}$, $c_2 \in \mathbb{C} \setminus \{0\}$ on the Riemann surface $\mathbb{C}$. This CMC 1 face has no singularities. So this example is indeed a complete spacelike CMC 1 immersion. [**Ak**] and [**Ra**, Theorem 7] independently showed that the only complete spacelike CMC 1 immersion in $\mathbb{S}^3_1$ is a flat totally umbilic immersion.



$\{z \in \mathbb{C} \,|\, {}^{|z|<5,}_{0 \le \arg z \le \pi}\}$. $\{z \in \mathbb{C} \,|\, {}^{|z|<10,}_{0 \le \arg z \le \pi}\}$.

Figure 6.1.1. Pictures of Example 6.1.1. The left-hand side is drawn with $c_1 = 1.2$, $c_2 = 1$ and the right-hand side is drawn with $c_1 = 0$, $c_2 = 1$.

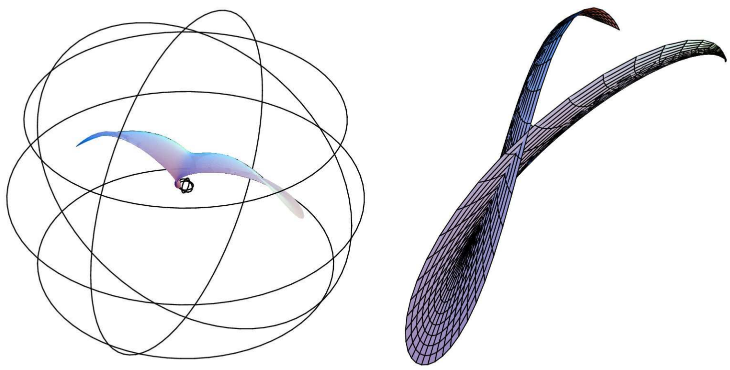
Example 6.1.2. The CMC 1 face associated to the Enneper cousin in $\mathbb{H}^3$ is given by the Weierstrass data $(g, \omega) = (z, c\,dz)$ with $c \in \mathbb{R} \setminus \{0\}$ on the Riemann surface $\mathbb{C}$. The induced metric $ds^2 = c^2(1 - |z|^2)^2 dz d\bar{z}$ degenerates where $|z| = 1$. Take a $p \in \mathbb{C}$ which satisfies $|p| = 1$. Define

$$\beta := \left(\frac{1 + |z|^2}{1 - |z|^2} \right)^2 .$$

Then

$$\lim_{\substack{z \to p \\ z \in W}} \beta ds^2 = 4c^2 dz d\bar{z}.$$

So all singularities are admissible and hence this is a CMC 1 face. Moreover, it is easily seen that this CMC 1 face is complete and of finite type. By Theorem 4.3.3, we see that ± 1, $\pm i$ are swallowtails, $\pm e^{\pm i\pi/4}$ are cuspidal cross caps, and other singular points are cuspidal edges. Since this CMC 1 face is simply-connected, the end of this CMC 1 face is an elliptic end. Since $\mathrm{Ord}_\infty Q = -4 < -2$, the end of this CMC 1 face is irregular. Hence this CMC 1 face does not satisfy equality in the inequality (5.3.5).



$\{z \in \mathbb{C} \mid |z| < 1.3\}$. $\quad$ $\{z \in \mathbb{C} \mid {}^{\,0.8<|z|<1.3}_{\pi-1<\arg z<\pi+1}\}$.

FIGURE 6.1.2. Pictures of Example 6.1.2, where $c = 1$.

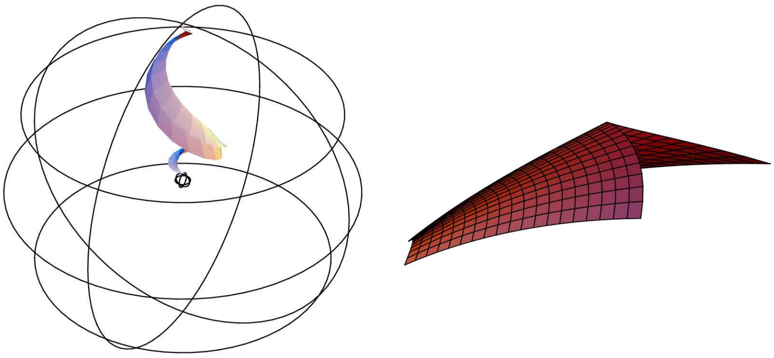
EXAMPLE 6.1.3. The CMC 1 face associated to the helicoid cousin in $\mathbb{H}^3$ is given by the Weierstrass data $(g, \omega) = (e^z, ice^{-z}dz)$ with $c \in \mathbb{R} \setminus \{0\}$ on the Riemann surface $\mathbb{C}$. Set $z = x + iy$. The induced metric $ds^2 = 4c^2 \sinh^2 x(dx^2 + dy^2)$ degenerates where $x = 0$. Take a $p \in \mathbb{C}$ which satisfies $\mathrm{Re}(p) = 0$. Define $\beta := \tanh^{-2} x$. Then

$$\lim_{\substack{z \to p \\ z \in W}} \beta ds^2 = 4c^2(dx^2 + dy^2).$$

So all singularities are admissible and hence this is a CMC 1 face. Since the singular set is non-compact, this CMC 1 face is neither complete nor of finite type.

6.2. Examples of genus 0 with two ends

To produce non-simply-connected CMC 1 faces, we first give the following definition:



$\left\{z \in \mathbb{C} \,\middle|\, {}^{-0.9<\mathrm{Re}z<0.9}_{-4\pi<\mathrm{Im}z<4\pi}\right\}$. $\quad$ $\left\{z \in \mathbb{C} \,\middle|\, {}^{-0.8<\mathrm{Re}z<0.8}_{-0.3<\mathrm{Im}z<0.3}\right\}$.

FIGURE 6.1.3. Pictures of Example 6.1.3, where $c = 1$.

DEFINITION 6.2.1. Let M be a Riemann surface and $\widetilde{M}$ the universal cover of M. A holomorphic null immersion $F : \widetilde{M} \to SL(2,\mathbb{C})$ satisfies the $SU(1,1)$ *condition* if the monodromy matrix Φ_γ of F with respect to γ as in Equation (5.1.1) is in $SU(1,1)$ for any loop γ in M.

Consider the Riemann surface $M = \mathbb{C} \setminus \{0\}$, so M is a twice punctured sphere. Define

$$G = z, \qquad Q = c\frac{dz^2}{z^2},$$

where $c \in \mathbb{C} \setminus \{0\}$. We will take the value c so that a holomorphic null immersion satisfying (3.2.9) will satisfy the $SU(1,1)$ condition. To do so, we first set

$$\mu = \sqrt{1 - 4c}.$$

Then we assume that $\mu \in \{z \in \mathbb{C} \setminus \{0\} \,|\, \mathrm{Re}(z) \geq 0\} \setminus \{1\}$. Direct computation gives:

LEMMA 6.2.2. (1) *For any $\mu \neq 0$, that is, for any $c \neq 1/4$, define $g(z) := z^\mu$. Then g satisfies Equation* (2.2.12).
(2) *When $\mu = 0$, that is, when $c = 1/4$, define $g(z) := \log z$. Then g satisfies Equation* (2.2.12).

So the general solution for (2.2.12) is given by

$$g(z) = \begin{cases} P \star \log z & \text{if } \mu = 0, \\ P \star z^\mu & \text{otherwise} \end{cases} \quad \text{for any} \quad P \in SL(2,\mathbb{C}).$$

First we consider the $\mu \neq 0$ case. Let $\varrho : \widetilde{M} \to M$ be the universal cover of M, and $F : \widetilde{M} \to SL(2,\mathbb{C})$ the solution of (3.2.9) which corresponds to $g_0 = -dF_{12}/dF_{11} = z^\mu$. In this case, Lee and Yang listed every P so that FP^{-1} satisfies the $SU(1,1)$ condition as follows:

Proposition 6.2.3 ([**LY**, Theorem 4]).
(1) *For $\mu \in \mathbb{N} \setminus \{1\}$, FP^{-1} satisfies the $SU(1,1)$ condition for any $P \in SL(2,\mathbb{C})$.*
(2) *For $\mu \in \mathbb{R}^+ \setminus \mathbb{N}$, FP^{-1} satisfies the $SU(1,1)$ condition if and only if*

$$P = S \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} \quad \text{or} \quad P = S \begin{pmatrix} 0 & ie^s \\ -ie^{-s} & 0 \end{pmatrix}$$

for any $S \in SU(1,1)$ and $s \in \mathbb{R}$.
(3) *For $\mu \in (\mathbb{N} \setminus \{1\}) \oplus i(\mathbb{R} \setminus \{0\})$, FP^{-1} satisfies the $SU(1,1)$ condition if and only if*

$$P = \frac{1}{\sqrt{2}} S \begin{pmatrix} e^{-i\phi} & e^{i\phi} \\ -e^{-i\phi} & e^{i\phi} \end{pmatrix}$$

for any $S \in SU(1,1)$ and $\phi \in \mathbb{R}$.
(4) *For $\mu \in (\mathbb{R}^+ \setminus \mathbb{N}) \oplus i(\mathbb{R} \setminus \{0\})$, FP^{-1} does not satisfy the $SU(1,1)$ condition for any $P \in SL(2,\mathbb{C})$.*

Note that in the first two cases, the ends are elliptic (see Figures 6.2.1 and 6.2.2), and in the third case, the ends are hyperbolic (see Figure 6.2.4).

Next we consider the case $\mu = 0$, and prove the following proposition as in the forthcoming paper [**FRUYY**]:

Proposition 6.2.4 ([**FRUYY**]). *For the case $\mu = 0$, FP^{-1} satisfies the $SU(1,1)$ condition if and only if*

$$P = \frac{i}{\sqrt{\lambda^2 + 1}} S \begin{pmatrix} \sqrt{\lambda} & \pm\sqrt{\lambda}^{-1} \\ \pm\sqrt{\lambda} & -\lambda\sqrt{\lambda} \end{pmatrix}$$

for any $S \in SU(1,1)$ and $\lambda \in \mathbb{R}^+$.

Proof. Let $F : \widetilde{M} \to SL(2,\mathbb{C})$ be the solution of (3.2.14) which corresponds to $g_0 = -dF_{12}/dF_{11} = \log z$. Let $\gamma : [0,1] \to M$ be a once wrapped loop around an end and τ the deck transformation of $\widetilde{M}$ associated to the homotopy class of γ. Define Φ by $F\Phi^{-1} = F \circ \tau$. Since $g_0 = \log z$, $g_0 \circ \tau = g_0 + 2\pi i$, we have

$$\Phi = \pm \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}.$$

Since $(FP^{-1}) \circ \tau = F\Phi^{-1}P^{-1}$, finding $P \in SL(2,\mathbb{C})$ so that FP^{-1} satisfying the $SU(1,1)$ condition is equivalent to finding $P \in SL(2,\mathbb{C})$ so that

$$P^{-1}e_3(P^{-1})^* = \begin{pmatrix} 1 & -2\pi i \\ 0 & 1 \end{pmatrix} P^{-1}e_3(P^{-1})^* \begin{pmatrix} 1 & 0 \\ 2\pi i & 1 \end{pmatrix}.$$

Since $P^{-1}e_3(P^{-1})^*$ is a Hermitian matrix with determinant -1, we see that

$$P^{-1}e_3(P^{-1})^* = \begin{pmatrix} p & \pm 1 \\ \pm 1 & 0 \end{pmatrix} =: X, \quad p \in \mathbb{R}.$$

X is conjugate to a diagonal matrix as follows:

$$X = \Lambda \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda^{-1} \end{pmatrix} \Lambda^*, \quad \text{where} \quad \Lambda = \frac{-i}{\sqrt{\lambda^2+1}} \begin{pmatrix} \lambda & \pm 1 \\ \pm 1 & -\lambda \end{pmatrix} \in SU(2),$$

and $\lambda = (p + \sqrt{p^2+4})/2 \in \mathbb{R}^+$ is an eigenvalue of X (the other eigenvalue is written as $-\lambda^{-1}$). Thus if

$$P^{-1} = \Lambda \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda}^{-1} \end{pmatrix} S^{-1}$$

for any $S \in SU(1,1)$, that is, if

$$P = S \begin{pmatrix} \sqrt{\lambda}^{-1} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} \Lambda^* = \frac{i}{\sqrt{\lambda^2+1}} S \begin{pmatrix} \sqrt{\lambda} & \pm\sqrt{\lambda}^{-1} \\ \pm\sqrt{\lambda} & -\lambda\sqrt{\lambda} \end{pmatrix}$$

for any $S \in SU(1,1)$, then FP^{-1} has monodromy in $SU(1,1)$ about γ. Since $M = \mathbb{C} \setminus \{0\}$, it is sufficient to consider only this single loop γ, and so FP^{-1} satisfies the $SU(1,1)$ condition. □

Therefore, if we set

$$g = S \star \frac{\log z \pm \lambda^{-1}}{\pm \log z - \lambda} \quad \text{for any} \quad S \in SU(1,1) \quad \text{and} \quad \lambda \in \mathbb{R}^+,$$

we have a CMC 1 face defined on M itself with two ends. Also, since $(FP^{-1}) \circ \tau = F\Phi^{-1}P^{-1} = (FP^{-1})(P\Phi^{-1}P^{-1})$ and $P\Phi^{-1}P^{-1}$ is conjugate to $\mathcal{P}$ in (5.1.2), we see that these ends are parabolic. Furthermore, direct computation shows that $|g(z)| = 1$ if and only if

$$|z| = \exp\left(\pm\frac{\lambda - \lambda^{-1}}{2}\right),$$

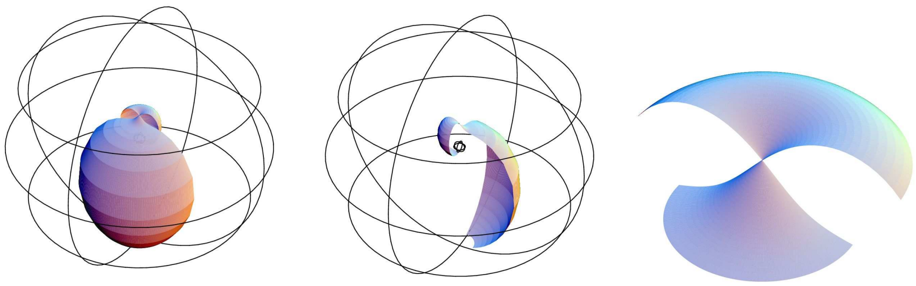
and hence the singular set is compact (see Figure 6.2.3).

Note that each example in this section, satisfying the $SU(1,1)$ condition, satisfies equality in the Osserman-type inequality.

6.3. Reducible CMC 1 faces

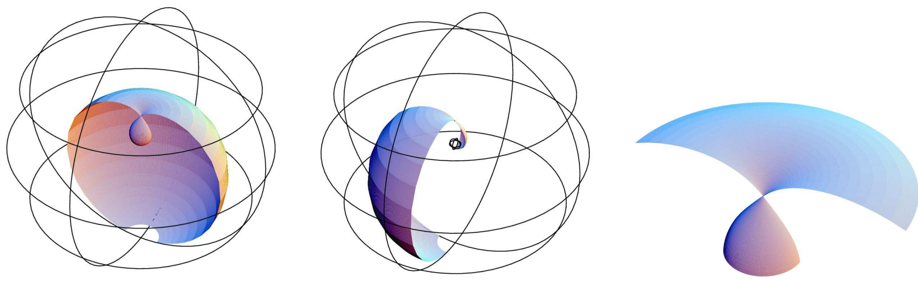
To produce further examples, we consider a relationship between CMC 1 faces and CMC 1 immersions in the hyperbolic space $\mathbb{H}^3$, and shall give a method for transferring from CMC 1 immersions in $\mathbb{H}^3$ to CMC 1 faces in $\mathbb{S}^3_1$.

Let $\hat{f} : M = \overline{M} \setminus \{p_1, \dots, p_n\} \to \mathbb{H}^3$ be a reducible CMC 1 immersion, whose first fundamental form $d\hat{s}^2$ has finite total curvature



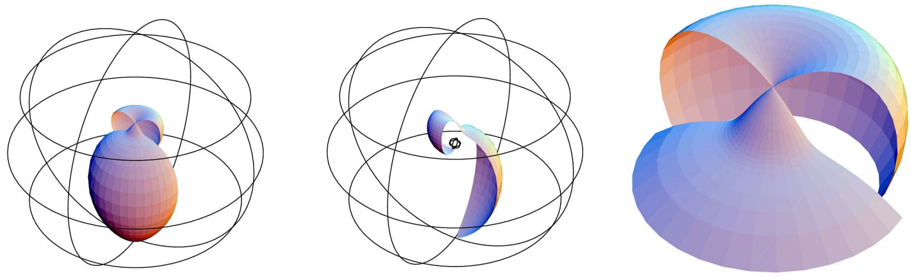
$\left\{z \in \mathbb{C} \,\middle|\, {e^{-5}<|z|<e^{5} \atop 0<\arg z<\pi}\right\}$. $\left\{z \in \mathbb{C} \,\middle|\, {e^{-5}<|z|<e^{5} \atop \pi<\arg z<(3/2)\pi}\right\}$. $\left\{z \in \mathbb{C} \,\middle|\, {e^{-2}<|z|<e^{2} \atop 0<\arg z<\pi}\right\}$.

FIGURE 6.2.1. Pictures of a CMC 1 face with two elliptic ends, where $\mu = 0.8$.



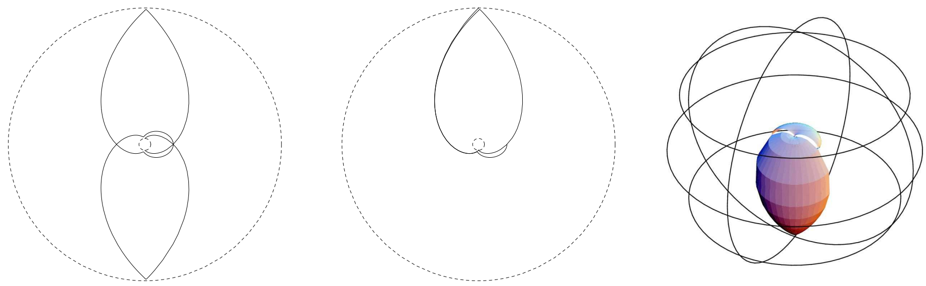
$\left\{z \in \mathbb{C} \,\middle|\, {e^{-5}<|z|<e^{5} \atop 0<\arg z<\pi}\right\}$. $\left\{z \in \mathbb{C} \,\middle|\, {e^{-5}<|z|<e^{5} \atop \pi<\arg z<(3/2)\pi}\right\}$. $\left\{z \in \mathbb{C} \,\middle|\, {e^{-2}<|z|<e^{2} \atop 0<\arg z<\pi}\right\}$.

FIGURE 6.2.2. Pictures of a CMC 1 face with two elliptic ends, where $\mu = 1.2$.



$\left\{z \in \mathbb{C} \,\middle|\, {e^{-5}<|z|<e^{5} \atop 0<\arg z<\pi}\right\}$. $\left\{z \in \mathbb{C} \,\middle|\, {e^{-5}<|z|<e^{5} \atop \pi<\arg z<(3/2)\pi}\right\}$. $\left\{z \in \mathbb{C} \,\middle|\, {e^{-2}<|z|<e^{2} \atop 0<\arg z<\pi}\right\}$.

FIGURE 6.2.3. Pictures of a CMC 1 face with two parabolic ends, where $\lambda = 1$.



$\{z \in \mathbb{R} \,|\, e^{-5\pi/2} < z < e^{5\pi/2}\}$. $\{z \in \mathbb{R} \,|\, 1 < z < e^{7\pi/2}\}$. $\left\{z \in \mathbb{C} \,\middle|\, {e^{-3\pi/2}<|z|<1 \atop 0<\arg z<\pi}\right\}$.

FIGURE 6.2.4. Pictures of a CMC 1 face with two hyperbolic ends and its profile curve, where $\mu = i$. In this case, the singular set is $\{|z| = e^{(m+1/2)\pi} \,|\, m \in \mathbb{Z}\}$, so the singular points accumulate to each end. We caution the reader that although the singular points might appear to be ends themselves (that is, appear to lie in the ideal boundary of the hollow ball model), they are in fact finite points (accumulating to the ends).

and is complete, where we define $\hat{f}$ to be *reducible* if there exists a holomorphic null lift F of $\hat{f}$ such that the image of the monodromy representation is in

$$U(1) = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \middle| \, \theta \in \mathbb{R} \right\}.$$

Let (g, ω) be the Weierstrass data associated to F, that is, (g, ω) satisfies

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega.$$

Then $|g|$ and $|\omega|$ are single-valued on M, as seen in the proof of Proposition 5.1.6. Assume that the absolute value of the secondary Gauss map is not equal to 1 at all ends $p_1, \dots, p_n$. Define $f := Fe_3F^*$. Then f is defined on M as well. Furthermore, we have the following proposition:

PROPOSITION 6.3.1. *The CMC 1 face $f : M \to \mathbb{S}^3_1$ defined as above, using $\hat{f}$ and its lift F with monodromy in $U(1)$, is complete and of finite type with only elliptic ends. Moreover, an end of $\hat{f}$ is embedded if and only if the corresponding end of f is embedded.*

PROOF. Fix an end p_j and assume $|g(p_j)| < 1$. Then we can take a neighborhood U_j such that $|g|^2 < 1 - \varepsilon$ holds on U_j, where $\varepsilon \in (0, 1)$ is a constant. In this case,

$$ds^2 = (1 - |g|^2)^2 \omega\bar{\omega} \geq \varepsilon^2 \omega\bar{\omega} \geq \frac{\varepsilon^2}{2}(1 + |g|^2)^2 \omega\bar{\omega} = \frac{\varepsilon^2}{2} d\hat{s}^2$$

holds on U_j. Since $d\hat{s}^2$ is complete at p_j, ds^2 is also complete. Moreover, the Gaussian curvatures K and $K_{d\hat{s}^2}$ satisfy

$$Kds^2 = \frac{4dgd\bar{g}}{(1-|g|^2)^2} \leq \left(\frac{2}{\varepsilon}-1\right)^2 \frac{4dgd\bar{g}}{(1+|g|^2)^2} = \left(\frac{2}{\varepsilon}-1\right)^2 (-K_{d\hat{s}^2})d\hat{s}^2.$$

Hence ds^2 is of finite total curvature at the end p_j.

On the other hand, if $|g(p_j)| > 1$, we can choose the neighborhood U_j such that $|g|^{-2} < 1-\varepsilon$ holds on U_j. Then

$$ds^2 = (1-|g|^{-2})^2|g|^4\omega\bar{\omega} \geq \varepsilon^2|g|^4\omega\bar{\omega} \geq \varepsilon^2(1+|g|^2)^2\omega\bar{\omega} = \varepsilon^2 d\hat{s}^2.$$

Hence ds^2 is complete at p_j. Moreover, since

$$Kds^2 = \frac{4dgd\bar{g}}{(1-|g|^2)^2} \leq \left(\frac{2}{\varepsilon}+1\right)^2 \frac{4dgd\bar{g}}{(1+|g|^2)^2} = \left(\frac{2}{\varepsilon}+1\right)^2 (-K_{d\hat{s}^2})d\hat{s}^2,$$

ds^2 is of finite total curvature. The proof of the final sentence of the proposition follows from the proof of Theorem 5.2.1, by showing $m_1 = 1$ for both f and $\hat{f}$. □

Moreover, [**UY1**, Theorem 3.3] shows that for each $\lambda \in \mathbb{R} \setminus \{0\}$, $(\lambda g, \lambda^{-1}\omega)$ induces a CMC 1 immersion $\hat{f}_\lambda : M \to \mathbb{H}^3$, where (g,ω) is stated in Proposition 6.3.1. Thus we have the following theorem:

THEOREM 6.3.2. *Let $\hat{f} : M \to \mathbb{H}^3$ be a reducible complete CMC 1 immersion of finite total curvature with n ends. Then there exists the holomorphic null lift F so that the image of the monodromy representation is in $U(1)$. Let (g,ω) be the Weierstrass data associated to F. Then there exist m $(0 \leq m \leq n)$ positive real numbers $\lambda_1, \dots, \lambda_m \in \mathbb{R}^+$ such that $f_\lambda : M \to \mathbb{S}^3_1$, induced from the Weierstrass data $(\lambda g, \lambda^{-1}\omega)$, is a complete CMC 1 face of finite type with only elliptic ends for any $\lambda \in \mathbb{R} \setminus \{0, \pm\lambda_1, \dots, \pm\lambda_m\}$.*

PROOF. Let $\hat{f}$, F and (g,ω) be as above and set $M = \overline{M}\setminus\{p_1,\dots,p_n\}$. Then by [**UY1**, Theorem 3.3], there exists a 1-parameter family of reducible complete CMC 1 immersions $\hat{f}_\lambda : M \to \mathbb{H}^3$ of finite total curvature with n ends. Define $\lambda_j \in \mathbb{R}^+ \cup \{0,\infty\}$ $(j = 1,\dots,m)$ as

$$\lambda_j = \begin{cases} 0 & \text{if } |g(p_j)| = \infty, \\ \infty & \text{if } |g(p_j)| = 0, \\ |g(p_j)|^{-1} & \text{otherwise.} \end{cases}$$

Then for any $\lambda \in (\mathbb{R}\cup\{\infty\})\setminus\{0, \pm\lambda_1, \dots, \pm\lambda_m\}$, $|\lambda g(p_j)| \neq 1$ for all p_j, and hence $f_\lambda : M \to \mathbb{S}^3_1$ induced from the Weierstrass data $(\lambda g, \lambda^{-1}\omega)$ is a complete CMC 1 face of finite type with only elliptic ends. □

Complete CMC 1 immersions with low total curvature and low dual total curvature in $\mathbb{H}^3$ were classified in [**RUY2, RUY3**]. Applying Theorem 6.3.2 to the reducible examples in their classification, we have the following:

Corollary 6.3.3. *There exist the following twelve types of complete CMC* 1 *faces* $f : M \to \mathbb{S}^3_1$ *of finite type with elliptic ends:*

$$\begin{array}{llll} \mathbf{O}(0), & \mathbf{O}(-5), & \mathbf{O}(-2,-3), & \mathbf{O}(-1,-1,-2), \\ \mathbf{O}(-4), & \mathbf{O}(-6), & \mathbf{O}(-2,-4), & \mathbf{O}(-1,-2,-2), \\ \mathbf{O}(-2,-2), & \mathbf{O}(-1,-4), & \mathbf{O}(-3,-3), & \mathbf{O}(-2,-2,-2), \end{array}$$

where f *is of* type $\mathbf{O}(d_1,\ldots,d_n)$ *when* $M = (\mathbb{C}\cup\{\infty\})\setminus\{p_1,\ldots,p_n\}$ *and* Q *has order* d_j *at each end* p_j*.*

Furthermore, reducible complete CMC 1 immersions of genus zero with an arbitrary number of regular ends and one irregular end and finite total curvature are constructed in [**MU**], using an analogue of the so-called UP-iteration. Applying Theorem 6.3.2 to their results, we have the following:

Corollary 6.3.4. *Set* $M = \mathbb{C}\setminus\{p_1,\ldots,p_n\}$ *for arbitrary* $n\in\mathbb{N}$*. Then there exist choices for* $p_1,\ldots,p_n$ *so that there exist complete CMC* 1 *faces* $f : M \to \mathbb{S}^3_1$ *of finite type with* n *regular elliptic ends and one irregular elliptic end.*

6.4. Examples of genus 1 with two ends

Consider the Riemann surface

(6.4.1)
$$M = \left\{ (z,w)\in(\mathbb{C}\cup\{\infty\})^2 \;\middle|\; w^2 = \frac{(z+1)(z-a)}{(z-1)(z+a)} \right\} \setminus \Big\{ (\infty,1),(\infty,-1) \Big\},$$

where $a > 1$. Then M is a twice punctured torus. Define

$$G = w, \qquad Q = \frac{c\,dz\,dw}{w} \tag{6.4.2}$$

for $c\in\mathbb{R}\setminus\{0\}$. The Riemannn surface M and (G,Q) are the Weierstrass data for a genus 1 catenoid. Let $F(z,w)\in SL(2,\mathbb{C})$ be the solution of Equation (3.2.14) with initial condition $F(0,1) = e_0$. Then $f = Fe_3F^*$ is a CMC 1 face in $\mathbb{S}^3_1$, and this CMC 1 face is defined on the universal cover $\widetilde{M}$ of M.

We do not yet know that f is well-defined on M itself. For this to happen, F must satisfy the $SU(1,1)$ condition. We satisfy the $SU(1,1)$ condition by changing the initial condition $F(0,1)$. It is enough to check the $SU(1,1)$ condition on the following three loops, since they generate the fundamental group of M (see Figure 6.4.1):

- The curve $\gamma_1 : [0,1]\to M$ starts at $\gamma_1(0) = (0,1)\in M$. Its first portion has z coordinate in the first quadrant of the z plane and ends at a point (z,w) where $z\in\mathbb{R}$ and $1 < z < a$. Its second portion starts at (z,w) and ends at $(0,-1)$ and has z coordinate in the fourth quadrant. Its third portion starts at $(0,-1)$ and ends at $(-z,1/w)$ and has z coordinate in the third quadrant. Its fourth and last portion starts at $(-z,1/w)$ and

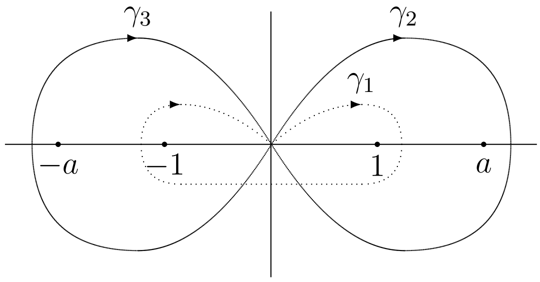


FIGURE 6.4.1. Projection on the z-plane of the curves γ_1, γ_2, and γ_3 which generate the fundamental group of M.

returns to the base point $\gamma_1(1) = (0, 1)$ and has z coordinate in the second quadrant.

- The curve $\gamma_2 : [0, 1] \to M$ starts at $\gamma_2(0) = (0, 1)$. Its first portion has z coordinate in the first quadrant and ends at a point (z, w) where $z \in \mathbb{R}$ and $z > a$. Its second and last portion starts at (z, w) and returns to $\gamma_2(1) = (0, 1)$ and has z coordinate in the fourth quadrant.
- The curve $\gamma_3 : [0, 1] \to M$ starts at $\gamma_3(0) = (0, 1)$. Its first portion has z coordinate in the third quadrant and ends at a point (z, w) where $z \in \mathbb{R}$ and $z < -a$. Its second and last portion starts at (z, w) and returns to $\gamma_3(1) = (0, 1)$ and has z coordinate in the second quadrant.

Let $\alpha_1 : [0, 1] \to M$ be a curve starting at $\alpha_1(0) = (0, 1)$ whose projection to the z-plane is an embedded curve in the first quadrant, and whose endpoint $\alpha_1(1)$ has a z coordinate so that $z \in \mathbb{R}$ and $1 < z < a$. Let $\alpha_2(t) : [0, 1] \to M$ be a curve starting at $\alpha_2(0) = (0, 1)$ whose projection to the z-plane is an embedded curve in the first quadrant, and whose endpoint $\alpha_2(1)$ has a z coordinate so that $z \in \mathbb{R}$ and $z > a$. With $F(0, 1) = e_0$, we solve Equation (3.2.14) along these two paths to find

$$F(\alpha_1(1)) = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \quad \text{and} \quad F(\alpha_2(1)) = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}.$$

Let τ_j be the deck transformation of $\widetilde{M}$ associated to the homotopy class of γ_j $(j = 1, 2, 3)$.

- Traveling about the loop γ_1, it follows from Lemmas 5.1 and 5.2 in [**RS**] that $F \circ \tau_1 = F\Phi_1$, where

$$\Phi_1 := \begin{pmatrix} \bar{A}_1 & -\bar{C}_1 \\ -\bar{B}_1 & \bar{D}_1 \end{pmatrix} \begin{pmatrix} D_1 & -C_1 \\ -B_1 & A_1 \end{pmatrix} \begin{pmatrix} \bar{D}_1 & \bar{B}_1 \\ \bar{C}_1 & \bar{A}_1 \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}.$$

- Traveling about the loop γ_2, it follows from Lemma 5.1 in [**RS**] that $F \circ \tau_2 = F\Phi_2$, where
$$\Phi_2 := \begin{pmatrix} \bar{D}_2 & -\bar{B}_2 \\ -\bar{C}_2 & \bar{A}_2 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}.$$
- Traveling about γ_3, $F \circ \tau_3 = F\Phi_3$, where
$$\Phi_3 := \begin{pmatrix} \bar{A}_2 & -\bar{C}_2 \\ -\bar{B}_2 & \bar{D}_2 \end{pmatrix} \begin{pmatrix} D_2 & C_2 \\ B_2 & A_2 \end{pmatrix}.$$

We now wish to change the initial condition from $F(0,1) = e_0$ to
$$F(0,1) = P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \in SL(2,\mathbb{C})$$
so that the $SU(1,1)$ conditions on all three loops γ_1, γ_2 and γ_3 will be solved. That is, we now find a constant matrix P so that
$$P^{-1}\Phi_1 P \quad \text{and} \quad P^{-1}\Phi_2 P \quad \text{and} \quad P^{-1}\Phi_3 P$$
are all in $SU(1,1)$.

To do this, we prepare several lemmas. First of all, we show the following two lemmas about the loops γ_2 and γ_3:

LEMMA 6.4.1. *Φ_2 and Φ_3 can be written as follows:*
$$\Phi_2 = \begin{pmatrix} \psi_{11} & i\psi_{12} \\ i\psi_{21} & \bar{\psi}_{11} \end{pmatrix}, \qquad \Phi_3 = \begin{pmatrix} \bar{\psi}_{11} & i\psi_{21} \\ i\psi_{12} & \psi_{11} \end{pmatrix},$$
where $\psi_{11} \in \mathbb{C}$ and $\psi_{12}, \psi_{21} \in \mathbb{R}$.

PROOF. By direct calculation and setting
$$\psi_{11} := A_2\bar{D}_2 - \bar{B}_2C_2, \quad i\psi_{12} := B_2\bar{D}_2 - \bar{B}_2D_2, \quad i\psi_{21} := \bar{A}_2C_2 - A_2\bar{C}_2,$$
we get the conclusion. □

Since $P \in SL(2,\mathbb{C})$, direct computation gives:

LEMMA 6.4.2. (1) *For $P^{-1}\Phi_2 P$ to be in $SU(1,1)$, we need*
(6.4.3)
$$\begin{cases} (P_{12}P_{21} - \overline{P_{12}P_{21}})(\psi_{11} - \bar{\psi}_{11}) \\ \qquad\qquad -(P_{11}P_{12} - \overline{P_{11}P_{12}})i\psi_{21} + (P_{21}P_{22} - \overline{P_{21}P_{22}})i\psi_{12} = 0, \\ (P_{11}P_{21} - \overline{P_{12}P_{22}})(\psi_{11} - \bar{\psi}_{11}) - (P_{11}^2 - \bar{P}_{12}^2)i\psi_{21} + (P_{21}^2 - \bar{P}_{22}^2)i\psi_{12} = 0. \end{cases}$$

(2) *For $P^{-1}\Phi_3 P$ to be in $SU(1,1)$, we need*
(6.4.4)
$$\begin{cases} (P_{12}P_{21} - \overline{P_{12}P_{21}})(\bar{\psi}_{11} - \psi_{11}) \\ \qquad\qquad +(P_{21}P_{22} - \overline{P_{21}P_{22}})i\psi_{21} - (P_{11}P_{12} - \overline{P_{11}P_{12}})i\psi_{12} = 0, \\ (P_{11}P_{21} - \overline{P_{12}P_{22}})(\bar{\psi}_{11} - \psi_{11}) + (P_{21}^2 - \bar{P}_{22}^2)i\psi_{21} - (P_{11}^2 - \bar{P}_{12}^2)i\psi_{12} = 0. \end{cases}$$

If

$$P_{11}P_{12} - \overline{P_{11}P_{12}} = P_{21}P_{22} - \overline{P_{21}P_{22}}, \tag{6.4.5}$$

$$P_{11}^2 - \bar{P}_{12}^2 = P_{21}^2 - \bar{P}_{22}^2 \tag{6.4.6}$$

hold, then Equations (6.4.3) and (6.4.4) are equivalent. But we do not want both $P_{11}^2 - \bar{P}_{12}^2$ and $P_{21}^2 - \bar{P}_{22}^2$ to be zero unless $P_{11}P_{21} - \overline{P_{12}P_{22}} = 0$.

Next, we show the following two lemmas about the loop γ_1:

LEMMA 6.4.3. *Φ_1 can be written as follows:*

$$\Phi_1 = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ -\bar{\varphi}_{12} & \varphi_{22} \end{pmatrix},$$

where $\varphi_{11}, \varphi_{22} \in \mathbb{R}$ and $\varphi_{12} \in \mathbb{C}$.

PROOF. By direct calculation and setting

$$\begin{aligned}
\varphi_{11} &:= |\bar{A}_1 D_1 + B_1 \bar{C}_1|^2 - (\bar{A}_1 C_1 + A_1 \bar{C}_1)^2, \\
\varphi_{22} &:= |\bar{A}_1 D_1 + B_1 \bar{C}_1|^2 - (\bar{B}_1 D_1 + B_1 \bar{D}_1)^2, \\
\varphi_{12} &:= (\bar{A}_1 D_1 + B_1 \bar{C}_1)(\bar{B}_1 D_1 + B_1 \bar{D}_1 - \bar{A}_1 C_1 - A_1 \bar{C}_1),
\end{aligned}$$

we get the conclusion. □

Direct computation gives:

LEMMA 6.4.4. *For $P^{-1}\Phi_1 P$ to be in $SU(1,1)$, we need*

$$\begin{cases} (\overline{P_{11}P_{22}} + P_{12}P_{21})\varphi_{11} - (P_{11}P_{22} + \overline{P_{12}P_{21}})\varphi_{22} \\ \qquad\qquad + (\overline{P_{11}P_{12}} + P_{21}P_{22})\varphi_{12} + (P_{11}P_{12} + \overline{P_{21}P_{22}})\bar{\varphi}_{12} = 0, \\ (P_{11}P_{21} + \overline{P_{12}P_{22}})(\varphi_{11} - \varphi_{22}) + (\bar{P}_{12}^2 + P_{21}^2)\varphi_{12} + (P_{11}^2 + \bar{P}_{22}^2)\bar{\varphi}_{12} = 0. \end{cases} \tag{6.4.7}$$

REMARK 6.4.5. Note that if we assume Equation (6.4.6), the second equation of (6.4.7) can be replaced by

$$(P_{11}P_{21} + \overline{P_{12}P_{22}})(\varphi_{11} - \varphi_{22}) + (P_{11}^2 + \bar{P}_{22}^2)(\varphi_{12} + \bar{\varphi}_{12}) = 0. \tag{6.4.8}$$

We set

$$P = P(\alpha, \beta) = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} \alpha & \varepsilon\beta \\ \alpha & -\varepsilon\beta \end{pmatrix}, \tag{6.4.9}$$

where $\alpha, \beta \in \mathbb{C}$ satisfy $\alpha\beta = -\varepsilon/2$, and ε is either $+1$ or -1. Then $\det P = 1$ and Equations (6.4.5) and (6.4.6) hold, and hence Equations (6.4.3) and (6.4.4) are equivalent. Furthermore, we see that the first equations of both (6.4.3) and (6.4.7) vanish. Thus Equations (6.4.3) and (6.4.4) reduce

$$(\alpha^2 + \bar{\beta}^2)(\psi_{11} - \bar{\psi}_{11}) + (\alpha^2 - \bar{\beta}^2)i(\psi_{12} - \psi_{21}) = 0 \tag{6.4.10}$$

and Equations (6.4.7) reduce to

$$(\alpha^2 - \bar{\beta}^2)(\varphi_{11} - \varphi_{22}) + (\alpha^2 + \bar{\beta}^2)(\varphi_{12} + \bar{\varphi}_{12}) = 0. \tag{6.4.11}$$

THEOREM 6.4.6. *Let* $(G,Q) = (w, cdzdw/w)$ *be the Weierstrass data on* M *defined as in* (6.4.1). *Let* $F : \widetilde{M} \to SL(2,\mathbb{C})$ *be the holomorphic null immersion so that* F *satisfies* (3.2.9) *with initial condition* $F(0,1) = P(\alpha,\beta)$ *as in* (6.4.9). *Then the following two conditions are equivalent:*

(1) F *satisfies the* $SU(1,1)$ *condition,*

(2) α *and* β *satisfy*

$$\begin{aligned} f_1 :&= -\frac{\bar{A}_1C_1 + A_1\bar{C}_1 + \bar{B}_1D_1 + B_1\bar{D}_1}{\bar{A}_1D_1 + A_1\bar{D}_1 + \bar{B}_1C_1 + B_1\bar{C}_1} \\ &= -\frac{\bar{A}_2C_2 - A_2\bar{C}_2 + \bar{B}_2D_2 - B_2\bar{D}_2}{\bar{A}_2D_2 - A_2\bar{D}_2 + \bar{B}_2C_2 - B_2\bar{C}_2} =: f_2 \end{aligned} \tag{6.4.12}$$

and the absolute value of this number is greater than 1.

PROOF. By (6.4.10), we have

$$\varepsilon\frac{\bar{\alpha}^2+\beta^2}{\bar{\alpha}^2-\beta^2} = -\frac{i(\psi_{12}-\psi_{21})}{\psi_{11}-\bar{\psi}_{11}} = -\frac{\bar{A}_2C_2 - A_2\bar{C}_2 + \bar{B}_2D_2 - B_2\bar{D}_2}{\bar{A}_2D_2 - A_2\bar{D}_2 + \bar{B}_2C_2 - B_2\bar{C}_2}.$$

Also, by (6.4.11), we have

$$\varepsilon\frac{\bar{\alpha}^2+\beta^2}{\bar{\alpha}^2-\beta^2} = -\frac{\varphi_{11}-\varphi_{22}}{\varphi_{12}+\bar{\varphi}_{12}} = -\frac{\bar{A}_1C_1 + A_1\bar{C}_1 + \bar{B}_1D_1 + B_1\bar{D}_1}{\bar{A}_1D_1 + A_1\bar{D}_1 + \bar{B}_1C_1 + B_1\bar{C}_1}.$$

Moreover, since $\alpha = -\varepsilon/2\beta$,

$$\varepsilon\frac{\bar{\alpha}^2+\beta^2}{\bar{\alpha}^2-\beta^2} = \varepsilon\frac{1+4|\beta|^4}{1-4|\beta|^4}$$

whose absolute value is greater than 1 for any $\beta \in \mathbb{C}$, proving the proposition. □

Therefore, if Equation (6.4.12) holds, we choose α and β and ε so that

$$f_1 = \varepsilon\frac{1+4|\beta|^4}{1-4|\beta|^4} = f_2$$

and then the $SU(1,1)$ condition is satisfied.

LEMMA 6.4.7. *If some* α, β *satisfy* (6.4.12), *we may assume* $\alpha,\beta \in \mathbb{R}$ *and that* (6.4.12) *still holds.*

PROOF. Since $\alpha\beta = -\varepsilon/2$, there exists $r > 0$ and $\theta \in [0,2\pi)$ so that

$$\alpha = re^{i\theta} \quad \text{and} \quad \beta = \frac{-\varepsilon}{2r}e^{-i\theta}.$$

Also, if $P^{-1}\Phi_jP \in SU(1,1)$ for $j = 1,2,3$, then $(PU)^{-1}\Phi_j(PU) \in SU(1,1)$ for $j = 1,2,3$. Thus, setting $U = \text{diag}(e^{-i\theta}, e^{i\theta})$, we see that

$$PU = \begin{pmatrix} re^{i\theta} & (-1/2r)e^{-i\theta} \\ re^{i\theta} & (1/2r)e^{-i\theta} \end{pmatrix}\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} = \begin{pmatrix} r & -1/2r \\ r & 1/2r \end{pmatrix}$$

and hence each entry of PU is real. $\square$

EXAMPLE 6.4.8. Now, in order to show the existence of a one-parameter family of weakly-complete CMC 1 faces of genus 1 with two ends which satisfy equality of the Osserman-type inequality, we find values $c \in \mathbb{R} \setminus \{0\}$ and $a > 1$ so that $|f_1| = |f_2| > 1$ and $f_1 = f_2$. By numerical experiments using Mathematica, we found such values (see Figure 6.4.2). Also, by numerical experiments with Mathematica, we see that the eigenvalues of $\Phi_3\Phi_2$ are in $\mathbb{S}^1$ for $c < 0$ (resp. $\mathbb{R} \setminus \{1\}$ for $c > 0$), so the ends are elliptic ends (resp. hyperbolic ends).

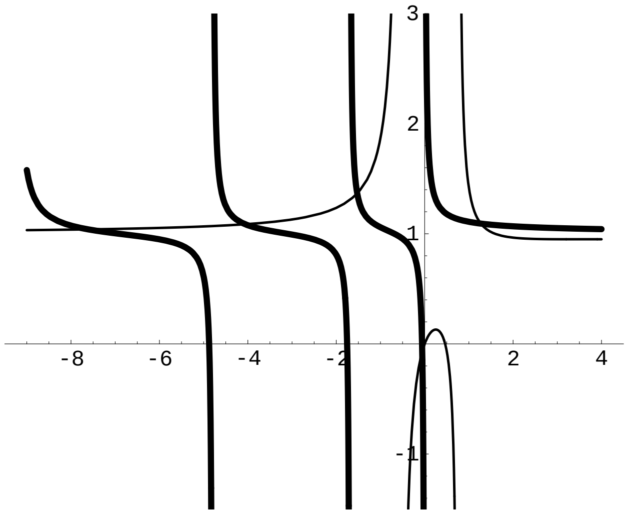


FIGURE 6.4.2. The function f_1 (thin curve) and f_2 (thick curve) when $a = 2$. The horizontal axis represents c, and the vertical axis represents f_1 and f_2. We see that f_1 and f_2 intersect 6 times for $c \in (-9, 4)$, at $c \approx -7.6119$, $c \approx -4.06015$, $c \approx -1.526035$, $c \approx -0.55$, $c \approx 1.26988$, and $f_1 = f_2 > 1$ except for $c \approx -0.55$.

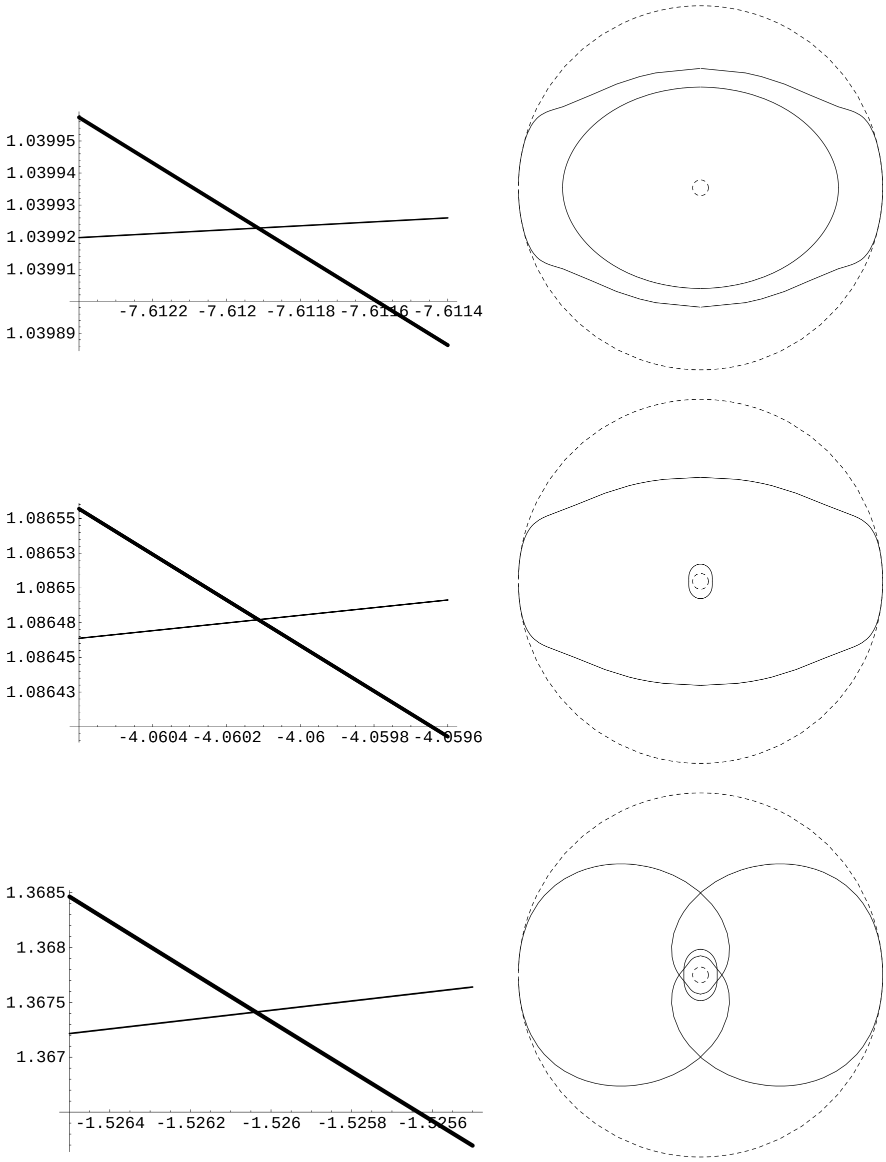


FIGURE 6.4.3. Left: The function f_1 (thin curve) and f_2 (thick curve) when $a = 2$. The horizontal axis represents c, and the vertical axis represents f_1 and f_2. We see that $f_1, f_2 > 1$ for $c \in (-7.6124, -7.6114)$ in the first row, $c \in (-4.0606, -4.0596)$ in the second row and $c \in (-1.5265, -1.5255)$ in the third row, and $f_1 = f_2$ at some such value of c in each case, and $a = 2 > 1$. Right: Symmetry curves in the CMC 1 face in Example 6.4.8 intersect the plane $\{(y_1, y_2, y_3) \in \mathscr{H} \mid y_2 = 0\}$, with $a = 2$ and $c = -7.6119$ (resp. $c = -4.06015$, $c = -1.526035$).

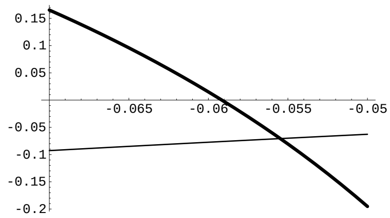


FIGURE 6.4.4. The function f_1 (thin curve) and f_2 (thick curve) when $a = 2$. The horizontal axis represents c, and the vertical axis represents f_1 and f_2. We see that $f_1 = f_2$ at some value of $c \in (-0.07, 0.05)$ but $|f_1| = |f_2| < 1$ at this value of c.

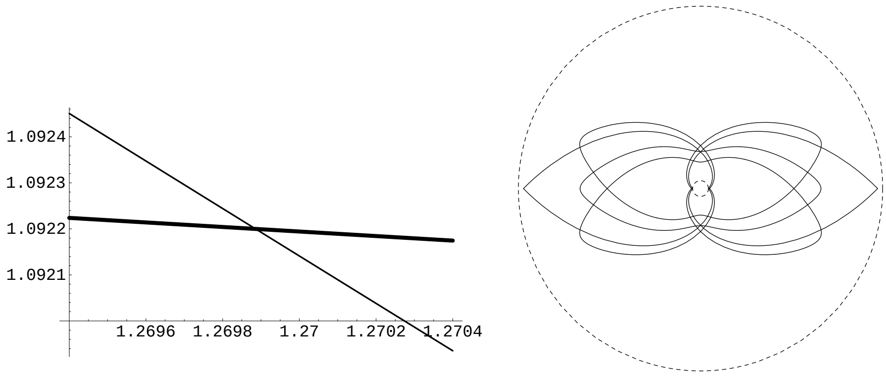


FIGURE 6.4.5. Left: The function f_1 (thin curve) and f_2 (thick curve) when $a = 2$. The horizontal axis represents c, and the vertical axis represents f_1 and f_2. We see that $f_1, f_2 > 1$ for $c \in (1.2694, 1.2704)$, and $f_1 = f_2$ at some such value of c, and $a = 2 > 1$. Right: Symmetry curves in the CMC 1 face in Example 6.4.8 intersect the plane $\{(y_1, y_2, y_3) \in \mathscr{H} \,|\, y_2 = 0\}$, with $a = 2$ and $c = 1.26988$.

APPENDIX A

Local surface theory in submanifolds of Lorentz 4-space

A.1. Local surface theory in Euclidean 3-space

Let $D \subset \mathbb{C}$ be a simply-connected domain (since we only study a local theory in this appendix, we always assume that D is simply-connected) with complex coordinate $z = x + iy$. Let $f : D \to \mathbb{R}^3$ an immersion. Without loss of generality we may assume f is conformal. Then there exists a smooth function $u : D \to \mathbb{R}$ so that

$$ds^2 = e^{2u} dz d\bar{z} = (dz, d\bar{z}) \mathrm{I} \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix},$$

where

$$\mathrm{I} = \begin{pmatrix} \langle f_z, f_z \rangle & \langle f_z, f_{\bar{z}} \rangle \\ \langle f_{\bar{z}}, f_z \rangle & \langle f_{\bar{z}}, f_{\bar{z}} \rangle \end{pmatrix} = \frac{1}{2} e^{2u} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Namely,

$$\langle f_z, f_z \rangle = \langle f_{\bar{z}}, f_z \rangle = 0 \quad \text{and} \quad \langle f_z, f_{\bar{z}} \rangle = \frac{1}{2} e^{2u},$$

where

$$f_z = \frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \quad \text{and} \quad f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right),$$

and $\langle \cdot, \cdot \rangle$ denotes the complex bilinear extension of the usual $T_p\mathbb{R}^3$ inner product for $p \in \mathbb{R}^3$. Note that for any vectors $\boldsymbol{v}, \boldsymbol{w} \in T_pD$,

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle = ds^2(\boldsymbol{v}, \boldsymbol{w}) = (v, \bar{v}) \mathrm{I} \begin{pmatrix} w \\ \bar{w} \end{pmatrix},$$

where $\boldsymbol{v} = v\partial_z + \bar{v}\partial_{\bar{z}}$ and $\boldsymbol{w} = w\partial_z + \bar{w}\partial_{\bar{z}}$. For each $p \in D$, $N(p) := (f_x(p) \times f_y(p))/\|f_x(p) \times f_y(p)\|$ defines a unit normal vector of f at p, where "$\times$" denotes the cross product of $T_p\mathbb{R}^3$. Then $N(p) \in T_p\mathbb{R}^3$ is orthogonal to the tangent plane $f_*(T_pD)$ of f at p. Note that $N : D \to \mathbb{S}^2$.

The second fundamental form h of f is defined by

$$h := -\langle df, dN \rangle = (dz, d\bar{z}) \mathrm{II} \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix},$$

where

$$\mathrm{I\!I} = \begin{pmatrix} \langle f_{zz}, N\rangle & \langle f_{z\bar{z}}, N\rangle \\ \langle f_{z\bar{z}}, N\rangle & \langle f_{\bar{z}\bar{z}}, N\rangle \end{pmatrix}.$$

Also, the shape operator S of f is defined by $S := \mathrm{I}^{-1}\mathrm{I\!I}$.

DEFINITION A.1.1. The mean curvature H of f and the Hopf differntial Q of f are defined as

$$H := \frac{1}{2}\mathrm{trace}S = 2e^{-2u}\langle f_{z\bar{z}}, N\rangle \quad \text{and} \quad Q = q dz^2 = \langle f_{zz}, N\rangle dz^2.$$

We set $\mathcal{F} := (f_z, f_{\bar{z}}, N)$ and call it the frame of f. Then we have

$$\mathcal{F}_z = \mathcal{F}\mathcal{U} \quad \text{and} \quad \mathcal{F}_{\bar{z}} = \mathcal{F}\mathcal{V},$$

where

$$\mathcal{U} = \begin{pmatrix} 2u_z & 0 & -H \\ 0 & 0 & -2e^{-2u}q \\ q & e^{2u}H/2 & 0 \end{pmatrix}$$

and

$$\mathcal{V} = \begin{pmatrix} 0 & 0 & -2e^{-2u}\bar{q} \\ 0 & 2u_{\bar{z}} & -H \\ e^{2u}H/2 & \bar{q} & 0 \end{pmatrix}.$$

These are equivalent to the following Gauss-Weingarten equations:

$$\begin{cases} f_{zz} = 2u_z f_z + qN, \\ f_{z\bar{z}} = \dfrac{1}{2}e^{2u}HN, \\ f_{\bar{z}\bar{z}} = 2u_{\bar{z}} f_{\bar{z}} + \bar{q}N, \end{cases} \quad \text{and} \quad \begin{cases} N_z = -Hf_z - 2e^{-2u}qf_{\bar{z}}, \\ N_{\bar{z}} = -2e^{-2u}\bar{q}f_z - Hf_{\bar{z}}. \end{cases}$$

Therefore

$$\mathrm{I\!I} = \begin{pmatrix} q & e^{2u}H/2 \\ e^{2u}H/2 & \bar{q} \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} H & 2e^{-2u}\bar{q} \\ 2e^{-2u}q & H \end{pmatrix},$$

and hence

$$h = q dz^2 + \bar{q} d\bar{z}^2 + e^{2u}H dz d\bar{z} = Q + \bar{Q} + H ds^2.$$

The Gaussian curvature K of f is defined by

$$K := \det S = H^2 - 4e^{-4u}q\bar{q}. \tag{A.1.1}$$

Note that for any vectors $\boldsymbol{v}, \boldsymbol{w} \in T_pD$,

$$h(\boldsymbol{v}, \boldsymbol{w}) = \langle S(\boldsymbol{v}), \boldsymbol{w}\rangle$$

holds, where

$$h(\boldsymbol{v}, \boldsymbol{w}) = (v, \bar{v})\mathrm{I\!I}\begin{pmatrix} w \\ \bar{w} \end{pmatrix},$$

($\boldsymbol{v} = v\partial_z + \bar{v}\partial_{\bar{z}}$, $\boldsymbol{w} = w\partial_z + \bar{w}\partial_{\bar{z}}$), and $S(\boldsymbol{v}) = \tilde{v}\partial_z + \bar{\tilde{v}}\partial_{\bar{z}}$, where

$$\begin{pmatrix} \tilde{v} \\ \bar{\tilde{v}} \end{pmatrix} = S\begin{pmatrix} v \\ \bar{v} \end{pmatrix}.$$

REMARK A.1.2. The third fundamental form $\langle dN, dN\rangle$ of f is given by

$$\langle dN, dN\rangle = (dz, d\bar{z})\mathrm{I\!I\!I}\begin{pmatrix} dz \\ d\bar{z}\end{pmatrix},$$

where

$$\begin{aligned}\mathrm{I\!I\!I} &= \begin{pmatrix}\langle N_z, N_z\rangle & \langle N_z, N_{\bar{z}}\rangle \\ \langle N_{\bar{z}}, N_z\rangle & \langle N_{\bar{z}}, N_{\bar{z}}\rangle\end{pmatrix} \\ &= \begin{pmatrix} 2Hq & e^{2u}H^2/2 + 2e^{-2u}q\bar{q} \\ e^{2u}H^2/2 + 2e^{-2u}q\bar{q} & 2H\bar{q}\end{pmatrix}.\end{aligned}$$

Then we have the following identity:

$$K\mathrm{I} - 2H\mathrm{I\!I} + \mathrm{I\!I\!I} = 0.$$

The Gauss-Codazzi equation, that is, the integrability condition $(\mathcal{F}_z)_{\bar{z}} = (\mathcal{F}_{\bar{z}})_z$, which is equivalent to

$$\mathcal{U}_{\bar{z}} - \mathcal{V}_z - [\mathcal{U}, \mathcal{V}] = 0, \tag{A.1.2}$$

has the following form:

$$2u_{z\bar{z}} - 2e^{-2u}q\bar{q} + \frac{1}{2}e^{2u}H^2 = 0, \tag{A.1.3}$$

$$q_{\bar{z}} = \frac{1}{2}e^{2u}H_z. \tag{A.1.4}$$

(A.1.3) is called the Gauss equation and (A.1.4) is called the Codazzi equation.

REMARK A.1.3. We set $\mathcal{A} := \mathcal{F}^{-1}d\mathcal{F}$. Then (A.1.2) is equivalent to

$$d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0.$$

$\mathcal{A}$ is called the Maurer-Cartan 1-form of $\mathcal{F}$.

By Equations (A.1.1) and (A.1.3), we have

$$K = -4e^{-2u}u_{z\bar{z}}. \tag{A.1.5}$$

Hence K depends only on u, that is, K is determined by only the first fundamental form, without requiring any knowledge of the second fundamental form, even though the definition of K uses the second fundamental form. This fact was first found by Gauss (in the year 1828) and he called it the *theorema egregium*, in Latin.

A.2. Local surface theory in hyperbolic 3-space

Hyperbolic 3-space is

$$\begin{aligned}\mathbb{H}^3 &= \mathbb{H}^3(-1) \\ &= \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1 \,|\, -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1,\ x_0 > 0\},\end{aligned}$$

with the metric induced from $\mathbb{R}^4_1$. $\mathbb{H}^3$ is a simply-connected complete 3-dimensional Riemannian manifold with constant sectional curvature -1.

Using the Hermitian matrix model as in Subsection 2.1.3, we can consider $\mathbb{R}^4_1$ to be the 2×2 self-adjoint matrices ($X^* = X$), by the identification as in (2.1.1). Then $\mathbb{H}^3$ is

$$\mathbb{H}^3 = \{X \mid X^* = X,\, \det X = 1,\, \mathrm{trace} X > 0\}$$

with the metric

$$\langle X, Y\rangle = -\frac{1}{2}\mathrm{trace}(Xe_2Y^te_2).$$

In particular, $|X|^2 = \langle X, X\rangle = -\det X$. In the same way as in Lemma 2.1.2, we can prove the following lemma:

LEMMA A.2.1. *$\mathbb{H}^3$ can be written as* $\mathbb{H}^3 = \{FF^* \mid F \in SL(2,\mathbb{C})\}$.

A.2.1. The Poincaré model. To visualize immersions in $\mathbb{H}^3$, there is a good model, called the *Poincaré model.* For any point

$$\begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \leftrightarrow (x_0, x_1, x_2, x_3) \in \mathbb{H}^3,$$

define

$$y_k = \frac{x_k}{1 + x_0}, \qquad k = 1, 2, 3.$$

Then $y_1^2 + y_2^2 + y_3^2 < 1$. The identification $(x_0, x_1, x_2, x_3) \leftrightarrow (y_1, y_2, y_3)$ is then a bijection from $\mathbb{H}^3$ to the unit open ball

$$D^3 = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1^2 + y_2^2 + y_3^2 < 1\}.$$

So $\mathbb{H}^3$ is identified with D^3. The metric induced on the Poincaré model from $\mathbb{H}^3$ is

$$ds^2 = \left(\frac{2}{1 - y_1^2 - y_2^2 - y_3^2}\right)^2 \sum_{k=1}^{3} dy_k^2.$$

A.2.2. Fundamental equations. Let $D \subset \mathbb{C}$ be a simply-connected domain with complex coordinate $z = x + iy$. Let $f : D \to \mathbb{H}^3$ an immersion. Without loss of generality we may assume f is conformal. Then there exists a smooth function $u : D \to \mathbb{R}$ so that

$$ds^2 = e^{2u}dzd\bar{z} = (dz, d\bar{z})\mathrm{I}\begin{pmatrix} dz \\ d\bar{z} \end{pmatrix},$$

where

$$\mathrm{I} = \begin{pmatrix} \langle f_z, f_z\rangle & \langle f_z, f_{\bar{z}}\rangle \\ \langle f_{\bar{z}}, f_z\rangle & \langle f_{\bar{z}}, f_{\bar{z}}\rangle \end{pmatrix} = \frac{1}{2}e^{2u}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

where $\langle\cdot,\cdot\rangle$ is extended linearly to the complexification of $T_p\mathbb{H}^3$ for $p \in \mathbb{H}^3$. For each $p \in D$, let $N(p)$ be a unit normal vector of f at p.

Then $N(p) \in T_p\mathbb{H}^3$ is orthogonal to the tangent plane $f_*(T_pD)$ of f at p. Then

$$N : D \to \mathbb{S}^3_1. \tag{A.2.1}$$

The second fundamental form h of f is defined by

$$h := -\langle df, dN\rangle = (dz, d\bar{z})\mathrm{I\!I}\begin{pmatrix} dz \\ d\bar{z}\end{pmatrix},$$

where

$$\mathrm{I\!I} = \begin{pmatrix} \langle f_{zz}, N\rangle & \langle f_{z\bar{z}}, N\rangle \\ \langle f_{z\bar{z}}, N\rangle & \langle f_{\bar{z}\bar{z}}, N\rangle \end{pmatrix}.$$

Also, the shape operator S of f is defined by $S := \mathrm{I}^{-1}\mathrm{I\!I}$. The mean curvature H of f and the Hopf differntial Q of f are defined as in Definition 2.2.1.

Let $\mathcal{F} := (f, f_z, f_{\bar{z}}, N)$ be the moving frame of f. Then we have

$$\mathcal{F}_z = \mathcal{F}\mathcal{U} \quad \text{and} \quad \mathcal{F}_{\bar{z}} = \mathcal{F}\mathcal{V},$$

where

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & e^{2u}/2 & 0 \\ 1 & 2u_z & 0 & -H \\ 0 & 0 & 0 & -2e^{-2u}q \\ 0 & q & e^{2u}H/2 & 0 \end{pmatrix}$$

and

$$\mathcal{V} = \begin{pmatrix} 0 & e^{2u}/2 & 0 & 0 \\ 0 & 0 & 0 & -2e^{-2u}\bar{q} \\ 1 & 0 & 2u_{\bar{z}} & -H \\ 0 & e^{2u}H/2 & \bar{q} & 0 \end{pmatrix}.$$

This is equivalent to the following Gauss-Weingarten equations:

$$\begin{cases} f_{zz} = 2u_z f_z + qN, \\ f_{z\bar{z}} = \dfrac{1}{2}e^{2u}f + \dfrac{1}{2}e^{2u}HN, \\ f_{\bar{z}\bar{z}} = 2u_{\bar{z}}f_{\bar{z}} + \bar{q}N, \end{cases} \quad \text{and} \quad \begin{cases} N_z = -Hf_z - 2e^{-2u}qf_{\bar{z}}, \\ N_{\bar{z}} = -2e^{-2u}\bar{q}f_z - Hf_{\bar{z}}. \end{cases}$$

Therefore

$$\mathrm{I\!I} = \begin{pmatrix} q & e^{2u}H/2 \\ e^{2u}H/2 & \bar{q} \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} H & 2e^{-2u}\bar{q} \\ 2e^{-2u}q & H \end{pmatrix},$$

and hence

$$h = q dz^2 + \bar{q}d\bar{z}^2 + e^{2u}Hdzd\bar{z} = Q + \bar{Q} + Hds^2.$$

The Gauss-Codazzi equation, that is, the integrability condition $(\mathcal{F}_z)_{\bar{z}} = (\mathcal{F}_{\bar{z}})_z$, which is equivalent to

$$\mathcal{U}_{\bar{z}} - \mathcal{V}_z - [\mathcal{U}, \mathcal{V}] = 0,$$

has the following form:

$$2u_{z\bar{z}} + \frac{1}{2}e^{2u}(H^2 - 1) - 2e^{-2u}q\bar{q} = 0, \tag{A.2.2}$$

$$H_z + 2e^{-2u}q_{\bar{z}} = 0. \tag{A.2.3}$$

(2.2.2) is called the Gauss equation and (2.2.3) is called the Codazzi equation.

The Gaussian curvature of $ds^2 = e^{2u}dzd\bar{z}$ is defined as $K = -4e^{-2u}u_{z\bar{z}}$ (See (A.1.5)), the Gauss equation (A.2.2) is written as

$$K = H^2 - 4e^{-4u}q\bar{q} - 1 = \det S - 1. \tag{A.2.4}$$

The third fundamental form $\langle dN, dN\rangle$ of f is given by

$$\langle dN, dN\rangle = (dz, d\bar{z})\mathrm{I\!I\!I}\begin{pmatrix} dz \\ d\bar{z}\end{pmatrix},$$

where

$$\begin{aligned}\mathrm{I\!I\!I} &= \begin{pmatrix} \langle N_z, N_z\rangle & \langle N_z, N_{\bar{z}}\rangle \\ \langle N_{\bar{z}}, N_z\rangle & \langle N_{\bar{z}}, N_{\bar{z}}\rangle \end{pmatrix} \\ &= \begin{pmatrix} 2Hq & e^{2u}H^2/2 + 2e^{-2u}q\bar{q} \\ e^{2u}H^2/2 + 2e^{-2u}q\bar{q} & 2H\bar{q}\end{pmatrix}.\end{aligned}$$

Then we have the following identity:

$$(K+1)\mathrm{I} - 2H\mathrm{I\!I} + \mathrm{I\!I\!I} = 0. \tag{A.2.5}$$

A.2.3. The 2 by 2 Lax pair for f. Now we use the Hermitian matrix model of $\mathbb{H}^3$. The following proposition is immediate:

PROPOSITION A.2.2. *If $F \in SL(2,\mathbb{C})$, then $\langle X, Y\rangle = \langle FXF^*, FYF^*\rangle$ for all X, Y in $\mathbb{H}^3$.*

We also have the following proposition:

PROPOSITION A.2.3. *There exists an $F \in SL(2,\mathbb{C})$ (unique up to sign $\pm F$) so that*

$$f = FF^*, \quad \frac{f_x}{|f_x|} = Fe_1F^*, \quad \frac{f_y}{|f_y|} = Fe_2F^*, \quad N = Fe_3F^*,$$

where $z = x + iy$.

Therefore, choosing F as in Proposition A.2.3, we have

$$f_x = e^uFe_1F^* \quad \text{and} \quad f_y = e^uFe_2F^*,$$

and so

$$f_z = e^uF\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}F^* \quad \text{and} \quad f_{\bar{z}} = e^uF\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}F^*.$$

Define the Lax pair U, V as

$$F^{-1}dF = Udz + Vd\bar{z}.$$

Then we have

$$U = \frac{1}{2}\begin{pmatrix} -u_z & e^u(1+H) \\ -2e^{-u}q & u_z \end{pmatrix} \quad \text{and} \quad V = \frac{1}{2}\begin{pmatrix} u_{\bar{z}} & 2e^{-u}\bar{q} \\ e^u(1-H) & -u_{\bar{z}} \end{pmatrix}.$$

A.3. Duality of surfaces in $\mathbb{H}^3$ and $\mathbb{S}^3_1$

This appendix is based on [**Kok2**]. As seen in Equations (2.2.1) and (A.2.1), immersions in $\mathbb{H}^3$ with singularities and spacelike immersions in $\mathbb{S}^3_1$ with singularities are related via their unit normal vector fields to each other. In this appendix we observe this correspondence more carefully.

Let $f : D \to \mathbb{H}^3$ be an immersion and $N : D \to \mathbb{S}^3_1$ its unit normal vector field. We denote by I, $\mathrm{I\!I}$ and $\mathrm{I\!I\!I}$ the first, second and third fundamental forms, respectively, that is,

$$\mathrm{I} = \langle df, df\rangle, \qquad \mathrm{I\!I} = -\langle df, dN\rangle, \qquad \mathrm{I\!I\!I} = \langle dN, dN\rangle.$$

(Note that we use I, $\mathrm{I\!I}$ and $\mathrm{I\!I\!I}$ as matrices in this thesis, but only in this appendix do we use these notations as forms.) Let K_f and H_f be the (intrinsic) Gaussian curvature and the mean curvature of f respectively. Then K_f and H_f satisfy

$$K_f = \det(\mathrm{I}^{-1}\mathrm{I\!I}) - 1, \qquad H_f = \frac{1}{2}\mathrm{trace}(\mathrm{I}^{-1}\mathrm{I\!I}).$$

Also, as seen in Equation (A.2.5), we have the following:

$$(K_f + 1)\,\mathrm{I} - 2H_f\,\mathrm{I\!I} + \mathrm{I\!I\!I} = 0.$$

Let K_N and H_N be the (intrinsic) Gaussian curvature and the mean curvature of N respectively. Then K_N and H_N satisfy

$$K_N = -\det(\mathrm{I\!I\!I}^{-1}\mathrm{I\!I}) + 1 = \frac{K_f}{K_f+1}, \qquad H_N = \frac{1}{2}\mathrm{trace}(\mathrm{I\!I\!I}^{-1}\mathrm{I\!I}) = \frac{H_f}{K_f+1}.$$

Thus we have the following proposition:

Proposition A.3.1. *For* $[\alpha : \beta] \in \mathbb{R}P^1$,

$$\alpha(H_f - 1) = \beta K_f \qquad \textit{if and only if} \qquad \alpha(H_N - 1) = (\beta - \alpha)K_N.$$

This proposition immediately implies the following:

Corollary A.3.2. (1) *Setting $\alpha = 0$, we see that f is flat if and only if N is flat.*

(2) *Setting $\alpha = 1$ and $\beta = 1/2$, we see that f has 1 as one of its principal curvatures if and only if N has 1 as one of its principal curvatures.*

(3) *Setting $\beta = 0$, we see that f is CMC 1 if and only if N is of harmonic mean curvature (HMC) 1, where N is HMC 1 if N satisfies $K_N = 1 - H_N$.*

(4) *Setting $\alpha = \beta = 1$, we see that f is HMC 1 if and only if N is CMC 1, where f is HMC 1 if f satisfies $K_f = H_f - 1$.*

REMARK A.3.3. HMC 1 immersions have the special geometric meaning that the harmonic mean value of the principal curvature is always 1. This is what motivates the terminology of HMC 1. See [**Kok1**].

By (4) in Corollary A.3.2, we see that the unit normal vector field of a CMC 1 face in $\mathbb{S}^3_1$ gives an HMC 1 immersion in $\mathbb{H}^3$ with singularities. Recently, Kokubu investigated HMC 1 surfaces in $\mathbb{H}^3$ with singularities [**Kok1**].

A.4. Local spacelike surface theory in Lorentz 3-space

Let $\mathbb{R}^3_1$ be the Lorentz 3-space with the Lorentz metric

$$\langle (x_0, x_1, x_2), (y_0, y_1, y_2) \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2.$$

Let $D \subset \mathbb{C}$ be a simply-connected domain with complex coordinate $z = x + iy$ and $f : D \to \mathbb{R}^3$ an spacelike immersion. Without loss of generality we may assume f is conformal. Then there exists a smooth function $u : D \to \mathbb{R}$ so that

$$ds^2 = e^{2u} dz d\bar{z} = (dz, d\bar{z}) \mathrm{I} \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix},$$

where

$$\mathrm{I} = \begin{pmatrix} \langle f_z, f_z \rangle & \langle f_z, f_{\bar{z}} \rangle \\ \langle f_{\bar{z}}, f_z \rangle & \langle f_{\bar{z}}, f_{\bar{z}} \rangle \end{pmatrix} = \frac{1}{2} e^{2u} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For each $p \in D$, let $N(p)$ be a unit normal vector of f at p. Then $N(p) \in T_p\mathbb{R}^3_1$ is orthogonal to the tangent plane $f_*(T_pD)$ of f at p. Note that N is timelike, that is, $\langle N, N \rangle = -1$, since f is spacelike. We choose N so that it is future pointing, that is, so that the first coordinate of N is positive. Then

(A.4.1)
$$N : D \to \mathbb{H}^2 := \{ n = (n_0, n_1, n_2) \in \mathbb{R}^3_1 \mid \langle n, n \rangle = -1 \ , \ \ n_0 > 0 \}.$$

The second fundamental form h of f is defined by

$$h := -\langle df, dN \rangle = (dz, d\bar{z}) \mathrm{II} \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix},$$

where

$$\mathrm{II} = \begin{pmatrix} \langle f_{zz}, N \rangle & \langle f_{z\bar{z}}, N \rangle \\ \langle f_{z\bar{z}}, N \rangle & \langle f_{\bar{z}\bar{z}}, N \rangle \end{pmatrix}.$$

Also, the shape operator S of f is defined by $S := \mathrm{I}^{-1}\mathrm{II}$. The mean curvature H of f and the Hopf differntial Q of f are defined as the same as in Definition 2.2.1.

We set $\mathcal{F} := (N, f_z, f_{\bar{z}})$ and call it the frame of f. Then we have

$$\mathcal{F}_z = \mathcal{F}\mathcal{U} \quad \text{and} \quad \mathcal{F}_{\bar{z}} = \mathcal{F}\mathcal{V},$$

where

$$\mathcal{U} = \begin{pmatrix} 0 & -q & -e^{2u}H/2 \\ -H & 2u_z & 0 \\ -2e^{-2u}q & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{V} = \begin{pmatrix} 0 & -e^{2u}H/2 & -\bar{q} \\ -2e^{-2u}\bar{q} & 0 & 0 \\ -H & 0 & 2u_{\bar{z}} \end{pmatrix}.$$

These are equivalent to the following Gauss-Weingarten equations:

$$\begin{cases} f_{zz} = 2u_z f_z - qN, \\ f_{z\bar{z}} = -\dfrac{1}{2}e^{2u}HN, \\ f_{\bar{z}\bar{z}} = 2u_{\bar{z}} f_{\bar{z}} - \bar{q}N, \end{cases} \quad \text{and} \quad \begin{cases} N_z = -Hf_z - 2e^{-2u}qf_{\bar{z}}, \\ N_{\bar{z}} = -2e^{-2u}\bar{q}f_z - Hf_{\bar{z}}. \end{cases}$$

Therefore

$$\mathrm{I\!I} = \begin{pmatrix} q & e^{2u}H/2 \\ e^{2u}H/2 & \bar{q} \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} H & 2e^{-2u}\bar{q} \\ 2e^{-2u}q & H \end{pmatrix},$$

and hence

$$h = qdz^2 + \bar{q}d\bar{z}^2 + e^{2u}Hdzd\bar{z} = Q + \bar{Q} + Hds^2.$$

The Gauss-Codazzi equation, that is, the integrability condition $(\mathcal{F}_z)_{\bar{z}} = (\mathcal{F}_{\bar{z}})_z$, which is equivalent to

$$\mathcal{U}_{\bar{z}} - \mathcal{V}_z - [\mathcal{U}, \mathcal{V}] = 0, \tag{A.4.2}$$

has the following form:

$$2u_{z\bar{z}} + 2e^{-2u}q\bar{q} - \frac{1}{2}e^{2u}H^2 = 0, \tag{A.4.3}$$

$$q_{\bar{z}} = \frac{1}{2}e^{2u}H_z. \tag{A.4.4}$$

(A.4.3) is called the Gauss equation and (A.4.4) is called the Codazzi equation.

The Gaussian curvature K of f is defined as $K = -4e^{-2u}u_{z\bar{z}}$ (see (A.1.5)). So the Gauss equation (A.4.3) is written as

$$K = -H^2 + 4e^{-4u}q\bar{q} = -\det S.$$

APPENDIX B

Further results

Here we describe further results in [**FSUY**].

B.1. Singularities of maxfaces

This appendix is based on Section 2 of [**FSUY**].

A holomorphic map $F = (F_1, F_2, F_3) : M \to \mathbb{C}^3$ of a Riemann surface M into the complex space form $\mathbb{C}^3$ is called *null* if $\sum_{j=1}^{3}(F_j)_z \cdot (F_j)_z$ vanishes. We consider two projections, the former is the projection into $\mathbb{R}^3$

$$p_E : \mathbb{C}^3 \ni (\zeta_1, \zeta_2, \zeta_3) \mapsto \mathrm{Re}(\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3,$$

and the latter one is the projection into $\mathbb{R}^3_1$

$$p_L : \mathbb{C}^3 \ni (\zeta_1, \zeta_2, \zeta_3) \mapsto \mathrm{Re}(-i\zeta_3, \zeta_1, \zeta_2) \in \mathbb{R}^3_1.$$

It is well-known that the projection of null holomorphic immersions into $\mathbb{R}^3$ by p_E gives conformal minimal immersions. Moreover, conformal minimal immersions are always given locally in such a manner.

On the other hand, the projection of null holomorphic immersions into $\mathbb{R}^3_1$ by p_L gives spacelike maximal surfaces with singularities, called *maxfaces* (see [**UY4**] for details). Moreover, [**UY4**] proves that maxfaces are all frontal maps and gives a necessary and sufficient condition for their singular points to be cuspidal edges and swallowtails. In this appendix, we shall give a necessary and sufficient condition for their singular points to be cuspidal cross caps and will show that generic singular points of maxfaces consist of cuspidal edges, swallowtails and cuspidal cross caps.

The following fact is known (see [**UY4**]):

Fact B.1.1. *Let $U \subset \mathbb{C}$ be a simply connected domain containing a base point z_0, and (g, ω) a pair of a meromorphic function and a holomorphic 1-form on U such that*

$$(1 + |g|^2)^2\omega\bar{\omega} \tag{B.1.1}$$

gives a Riemannian metric on U. Then

$$f(z) := \mathrm{Re}\int_{z_0}^{z}\bigl(-2g, 1 + g^2, i(1 - g^2)\bigr)\omega \tag{B.1.2}$$

gives a maxface in $\mathbb{R}^3_1$. Moreover, any maxfaces are locally obtained in this manner.

The first fundamental form (that is, the induced metric) of f in (B.1.2) is given by

$$ds^2 = (1-|g|^2)^2\omega\bar{\omega}.$$

In particular, $z \in U$ is a singular point of f if and only if $|g(z)| = 1$, and at $f \colon U \setminus \{|g| = 1\} \to \mathbb{R}^3_1$ is a spacelike maximal (that is, vanishing mean curvature) immersion. The meromorphic function g can be identified with the Lorentzian Gauss map. We call the pair (g, ω) the *Weierstrass data* of f. In [**UY4**], Umehara and Yamada proved that *f is a front on a neighborhood of a given singular point $z = p$ if and only if* $\mathrm{Re}\big(dg/(g^2\omega)\big) \neq 0$. Moreover, the following assertions are proved in [**UY4**]:

FACT B.1.2 ([**UY4**, Theorem 3.1]). *Let U be a domain of the complex plane $(\mathbb{C}, z)$ and $f : U \to \mathbb{R}^3_1$ a maxface constructed from the Weierstrass data $(g, \omega = \hat{\omega}\, dz)$, where $\hat{\omega}$ is a holomorphic function on U. Then f is a frontal map into $\mathbb{R}^3_1$ (which is identified with $\mathbb{R}^3$). Take an arbitrary point $p \in U$. Then p is a singular point of f if and only if $|g(p)| = 1$, and f is a front at a singular point p if and only if $\mathrm{Re}\big(g'/(g^2\hat{\omega})\big) \neq 0$ holds at p, where $' = d/dz$. Suppose now $\mathrm{Re}\big(g'/(g^2\hat{\omega})\big) \neq 0$ at a singular point p. Then*

(1) *f is $\mathcal{A}$-equivalent to a cuspidal edge at p if and only if*

$$\mathrm{Im}\left(\frac{g'}{g^2\hat{\omega}}\right) \neq 0,$$

and

(2) *f is $\mathcal{A}$-equivalent to a swallowtail at p if and only if*

$$\frac{g'}{g^2\hat{\omega}} \in \mathbb{R}\setminus\{0\} \qquad \text{and} \qquad \mathrm{Re}\left[\frac{g}{g'}\left(\frac{g'}{g^2\hat{\omega}}\right)'\right] \neq 0.$$

The statements of Theorem 3.1 of [**UY4**] are criteria to be *locally diffeomorphic* to a cuspidal edge or a swallowtail. However, in this case, local diffeomorphicity implies A-equivalency. See the appendix of [**KRSUY**]. We shall prove the following:

THEOREM B.1.3. *Let U be a domain of the complex plane $(\mathbb{C}, z)$ and $f \colon U \to \mathbb{R}^3_1$ a maxface constructed from the Weierstrass data $(g, \omega = \hat{\omega}\, dz)$, where $\hat{\omega}$ is a holomorphic function on U. Take an arbitrary singular point $p \in U$. Then f is $\mathcal{A}$-equivalent to a cuspidal cross cap at p if and only if*

$$\frac{g'}{g^2\hat{\omega}} \in i\mathbb{R}\setminus\{0\} \qquad \text{and} \qquad \mathrm{Im}\left[\frac{g}{g'}\left(\frac{g'}{g^2\hat{\omega}}\right)'\right] \neq 0,$$

where $' = d/dz$.

PROOF. We identify $\mathbb{R}^3_1$ with the Euclidean 3-space $\mathbb{R}^3$. Let f be a maxface as in (B.1.2). Then

$$N := \frac{1}{\sqrt{(1+|g|^2)^2+4|g|^2}}(1+|g|^2, 2\operatorname{Re} g, 2\operatorname{Im} g)$$

is the unit normal vector field of f with respect to the Euclidean metric of $\mathbb{R}^3$. Let $p \in U$ be a singular point of f, that is, $|g(p)| = 1$ holds. Since (B.1.1) gives a Riemannian metric on U, ω does not vanish at p. Here,

$$\lambda = \det(f_u, f_v, N) = (|g|^2-1)|\hat{\omega}|^2\sqrt{(1+|g|^2)^2+4|g|^2},$$

under the complex coordinate $z = u+iv$ on U. Then the singular point p is non-degenerate if and only if $dg \neq 0$.

The singular direction ξ and the null direction η are given by $\xi = i\overline{(g'/g)}$, and $\eta = i/(g\hat{\omega})$, respectively. Thus, we can parametrize the singular curve $\gamma(t)$ as

$$\dot{\gamma}(t) = i\overline{\left(\frac{g'}{g}\right)}\bigl(\gamma(t)\bigr) \qquad \left(\cdot = \frac{d}{dt}\right) \tag{B.1.3}$$

(see the proof of Theorem 3.1 in [**UY4**]). Then $\dot{\gamma}$ and η are transversal if and only if

$$\det(\xi,\eta) = \operatorname{Im}\bar{\xi}\eta = \operatorname{Im}\left(\frac{g'}{g^2\hat{\omega}}\right) \neq 0.$$

On the other hand, one can compute ψ as in Theorem 4.1.2 as

$$\psi = \det\bigl(f_*\dot{\gamma}, dN(\eta), N\bigr) = \operatorname{Re}\left(\frac{g'}{g^2\omega}\right)\cdot\psi_0,$$

where ψ_0 is a smooth function on a neighborhood of p such that $\psi_0(p) \neq 0$. Then the second condition of Theorem 4.1.2 is written as

$$\operatorname{Re}\left(\frac{g'}{g^2\hat{\omega}}\right) = 0 \qquad \text{and} \qquad \operatorname{Im}\left[\left(\frac{g'}{g^2\hat{\omega}}\right)'\overline{\left(\frac{g'}{g}\right)}\right] \neq 0.$$

Here, we used the relation $d/dt = i[\overline{(g'/g)}(\partial/\partial z) - (g'/g)(\partial/\partial\bar{z})]$, which comes from (B.1.3). Using the relation $\overline{(g'/g)}$ equals g/g' times a real valued function, we have the conclusion. □

EXAMPLE B.1.4. The Lorentzian Enneper surface is a maxface

$$f\colon \mathbb{C} \to \mathbb{R}^3_1$$

with the Weierstrass data $(g,\omega) = (z, dz)$ (see [**UY4**, Example 5.2]), whose set of singularities is $\{z \mid |z| = 1\}$. As pointed out in [**UY4**], Fact B.1.2 implies that the points of the set

$$\{z \mid |z| = 1\} \setminus \bigl\{\pm 1, \pm i, \pm e^{\pm i\pi/4}\bigr\}$$

are cuspidal edges and the points ± 1, $\pm i$ are swallowtails. Moreover, using Theorem B.1.3, we deduce that the four points $\pm e^{\pm i\pi/4}$ are cuspical cross caps.

We take a holomorphic function h defined on a simply connected domain $U \subset \mathbb{C}$. Then there is a maxface f_h with Weierstrass data $(g = e^h, \omega = dz)$, where z is a complex coordinate of U. Let $\mathcal{O}(U)$ be the set of holomorphic functions on U, which is endowed with the compact open C^∞-topology. Since the criteria for cuspidal edges, swallowtails and cuspidal cross caps in terms of (g, ω) are exactly the same as in the case of CMC 1 faces, we have the following:

COROLLARY B.1.5. *Let $U \subset \mathbb{C}$ be a simply connected domain and K an arbitrary compact set, and let $S(K)$ be the subset of $\mathcal{O}(U)$ consisting of $h \in \mathcal{O}(U)$ such that the singular points of f_h are cuspidal edges, swallowtails or cuspidal cross caps. Then $S(K)$ is an open and dense subset of $\mathcal{O}(U)$.*

B.2. A criterion for $5/2$-cusps

This appendix is based on Appendix A of [**FSUY**].

Let $I \subset \mathbb{R}$ be an open interval and $\gamma : I \to \mathbb{R}^2$ a C^∞-map. A point $t_0 \in I$ is called a *regular point* if the derivative $\gamma'(t_0)$ does not vanish. If $\gamma(t_0)$ is not a regular point, it is called a *singular point*. The singular point t_0 is called a $(2n+1)/2$-cusp ($n = 1, 2, 3, \dots$) if there exists a local diffeomorphism φ from $(\mathbb{R}, t_0)$ to $(\mathbb{R}, 0)$ and a local diffeomorphism Φ of $\bigl(\mathbb{R}^2, \gamma(t_0)\bigr)$ to $(\mathbb{R}^2, 0)$ such that

$$\Phi \circ \gamma \circ \varphi^{-1}(t) = (t^2, t^{2n+1}),$$

where by "a local diffeomorphism Φ from (M, p) to (N, q)", we mean a local diffeomorphism Φ with $\Phi(p) = q$.

In this appendix, we shall introduce a criterion for $3/2$-cusps and $5/2$-cusps. The former one is well-known, however, the latter one seems to be not so familiar.

PROPOSITION B.2.1. *A singular point $t_0 \in I$ of a plane curve $\gamma(t)$ is a $3/2$-cusp if and only if $\det\bigl(\gamma''(t_0), \gamma'''(t_0)\bigr)$ does not vanish.*

PROPOSITION B.2.2. *A singular point $t_0 \in I$ of a plane curve $\gamma(t)$ is a $5/2$-cusp if and only if $\det\bigl(\gamma''(t_0), \gamma'''(t_0)\bigr) = 0$ and*

(B.2.1)

$$3\gamma''(t_0) \det\bigl(\gamma''(t_0), \gamma^{(5)}(t_0)\bigr) - 10\gamma'''(t_0) \det\bigl(\gamma''(t_0), \gamma^{(4)}(t_0)\bigr) \neq 0.$$

Before proving these two propositions, we prove the following lemma:

LEMMA B.2.3. *Suppose that $\gamma(t) = \bigl(u(t), v(t)\bigr)$ has a singularity at $t = 0$ and*

$$u(0) = u'(0) = 0, \qquad v(0) = v'(0) = \cdots = v^{(2n)}(0) = 0.$$

Then $t = 0$ *is a* $(2n+1)/2$*-cusp if and only if* $u''(0) \neq 0$ *and* $v^{(2n+1)}(0) \neq 0$.

PROOF. By the Hadamard lemma, we can write

$$u(t) = t^2\tilde{u}(t), \qquad v(t) = t^{2n+1}\tilde{v}(t),$$

where $\tilde{u}(t)$ and $\tilde{v}(t)$ are smooth functions. Then $u''(0) \neq 0$ and $v^{(2n+1)}(0) \neq 0$ if and only if $\tilde{u}(0) \neq 0$ and $\tilde{v}(0) \neq 0$.

If $\tilde{u}(0) = 0$, we have $\gamma''(0) = 0$. In this case, $t = 0$ is obviously not a $(2n+1)/2$-cusp. So we may assume that $\tilde{u}(0) \neq 0$. Then $t \mapsto t\sqrt{\tilde{u}(t)}$ is a local diffeomorphism around the origin on $\mathbb{R}$, so we may replace $t\sqrt{u(t)}$ by t. Thus we have

$$\gamma(t) = \bigl(t^2, t^{2n+1}\tilde{v}(t)\bigr).$$

If $\tilde{v}(0) = 0$, then $t = 0$ is obviously not a $(2n+1)/2$-cusp, so we may assume that $\tilde{v}(0) \neq 0$. Now we set

$$a(t) = \frac{\tilde{v}(t) + \tilde{v}(-t)}{2}, \qquad b(t) = \frac{\tilde{v}(t) - \tilde{v}(-t)}{2}.$$

Then by the Whitney lemma, there exist C^∞-functions $\tilde{a}(t)$ and $\tilde{b}(t)$ such that $a(t) = \tilde{a}(t^2)$ and $b(t) = t\tilde{b}(t^2)$. Then we have

$$\gamma(t) = \bigl(t^2, t^{2n+1}\tilde{a}(t^2) + t^{2n+2}\tilde{b}(t^2)\bigr).$$

Since $\tilde{v}(0) \neq 0$, we have $\tilde{a}(0) \neq 0$. Then the map

$$\Phi : (x, y) \longmapsto \bigl(x, ya(x) + x^{n+1}b(x)\bigr)$$

gives a local diffeomorphism around the origin on $\mathbb{R}^2$, and $\Phi\circ(t^2, t^{2n+1}) = \gamma(t)$ holds. Thus the necessary and sufficient condition for $t = 0$ to be a $(2n+1)/2$-cusp is that $\tilde{u}(0) \neq 0$ and $\tilde{v}(0) \neq 0$. □

PROOF OF PROPOSITION B.2.1. We may assume that $t_0 = 0$ and $\gamma(0) = \gamma'(0) = 0$. Then we can write $\gamma(t) = t^2\bigl(a(t), b(t)\bigr)$ by Hadamard lemma, where $a(t)$ and $b(t)$ are smooth functions. If $b(0) = 0$, we can apply Lemma B.2.3 and $t = 0$ is a 3/2-cusp if and only if $a(0)b'(0) \neq 0$, which is equivalent to $\det\bigl(\gamma''(0), \gamma'''(0)\bigr) \neq 0$.

If $a(0) = 0$, we may switch the roles of the x-axis and the y-axis, and reduce to the case $b(0) = 0$. Thus we may assume that $a(0) \neq 0$. Consider a map

$$\Phi : \mathbb{R}^2 \ni (x, y) \longmapsto \left(x, y - \frac{b(0)}{a(0)}x\right) \in \mathbb{R}^2,$$

which gives a linear isomorphism on $\mathbb{R}^2$, and $\Phi \circ \gamma(t)$ satisfies the condition of Lemma B.2.3 and is a 3/2-cusp if and only if $a(0)b'(0) - a'(0)b(0) \neq 0$. Since $\det\bigl((\Phi \circ \gamma)''(0), (\Phi \circ \gamma)'''(0)\bigr) = 8(a(0)b'(0) - a'(0)b(0))$, we have the conclusion. □

Proof of Proposition B.2.2. As in the proof of Proposition B.2.1, we assume $t_0 = 0$, $\gamma(0) = \gamma'(0) = 0$, and write $\gamma(t) = t^2\big(a(t), b(t)\big)$. If the singular point is a $5/2$-cusp, then $\gamma''(0) \neq 0$. On the other hand, if $\gamma''(0) = 0$, then (B.2.1) does not hold. So without loss of generality, we may assume that $a(0) \neq 0$.

Consider a local diffeomorphism

$$\Phi : (x, y) \longmapsto \left(x, y - \frac{b(0)}{a(0)}x\right).$$

Then

$$\hat{\gamma}(t) := \Phi \circ \gamma(t) = t^2 \left(a(t), b(t) - \frac{b(0)}{a(0)}a(t)\right).$$

For the sake of simplicity, we set

$$\beta(t) := b(t) - \frac{b(0)}{a(0)}a(t). \tag{B.2.2}$$

By Proposition B.2.1, the condition $\det\big(\gamma''(0), \gamma'''(0)\big) = 0$ is independent of A-equivalency. So if the singular point is a $5/2$-cusp, then $\det\big(\hat{\gamma}''(0), \hat{\gamma}'''(0)\big) = 0$ and $\beta(0) = \beta'(0) = 0$. By the Hadamard lemma, there exists a C^∞-function $\tilde{\beta}(t)$ such that $\beta(t) = t^2\tilde{\beta}(t)$. Consider the coordinate change

$$\Psi : (x, y) \longmapsto \left(x, y - \frac{\tilde{\beta}(0)}{a(0)^2}x^2\right)$$

on $\mathbb{R}^2$ around the origin. Then

$$\Psi \circ \hat{\gamma}(t) = \left(t^2 a(t), t^4 (\tilde{\beta}(t) - \frac{\tilde{\beta}(0)}{a(0)^2}a(t)^2)\right).$$

Since $\tilde{\beta}(t) - \big(\tilde{\beta}(0)/a(0)^2\big)a(t)^2$ vanishes at $t = 0$, Lemma B.2.3 yields that $t = 0$ is a $5/2$-cusp if and only if

$$a(0)\tilde{\beta}'(0) - 2\tilde{\beta}(0)a'(0) \neq 0,$$

which is equivalent to (B.2.1), because we have by (B.2.2) that

$$\tilde{\beta}(0) = \frac{1}{2}\left(b''(0) - \frac{b(0)}{a(0)}a''(0)\right), \qquad \tilde{\beta}'(0) = \frac{1}{6}\left(b'''(0) - \frac{b(0)}{a(0)}a'''(0)\right),$$

and

$$\begin{aligned}
a(0) &= \frac{x''(0)}{2}, \quad a'(0) = \frac{x'''(0)}{6}, \quad a''(0) = \frac{x^{(4)}(0)}{12}, \quad a'''(0) = \frac{x^{(5)}(0)}{20}, \\
b(0) &= \frac{y''(0)}{2}, \quad b'(0) = \frac{y'''(0)}{6}, \quad b''(0) = \frac{y^{(4)}(0)}{12}, \quad b'''(0) = \frac{y^{(5)}(0)}{20},
\end{aligned}$$

where $\gamma(t) = (x(t), y(t))$. In fact, the first component of (B.2.1) is proportional to

$$\big(a(0)b'''(0) - a'''(0)b(0)\big)a(0) - \big(a(0)b''(0) - a''(0)b(0)\big)a'(0)$$

which is coincides with $(a(0)\tilde{\beta}'(0)-2\tilde{\beta}(0)a'(0))a(0)$. On the other hand the second component is proportional to

$$(a(0)b'''(0) - a'''(0)b(0))b(0) - 6(a(0)b''(0) - a''(0)b(0))b'(0).$$

Since $\det(\gamma''(0), \gamma'''(0)) = 0$, $(a(0), a'(0))$ is proportional to $(b(0), b'(0))$ and the second component is proportional to the first one. $\square$

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